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EXTREMAL PROBLEMS ON THE HYPERCUBE AND THE CODEGREE TURÁN DENSITY OF COMPLETE r-GRAPHS

ALEXANDER SIDORENKO*

Abstract. Let G be a finite abelian group, and r be a multiple of its exponent. The generalized Erdős–Ginzburg–Ziv constant $s_r(G)$ is the smallest integer s such that every sequence of length s over G has a zero-sum subsequence of length r. We show that $s_{2m}(\mathbb{Z}_2^d) \leq C_m 2^{d/m} + O(1)$ when $d \to \infty$, and $s_{2m}(\mathbb{Z}_2^d) \geq 2^{d/m} + 2m - 1$ when $d = km$. We use results on $s_r(G)$ to prove new bounds for the codegree Turán density of complete r-graphs.

Key words. Turán density, codegree, Sidon set, zero-sum subsequence, Erdős–Ginzburg–Ziv constant

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1. Introduction. In this paper, we consider three problems: the Sidon problem for $\mathbb{Z}_2^d$ (section 3), the generalized Erdős–Ginzburg–Ziv problem (section 4), and the codegree Turán problem for complete r-graphs (sections 2 and 5). Sections 3 and 4 can be read independently from the rest of the article. In the proof of Theorem 4.4, we use the notion of r-graphs. The necessary definitions and notation are given below.

An *r-graph* is a pair $H = (V(H), E(H))$ where $V(H)$ is a finite set of vertices, and the edge set $E(H)$ is a collection of r-subsets of $V(H)$. We denote $\mathrm{v}(H) = |V(H)|$ and $\mathrm{e}(H) = |E(H)|$. The *independence number* $\alpha(H)$ is the maximum size of a subset of $V(H)$ which contains no edges of H. The *degree* of a subset $A \subseteq V(H)$ is the number of edges of H which contain A. For $0 \leq l \leq r$, let $\Delta_l(H)$ denote the maximum degree among of l-subsets of $V(H)$. Notice that $\Delta_0(H) = \mathrm{e}(H)$ and

$$\frac{\Delta_0(H)}{\binom{n}{r}} \;\leq\; \frac{\Delta_1(H)}{\binom{n-1}{r-1}} \;\leq\; \cdots \;\leq\; \frac{\Delta_{r-1}(H)}{\binom{n-(r-1)}{1}} \,. \tag{1.1}$$

2. Codegree Turán density. The classical Turán number, $T(n,k,r)$, is the minimum number of edges in an n-vertex r-graph H with $\alpha(H) < k$. Correspondingly, $\binom{n}{r} - T(n,k,r)$ is the largest number of edges in an n-vertex r-graph that does not contain a complete subgraph on k vertices. There exists the limit $t(k,r) = \lim_{n\to\infty} T(n,k,r)/\binom{n}{r}$. The exact values of Turán numbers for $r = 2$ were found by Mantel [22] in the case $k = 3$, and by Turán [29] for all k. In particular, $t(k,2) = 1/(k-1)$. For $k > r > 2$, not a single value $t(k,r)$ is known. For details, see surveys [16, 26].

One of the ways to generalize the Turán numbers is

$$T_l(n,k,r) \;=\; \min\{\Delta_l(H) :\; \mathrm{v}(H) = n,\; \alpha(H) < k\} \,.$$

Notice that $T(n,k,r) = T_0(n,k,r)$. Lo and Markström [20] proved the existence of the limit $t_l(k,r) \;=\; \lim_{n\to\infty} T_l(n,k,r) \Big/ \binom{n-l}{r-l}$. Inequalities (1.1) imply

$$t_0(k,r) \;\leq\; t_1(k,r) \;\leq\; \ldots \;\leq\; t_{r-1}(k,r) \,.$$

The case $l = 1$ is known as a *Zarankiewicz type problem* (see [28, Chapter 3]), and $t_1(k,r) = t_0(k,r) = t(k,r)$. The problem of determining $t_l(k,r)$ has been studied

*sidorenko.ny@gmail.com.com

in [3, 8, 20, 23] in its complimentary form (see also Chapter 13.2 of survey [16]). In notation of [20], $t_l(k,r) = 1 - \pi_l\,(K_k^r)$. The case $l = r-1$ was first introduced by Mubayi and Zhao [23] under the name of *codegree density.* Lo and Markström [20] proved that for all $l = 1, 2, \ldots, r-1$,

$$t_l(k,r) \;\leq\; t_{l-1}(k-1, r-1)\;. \tag{2.1}$$

To simplify notation for the codegree density, we define $\tau(k,r) = t_{r-1}(k,r)$. The known upper bounds for $\tau(k,r)$ follow from (2.1) and upper bounds for the classical Turán density: $\tau(k,r) \leq 1/(k-r+1)$. Czygrinow and Nagle [3] conjectured that $\tau(4,3) = 1/2$. Lo and Markström [20] extended this conjecture to $\tau(r+1,r) = 1/2$.

We will prove upper bounds on the codegree density which are significantly better than $\tau(k,r) \leq 1/(k-r+1)$.

In sections 3 and 4 of this article, we study Sidon sets and zero-sum-free sequences in group $\mathbb{Z}_2^d$. The results of sections 3 and 4 are used in section 5 to obtain new upper bounds for $\tau(k,r)$ when $k-r$ is small. In particular, for $r = 3$, we prove

$$\tau(2a_d + 1, 3) \;\leq\; 3^{-d}\;, \tag{2.2}$$

where a_d is the maximum size of a cap in the affine geometry $AG(d,3)$ ($a_2 = 4,\ a_3 = 9,\ a_4 = 20,\ a_5 = 45,\ a_6 = 112$). For $r \geq 4$, we prove

$$\tau(r+2,r) \leq 1/4, \quad \tau(r+3,r) \leq 1/8, \quad \tau(r+5,r) \leq 1/16, \tag{2.3}$$

and in general,

$$\tau(r + b_d, r) \;\leq\; 2^{-d}\;, \tag{2.4}$$

where $b_d = \lfloor \left(2^{d+1} - 7/4\right)^{1/2} - 1/2 \rfloor$. Notice that $d = 1$ in (2.4) gives $\tau(r+1,r) \leq 1/2$ which is in line with the conjecture of Lo and Markström.

3. Sidon problem for $\mathbb{Z}_2^d$. A *Sidon set* A in an abelian group G is a set with the property that all pairwise sums of its elements are different (see [1]). If G is finite, let $\beta(G)$ denote the largest size of its Sidon set. Obviously, $\binom{\beta(G)}{2} \leq |G|$.

We denote by $\mathbb{Z}_k^d$ the group of d-dimensional vectors over $\mathbb{Z}_k$.

THEOREM 3.1.

$$\beta\left(\mathbb{Z}_2^d\right) \;\leq\; \sqrt{2^{d+1} - \frac{7}{4}} + \frac{1}{2}\;.$$

Proof. Let A be a Sidon set in $\mathbb{Z}_2^d$. Since two unequal elements can not have zero sum, $\binom{|A|}{2} \leq 2^d - 1$ which results in $|A| \leq (2^{d+1} - 7/4)^{1/2} + 1/2$. ∎

THEOREM 3.2 ([19]). *For even values of d,*

$$\beta\left(\mathbb{Z}_2^d\right) \;\geq\; 2^{d/2}\;.$$

THEOREM 3.3. $\beta\left(\mathbb{Z}_2^1\right) = 2,\ \beta\left(\mathbb{Z}_2^2\right) = 3,\ \beta\left(\mathbb{Z}_2^3\right) = 4,$ *and* $\beta\left(\mathbb{Z}_2^4\right) = 6.$

Proof. Let A_d be the set of vectors from $\mathbb{Z}_2^d$ with at most one non-zero component. This is a Sidon set, and $|A_d| = d+1$ provides a lower estimate for $d \leq 3$. For $d = 4$, A_4 with the addition of vector $(1,1,1,1)$ demonstrates that $\beta\left(\mathbb{Z}_2^4\right) \geq 6$. The matching upper bounds follow from Theorem 3.1. ∎

4. Zero-sum-free sequences in $\mathbb{Z}_2^d$. Let G be a finite abelian group with exponent $\exp(G)$ (that is the least common multiple of the orders of its elements). The Erdős–Ginzburg–Ziv constant $s(G)$ is the smallest integer s such that every sequence of length s over G has a zero-sum subsequence of length $\exp(G)$ (see [4, 5, 9, 12, 14, 17, 25]). In 1961, Erdős, Ginzburg, and Ziv [6] proved $s(\mathbb{Z}_k) = 2k-1$. Kemnitz' conjecture, $s(\mathbb{Z}_k^2) = 4k-3$ (see [17]), was open for more than twenty years and finally was proved by Reiher [25] in 2007.

Harborth [14] introduced constant $g(G)$ which is the smallest integer g such that every subset of size g in G contains $\exp(G)$ elements with zero sum. When $\exp(G) = 3$, the sum of three elements of G is zero if and only if they form an arithmetic progression. It is known that

$$s(\mathbb{Z}_3^d) \;=\; 2g(\mathbb{Z}_3^d) - 1\,, \tag{4.1}$$

and $a_d = g(\mathbb{Z}_3^d) - 1$ is the maximum size of a cap in the affine geometry $AG(d,3)$ (see [4]). The known exact values (see [4, 24]) are $a_2 = 4$, $a_3 = 9$, $a_4 = 20$, $a_5 = 45$, $a_6 = 112$. Ellenberg and Gijswijt [5] proved $g(\mathbb{Z}_3^d) - 1 \le \eta^d$, where $\eta = (3/8)\sqrt[3]{207+33\sqrt{33}} < 2.756$. Consequently,

$$s(\mathbb{Z}_3^d) \;\le\; 2\eta^d + 1\,. \tag{4.2}$$

The following generalization of the Erdős–Ginzburg–Ziv constant was introduced by Gao [12]. If r is a multiple of $\exp(G)$ then $s_r(G)$ denotes the smallest integer s such that every sequence of length s over G has a zero-sum subsequence of length r. (Notice that if r is not a multiple of $\exp(G)$ then there is an element $x \in G$ whose order is not a divisor of r, and the infinite sequence $x, x, x, \ldots$ contains no zero-sum subsequence of length r.) Obviously, $s_{\exp(G)}(G) = s(G)$. Constants $s_r(G)$ were studied in [2, 10, 11, 12, 13, 15, 18]. In the case when k is a power of a prime, Gao proved $s_{km}(\mathbb{Z}_k^d) = km + (k-1)d$ for $m \ge k^{d-1}$ (see [11, 18]) and conjectured that

$$s_{km}(\mathbb{Z}_k^d) \;=\; km + (k-1)d \quad \text{for } km > (k-1)d\,. \tag{4.3}$$

The Harborth constant $g(G)$ allows a similar generalization. We say that $A \subseteq G$ is a *zero-free set of rank* r if the sum of any r distinct elements of A is non-zero. When r is a multiple of $\exp(G)$, we denote the largest size of such set by $\beta_r(G)$. Obviously, $\beta_{\exp(G)}(G) = g(G) - 1$.

In section 3, we studied $\beta(\mathbb{Z}_2^d)$, the largest size of a Sidon set in $\mathbb{Z}_2^d$. It is easy to see that a zero-free set of rank 4 in $\mathbb{Z}_2^d$ is the same as a Sidon set. Hence, $\beta_4(\mathbb{Z}_2^d) = \beta(\mathbb{Z}_2^d)$. Note that a zero-free set of rank $2m$ in $\mathbb{Z}_2^d$, where $m \ge 3$, may contain different m-subsets with the same sum, for example, $x_1 + x_2 + x_3 = x_1 + x_4 + x_5$. Nevertheless, we will prove that both $\beta_{2m}(\mathbb{Z}_2^d)$ and $s_{2m}(\mathbb{Z}_2^d)$ are of order $2^{d/m}$ as $d \to \infty$.

Theorem 4.1.

$$s_{2m}(\mathbb{Z}_2^d) \;\le\; \beta_{2m}(\mathbb{Z}_2^d) \;+\; 2m - 1\,.$$

Proof. Consider a sequence S of length $\beta + 2m - 1$ over $\mathbb{Z}_2^d$ where $\beta = \beta_{2m}(\mathbb{Z}_2^d)$. We are going to show that S contains a zero-sum subsequence of size $2m$. For each $x \in \mathbb{Z}_2^d$, denote by $k(x)$ the number of appearances of x in S. Let B be the set of elements $x \in \mathbb{Z}_2^d$ such that $k(x) \ge 1$. If $|B| > \beta$, a zero-sum subsequence exists by the definition of $\beta_{2m}(\mathbb{Z}_2^d)$. We may assume $|B| \le \beta$. Let $k'(x)$ be the largest even number that does not exceed $k(x)$. Then

$$\sum_{x\in B} k'(x) \ge \sum_{x\in B}(k(x)-1) = \sum_{x\in B} k(x) - |B| = (\beta+2m-1) - |B| \ge 2m-1.$$

Since the values of $k'(x)$ are even, $\sum_{x\in B} k'(x) \ge 2m$. Select a set of even numbers $k''(x)$ such that $k''(x) \le k'(x)$ and $\sum_{x\in B} k''(x) = 2m$. Then $k''(x)$ appearances of every $x \in B$ in S constitute a zero-sum subsequence of length $2m$. ∎

From Theorems 3.1 and 4.1 we get

Corollary 4.2.

$$s_4(\mathbb{Z}_2^d) \ \le\ \sqrt{2^{d+1} - \frac{7}{4}} + \frac{7}{2}\ .$$

Theorem 4.3.

$$s_4(\mathbb{Z}_2^d) \ =\ \beta(\mathbb{Z}_2^d)\ +\ 3\ .$$

Proof. Let $A = \{x_1, x_2, \ldots, x_\beta\}$ be a Sidon set in $\mathbb{Z}_2^d$ where $\beta = \beta(\mathbb{Z}_2^d)$. Notice that in the sequence $x_1, x_2, \ldots, x_\beta$ all subsequences of size 2 and 4 have non-zero sums. Consider the sequence $x_1, x_2, \ldots, x_\beta, x_{\beta+1}, x_{\beta+2}$ where $x_{\beta+2} = x_{\beta+1} = x_\beta$. All 4-element subsequences of this sequence will have non-zero sums. Hence, $s_4(\mathbb{Z}_2^d) \ge \beta(\mathbb{Z}_2^d) + 3$. The opposite inequality follows from Theorem 4.1. ∎

Theorem 4.4. *For each m, there is a constant C_m such that*

$$\beta_{2m}(\mathbb{Z}_2^d) \ \le\ C_m 2^{d/m} + O(1) \ \text{ as }\ d \to \infty\ .$$

A subset of edges in an r-graph is called *independent* if they are pairwise disjoint. In order to prove Theorem 4.4, we need the following two lemmas.

Lemma 4.5. *If an r-graph H has no more than λ independent edges, then* $\mathrm{e}(H) \le \lambda \cdot (1 + r \cdot (\Delta_1(H) - 1))$.

Proof. We will use induction on λ. The basis for $\lambda = 0$ is trivial. Suppose, the statement of the lemma holds for $\lambda < k$. We will show that it holds for $\lambda = k$ as well. Select an arbitrary edge A in H and remove r vertices that form A together with all edges that intersect A. The resulting r-graph H_1 has no more than $k-1$ independent edges, hence $\mathrm{e}(H_1) \le (k-1)(1 + r \cdot (\Delta_1(H_1) - 1))$. The number of edges we have removed is at most $1 + r \cdot (\Delta_1(H) - 1)$, hence $\mathrm{e}(H) \le \mathrm{e}(H_1) + 1 + r \cdot (\Delta_1(H) - 1) \le k \cdot (1 + r \cdot (\Delta_1(H) - 1))$. ∎

Lemma 4.6 (The Erdős–Ko–Rado theorem [7]). *Let H be an r-graph with $n \ge 2r$ vertices. If every pair of edges in H has non-empty intersection, then* $\mathrm{e}(H) \le \binom{n-1}{r-1}$.

Proof of Theorem 4.4. For a subset $X \subset \mathbb{Z}_2^d$, let $\Sigma(X)$ denote the sum of its elements. Let $n = \beta_{2m}(\mathbb{Z}_2^d)$, and $A \subset \mathbb{Z}_2^d$ be a zero-free set of rank $2m$ and size n. For each $r = 2, 3, \ldots, m$, let $q(r)$ denote the integer $q \in \{0, 1, \ldots, r-1\}$ such that $m+q \equiv 0 \pmod{r}$. Denote $\lambda_r = 2(m + q(r))/r + 2r - q(r) - 3$ if $q(r) > 0$, and $\lambda_r = 2m/r - 1$ if $q(r) = 0$. It is easy to see that λ_r is a positive integer. We say that an r-subset $X \subseteq A$ is *exceptional* if $q(r) > 0$ and there exist r-subsets $X_1, X_2, \ldots, X_{\lambda_r} \subseteq A$ such that $X_1, X_2, \ldots, X_{\lambda_r}, X$ are pairwise disjoint and $\Sigma(X_1) = \Sigma(X_2) = \ldots = \Sigma(X_{\lambda_r}) = \Sigma(X)$.

Our first step will be to prove that if $q(r) > 0$ then two exceptional r-subsets can not have intersection of size $q(r)$. Indeed, let X and Y be exceptional r-subsets and $|X \cap Y| = q(r)$. There exist r-subsets $X_1, X_2, \ldots, X_\lambda$ and $Y_1, Y_2, \ldots, Y_\lambda$ such that $X_1, X_2, \ldots, X_\lambda, X$ are pairwise disjoint, $Y_1, Y_2, \ldots, Y_\lambda, Y$ are pairwise disjoint, $\Sigma(X_1) = \Sigma(X_2) = \ldots = \Sigma(X_\lambda) = \Sigma(X)$, and $\Sigma(Y_1) = \Sigma(Y_2) = \ldots = \Sigma(Y_\lambda) = \Sigma(Y)$, where $\lambda = \lambda_r$. It is possible that $\Sigma(X) = \Sigma(Y)$ and $X_i = Y_j$ for some i, j. Notice that $X - Y$ can intersect at most $r - q(r)$ subsets among $Y_1, Y_2, \ldots, Y_\lambda$. As $\lambda > r - q(r)$, there is an index j such that $X \cap Y_j = \emptyset$. Similarly, $Y - X$ can intersect at most $r - q(r)$ subsets among $X_1, X_2, \ldots, X_\lambda$. Also, Y_j can intersect at most r subsets among $X_1, X_2, \ldots, X_\lambda$. Since $\lambda - (r - q(r)) - r = 2k - 3$ with $k = (m + q(r))/r$, there exist $2k-3$ indices $1 \le i_1 < i_2 < \ldots < i_{2k-3} \le \lambda$ such that $\left(X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_{2k-3}}\right) \cap (Y \cup Y_j) = \emptyset$. Among $2k$ subsets $X, X_{i_1}, X_{i_2}, \ldots, X_{i_{2k-3}}, Y, Y_j$, the only pair with non-empty intersection is $\{X, Y\}$. Let

$$B \;=\; (X \cup X_{i_1} \cup X_{i_2} \cup \ldots \cup X_{i_{2k-3}} \cup Y \cup Y_j) \;-\; (X \cap Y)\,.$$

Then $|B| = 2kr - 2|X \cap Y| = 2kr - 2q(r) = 2m$ and $\Sigma(B) = (2k-2)\Sigma(X) + 2\Sigma(Y) - 2\Sigma(X \cap Y) = 0$ which contradicts the assumption that A is a zero-free set of rank $2m$.

Our second step is to obtain an upper bound on the number of exceptional r-subsets. Fix $B \subset A$ where $|B| = q(r) > 0$ and consider a family $\mathcal{F}_B$ of subsets $F \subset A - B$ such that $|F| = r - q(r)$ and $F \cup B$ is an exceptional r-subset. Then any two members of $\mathcal{F}_B$ must have non-empty intersection. Since $n = \beta_{2m}(\mathbb{Z}_2^d) \ge 2m - 1$ and $r < m$, we have $|A - B| = n - q(r) \ge 2(r - q(r))$. By Lemma 4.6, $|\mathcal{F}_B| \le \binom{|A-B|-1}{r-q(r)-1} = \binom{n-q(r)-1}{r-q(r)-1}$. Then the total number of exceptional r-subsets is at most

$$\binom{n}{q(r)}\binom{n-q(r)-1}{r-q(r)-1} \;=\; \frac{r-q(r)}{n-q(r)}\binom{n}{r}\binom{r}{q(r)}\,,$$

which is a polynomial in n of degree $r - 1$.

In the case $q(r) > 0$, let G_r denote an m-graph with vertex-set A where an m-subset $B \subseteq A$ is an edge if B contains an exceptional r-subset. Then $\mathrm{e}(G_r) \le \frac{r-q(r)}{n-q(r)}\binom{n}{r}\binom{r}{q(r)}\binom{n-r}{m-r}$. Denote

$$P_m(n) \;=\; \sum_{\substack{r=2\\ q(r)>0}}^{m-1} \frac{r-q(r)}{n-q(r)}\binom{n}{r}\binom{n-r}{m-r}\binom{r}{q(r)}\,.$$

Notice that $P_m(n)$ is a polynomial in n of degree at most $m-1$. For $r = 1, 2, \ldots, m$ and $z \in \mathbb{Z}_2^d$, we denote by $H_r(z)$ an r-graph with vertex set A whose edges are r-subsets X such that $\Sigma(X) = z$ and X does not contain an exceptional subset. Notice that

$$\sum_{z \in \mathbb{Z}_2^d} \mathrm{e}(H_m(z)) \;\ge\; \binom{n}{m} - \sum_{\substack{r=2\\ q(r)>0}}^{m-1} \mathrm{e}(G_r) \;\ge\; \binom{n}{m} \;-\; P_m(n)\,.$$

As the third step, we will obtain an upper bound on $\mathrm{e}(H_r(z))$. Let $N_1 = 1$ and $N_r = \lambda_r \cdot (1 + r \cdot (N_{r-1} - 1))$ for $r = 2, 3, \ldots, m$. We are going to prove $\mathrm{e}(H_r(z)) \le N_r$ for every $r \le m$ and every $z \in \mathbb{Z}_2^d$. We will use induction on r. The case $r = 1$ serves as the induction base. Indeed, $H_1(z)$ has either 1 edge (that is z itself) if $z \in A$, or no edges if $z \notin A$. Now we will prove the induction step from $r-1$ to r. Notice that the degree of vertex x in $H_r(z)$ is at most $\mathrm{e}(H_{r-1}(z + x)) \le N_{r-1}$. If

$q(r) > 0$ then $H_r(z)$ has no exceptional r-subset as its edge, hence, it has at most λ_r independent edges. If $q(r) = 0$, then r is a divisor of m, and $H_r(z)$ can not have $2m/r = \lambda_r + 1$ independent edges: their union would be an $(2m)$-subset with zero sum. We apply Lemma 4.5 to $H_r(z)$ with $\lambda = \lambda_r$ and $\Delta_1(H_r(z)) \le N_{r-1}$, to get $\mathrm{e}(H_r(z)) \le \lambda_r \cdot (1 + r \cdot (N_{r-1} - 1)) = N_r$.

We recall that $\sum_{z\in\mathbb{Z}_2^d} \mathrm{e}(H_m(z)) \ge \binom{n}{m} - P_m(n)$, where $P_m(n)$ is a polynomial of order less than m. On the other hand, $\mathrm{e}(H_m(z)) \le N_m$ for every $z \in \mathbb{Z}_2^d$. Therefore, $\binom{n}{m} - P_m(n) \le 2^d N_m$. Since $n = \beta_{2m}(\mathbb{Z}_2^d)$, we get $\beta_{2m}(\mathbb{Z}_2^d) \le \left(2^d m! N_m\right)^{1/m} + O(1)$ as $d \to \infty$. ∎

Remark 4.7. In the proof of Theorem 4.4, one may estimate $\lambda_r < 2\left(\frac{m}{r} + r\right)$, and hence, $(C_m)^m < m! \prod_{r=2}^m r\lambda_r < m! \prod_{r=2}^m 2(m + r^2)$. This implies $C_m = O(m^3)$ as $m \to \infty$. For small m, $C_3 = 60^{1/3} < 3.9149$ and $C_4 = 3288^{1/4} < 7.5724$.

The next result is a generalization of Theorem 3.2.

THEOREM 4.8. *If d is a multiple of m then*

$$\beta_{2m}(\mathbb{Z}_2^d) \;\ge\; 2^{d/m}\,, \qquad s_{2m}(\mathbb{Z}_2^d) \;\ge\; 2^{d/m} + 2m - 1\,.$$

Proof. Let $d = m \cdot k$. Since $\mathbb{Z}_2^k$ is the additive group of $\mathbb{GF}\left(2^k\right)$, the elements of $\mathbb{Z}_2^{mk}$ can be represented by vectors $(x_1, x_2, \ldots, x_m)$ where $x_i \in \mathbb{GF}\left(2^k\right)$. Let A be a set of 2^k vectors $(x, x^3, x^5, \ldots, x^{2m-1})$ where $x \in \mathbb{GF}\left(2^k\right)$. We are going to prove that A is a zero-free set of rank $2n$ for each $n = 1, 2, \ldots, m$. Indeed, suppose that $x_1, x_2, \ldots, x_{2n} \in \mathbb{GF}\left(2^k\right)$ and $\sum_{i=1}^{2n} (x_i)^r = 0$ for every odd $r \le 2n - 1$. We need to show that there are i, j such that $x_i = x_j$, $i \ne j$. As $\left(\sum_{i=1}^{2n} (x_i)^r\right)^2 = \sum_{i=1}^{2n} (x_i)^{2r}$, we get $\sum_{i=1}^{2n} (x_i)^r = 0$ for all $r \le 2n - 1$. Let $M = [M_{ij}]$ be a square matrix of order $2n$ over $\mathbb{GF}\left(2^k\right)$, where $M_{ij} = (x_i)^{j-1}$. Notice that $(1, 1, \ldots, 1) \cdot M = (0, 0, \ldots, 0)$, so $\det(M) = 0$ (where 0 and 1 are elements of $\mathbb{GF}\left(2^k\right)$). On the other hand, $\det(M) = \prod_{1\le i<j\le 2n} (x_i - x_j)$ which means that there are i, j such that $x_i = x_j$, $i \ne j$. We have proved by now that $\beta_{2m}(\mathbb{Z}_2^{mk}) \ge 2^k$.

To prove the lower bound for $s_{2m}(\mathbb{Z}_2^{mk})$, select an element $a \in A$ and consider a sequence S of length $2^k + 2m - 2$ where a appears $2m - 1$ times and each other element from A appears once. We claim that S does not contain a zero-sum subsequence of length $2m$. Indeed, suppose that such a subsequence S' exists, and let t be the number of appearances of a in it. Let $2s$ be the largest even number that does not exceed t. Let S'' be obtained from S by removing $2s$ copies of a. Then S'' is a zero-sum subsequence of length $2m - 2s$ which does not contain multiple copies of the same element. It contradicts with the fact that A is a zero-free set of rank $2(m - s)$. Therefore, $s_{2m}(\mathbb{Z}_2^{mk}) > 2^k + 2m - 2$. ∎

5. Bounds for codegree Turán densities. Let G be a finite abelian group, and r be a multiple of its exponent. In section 4, we defined $s_r(G)$ as the smallest integer s such that every sequence of length s over G contains a zero-sum subsequence of length r.

THEOREM 5.1. *If G is a finite abelian group and r is a multiple of* $\exp(G)$ *then*

$$\tau\big(s_r(G),\ r\big) \;\le\; \frac{1}{|G|}\,.$$

Proof. Let H_n be an r-graph with n vertices that are divided into $|G|$ baskets of almost equal sizes, each basket is associated with an element of G, and r vertices form an edge when the sum of their associated elements is zero. The degrees of all $(r-1)$-subsets of $V(H_n)$ are $|G|^{-1}n + O(1)$ as $n \to \infty$. By the definition of $s_r(G)$, any subset of vertices of size $s_r(G)$ contains an edge of H_n. ∎

Theorem 5.1 provides the strongest results when r is small and $|G|$ is large. The best cases are $G = \mathbb{Z}_3^d$ with $r = 3$, and $G = \mathbb{Z}_2^d$ with even values of r. When $G = \mathbb{Z}_2^d$ and $r = 4$, Theorem 5.1 and Corollary 4.2 yield

$$\tau\left(\left\lfloor\sqrt{2^{d+1} - \frac{7}{4}} + \frac{7}{2}\right\rfloor, 4\right) \leq 2^{-d}, \tag{5.1}$$

as well as

$$\tau(k,4) \leq 2k^{-2} + O\left(k^{-3}\right) \quad \text{as } k \to \infty .$$

By combining (2.1) and (5.1), we obtain (2.3) and (2.4). Theorems 4.1, 4.4 and 5.1, together with (2.1), yield for $r \geq 4$

$$\tau(k,r) \leq O\left(k^{-\lfloor r/2 \rfloor}\right) \quad \text{as } k \to \infty .$$

As $s_3(\mathbb{Z}_3^d) = s(\mathbb{Z}_3^d)$, Theorem 5.1 together with (4.1) yield (2.2). Theorem 5.1 together with (4.2) yield $\tau(k,3) \leq O\left(k^{-\ln(3)/\ln(\eta)}\right)$ where $\eta = (3/8)\sqrt[3]{207 + 33\sqrt{33}}$. As $\ln(3)/\ln(\eta) > 1.084$, it results in

$$\tau(k,3) = o\left(k^{-1.084}\right) \quad \text{as } k \to \infty .$$

Recently, Lo and Zhao [21] proved that for each $r \geq 3$,

$$c_1 \frac{\ln k}{k^{r-1}} \leq \tau(k,r) \leq c_2 \frac{\ln k}{k^{r-1}} \quad \text{as } k \to \infty . \tag{5.2}$$

The upper estimate in (5.2) is better than our asymptotic bounds. Nevertheless, in the case when $k - r$ is small, our bounds (2.2) and (2.3) are still better.

Very recently, Gao's conjecture (4.3) was proved in [27] for $k = 2$. As a consequence, we may derive from Theorem 5.1 that

$$\tau(r+d, r) \leq 2^{-d} \quad \text{for } r \geq 2\lceil d/2 \rceil .$$

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