# CONSTRUCTION OF 2-PEAKON SOLUTIONS AND ILL-POSEDNESS FOR THE NOVIKOV EQUATION 

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#### Abstract

For the Novikov equation, on both the line and the circle, we construct a 2-peakon solution with an asymmetric antipeakon-peakon initial profile whose $H^{s}$-norm for $s<3 / 2$ is arbitrarily small. Immediately after the initial time, both the antipeakon and peakon move in the positive direction, and a collision occurs in arbitrarily small time. Moreover, at the collision time the $H^{s}$-norm of the solution becomes arbitrarily large when $5 / 4<s<3 / 2$, thus resulting in norm inflation and ill-posedness. However, when $s<5 / 4$, the solution at the collision time coincides with a second solitary antipeakon solution. This scenario thus results in nonuniqueness and ill-posedness. Finally, when $s=5 / 4$ ill-posedness follows either from a failure of convergence or a failure of uniqueness. Considering that the Novikov equation is well-posed for $s>3 / 2$, these results put together establish $3 / 2$ as the critical index of well-posedness for this equation. The case $s=3 / 2$ remains an open question.


## 1. Introduction and Results

We consider the Cauchy problem for the Novikov equation (NE) on the line and the circle

$$
\begin{align*}
& u_{t}+u^{2} u_{x}+\partial_{x} D^{-2}\left[u^{3}+\frac{3}{2} u u_{x}^{2}\right]+D^{-2}\left[\frac{1}{2} u_{x}^{3}\right]=0,  \tag{1.1}\\
& u(x, 0)=u_{0}(x), \quad x \in \mathbb{R} \text { or } \mathbb{T}, \quad t \in \mathbb{R}, \tag{1.2}
\end{align*}
$$

where $D^{-2}$ is the Bessel potential $D^{-2}=\left(1-\partial_{x}^{2}\right)^{-1}$, and construct specific 2-peakon solutions $u(t)$ that collide at a finite time $T$ in such a way as to give rise to the phenomenon of norminflation. In particular, the norm-inflation generated by these 2-peakon collisions occur in Sobolev spaces $H^{s}$ with exponents between $5 / 4$ and $3 / 2$. As such, we will refer to $3 / 2$ as the critical exponent for well-posedness, as well-posedness has been proven for exponents greater than $3 / 2$ (see [HH2]). For exponents $s$ less than $5 / 4$, the collision of the 2-peakons in fact converges to a single antipeakon $u(T)$, which can be thought of as a superposition of both peakons. This scenario allows us to demonstrate non-uniqueness. Taken together, these results prove that NE is ill-posed in $H^{s}$ for $s<3 / 2$.

We recall that NE is well-posed in the sense of Hadamard (see [H]) in Sobolev spaces $H^{s}$ with exponents $s>3 / 2$ (see [HH2]). More precisely, if $u_{0}$ belongs to the Sobolev space $H^{s}$ on the circle or the line, then there exists $T_{s}=T_{s}\left(\left\|u_{0}\right\|_{H^{s}}\right)>0$ and a unique solution $u \in C\left(\left[0, T_{s}\right] ; H^{s}\right)$ of the Cauchy problem for the Novikov equation (1.1)-(1.2) satisfying the

[^0]following estimate
\[

$$
\begin{equation*}
\|u(t)\|_{H^{s}} \leq 2\left\|u_{0}\right\|_{H^{s}}, \quad \text { for } 0 \leq t \leq T_{s}, \quad \text { with } \quad T_{s}=\frac{1}{4 c_{s}\left\|u_{0}\right\|_{H^{s}}^{2}}, \tag{1.3}
\end{equation*}
$$

\]

where $c_{s}>0$ is a constant depending on $s$. Furthermore, the data-to-solution map $u(0) \mapsto u(t)$ is continuous but not uniformly continuous.

The Novikov equation is an integrable equation and its local form,

$$
\begin{equation*}
\left(1-\partial_{x}^{2}\right) u_{t}=u^{2} u_{x x x}+3 u u_{x} u_{x x}-4 u^{2} u_{x} \tag{1.4}
\end{equation*}
$$

was derived by Vladimir Novikov [ N ] in his attempt to classify all integrable Camassa-Holmtype equations with quadratic and cubic nonlinearities of the form $\left(1-\partial_{x}^{2}\right) u_{t}=P\left(u, u_{x}, u_{x x}, \ldots\right)$, where $P$ is a polynomial of $u$ and its derivatives. The Lax pair for NE was derived by Hone and Wang in $[\mathrm{HW}]$ and is given by the equations

$$
\left(\begin{array}{l}
\psi_{1}  \tag{1.5a}\\
\psi_{2} \\
\psi_{3}
\end{array}\right)_{x}=U(m, \lambda)\left(\begin{array}{l}
\psi_{1} \\
\psi_{2} \\
\psi_{3}
\end{array}\right), \quad\left(\begin{array}{l}
\psi_{1} \\
\psi_{2} \\
\psi_{3}
\end{array}\right)_{t}=V(m, u, \lambda)\left(\begin{array}{l}
\psi_{1} \\
\psi_{2} \\
\psi_{3}
\end{array}\right)
$$

where $m=u-u_{x x}$ and the matrices $U$ and $V$ are defined by

$$
U(m, \lambda)=\left(\begin{array}{rcr}
0 & \lambda m & 1  \tag{1.5b}\\
0 & 0 & \lambda m \\
1 & 0 & 0
\end{array}\right), V(m, u, \lambda)=\left(\begin{array}{lcr}
\frac{1}{3 \lambda^{2}}-u u_{x} & \frac{u_{x}}{\lambda}-\lambda m u^{2} & u_{x}^{2} \\
\frac{u}{\lambda} & -\frac{2}{3 \lambda^{2}} & -\frac{u_{x}}{\lambda}-\lambda m u^{2} \\
-u^{2} & \frac{u}{\lambda} & \frac{1}{3 \lambda^{2}}+u u_{x}
\end{array}\right)
$$

The Novikov equation possesses peakon traveling wave solutions [HM],[HLS], [GH], which on the real line are given by the formula

$$
\begin{equation*}
u(x, t)= \pm \sqrt{c} e^{-|x-c t|} \tag{1.6}
\end{equation*}
$$

where $c>0$ is the wave speed. On the circle, the peakon solutions are given by the formula

$$
\begin{equation*}
u(x, t)=\frac{\sqrt{c}}{\cosh (\pi)} \cosh \left([x-c t]_{p}-\pi\right), \quad \text { where } \quad[x-c t]_{p} \doteq x-c t-2 \pi\left[\frac{x-c t}{2 \pi}\right] \tag{1.7}
\end{equation*}
$$

In fact, the Novikov equation possess multi-peakon traveling wave solutions on both the line and the circle $[\mathrm{HM}],[\mathrm{HLS}],[\mathrm{GH}]$. More precisely, on the line the $n$-peakon,

$$
\begin{equation*}
u(x, t)=\sum_{j=1}^{n} p_{j}(t) e^{-\left|x-q_{j}(t)\right|} \tag{1.8}
\end{equation*}
$$

is a solution to NE if and only if the positions $\left(q_{1}, \cdots, q_{n}\right)$ and the momenta $\left(p_{1}, \cdots, p_{n}\right)$ satisfy the following system of $2 n$ differential equations

$$
\left\{\begin{align*}
\frac{d q_{j}}{d t} & =u^{2}\left(q_{j}\right)  \tag{1.9}\\
\frac{d p_{j}}{d t} & =-u\left(q_{j}\right) u_{x}\left(q_{j}\right) p_{j}
\end{align*}\right.
$$

The description of the periodic $n$-peakon is similar. Furthermore, NE solutions conserve the $H^{1}$-norm, that is

$$
\begin{equation*}
\int_{\mathbb{R} \text { or } \mathbb{T}}\left[u^{2}(t)+u_{x}^{2}(t)\right] d x=\int_{\mathbb{R} \text { or } \mathbb{T}}\left[u^{2}(0)+u_{x}^{2}(0)\right] d x . \tag{1.10}
\end{equation*}
$$

Next, we state our first result that gives the basic properties of the 2-peakon solutions, which are constructed here and are needed for proving the ill-posedness of NE below 3/2.

Theorem 1. For any $\varepsilon>0$ there exists a $T>0$ for which the NE Cauchy problem on the line and the circle (1.1)-(1.2) has a 2-peakon solution $u \in C\left([0, T] ; H^{s}\right)$ such that its lifespan and its initial size satisfy the estimates

$$
\begin{align*}
& \text { Lifespan }=T<\varepsilon,  \tag{1.11}\\
& \left\|u_{0}\right\|_{H^{s}}<\varepsilon \tag{1.12}
\end{align*}
$$

while as $t$ approaches the lifespan $T$ the $H^{s}$ norm of the solution $u(t)$ satisfies the estimates

$$
\lim _{t \rightarrow T}\|u(t)\|_{H^{s}}= \begin{cases}\infty \quad(\text { norm inflation }), & 5 / 4<s<3 / 2  \tag{1.13}\\ \text { may not exist, } & s=5 / 4 \\ C_{s}, \quad \text { for some } C_{s}>0, & s<5 / 4\end{cases}
$$

Moreover, when $s<5 / 4$ then $u(t)$ converges to an antipeakon $u(T)=-\sqrt{c_{T}} e^{-\left|x-q_{T}\right|}$, for some $c_{T}>0$ and $q_{T}>0$, with $\|u(T)\|_{H^{s}}=C_{s}$.

This theorem is a very interesting result in its own right. Unlike the Camassa-Holm (CH) equation (see [CH], [FF], [L1], [MN])

$$
\begin{equation*}
\left(1-\partial_{x}^{2}\right) u_{t}=u u_{x x x}+2 u_{x} u_{x x}-3 u u_{x} \tag{1.14}
\end{equation*}
$$

and the Degasperis-Procesi (DP) equation (see [DP], [HS], [LS], [L2], [DHH])

$$
\begin{equation*}
\left(1-\partial_{x}^{2}\right) u_{t}=u u_{x x x}+3 u_{x} u_{x x}-4 u u_{x}, \tag{1.15}
\end{equation*}
$$

for which we can construct special symmetric 2-peakon solutions, called peakon-antipeakons, of the form

$$
\begin{equation*}
u(x, t)=p(t) e^{-|x+q(t)|}-p(t) e^{-|x-q(t)|}, \tag{1.16}
\end{equation*}
$$

this is impossible for NE. Peakon-antipeakon solutions, which are convenient to work with, are possible for CH and DP because these equations contain a symmetry that allows us to reduce the corresponding to (1.9) ODE system for the positions and the momenta via $p=p_{1}=-p_{2}$ and $q=q_{1}=-q_{2}$. This symmetry causes the peak and antipeak to move against each other and collide in finite time (see [HHG], [HGH], [By]). Such a construction is not possible for NE because by equations (1.9) we have $\frac{d q_{j}}{d t} \geq 0$ for all positions $q_{j}$. Thus, we see that for NE all the peaks and antipeaks move in the same direction. Therefore collision can occur only if the peakon that follows moves faster than the one ahead of it, and eventually overtakes it. For this scenario to happen we must break symmetry and solve the full system of the four highly nonlinear differential equation defined by system (1.9) for $n=2$ with appropriate initial data. This procedure involves several novel ideas which are described in the Sections 2 and 7. The results are summarized in Theorems 3 and 6.

Next, using Theorem 1 we obtain the following ill-posedness result for NE.
Theorem 2. The Cauchy problem for the Novikov equation on the line and the circle (1.1)(1.2) is ill-posed in Sobolev spaces $H^{s}$ for $s<3 / 2$. More precisely, if $5 / 4<s<3 / 2$ then the data-to-solution map is not continuous while if $s<5 / 4$ then solution is not unique. When $s=5 / 4$ then either continuity or uniqueness fails.

As we have mentioned before, this theorem combined with the well-posedness result for NE in $H^{s}, s>3 / 2$, proved in [HH2], completes the well-posedness picture of NE in Sobolev spaces, except for $s=3 / 2$, which remains an open question. It is worth comparing the ill-posedness of NE, which has cubic nonlinearities, with those of CH and DP , the two integrable equations of the same type but with quadratic nonlinearities, which are both well-posed in $H^{s}, s>3 / 2$. Defining the "inflation index" to be the Sobolev exponent $s_{i}$ such that there is norm inflation (which implies discontinuity of the data-to-solution map) for all $s_{i}<s<3 / 2$, we have the following observations. For CH the inflation index $s_{i}=1$ and coincides with the index of the $H^{1}$-norm, which is the most important conserved quantity of CH. For $s<1$ the peakonantipeakon traveling wave solution (1.16) for CH converges in $H^{s}$ to $u(T)=0$ as $t$ approaches the collision time $T$, giving rise to another solution (namely the trivial solution) demonstrating ill-posedness due to failure of uniqueness. However, for DP the inflation index $s_{i}=1 / 2$. When $s<1 / 2$, then the corresponding peakon-antipeakon traveling wave solution for DP converges in $H^{s}$ to a function, which gives rise to another kind of DP solution, called "shock peakon" that results to failure of uniqueness (see [HHG]). From our results above we see that the the inflation index for NE is $5 / 4$, which is a very interesting number and which follows from the limiting behavior of the momenta $p_{1}(t)$ and $p_{2}(t)$ as $t$ approaches the collision time (see Theorem 3). For $s<5 / 4$ it is shown that the 2 -peakon solution (2.1) constructed in Section 2 converges in $H^{s}$ to an antipeakon, which gives rise to an antipeakon traveling wave solution demonstrating failure of uniqueness (see Proposition 5).

Finally, we mention that the method used here for proving ill-posedness for NE is similar to that used by many authors for other nonlinear evolution equations. For example, Bourgain and Pavlovic in $[\mathrm{BP}]$ proved ill-posedness for the 3D Navier-Stokes equations in Besov spaces in the sense of norm inflation. Similar methods for establishing ill-posedness for dispersive equations have been used by Kenig, Ponce and Vega [KPV] and Christ, Colliander and Tao [CCT]. The ill-posedness for the generalized KdV and nonlinear Schrödinger equations in Sobolev spaces has been tackled in [BKPSV]. The Euler equation in Sobolev spaces is examined in [BL1], where a norm-inflation result for the related vorticity equation provides the foundation for its ill-posedness. For the ill-posedness of the Burgers equation in $H^{3 / 2}$ we refer to Linares, Pilod and Saut [LPS]. For more results on traveling wave solutions, well-posedness and other analytic and geometric properties of nonlinear evolution equations we refer the reader to the following works and the references therein [BC], [BL2], [CHT], [CL], [CM], [DGH], [EEP], [ELY], [EY], [H], [HK], [HKM], [HMP], [KL], [KT], [LO], [Mc], [MST], [Ti], [W].

This paper is organized as follows. In Section 2, we construct the 2-peakon solutions on the line having the properties described in Theorem 1. We begin with the system of the four differential equations defined by (1.9) when $n=2$ and after making the change of the dependent variables $q=q_{2}-q_{1}, p=p_{2}-p_{1}, w=p_{2}+p_{1}$, and $z=p_{1} p_{2}$, we solve the resulting system and find explicit formulas for $p, w$ and $z$ in terms of $q$ (see Proposition 1). For $q=q(t)$ we obtain a rather complicated autonomous differential equation, which can be dominated by a simpler one for which we can prove, by a comparison argument, that $q$ becomes zero (collision) in finite time. Also, a precise estimate of the collision time is derived. This is contained in Proposition 2. In Section 3, we estimate the $H^{s}$-norm of the 2-peakon solutions constructed earlier (see Proposition 3), and in section 4 we choose the parameter appropriately so that
both the lifespan (collision time) and the size of the 2-peakon solution at the initial time are small. In Section 5, we prove norm-inflation and illposedness for $5 / 4<s<3 / 2$. Then, in Section 6 , we prove non-uniqueness for $s<5 / 4$ by showing that our 2-peakon solution $u(t)$ converges in $H^{s}$ to an antipeakon $u(T)$, which gives rise to a second solution for NE having the same initial data. Also, we explain the ill-posedness of NE for $s=5 / 4$. Finally, in Section 7 we prove our results on the circle. We use analogous arguments to those used on the line, with the necessary modifications to account for the periodic environment. A detailed outline of the periodic case can be found in Subsection 7.1.

## 2. Construction of 2-PEAKOn SOLUTIONS

It can be shown (see $[\mathrm{HW}],[\mathrm{GH}]$ ) that the 2 -peakon

$$
\begin{equation*}
u(x, t)=p_{1}(t) e^{-\left|x-q_{1}(t)\right|}+p_{2}(t) e^{-\left|x-q_{2}(t)\right|} \tag{2.1}
\end{equation*}
$$

is a solution of NE if the positions $q_{1}, q_{2}$ and the momenta $p_{1}, p_{2}$ satisfy the following system of the four differential equations

$$
\begin{align*}
& q_{1}^{\prime}=\left(p_{1}+p_{2} e^{-\left|q_{1}-q_{2}\right|}\right)^{2} \\
& q_{2}^{\prime}=\left(p_{1} e^{-\left|q_{2}-q_{1}\right|}+p_{2}\right)^{2}  \tag{2.2}\\
& p_{1}^{\prime}=p_{1} p_{2}\left(p_{1}+p_{2} e^{-\left|q_{1}-q_{2}\right|}\right) \cdot \operatorname{sgn}\left(q_{1}-q_{2}\right) e^{-\left|q_{1}-q_{2}\right|} \\
& p_{2}^{\prime}=p_{1} p_{2}\left(p_{1} e^{-\left|q_{2}-q_{1}\right|}+p_{2}\right) \cdot \operatorname{sgn}\left(q_{2}-q_{1}\right) e^{-\left|q_{2}-q_{1}\right|}
\end{align*}
$$

where $\operatorname{sgn}(x)$ is the standard sign function defined to be 1 if $x>0,-1$ if $x<0$, and 0 if $x=0$. At this point we make our first observation. Since $q_{1}^{\prime} \geq 0$ and $q_{2}^{\prime} \geq 0$, both position are increasing with time. Therefore we cannot have the "typical" peakon-antipeakon collision which is created from the peakon traveling in the positive direction and antipeakon traveling in the negative direction as observed in the cases of the CH and DP equations. Also, we note that by translation we may assume that the initial positions $q_{1}$ and $q_{2}$ are symmetric, that is

$$
\begin{equation*}
q_{1}(0)=-a \quad \text { and } \quad q_{2}(0)=a, \quad \text { for some } a>0 \tag{2.3}
\end{equation*}
$$

and, at least for a while, the difference of the positions is positive, that is

$$
\begin{equation*}
q(t)=q_{2}(t)-q_{1}(t)>0 \tag{2.4}
\end{equation*}
$$

Thus, the last system takes the following simpler form

$$
\begin{align*}
q_{1}^{\prime} & =\left(p_{1}+p_{2} e^{-q}\right)^{2} \\
q_{2}^{\prime} & =\left(p_{1} e^{-q}+p_{2}\right)^{2}  \tag{2.5}\\
p_{1}^{\prime} & =-p_{1} p_{2}\left(p_{1}+p_{2} e^{-q}\right) e^{-q} \\
p_{2}^{\prime} & =p_{1} p_{2}\left(p_{1} e^{-q}+p_{2}\right) e^{-q}
\end{align*}
$$

Furtheremore, we shall assume that at time $t=0$ the initial momenta are

$$
\begin{equation*}
p_{2}(0)=b \gg 1, \quad p_{1}(0)=-(b+\delta), \quad \delta>0 \tag{2.6}
\end{equation*}
$$

That is, the initial profile $u_{0}(x)=u(x, 0)$ is the following asymetric antipeakon-peakon

$$
\begin{equation*}
u_{0}(x)=-(b+\delta) e^{-|x+a|}+b e^{-|x-a|} \tag{2.7}
\end{equation*}
$$

which is displayed in the Figure 1.


Figure 1: Initial profile $u_{0}(x)$

Next, we shall solve the system of differential equations (2.5) with initial data the antipeakonpeakon (2.7) and prove that there is a collision in finite time. To demonstrate this claim, it is more convenient to work with the following new dependent variables

$$
\begin{align*}
q(t) & =q_{2}(t)-q_{1}(t), & q(0) & =2 a>0 \\
p(t) & =p_{2}(t)-p_{1}(t), & p(0) & =2 b+\delta>0 \\
w(t) & =p_{2}(t)+p_{1}(t), & w(0) & =-\delta<0  \tag{2.8}\\
z(t) & =p_{2}(t) \cdot p_{1}(t), & z(0) & =-b(b+\delta)<0
\end{align*}
$$

Deriving equations for $q, p, w$ and $z$. Subtracting the first equation of the system (2.5) from the second we have

$$
\begin{align*}
q^{\prime} & =\left(p_{1} e^{-q}+p_{2}\right)^{2}-\left(p_{1}+p_{2} e^{-q}\right)^{2} \\
& =\left(p_{2}-p_{1}\right)\left(p_{2}+p_{1}\right)\left(1-e^{-2 q}\right) \\
& =p w\left(1-e^{-2 q}\right) \tag{2.9}
\end{align*}
$$

Next, we shall try to form differential equations for $p$ and $w$ using the system (2.2). Assuming $p_{1}<0$ and $p_{2}>0$, at least for some time, for $p$ we have

$$
\begin{align*}
p^{\prime} & =p_{1} p_{2}\left(p_{1} e^{-q}+p_{2}\right) e^{-q}+p_{1} p_{2}\left(p_{1}+p_{2} e^{-q}\right) e^{-q} \\
& =p_{1} p_{2}\left(p_{2}+p_{1}\right) e^{-q}\left(1+e^{-q}\right) \\
& =z w e^{-q}\left(1+e^{-q}\right) \tag{2.10}
\end{align*}
$$

For $w$ we have

$$
\begin{align*}
w^{\prime} & =\left(p_{1} e^{-q}+p_{2}\right) e^{-q}-p_{1} p_{2}\left(p_{1}+p_{2} e^{-q}\right) e^{-q} \\
& =p_{1} p_{2}\left(p_{2}-p_{1}\right) e^{-q}\left(1-e^{-q}\right) \\
& =z p e^{-q}\left(1-e^{-q}\right) \tag{2.11}
\end{align*}
$$

Finally, for $z$ we have

$$
\begin{align*}
z^{\prime} & =p_{2}^{\prime} \cdot p_{1}+p_{2} \cdot p_{1}^{\prime} \\
& =p_{1}^{2} p_{2}\left(p_{1} e^{-q}+p_{2}\right) e^{-q}-p_{1} p_{2}^{2}\left(p_{1}+p_{2} e^{-q}\right) e^{-q} \\
& =p_{1} p_{2} e^{-q}\left[\left(p_{1}^{2}-p_{2}^{2}\right) e^{-q}\right] \\
& =-z w p e^{-2 q} \tag{2.12}
\end{align*}
$$

To summarize, we have the following system for $q, p, w$ and $z$

$$
\begin{align*}
& q^{\prime}=p w\left(1-e^{-2 q}\right), \quad q_{0}=q(0)=2 a>0, \\
& p^{\prime}=z w e^{-q}\left(1+e^{-q}\right), \quad p_{0}=p(0)=2 b+\delta>0, \\
& w^{\prime}=z p e^{-q}\left(1-e^{-q}\right), \quad w_{0}=w(0)=-\delta<0,  \tag{2.13}\\
& z^{\prime}=-z w p e^{-2 q}, \quad z_{0}=z(0)=-b(b+\delta)<0 .
\end{align*}
$$

In the following result we derive explicit formulas for $p, w$ and $z$ in terms of $q$. For $q$, we derive an autonomous differential equation, which in turn, is dominated by a simpler such equation.

Proposition 1 (Solutions of transformed 2-peakon system). The system of differential equations (2.13) has a unique smooth solution $(q(t), p(t), w(t), z(t))$ in an interval $[0, T)$, for some $T>0$, such that $z=z(t)$ is decreasing and in terms of $q$ is expressed by the formula

$$
\begin{equation*}
z=\frac{-z_{1}}{\left(1-e^{-2 q}\right)^{1 / 2}}<0, \quad \text { where } \quad z_{1}=b(b+\delta)\left(1-e^{-2 q_{0}}\right)^{1 / 2} \tag{2.14}
\end{equation*}
$$

$p=p(t)$ is decreasing and as a function of $q$ is expressed by the formula

$$
\begin{equation*}
p=\left(p_{0}^{2}+2 z_{1}\left[\frac{1+e^{-q}}{\sqrt{1-e^{-2 q}}}-\frac{1+e^{-q_{0}}}{\sqrt{1-e^{-2 q_{0}}}}\right]\right)^{1 / 2}>0 \tag{2.15}
\end{equation*}
$$

and $w=w(t)$ is decreasing and as a function of $q$ is expressed by the formula

$$
\begin{equation*}
w(t)=-\left(w_{0}^{2}+2 z_{1}\left[\frac{\sqrt{1-e^{-2 q_{0}}}}{1+e^{-q_{0}}}-\frac{\sqrt{1-e^{-2 q}}}{1+e^{-q}}\right]\right)^{1 / 2}<0 \tag{2.16}
\end{equation*}
$$

The difference of the positions $q=q(t)$ is decreasing and satisfies the initial value problem

$$
\begin{align*}
q^{\prime}=-f(q) \doteq- & \left(w_{0}^{2}+2 z_{1}\left[\frac{\sqrt{1-e^{-2 q_{0}}}}{1+e^{-q_{0}}}-\frac{\sqrt{1-e^{-2 q}}}{1+e^{-q}}\right]\right)^{\frac{1}{2}} \\
& \cdot\left(p_{0}^{2}+2 z_{1}\left[\frac{1+e^{-q}}{\sqrt{1-e^{-2 q}}}-\frac{1+e^{-q_{0}}}{\sqrt{1-e^{-2 q_{0}}}}\right]\right)^{\frac{1}{2}} \cdot\left(1-e^{-2 q}\right)  \tag{2.17}\\
q(0)=q_{0}=2 a> & 0
\end{align*}
$$

Furthermore, the initial value problem (2.17) for $q$ is dominated by the simpler initial value problem

$$
\begin{equation*}
q^{\prime}=-g(q) \doteq-q_{1}\left(1-e^{-2 q}\right)^{3 / 4}, \quad 0<q(0)=2 a<1 / 2 \tag{2.18}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{1}=\delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4} \tag{2.19}
\end{equation*}
$$

Proof. We begin by expressing $z$ in terms of $q$. Using the equation for $z^{\prime}$ and $q^{\prime}$, we find

$$
\frac{z^{\prime}}{q^{\prime}}=\frac{-z p w e^{-2 q}}{p w\left(1-e^{-2 q}\right)}, \quad \text { or } \quad \frac{z^{\prime}}{z}=\frac{-e^{-2 q} q^{\prime}}{\left(1-e^{-2 q}\right)}
$$

Since $z(0)<0$ we shall assume that $z(t)$ will remain negative. Therefore, from the last relation we have

$$
\frac{d}{d t}[\ln (-z)]=-\frac{1}{2} \frac{d}{d t}\left[\ln \left(1-e^{-2 q}\right)\right] .
$$

Integrating from 0 to $t$ gives

$$
\ln \left[\frac{z(t)}{z_{0}}\right]=-\frac{1}{2} \ln \left[\frac{1-e^{-2 q}}{1-e^{-2 q_{0}}}\right] .
$$

Finally, solving for $z$ gives formula (2.14), which expresses $z$ in terms of $q$.
Next we express $p$ in terms of $q$. For this we divide the equation for $p^{\prime}$ by the equation for $q^{\prime}$ and we get

$$
\frac{p^{\prime}}{q^{\prime}}=\frac{z w e^{-q}\left(1+e^{-q}\right)}{p w\left(1-e^{-2 q}\right)}, \quad \text { or } \quad p p^{\prime}=z \cdot \frac{e^{-q}\left(1+e^{-q}\right) q^{\prime}}{\left(1-e^{-2 q}\right)}
$$

Substituting into the above relation the formula for $z$ given by (2.14), we have

$$
\begin{equation*}
p p^{\prime}=\frac{-z_{1}}{\left(1-e^{-2 q}\right)^{1 / 2}} \cdot \frac{e^{-q}\left(1+e^{-q}\right) q^{\prime}}{\left(1-e^{-2 q}\right)}=\frac{-z_{1}\left(1+e^{-q}\right) e^{-q} q^{\prime}}{\left(1-e^{-2 q}\right)^{3 / 2}} . \tag{2.20}
\end{equation*}
$$

Furthermore, by making the change of variables, $u=e^{-q(t)}$, we have $d u=-e^{-q(t)} q^{\prime}(t) d t$ and

$$
\int \frac{-\left(1+e^{-q}\right) e^{-q} q^{\prime}}{\left(1-e^{-2 q}\right)^{3 / 2}} d t=\int \frac{1+u}{\left(1-u^{2}\right)^{3 / 2}} d u=\frac{1+u}{\left(1-u^{2}\right)^{1 / 2}}+C=\frac{1+e^{-q(t)}}{\left(1-e^{-2 q(t)}\right)^{1 / 2}}+C
$$

Therefore, relation (2.20) reads as

$$
\begin{equation*}
\frac{d}{d t}\left[\frac{1}{2} p^{2}\right]=z_{1} \frac{d}{d t}\left[\frac{1+e^{-q}}{\sqrt{1-e^{-2 q}}}\right] \tag{2.21}
\end{equation*}
$$

Integrating (2.21) from 0 to $t$ gives

$$
\frac{1}{2}\left[p^{2}(t)-p_{0}^{2}\right]=z_{1}\left[\frac{1+e^{-q(t)}}{\sqrt{1-e^{-2 q(t)}}}-\frac{1+e^{-q_{0}}}{\sqrt{1-e^{-2 q_{0}}}}\right]
$$

which, when solved for $p$, gives formula (2.15), which expresses $p$ in terms of $q$.
Finally, we express $w$ in terms of $q$. Dividing the equation for $w^{\prime}$ by the equation for $q^{\prime}$ gives

$$
\frac{w^{\prime}}{q^{\prime}}=\frac{z p e^{-q}\left(1-e^{-q}\right)}{p w\left(1-e^{-2 q}\right)}, \quad \text { or } \quad w w^{\prime}=z \cdot \frac{e^{-q}\left(1-e^{-q}\right) q^{\prime}}{\left(1-e^{-2 q}\right)}
$$

Now, substituting the formula for $z$ given by (2.14) into the above relation, we get

$$
\begin{equation*}
w w^{\prime}=\frac{-z_{1}}{\left(1-e^{-2 q}\right)^{1 / 2}} \cdot \frac{e^{-q}\left(1-e^{-q}\right) q^{\prime}}{\left(1-e^{-2 q}\right)}=\frac{-z_{1}\left(1-e^{-q}\right) e^{-q} q^{\prime}}{\left(1-e^{-2 q}\right)^{3 / 2}} . \tag{2.22}
\end{equation*}
$$

Furthermore, making again the change of variables $u=e^{-q(t)}$, we have

$$
\int \frac{-\left(1-e^{-q}\right) e^{-q} q^{\prime}}{\left(1-e^{-2 q}\right)^{3 / 2}} d t=\int \frac{1-u}{\left(1-u^{2}\right)^{3 / 2}} d u=-\frac{\sqrt{1-u^{2}}}{1+u}+C=-\frac{\sqrt{1-e^{-2 q(t)}}}{1+e^{-q(t)}}+C .
$$

Therefore, relation (2.22) reads as follows

$$
\begin{equation*}
\frac{d}{d t}\left[\frac{1}{2} w^{2}\right]=-z_{1} \frac{d}{d t}\left[\frac{\sqrt{1-e^{-2 q(t)}}}{1+e^{-q(t)}}\right] . \tag{2.23}
\end{equation*}
$$

Integrating (2.23) from 0 to $t$ gives

$$
\frac{1}{2}\left[w^{2}(t)-w_{0}^{2}\right]=z_{1}\left[\frac{\sqrt{1-e^{-2 q_{0}}}}{1+e^{-q_{0}}}-\frac{\sqrt{1-e^{-2 q(t)}}}{1+e^{-q(t)}}\right] .
$$

Solving for $w$ while taking into consideration that $w(t)<0$ in the choice of sign, gives formula (2.16), which expresses $w$ in terms of $q$.

Concerning the differential equation for $q$, we begin from its equation $q^{\prime}=w p\left(1-e^{-2 q}\right)$ and substituting for $w$ and $p$ their expressions (2.16) and (2.15), we obtain the desired autonomous initial value problem (2.16). Next, we observe that

$$
\begin{equation*}
\frac{\sqrt{1-e^{-2 q_{0}}}}{1+e^{-q_{0}}}-\frac{\sqrt{1-e^{-2 q}}}{1+e^{-q}} \geq 0, \quad 0 \leq q \leq q_{0} \tag{2.24}
\end{equation*}
$$

and also that

$$
\begin{equation*}
p_{0}^{2}-2 z_{1} \frac{1+e^{-q_{0}}}{\sqrt{1-e^{-2 q_{0}}}} \geq 0 \Longleftrightarrow \frac{(2 b+\delta)^{2}}{2 b(b+\delta)} \geq 1+e^{-q_{0}} \tag{2.25}
\end{equation*}
$$

In fact, condition (2.25) is implied by the stronger condition

$$
\frac{(2 b+\delta)^{2}}{2 b(b+\delta)} \geq 2 \Longleftrightarrow 4 b^{2}+4 b \delta+\delta^{2}>4 b^{2}+4 b \delta \Longleftrightarrow \delta^{2}>0, \quad \text { which is true. }
$$

Now, using (2.24) and (2.25) we see that the function $f(q)$ in the right-hand side of the differential equation (2.17) can be bounded from below by

$$
\begin{aligned}
f(q) & \geq\left(w_{0}^{2}\right)^{\frac{1}{2}} \cdot\left(2 z_{1}\left[\frac{1+e^{-q}}{\sqrt{1-e^{-2 q}}}\right]\right)^{\frac{1}{2}} \cdot\left(1-e^{-2 q}\right) \\
& =\delta \cdot\left(2 b(b+\delta)\left(1-e^{-2 q_{0}}\right)^{1 / 2}\right)^{\frac{1}{2}}\left(\left[\frac{1+e^{-q}}{\sqrt{1-e^{-2 q}}}\right]\right)^{\frac{1}{2}} \cdot\left(1-e^{-2 q}\right) .
\end{aligned}
$$

Using the bounds $1+e^{-q}>1$ and $1-e^{-2 q_{0}} \geq q_{0}$, for $0 \leq q_{0} \leq 1 / 2$, which follow from the following simple but useful approximation

$$
\frac{x}{2} \leq 1-e^{-x} \leq x \Longleftrightarrow 1-e^{-x} \simeq x, \quad \text { if } \quad 0 \leq x \leq 1,
$$

we have

$$
f(q) \geq \delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4} \cdot\left(1-e^{-2 q}\right)^{3 / 4} \doteq g(q)
$$

Therefore, defining $q_{1} \doteq \delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4}$ we see that the complicated initial value problem for $q$ given in (2.17) is dominated by the simpler one shown in (2.18).

Next we move our attention to the study of the solution $q(t)$ of the initial value problem stated in Proposition 1. From the formulas for $p$ and $w$, we see that they blow-up at a zero of $q$. Therefore, the lifespan of our 2-peakon solution is equal to the first such zero. The following result, which is applicable to the simpler dominant initial value problem (2.18) proves existence of a zero and provides an estimate for it size in terms of the initial data.

Proposition 2 (Zero of $q$ ). If $r<1$ then for given $q_{0} \in(0,1 / 2)$ and $q_{1}>0$ the solution to the initial value problem

$$
\begin{equation*}
\frac{d q}{d t}=-g_{r}(q) \doteq-q_{1}\left(1-e^{-2 q}\right)^{r}, \quad q(0)=q_{0}, \tag{2.26}
\end{equation*}
$$

which begins positive and is decreasing, becomes zero in finite time $T$ given by

$$
\begin{equation*}
T=\int_{0}^{q_{0}} \frac{d q}{g_{r}(q)}=\frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{\left(1-e^{-2 q}\right)^{r}} \simeq \frac{1}{1-r} \frac{q_{0}^{1-r}}{q_{1}} . \tag{2.27}
\end{equation*}
$$

A key ingredient in proving Proposition 2 is the following elementary result that compares solutions of the initial value problem (2.26) for different values of $r$. It states that a bigger $r$ correspond to a bigger solution.

Lemma 1 (Comparison principle). If $r_{1}$ and $r_{2}$ are two values of $r$ such that $r_{1} \leq r_{2}$, then the corresponding solutions $q_{r_{1}}(t)$ and $q_{r_{2}}(t)$ to the initial value problem (2.26) with the same initial data $q_{0}$ satisfy $q_{r_{1}}(t) \leq q_{r_{2}}(t)$. That is,

$$
r_{1} \leq r_{2} \quad \Longrightarrow \quad q_{r_{1}}(t) \leq q_{r_{2}}(t)
$$

Proof. It follows from the fact that $r_{1} \leq r_{2}$ implies

$$
-q_{1}\left(1-e^{-2 q}\right)^{r_{1}} \leq-q_{1}\left(1-e^{-2 q}\right)^{r_{2}} .
$$

Remark. We note that for $r \geq 1$ the solution to the initial value problem (2.26) has no zero. In fact, for $r=1$ it reads as follows

$$
\frac{d q}{d t}=-q_{1}\left(1-e^{-2 q}\right), \quad q(0)=q_{0} .
$$

Integrating this equation, gives the explicit formula

$$
q(t)=\frac{1}{2} \ln \left[1+\left(e^{2 q_{0}}-1\right) e^{-q_{1} t}\right] \doteq q_{1}(t) .
$$

From this formula we see that the solution $q(t)$ exists for all $t \geq 0$, is positive for all times and decreases to zero as $t$ goes to $\infty$. Thus when $r=1$ then $q_{1}(t)$ has no zero in finite time. Since, by the comparison principle the solution $q_{r}(t)$ that corresponds to an $r>1$ is greater to $q_{1}(t)$, we conclude that $q_{r}(t)$ has no zero in finite time if $r>1$. Therefore, the lifespan $T$ is equal to $\infty$ if $r \geq 1$.

Proof of Proposition 2. We begin with the case $r \leq 0$. When $r=0$ then our initial value problem (2.17) become the following simple one $q^{\prime}(t)=-q_{1}, q(0)=q_{0}$, whose solution is

$$
q(t)=q_{0}-q_{1} t \doteq q_{0}(t)
$$

which has a zero at $T=q_{0} / q_{1}$. Thus, by the comparison Lemma 1 the solution $q_{r}(t)$ that corresponds to an $r<0$ is smaller to $q_{0}(t)$, and therefore has a zero in finite time. In fact, it is smaller than $q_{0} / q_{1}$. This proves existence of zero for $q_{r}(t)$ when $r \leq 0$.

Existence of a zero for $q(t)$ if $0<r<1$ : To prove existence of zero of $q_{r}(t)$ for $0<r<1$, it suffices to do so under the additional condition

$$
\begin{equation*}
r \neq \frac{n-1}{n}, \quad \text { for all } \quad n=1,2,3, \cdots \tag{2.28}
\end{equation*}
$$

In fact, if $r$ were of the form $\frac{n-1}{n}$ then we could choose another $r_{2} \in(0,1)$ which is not of this form and $r<r_{2}$. Then, by the comparison Lemma 1, proving the existence of a zero for $q_{r_{2}}(t)$ implies existence of a zero for $q_{r}(t)$. So, from now on we shall assume that $r$ satisfies condition (2.28). Therefore, there is a positive integer $n \geq 2$ such that

$$
\begin{equation*}
\frac{n-2}{n-1}<r<\frac{n-1}{n} . \tag{2.29}
\end{equation*}
$$

It turns out that for proving existence of a zero of $q=q_{r}(t)$, we need its $n$-th order Taylor polynomial approximation at $t=0$. Differentiating equation (2.17) $n$ times, we arrive at the formula

$$
\begin{equation*}
q^{(n)}(t)=q_{1}^{n} c_{n}(r)\left(1-e^{-2 q(t)}\right)^{n r-(n-1)}+q_{1}^{n} \sum_{j=1}^{n-1} c_{j}(r)\left(1-e^{-2 q(t)}\right)^{n r-(j-1)}, \tag{2.30}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n}(r)=(-1)^{n} 2^{n-1} r(2 r-1) \cdots([n-1] r-[n-2]) \tag{2.31}
\end{equation*}
$$

and $c_{j}(r)$ for $j=1, \cdots, n-1$ are coefficients depending on $r$. Also, we obtain the following formula for the $(n+1)$-th derivative of $q$

$$
\begin{equation*}
q^{(n+1)}(t)=q_{1}^{n+1} c_{n+1}(r)\left(1-e^{-2 q(t)}\right)^{(n+1) r-n}+q_{1}^{n+1} \sum_{j=1}^{n} c_{j}(r)\left(1-e^{-2 q(t)}\right)^{(n+1) r-(j-1)}, \tag{2.32}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{n+1}(r)=(-1)^{n+1} 2^{n} r(2 r-1) \cdots(n r-[n-1]), \tag{2.33}
\end{equation*}
$$

and again $c_{j}(r)$ for $j=1, \cdots, n$ are coefficients depending on $r$. Therefore, the $n$-th order Taylor polynomial approximation of $q(t)$ at $t=0$ is given by

$$
\begin{equation*}
q(t)=q_{0}+q^{\prime}(0) t+\frac{q^{\prime \prime}(0)}{2!} t^{2}+\frac{q^{(3)}(0)}{3!} t^{3}+\cdots+\frac{q^{(n)}(0)}{n!} t^{n}+\frac{q^{(n+1)}(\tau)}{(n+1)!} t^{n+1} \tag{2.34}
\end{equation*}
$$

where $0 \leq \tau \leq t$. Next, we shall show that the coeficients $c_{n}(r)$ and $c_{n+1}(r)$ defined by (2.31) and (2.33) have the same sign, which is the key ingredient for proving the existence of a zero for $q(t)$. We prove this claim by considering the two cases possible, $n$ even and $n$ odd. We begin with the case of $n$ even. In this case, using the first part of inequality (2.29) that $r$ satisfies, we see that $(n-1) r>n-2$ and this implies that $c_{n}(r)$ is a positive number. Also, using the second part of inequality (2.29) we see that $n r<n-1$, which implies that $c_{n+1}(r)>0$ is a positive number too. Furthermore, in the expression of $q^{(n)}(0)$ the first term $q_{1}^{n} c_{n}(r)\left(1-e^{-2 q_{0}}\right)^{n r-(n-1)}$ is the dominant term for $q_{0}$ small enough since the exponent
$n r-(n-1)$ is negative while the exponents of $\left(1-e^{-2 q_{0}}\right)$ appearing in all other terms of the sum (2.30) are positive. Thus we can conclude

$$
\begin{equation*}
q^{(n)}(0)>0, \quad \text { if } n \text { is even and } q_{0} \text { is small enough. } \tag{2.35}
\end{equation*}
$$

Similarly, in the expression of $q^{(n+1)}(t)$, the first term $q_{1}^{n+1} c_{n+1}(r)\left(1-e^{-2 q(t)}\right)^{(n+1) r-n}$ is the dominant term for $q_{0}$ small enough, since the exponent $(n+1) r-n$ is negative while all the exponents of $\left(1-e^{-2 q(t)}\right)$ appearing in the sum (2.32) are positive, except the one that corresponds to $j=n$ which has exponent $(n+1) r-(n-1)$ which is negative. However, $\left(1-e^{-2 q(t)}\right)^{(n+1) r-(n-1)}$ is dominated by $\left(1-e^{-2 q(t)}\right)^{(n+1) r-n}$, for $q(t) \leq q_{0}$ small enough. Thus, we also have

$$
\begin{equation*}
q^{(n+1)}(\tau)>0, \quad \text { if } n \text { is even and } q_{0} \text { is small enough. } \tag{2.36}
\end{equation*}
$$

In the case that $n \geq 2$ is an odd positive integer then the signs change due to the fact $(-1)^{n}=-1$ and using the same reasoning as in the even case we obtain that

$$
\begin{equation*}
q^{(n)}(0)<0, \quad \text { if } n \text { is odd and } q_{0} \text { is small enough, } \tag{2.37}
\end{equation*}
$$

and

$$
\begin{equation*}
q^{(n+1)}(\tau)<0, \quad \text { if } n \text { is odd and } q_{0} \text { is small enough. } \tag{2.38}
\end{equation*}
$$

Now, we are ready to prove the existence of zero for $q(t)$. First we consider the case that $n$ is an odd number. Then, using the $n$-th order Taylor polynomial approximation (2.34) and the conditions (2.37), (2.38) we obtain that

$$
q(t) \leq q_{0}+q^{\prime}(0) t+\frac{q^{\prime \prime}(0)}{2!} t^{2}+\frac{q^{(3)}(0)}{3!} t^{3}+\cdots+\frac{q^{(n)}(0)}{n!} t^{n}, \quad \text { for all } \quad t \geq 0 .
$$

Furthermore, since for large $t$ the term $\frac{q^{(n)}(0)}{n!} t^{n}$ dominates and $q^{(n)}(0)<0$ we have that the $n$-th order Taylor polynomial approximation of $q(t)$ will become negative, thus crossing the $t$-axis. This forces $q(t)$ to have a zero at some positive time $T$, which is the desired conclusion. Finally, we prove the existence of zero for $q(t)$ in the even case. This is done by contradiction. In fact, if $q(t)>0$ for all $t>0$ then our differential equation $q^{\prime}(t)=-q_{1}\left(1-e^{-2 q(t)}\right)^{r}$ implies that $q(t)$ is decreasing for all $t>0$ and therefore

$$
\begin{equation*}
q(t) \leq q_{0} \quad \text { for all } \quad t \geq 0 . \tag{2.39}
\end{equation*}
$$

However, if $n$ is even, then using the $n$-th order Taylor polynomial approximation of $q(t)$ at $t=0$, which is given by (2.34), and conditions (2.35) and (2.36), we have that

$$
\begin{equation*}
q(t) \geq q_{0}+q^{\prime}(0) t+\frac{q^{\prime \prime}(0)}{2!} t^{2}+\frac{q^{(3)}(0)}{3!} t^{3}+\cdots+\frac{q^{(n)}(0)}{n!} t^{n}, \quad \text { for all } \quad t \geq 0 \tag{2.40}
\end{equation*}
$$

Inequality (2.40) leads to a contradiction because for large $t$ the term $\frac{q^{(n)}(0)}{n!} t^{n}$ dominates the Taylor polynomial approximation. Thus, there is some large time $T>0$ such that

$$
\begin{equation*}
q(t) \geq q_{0}+\frac{1}{2} \frac{q^{(n)}(0)}{n!} T^{n}>2 q_{0}, \quad 0 \leq t \leq T \tag{2.41}
\end{equation*}
$$

which contradicts inequality (2.39). This argument completes the proof of the existence of zero for $q(t)$ when $0<r<1$.

Estimating the zero $T$ of the position $q(t)$ when $r<1$. Let $T$ be the zero of the solution $q(t)$ of our initial value problem (2.17), which is: $q^{\prime}(t)=-q_{1}\left(1-e^{-2 q(t)}\right)^{r}, q(0)=q_{0}$. Integrating it from 0 to $T$ we have

$$
\int_{0}^{T} \frac{q^{\prime}(t)}{\left(1-e^{-2 q(t)}\right)^{r}} d t=-q_{1} T
$$

Then, making the substitution $q=q(t)$, and using the initial and terminal conditions $q(0)=q_{0}$ and $q(T)=0$, we obtain the following formula for $T$

$$
\begin{equation*}
T=\frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{\left(1-e^{-2 q}\right)^{r}} T \simeq \frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{q^{r}}=\frac{1}{1-r} \frac{q_{0}^{1-r}}{q_{1}} \tag{2.42}
\end{equation*}
$$

Above, we used the estimate $q \leq 1-e^{-2 q} \leq 2 q$, if $0 \leq q \leq \frac{1}{2}$. This completes the proof of Proposition 2.

Applying Proposition 2 with $r=3 / 4$ we obtain the following result for the the zero the initial value problem (2.17) and the lifespan of our 2-peakon solution $u$.

Corollary 1 (Zero of $q$ and lifespan of $u$ ). If $0<q_{0}<1 / 2$ and $b, \delta$ satisfy condition (2.6) then the solution to the initial value problem (2.17) begins positive, is decreasing, and becomes zero in finite time $T$ given by

$$
\begin{equation*}
T=\int_{0}^{q_{0}} \frac{d q}{f(q)} \leq \frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{\left(1-e^{-2 q}\right)^{3 / 4}} \simeq \frac{q_{0}^{1 / 4}}{q_{1}} \simeq \frac{q_{0}^{1 / 4}}{\delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4}} \simeq \frac{1}{\delta \sqrt{2 b(b+\delta)}} \tag{2.43}
\end{equation*}
$$

Proof. The existence and uniqueness of the solution follows from the fundamental ODE theorem since $f(q)$ is a smooth function. That $q(t)$ is decreasing follows from the fact that $q^{\prime}=-f(q)<0$. Finally, that $q(t)$ becomes zero in finite time follows from the fact that our initial value problem (2.17) is dominated by the initial value problem (2.18) for which Proposition 2 is applicable with $r=3 / 4$. Therefore, estimate (2.27) gives (2.43), and this completes the proof of the lemma.

The properties of our special 2-peakon solutions are summarized in the following Theorem and are a consequence of Proposition 1 and Corollary 1.

Theorem 3 (Construction of 2-peakon solutions). For given $0<a \leq 1 / 4$ and $b, \delta$ satisfying condition (2.6) the initial value problem for the positions $q_{1}, q_{2}$ and the momenta $p_{1}, p_{2}$,

$$
\begin{array}{ll}
q_{1}^{\prime}=\left(p_{1}+p_{2} e^{-q}\right)^{2}, & q_{1}(0)=-a \\
q_{2}^{\prime}=\left(p_{1} e^{-q}+p_{2}\right)^{2}, & q_{2}(0)=a>0  \tag{2.44}\\
p_{1}^{\prime}=-p_{1} p_{2}\left(p_{1}+p_{2} e^{-q}\right) e^{-q}, & p_{1}(0)=-b-\delta \\
p_{2}^{\prime}=p_{1} p_{2}\left(p_{1} e^{-q}+p_{2}\right) e^{-q}, & p_{2}(0)=b
\end{array}
$$

has a unique smooth solution $\left(q_{1}, q_{2}, p_{1}, p_{2}\right)(t)$ with a finite lifespan $T$, which is the zero of $q=q_{2}-q_{1}$, and which satisfies the estimate

$$
\begin{equation*}
T \lesssim \frac{1}{\delta \sqrt{2 b(b+\delta)}} \tag{2.45}
\end{equation*}
$$

Furthermore, we have

$$
\begin{aligned}
& p_{1}=\frac{w-p}{2}<0, \text { decreasing } \\
& \lim _{t \rightarrow T^{-}} p_{1}(t)=-\infty, \text { and }-p_{1} \simeq p \simeq q^{-1 / 4},
\end{aligned}
$$

and

$$
\begin{aligned}
& p_{2}=\frac{w+p}{2}>0, \text { increasing } \\
& \lim _{t \rightarrow T^{-}} p_{2}(t)=\infty, \text { and } p_{2} \simeq p \simeq q^{-1 / 4},
\end{aligned}
$$

where $p$ and $w$ are given in Proposition 1. Also, $w=p_{1}+p_{2}$ is decreasing from $w_{0}<0$ to $w_{T}$, where $w_{T} \doteq \lim _{t \rightarrow T^{-}} w(t)$, that is

$$
\begin{equation*}
w_{T}=-\left(\delta^{2}+2 b(b+\delta)\left(1-e^{-2 a}\right)\right)^{\frac{1}{2}} \tag{2.46}
\end{equation*}
$$

Finally, the 2-peakon

$$
u(x, t)=p_{1}(t) e^{-\left|x-q_{1}(t)\right|}+p_{2}(t) e^{-\left|x-q_{2}(t)\right|},
$$

is $N E$ solution for $x \in \mathbb{R}, 0<t<T$, with the following asymmetric antipeakon-peakon initial profile

$$
u(x, 0)=-(b+\delta) e^{-|x+a|}+b e^{-|x-a|}, x \in \mathbb{R}
$$



Figure 2: Graphs of $p_{1}, p_{2}, q$ and $w$

## 3. Calculating the Norm

Proposition 3. Let $u(t)$ be the two-peakon solution to the $N E$ equation. Then on $[0, T)$ we have

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=16 r(t) p_{1}^{2}(t) Q_{s}(q)+4 c_{s}(1-r(t))^{2} p_{1}^{2}(t), \quad \text { with } \quad r(t) \doteq-\frac{p_{2}(t)}{p_{1}(t)} \tag{3.1}
\end{equation*}
$$

where $c_{s}=\int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} d \xi$ and $Q_{s}(q)$, which is given below, satisfies the estimates:

$$
Q_{s}(q) \doteq \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} \sin ^{2}\left(\frac{q \xi}{2}\right) d \xi \simeq \begin{cases}q^{3-2 s}, & 1 / 2<s<3 / 2  \tag{3.2}\\ q^{2} \cdot \ln (1 / q), & s=1 / 2 \\ q^{2}, & s<1 / 2\end{cases}
$$

Proof. Since $\widehat{e^{-|x|}}(\xi)=2 /\left(1+\xi^{2}\right)$ we have that the Fourier transform of

$$
u(x, t)=p_{1} e^{-\left|x-q_{1}\right|}+p_{2} e^{-\left|x-q_{2}\right|}
$$

is given by

$$
\widehat{u}(\xi, t)=\frac{2 p_{1} e^{-i \xi q_{1}}}{1+\xi^{2}}+\frac{2 p_{2} e^{-i \xi q_{2}}}{1+\xi^{2}}=\frac{2}{1+\xi^{2}} \cdot p_{1} e^{-i \xi q_{1}} \cdot\left(1+\frac{p_{2}}{p_{1}} e^{-i \xi q}\right) .
$$

Taking the square of the $H^{s}$ norm of this quantity and factoring out $p_{1}^{2}$, we obtain

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=4 p_{1}^{2} \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2}\left|1+\frac{p_{2}}{p_{1}} e^{-i \xi q}\right|^{2} d \xi . \tag{3.3}
\end{equation*}
$$

Using Proposition 1 we see that

$$
r=r(t) \doteq-\frac{p_{2}(t)}{p_{1}(t)}=\frac{p+w}{p-w}<1, \quad \text { and } \quad r(t) \nearrow 1 \text { as } t \nearrow T .
$$

Next, using $r$ we write (3.3) as follows

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=4 p_{1}^{2} \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2}\left|1-r e^{-i \xi q}\right|^{2} d \xi \tag{3.4}
\end{equation*}
$$

Expanding out the square under the integral in (3.4), we have

$$
\begin{equation*}
\left|r e^{i q \xi}-1\right|^{2}=(1-r)^{2}+4 r \sin ^{2}\left(\frac{q \xi}{2}\right) \tag{3.5}
\end{equation*}
$$

Substituting (3.5) into (3.4)

$$
\|u(t)\|_{H^{s}}^{2}=16 r p_{1}^{2} \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} \sin ^{2}\left(\frac{q \xi}{2}\right) d \xi+4(1-r)^{2} p_{1}^{2} \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} d \xi
$$

or

$$
\|u(t)\|_{H^{s}}^{2}=16 r p_{1}^{2} Q_{s}(q)+4 c_{s}(1-r)^{2} p_{1}^{2},
$$

where

$$
\begin{equation*}
c_{s}=\int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} d \xi \quad \text { and } \quad Q_{s}(q)=\int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s-2} \sin ^{2}\left(\frac{q \xi}{2}\right) d \xi \tag{3.6}
\end{equation*}
$$

Now, we see that to prove Proposition 3 it suffices to show that for $0<q<1 / 8$ we have the following estimate

$$
Q_{s}(q) \simeq \begin{cases}q^{3-2 s}, & 1 / 2<s<3 / 2 \\ q^{2} \cdot \ln (1 / q), & s=1 / 2 \\ q^{2}, & s<1 / 2\end{cases}
$$

Starting with the integrand for $Q_{s}$ from (3.6) and making the change of variables $x=q \xi$, which gives $d x=q d \xi$, we can write $Q_{s}(q)$ as

$$
\begin{equation*}
Q_{s}(q)=2 q^{3-2 s} \int_{0}^{\infty}\left(q^{2}+x^{2}\right)^{s-2} \sin ^{2}(x / 2) d x=2 q^{3-2 s}\left[I_{1}+I_{2}\right] \tag{3.7}
\end{equation*}
$$

where

$$
I_{1} \doteq \int_{0}^{1} \frac{x^{2}}{\left(q^{2}+x^{2}\right)^{2-s}} d x \quad \text { and } \quad I_{2} \doteq \int_{1}^{\infty} \frac{\sin ^{2}(x / 2)}{\left(q^{2}+x^{2}\right)^{2-s}} d x
$$

If $s<3 / 2$ then the integral $I_{2}$ is bounded since

$$
\begin{equation*}
I_{2}=\int_{1}^{\infty} \frac{\sin ^{2}(x / 2)}{\left(q^{2}+x^{2}\right)^{2-s}} d x \lesssim \int_{1}^{\infty} x^{2 s-4} d x=\frac{1}{3-2 s} \tag{3.8}
\end{equation*}
$$

Also, when $s>1 / 2$ we have the following upper bound for $I_{1}$

$$
\begin{equation*}
I_{1} \leq \int_{0}^{1} x^{2 s-2} d x=\frac{1}{2 s-1} . \tag{3.9}
\end{equation*}
$$

Furthermore, for any $s<2$ we have

$$
\begin{equation*}
I_{1} \geq 2^{s-2} \int_{0}^{1} x^{2} d x=2^{s-2} \cdot \frac{1}{3} \tag{3.10}
\end{equation*}
$$

Combining (3.10), (3.8), and (3.7) gives

$$
\left\|f_{q}\right\|_{H^{s}} \simeq q^{\frac{3}{2}-s}, \quad \text { if } \quad 1 / 2<s<3 / 2
$$

The case $s<1 / 2$ : Since $I_{1}+I_{2}$ is bounded below by $I_{1}$ and $0<y<1$ we have

$$
\begin{equation*}
I_{1}+I_{2} \gtrsim \int_{0}^{q} \frac{x^{2}}{\left(q^{2}+x^{2}\right)^{2-s}} d x \gtrsim q^{2 s-4} \int_{0}^{q} x^{2} d x \simeq q^{2 s-1} . \tag{3.11}
\end{equation*}
$$

Combining (3.11) and (3.7) gives

$$
\begin{equation*}
Q_{s}(q) \gtrsim q^{3-2 s} \cdot q^{2 s-1}=q^{2}, \quad \text { if } \quad s<1 / 2 . \tag{3.12}
\end{equation*}
$$

To prove the reverse of inequality (3.12) we obtain an upper bound for $I_{1}$. For this argument, we let $z=x / y$ and get

$$
I_{1} \doteq \int_{0}^{1} \frac{x^{2}}{\left(q^{2}+x^{2}\right)^{2-s}} d x=q^{2 s-1} \int_{0}^{1 / q} \frac{z^{2}}{\left(1+z^{2}\right)^{2-s}} d z \leq q^{2 s-1} \int_{0}^{\infty} \frac{1}{\left(1+z^{2}\right)^{1-s}} d z .
$$

Since the last integral converges if $2(1-s)>1$, which is equivalent to $s<1 / 2$, we see that it is equal to a finite constant $c_{s}$. Combining this fact together with (3.7) and (3.8) we have

$$
Q_{s}(q) \lesssim q^{3-2 s}\left[q^{2 s-1}+1\right] \lesssim q^{2}, \quad \text { if } \quad s<1 / 2,
$$

which together with (3.11) gives

$$
Q_{s}(q) \simeq y^{2}, \quad \text { if } \quad s<1 / 2
$$

The case $s=1 / 2$ : We observe that

$$
I_{1}=\int_{0}^{1} \frac{x^{2}}{\left(q^{2}+x^{2}\right)^{3 / 2}} d x=\ln \left(\sqrt{q^{2}+1}+1\right)-\frac{1}{\sqrt{q^{2}+1}}+\ln (1 / q) .
$$

Upper Bound. From here we begin by removing the middle term and using the fact that $y<1 / 4$ in the first term. We get

$$
\begin{equation*}
I_{1} \leq \ln (\sqrt{2}+1)+\ln (1 / q) . \tag{3.13}
\end{equation*}
$$

Substituting (3.13) back into (3.7) and taking into account estimate (3.8) for $I_{2}$ we have

$$
\begin{equation*}
Q_{\frac{1}{2}}(s) \lesssim q^{2} \ln (1 / q) . \tag{3.14}
\end{equation*}
$$

Lower Bound. Using the fact that $2 \ln (1 / q)>1$ we have

$$
I_{1} \geq \ln (1 / q)+(\ln (2)-1)[2 \ln (1 / q)]=(2 \ln (2)-1) \ln (1 / q) .
$$

We therefore arrive at

$$
\begin{equation*}
Q_{\frac{1}{2}}(s) \gtrsim q^{2} \cdot I_{1} \gtrsim q^{2} \ln (1 / q) . \tag{3.15}
\end{equation*}
$$

Putting these upper and lower bounds (3.14) and (3.15) together and taking the square root of both sides of the equation gives the desired result of

$$
Q_{\frac{1}{2}}(s) \simeq y^{2} \cdot \ln (1 / q)
$$

## 4. Small lifespan and initial data

We begin by assuming that

$$
p_{2}(0)=b \gg 1 \text { and }-p_{1}(0)=b+\delta, \delta>0,
$$

so that the conditions for the existence of our 2-peakon with the lifespan estimate (2.43) hold. Then, we have the following.

Lifespan Estimate. For given $\varepsilon>0$, we need to find $b>1$ such that $T<\varepsilon$. Since, by Proposition 1 we have

$$
T \lesssim \frac{1}{\delta \sqrt{2 b(b+\delta)}} \leq \frac{1}{\delta b},
$$

we must have

$$
\begin{equation*}
\frac{1}{\delta b} \leq \varepsilon \Longleftrightarrow b \geq \delta^{-1} \varepsilon^{-1} \tag{4.1}
\end{equation*}
$$

Initial Data Estimate. Now, for the same $\varepsilon>0$ we need to find $q_{0}<1 / 8$ such that $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$. For this argument we use Proposition 3, from which we have

$$
\begin{aligned}
\|u(0)\|_{H^{s}}^{2} & =16 r(0) p_{1}^{2}(0) Q_{s}\left(q_{0}\right)+4 c_{s}(1-r(0))^{2} p_{1}^{2}(0), \quad r(t) \doteq-\frac{p_{2}(t)}{p_{1}(t)} \\
& =16 b(b+\delta) Q_{s}\left(q_{0}\right)+4 c_{s} \delta^{2}
\end{aligned}
$$

which in turn gives

$$
\|u(0)\|_{H^{s}}^{2} \leq 32 b^{2} Q_{s}\left(q_{0}\right)+4 c_{s} \delta^{2} .
$$

Case $1 / 2<s<3 / 2$ : Then by Proposition 3 we have $Q_{s}\left(q_{0}\right) \lesssim q_{0}^{3-2 s}$ and therefore

$$
\|u(0)\|_{H^{s}}^{2} \leq C_{s} b^{2} q_{0}^{3-2 s}+4 c_{s} \delta^{2} .
$$

To demonstrate $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$, it suffices to choose $q_{0}$ and $\delta$ such that $C_{s} b^{2} q_{0}^{3-2 s}+4 c_{s} \delta^{2} \leq \varepsilon^{2}$, or

$$
4 c_{s} \delta^{2} \leq \frac{\varepsilon^{2}}{2} \quad \text { and } \quad C_{s} b^{2} q_{0}^{3-2 s} \leq \frac{\varepsilon^{2}}{2}
$$

The first inequality holds if

$$
\begin{equation*}
\delta \leq \frac{\varepsilon}{2 \sqrt{2 c_{s}}} . \tag{4.2}
\end{equation*}
$$

Taking into consideration (4.2) and (4.1), the second inequality holds if

$$
q_{0}^{3-2 s} \leq \frac{\varepsilon^{2}}{2 C_{s} b^{2}} \leq \frac{\varepsilon^{2}}{2 C_{s} \delta^{-2} \varepsilon^{-2}}=\frac{\delta^{2} \varepsilon^{4}}{2 C_{s}} \leq \frac{\varepsilon^{2} \varepsilon^{4}}{8 c_{s} \cdot 2 C_{s}},
$$

or

$$
q_{0} \leq\left(\frac{\varepsilon^{6}}{16 c_{s} C_{s}}\right)^{\frac{1}{3-2 s}}
$$

Case $s \leq 1 / 2$ : For such a Sobolev exponent $s$ we have $\|u(0)\|_{H^{s}} \leq\|u(0)\|_{H^{1}}$. This combined with Proposition 3, which tells us that $Q_{1}\left(q_{0}\right) \lesssim q_{0}$, gives

$$
\|u(0)\|_{H^{s}}^{2} \leq\|u(0)\|_{H^{1}}^{2} \leq C_{1} b^{2} q_{0}+4 c_{1} \delta^{2} .
$$

Thus $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$ if $q_{0}$ and $\delta$ satisfy the inequalities

$$
4 c_{1} \delta^{2} \leq \frac{\varepsilon^{2}}{2} \quad \text { and } \quad C_{1} b^{2} q_{0} \leq \frac{\varepsilon^{2}}{2}
$$

These inequality holds if

$$
\delta \leq \frac{\varepsilon}{2 \sqrt{2 c_{1}}} \quad \text { and } \quad q_{0} \leq \frac{\varepsilon^{6}}{16 c_{s} C_{s}}
$$

## 5. Norm-Inflation and illposedness for $5 / 4<s<3 / 2$

From Proposition 3 we have

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=16 r(t) p_{1}^{2}(t) Q_{s}(q)+4 c_{s} p_{1}^{2}(t)(1-r(t))^{2} \tag{5.1}
\end{equation*}
$$

where the estimate for $Q_{s}$ is given in (3.2). Also, using Theorem 3 we have

$$
p_{1}^{2}(t) \simeq q^{-1 / 2}(t), \quad \text { and } \quad p_{2}^{2}(t) \simeq q^{-1 / 2}(t), \quad \text { for } t \text { close to } T .
$$

Next, we see that

$$
r=r(t) \doteq \frac{p_{2}(t)}{-p_{1}(t)} \simeq \frac{q^{-1 / 4}}{q^{-1 / 4}} \simeq 1, \quad \text { as } t \nearrow T,
$$

and

$$
p_{1}(t)(1-r(t))=p_{1}(t)\left(1+\frac{p_{2}(t)}{p_{1}(t)}\right)=p_{2}(t)+p_{1}(t)=w(t) .
$$

Also, we have

$$
\begin{equation*}
\lim _{t \rightarrow T} p_{1}^{2}(t)(1-r(t))^{2}=\lim _{t \rightarrow T} w^{2}(t)=\delta^{2}+2 b(b+\delta) \cdot\left(1-e^{-q_{0}}\right) \tag{5.2}
\end{equation*}
$$

Therefore, the first term of (5.1) can be estimated by

$$
16 r(t) p_{1}^{2}(t) Q_{s}(q) \simeq \begin{cases}q^{\frac{5}{2}-2 s}, & 1 / 2<s<3 / 2 \\ q^{\frac{3}{2}} \cdot \ln (1 / q), & s=1 / 2 \\ q^{\frac{3}{2}}, & s<1 / 2\end{cases}
$$

Combining the last estimate with the fact $\frac{5}{2}-2 s=0 \Longleftrightarrow s=\frac{5}{4}$ we see that

$$
\lim _{t \rightarrow T} 16 r(t) p_{1}^{2}(t) Q_{s}(q)= \begin{cases}\infty \quad \text { (inflation) }, & 5 / 4<s<3 / 2  \tag{5.3}\\ \text { may not exist, } & s=5 / 4 \\ 0, & s<5 / 4\end{cases}
$$

Finally, using the limits (5.3) and (5.2) from formula (5.1) we conclude that

$$
\lim _{t \rightarrow T}\|u(t)\|_{H^{s}}^{2}= \begin{cases}\infty \quad \text { (inflation), } & 5 / 4<s<3 / 2  \tag{5.4}\\ \text { may not exist, } & s=5 / 4 \\ 4 c_{s}\left[\delta^{2}+2 b(b+\delta) \cdot\left(1-e^{-q_{0}}\right)\right], & s<5 / 4\end{cases}
$$

Therefore when $5 / 4<s<3 / 2$ we have norm inflation and ill-posedness for the Novikov equation.

## 6. Non-Uniqueness for $s<5 / 4$

In this section, we prove that once we take the Sobolev exponent to be less that $5 / 4$, the Novikov equation admits non-unique solutions.

Theorem 4 (Non-uniqueness). For $s<5 / 4$ NE admits non-unique solutions.
Our proof of non-uniqueness revolves around examining the behavior of the limit as $t \rightarrow T^{-}$ of the 2-peakon solution $u$ with initial data given in (2.7). Once we take the Sobolev exponent to be $s<5 / 4$, this limit exists, and it is a single antipeakon. The non-uniqueness then can be realized by taking a single antipeakon traveling wave that which at time $T$ has the same profile as $\lim _{t \rightarrow T^{-}} u(x, t)$. From this point, a change of variables can recast this scenario as two solutions arising from the same initial data. To proceed with this argument, we begin by examining the pointwise limit, then the $L^{r}$ limit, and finally use these results in addition to the generalized Dominated Convergence Theorem to establish the $H^{s}$ limit.

Proposition 4 (Pointwise limit). For each $x \in \mathbb{R}$ we have

$$
\begin{equation*}
\lim _{t \rightarrow T^{-}} u(x, t)=w_{T} e^{-\left|x-q_{T}\right|} \doteq v_{T}(x) \tag{6.1}
\end{equation*}
$$

where $w_{t}$ is given by $(2.46)$, that is $w_{T}=-\left(\delta^{2}+2 b(b+\delta)\left(1-e^{-2 a}\right)\right)^{\frac{1}{2}}<0$, and

$$
q_{T} \doteq \lim _{t \rightarrow T^{-}} q_{1}(t)=\lim _{t \rightarrow T^{-}} q_{2}(t)
$$

For proving Proposition 4 we shall need the following elementary result.
Lemma 2. Given our functions $p_{1}, p_{2}$ and $q=q_{1}-q_{1}$ the following limits hold as $t \rightarrow T^{-}$:

$$
\begin{equation*}
\lim _{t \rightarrow T^{-}} p_{j}(t)\left(1-e^{-q(t)}\right)=0, \quad j=1,2 \tag{6.2}
\end{equation*}
$$

Proof. Using the estimates

$$
p_{1}^{2} \simeq q^{-1 / 2} \quad \text { and } \quad p_{2}^{2} \simeq q^{-1 / 2}
$$

and the inequality $1-e^{-x}<x$ for $x \in[0,1]$ we have

$$
\lim _{t \rightarrow T^{-}}\left|p_{j}(t)\left(1-e^{-q(t)}\right)\right| \leq \lim _{t \rightarrow T^{-}}\left|p_{j}(t)\right| \cdot|q(t)| \lesssim \lim _{t \rightarrow T^{-}}\left|q^{-1 / 4}(t)\right| \cdot|q(t)|=0
$$

Proof of Proposition 4. As we are working with a pointwise limit, we consider the cases $x \geq q_{T}$ and $x<q_{T}$ separately so that we can evaluate the absolute values $\left|x-q_{j}\right|$ in the definition of the 2 -peakon solution $u$.

Case $x \geq q_{T}$. Since $q_{1} \leq q_{2} \leq q_{T}$, we have $x-q_{j} \geq 0$ and therefore

$$
u(x, t)=p_{1}(t) e^{-\left|x-q_{1}(t)\right|}+p_{2}(t) e^{-\left|x-q_{2}(t)\right|}=e^{-x} \cdot\left(p_{1}(t) e^{q_{1}(t)}+p_{2}(t) e^{q_{2}(t)}\right)
$$

Next, we will rewrite $u$ in such a way so as to utilize Lemma 2. We have

$$
u(x, t)=e^{-x} \cdot e^{q_{2}(t)} \cdot\left(-p_{1}(t)\left(1-e^{-q(t)}\right)+w(t)\right)
$$

Finally taking the limit as $t \rightarrow T^{-}$of $u$ and using (6.2) we get

$$
\begin{equation*}
\lim _{t \rightarrow T^{-}} u(x, t)=w_{T} \cdot e^{-x+q_{T}}, \quad x \geq q_{T} . \tag{6.3}
\end{equation*}
$$

Case $x<q_{T}$. We follow essentially the same strategy as in the previous case, simply correcting for signs. Since $x$ is fixed and $q_{1} \leq q_{2} \leq q_{T}$, we see that after some time $t_{0}$ we must have $x<q_{1}(t) \leq q_{2}(t) \leq q_{T}$. Therefore, for $t>t_{0}$ we have $x-q_{j}<0$ and $u$ can be written as

$$
u(x, t)=e^{x} \cdot e^{-q_{1}(t)}\left(w(t)-p_{2}(t)\left(1-e^{-q(t)}\right)\right) .
$$

Thus taking the limit as $t \rightarrow T^{-}$of $u$ and using again Lemma 2 we obtain

$$
\begin{equation*}
\lim _{t \rightarrow T^{-}} u(x, t)=w_{T} \cdot e^{x-q_{T}}, \quad x<q_{T} . \tag{6.4}
\end{equation*}
$$

Combining (6.3) and (6.4) we conclude that the 2-peakon solution $u(t)$ has a limit as $t \rightarrow T^{-}$, which is given by the antipeakon (6.1).

We next examine the the limit of $u$ in $L^{r}$ topology.
Proposition 5 (Convergence in $L^{r}$ ). For our antipeakon-peakon solution u to NE, we have

$$
\lim _{t \rightarrow T^{-}}\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}=0, \quad \text { for } \quad 1 \leq r<4 .
$$

Proof. As we will need to evaluate the absolute values in the exponents, we note that the order of the peaks positions of $u(x, t)$ and $v_{T}(x)$ is $q_{1}(t)<q_{2}(t)<q_{T}$. We now expand the $L^{r}$ norm as

$$
\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}^{r} \doteq I_{1}(t)+I_{2}(t)+I_{3}(t)+I_{4}(t) .
$$

where the integrals $I_{j}(t)$ have their domains determined by $q_{1}<q_{2}<q_{T}$, that is

$$
\begin{array}{ll}
I_{1}(t) \doteq \int_{-\infty}^{q_{1}(t)}\left|u(x, t)-v_{T}(x)\right|^{r} d x, & I_{3}(t) \doteq \int_{q_{2}(t)}^{q_{T}}\left|u(x, t)-v_{T}(x)\right|^{r} d x \\
I_{2}(t) \doteq \int_{q_{1}(t)}^{q_{2}(t)}\left|u(x, t)-v_{T}(x)\right|^{r} d x, & I_{4}(t) \doteq \int_{q_{T}}^{\infty}\left|u(x, t)-v_{T}(x)\right|^{r} d x
\end{array}
$$

Evaluating $I_{1}$. Calculating the integral, we have

$$
I_{1}(t)=\frac{e^{r q_{1}(t)}}{r} \cdot\left|p_{1} e^{-q_{1}(t)}+p_{2} e^{-q_{2}(t)}-w_{T} e^{-q_{T}}\right|^{r} .
$$

In order to proceed with evaluating the limit, we observe the following identity

$$
p_{1} e^{-q_{1}(t)}+p_{2} e^{-q_{2}(t)}-w_{T} e^{-q_{T}}=e^{-q_{2}} p_{1}\left(e^{q}-1\right)+w(t) e^{-q_{2}(t)}-w_{T} e^{-q_{T}} .
$$

We can now evaluate the limit as

$$
\lim _{t \rightarrow T^{-}} I_{1}(t)=\lim _{t \rightarrow T^{-}} \frac{e^{r q_{1}(t)}}{r} \cdot\left|e^{-q_{2}} p_{1}\left(e^{q}-1\right)+\left[w(t) e^{-q_{2}(t)}-w_{T} e^{-q_{T}}\right]\right|^{r}=0 .
$$

Evaluating $I_{2}$. Using the Jensen's inequality $\left|a_{1}+\cdots+a_{n}\right|^{r} \leq n^{r}\left(\left|a_{1}\right|^{r}+\cdots+\left|a_{n}\right|^{r}\right)$ together with $e^{-\left|x-q_{j}(t)\right|} \leq 1, \quad e^{-\left|x-q_{T}(t)\right|} \leq 1$ and $\left|p_{j}\right| \simeq q^{-1 / 4}$, we have

$$
\lim _{t \rightarrow T^{-}} I_{2}(t) \lesssim \lim _{t \rightarrow T^{-}}\left(q^{1-\frac{r}{4}}(t)+q(t)\left|w_{T}\right|^{r}\right)=0 \quad(\text { assuming } r<4)
$$

Evaluating $I_{3}$. After evaluating the absolute values inside the exponential, and using the identity

$$
p_{1} e^{-x+q_{1}}+p_{2} e^{-x+q_{2}}-w_{T} e^{x-q_{T}}=e^{-x} e^{q_{2}} p_{1}\left(e^{-q}-1\right)+e^{-x} w e^{q_{2}}-e^{x} w_{T} e^{-q_{T}},
$$

an application of Jensen's inequality gives us

$$
I_{3}(t) \lesssim \int_{q_{2}(t)}^{q_{T}}\left|e^{-x} e^{q_{2}} p_{1}(t)\left(e^{-q(t)}-1\right)\right|^{r} d x+\int_{q_{2}(t)}^{q_{T}}\left|e^{-x} w(t) e^{q_{2}(t)}-e^{x} w_{T} e^{-q_{T}}\right|^{r} d x .
$$

We see that for the first term in this sum, we have

$$
\int_{q_{2}(t)}^{q_{T}}\left|e^{-x} e^{q_{2}(t)} p_{1}(t)\left(e^{-q(t)}-1\right)\right|^{r} d x, \lesssim\left(q_{T}-q_{2}(t)\right) \cdot\left|e^{q_{2}(t)} p_{1}(t)\left(e^{-q(t)}-1\right)\right|^{r}
$$

For the second term of this sum, we use the fact that $\left|e^{-x} w e^{q_{2}}-e^{x} w_{T} e^{-q_{T}}\right|^{r} \lesssim 1$ and Hölder's inequality to get

$$
\int_{q_{2}(t)}^{q_{T}}\left|e^{-x} w(t) e^{q_{2}(t)}-e^{x} w_{T} e^{-q_{T}}\right|^{r} d x \lesssim\left(q_{T}-q_{2}(t)\right)
$$

Putting these estimates together, we can now evaluate the limit of $I_{3}$ as $t \rightarrow T$ via

$$
\lim _{t \rightarrow T} I_{3} \lesssim \lim _{t \rightarrow T}\left(\left(q_{T}-q_{2}(t)\right) \cdot\left|e^{q_{2}(t)} p_{1}(t)\left(e^{-q(t)}-1\right)\right|^{r}+\left(q_{T}-q_{2}(t)\right)=0\right.
$$

Evaluating $I_{4}$. This term is handled in precisely the same fashion as $I_{1}$. Performing the integration gives us

$$
I_{4}(t)=\frac{e^{-r q_{T}}}{r} \cdot\left|p_{1} e^{q_{1}(t)}+p_{2} e^{q_{2}(t)}-w_{T} e^{q_{T}}\right|^{r} .
$$

Rewriting the expression inside of the absolute value gives us

$$
p_{1} e^{q_{1}(t)}+p_{2} e^{q_{2}(t)}-w_{T} e^{q_{T}}=e^{q_{2}(t)} p_{1}\left(e^{-q(t)}-1\right)+w(t) e^{q_{2}(t)}-w_{T} e^{q_{T}} .
$$

Therefore, using the above identity along with the triangle inequality yields

$$
\lim _{t \rightarrow T^{-}} I_{4}(t) \leq \lim _{t \rightarrow T^{-}} \frac{e^{-r q_{T}}}{r} \cdot\left(\left|e^{q_{2}(t)} p_{1}\left(e^{-q(t)}-1\right)\right|+\left|w(t) e^{q_{2}(t)}-w_{T} e^{q_{T}}\right|\right)^{r}=0
$$

Summarizing the $L^{r}$ convergence, $1 \leq r<4$. As we have computed $\lim _{t \rightarrow T^{-}} I_{j}(t)=0$ for $j=1,2,3,4$ it immediately follows that

$$
\lim _{t \rightarrow T^{-}}\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}^{r}=\lim _{t \rightarrow T^{-}}\left(I_{1}(t)+I_{2}(t)+I_{3}(t)+I_{4}(t)\right)=0 .
$$

Corollary 2. As $t$ goes to $T$ our 2-peakon solution $u(t)$ converges in $H^{s}, s \leq 0$, to the antipeakon $v_{T}=w_{T} e^{-\left|x-q_{T}\right|}$.

Now that we have successfully established pointwise and $L^{r}$ convergence, we are ready to move on to a much stronger result that is of interest in and of itself. As $t \rightarrow T$, the antipeakon-peakon solution converges to a single solitary antipeakon in $H^{s}$, for $s<5 / 4$.

Theorem 5 (Convergence in $H^{s}$ ). For $s<5 / 4$, our 2-peakon solution $u(t)$ converges to the antipeakon $v_{T}$ in $H^{s}$, i.e.

$$
\begin{equation*}
\lim _{t \rightarrow T}\left\|u(t)-v_{T}\right\|_{H^{s}}=0 \tag{6.5}
\end{equation*}
$$

Proof. We will begin by simplifying the $H^{s}$ norm of $u(x, t)-v_{T}(x)$. We have

$$
\left\|u(t)-v_{T}\right\|_{H^{s}}^{2}=\int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s}\left|\widehat{u}(\xi, t)-\widehat{v_{T}}(\xi)\right|^{2} d \xi
$$

Our objective, when taking the limit, will be to move the limit inside of the integral. Thus, the first thing we should verify is whether pointwise, we have

$$
\left(1+\xi^{2}\right)^{s}\left|\widehat{u}(\xi, t)-\widehat{v_{T}}(\xi)\right|^{2} \rightarrow 0 \quad \text { as } \quad t \rightarrow T^{-}
$$

We have

$$
|\widehat{u}(\xi, t)-\widehat{u}(\xi, T)| \leq\|u(x, t)-u(x, T)\|_{L^{1}}
$$

As we have proved that $u(x, t) \rightarrow u(x, T)$ in $L^{1}$, we have

$$
\lim _{t \rightarrow T^{-}}\left(1+\xi^{2}\right)^{s}\left|\widehat{u}(\xi, t)-\widehat{v_{T}}(\xi)\right|^{2}=0
$$

Next, we will define the bounding functions that will allow us to apply the generalized Dominated Convergence Theorem (gDCT). We set

$$
f_{t}(\xi) \doteq\left(1+\xi^{2}\right)^{s}\left|\widehat{u}(\xi, t)-\widehat{v_{T}}(\xi)\right|^{2} \leq 4\left(1+\xi^{2}\right)^{s}\left(|\widehat{u}(\xi, t)|^{2}+\left|\widehat{v_{T}}(\xi)\right|^{2}\right) \doteq g_{t}(\xi)
$$

Next, we need to establish that the $g_{t}$ 's, have a pointwise limit $g$. The most obvious candidate for $g$ is

$$
g(\xi)=8\left(1+\xi^{2}\right)^{s}\left|\widehat{v}_{T}(\xi)^{2}\right|
$$

Indeed, using the laws of limits, we have $\widehat{u}(\xi, t) \rightarrow \widehat{v}_{T}(\xi)$ pointwise in $\xi$ implies $g_{t} \rightarrow g$ pointwise in $\xi$. To finish satisfying the hypotheses of gDCT , we must now establish the integral properties of the $g_{t}$ 's. We have

$$
\begin{aligned}
\lim _{t \rightarrow T} \int_{\mathbb{R}} g_{t}(\xi) d \xi & =\lim _{t \rightarrow T} 4 \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s}|\widehat{u}(\xi, t)|^{2} d \xi+\lim _{t \rightarrow T} 4 \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s}\left|\widehat{v_{T}}(\xi)\right|^{2} d \xi \\
& \stackrel{(5.4)}{=} 32 c_{s} w_{T}^{2}
\end{aligned}
$$

where the left limit uses the hypothesis that $s<5 / 4$. Furthermore, $g$ is integrable and

$$
\int_{\mathbb{R}} g(\xi) d \xi=\int_{\mathbb{R}} 8\left|v_{T}(\xi)^{2}\right| d \xi=8 \cdot\left\|v_{T}\right\|_{H^{s}}^{2}=8 \cdot 4 c_{s} w_{T}^{2}=32 c_{s} w_{T}^{2}
$$

Therefore, we have $\int g_{t} \rightarrow \int g$. We now see that the hypotheses for gDCT are satisfied. Thus, we can conclude

$$
\lim _{t \rightarrow T} \int_{\mathbb{R}} f_{t}(\xi) d \xi=\int_{\mathbb{R}} f(x) d \xi
$$

which written more explicitly tells us that

$$
\lim _{t \rightarrow T} \int_{\mathbb{R}}\left(1+\xi^{2}\right)^{s}\left|\widehat{u}(\xi, t)-\widehat{v_{T}}(\xi)\right|^{2} d \xi=0
$$

Thus, we can conclude that as $t \rightarrow T$, we have $u(t) \rightarrow v_{T}$ in $H^{s}$.
Proof of Theorem 4. Translating the NE 1-peakon solution (1.6) by $x_{0}$ and choosing the minus sign we obtain the following antipeakon solution for NE

$$
v(x, t)=-\sqrt{c} e^{-\left|\left(x-x_{0}\right)-c t\right|}, \quad \text { for any } \quad c>0 \quad \text { and } \quad x_{0} \in \mathbb{R}
$$

Choosing

$$
c=w_{T}^{2}, \quad \text { and } \quad x_{0}=q_{T}-w_{T}^{2} T
$$

we obtain the NE antipeakon solution

$$
v(x, t)=-\sqrt{w_{T}^{2}} e^{-\left|\left(x-q_{T}+w_{T}^{2} T\right)-w_{T}^{2} t\right|} .
$$

Since at $t=T$ we have

$$
v(x, T)=w_{T} e^{-\left|x-q_{T}\right|}=u(x, T)
$$

we see that we have constructed two different NE solutions, which belong in $H^{s}, s<5 / 4$, and agree at $t=T$. From here, a change of variables can recast these two solutions as initial value problems at time $t=0$. This proves failure of uniqueness in this range of Sobolev spaces.

The case $s=5 / 4$ : If $s=5 / 4$ then there are two possibilities. Either our 2-peakon solution $u(t)$ does not converge in $H^{5 / 4}$ in which case we can prove (by a standard argument) that continuity of the solution map fails, or $u(t)$ converges in $H^{5 / 4}$ and has limit $u(T)$ (since this is the limit for lower Sobolev exponents). In the second case, we have non-uniqueness like in Theorem 4. This result completes the proof of both of Theorems 1 and 2 in the non-periodic case.

## 7. The Periodic Case

7.1. Outline of the proofs in the periodic case. The proofs of Theorems 1 and 2 have been demonstated on the line and we now present these proofs on the circle, $\mathbb{T}=\mathbb{R} / 2 \pi \mathbb{Z}$. The key ingredient is using a periodic version of the peakon. In Subsection 7.2, we construct the 2-peakon solutions on the circle having the properties described in Theorem 1. In Subsection 7.3, we estimate the $H^{s}$-norm of the 2-peakon solutions and in Subsection 7.4 we choose the parameters so that both the lifespan and the size of the 2-peakon solution at the initial time are simultaneously small. In Subsection 7.5, we prove norm-inflation and illposedness for $5 / 4<s<3 / 2$. Finally, in Subsection 7.6, we prove non-uniqueness for $s<5 / 4$ and explain the ill-posedness of NE for $s=5 / 4$.
7.2. Construction of 2-peakon solutions on the circle. The 2-peakon solutions to the periodic version of Novikov's equation are similar to those on the real-line with the caveat that the peak is generated by periodizing the hyperbolic cosine rather that using the exponential of the negative absolute value. The following equations are taken from [GH] and [HM] and can also be derived in a straightforward fashion.

The periodic Novikov 2-peakon solutions are of the form

$$
\begin{equation*}
u(x, t)=p_{1}(t) \cosh \left(\left[x-q_{1}(t)\right]_{p}-\pi\right)+p_{2}(t) \cosh \left(\left[x-q_{2}(t)\right]_{p}-\pi\right), \tag{7.1}
\end{equation*}
$$

where $[\cdot]_{p}$ periodizes our function and is defined by the floor

$$
\begin{equation*}
[x]_{p}=x-2 \pi\left\lfloor\frac{x}{2 \pi}\right\rfloor . \tag{7.2}
\end{equation*}
$$

We see that $u$ solves NE if the momenta $p_{1}, p_{2}$ and the positions $q_{1}, q_{2}$ satisfy the following system of ODEs, which can be obtained by using Theorem 1.2 of [HM] with the choice of
parameters $a=0, b=3$. The $4 \times 4$ system we get is

$$
\begin{align*}
q_{1}^{\prime} & =p_{1}^{2}\left[1+\sinh ^{2} \pi\right]+2 p_{1} p_{2} \cosh \pi \cosh \left(\left[q_{1}-q_{2}\right]_{p}-\pi\right)+p_{2}^{2}\left[1+\sinh ^{2}\left(\left[q_{1}-q_{2}\right]_{p}-\pi\right)\right], \\
q_{2}^{\prime} & =p_{2}^{2}\left[1+\sinh ^{2} \pi\right]+2 p_{1} p_{2} \cosh \pi \cosh \left(\left[q_{2}-q_{1}\right]_{p}-\pi\right)+p_{1}^{2}\left[1+\sinh ^{2}\left(\left[q_{2}-q_{1}\right]_{p}-\pi\right)\right],  \tag{7.3}\\
p_{1}^{\prime} & =-p_{1} p_{2} \sinh \left(\left[q_{1}-q_{2}\right]_{p}-\pi\right)\left[p_{1} \cosh \pi+p_{2} \cosh \left(\left[q_{1}-q_{2}\right]_{p}-\pi\right)\right], \\
p_{2}^{\prime} & =-p_{1} p_{2} \sinh \left(\left[q_{2}-q_{1}\right]_{p}-\pi\right)\left[p_{2} \cosh \pi+p_{1} \cosh \left(\left[q_{2}-q_{1}\right]_{p}-\pi\right)\right] .
\end{align*}
$$

Setting

$$
E(x) \doteq \frac{1}{\cosh \pi} \cdot \cosh \left([x]_{p}-\pi\right), \quad E^{\prime}(x) \doteq \frac{1}{\cosh \pi} \cdot \sinh \left([x]_{p}-\pi\right),
$$

and using $q=q_{2}-q_{1}$, our system can be written in the more compact form

$$
\begin{align*}
q_{1}^{\prime} & =\cosh ^{2} \pi \cdot\left(p_{1}+p_{2} E(q)\right)^{2}, \\
q_{2}^{\prime} & =\cosh ^{2} \pi \cdot\left(p_{1} E(q)+p_{2}\right)^{2}, \\
p_{1}^{\prime} & =\cosh ^{2} \pi \cdot p_{1} p_{2}\left(p_{1}+p_{2} E(q)\right) E^{\prime}(q),  \tag{7.4}\\
p_{2}^{\prime} & =-\cosh ^{2} \pi \cdot p_{1} p_{2}\left(p_{1} E(q)+p_{2}\right) E^{\prime}(q) .
\end{align*}
$$

Initial Data. From this point, we make the same initial data assumptions as in the real-line case. We take the positions the positions, $q_{1}$ and $q_{2}$ at time $t=0$ to be

$$
q_{1}(0)=-a \quad \text { and } \quad q_{2}=a, \quad \text { for some } a>0 .
$$

For the initial momenta, we shall assume that at time $t=0$ that

$$
\begin{equation*}
p_{2}(0)=b \gg 1, \quad p_{1}(0)=-(b+\delta), \quad \delta>0 . \tag{7.5}
\end{equation*}
$$

With these assumptions, the initial profile $u_{0}(x)=u(x, 0)$ is the asymmetric periodic antipeakonpeakon

$$
\begin{equation*}
u_{0}(x)=-(b+\delta) \cosh \left([x+a]_{p}-\pi\right)+b \cosh \left([x-a]_{p}-\pi\right) . \tag{7.6}
\end{equation*}
$$

This initial profile for $u$ is displayed in the Figure 3.


Figure 3: Initial profile $u_{0}(x)$

Following the intuition we developed in the real-line case, we again will examine the ODEsystem (7.4) in the derived variables $p, q, w, z$ given by

$$
\begin{array}{rlrl}
q(t) & =q_{2}(t)-q_{1}(t), & q(0) & =2 a>0 \\
p(t) & =p_{2}(t)-p_{1}(t), & p(0)=2 b+\delta>0, \\
w(t) & =p_{2}(t)+p_{1}(t), & w(0)=-\delta<0,  \tag{7.7}\\
z(t) & =p_{2}(t) \cdot p_{1}(t), & z(0)=-b(b+\delta)<0 .
\end{array}
$$

Deriving equations for $q, p, w$ and $z$ on the circle. Beginning with $q$, we follow the same strategy as in the non-periodic case. We see that

$$
\begin{align*}
q^{\prime} & =\cosh ^{2} \pi \cdot\left(p_{1} E(q)+p_{2}\right)^{2}-\cosh ^{2} \pi \cdot\left(p_{1} E(q)+p_{2}\right)^{2}, \\
& =\cosh ^{2} \pi \cdot\left(p_{2}-p_{1}\right)\left(p_{2}+p_{1}\right)\left(1-E^{2}(q)\right), \\
& =\cosh ^{2} \pi \cdot p w\left(1-E^{2}(q)\right) . \tag{7.8}
\end{align*}
$$

The computations for $p, w$ and $z$ follow the same strategy, and we arrive at the $4 \times 4$ system

$$
\begin{array}{ll}
q^{\prime}=\cosh ^{2} \pi \cdot p w\left(1-E^{2}(q)\right), & q(0)=2 a>0, \\
p^{\prime}=-\cosh ^{2} \pi \cdot w z(1+E(q)) E^{\prime}(q), & p(0)=2 b+\delta>0, \\
w^{\prime}=-\cosh ^{2} \pi \cdot z p(1-E(q)) E^{\prime}(q), & w(0)=-\delta<0,  \tag{7.9}\\
z^{\prime}=\cosh ^{2} \pi \cdot z w p E(q) E^{\prime}(q), & z(0)=-b(b+\delta)<0 .
\end{array}
$$

This derived system of ODEs is more easily manipulated than the original $4 \times 4$ system, and we are now ready to tackle Proposition 1 in the periodic setting.

Proposition 6 (Periodic version of Proposition 1). The system of differential equations (7.9) has a unique smooth solution $(q(t), p(t), w(t), z(t))$ in an interval $[0, T)$, for some $T>0$, such that $z=z(t)$ is decreasing and in terms of $q$ is expressed by the formula

$$
\begin{equation*}
z=\frac{-z_{1}}{\left(1-E^{2}(q)\right)^{1 / 2}}<0, \quad \text { where } \quad z_{1}=b(b+\delta)\left(1-E^{2}\left(q_{0}\right)\right)^{1 / 2}>0 \tag{7.10}
\end{equation*}
$$

$p=p(t)$ is decreasing and as a function of $q$ is expressed by the formula

$$
\begin{equation*}
p(t)=\left(p_{0}^{2}+2 z_{1}\left[\frac{1+E(q(t))}{\sqrt{1-E^{2}(q(t))}}-\frac{1+E\left(q_{0}\right)}{\sqrt{1-E^{2}\left(q_{0}\right)}}\right]\right)^{1 / 2}>0 \tag{7.11}
\end{equation*}
$$

and $w=w(t)$ is decreasing and as a function of $q$ is expressed by the formula

$$
\begin{equation*}
w(t)=-\left(w_{0}^{2}+2 z_{1}\left[\frac{\sqrt{1-E^{2}\left(q_{0}\right)}}{1+E\left(q_{0}\right)}-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}\right]\right)^{1 / 2}<0 . \tag{7.12}
\end{equation*}
$$

The difference of the positions $q=q(t)$ is decreasing and satisfies the initial value problem

$$
\begin{align*}
& q^{\prime}=-f(q) \doteq- \cosh ^{2} \pi \cdot\left(w_{0}^{2}+2 z_{1}\left[\frac{\sqrt{1-E^{2}\left(q_{0}\right)}}{1+E\left(q_{0}\right)}-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}\right]\right)^{1 / 2}  \tag{7.13}\\
& \cdot\left(p_{0}^{2}+2 z_{1}\left[\frac{1+E(q(t))}{\sqrt{1-E^{2}(q(t))}}-\frac{1+E\left(q_{0}\right)}{\sqrt{1-E^{2}\left(q_{0}\right)}}\right]\right)^{1 / 2} \cdot\left(1-E^{2}(q)\right) \\
& q(0)=q_{0}=2 a>0
\end{align*}
$$

Furthermore, the initial value problem (7.13) is dominated by the simpler initial value problem

$$
q^{\prime}=-g(q) \doteq-q_{1}\left(1-e^{-2 q}\right)^{3 / 4}, \quad 0<q(0)=2 a<1 / 2
$$

where

$$
q_{1}=\delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4}
$$

Proof. We begin by solving for $p, w$ and $z$ in terms of $q$. After this task is completed, we can form an autonomous equation for $q$ by substituting in these results.

Expressing $z$ in terms of $q$. Using the equation for $z^{\prime}$ and $q^{\prime}$ we find

$$
\frac{z^{\prime}}{q^{\prime}}=\frac{\cosh ^{2} \pi \cdot z w p E(q) E^{\prime}(q)}{\cosh ^{2} \pi \cdot p w\left(1-E^{2}(q)\right)}=\frac{z E(q) E^{\prime}(q)}{1-E^{2}(q)} \quad \text { or } \quad \frac{z^{\prime}}{z}=\frac{E(q) E^{\prime}(q) q^{\prime}}{1-E^{2}(q)} .
$$

Since $z(0)<0$ we shall assume that $z(t)$ will remain negative. Therefore, from the last relation we have

$$
\frac{d}{d t}[\ln (-z)]=-\frac{1}{2} \frac{d}{d t}\left[\ln \left(1-E^{2}(q)\right)\right] .
$$

Integrating this equation from 0 to $t$ gives

$$
\ln \left[\frac{z(t)}{z_{0}}\right]=-\frac{1}{2} \ln \left[\frac{1-E^{2}(q(t))}{1-E^{2}\left(q_{0}\right)}\right] .
$$

Solving for $z(t)$, we find formula (7.10) for $z$ in terms of $q$.
Expressing $w$ in terms of $q$. Dividing the equation for $w^{\prime}$ by the equation for $q^{\prime}$ we have

$$
\begin{align*}
& \frac{w^{\prime}}{q^{\prime}}=\frac{-\cosh ^{2} \pi \cdot z p(1-E(q)) E^{\prime}(q)}{\cosh ^{2} \pi \cdot p w\left(1-E^{2}(q)\right)}=\frac{-z(1-E(q)) E^{\prime}(q)}{w\left(1-E^{2}(q)\right)},  \tag{7.14}\\
& \text { or } \quad w w^{\prime}=-z \cdot \frac{(1-E(q)) E^{\prime}(q) q^{\prime}}{1-E^{2}(q)} \tag{7.15}
\end{align*}
$$

Substituting the formula for $z$ given by (7.10) into the above equation gives us

$$
\begin{equation*}
w w^{\prime}=\frac{z_{1}}{\left(1-E^{2}(q)\right)^{1 / 2}} \cdot \frac{(1-E(q)) E^{\prime}(q) q^{\prime}}{1-E^{2}(q)}=\frac{z_{1}(1-E(q)) E^{\prime}(q) q^{\prime}}{\left(1-E^{2}(q)\right)^{3 / 2}} . \tag{7.16}
\end{equation*}
$$

Making the change of variables $u=E(q(t)), d u=E^{\prime}(q(t)) q^{\prime}(t) d t$, we obtain

$$
\int \frac{(1-E(q)) E^{\prime}(q) q^{\prime}}{\left(1-E^{2}(q)\right)^{3 / 2}} d t=\int \frac{1-u}{\left(1-u^{2}\right)^{3 / 2}} d u=-\frac{\sqrt{1-u^{2}}}{1+u}+C=-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}+C .
$$

Therefore, relation (7.16) reads as

$$
\begin{equation*}
\frac{d}{d t}\left[\frac{1}{2} w^{2}\right]=-z_{1} \frac{d}{d t}\left[\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}\right] . \tag{7.17}
\end{equation*}
$$

Integrating this equation from 0 to $t$ gives

$$
\begin{equation*}
\frac{1}{2}\left[w^{2}(t)-w_{0}^{2}\right]=z_{1}\left[\frac{\sqrt{1-E^{2}\left(q_{0}\right)}}{1+E\left(q_{0}\right)}-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}\right] . \tag{7.18}
\end{equation*}
$$

We are thus able to solve for $w(t)$ in terms of $q(t)$, which gives us formula (7.12).

Expressing $p$ in terms of $q$. Dividing the equation for $p^{\prime}$ by the equation for $q^{\prime}$ we have

$$
\begin{align*}
& \frac{p^{\prime}}{q^{\prime}}=\frac{\left.-\cosh ^{2} \pi \cdot w z(1+E(q)) E^{\prime}(q)\right)}{\cosh ^{2} \pi \cdot p w\left(1-E^{2}(q)\right)}=\frac{\left.-z(1+E(q)) E^{\prime}(q)\right)}{p\left(1-E^{2}(q)\right)}  \tag{7.19}\\
& \text { or } \quad p p^{\prime}=-z \cdot \frac{(1+E(q)) E^{\prime}(q) q^{\prime}}{1-E^{2}(q)} \tag{7.20}
\end{align*}
$$

Substituting in the above relation the formula for $z$ given by (7.10) we have

$$
\begin{equation*}
p p^{\prime}=\frac{z_{1}}{\left(1-E^{2}(q)\right)^{1 / 2}} \cdot \frac{(1+E(q)) E^{\prime}(q) q^{\prime}}{1-E^{2}(q)}=\frac{z_{1}(1+E(q)) E^{\prime}(q) q^{\prime}}{\left(1-E^{2}(q)\right)^{3 / 2}} \tag{7.21}
\end{equation*}
$$

Next, we make the change of variables $u=E(q(t)), d u=E^{\prime}(q(t)) q^{\prime}(t) d t$ and get

$$
\int \frac{(1+E(q)) E^{\prime}(q) q^{\prime}}{\left(1-E^{2}(q)\right)^{3 / 2}} d t=\int \frac{1+u}{\left(1-u^{2}\right)^{3 / 2}} d u=\frac{1+u}{\left(1-u^{2}\right)^{1 / 2}}+C=\frac{1+E(q)}{\left(1-E^{2}(q)\right)^{1 / 2}}+C
$$

Therefore, relation (7.21) reads as follows

$$
\frac{d}{d t}\left[\frac{1}{2} p^{2}\right]=z_{1} \frac{d}{d t}\left[\frac{1+E(q)}{\sqrt{1-E^{2}(q)}}\right]
$$

Integrating this equation from 0 to $t$ gives us

$$
\frac{1}{2}\left[p^{2}(t)-p_{0}^{2}\right]=z_{1}\left[\frac{1+E(q(t))}{\sqrt{1-E^{2}(q(t))}}-\frac{1+E\left(q_{0}\right)}{\sqrt{1-E^{2}\left(q_{0}\right)}}\right]
$$

and we are able to solve for $p(t)$ and obtain formula (7.11).
Solving the $q$ ODE. Starting with the differential equation for $q$, which is $q^{\prime}=\cosh ^{2} \pi$. $p w\left(1-E^{2}(q)\right)$, we substitute in for $w$ and $p$ their expressions (7.12) and (7.11) respectively. We consequently obtain the following autonomous differential equation for $q$

$$
\begin{array}{rl}
q^{\prime}=-f(q) \doteq & \cosh ^{2} \pi \cdot\left\{-\left(w_{0}^{2}+2 z_{1}\left[\frac{\sqrt{1-E^{2}\left(q_{0}\right)}}{1+E\left(q_{0}\right)}-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))}\right]\right)^{1 / 2}\right\}  \tag{7.22}\\
& \cdot\left(p_{0}^{2}+2 z_{1}\left[\frac{1+E(q(t))}{\sqrt{1-E^{2}(q(t))}}-\frac{1+E\left(q_{0}\right)}{\sqrt{1-E^{2}\left(q_{0}\right)}}\right]\right)^{1 / 2} \cdot\left(1-E^{2}(q)\right) \\
q(0)=q_{0}=2 a>0 & 0
\end{array}
$$

Next, we observe that

$$
\begin{equation*}
\frac{\sqrt{1-E^{2}\left(q_{0}\right)}}{1+E\left(q_{0}\right)}-\frac{\sqrt{1-E^{2}(q(t))}}{1+E(q(t))} \geq 0, \quad 0 \leq q \leq q_{0}<\pi . \tag{7.23}
\end{equation*}
$$

This inequality follows from the fact that

$$
\left(\frac{\sqrt{1-E^{2}(x)}}{1+E(x)}\right)^{\prime}=\frac{-E^{\prime}(x)}{2 \sqrt{1+E(x)}}
$$

Since the denominator is always positive, the sign of this derivative is controlled by numerator, $-E^{\prime}(x)=-\frac{1}{\cosh (\pi)} \sinh \left([x]_{p}-\pi\right)$, which is positive for $x \in[0, \pi)$. Next, we have

$$
\begin{equation*}
p_{0}^{2}-2 z_{1} \frac{1+E\left(q_{0}\right)}{\sqrt{1-E^{2}\left(q_{0}\right)}} \geq 0 \Longleftrightarrow \frac{(2 b+\delta)^{2}}{2 b(b+\delta)} \geq 1+E\left(q_{0}\right) \tag{7.24}
\end{equation*}
$$

Our choice of initial data allows for the inequality $1+E\left(q_{0}\right) \leq 2$, and we have

$$
\frac{(2 b+\delta)^{2}}{2 b(b+\delta)} \geq 2 \Longleftrightarrow 4 b^{2}+4 b \delta+\delta^{2}>4 b^{2}+4 b \delta \Longleftrightarrow \delta^{2}>0, \quad \text { which is true. }
$$

Now, using (7.23) and (7.24) we see that the function $f(q)$ in the right-hand side of the differential equation (7.22) can be bounded from below as follows

$$
f(q) \geq \cosh ^{2} \pi \cdot \delta\left(2 b(b+\delta)\left(1-E^{2}\left(q_{0}\right)\right)^{1 / 2}\right)^{\frac{1}{2}}\left(\left[\frac{1+E(q(t))}{\sqrt{1-E^{2}(q(t))}}\right]\right)^{\frac{1}{2}} \cdot\left(1-E^{2}(q)\right)
$$

To continue in our objective of finding a simpler dominating function for $f$, analogous to the strategy in the real-line case of this proof, we use the fact that $E(q) \geq 0$ in conjunction with the following lemma.

Lemma 3. For $c \geq 2 \cosh ^{2}(\pi) / \sinh (2 \pi-1)$, and $x \in[0,1 / 2]$,

$$
\begin{equation*}
c\left(1-E^{2}(x)\right) \geq 1-e^{-2 x} . \tag{7.25}
\end{equation*}
$$

Furthermore, we have the inequality

$$
\begin{equation*}
1-E^{2}\left(q_{0}\right) \geq \frac{1}{3} q_{0} . \tag{7.26}
\end{equation*}
$$

In particular, we will take $c=3$ in later computations.
Proof. Define the function

$$
f(x) \doteq\left(1-e^{-2 x}\right)-\left(c \cdot\left[1-E^{2}(x)\right]\right)
$$

Computing the derivative of $f(x)$ shows that it will be negative for $x \in(0,1 / 2]$, and

$$
c \geq \frac{2 \cosh ^{2} \pi}{\sinh (2 \pi-1)}
$$

As the (7.25) has been established, we now move onto proving (7.26). This inequality is obtained by applying our first inequality and then using the exponential inequality. We get

$$
1-E^{2}\left(q_{0}\right) \geq \frac{1}{3}\left(1-e^{-2 q_{0}}\right) \geq \frac{1}{3} q_{0} .
$$

With the above lemma, we are now ready to return to the proof of the proposition.
Dominating Equation (periodic version). Using the above inequalities, and following the same strategy as in the non-periodic case, we obtain

$$
f(q) \geq\left[\frac{\sqrt{2} \cosh ^{2} \pi}{3} \cdot \delta \cdot \sqrt{b(b+\delta)} \cdot q_{0}^{1 / 4}\right] \cdot\left(1-e^{-2 q}\right)^{3 / 4}
$$

Since $\frac{\sqrt{2} \cosh ^{2} \pi}{3} \geq 1$, we can remove this factor as we are bounding from below. Consequently, $f(q)$ has precisely the same lower bound as in the real-case give by

$$
f(q) \geq \delta \cdot \sqrt{b(b+\delta)} \cdot q_{0}^{1 / 4} \cdot\left(1-e^{-2 q}\right)^{3 / 4}=q_{1}\left(1-e^{-2 q}\right)^{3 / 4}
$$

where the constant $q_{1}$ is given by

$$
q_{1}=\delta \cdot \sqrt{b(b+\delta)} \cdot q_{0}^{1 / 4}
$$

Thus, we see that the complicated initial value problem for $q(7.22)$ is dominated by

$$
\begin{equation*}
q^{\prime}=-q_{1}\left(1-e^{-2 q}\right)^{3 / 4}, \quad q(0)=q_{0}=2 a>0 . \tag{7.27}
\end{equation*}
$$

This ODE is precisely the same as the one derived in the real-line case. Therefore we can immediately arrive at the same conclusions for $q$.

Proposition 7 (Periodic version of Proposition 2). If $r<1$ then for given $q_{0} \in(0,1 / 2)$ and $q_{1}>0$ the solutions to the initial value problem

$$
\begin{equation*}
\frac{d q}{d t}=-g_{r}(q) \doteq-q_{1}\left(1-e^{-2 q}\right)^{r}, \quad q(0)=q_{0}, \tag{7.28}
\end{equation*}
$$

which begins positive and is decreasing, becomes zero in finite time $T$ given by

$$
\begin{equation*}
T=\int_{0}^{q_{0}} \frac{d q}{g_{r}(q)}=\frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{\left(1-e^{-2 q}\right)^{r}} \simeq \frac{1}{1-r} \frac{q_{0}^{1-r}}{q_{1}} . \tag{7.29}
\end{equation*}
$$

Corollary 3 (Periodic version of Corollary 1). If $0<q_{0}<1 / 2$ and $b>1, \delta>0$ satisfy condition (7.5) then the solution to the initial value problem (7.13) begins positive, is decreasing, and becomes zero in finite time $T$ given by

$$
\begin{equation*}
T=\int_{0}^{q_{0}} \frac{d q}{f(q)} \leq \frac{1}{q_{1}} \int_{0}^{q_{0}} \frac{d q}{\left(1-e^{-2 q}\right)^{3 / 4}} \simeq \frac{q_{0}^{1 / 4}}{q_{1}} \simeq \frac{q_{0}^{1 / 4}}{\delta \sqrt{2 b(b+\delta)} \cdot q_{0}^{1 / 4}} \simeq \frac{1}{\delta \sqrt{2 b(b+\delta)}} . \tag{7.30}
\end{equation*}
$$

We summarize the above results in the following Theorem.
Theorem 6 (Periodic version of Theorem 3). For given $0<a \leq 1 / 4, b>1$ and $\delta>0$ satisfying condition (7.5), the initial value problem for the positions $q_{1}, q_{2}$ and the momenta $p_{1}, p_{2}$

$$
\begin{align*}
q_{1}^{\prime}=\cosh ^{2} \pi \cdot\left(p_{1}+p_{2} E(q)\right)^{2}, & q_{1}(0)=-a, \\
q_{2}^{\prime}=\cosh ^{2} \pi \cdot\left(p_{1} E(q)+p_{2}\right)^{2}, & q_{2}(0)=a, \\
p_{1}^{\prime}=\cosh ^{2} \pi \cdot p_{1} p_{2}\left(p_{1}+p_{2} E(q)\right) E^{\prime}(q), & p_{1}(0)=-(b+\delta),  \tag{7.31}\\
p_{2}^{\prime}=-\cosh ^{2} \pi \cdot p_{1} p_{2}\left(p_{1} E(q)+p_{2}\right) E^{\prime}(q), & p_{2}(0)=b,
\end{align*}
$$

has a unique smooth solution $\left(q_{1}(t), q_{2}(t), p_{1}(t), p_{2}(t)\right)$ with a finite lifespan $T$, which is the zero of $q=q_{2}-q_{1}$, satisfying the estimate (7.30) and such that

$$
\begin{aligned}
& p_{1}=\frac{w-p}{2}<0, \text { decreasing, } \lim _{t \rightarrow T^{-}} p_{1}(t)=-\infty, \text { and }-p_{1} \simeq p \simeq q^{-1 / 4}, \text { and } \\
& p_{2}=\frac{w+p}{2}>0, \text { increasing, } \lim _{t \rightarrow T^{-}} p_{2}(t)=\infty, \text { and } p_{2} \simeq p \simeq q^{-1 / 4},
\end{aligned}
$$

where $p$ and $w$ are given in Proposition 6. Also, $w=p_{1}+p_{2}$ is decreasing from $w_{0}<0$ to $w_{T}$, where

$$
w_{T} \doteq \lim _{t \rightarrow T^{-}} w(t)=-\left(\delta^{2}+2 b(b+\delta)(1-E(2 a))\right)^{\frac{1}{2}}
$$

Finally, the 2-peakon

$$
u(x, t)=p_{1}(t) \cosh \left(\left[x-q_{1}(t)\right]_{p}-\pi\right)+p_{2}(t) \cosh \left(\left[x-q_{2}(t)\right]_{p}-\pi\right), x \in \mathbb{T}, 0<t<T .
$$

is a solution to NE with following the asymmetric antipeakon-peakon initial profile

$$
u(x, 0)=-(b+\delta) \cosh \left([x+a]_{p}-\pi\right)+b \cosh \left([x-a]_{p}-\pi\right),
$$

The quantities $p_{1}, p_{2}, q$ and $w$ have similar properties to their analogues defined on the line, and we refer to Figure 2 for a visualization of them.
7.3. Calculating the Norm on the circle. We begin with the following proposition which summarizes the calculation of the $H^{s}$ norm of $u$. This computation is nearly identical to non-periodic case with the exception of an extra factor of $\sinh ^{2} \pi$.

Proposition 8 (Periodic version of Proposition 3). Let $u(t)$ be the two-peakon solution (7.1) to the NE equation. Then on $[0, T)$ we have

$$
\begin{aligned}
& \|u(t)\|_{H^{s}}^{2}=16 \sinh ^{2} \pi \cdot r(t) p_{1}^{2}(t) Q_{s}(q)+4 \sinh ^{2} \pi \cdot c_{s}(1-r(t))^{2} p_{1}^{2}(t) \\
& \text { with } \quad r(t) \doteq-\frac{p_{2}(t)}{p_{1}(t)}
\end{aligned}
$$

where $c_{s}=\sum_{-\infty}^{\infty}\left(1+n^{2}\right)^{s-2}$ and $Q_{s}(q)$, which is given below, satisfies the estimates:

$$
Q_{s}(q) \doteq \sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s-2} \sin ^{2}\left(\frac{q n}{2}\right) \simeq \begin{cases}q^{3-2 s}, & 1 / 2<s<3 / 2 \\ q^{2} \cdot \ln (1 / q), & s=1 / 2 \\ q^{2}, & s<1 / 2\end{cases}
$$

Proof. We begin by noting that the Fourier transform of $E$ is calculated as

$$
\widehat{E}(n)=\left(2 \cdot \frac{\sinh (\pi)}{\cosh (\pi)}\right) \cdot \frac{1}{1+n^{2}} .
$$

Recalling that the 2-peakon $u$ can be written as

$$
u(x, t)=\cosh \pi \cdot\left(p_{1}(t) E\left(x-q_{1}(t)\right)+p_{2}(t) E\left(x-q_{2}(t)\right)\right)
$$

we can express the Fourier transform of $u$ as

$$
\widehat{u}(n, t)=\frac{2 \sinh \pi}{1+n^{2}} \cdot p_{1} e^{-i n q_{1}} \cdot\left(1+\frac{p_{2}}{p_{1}} e^{-i n q}\right) .
$$

Taking the square of the $H^{s}$ norm of this quantity, we obtain

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=\sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s}|\widehat{u}(n, t)|^{2}=4 \sinh ^{2} \pi \cdot p_{1}^{2} \sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s-2}\left|1+\frac{p_{2}}{p_{1}} e^{-i n q}\right|^{2} . \tag{7.32}
\end{equation*}
$$

Using Proposition 6 we see that

$$
\begin{equation*}
r=r(t) \doteq-\frac{p_{2}(t)}{p_{1}(t)}=\frac{p+w}{p-w}<1, \quad \text { and } \quad r(t) \nearrow 1 \text { as } t \nearrow T . \tag{7.33}
\end{equation*}
$$

Using $r$ we write (7.32) as follows

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=4 \sinh ^{2} \pi \cdot p_{1}^{2} \sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s-2}\left|1-r e^{-i n q}\right|^{2} \tag{7.34}
\end{equation*}
$$

Expanding out the square of the absolute value inside of the sum (7.34), we have

$$
\left|r e^{i q n}-1\right|^{2}=(1-r)^{2}+4 r \sin ^{2}\left(\frac{q n}{2}\right) .
$$

We therefore obtain the formula

$$
\|u(t)\|_{H^{s}}^{2}=16 \sinh ^{2} \pi \cdot r p_{1}^{2} Q_{s}(q)+4 \sinh ^{2} \pi \cdot c_{s}(1-r)^{2} p_{1}^{2}
$$

where

$$
c_{s}=\sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s-2} \quad \text { and } \quad Q_{s}(q)=\sum_{n=-\infty}^{\infty}\left(1+n^{2}\right)^{s-2} \sin ^{2}\left(\frac{q n}{2}\right)
$$

From this point, we note that $Q_{s}$ has already been estimated in this periodic setting in [HHG]. Using 4.25 from [HHG], and noting 4.28, where the norm is expanded into the sum of the squares of sines, we have

$$
Q_{s} \simeq \sum_{n=1}^{\infty} \sin ^{2}\left(\frac{q n}{2}\right)\left(1+n^{2}\right)^{s-2} \simeq \begin{cases}q^{3 / 2-s}, & 1 / 2<s<3 / 2  \tag{7.35}\\ q \sqrt{\ln (1 / q)}, & s=1,2 \\ q, & s<1 / 2\end{cases}
$$

7.4. Small lifespan and initial data on the circle. This section follows the same argument as in the real-line case, with the exception of an extra factor of $\sinh ^{2} \pi$ stemming from the periodic version of the norm-estimates. We begin by assuming that

$$
p_{2}(0)=b \gg 1 \text { and }-p_{1}(0)=b+\delta, \delta>0
$$

so that the conditions for the existence of our 2-peakon with the lifespan estimate (7.30) hold.
Lifespan Estimate. For given $\varepsilon>0$, we need to find $b>1$ such that $T<\varepsilon$. Since, by Proposition 6 we have

$$
T \lesssim \frac{1}{\delta \sqrt{2 b(b+\delta)}} \leq \frac{1}{\delta b}
$$

we must have

$$
\begin{equation*}
\frac{1}{\delta b} \leq \varepsilon \Longleftrightarrow b \geq \delta^{-1} \varepsilon^{-1} \tag{7.36}
\end{equation*}
$$

Initial Data Estimate. Now, for the same $\varepsilon>0$ we need to find $q_{0}<1 / 8$ such that $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$. For this we use Proposition 8 , from which we have, recalling that $r(t)=-\frac{p_{2}(t)}{p_{1}(t)}$,

$$
\|u(0)\|_{H^{s}}^{2}=16 \sinh ^{2} \pi \cdot b(b+\delta) Q_{s}\left(q_{0}\right)+4 \sinh ^{2} \pi \cdot c_{s} \delta^{2}
$$

This identity implies

$$
\|u(0)\|_{H^{s}}^{2} \leq 32 \sinh ^{2} \pi \cdot b^{2} Q_{s}\left(q_{0}\right)+4 \sinh ^{2} \pi \cdot c_{s} \delta^{2}
$$

Case $1 / 2<s<3 / 2$ : Then by Proposition 8 we have $Q_{s}\left(q_{0}\right) \lesssim q_{0}^{3-2 s}$ and therefore

$$
\|u(0)\|_{H^{s}}^{2} \leq C_{s} \sinh ^{2} \pi \cdot b^{2} q_{0}^{3-2 s}+4 \sinh ^{2} \pi \cdot c_{s} \delta^{2}
$$

For having $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$ it suffices to choose $q_{0}$ and $\delta$ such that $C_{s} \sinh ^{2} \pi \cdot b^{2} q_{0}^{3-2 s}+4 \sinh ^{2} \pi$. $c_{s} \delta^{2} \leq \varepsilon^{2}$, or

$$
4 \sinh ^{2} \pi \cdot c_{s} \delta^{2} \leq \frac{\varepsilon^{2}}{2} \quad \text { and } \quad C_{s} \sinh ^{2} \pi \cdot b^{2} q_{0}^{3-2 s} \leq \frac{\varepsilon^{2}}{2}
$$

The first inequality holds if

$$
\begin{equation*}
\delta \leq \frac{\varepsilon}{2 \sinh \pi \sqrt{2 c_{s}}} \tag{7.37}
\end{equation*}
$$

Taking into consideration (7.37) and (7.36), the second inequality holds if

$$
q_{0}^{3-2 s} \leq \frac{\varepsilon^{2}}{2 C_{s} \sinh ^{2} \pi \cdot b^{2}} \leq \frac{\varepsilon^{2}}{2 C_{s} \delta^{-2} \varepsilon^{-2}}=\frac{\delta^{2} \varepsilon^{4}}{2 C_{s}} \leq \frac{\varepsilon^{2} \varepsilon^{4}}{8 c_{s} \cdot 2 C_{s}}
$$

or

$$
q_{0} \leq\left(\frac{\varepsilon^{6}}{16 c_{s} C_{s}}\right)^{\frac{1}{3-2 s}}
$$

Case $s \leq 1 / 2$ : For a such Sobolev exponent $s$ we have $\|u(0)\|_{H^{s}} \leq\|u(0)\|_{H^{1}}$. This combined with Proposition 8 , which tells us that $Q_{1}\left(q_{0}\right) \lesssim q_{0}$, gives

$$
\|u(0)\|_{H^{s}}^{2} \leq\|u(0)\|_{H^{1}}^{2} \leq C_{1} b^{2} q_{0}+4 c_{1} \delta^{2}
$$

Thus $\left\|u_{0}\right\|_{H^{s}}<\varepsilon$ if $q_{0}$ and $\delta$ satisfy the inequalities $4 c_{1} \delta^{2} \leq \frac{\varepsilon^{2}}{2}$ and $C_{1} b^{2} q_{0} \leq \frac{\varepsilon^{2}}{2}$. These inequality holds if

$$
\delta \leq \frac{\varepsilon}{2 \sqrt{2 c_{1}}} \quad \text { and } \quad q_{0} \leq \frac{\varepsilon^{6}}{16 c_{s} C_{s}}
$$

7.5. Norm-Inflation and illposedness on the circle. From Proposition 8 we have

$$
\begin{equation*}
\|u(t)\|_{H^{s}}^{2}=16 \sinh ^{2} \pi \cdot r(t) p_{1}^{2}(t) Q_{s}(q)+4 \sinh ^{2} \pi \cdot c_{s} p_{1}^{2}(t)(1-r(t))^{2} \tag{7.38}
\end{equation*}
$$

We see that the same argument as in Section 5 holds, with the simple inclusion of a factor of $\sinh ^{2} \pi$. Following these arguments, we see that

$$
\lim _{t \rightarrow T}\|u(t)\|_{H^{s}}^{2}= \begin{cases}\infty \quad(\text { inflation }), & 5 / 4<s<3 / 2  \tag{7.39}\\ \text { may not exist, } & s=5 / 4 \\ 4 \sinh ^{2} \pi \cdot c_{s}\left[\delta^{2}+2 b(b+\delta) \cdot\left(1-e^{-q_{0}}\right)\right], & s<5 / 4\end{cases}
$$

Therefore when $5 / 4<s<3 / 2$ we have norm inflation and ill-posedness for the Novikov equation.
7.6. Non-Uniqueness for $s<5 / 4$ on the circle. As in the case on the line, the NE admits non-unique solutions once we take the Sobolev exponent $s<5 / 4$. This is an equally interesting result as the periodic 2 -peakons maintain the same collision properties as non-periodic ones.

Theorem 7 (Nonuniqueness - Periodic version of Theorem 4). For $s<5 / 4$ NE admits nonunique solutions.

Our proof of non-uniqueness in the periodic setting again follows the same strategy used in the real-line case. We again examine the behavior of the limit as $t \rightarrow T^{-}$of the 2-peakon solution $u$ with initial data given in (7.6). Once this limit has been established in the desired ways, the same argument as in the real-line case implies non-uniqueness.

Proposition 9 (Pointwise limit - Periodic version of Proposition 4). For each $x \in \mathbb{R}$ we have

$$
\begin{equation*}
\lim _{t \rightarrow T} u(x, t)=w_{T} \cosh \left(\left[x-q_{T}\right]_{p}-\pi\right) \doteq v_{T}(x) \tag{7.40}
\end{equation*}
$$

where

$$
\begin{equation*}
q_{T} \doteq \lim _{t \rightarrow T^{-}} q_{1}(t)=\lim _{t \rightarrow T^{-}} q_{2}(t) \quad \text { and } \quad w_{T} \doteq \lim _{t \rightarrow T^{-}} w(t) . \tag{7.41}
\end{equation*}
$$

Remark. We can avoid the multiple cases needed in the real-line version of this proof as we do not need to expand out an absolute value. Here, as we are using the hyperbolic cosine, we will have both $e^{x}$ and $e^{-x}$ present thus avoiding the need to break into cases.

Proof. Our solution $u$ is a $2 \pi$-periodic function, and we will restrict our attention to the interval $[0,2 \pi]$. As we know that the limits of $q_{1}$ and $q_{2}$ exists, we will further restrict our attention to after some time $t_{0}>0$ such that these position function remain within a single period. This will avoid any complications of moving between periods which will require using the floor function in our definition. Using the exponential definition of the hyperbolic cosine, we get

$$
u(x, t)=\frac{1}{2}\left[e^{\pi-x}\left(p_{1} e^{q_{1}}+p_{2} e^{q_{2}}\right)+e^{-\pi+x}\left(p_{1} e^{-q_{1}}+p_{2} e^{-q_{2}}\right)\right] .
$$

Rewriting this expression to generate terms containing $w$ gives us

$$
\begin{equation*}
u(x, t)=\frac{1}{2}\left[e^{\pi-x}\left(e^{q_{2}} p_{1}\left(e^{-q}-1\right)+w e^{q_{2}}\right)+e^{-\pi+x}\left(w e^{-q_{1}}+e^{-q_{1}} p_{2}\left(e^{-q}-1\right)\right)\right] \tag{7.42}
\end{equation*}
$$

Taking the limit as $t \rightarrow T^{-}$of (7.42), and using the limit established in Lemma 2, we obtain

$$
\lim _{t \rightarrow T^{-}} u(x, t)=w_{T} \cosh \left(\left[x-q_{T}\right]_{p}-\pi\right)
$$

We next demonstrate that $u$ converges to $v_{T}$ as $t \rightarrow T^{-}$in $L^{r}$.
Proposition 10 (Convergence in $L^{r}$ - Periodic version of Proposition 5). For our antipeakonpeakon solution u to NE, we have

$$
\begin{equation*}
\lim _{t \rightarrow T^{-}}\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}=0 \tag{7.43}
\end{equation*}
$$

Proof. The same remarks that we made for the pointwise proof apply here as to taking a $t_{0}>0$ such that $q_{1}$ and $q_{2}$ lie within a single $2 \pi$ period after time $t_{0}$. Analogous to the pointwise limit, as we have both $e^{x}$ and $e^{-x}$ present in our hyperbolic cosines, we will not have to break our argument into cases in order to simplify the absolute values. This fact also allows us to bypass the restriction $1 \leq r<4$ as we do not cut the domain of the integration, creating the situation we saw in the real-line case on the sub-integral on $\left[q_{1}, q_{2}\right]$. After rewriting the hyperbolic cosines in their exponential form we get

$$
\begin{aligned}
& \left.\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}^{r}=\frac{1}{2} \int_{0}^{2 \pi} \right\rvert\, e^{x-\pi}\left(p_{1} e^{-q_{1}}+p_{2} e^{-q_{2}}-w_{T} e^{-q_{T}}\right) \\
&+\left.e^{\pi-x}\left(p_{1} e^{q_{1}}+p_{2} e^{q_{2}}-w_{T} e^{q_{T}}\right)\right|^{r} d x
\end{aligned}
$$

Using Jensen's inequality, and evaluating the resulting integrals, we get

$$
\begin{aligned}
& \frac{1}{2} \int_{0}^{2 \pi}\left|e^{x-\pi}\left(p_{1} e^{-q_{1}}+p_{2} e^{-q_{2}}-w_{T} e^{-q_{T}}\right)+e^{\pi-x}\left(p_{1} e^{q_{1}}+p_{2} e^{q_{2}}-w_{T} e^{q_{T}}\right)\right|^{r} d x \\
& \quad \leq \frac{e^{r \pi}-e^{-r \pi}}{r} \cdot\left|p_{1} e^{-q_{1}}+p_{2} e^{-q_{2}}-w_{T} e^{-q_{T}}\right|^{r}+\frac{e^{r \pi}-e^{-r \pi}}{r} \cdot\left|p_{1} e^{q_{1}}+p_{2} e^{q_{2}}-w_{T} e^{q_{T}}\right|^{r} .
\end{aligned}
$$

Using Lemma 2, we have

$$
\lim _{t \rightarrow T^{-}}\left|p_{1} e^{q_{1}}+p_{2} e^{q_{2}}-w_{T} e^{q_{T}}\right|^{r}=0 \quad \text { and } \quad \lim _{t \rightarrow T^{-}}\left|p_{1} e^{q_{1}}+p_{2} e^{q_{2}}-w_{T} e^{q_{T}}\right|^{r}=0 .
$$

Therefore applying the limit as $t \rightarrow T^{-}$we get

$$
\lim _{t \rightarrow T^{-}}\left\|u(x, t)-v_{T}(x)\right\|_{L^{r}}^{r}=0
$$

Now that we have successfully established pointwise and $L^{r}$ convergence, we will use these results to establish $H^{s}$ by using the Dominated Convergence Theorem.

Theorem 8 (Convergence in $H^{s}$ - Periodic version of Theorem 5). For $s<5 / 4$, our antipeakonpeakon solution $u$ converges to $v_{T}$ in $H^{s}$, i.e.

$$
\begin{equation*}
\lim _{t \rightarrow T}\left\|u(x, t)-v_{T}(x)\right\|_{H^{s}}=0 \tag{7.44}
\end{equation*}
$$

Proof. From the definition of the $H^{s}$ norm of $u(x, t)-v_{T}(x)$, we have

$$
\left\|u(x, t)-v_{T}(x)\right\|_{H^{s}}^{2}=\sum_{n \in \mathbb{Z}}\left(1+n^{2}\right)^{s}\left|\widehat{u}(n, t)-\widehat{v_{T}}(n)\right|^{2} .
$$

Our objective, when taking the limit, will be to move the limit inside of the integral. Thus, we begin by examining the limit of the summand. As we have already established the convergence of $u$ to $v_{T}$ in $L^{1}$, via Proposition 10, we see that the inequality

$$
|\widehat{u}(n, t)-\widehat{u}(n, T)| \leq\|u(x, t)-u(x, T)\|_{L^{1}},
$$

implies that

$$
\lim _{t \rightarrow T^{-}}\left(1+n^{2}\right)^{s}\left|\widehat{u}(n, t)-\widehat{v_{T}}(n)\right|^{2}=0 .
$$

Next, we will define the bounding sequences that will allow us to apply the generalized Dominated Convergence Theorem. We set

$$
f_{t}(n) \doteq\left(1+n^{2}\right)^{s}\left|\widehat{u}(n, t)-\widehat{v_{T}}(n)\right|^{2} \leq 4\left(1+n^{2}\right)^{s}\left(|\widehat{u}(n, t)|^{2}+\left|\widehat{v_{T}}(n)\right|^{2}\right) \doteq g_{t}(n)
$$

We need to establish that the $g_{t}$ 's, have a pointwise limit $g$, i.e. for each $n \in \mathbb{Z}, \lim _{t \rightarrow T^{-}} g_{t}(n)=$ $g(n)$. The most obvious candidate for $g$ is

$$
g(n)=8\left(1+n^{2}\right)^{s}\left|\widehat{v}_{T}(n)\right|^{2} \quad \text { where } \quad \sum_{n \in \mathbb{Z}} g(n)=32 \cosh ^{2} \pi \cdot c_{s} w_{T} .
$$

Indeed, we have using the laws of limits, $\widehat{u}(n, t) \rightarrow \widehat{v}_{T}(n)$ for each $n$ implies $g_{t} \rightarrow g$ for each $n$. To finish satisfying the hypotheses of the generalized Dominated Convergence theorem, we must now establish the sum properties of the $g_{t}$ 's. We have

$$
\lim _{t \rightarrow T} \sum_{n \in \mathbb{Z}} g_{t}(n)=32 \sinh ^{2} \pi \cdot c_{s} w_{T}
$$

where the left limit uses the $5 / 4$-hypothesis (1.13). We now see that the hypotheses for the generalized Dominated Convergence Theorem are satisfied. Thus, we can conclude that as $t \rightarrow T$, we have $u(x, t) \rightarrow v_{T}$ in $H^{s}$.

Proof of Theorem 7. Translating the NE 1-peakon solution by $x_{0}$ and choosing the minus sign we obtain the following antipeakon solution for NE

$$
v(x, t)=-\sqrt{c} \cosh \left(\left[x-x_{0}-c t\right]_{p}-\pi\right), \quad \text { for any } \quad c>0 \quad \text { and } \quad x_{0} \in \mathbb{T} .
$$

As in the real-line case, we choose $c=w_{T}^{2}$ and $x_{0}=q_{T}-w_{T}^{2} T$, and obtain the antipeakon solution $v(x, t)=w_{T} \cosh \left(\left[x-x_{0}-w_{T}^{2} t\right]-\pi\right)$. Since at $t=T$ we have

$$
v(x, T)=w_{T} \cosh \left(\left[x-q_{T}\right]_{p}-\pi\right)=u(x, T),
$$

we see that we have constructed two different NE solutions, which belong in $H^{s}, s<5 / 4$, and agree at $t=T$. From here, a change of variables can recast these two solutions as stemming from the same initial data at time $t=0$. This scenario proves failure of uniqueness in this range of Sobolev spaces.

The case $s=5 / 4$ : The argument for ill-posedness in this case is precisely the same as that in the non-periodic case. This result completes the proof of both of Theorems 1 and 2 in the periodic case.

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