# A DISTRIBUTED ADMM-LIKE METHOD FOR RESOURCE SHARING OVER TIME-VARYING NETWORKS* 

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#### Abstract

We consider cooperative multi-agent resource sharing problems over time-varying communication networks, where only local communications are allowed. The objective is to minimize the sum of agent-specific composite convex functions subject to a conic constraint that couples agents' decisions. We propose a distributed primal-dual algorithm DPDA-D to solve the saddle point formulation of the sharing problem on time-varying (un)directed communication networks; and we show that primal-dual iterate sequence converges to a point defined by a primal optimal solution and a consensual dual price for the coupling constraint. Furthermore, we provide convergence rates for suboptimality, infeasibility and consensus violation of agents' dual price assessments; examine the effect of underlying network topology on the convergence rates of the proposed decentralized algorithm; and compare DPDA-D with centralized methods on the basis pursuit denoising and multi-channel power allocation problems.


Key words. multi-agent distributed optimization, primal-dual method, resource sharing problem, convex optimization, convergence rate

AMS subject classifications. 90C25, 90C46, 68W15

1. Introduction. Let $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{R}_{+}}$denote a time-varying graph of $N$ computing nodes. More precisely, for $t \geq 0$, the graph has the form $\mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$, where $\mathcal{N} \triangleq$ $\{1, \ldots, N\}$ and $\mathcal{E}^{t} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of directed edges at time $t$. Suppose each node $i \in \mathcal{N}$ has a private constraint function $g_{i}: \mathbb{R}^{n_{i}} \rightarrow \mathbb{R}^{m}$ and a private cost function $\varphi_{i}: \mathbb{R}^{n_{i}} \rightarrow \mathbb{R} \cup\{+\infty\}$ such that

$$
\begin{equation*}
\varphi_{i}\left(\xi_{i}\right) \triangleq \rho_{i}\left(\xi_{i}\right)+f_{i}\left(\xi_{i}\right), \tag{1.1}
\end{equation*}
$$

where $\rho_{i}: \mathbb{R}^{n_{i}} \rightarrow \mathbb{R} \cup\{+\infty\}$ is a proper, closed convex function (possibly non-smooth), $f_{i}: \mathbb{R}^{n_{i}} \rightarrow \mathbb{R}$ is a smooth convex function. Assuming each node $i \in \mathcal{N}$ has only access to $\varphi_{i}, g_{i}$ and a closed convex cone $\mathcal{K} \subseteq R^{m}$, consider the following problem:

$$
\begin{equation*}
\min _{\boldsymbol{\xi} \in \mathbb{R}^{n}} \varphi(\boldsymbol{\xi}) \triangleq \sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right) \quad \text { s.t. } \quad g(\boldsymbol{\xi}) \triangleq \sum_{i \in \mathcal{N}} g_{i}\left(\xi_{i}\right) \in-\mathcal{K} \tag{1.2}
\end{equation*}
$$

where $\xi_{i} \in \mathbb{R}^{n_{i}}$ denotes the local decision of node $i \in \mathcal{N}$ and $n \triangleq \sum_{i \in \mathcal{N}} n_{i}$.
Assumption 1. For all $i \in \mathcal{N}$, the function $f_{i}$ is differentiable on an open set containing dom $\rho_{i}$, and $\nabla f_{i}$ is Lipschitz with constant $L_{f_{i}}$; the prox map of $\rho_{i}$,

$$
\begin{equation*}
\operatorname{prox}_{\rho_{i}}\left(\xi_{i}\right) \triangleq \underset{x_{i} \in \mathbb{R}^{n_{i}}}{\operatorname{argmin}}\left\{\rho_{i}\left(x_{i}\right)+\frac{1}{2}\left\|x_{i}-\xi_{i}\right\|^{2}\right\}, \tag{1.3}
\end{equation*}
$$

is efficiently computable, where $\|$.$\| denotes the Euclidean norm. Moreover, g_{i}$ is $\mathcal{K}$ convex [6, Chapter 3.6.2], Lipschitz continuous with constant $C_{g_{i}}$, and has a Lipschitz continuous Jacobian, $\mathbf{J} g_{i}$, with constant $L_{g_{i}}$.
In this paper, we design a distributed algorithm for solving (1.2) and provide a unified approach for analyzing the convergence behavior of the proposed method, regardless of whether the communications over the time-varying graph $\left\{\mathcal{G}^{t}\right\}$ are unidirectional

[^0]or bidirectional. To this aim, we need some definitions and assumptions related to the time-varying graph $\left\{\mathcal{G}^{t}\right\}$. To unify the notation, we assume all edges are directed, and consider undirected graphs as a special case of directed graphs.

Definition 1. For any $t \geq 0, \mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$ is a directed graph; let $\mathcal{N}_{i}^{t, \text { in }} \triangleq\{j \in$ $\left.\mathcal{N}:(j, i) \in \mathcal{E}^{t}\right\} \cup\{i\}$ and $\mathcal{N}_{i}^{t, \text { out }} \triangleq\left\{j \in \mathcal{N}:(i, j) \in \mathcal{E}^{t}\right\} \cup\{i\}$ denote the in-neighbors and out-neighbors of node $i \in \mathcal{N}$ at time $t$, respectively; and let $d_{i}^{t} \triangleq\left|\mathcal{N}_{i}^{t, \text { out }}\right|-1$ be the out-degree of node $i \in \mathcal{N} . \mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$ is called undirected when $(i, j) \in \mathcal{E}^{t}$ if and only if $(j, i) \in \mathcal{E}^{t}$. For undirected $\mathcal{G}^{t}$, let $\mathcal{N}_{i}^{t} \triangleq \mathcal{N}_{i}^{t, \text { in }} \backslash\{i\}=\mathcal{N}_{i}^{t, \text { out }} \backslash\{i\}$ denote the neighbors of $i \in \mathcal{N}$, and $d_{i}^{t} \triangleq\left|\mathcal{N}_{i}^{t}\right|$ represents the degree of node $i \in \mathcal{N}$ at time $t$.

Assumption 2. When $\mathcal{G}^{t}$ is a (general) directed graph, node $i \in \mathcal{N}$ can receive data from $j \in \mathcal{N}$ only if $j \in \mathcal{N}_{i}^{t, \text { in }}$, i.e., $(j, i) \in \mathcal{E}^{t}$, and can send data to $j \in \mathcal{N}$ only if $j \in \mathcal{N}_{i}^{t, \text { out }}$, i.e., $(i, j) \in \mathcal{E}^{t}$; on the other hand, when $\mathcal{G}^{t}$ is undirected, node $i \in \mathcal{N}$ can send and receive data to and from $j \in \mathcal{N}$ at time $t$ only if $j \in \mathcal{N}_{i}^{t}$, i.e., $(i, j) \in \mathcal{E}^{t}$.

Our objective is to solve (1.2) in a decentralized fashion using the computing nodes in $\mathcal{N}$ while the information exchange among the nodes is restricted to edges in $\mathcal{E}^{t}$ for $t \geq 0$ according to Assumption 2. We are interested in designing algorithms which can distribute the computation over the nodes such that each node's computation is based on the local topology of $\mathcal{G}^{t}$ and information only available to that node.

Decentralized optimization over communication networks has drawn attention from a wide range of application areas: coordination and control in multirobot networks, parameter estimation in wireless sensor networks, processing distributed big data in machine learning, and distributed power control in cellular networks, to name a few. In these examples, the network size can be prohibitively large for centralized optimization, which requires a fusion center that collects the physically distributed data and runs a centralized optimization method. This process has expensive communication overhead, requires large enough memory to store and process the data, and also may violate data privacy in case agents are not willing to share their data even though they are collaborative [37]. Therefore, a common objective of today's big-data networks is to use decentralized optimization techniques to avoid expensive communication overhead required by the centralized setting and to enhance the data privacy. The communication networks in these application areas may be directed, i.e., communication links can be unidirectional, and/or the network may be time-varying, e.g., communication links in a wireless network can be on/off over time due to failures or the links may exist among agents depending on their inter-distances.

In the remainder of this section, as a brief preliminary, we discuss the primaldual algorithm (PDA) proposed in [8] to solve convex-concave saddle-point problems with a bilinear coupling term, explain its connections to ADMM-like algorithms, and briefly discuss some recent work related to ours. It is worth noting that the saddle point ( SP ) problem formulation of (1.2) contains a coupling term that is not bilinear due to nonlinear $\left\{g_{i}\right\}_{i \in \mathcal{N}}$; therefore, PDA is not applicable. Next, in Section 2, we propose DPDA-D, a new distributed algorithm based on PDA and extending it to handle nonlinear constraints, for solving the SP formulation of the multi-agent sharing problem in (1.2) when the topology of the connectivity graph is time-varying with (un)directed communication links, and we state the main theorem establishing the convergence properties of DPDA-D; and in Section 3, we provide the proof of the main theorem. Subsequently, in Sections 4, 5 and 6, we discuss certain details related to the applicability of the method in practice. In Section 7, we compare our method with Prox-JADMM [14] on the basis pursuit denoising problem, and with Mirror-
prox [23] on the multi-channel power allocation problem; and finally, in Section 8 we state our concluding remarks and briefly discuss potential future work.
1.1. Preliminary. In this paper, we study an inexact variant of the primaldual algorithm (PDA) proposed in [8], extending it to handle nonlinear constraints, to solve the SP formulation of (1.2) in a decentralized manner over a time-varying communication network. There has been active research on efficient algorithms for convex-concave saddle point problems $\min _{\mathbf{x}} \max _{\mathbf{y}} \mathcal{L}(\mathbf{x}, \mathbf{y})$, e.g., [7, 13, 22, 34]. PDA [8] also belongs to this family and is proposed for the convex-concave SP problem:

$$
\begin{equation*}
\min _{\mathbf{x} \in \mathcal{X}} \max _{\mathbf{y} \in \mathcal{Y}} \mathcal{L}(\mathbf{x}, \mathbf{y}) \triangleq \Phi(\mathbf{x})+\langle T(\mathbf{x}), \mathbf{y}\rangle-h(\mathbf{y}) \tag{1.4}
\end{equation*}
$$

where $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional vector spaces, $\Phi(\mathbf{x}) \triangleq \rho(\mathbf{x})+f(\mathbf{x}), \rho$ and $h$ are possibly non-smooth convex functions, $f$ is a convex function and has a Lipschitz continuous gradient defined on $\operatorname{dom} \rho$ with Lipschitz constant $L$, and $T: \mathcal{X} \rightarrow \mathcal{Y}$ is a linear map. Briefly, given $\mathbf{x}^{0} \in \mathcal{X}, \mathbf{y}^{0} \in \mathcal{Y}$ and algorithm parameters $\nu_{x}, \nu_{y}>0$, PDA consists of two proximal-gradient steps that can be written as:

$$
\begin{align*}
& \mathbf{x}^{k+1} \leftarrow \underset{\mathbf{x} \in \mathcal{X}}{\operatorname{argmin}} \rho(\mathbf{x})+f\left(\mathbf{x}^{k}\right)+\left\langle\nabla f\left(\mathbf{x}^{k}\right), \mathbf{x}-\mathbf{x}^{k}\right\rangle+\left\langle T(\mathbf{x}), \mathbf{y}^{k}\right\rangle+\frac{1}{\nu_{x}} D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)  \tag{1.5a}\\
& \mathbf{y}^{k+1} \leftarrow \underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmin}} h(\mathbf{y})-\left\langle 2 T\left(\mathbf{x}^{k+1}\right)-T\left(\mathbf{x}^{k}\right), \mathbf{y}\right\rangle+\frac{1}{\nu_{y}} D_{y}\left(\mathbf{y}, \mathbf{y}^{k}\right),
\end{align*}
$$

where $D_{x}$ and $D_{y}$ are Bregman distance functions corresponding to some continuously differentiable strongly convex functions $\psi_{x}$ and $\psi_{y}$ such that $\operatorname{dom} \psi_{x} \supset \operatorname{dom} \rho$ and $\operatorname{dom} \psi_{y} \supset \operatorname{dom} h$. In particular, $D_{x}(\mathbf{x}, \overline{\mathbf{x}}) \triangleq \psi_{x}(\mathbf{x})-\psi_{x}(\overline{\mathbf{x}})-\left\langle\nabla \psi_{x}(\overline{\mathbf{x}}), \mathbf{x}-\overline{\mathbf{x}}\right\rangle$, and $D_{y}$ is defined similarly. Abusing the notation, below we use $T$ also to denote the corresponding matrix, i.e., $T(\mathbf{x})=T \mathbf{x}$.

In [8], it is shown that, when the convexity modulus for $\psi_{x}$ and $\psi_{y}$ is 1 , if $\nu_{x}, \nu_{y}>0$ are chosen such that $\left(\frac{1}{\nu_{x}}-L\right) \frac{1}{\nu_{y}} \geq \sigma_{\max }^{2}(T)$, then for any $\mathbf{x}, \mathbf{y} \in \mathcal{X} \times \mathcal{Y}$,

$$
\begin{equation*}
\mathcal{L}\left(\overline{\mathbf{x}}^{K}, \mathbf{y}\right)-\mathcal{L}\left(\mathbf{x}, \overline{\mathbf{y}}^{K}\right) \leq \frac{1}{K}\left(\frac{1}{\nu_{x}} D_{x}\left(\mathbf{x}, \mathbf{x}^{0}\right)+\frac{1}{\nu_{y}} D_{y}\left(\mathbf{y}, \mathbf{y}^{0}\right)-\left\langle T\left(\mathbf{x}-\mathbf{x}^{0}\right), \mathbf{y}-\mathbf{y}^{0}\right\rangle\right), \tag{1.6}
\end{equation*}
$$

holds for all $K \geq 1$, where $\overline{\mathbf{x}}^{K} \triangleq \frac{1}{K} \sum_{k=1}^{K} \mathbf{x}^{k}$ and $\overline{\mathbf{y}}^{K} \triangleq \frac{1}{K} \sum_{k=1}^{K} \mathbf{y}^{k}$.
It is worth mentioning the connection between PDA and the alternating direction method of multipliers (ADMM). Indeed, when implemented on $\min _{\mathbf{v} \in \mathcal{X}^{*}, \mathbf{y} \in \mathcal{Y}}\left\{\Phi^{*}(\mathbf{v})+\right.$ $\left.h(\mathbf{y}): \mathbf{v}+T^{\top} \mathbf{y}=\mathbf{0}\right\}$, preconditioned ADMM is equivalent to PDA [7, 8], where $\mathcal{X}^{*}$ denotes the dual space and $\Phi^{*}$ is the convex conjugate of $\Phi$. There is also a strong connection between the linearized ADMM algorithm proposed by Aybat et al. in [4] and PDA proposed in [8] - for details of these relations, see Section 1.4.

Notation. $\|\cdot\|$ denotes the Euclidean or the spectral norm depending on its $\operatorname{argument}$, i.e., for a matrix $R,\|R\|=\sigma_{\max }(R)$. Given a convex set $\mathcal{S}$, let $\sigma_{\mathcal{S}}(\cdot)$ denote its support function, i.e., $\sigma_{\mathcal{S}}(\theta) \triangleq \sup _{w \in \mathcal{S}}\langle\theta, w\rangle$, let $\mathbb{1}_{S}(\cdot)$ denote the indicator function of $\mathcal{S}$, i.e., $\mathbb{1}_{S}(w)=0$ for $w \in \mathcal{S}$ and equal to $+\infty$ otherwise, and let $\mathcal{P}_{\mathcal{S}}(w) \triangleq$ $\operatorname{argmin}\{\|v-w\|: v \in \mathcal{S}\}$ denote the Euclidean projection onto $\mathcal{S}$. For a closed convex set $\mathcal{S}$, we define the distance function as $d_{\mathcal{S}}(w) \triangleq\left\|\mathcal{P}_{\mathcal{S}}(w)-w\right\|$. Given a convex cone $\mathcal{K} \in \mathbb{R}^{m}$, let $\mathcal{K}^{*}$ denote its dual cone, i.e., $\mathcal{K}^{*} \triangleq\left\{\theta \in \mathbb{R}^{m}:\langle\theta, w\rangle \geq 0 \forall w \in \mathcal{K}\right\}$, and $\mathcal{K}^{\circ} \triangleq-\mathcal{K}^{*}$ denote the polar cone of $\mathcal{K}$. Note that for any cone $\mathcal{K} \in \mathbb{R}^{m}, \sigma_{\mathcal{K}}(\theta)=0$ for $\theta \in \mathcal{K}^{\circ}$ and equal to $+\infty$ if $\theta \notin \mathcal{K}^{\circ}$, i.e., $\sigma_{\mathcal{K}}(\theta)=\mathbb{1}_{\mathcal{K}^{\circ}}(\theta)$ for all $\theta \in \mathbb{R}^{m}$. Given a convex function $h: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$, its convex conjugate is $h^{*}(w) \triangleq \sup _{\theta \in \mathbb{R}^{n}}\langle w, \theta\rangle-h(\theta)$, and for differentiable $h: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}, \mathbf{J} h: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m \times n}$ denotes the Jacobian of $h$. Throughout the paper, $\otimes$ denotes the Kronecker product, $\Pi$ denotes the Cartesian product, and $\mathbf{I}_{n}$ is the $n \times n$ identity matrix. $Q$-norm is defined as $\|z\|_{Q} \triangleq\left(z^{\top} Q z\right)^{\frac{1}{2}}$ for any positive definite matrix $Q$.
1.2. Our Previous Work on Resource Sharing. In [1], we considered (1.2) when $g_{i}(\xi)=r_{i}-R_{i} \xi_{i}$ is affine for $i \in \mathcal{N}$, over a static and undirected communication network $\mathcal{G}=(\mathcal{N}, \mathcal{E})$ as a dual consensus problem. Using Lagrangian duality, we reformulated it as an SP problem, $\min _{\xi} \max _{y \in \mathcal{K}^{\circ}} \sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right)+\left\langle\sum_{i \in \mathcal{N}} R_{i} \xi_{i}-r_{i}, y\right\rangle$ which can be written in a distributed form through creating local copies of dual variable $y \in \mathbb{R}^{n}$ as $(P): \min _{\boldsymbol{\xi}} \operatorname{maxy}\left\{\sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right)+\left\langle R_{i} \xi_{i}-r_{i}, y_{i}\right\rangle: y_{i} \in \mathcal{K}^{\circ} \forall i \in\right.$ $\left.\mathcal{N}, y_{i}=y_{j} \forall(i, j) \in \mathcal{E}\right\}$, where $\boldsymbol{\xi}=\left[\xi_{i}\right]_{i \in \mathcal{N}}$ and $\mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}}$. Using $M$, the edge-node incidence matrix of $\mathcal{G}$, the consensus constraints $y_{i}=y_{j}$ for $(i, j) \in \mathcal{E}$ can be written as $M \mathbf{y}=\mathbf{0}$. Furthermore, by dualizing the consensus constraints, we obtain another SP problem, equivalent to $(P)$, in the form of (1.4):

$$
\begin{equation*}
\min _{\boldsymbol{\xi}} \max _{\mathbf{y} \in \Pi_{i \in \mathcal{N}} \mathcal{K}^{\circ}} \min _{\mathbf{w}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})=\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y} \in \Pi_{i \in \mathcal{N}} \mathcal{K}^{\circ}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y}), \tag{1.7}
\end{equation*}
$$

where $\mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y}) \triangleq \sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right)+\left\langle R_{i} \xi_{i}-r_{i}, y_{i}\right\rangle-\langle\mathbf{w}, M \mathbf{y}\rangle$. The equality in (1.7) holds as long as $\mathcal{K}$ is a pointed cone - hence $\operatorname{int}\left(\mathcal{K}^{\circ}\right) \neq \emptyset$; therefore, for each fixed $\boldsymbol{\xi}$, inner $\max _{\mathbf{y}}$ and $\min _{\mathbf{w}}$ can be interchanged. The saddle-point problem on the right side of (1.7) is special case of (1.4) with a separable structure. Exploiting this special structure, we customized PDA in (1.5) and proposed Algorithm DPDA-S. In [1] we showed that Algorithm DPDA-S can solve the sharing problem (1.2) with an affine conic constraint in a decentralized way and established its convergence properties provided that the node-specific primal-dual step-sizes $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ and the algorithm parameter $\gamma>0$ satisfy $\frac{1}{\tau_{i}}>L_{f_{i}}$, and $\left(\frac{1}{\tau_{i}}-L_{f_{i}}\right)\left(\frac{1}{\kappa_{i}}-2 \gamma d_{i}\right) \geq\left\|R_{i}\right\|^{2}$, for all $i \in \mathcal{N}$, where $d_{i}$ denotes the degree of $i \in \mathcal{N}$ for the static $\mathcal{G}$. Our result in [1] refines the error bound in (1.6) and establishes $\mathcal{O}(1 / k)$ ergodic rate in terms of suboptimality and infeasibility of the DPDA-S iterate sequence - see Theorem 2 in [1].

The arguments used for proving Theorem 2 in [1] cannot be used for the timevarying directed communication network setting considered in this paper since the undirected network is encoded through the use of $M \mathbf{y}=\mathbf{0}$ constraint. However, when the topology is time-varying or when the edges are directed, it is not immediately clear how one can represent this problem as an SP problem. To extend our previous results to a more general setting of time-varying topology with possibly directed edges, in this paper we develop a new SP formulation that can impose consensus over the dual variables while the formulation is independent of the changing topology. Finally, the new method can also handle nonlinear conic constraints on resource sharing in (1.2).
1.3. Related Work. Now we briefly review some recent work on the distributed resource sharing problem. From the application perspective, algorithms and their basic convergence analysis have been studied for the economic dispatch problem (EDP), e.g., [40] for power-flow networks and [21, 46] for smart-grids. The variants of EDP considered in $[21,40,46]$ are special cases of (1.2). In particular, each node $i \in \mathcal{N}$ has a convex objective function $f_{i}$, usually a quadratic function; $\rho_{i}\left(\xi_{i}\right)=\mathbb{1}_{\mathcal{X}_{i}}\left(\xi_{i}\right)$, where $\mathcal{X}_{i}$ is a local simple convex constraint set, $g_{i}\left(\xi_{i}\right)=\xi_{i}-r_{i}$ and $\mathcal{K}=\{\mathbf{0}\}$. In [40], the aim is to optimize the total power generation cost in a DC power-flow model, [21, 46] also study a similar problem considering random wind power injection - both papers establish basic convergence results without any rate guarantees. Distributed resource allocation problem can also arise in controlling and coordinating internet services over hybrid edge-cloud networks; for which a distributed ADMM algorithm is proposed in [24] to solve a problem in form of (1.2) with $\mathcal{K}=\{\mathbf{0}\}$ and $g_{i}\left(\xi_{i}\right)=\xi_{i}-r_{i}$. [45] studies EDP considering communication delays in directed time-varying network topology, and an algorithm based on push-sum protocol is proposed.

From the theoretical point of view, there has been active research on distributed resource allocation problem. In [15], a distributed Lagrangian method (DLM) has been proposed for solving a particular case of (1.2) on a static network; more precisely, the objective is to minimize sum of local convex functions subject to local convex compact sets and a coupling constraint of the form $\sum_{i \in \mathcal{N}} \xi_{i}-r_{i}=\mathbf{0}$. In [15], the authors establish convergence rate of $\mathcal{O}(\log (k) / \sqrt{k})$ for the dual function values estimated at the time-weighted average of dual iterates. Reference [16] gives a gradient balancing protocol to solve (1.2) in which $\rho_{i}(\cdot)=0, g_{i}\left(\xi_{i}\right)=\xi_{i}-r_{i}$ and $\mathcal{K}=\{\mathbf{0}\}$. The authors show that the generated sequence $\boldsymbol{\xi}^{k}=\left[\xi_{i}^{k}\right]_{i \in \mathcal{N}}$ satisfies $\sum_{i \in \mathcal{N}} f_{i}\left(\xi_{i}^{k}\right)-\varphi^{*} \leq \mathcal{O}(1 / k)$ and is feasible for all $k$ under the assumption that the initial point $\boldsymbol{\xi}^{0}=\left[\xi_{i}^{0}\right]_{i \in \mathcal{N}}$ is feasible $-\varphi^{*}$ denotes the optimal value; moreover, a linear rate is established when each $f_{i}$ is strongly convex. For a similar formulation as in [16], an asynchronous gradient-descent method is proposed in [26] for time-varying undirected communication networks; the proposed algorithm produces a feasible iterate sequence such that $\min _{\ell=1, \ldots, k} \max _{i, j \in \mathcal{N}}\left\|\nabla f_{i}\left(\xi_{i}^{\ell}\right)-\nabla f_{i}\left(\xi_{j}^{\ell}\right)\right\| \leq \mathcal{O}(1 / \sqrt{k})$ when each $f_{i}$ is convex and has a Lipschitz gradient. However, none of these methods can solve (1.2) in its full generality over a time-varying and directed communication network.

In [10], a method based on ADMM is proposed to reduce the computational work of ADMM due to exact minimizations in each iteration. First, a dual consensus ADMM is proposed for solving (1.2) over an undirected static network in a distributed fashion when $\mathcal{K}=\{\mathbf{0}\}, g_{i}\left(\xi_{i}\right)=R_{i} \xi_{i}-r_{i}$, and $\varphi_{i}\left(\xi_{i}\right)=\rho_{i}\left(\xi_{i}\right)+f_{i}\left(A_{i} \xi_{i}\right)$ for $\rho_{i}$ and $f_{i}$ as in (1.1). To avoid exact minimizations in ADMM, an inexact variant taking proximal-gradient steps is analyzed. Convergence of primal-dual sequence is shown when each $f_{i}$ is strongly convex - without a rate result; and a linear rate is established in the absence of the non-smooth $\rho_{i}$, i.e., $\varphi_{i}\left(\xi_{i}\right)=f_{i}\left(A_{i} \xi_{i}\right)$, and assuming each $A_{i}$ has full column-rank and $f_{i}$ is strongly convex, i.e., $\varphi_{i}$ is strongly convex.

In [9], a proximal dual consensus ADMM method, PDC-ADMM, is proposed by Chang to minimize $\sum_{i \in \mathcal{N}} \varphi_{i}$ subject to coupling equality and agent-specific constraints over both static and time-varying undirected networks - for the time-varying topology, they assumed that agents are on/off and communication links fail randomly with certain probabilities. The goal in the paper is to solve $\min _{\boldsymbol{\xi}}\left\{\sum_{i} \varphi_{i}\left(\xi_{i}\right)\right.$ : $\left.\sum_{i \in \mathcal{N}} R_{i} \xi_{i}=r, \xi_{i} \in \mathcal{X}_{i}, i \in \mathcal{N}\right\}$ where $\varphi_{i}$ is closed convex, $\mathcal{X}_{i}=\left\{\xi_{i} \in \mathcal{S}_{i}: C_{i} \xi_{i} \leq d_{i}\right\}$ and $\mathcal{S}_{i}$ is a convex compact set for each $i \in \mathcal{N}$. The polyhedral constraints $\xi_{i} \in \mathcal{X}_{i}$ are handled using a penalty formulation without requiring projections onto them. It is shown that both for static and time-varying cases, PDC-ADMM have $\mathcal{O}(1 / k)$ ergodic convergence rate in the mean for suboptimality and infeasibility; that said, in each iteration, costly exact minimizations involving $\varphi_{i}$ are needed. To alleviate this burden, Chang also proposed an inexact PDC-ADMM taking prox-gradient steps when $\varphi_{i}\left(\xi_{i}\right)=\rho_{i}\left(\xi_{i}\right)+f_{i}\left(A_{i} \xi_{i}\right)$ and $A_{i}$ is a linear map for each $i \in \mathcal{N}$, and showed $\mathcal{O}(1 / k)$ ergodic convergence rate when each $f_{i}$ is strongly convex and differentiable with a Lipschitz continuous gradient for $i \in \mathcal{N}$.

In [11], a consensus-based distributed primal-dual perturbation (PDP) algorithm using a diminishing step-size sequence is proposed. The objective is to minimize a composition of a global network function (smooth) with the sum of local objective functions (smooth), i.e., $\mathcal{F}\left(\sum_{i \in \mathcal{N}} f_{i}(x)\right)$, subject to local compact sets and an inequality constraint, $\sum_{i \in \mathcal{N}} g_{i}(x) \leq 0$, over a time-varying directed network. It is shown that the primal-dual iterate sequence converges to an optimal primal-dual solution; however, no rate result is provided.

There are fewer papers on resource allocation over time-varying directed networks. [20] considers a special case of (1.2) with $\mathcal{K}=\{\mathbf{0}\}, g_{i}\left(\xi_{i}\right)=\xi_{i}-r_{i}, f_{i}$ is convex, and $\rho_{i}\left(\xi_{i}\right)=\mathbb{1}_{\mathcal{X}_{i}}\left(\xi_{i}\right)$ where $\mathcal{X}_{i}$ is convex and compact for $i \in \mathcal{N}$. Assuming a Slater point exists which implies boundedness of dual optimal set, the authors proved $\mathcal{O}(\log (k) / \sqrt{k})$ rate result. Reference [44] has the same setting in [20] with $\mathcal{X}_{i}=\left[\underline{\xi}_{i}, \bar{\xi}_{i}\right]$. Assuming each $f_{i}$ is smooth and strongly convex, a distributed method is proposed and its convergence is shown without providing a rate result. Finally, while we were preparing this paper, we became aware of a recent work [28, 32]. [28] also uses Fenchel conjugation and dual consensus formulation to decompose separable constraints. A distributed algorithm on time-varying balanced ${ }^{1}$ directed communication networks is proposed for solving saddle-point problems subject to consensus constraints. Assuming each agents' local iterates and subgradient sets are uniformly bounded, it is shown that the ergodic average of primal-dual sequence converges with $\mathcal{O}(1 / \sqrt{k})$ rate in terms of saddle-point evaluation error; however, when the method is applied to constrained optimization problems, no rate in terms of suboptimality and infeasibility is provided. The other recent work in [32] investigates the connection between decentralized resource allocation problem and decentralized consensus optimization problem where the objective is to minimize sum of convex functions subject to local closed convex sets and $\sum_{i \in \mathcal{N}} \xi_{i}-r_{i}=\mathbf{0}$ over static undirected networks. Utilizing the mirror relationship between the optimality conditions of these problems, they proposed a method for solving the decentralized resource allocation problem and proved $o(1 / k)$ rate of convergence in terms of squared residuals of first-order optimality conditions.
1.4. Connection of PDA to ADMM. Consider the following convex-concave saddle point problem:

$$
\begin{equation*}
\min _{\mathbf{x}} \max _{\mathbf{y}} \rho(\mathbf{x})+\langle T \mathbf{x}, \mathbf{y}\rangle-h(\mathbf{y}), \tag{1.8}
\end{equation*}
$$

where $\rho$ and $h$ are closed convex functions, and $T$ is a linear map. The corresponding primal minimization problem take the form of $\min _{\mathbf{x}} \rho(\mathbf{x})+h^{*}(T \mathbf{x})$, or equivalently

$$
\begin{equation*}
\min _{\mathbf{x}, \mathbf{w}} \rho(\mathbf{x})+h^{*}(\mathbf{w}) \quad \text { s.t. } \quad T \mathbf{x}-\mathbf{w}=0 \tag{1.9}
\end{equation*}
$$

Moreover, the dual of (1.9) can be written as follows:

$$
\begin{equation*}
\min _{\mathbf{v}, \mathbf{y}} \rho^{*}(\mathbf{v})+h(\mathbf{y}) \quad \text { s.t. } \quad \mathbf{v}+T^{\top} \mathbf{y}=0 \tag{1.10}
\end{equation*}
$$

In an earlier paper [7], Chambolle and Pock proposed a primal-dual algorithm for solving (1.9). Given $\theta \in(0,1]$, in each iteration two proximal steps are computed as follows:

$$
\begin{align*}
& \mathbf{y}^{k+1} \leftarrow \operatorname{prox}_{\nu_{y} h}\left(\mathbf{y}^{k}+\nu_{y} T \mathbf{z}^{k}\right)  \tag{1.11a}\\
& \mathbf{x}^{k+1} \leftarrow \operatorname{prox}_{\nu_{x} \rho}\left(\mathbf{x}^{k}-\nu_{x} T^{\top} \mathbf{y}^{k+1}\right)  \tag{1.11b}\\
& \mathbf{z}^{k+1} \leftarrow \mathbf{x}^{k+1}+\theta\left(\mathbf{x}^{k+1}-\mathbf{x}^{k}\right) \tag{1.11c}
\end{align*}
$$

For $\theta=1$, by reindexing the $\left\{\mathbf{y}^{k}\right\}_{k}$ iterate sequence, the iterations in (1.11) can be written as: $\mathbf{x}^{k+1}=\operatorname{prox}_{\nu_{x} \rho}\left(\mathbf{x}^{k}-\nu_{x} T^{\top} \mathbf{y}^{k}\right)$ and $\mathbf{y}^{k+1}=\operatorname{prox}_{\nu_{y} h}\left(\mathbf{y}^{k}+\nu_{y} T\left(2 \mathbf{x}^{k+1}-\mathbf{x}^{k}\right)\right)$ - note that this is the same iterate sequence generated by (1.5) implemented on (1.8) when both $D_{x}$ and $D_{y}$ are chosen as $\frac{1}{2}\|\cdot\|^{2}$. In particular, (1.11) proposed in [7] is a special case of (1.5) proposed in [8].

[^1]When ADMM [18] is applied on problem (1.10), it generates the following iterates:

$$
\begin{align*}
& \mathbf{y}^{k+1} \leftarrow \underset{\mathbf{y}}{\operatorname{argmin}}\left\{h(\mathbf{y})-\left\langle T^{\top} \mathbf{y}, \mathbf{x}^{k}\right\rangle+\frac{c^{\prime}}{2}\left\|T^{\top} \mathbf{y}+\mathbf{v}^{k}\right\|^{2}\right\}  \tag{1.12a}\\
& \mathbf{v}^{k+1} \leftarrow \underset{\mathbf{v}}{\operatorname{argmin}}\left\{\rho^{*}(\mathbf{v})-\left\langle\mathbf{v}, \mathbf{x}^{k}\right\rangle+\frac{c^{\prime}}{2}\left\|T^{\top} \mathbf{y}^{k+1}+\mathbf{v}\right\|^{2}\right\}  \tag{1.12b}\\
& \mathbf{x}^{k+1} \leftarrow \mathbf{x}^{k}-c^{\prime}\left(T^{\top} \mathbf{y}^{k+1}+\mathbf{v}^{k+1}\right) \tag{1.12c}
\end{align*}
$$

for some penalty parameter $c^{\prime}>0$. It is shown in [7] that when $T=\mathbf{I}, \theta=1$, and $\nu_{x}=c^{\prime}, \nu_{y}=\frac{1}{c^{\prime}}$ for $c^{\prime}>0$, the algorithm in (1.11) is equivalent to the ADMM implementation in (1.12).

Let $M_{1}, M_{2}$ be positive semidefine matrices. Alternating direction proximal method of multipliers (AD-PMM), proposed in [41] to solve (1.9), computes the iterates as follows:

$$
\begin{align*}
& \mathbf{x}^{k+1} \leftarrow \underset{\mathbf{x}}{\operatorname{argmin}}\left\{\rho(\mathbf{x})+\frac{c}{2}\left\|T \mathbf{x}-\mathbf{w}^{k}+c^{-1} \mathbf{y}^{k}\right\|^{2}+\frac{1}{2}\left\|\mathbf{x}-\mathbf{x}^{k}\right\|_{M_{1}}^{2}\right\}  \tag{1.13a}\\
& \mathbf{w}^{k+1} \leftarrow \underset{\mathbf{y}}{\operatorname{argmin}}\left\{h^{*}(\mathbf{w})+\frac{c}{2}\left\|T \mathbf{x}^{k+1}-\mathbf{w}+c^{-1} \mathbf{y}^{k}\right\|^{2}+\frac{1}{2}\left\|\mathbf{w}-\mathbf{w}^{k}\right\|_{M_{2}}^{2}\right\}  \tag{1.13b}\\
& \mathbf{y}^{k+1} \leftarrow \mathbf{y}^{k}+c\left(T \mathbf{x}^{k+1}-\mathbf{w}^{k+1}\right) \tag{1.13c}
\end{align*}
$$

Another variant of (1.11) with the same convergence guarantees can be obtained by simply replacing (1.11c) with $\mathbf{z}^{k+1}=\mathbf{y}^{k+1}+\theta\left(\mathbf{y}^{k+1}-\mathbf{y}^{k}\right)$ and switching the order of updates in (1.11b) and (1.11a), i.e., $\mathbf{x}^{k+1}$ is computed before $\mathbf{y}^{k+1}$. When $\theta=1$, the iterations for this variant can be written as: $\mathbf{x}^{k+1}=\operatorname{prox}_{\nu_{x} \rho}\left(\mathbf{x}^{k}-\nu_{x} T^{\top} \mathbf{z}^{k}\right)$, $\mathbf{y}^{k+1}=\operatorname{prox}_{\nu_{y} h}\left(\mathbf{y}^{k}+\nu_{y} T \mathbf{x}^{k+1}\right), \mathbf{z}^{k+1}=2 \mathbf{x}^{k+1}-\mathbf{x}^{k}$. According to Proposition 3.1 in [41], the iterates generated by this variant of (1.11) for $\nu_{y}=c>0$ is equivalent to those generated by (1.13) when $M_{1}=\nu_{x}^{-1} \mathbf{I}_{n}-c T^{\top} T$ and $M_{2}=\mathbf{0}$.

In [4], Aybat et al. proposed proximal-gradient ADMM (PG-ADMM) for solving $\min _{\mathbf{x}, \mathbf{w}}\left\{h^{*}(\mathbf{w})+\sum_{i=1}^{N} \varphi_{i}\left(x_{i}\right): A_{i} x_{i}+B_{i} \mathbf{w}=b_{i}, \quad i=1, \ldots N\right\}$, where $\mathbf{x}=\left[x_{i}\right]_{i=1}^{N}$ for $x_{i} \in \mathbb{R}^{n_{i}}, \varphi_{i}=\rho_{i}+f_{i}$ is composite convex as in (1.1) and $h$ is a closed convex function. PG-ADMM is only different from ADMM in $x_{i}$-subproblems where $x_{i}^{k+1}$ is computed by minimizing the linear approximation of the augmented Lagrangian (AL) function after fixing $\mathbf{w}$ at $\mathbf{w}^{k}$ and linearizing the whole smooth part of the AL including $f_{i}$ around $\mathbf{x}^{k}$ - this leads to taking a prox-gradient step to compute each $x_{i}$ iterate. In the extreme case that $\rho_{i}=0$ for $i \in \mathcal{N}$, PG-ADMM reduces to GADM, studied in [27] and [19] - GADM takes gradient steps to compute $x_{i}$; Gao et al. [19] prove the $O(1 / t)$ convergence rate for GADM. PG-ADMM has also $\mathcal{O}(1 / t)$ rate and it can be viewed as an extension of G-ADMM where convex $\rho_{i}$ 's are allowed.

There is a strong connection between PG-ADMM and PDA in [8]. For all $\mathbf{x} \in$ $\mathcal{X}$, let $\Phi(\mathbf{x})=\rho(\mathbf{x})+f(\mathbf{x})$ as in (1.1) such that $\nabla f$ is Lipschitz with constant $L$; implementing PG-ADMM on $\min _{\mathbf{x}, \mathbf{w}}\left\{\Phi(\mathbf{x})+h^{*}(\mathbf{w}): T \mathbf{x}-\mathbf{w}=\mathbf{0}\right\}$ generates the following iterate sequence:

$$
\begin{align*}
& \mathbf{x}^{k+1} \leftarrow \operatorname{prox}_{\tau \rho}\left(\mathbf{x}^{k}-\tau\left[\nabla f\left(\mathbf{x}^{k}\right)+T^{\top}\left(\mathbf{y}^{k}+c\left(T \mathbf{x}^{k}-\mathbf{w}^{k}\right)\right)\right]\right)  \tag{1.14a}\\
& \mathbf{w}^{k+1} \leftarrow \operatorname{argmin}\left\{h^{*}(\mathbf{w})+\frac{c}{2}\left\|T \mathbf{x}^{k+1}-\mathbf{w}+c^{-1} \mathbf{y}^{k}\right\|^{2}\right\}  \tag{1.14b}\\
& \mathbf{y}^{k+1} \leftarrow \mathbf{y}^{k}+c\left(T \mathbf{x}^{k+1}-\mathbf{w}^{k+1}\right) \tag{1.14c}
\end{align*}
$$

For PG-ADMM iterate sequence, the suboptimality and infeasibility converges to 0 in the ergodic sense for any $c>0$ when $\frac{1}{\tau} \geq L+c\|T\|^{2}$. Note (1.14a) can be
rewritten as $\mathbf{x}^{k+1}=\operatorname{prox}_{\tau \rho}\left(\mathbf{x}^{k}-\tau\left[\nabla f\left(\mathbf{x}^{k}\right)+T^{\top}\left(2 \mathbf{y}^{k}-\mathbf{y}^{k-1}\right)\right]\right)$. Using Moreau proximal decomposition on $\mathbf{w}$-updates in (1.14b), we get

$$
\begin{equation*}
\mathbf{w}^{k+1}=T \mathbf{x}^{k+1}+c^{-1} \mathbf{y}^{k}-\frac{1}{c} \operatorname{prox}_{c h}\left(\mathbf{y}^{k}+c T \mathbf{x}^{k+1}\right) . \tag{1.15}
\end{equation*}
$$

Combining (1.15) and (1.14c) shows $\mathbf{y}^{k+1}=\operatorname{prox}_{c h}\left(\mathbf{y}^{k}+c T \mathbf{x}^{k+1}\right)$. Thus, (1.14) can be written as

$$
\begin{align*}
& \mathbf{x}^{k+1} \leftarrow \operatorname{prox}_{\tau \rho}\left(\mathbf{x}^{k}-\tau\left[\nabla f\left(\mathbf{x}^{k}\right)+T^{\top}\left(2 \mathbf{y}^{k}-\mathbf{y}^{k-1}\right)\right]\right)  \tag{1.16a}\\
& \mathbf{y}^{k+1} \leftarrow \operatorname{prox}_{c h}\left(\mathbf{y}^{k}+c T \mathbf{x}^{k+1}\right) \tag{1.16b}
\end{align*}
$$

The iterative scheme in (1.16) is a variant of PDA iterations in [8]. In particular, PGADMM as written in (1.16) generates the same iterate sequence as $\left(\mathbf{x}^{k+1}, \mathbf{y}^{k+1}\right)=$ $\mathcal{P} \mathcal{D}_{\tau, c}\left(\mathbf{x}^{k}, \mathbf{y}^{k}, \mathbf{x}^{k+1}, 2 \mathbf{y}^{k}-\mathbf{y}^{k-1}\right)$ in [8] when the Bregman functions $D_{x}$ and $D_{y}$ chosen as $\frac{1}{2}\|\cdot\|^{2}$. Moreover, one can easily prove that Theorem 1 in [8] is still true for this variant of PDA for any $\tau, c>0$ such that $\left(\frac{1}{\tau}-L\right) \frac{1}{c} \geq\|T\|^{2}$.

## 2. A Distributed Algorithm for Time-varying Network Topology. In

 this section, we develop a distributed algorithm for solving (1.2) when the communication network topology is time-varying, under the following assumption.Assumption 3. A primal-dual solution to (1.2) exists and the duality gap is 0.
Clearly this assumption holds if a Slater point for (1.2) exists, i.e., there exists some $\overline{\boldsymbol{\xi}} \in \operatorname{relint}(\operatorname{dom} \varphi \cap \operatorname{dom} g)$ such that $g(\overline{\boldsymbol{\xi}}) \in \operatorname{int}(-\mathcal{K})$. Existence of a Slater point is also assumed in many related papers, e.g., [11, 20, 28, 32, 34]. When $\mathcal{K}=\{\mathbf{0}\}$ and $g_{i}(\xi)=R_{i} \xi-r_{i}$ for $i \in \mathcal{N}$, Assumption 3 trivially holds if there exists some $\overline{\boldsymbol{\xi}} \in \operatorname{relint}(\operatorname{dom} \varphi)$ that is feasible, i.e., $\sum_{i \in \mathcal{N}} R_{i} \bar{\xi}_{i}-r_{i}=\mathbf{0}$.

Since $\mathbb{1}_{\mathcal{K}}(\cdot)=\sup _{y \in \mathbb{R}^{m}}\left\{\langle y, \cdot\rangle-\sigma_{\mathcal{K}}(y)\right\}$, one can reformulate (1.2) as

$$
\begin{equation*}
\min _{\xi} \max _{y \in \mathbb{R}^{m}}\left\{\sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right)-\left\langle\sum_{i \in \mathcal{N}} g_{i}\left(\xi_{i}\right), y\right\rangle-\sigma_{\mathcal{K}}(y)\right\} \tag{2.1}
\end{equation*}
$$

According to Assumption 3, a dual optimal solution $y^{*} \in \mathcal{K}^{\circ}$ exists and the duality gap is 0 for (1.2). Suppose each node $i \in \mathcal{N}$ has its own estimate $y_{i} \in \mathbb{R}^{m}$ of a dual optimal solution; and $\mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}}$ denotes these estimates in long-vector form. We define the consensus set as

$$
\begin{equation*}
\mathcal{C} \triangleq\left\{\mathbf{y} \in \mathbb{R}^{m|\mathcal{N}|}: \exists \bar{y} \in \mathbb{R}^{m} \text { s.t. } y_{i}=\bar{y} \quad \forall i \in \mathcal{N}\right\} \tag{2.2}
\end{equation*}
$$

Suppose we are given a (possibly trivial) bound $B \in(0, \infty]$ such that $\left\|y^{*}\right\| \leq B$. For instance, if a Slater point is available, then a nontrivial bound $B \in(0, \infty)$ on dual solutions can be obtained by solving a convex problem in a distributed way; on the other hand, when Slater condition holds for (1.2) but a Slater point is not available, then the nodes can collectively compute a Slater point - see Section 6. Let $\mathcal{B}_{0} \triangleq\left\{y \in \mathbb{R}^{m}:\|y\| \leq 2 B\right\}$ and $\mathcal{B} \triangleq \Pi_{i \in \mathcal{N}} \mathcal{B}_{0}$, i.e., $\mathcal{B}=\left\{\mathbf{y}:\left\|y_{i}\right\| \leq 2 B, i \in \mathcal{N}\right\}$. Finally, we also define the bounded consensus set,

$$
\begin{equation*}
\tilde{\mathcal{C}} \triangleq \mathcal{C} \cap \mathcal{B}=\left\{\mathbf{y} \in \mathbb{R}^{m|\mathcal{N}|}: \exists \bar{y} \in \mathcal{B}_{0} \subset \mathbb{R}^{m} \text { s.t. } y_{i}=\bar{y} \quad \forall i \in \mathcal{N}\right\} . \tag{2.3}
\end{equation*}
$$

We can equivalently reformulate (2.1) as the following dual consensus problem:

$$
\begin{equation*}
\min _{\boldsymbol{\xi}} \max _{\mathbf{y} \in \widetilde{\mathcal{C}}} L(\boldsymbol{\xi}, \mathbf{y}) \triangleq \sum_{i \in \mathcal{N}}\left(\varphi_{i}\left(\xi_{i}\right)-\left\langle g_{i}\left(\xi_{i}\right), y_{i}\right\rangle-\sigma_{\mathcal{K}}\left(y_{i}\right)\right), \tag{2.4}
\end{equation*}
$$

i.e., any saddle point of (2.4) is also a saddle point of (2.1), which follows from the definitions of $\sigma_{\mathcal{K}}(\cdot)$ and $\widetilde{\mathcal{C}}$. Define $\mathcal{L}: \mathbb{R}^{n} \times \mathbb{R}^{m|\mathcal{N}|} \times \mathbb{R}^{m|\mathcal{N}|} \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ such that

$$
\begin{equation*}
\mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y}) \triangleq \sum_{i \in \mathcal{N}}\left(\varphi_{i}\left(\xi_{i}\right)-\left\langle g_{i}\left(\xi_{i}\right), y_{i}\right\rangle-\sigma_{\mathcal{K}}\left(y_{i}\right)\right)-\langle\mathbf{w}, \mathbf{y}\rangle+\sigma_{\tilde{\mathcal{C}}}(\mathbf{w})-\mathbb{1}_{\mathcal{B}}(\mathbf{y}) . \tag{2.5}
\end{equation*}
$$

Note that for any $\boldsymbol{\xi} \in \operatorname{dom} \varphi$, we have $\max _{\mathbf{y} \in \widetilde{\mathcal{C}}} L(\boldsymbol{\xi}, \mathbf{y})=\max _{\mathbf{y}} \min _{\mathbf{w}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})$; hence, (2.4) can be equivalently written as follows:

$$
\begin{equation*}
\min _{\boldsymbol{\xi}}\left\{\max _{\mathbf{y}} \min _{\mathbf{w}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})\right\}=\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y}) \tag{2.6}
\end{equation*}
$$

where interchanging $\max _{\mathbf{y}}$ and $\min _{\mathbf{w}}$ is trivially justified when $\mathcal{B}$ is bounded; in case $B=+\infty$, i.e., $\mathcal{B}_{0}=\mathbb{R}^{m}$, one can directly verify that $\min _{\mathbf{w}} \max _{\mathbf{y}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})=$ $\min _{\mathbf{w}} \max _{\mathbf{y}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})$ and is equal to $\varphi(\boldsymbol{\xi})$ if $g(\boldsymbol{\xi}) \in-\mathcal{K}$, and $+\infty$ otherwise.

Since we can equivalently solve $\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})$ in (2.6) to solve (1.2), we next generalize PDA iterations in (1.5a)-(1.5b) to solve this saddle-point problem.

Definition 2. Let $\mathcal{X} \triangleq \Pi_{i \in \mathcal{N}} \mathbb{R}^{n_{i}} \times \Pi_{i \in \mathcal{N}} \mathbb{R}^{m}$ and $\mathcal{X} \ni \mathbf{x}=\left[\boldsymbol{\xi}^{\top} \mathbf{w}^{\top}\right]^{\top}$ for $\boldsymbol{\xi}=$ $\left[\xi_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n}$ and $\mathbf{w} \in \mathbb{R}^{n_{0}}$, where $n \triangleq \sum_{i \in \mathcal{N}} n_{i}$ and $n_{0} \triangleq m|\mathcal{N}|$; let $\mathcal{Y} \triangleq \Pi_{i \in \mathcal{N}} \mathbb{R}^{m}$ and $\mathcal{Y} \ni \mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}$. Given parameters $\gamma>0$, and $\tau_{i}, \kappa_{i}>0$ for $i \in \mathcal{N}$, let $\mathbf{D}_{\gamma} \triangleq \frac{1}{\gamma} \mathbf{I}_{n_{0}}, \mathbf{D}_{\tau} \triangleq \operatorname{diag}\left(\left[\frac{1}{\tau_{i}} \mathbf{I}_{n_{i}}\right]_{i \in \mathcal{N}}\right)$, and $\mathbf{D}_{\kappa} \triangleq \operatorname{diag}\left(\left[\frac{1}{\kappa_{i}} \mathbf{I}_{m}\right]_{i \in \mathcal{N}}\right)$. Defining $\psi_{x}(\mathbf{x}) \triangleq$ $\frac{1}{2} \boldsymbol{\xi}^{\top} \mathbf{D}_{\tau} \boldsymbol{\xi}+\frac{1}{2} \mathbf{w}^{\top} \mathbf{D}_{\gamma} \mathbf{w}$ and $\psi_{y}(\mathbf{y}) \triangleq \frac{1}{2} \mathbf{y}^{\top} \mathbf{D}_{\kappa} \mathbf{y}$ leads to the following Bregman distance functions: $D_{x}(\mathbf{x}, \overline{\mathbf{x}})=\frac{1}{2}\|\boldsymbol{\xi}-\overline{\boldsymbol{\xi}}\|_{\mathbf{D}_{\tau}}^{2}+\frac{1}{2}\|\mathbf{w}-\overline{\mathbf{w}}\|_{\mathbf{D}_{\gamma}}^{2}$, and $D_{y}(\mathbf{y}, \overline{\mathbf{y}})=\frac{1}{2}\|\mathbf{y}-\overline{\mathbf{y}}\|_{\mathbf{D}_{\kappa}}^{2}$. To simplify notation, also define $\mathcal{Z} \triangleq \mathcal{X} \times \mathcal{Y}$ and $\mathcal{Z} \ni \mathbf{z}=\left[\mathbf{x}^{\top} \mathbf{y}^{\top}\right]^{\top}$.

Definition 3. Suppose $\varphi_{i}=\rho_{i}+f_{i}$ is a composite convex function defined as in (1.1) for $i \in \mathcal{N}$. Let $\Phi(\mathbf{x}) \triangleq \rho(\mathbf{x})+f(\mathbf{x})$ and $h(\mathbf{y}) \triangleq \sum_{i \in \mathcal{N}} h_{i}\left(y_{i}\right)$ for all $\mathbf{y} \in \mathcal{Y}$, where $\rho(\mathbf{x}) \triangleq \sigma_{\widetilde{\mathcal{C}}}(\mathbf{w})+\sum_{i \in \mathcal{N}} \rho_{i}\left(\xi_{i}\right), f(\mathbf{x}) \triangleq \sum_{i \in \mathcal{N}} f_{i}\left(\xi_{i}\right)$ and $h_{i}\left(y_{i}\right) \triangleq \sigma_{\mathcal{K}}\left(y_{i}\right)+\mathbb{1}_{\mathcal{B}_{0}}\left(y_{i}\right)$ for $i \in \mathcal{N}$. Let $G: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n_{0}}$ such that $G(\boldsymbol{\xi}) \triangleq\left[g_{i}\left(\xi_{i}\right)\right]_{i \in \mathcal{N}}$ for all $\mathbf{x} \in \mathcal{X}$ and define $T: \mathbb{R}^{n} \times \mathbb{R}^{n_{0}} \rightarrow \mathbb{R}^{n_{0}}$ such that $T(\mathbf{x}) \triangleq-G(\boldsymbol{\xi})-\mathbf{w}$; hence, $\mathbf{J} T(\mathbf{x})=\left[-\mathbf{J} G(\boldsymbol{\xi})-\mathbf{I}_{n_{0}}\right]$. With the aim of solving (1.2) as an SP problem, let $\Phi, h$, and $T$ be as given in Definition 3, and consider $\min _{\mathbf{x} \in \mathcal{X}} \max _{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x})+\langle T(\mathbf{x}), \mathbf{y}\rangle-h(\mathbf{y})$. Hence, given the initial iterates $\boldsymbol{\xi}^{0}, \mathbf{w}^{0}, \mathbf{y}^{0}$ and parameters $\gamma>0, \tau_{i}, \kappa_{i}>0$ for $i \in \mathcal{N}$, choosing Bregman functions $D_{x}$ and $D_{y}$ as in Definition 2, and setting $\nu_{x}=\nu_{y}=1$, we propose a modified version of PDA iterations to handle nonlinear $T(\cdot)$; indeed, after linearizing $T(\mathbf{x})$ around $\mathbf{x}^{k}$ in (1.5a), the iterations in (1.5) can be written as follows for $k \geq 0$ :

$$
\begin{align*}
& \mathbf{w}^{k+1} \leftarrow \underset{\mathbf{w}}{\operatorname{argmin}} \sigma_{\widetilde{\mathcal{C}}}(\mathbf{w})-\left\langle\mathbf{y}^{k}, \mathbf{w}\right\rangle+\frac{1}{2 \gamma}\left\|\mathbf{w}-\mathbf{v}^{k}\right\|^{2},  \tag{2.7a}\\
& \mathbf{v}^{k+1} \leftarrow \mathbf{w}^{k+1},  \tag{2.7b}\\
& \xi_{i}^{k+1} \leftarrow \underset{\xi_{i}}{\operatorname{argmin}} \rho_{i}\left(\xi_{i}\right)+\left\langle\nabla f_{i}\left(\xi_{i}^{k}\right)-\mathbf{J} g_{i}\left(\xi_{i}^{k}\right)^{\top} y_{i}^{k}, \xi_{i}-\xi_{i}^{k}\right\rangle+\frac{1}{2 \tau_{i}}\left\|\xi_{i}-\xi_{i}^{k}\right\|^{2}, i \in \mathcal{N},  \tag{2.7c}\\
& y_{i}^{k+1} \leftarrow \underset{y_{i} \in \mathcal{K}^{\circ} \cap \mathcal{B}_{0}}{\operatorname{argmin}}\left\langle 2 g_{i}\left(\xi_{i}^{k+1}\right)-g_{i}\left(\xi_{i}^{k}\right)+2 v_{i}^{k+1}-v_{i}^{k}, y_{i}\right\rangle+\frac{1}{2 \kappa_{i}}\left\|y_{i}-y_{i}^{k}\right\|^{2}, i \in \mathcal{N}, \tag{2.7~d}
\end{align*}
$$

where we initialize $\mathbf{v}^{0}=\mathbf{w}^{0}$. The reason we introduced an auxiliary sequence $\left\{\mathbf{v}^{k}\right\}_{k \geq 0}$ such that $\mathbf{v}^{k}=\left[v_{i}^{k}\right]_{i \in \mathcal{N}}$ will be explained shortly. Briefly, in its currently stated form, the computation in (2.7) can be considered as linearized PDA iterations $-T(\cdot)$ in (1.5a)-(1.5b) is linearized around $\mathbf{x}^{k}$; however, this naive scheme is not suitable for our purposes, i.e., the $\mathbf{w}^{k+1}$ update in (2.7a) is not practical to be computed in a distributed manner. Therefore, instead of setting $\mathbf{v}^{k+1}$ to $\mathbf{w}^{k+1}$, we will replace (2.7b) and assign $\mathbf{v}^{k+1}$ to an approximation of $\mathbf{w}^{k+1}$ such that this approximation can be efficiently computed in a distributed way - this modified version of (2.7) will be analyzed as an inexact variant of linearized PDA.

Using the extended Moreau decomposition for proximal operators, for $k \geq 0$,

$$
\begin{align*}
\mathbf{w}^{k+1} & =\underset{\mathbf{w}}{\operatorname{argmin}} \sigma_{\widetilde{\mathcal{C}}}(\mathbf{w})+\frac{1}{2 \gamma}\left\|\mathbf{w}-\left(\mathbf{v}^{k}+\gamma \mathbf{y}^{k}\right)\right\|^{2}=\operatorname{prox}_{\gamma \sigma_{\widetilde{\mathcal{C}}}}\left(\mathbf{v}^{k}+\gamma \mathbf{y}^{k}\right), \\
& =\gamma\left[\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}-\mathcal{P}_{\widetilde{\mathcal{C}}}\left(\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)\right] . \tag{2.8}
\end{align*}
$$

For an arbitrary $\mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}, \mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{y})$ can be computed as $\mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{y})=\mathbf{1} \otimes$ $\operatorname{argmin}_{x \in \mathcal{B}_{0}} \sum_{i \in \mathcal{N}}\left\|x-y_{i}\right\|^{2}=\mathbf{1} \otimes \operatorname{argmin}_{x \in \mathcal{B}_{0}}\left\|x-\frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N}} y_{i}\right\|^{2}$, where $\mathbf{1} \in \mathbb{R}^{|\mathcal{N}|}$ denotes the vector of all ones. Hence, we can write $\mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{y})=\mathcal{P}_{\mathcal{B}}\left(\left(W \otimes \mathbf{I}_{m}\right) \mathbf{y}\right)$, where $W \triangleq \frac{1}{|\mathcal{N}|} \mathbf{1 1}^{\top} \in \mathbb{R}^{|\mathcal{N}| \times|\mathcal{N}|}$. Equivalently,

$$
\begin{equation*}
\mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{y})=\mathcal{P}_{\mathcal{B}}(\mathbf{1} \otimes p(\mathbf{y})), \quad \text { where } \quad p(\mathbf{y}) \triangleq \frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N}} y_{i} \tag{2.9}
\end{equation*}
$$

Note that $\mathcal{P}_{\mathcal{B}}(\mathbf{y})=\mathbf{y}$ for all $\mathbf{y} \in \mathcal{Y}$ when $B=\infty$; and for $B<\infty, \mathcal{P}_{\mathcal{B}}(\cdot)$ is easy to compute locally since $\mathcal{B}=\Pi_{i \in \mathcal{N}} \mathcal{B}_{0}$ and $\mathcal{P}_{\mathcal{B}_{0}}(y)=y \min \{1,2 B /\|y\|\}$ for $y \in \mathbb{R}^{m}$. Furthermore, $\boldsymbol{\xi}$-step and $\boldsymbol{y}$-step of the PDA implementation in (2.7) can also be computed locally at each node; however, computing $\mathbf{w}^{k+1}$ requires communication among the nodes. Indeed, evaluating the average operator $p(\cdot)$ is not a simple operation in a decentralized computational setting which only allows for communication among neighboring nodes - see Assumption 2. To overcome the issue with decentralized computation of the averaging operator $p(\cdot)$, we will use multi-communication rounds to approximate $p(\cdot)$, and analyze the resulting primal-dual iterations as an inexact primal-dual algorithm. In [12], the idea of using multi-communication rounds has also been exploited within a distributed primal algorithm for unconstrained consensus optimization problems over undirected communication networks.

We define a communication round as an operation over $\mathcal{G}^{t}$ such that every node simultaneously sends and receives data to and from its neighboring nodes according to Assumption 2 - the details of this operation will be discussed shortly. We assume that communication among neighbors occurs instantaneously, and nodes operate synchronously; and we further assume that for each iteration $k \geq 0$, there exists an approximate averaging operator $\mathcal{R}^{k}(\cdot)$ which can be computed in a decentralized fashion and approximates $\mathcal{P}_{\widetilde{\mathcal{C}}}(\cdot)$ with decreasing approximation error in $k$.

Assumption 4. Given a time-varying network $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{R}_{+}}$such that $\mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$ for $t \geq 0$, suppose that there is a global clock known to all $i \in \mathcal{N}$. Assume that the local operations in (2.7c) and (2.7d) can be completed between two tics of the clock for all $i \in \mathcal{N}$ and $k \geq 1$, and every time the clock ticks a communication round with instantaneous messaging between neighboring nodes takes place subject to Assumption 2. Suppose that for each $k \geq 0$ there exists $\mathcal{R}^{k}(\cdot)=\left[\mathcal{R}_{i}^{k}(\cdot)\right]_{i \in \mathcal{N}}$ such that $\mathcal{R}_{i}^{k}(\cdot)$ can be computed with local information available to node $i \in \mathcal{N}$ and decentralized computation of $\mathcal{R}^{k}$ requires $q_{k}$ communication rounds. Furthermore, we assume that there exist $\Gamma>0$ and $\alpha \in(0,1)$ such that $N \Gamma \geq 1$ and for all $k \geq 0, \mathcal{R}^{k}$ satisfies

$$
\begin{equation*}
\mathcal{R}^{k}(\mathbf{w}) \in \mathcal{B}, \quad\left\|\mathcal{R}^{k}(\mathbf{w})-\mathcal{P}_{\tilde{\mathcal{C}}}(\mathbf{w})\right\| \leq N \Gamma \alpha^{q_{k}}\|\mathbf{w}\|, \quad \forall \mathbf{w} \in \mathbb{R}^{n_{0}} . \tag{2.10}
\end{equation*}
$$

The "unit time" is defined to be the length of the interval between two tics of the clock. The assumption that every node $i \in \mathcal{N}$ can finish its $\xi_{i}$ and $y_{i}$ updates in one unit time is mainly for the sake of notational simplicity throughout the analysis. All of our results still hold as long as there exists a uniform bound $\Delta \in \mathbb{Z}_{+}$such that the local operations in (2.7c) and (2.7d) can be completed in $\Delta$ unit time for all $i \in \mathcal{N}$ and $k \geq 1$. In the rest, we assume that $\Delta=1$ as in Assumption 4.

Consider the $k$-th iteration of PDA as shown in (2.7). Instead of setting $\mathbf{v}^{k+1}$ to $\mathbf{w}^{k+1}$ as in (2.7b), we propose approximating $\mathbf{w}^{k+1}$ using the inexact averaging operator $\mathcal{R}^{k}(\cdot)=\left[\mathcal{R}_{i}^{k}(\cdot)\right]_{i \in \mathcal{N}}$ of Assumption 4 and set $\mathbf{v}^{k+1}$ to this approximation. This way, we can skip (2.7a) step and avoid explicitly computing $\mathbf{w}^{k+1}$ as in (2.8) which requires using the exact averaging to compute $\mathcal{P}_{\tilde{\mathcal{C}}}(\cdot)$. More precisely, to obtain an inexact variant of (2.7), we replace (2.7b) with the following:

$$
\begin{equation*}
\mathbf{v}^{k+1} \leftarrow \gamma\left[\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}-\mathcal{R}^{k}\left(\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)\right] . \tag{2.11}
\end{equation*}
$$

Thus, PDA iterations in (2.7), for solving the saddle-point formulation, $\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y}}$ $\mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})$, of the distributed resource allocation problem in (1.2), can be computed inexactly, but in decentralized way for a time-varying connectivity network $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$ provided that $\mathcal{R}^{k}$ satisfying Assumption 4 exists for $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$. We call this inexact version of the linearized PDA as Algorithm DPDA-D and the node-specific computations of DPDA-D are displayed in Fig. 2.1 below. Indeed, the iterate sequence $\left\{\boldsymbol{\xi}^{k}, \mathbf{v}^{k}, \mathbf{y}^{k}\right\}_{k \geq 0}$ generated by Algorithm DPDA-D is the same sequence generated by the recursion in (2.11), (2.7c), and (2.7d). As emphasized previously, the sequence $\left\{\mathbf{w}^{k}\right\}_{k \geq 0}$ will not be explicitly computed, instead we will use it in the analysis of the inexact algorithm. Next, we discuss the existence of inexact average operators $\mathcal{R}^{k}$ satisfying Assumption 4 under various assumptions on time-varying network $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$.

```
Algorithm DPDA-D \(\left(\boldsymbol{\xi}^{0}, \gamma,\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}\right)\)
Initialization: \(v_{i}^{0} \leftarrow \mathbf{0}, \quad y_{i}^{0} \leftarrow \mathbf{0} \quad i \in \mathcal{N}\)
Iteration \(k:(k \geq 0)\)
    1. \(v_{i}^{c} \leftarrow \frac{1}{\gamma} v_{i}^{k}+y_{i}^{k}, \quad v_{i}^{k+1} \leftarrow \gamma v_{i}^{c}-\gamma \mathcal{R}_{i}^{k}\left(\mathbf{v}^{c}\right), \quad i \in \mathcal{N}, \quad\) where \(\quad \mathbf{v}^{c}=\left[v_{i}^{c}\right]_{i \in \mathcal{N}}\)
    2. \(\xi_{i}^{k+1} \leftarrow \operatorname{prox}_{\tau_{i} \rho_{i}}\left(\xi_{i}^{k}-\tau_{i}\left(\nabla f_{i}\left(\xi_{i}^{k}\right)-\mathbf{J} g_{i}\left(\xi_{i}^{k}\right)^{\top} y_{i}^{k}\right)\right), \quad i \in \mathcal{N}\)
3. \(y_{i}^{k+1} \leftarrow \mathcal{P}_{\mathcal{K}} \cap^{\circ} \mathcal{B}_{0}\left(y_{i}^{k}-\kappa_{i}\left(2 g_{i}\left(\xi_{i}^{k+1}\right)-g_{i}\left(\xi_{i}^{k}\right)+\left(2 v_{i}^{k+1}-v_{i}^{k}\right)\right)\right), \quad i \in \mathcal{N}\)
```

Fig. 2.1. Distributed Primal-Dual Algorithm for Time-Varying $\left\{\mathcal{G}^{t}\right\}$ (DPDA-D)
2.1. Inexact averaging operators. Let $t_{k} \in \mathbb{Z}_{+}$be the total number of communication rounds done before the $k$-th iteration of DPDA-D, in Figure 2.1, and let $q_{k} \in \mathbb{Z}_{+}$be the number of communication rounds to be performed within the $k$-th iteration while evaluating $\mathcal{R}^{k}$. According to Assumption 4, each node $i \in \mathcal{N}$ can finish $\xi_{i}^{k+1}$ and $y_{i}^{k+1}$ computation within one unit time, i.e., between two consecutive tics of the clock, for all $k \geq 0$, and communication rounds occur every time the global clock tics; hence, $\mathcal{G}^{t}$ represents the connectivity network at the time of $t$-th communication round for all $t \in \mathbb{Z}_{+}$. Thus, only $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{Z}_{+}}$among $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{R}_{+}}$is relevant since the topology of the time-varying network is only pertinent at those times when communication happens among neighboring nodes. For implementation in practice, it is sufficient for each node to count the number of global clock tics since the last update.

Definition 4. Let $V^{t} \in \mathbb{R}^{|\mathcal{N}| \times|\mathcal{N}|}$ be a matrix encoding the topology of $\mathcal{G}^{t}=$ $\left(\mathcal{N}, \mathcal{E}^{t}\right)$ in some way for $t \in \mathbb{Z}_{+}$. We define $W^{t, s} \triangleq V^{t} V^{t-1} \ldots V^{s+1}$ for any $t, s \in \mathbb{Z}_{+}$ such that $t \geq s+1$.

Let $\left\{\mathcal{G}^{t}\right\}$ be a time-varying directed graph; we adopt the information exchange model in [31] satisfying the assumptions stated in Assumption 5.

Assumption 5. For all $t \in \mathbb{Z}_{+}:$(i) every $i \in \mathcal{N}$ knows $\mathcal{N}_{i}^{t, \text { out }}$ and there exists $\zeta \in(0,1)$ such that for $i \in \mathcal{N}, V_{i j}^{t} \geq \zeta$ if $j \in \mathcal{N}_{i}^{t, \text { in }}$, and $V_{i j}^{t}=0$ otherwise. (ii) $\mathcal{G}^{t}$ is $M$-strongly-connected, i.e., there exist an $\mathbb{Z} \ni M \geq 1$ (possibly unknown to nodes) such that the graph with edge set $\mathcal{E}_{M}^{k}=\bigcup_{t=k M}^{(k+1) M-1} \mathcal{E}^{t}$ is strongly connected for $k \in \mathbb{Z}_{+}$.
2.1.1. Undirected $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{Z}_{+}}$. Let $\left\{\mathcal{G}^{t}\right\}$ be a time-varying undirected graph; $\mathcal{N}_{i}^{t}$ is defined as in Definition 1, and $d_{i}^{t}=\left|\mathcal{N}_{i}^{t}\right|$ for $i \in \mathcal{N}$. For undirected case, we assume $\left\{V^{t}\right\}_{t \in \mathbb{Z}_{+}}$is doubly stochastic and satisfies Assumption 5. For instance, $V^{t}$ can be set as the Metropolis edge weight matrix [5] corresponding to $\mathcal{G}^{t}$, i.e., for each $i \in \mathcal{N}$ set $V_{i j}^{t}=\left(\max \left\{d_{i}^{t}, d_{j}^{t}\right\}+1\right)^{-1}$ for $j \in \mathcal{N}_{i}^{t}, V_{i j}^{t}=0$ for $j \notin \mathcal{N}_{i}^{t} \cup\{i\}$ and
$V_{i i}^{t}=1-\sum_{j \in \mathcal{N}_{i}^{t}} V_{i j}^{t}$. Suppose that there exists $d_{\max }$ such that $d_{i}^{t} \leq d_{\max }$ for all $i \in \mathcal{N}$ and $t \in \mathbb{Z}_{+}$. Under this assumption, it is trivial to check $\zeta=\left(d_{\max }+1\right)^{-1}$.

For $V^{t}$ satisfying $(\boldsymbol{i})$ in Assumption 5, given any $w \in \mathbb{R}^{|\mathcal{N}|}$, the matrix-vector multiplication $V^{t} w \in \mathbb{R}^{|\mathcal{N}|}$ can be computed in a distributed way, i.e., the $i$-th component $\left(V^{t} w\right)_{i}=\sum_{j \in \mathcal{N}_{i} \cup\{i\}} V_{i j}^{t} w_{j}$ can be computed at node $i \in \mathcal{N}$ requiring only local communication of $i$ with nodes in $\mathcal{N}_{i}^{t}$. The next result shows how this distributed operation can be used to approximate the average - also see [33].

Lemma 2.1. Let $\left\{V^{t}\right\}_{t \in \mathbb{Z}_{+}}$be a sequence of doubly stochastic matrices satisfying Assumption 5. For any $s, t \in \mathbb{Z}_{+}$such that $t \geq s,\left\|\left(W^{t, s} \otimes \mathbf{I}_{m}\right) \mathbf{w}-\mathbf{1} \otimes p(\mathbf{w})\right\| \leq$ $\frac{8}{7} \alpha^{t-s}\|\mathbf{w}\|$ for any $\mathbf{w}=\left[w_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}$, where $\alpha=\left(1-\frac{\zeta}{2 N^{2}}\right)^{\frac{1}{2 M}}$.

Proof. The proof immediately follows from Lemma 5 in [31].
For $\mathbf{w}=\left[w_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}$ such that $w_{i} \in \mathbb{R}^{m}$ for $i \in \mathcal{N}$, define

$$
\begin{equation*}
\mathcal{R}^{k}(\mathbf{w}) \triangleq \mathcal{P}_{\mathcal{B}}\left(\left(W^{t_{k}+q_{k}, t_{k}} \otimes \mathbf{I}_{m}\right) \mathbf{w}\right) \tag{2.12}
\end{equation*}
$$

to approximate $\mathcal{P}_{\widetilde{\mathcal{C}}}(\cdot)$ in (2.8). Note that $\mathcal{R}^{k}(\cdot)$ can be computed in a distributed fashion requiring $q_{k}$ communications with the neighbors for each node. In particular, components of $\mathcal{R}^{k}(\mathbf{w})$ can be computed at each node as $\mathcal{R}^{k}(\mathbf{w})=\left[\mathcal{R}_{i}^{k}(\mathbf{w})\right]_{i \in \mathcal{N}}$ such that $\mathcal{R}_{i}^{k}(\mathbf{w}) \triangleq \mathcal{P}_{\mathcal{B}_{0}}\left(\sum_{j \in \mathcal{N}_{i} \cup\{i\}} W_{i j}^{t_{k}+q_{k}, t_{k}} w_{j}\right)$. Moreover, the approximation error, $\mathcal{R}^{k}(\mathbf{w})-\mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{w})$, for any $\mathbf{w}$ can be bounded as in (2.10) using the non-expansivity of $\mathcal{P}_{\mathcal{B}}$ and Lemma 2.1. More precisely, $\mathcal{R}^{k}$ defined in (2.12) satisfies Assumption 4.
2.1.2. Directed $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{Z}_{+}}$. Let $\left\{\mathcal{G}^{t}\right\}$ be a time-varying directed graph, and $\mathcal{N}_{i}^{t, \text { in }}, \mathcal{N}_{i}^{t, \text { out }}$ be defined as in Definition 1 for $i \in \mathcal{N}$. Recall $d_{i}^{t}=\left|\mathcal{N}_{i}^{t, \text { out }}\right|-1$. Since the definition of $\widetilde{\mathcal{C}}$ in (2.3) does not depend on the topology of the network, using the push-sum protocol [25] within DPDA-D, one can also handle time-varying directed communication networks. Indeed, given any $\mathbf{w}=\left[w_{i}\right]_{i \in \mathcal{N}}$, nodes can inexactly compute $\mathcal{P}_{\widetilde{\mathcal{C}}}(\mathbf{w})$ in a distributed fashion with increasing approximation quality; consider the weight-matrix sequence $\left\{V^{t}\right\}_{t \in \mathbb{Z}_{+}}$: for any $t \geq 0$,

$$
\begin{equation*}
V_{i j}^{t}=\frac{1}{d_{j}^{t}+1} \text { if } j \in \mathcal{N}_{i}^{t, \text { in }} ; \quad V_{i j}^{t}=0 \quad \text { if } j \notin \mathcal{N}_{i}^{t, \text { in }}, \quad i \in \mathcal{N} \tag{2.13}
\end{equation*}
$$

For $\mathbf{w}=\left[w_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}$ such that $w_{i} \in \mathbb{R}^{m}$ for $i \in \mathcal{N}$, define

$$
\begin{equation*}
\mathcal{R}^{k}(\mathbf{w}) \triangleq \mathcal{P}_{\mathcal{B}}\left(\operatorname{diag}\left(W^{t_{k}+q_{k}, t_{k}} \mathbf{1}_{N} \otimes \mathbf{I}_{m}\right)^{-1}\left(W^{t_{k}+q_{k}, t_{k}} \otimes \mathbf{I}_{m}\right) \mathbf{w}\right) \tag{2.14}
\end{equation*}
$$

to approximate $\mathcal{P}_{\widetilde{\mathcal{C}}}(\cdot)$ in (2.8). $\mathcal{R}^{k}(\cdot)$ can be computed in a distributed fashion requiring $q_{k}$ communication rounds, and is a compact representation of push-sum operation.

Lemma 2.2. Consider $\mathcal{R}^{k}$ defined in (2.14) for $k \geq 0$. Assuming $\left\{\mathcal{G}^{t}\right\}_{t \in \mathbb{Z}_{+}}$is uniformly strongly connected (M-strongly connected), (2.10) holds for some $\Gamma>0$ and $\alpha \in(0,1)$ such that $\Gamma \leq 8 N^{N M}$ and $\alpha \leq\left(1-\frac{1}{N^{N M}}\right)^{\frac{1}{M}}$.

Proof. The result follows from the proof Lemma 1 in [29].
3. Convergence of Algorithm DPDA-D. Define $\bar{C}_{g} \triangleq \sum_{i \in \mathcal{N}} C_{g_{i}} / N$ and $\bar{R}_{x} \triangleq \max \left\{\left\|\boldsymbol{\xi}^{*}-\boldsymbol{\xi}^{0}\right\|_{\mathbf{L}_{g}},\left\|\boldsymbol{\xi}^{*}-\boldsymbol{\xi}^{0}\right\|_{\mathbf{L}^{\prime}}\right\} / \sqrt{N}$, where $\mathbf{L}^{\prime} \triangleq \operatorname{diag}\left(\left[\left(1+L_{f_{i}}+C_{g_{i}}\right) \mathbf{I}_{n_{i}}\right]_{i \in \mathcal{N}}\right)$ and $\mathbf{L}_{g} \triangleq \operatorname{diag}\left(\left[\left(L_{g_{i}}\right) \mathbf{I}_{n_{i}}\right]_{i \in \mathcal{N}}\right)$.

Theorem 3.1. Suppose Assumptions 1, 2, 3 and 4 hold. For any $\gamma>0$, let the primal-dual step-sizes $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ be chosen such that for some $\beta>0$,

$$
\begin{equation*}
\tau_{i}=\left(\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}+C_{g_{i}}\right)^{-1}, \quad \kappa_{i}=\left(C_{g_{i}}+\frac{5 \gamma}{2}\right)^{-1}, \quad \forall i \in \mathcal{N} \tag{3.1}
\end{equation*}
$$

Given $B \in(0, \infty]$, starting from $\mathbf{v}^{0}=\mathbf{y}^{0}=\mathbf{0}$ and an arbitrary $\boldsymbol{\xi}^{0}$, let $\left\{\left(\boldsymbol{\xi}^{k}, \mathbf{v}^{k}\right)\right\}_{k \geq 0}$ be the primal, and $\left\{\mathbf{y}^{k}\right\}_{k \geq 0}$ be the dual iterate sequence generated by Algorithm DPDA$D$, displayed in Fig. 2.1, using $q_{k} \in \mathbb{Z}_{+}$communication rounds for the $k$-th iteration such that $C_{0} \triangleq \sum_{k=0}^{\infty} \alpha^{q_{k}}(k+1)<\infty$. For any $\gamma>0$, if $\beta>0$ is chosen as discussed below, then $\left\{\left(\boldsymbol{\xi}^{k}, \mathbf{y}^{k}\right)\right\}_{k \geq 0}$ converges to $\left(\boldsymbol{\xi}^{*}, \mathbf{y}^{*}\right)$ such that $\mathbf{y}^{*}=\mathbf{1} \otimes y^{*}$ and $\left(\boldsymbol{\xi}^{*}, y^{*}\right)$ is an optimal primal-dual solution to (1.2). Moreover, both infeasibility, $F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right)$, and suboptimality, $\left|\varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)\right|$ are $\mathcal{O}(1 / K)$, i.e., for all $K \geq 1$ :

$$
\begin{align*}
& F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right) \triangleq d_{\mathcal{C}}\left(\overline{\mathbf{y}}^{K}\right)+\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) \leq \frac{\Lambda(\gamma, \beta)}{K}  \tag{3.2}\\
& 0 \leq \varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)+\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) \leq \frac{\Lambda(\gamma, \beta)}{K}-F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right) \tag{3.3}
\end{align*}
$$

for some $\Lambda(\gamma, \beta) \in \mathbb{R}_{+}$, where $\overline{\boldsymbol{\xi}}^{K}=\frac{1}{K} \sum_{k=1}^{K} \boldsymbol{\xi}^{k}$ and $\overline{\mathbf{y}}^{K}=\frac{1}{K} \sum_{k=1}^{K} \mathbf{y}^{k}$ for $K \geq 1$.
(CASE 1): If a dual bound is known, i.e., $B<\infty$, then (3.2) and (3.3) hold for $\beta=2 B$; moreover, setting the free parameter $\gamma=\left(N^{3 / 2} \Gamma C_{0} B\right)^{-1}$ gives

$$
\begin{equation*}
\Lambda(\gamma, \beta)=\mathcal{O}\left(N B\left(\bar{R}_{x}^{2}+\bar{C}_{g} B\right)+N^{3 / 2} \Gamma C_{0} B\right) \tag{3.4}
\end{equation*}
$$

(CASE 2): If the dual bound does not exist, set $B=\infty$ within DPDA-D. Assuming $q_{k} \geq \log _{1 / \alpha}(24 N \Gamma(k+1))$ for $k \geq 0$, there exists $\bar{\beta}>0$ such that (3.2) and (3.3) hold for all $\beta \geq \bar{\beta}$; moreover, selecting $\gamma=N^{\frac{3}{2}} \Gamma C_{0} \bar{R}_{x}^{2}$ gives $\Lambda(\gamma, \beta)=$ $\mathcal{O}\left(N^{\frac{9}{2}} \Gamma^{3} C_{0}^{3} \bar{R}_{x}^{2} \max \left\{1,\left\|y^{*}\right\|^{2}\right\}\right)$. Finally, when $g_{i}$ is affine for $i \in \mathcal{N}$ and $\left\{\tau_{i}\right\}$ are independent of $\beta, \gamma=\left(N^{\frac{3}{2}} \Gamma C_{0}\right)^{-1}$ leads to $\Lambda(\gamma, \beta)=\mathcal{O}\left(N^{3} \Gamma^{2} C_{0}^{2}\left(\bar{R}_{x}^{2}+\bar{C}_{g} \max \left\{1,\left\|y^{*}\right\|^{2}\right\}\right)\right)$.

Remark 3.1. We assume agents know $q_{k}$ as a function of $k$ at the initialization; hence, synchronicity can be achieved among nodes if simply each node counts the number of times the global clock tics, where at each tic one communication round occurs according to Assumption 4.

Remark 3.2. Suppose we are given $(0,1) \ni \bar{\alpha} \geq \alpha$. For any $c>0$, choosing $q_{k}=\left\lceil(2+c) \log _{\frac{1}{\bar{\alpha}}}(k+1)\right\rceil$ for $k \geq 0$ satisfies the condition in Theorem 3.1, i.e., $C_{0}=\sum_{k=0}^{\infty} \alpha^{q_{k}}(k+1) \leq \frac{1}{c}+1$. Moreover, this choice of $\left\{q_{k}\right\}_{k \in \mathbb{Z}_{+}}$implies that the total number of communication rounds right before the $K$-th iteration is equal to $t_{K}=\sum_{k=0}^{K-1} q_{k}=(2+c)\left[(K-1) \log _{\frac{\bar{\alpha}}{}}(K)+\log _{\frac{1}{\bar{\alpha}}}(e)\right]$ where $e$ is Euler's number.

Corollary 3.2. Under the premise of Theorem 3.1, let $\left\{\mathcal{G}^{t}\right\}$ be an undirected time-varying graph and $\left\{q_{k}\right\}$ be as in Remark 3.2 with $(0,1) \ni \bar{\alpha}=\iota \alpha$ for some $\iota>1$. Let $Q(\epsilon)$ be the total number of communications needed to compute an $\epsilon$ optimal and $\epsilon$-feasible solution $\left(\boldsymbol{\xi}^{\epsilon}, \mathbf{y}^{\epsilon}\right)$ for $\gamma=1 / \mathcal{O}(\sqrt{N})$, i.e., $F\left(\boldsymbol{\xi}^{\epsilon}, \mathbf{y}^{\epsilon}\right)<\epsilon$ and $\left|\varphi\left(\boldsymbol{\xi}^{\epsilon}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)\right|<\epsilon$. If a dual bound $B<\infty$ is known, then $Q(\epsilon)=\mathcal{O}\left(\frac{N^{4}}{\epsilon} \log \left(\frac{N}{\epsilon}\right)\right)$. If a Slater point does not exist, i.e., $B=\infty$, then $Q(\epsilon)=\mathcal{O}\left(\frac{N^{4.5}}{\epsilon} \log \left(\frac{N^{1.5}}{\epsilon}\right)\right)$; moreover, $Q(\epsilon)=\mathcal{O}\left(\frac{N^{4}}{\epsilon} \log \left(\frac{N}{\epsilon}\right)\right)$ is achieved when $g_{i}$ is an affine function for $i \in \mathcal{N}$.

Proof. Theorem 3.1 implies that $\left(\boldsymbol{\xi}^{\epsilon}, \mathbf{y}^{\epsilon}\right)$ can be computed in $K^{\epsilon}=\Lambda(\gamma, \beta) / \epsilon$ DPDA-D iterations which requires $t_{K^{\epsilon}}=\mathcal{O}\left(K^{\epsilon} \log \left(K^{\epsilon}\right) / \log \left(\frac{1}{\alpha}\right)\right)$ communications in total - see Remark 3.2. Lemma 2.1 implies that $\Gamma=1 / N$; hence, setting $\gamma$ as described in Theorem 3.1, we bound $\Lambda(\gamma, \beta)$ with $\mathcal{O}(N)$ for CASE $1, \mathcal{O}\left(N^{1.5}\right)$ for CASE 2 in general and with $\mathcal{O}(N)$ when $g_{i}$ 's are linear. Thus, the result follows from $\log \left(\frac{1}{\alpha}\right) \geq \zeta / N^{2}$, where $\zeta$ can be as small as $\mathcal{O}(1 / N)$.
Note that when $\left\{\mathcal{G}^{t}\right\}$ is a general time-varying directed graph, we employ push-sum protocol with $\Gamma=N^{N M}$ (see Lemma 2.2) which leads to exponential $\mathcal{O}(1)$ bounds,
e.g., $\Lambda(\gamma, \beta)=\mathcal{O}\left(N^{N M+\frac{3}{2}} B\right)$ for CASE 1. To our best knowledge, polynomial bounds for directed graphs in $N$ is still an open question [30]. That said, setting $\left\{q_{k}\right\}$ as in Remark 3.2, DPDA-D can compute an $\epsilon$-solution in $\mathcal{O}\left(\frac{1}{\epsilon} \log \left(\frac{1}{\epsilon}\right)\right)$ communications even for general directed graphs and choosing $q_{k}=(2+c) \log _{\frac{1}{\bar{\alpha}}}(k+1)$ in CASE 1 leads to $\mathcal{O}(1)$ constant $N^{2 N M+1.5}$ which is better than $\mathcal{O}\left(\frac{1}{\epsilon^{2}} \log ^{2}\left(\frac{1}{\epsilon}\right)\right)$ result in [20, 29] with $\mathcal{O}(1)$ constant of $N^{2 N M+2}$. The method in [20] has $\log (k) / \sqrt{k}$ rate and requires exact minimization of convex $f_{i}$ over compact $\mathcal{X}_{i}$ at each iteration. The method in [29] can be used to solve the dual of (1.2) when $\rho_{i}$ is the indicator function of some compact convex set $\mathcal{X}_{i}$ and $f_{i}$ is convex for $i \in \mathcal{N}$; but, the subproblem that needs to be solved at each iteration is fairly complicated as in [20].

Remark 3.3. Since $\sum_{k=1}^{\infty} \alpha^{\sqrt[p]{k}} k<\infty$ for any $p \geq 1$, if one chooses $q_{k}=\sqrt[p]{k}$ for $k \geq 1$, then $t_{K}=\sum_{k=0}^{K-1} q_{k}=\mathcal{O}\left(K^{1+1 / p}\right)$. This choice of $\left\{q_{k}\right\}_{k \in \mathbb{Z}_{+}}$, unlike the one in Remark 3.2, is independent of the parameter $\alpha \in(0,1)$; but leads to a larger $C_{0}=\sum_{k=0}^{\infty} \alpha^{q_{k}}(k+1)=\mathcal{O}\left(\alpha / \log ^{2 p}(\alpha)\right)$ for $\alpha \in(1 / e, 1)$. On the other hand, apriori running DPDA-D, a practical way to estimate $\alpha \in(0,1)$ is to run an average consensus iterations with a random initialization until iterates stagnate around the average; this leads to a rate coefficient $\alpha_{i}$ for $i \in \mathcal{N}$. Next, nodes can do a max consensus to compute $\bar{\alpha}=\max _{i \in \mathcal{N}} \alpha_{i}$ and use it to set $q_{k}=(2+c) \log _{\frac{1}{\alpha}}(k+1)$.

Remark 3.4. Suppose the dual bound is not available. If $q_{k}=(2+c) \log _{1 / \bar{\alpha}}(k+1)$ for some $c>0$ and $(0,1) \ni \bar{\alpha} \geq \alpha$, then $q_{k} \geq \log _{1 / \alpha}(24 N \Gamma(k+1)$ ) for all $k \geq \tilde{K} \triangleq$ $\left\lceil(24 N \Gamma)^{1 /(1+c)}\right\rceil$. If $q_{k}=\sqrt[p]{k}$ for some $p \geq 1$, then $q_{k} \geq \log _{1 / \alpha}(24 N \Gamma(k+1))$ for all $k \geq \tilde{K}=\left\lceil\left(\log _{1 / \alpha}(24 N \Gamma)+p \log _{1 / \alpha} p\right)^{p}\right\rceil$. Hence, the rate results of Theorem 3.1 will hold after the transient period of $\tilde{K}$ iterations.
3.1. Auxiliary results to prove Theorem 3.1. Let $\left\{\boldsymbol{\xi}^{k}, \mathbf{v}^{k}, \mathbf{y}^{k}\right\}_{k \geq 0}$ be the iterate sequence generated by DPDA-D as shown in Figure 2.1 and $\left\{\mathbf{w}^{k}\right\}_{k \geq 0}$ be the auxiliary sequence where $\mathbf{w}^{k}$ is given in (2.8) for $k \geq 1$ and we set $\mathbf{w}^{0} \triangleq \mathbf{v}^{0}=\mathbf{0}$. We first define the error sequence $\left\{\mathbf{e}^{k}\right\}_{k \geq 0}$ : let $\mathbf{e}^{k} \triangleq\left(\mathbf{v}^{k}-\mathbf{w}^{k}\right) / \gamma$ for all $k \geq 0$; hence, $\mathbf{e}^{0}=\mathbf{e}^{1}=\mathbf{0}$ and for $k \geq 0$, we have

$$
\begin{equation*}
\mathbf{e}^{k+1}=\mathcal{P}_{\tilde{\mathcal{C}}}\left(\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)-\mathcal{R}^{k}\left(\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right) . \tag{3.5}
\end{equation*}
$$

In order to prove Theorem 3.1, we first prove Lemma 3.3 which help us to bound $\mathcal{L}\left(\boldsymbol{\xi}^{k}, \mathbf{v}^{k}, \mathbf{y}\right)-\mathcal{L}\left(\boldsymbol{\xi}, \mathbf{v}, \mathbf{y}^{k}\right)$ for any given $(\boldsymbol{\xi}, \mathbf{v}, \mathbf{y}) \in \mathcal{Z}$ and $k \geq 1$, where $\mathcal{L}$ is defined in (2.5); and then we provide a few other technical results which will be used together with Lemma 3.3 to show the asymptotic convergence of $\left\{\boldsymbol{\xi}^{k}, \mathbf{v}^{k}, \mathbf{y}^{k}\right\}$ in Theorem 3.1.

Definition 5. Let $\mathbf{D}_{\gamma}$ and $\mathbf{D}_{\kappa}$ be the diagonal matrices given in Definition 2. Define a diagonal matrix $\mathbf{C} \triangleq \operatorname{diag}\left(\left[C_{g_{i}}\right]_{i \in \mathcal{N}}\right)$, and $H \triangleq\left[\begin{array}{ll}\mathbf{C} & \mathbf{I}_{N}\end{array}\right]$. Given some $\beta>0$, define the symmetric matrix $\overline{\mathbf{Q}}(\beta) \triangleq\left[\begin{array}{cc}\overline{\mathbf{D}}(\beta) & -H^{\top} \\ -H & \overline{\mathbf{D}}_{\kappa}\end{array}\right]$, where $\overline{\mathbf{D}}(\beta) \triangleq\left[\begin{array}{cc}\overline{\mathbf{D}}_{\tau}(\beta) & \mathbf{0} \\ \mathbf{0} & \frac{1}{\gamma} \mathbf{I}_{N}\end{array}\right]$, $\overline{\mathbf{D}}_{\tau}(\beta) \triangleq \operatorname{diag}\left(\left[\frac{1}{\tau_{i}}-\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}\right]_{i \in \mathcal{N}}\right)$ and $\overline{\mathbf{D}}_{\kappa} \triangleq \operatorname{diag}\left(\left[\frac{1}{\kappa_{i}}\right]_{i \in \mathcal{N}}\right)$. Let $\mathbf{u}: \mathcal{Z} \times \mathcal{Z} \rightarrow$ $\mathbb{R}^{3 N}$ such that $\mathbf{u}(\mathbf{z}, \overline{\mathbf{z}}) \triangleq\left[\left[\left\|\xi_{i}-\bar{\xi}_{i}\right\|\right]_{i \in \mathcal{N}}^{\top} \quad\left[\left\|w_{i}-\bar{w}_{i}\right\|\right]_{i \in \mathcal{N}}^{\top} \quad\left[\left\|y_{i}-\bar{y}_{i}\right\|\right]_{i \in \mathcal{N}}^{\top}\right]^{\top} \in \mathbb{R}^{3 N}$.

Lemma 3.3. Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be the spaces defined in Definition 2. Suppose $\left\{\tilde{\mathbf{x}}^{k}\right\}_{k \geq 0} \subset \mathcal{X}$ be the primal and $\left\{\mathbf{y}^{k}\right\}_{k \geq 0} \subset \mathcal{Y}$ be the dual iterate sequences generated by Algorithm DPDA-D in Fig. 2.1, using some positive stepsizes: $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ and $\gamma$, and initializing from an arbitrary $\boldsymbol{\xi}^{0}$ and $\mathbf{v}^{0}=\mathbf{y}^{0}=\mathbf{0}$, where $\tilde{\mathbf{x}}^{k}=\left[\boldsymbol{\xi}^{k^{\top}} \mathbf{v}^{k^{\top}}\right]^{\top}$ for $k \geq 0$. Define $\left\{\mathbf{x}^{k}\right\}$ and $\left\{\mathbf{z}^{k}\right\}$ such that $\mathbf{x}^{k}=\left[\boldsymbol{\xi}^{k^{\top}} \mathbf{w}^{k \top}\right]^{\top} \in \mathcal{X}$ and $\mathbf{z}^{k}=\left[\mathbf{x}^{k^{\top}} \mathbf{y}^{k^{\top}}\right]^{\top} \in$
$\mathcal{Z}$ for $k \geq 0$. Let $\left\{\beta_{k}\right\}_{k \geq 0}$ be such that $\beta_{k} \geq \max _{i \in \mathcal{N}}\left\|y_{i}^{k}\right\|$ for $k \geq 0$, then for any $\mathbf{x}=\left[\boldsymbol{\xi}^{\top} \mathbf{w}^{\top}\right]^{\top} \in \mathcal{X}$, and $\mathbf{y} \in \mathcal{Y},\left\{\mathbf{z}^{k}\right\}_{k \geq 0} \subset \mathcal{Z}$ satisfies

$$
\begin{align*}
\mathcal{L}\left(\mathbf{x}^{k+1}, \mathbf{y}\right)-\mathcal{L}\left(\mathbf{x}, \mathbf{y}^{k+1}\right) \leq & {\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{k}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{k}\right), \mathbf{y}-\mathbf{y}^{k}\right\rangle\right] } \\
& -\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{k+1}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{k+1}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{k+1}\right), \mathbf{y}-\mathbf{y}^{k+1}\right\rangle\right] \\
& +E^{k+1}(\mathbf{z})-\frac{1}{2} \mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right)^{\top} \overline{\mathbf{Q}}\left(\beta_{k}\right) \mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right), \quad \forall k \geq 0, \tag{3.6}
\end{align*}
$$

where $\mathbf{u}(\cdot, \cdot)$ is given in Definition $5, \mathbf{z}^{k}=\left[\mathbf{x}^{k^{\top}} \mathbf{y}^{k^{\top}}\right]^{\top}$, $D_{x}$ and $D_{y}$ are Bregman functions in Definition 2, $T(\cdot)$ is given in Definition 3, $E^{k+1}(\mathbf{z}) \triangleq\left\|\mathbf{e}^{k}\right\|\left\|\mathbf{w}-\mathbf{w}^{k+1}\right\|+$ $\gamma\left\|2 \mathbf{e}^{k+1}-\mathbf{e}^{k}\right\|\left\|\mathbf{y}-\mathbf{y}^{k+1}\right\|$ and $\mathbf{e}^{k} \triangleq\left(\mathbf{v}^{k}-\mathbf{w}^{k}\right) / \gamma$ for $k \geq 0$.

Proof. Given $\left\{\mathbf{v}^{k}\right\}_{k \geq 0}$ generated as in Fig. 2.1, let $\left\{\mathbf{w}^{k}\right\}_{k \geq 0}$ sequence be defined according to (2.7a) - recall that $\left\{\mathbf{w}^{k}\right\}_{k \geq 0}$ sequence is never actually computed in practice; this sequence will help us in our analysis of DPDA-D.

Let $\Phi, h$, and possibly nonlinear map $T(\cdot)$ be as given in Definition 3; hence, our objective is to compute a saddle-point for $\min _{\mathbf{x} \in \mathcal{X}} \max _{\mathbf{y} \in \mathcal{Y}} \Phi(\mathbf{x})+\langle T(\mathbf{x}), \mathbf{y}\rangle-h(\mathbf{y})$ to solve (1.2). Using this notation, and the fact that $\mathbf{v}^{k}=\mathbf{w}^{k}+\gamma \mathbf{e}^{k}$ for $k \geq 0$, we can represent $\left\{\boldsymbol{\xi}^{k}\right\},\left\{\mathbf{w}^{k}\right\}$ and $\left\{\mathbf{y}^{k}\right\}$ sequences in a more compact form as follows:
(3.7a) $\mathbf{x}^{k+1}=\underset{\mathbf{x} \in \mathcal{X}}{\operatorname{argmin}} \rho(\mathbf{x})+f\left(\mathbf{x}^{k}\right)+\left\langle\nabla f\left(\mathbf{x}^{k}\right)+\mathbf{J} T\left(\mathbf{x}^{k}\right)^{\top} \mathbf{y}^{k}+U \mathbf{e}^{k}, \mathbf{x}-\mathbf{x}^{k}\right\rangle+D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)$,
$(3.7 \mathrm{~b}) \mathbf{y}^{k+1}=\underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmin}} h(\mathbf{y})-\left\langle 2 T\left(\mathbf{x}^{k+1}\right)-T\left(\mathbf{x}^{k}\right)-\gamma\left(2 \mathbf{e}^{k+1}-\mathbf{e}^{k}\right), \mathbf{y}\right\rangle+D_{y}\left(\mathbf{y}, \mathbf{y}^{k}\right)$,
where $U=\left[\begin{array}{ll}\mathbf{0} & \mathbf{I}_{n_{0}}\end{array}\right]^{\top} \in \mathbb{R}^{\left(n+n_{0}\right) \times n_{0}}$ and $\left\{\mathbf{v}^{k}\right\}$ is updated according to (2.11). Let $S_{1}^{k}(\mathbf{w}) \triangleq\left\langle\mathbf{e}^{k}, \mathbf{w}-\mathbf{w}^{k+1}\right\rangle$. Since $\rho$ is a proper, closed, convex function and $D_{x}$ is a Bregman function, Property 1 in [42] applied to (3.7a) implies that for any $\mathbf{x} \in \mathcal{X}$,

$$
\begin{align*}
& \rho(\mathbf{x})-\rho\left(\mathbf{x}^{k+1}\right)+\left\langle\nabla f\left(\mathbf{x}^{k}\right)+\mathbf{J} T\left(\mathbf{x}^{k}\right)^{\top} \mathbf{y}^{k}, \mathbf{x}-\mathbf{x}^{k+1}\right\rangle \geq  \tag{3.8}\\
& D_{x}\left(\mathbf{x}, \mathbf{x}^{k+1}\right)-D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)+D_{x}\left(\mathbf{x}^{k+1}, \mathbf{x}^{k}\right)-S_{1}^{k}(\mathbf{w})
\end{align*}
$$

Moreover, convexity of $f_{i}$ and Lipschitz continuity of $\nabla f_{i}$ implies that for any $\xi_{i} \in \mathbb{R}^{n_{i}}$,

$$
f_{i}\left(\xi_{i}\right) \geq f_{i}\left(\xi_{i}^{k}\right)+\left\langle\nabla f_{i}\left(\xi_{i}^{k}\right), \xi_{i}-\xi_{i}^{k}\right\rangle \geq f_{i}\left(\xi_{i}^{k+1}\right)+\left\langle\nabla f_{i}\left(\xi_{i}^{k}\right), \xi_{i}-\xi_{i}^{k+1}\right\rangle-\frac{L_{f_{i}}}{2}\left\|\xi_{i}^{k+1}-\xi_{i}^{k}\right\|^{2}
$$

Similarly, since $-y_{i}^{k} \in \mathcal{K}^{*}, \mathcal{K}$-convexity of $g_{i}$ and Lipschitz continuity of $\mathbf{J} g_{i}$ imply

$$
\begin{aligned}
-\left\langle g_{i}\left(\xi_{i}\right), y_{i}^{k}\right\rangle & \geq-\left\langle g_{i}\left(\xi_{i}^{k}\right), y_{i}^{k}\right\rangle-\left\langle\mathbf{J} g_{i}\left(\xi_{i}^{k}\right)^{\top} y_{i}^{k}, \xi_{i}-\xi_{i}^{k}\right\rangle \\
& \geq-\left\langle g_{i}\left(\xi_{i}^{k+1}\right), y_{i}^{k}\right\rangle-\left\langle\mathbf{J} g_{i}\left(\xi_{i}^{k}\right)^{\top} y_{i}^{k}, \xi_{i}-\xi_{i}^{k+1}\right\rangle-\frac{\beta_{k} L_{g_{i}}}{2}\left\|\xi_{i}^{k+1}-\xi_{i}^{k}\right\|^{2}
\end{aligned}
$$

Summing the last two inequalities first for each $i$, then summing over $i \in \mathcal{N}$, and combining the sum with (3.8), we get

$$
\begin{align*}
\Phi(\mathbf{x})-\Phi\left(\mathbf{x}^{k+1}\right)+\langle T(\mathbf{x})- & \left.T\left(\mathbf{x}^{k+1}\right), \mathbf{y}^{k}\right\rangle \geq  \tag{3.9}\\
& D_{x}\left(\mathbf{x}, \mathbf{x}^{k+1}\right)-D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)+\frac{1}{2}\left\|\mathbf{x}^{k+1}-\mathbf{x}^{k}\right\|_{\tilde{\mathbf{D}}^{k}}^{2}-S_{1}^{k}(\mathbf{w})
\end{align*}
$$

where $\tilde{\mathbf{D}}^{k} \triangleq\left[\begin{array}{cc}\tilde{\mathbf{D}}_{\tau}^{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\gamma}\end{array}\right]$ and $\tilde{\mathbf{D}}_{\tau}^{k} \triangleq \operatorname{diag}\left(\left[\left(\frac{1}{\tau_{i}}-\left(L_{f_{i}}+\beta_{k} L_{g_{i}}\right)\right) \mathbf{I}_{n_{i}}\right]_{i \in \mathcal{N}}\right)$.

Finally, since $h$ is a proper, closed, convex function and $D_{y}$ is a Bregman function, Property 1 in [42] applied to (3.7b) implies

$$
\begin{align*}
& h(\mathbf{y})-h\left(\mathbf{y}^{k+1}\right)-\left\langle 2 T\left(\mathbf{x}^{k+1}\right)-T\left(\mathbf{x}^{k}\right), \mathbf{y}-\mathbf{y}^{k+1}\right\rangle \geq  \tag{3.10}\\
& D_{y}\left(\mathbf{y}, \mathbf{y}^{k+1}\right)-D_{y}\left(\mathbf{y}, \mathbf{y}^{k}\right)+\frac{1}{2}\left\|\mathbf{y}^{k+1}-\mathbf{y}^{k}\right\|_{\mathbf{D}_{\kappa}}^{2}-S_{2}^{k}(\mathbf{y})
\end{align*}
$$

where $S_{2}^{k}(\mathbf{y})=\gamma\left\langle\left(2 \mathbf{e}^{k+1}-\mathbf{e}^{k}\right), \mathbf{y}-\mathbf{y}^{k+1}\right\rangle$. Summing (3.9) and (3.10), and rearranging the terms yields

$$
\begin{aligned}
& \mathcal{L}\left(\mathbf{x}^{k+1}, \mathbf{y}\right)-\mathcal{L}\left(\mathbf{x}, \mathbf{y}^{k+1}\right) \leq S^{k}(\mathbf{z})+\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{k}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{k}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{k}\right), \mathbf{y}-\mathbf{y}^{k}\right\rangle\right] \\
& -\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{k+1}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{k+1}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{k+1}\right), \mathbf{y}-\mathbf{y}^{k+1}\right\rangle\right], \\
& S^{k}(\mathbf{z}) \triangleq S_{1}^{k}(\mathbf{w})+S_{2}^{k}(\mathbf{y})+\left\langle T\left(\mathbf{x}^{k+1}\right)-T\left(\mathbf{x}^{k}\right), \mathbf{y}^{k+1}-\mathbf{y}^{k}\right\rangle-\frac{1}{2}\left\|\mathbf{x}^{k+1}-\mathbf{x}^{k}\right\|_{\tilde{\mathbf{D}}^{k}}^{2}-\frac{1}{2}\left\|\mathbf{y}^{k+1}-\mathbf{y}^{k}\right\|_{\mathbf{D}_{\kappa}}^{2} .
\end{aligned}
$$

Using Cauchy Schwartz inequality and Lipschitz continuity of $g_{i}$ for all $i \in \mathcal{N}$, one can bound $S^{k}(\mathbf{z})$ as follows:

$$
S^{k}(\mathbf{z}) \leq\left\|\mathbf{e}^{k}\right\|\left\|\mathbf{w}-\mathbf{w}^{k+1}\right\|+\gamma\left\|2 \mathbf{e}^{k+1}-\mathbf{e}^{k}\right\|\left\|\mathbf{y}-\mathbf{y}^{k+1}\right\|-\frac{1}{2} \mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right)^{\top} \overline{\mathbf{Q}}\left(\beta_{k}\right) \mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right)
$$

for all $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}$, and $k \geq 0$.
Given some $\beta>0$, next lemma gives a sufficient condition on the local step-sizes for $\overline{\mathbf{Q}}(\beta)$ to be positive (semi)-definite.

Lemma 3.4. Consider $\overline{\mathbf{Q}}(\beta)$ given in Definition 5 for some $\beta>0$. If positive $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ and $\gamma$ satisfy $\tau_{i} \leq \frac{1}{\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}}, \kappa_{i} \leq \frac{1}{\gamma}$ and $\left(\frac{1}{\tau_{i}}-\max \left\{1, L_{f_{i}}+\right.\right.$ $\left.\left.\beta L_{g_{i}}\right\}\right)\left(\frac{1}{\kappa_{i}}-\gamma\right)>C_{g_{i}}^{2}$ for all $i \in \mathcal{N}$, then $\overline{\mathbf{Q}}(\beta) \succ \mathbf{0}$. Moreover, $\overline{\mathbf{Q}}(\beta) \succeq \mathbf{0}$ if the strict inequalities in the last condition are relaxed to $\geq$-relation for some $i \in \mathcal{N}$.

Proof. Given a permutation matrix $\mathbf{P} \triangleq\left[\begin{array}{ccc}\mathbf{I}_{N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{N} \\ \mathbf{0} & \mathbf{I}_{N} & \mathbf{0}\end{array}\right], \overline{\mathbf{Q}}(\beta) \succ \mathbf{0}$ is equivalent to $\mathbf{P} \overline{\mathbf{Q}}(\beta) \mathbf{P}^{-1} \succ \mathbf{0}$. Since $\gamma>0$, Schur complement condition implies

$$
\mathbf{P} \overline{\mathbf{Q}}(\beta) \mathbf{P}^{-1}=\left[\begin{array}{ccc}
\overline{\mathbf{D}}_{\tau}(\beta) & -\mathbf{C} & \mathbf{0}  \tag{3.11}\\
-\mathbf{C} & \overline{\mathbf{D}}_{\kappa} & -\mathbf{I}_{N} \\
\mathbf{0} & -\mathbf{I}_{N} & \frac{1}{\gamma} \mathbf{I}_{N}
\end{array}\right] \succ \mathbf{0} \Leftrightarrow\left[\begin{array}{cc}
\overline{\mathbf{D}}_{\tau}(\beta) & -\mathbf{C} \\
-\mathbf{C} & \overline{\mathbf{D}}_{\kappa}
\end{array}\right]-\gamma\left[\begin{array}{cc}
\mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{N}
\end{array}\right] \succ \mathbf{0} .
$$

Note $\overline{\mathbf{D}}_{\tau}(\beta) \succ 0$; hence, using Schur complement again, one can conclude that the condition on the right-hand-side of (3.11) holds if and only if $\overline{\mathbf{D}}_{\kappa}-\gamma \mathbf{I}_{N}-\mathbf{C} \overline{\mathbf{D}}_{\tau}(\beta)^{-1} \mathbf{C} \succ \mathbf{0}$, equivalently $\left(\frac{1}{\kappa_{i}}-\gamma\right)-\left(\frac{1}{\tau_{i}}-\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}\right)^{-1} C_{g_{i}}^{2}>0$ for all $i \in \mathcal{N}$. Hence, the conditions in Lemma 3.4 are both necessary and sufficient for $\overline{\mathbf{Q}}(\beta) \succ \mathbf{0}$. If the strict inequalities in the last condition are relaxed to include equality for some $i \in \mathcal{N}$, then it is sufficient for $\overline{\mathbf{Q}}(\beta) \succeq \mathbf{0}$.
Note if $\left\{\mathbf{y}^{k}\right\} \subseteq \mathcal{B}$, then we can set $\beta^{k}=2 B$ for all $k \geq 0$; hence, Lemma 3.4 implies that if the local step-size condition in (3.1) holds (possibly with equality for some $i \in \mathcal{N}$ ), then $\overline{\mathbf{Q}}\left(\beta^{k}\right)$ in (3.6) is positive (semi)-definite for all $k \geq 0$, which helps to simplify the analysis of Theorem 3.1.
3.2. Proof of Theorem 3.1. Using the two technical lemmas in the Appendix 9, we are ready to prove Theorem 3.1. The proof is divided into three subsections where we first show that the dual iterate sequence $\left\{\mathbf{y}^{k}\right\}$ stays bounded even if a dual bound is not provided, i.e., $B=\infty$; second, we prove the convergence of the iterate sequences; finally, we provide rate statements for the infeasibility and suboptimality.

Under Assumption 3, a saddle point $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ for $\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y}} \mathcal{L}(\boldsymbol{\xi}, \mathbf{w}, \mathbf{y})$ exists, where $\mathcal{L}$ is given in (2.5); moreover, any saddle point $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ satisfies that $\mathbf{y}^{*}=\mathbf{1} \otimes y^{*}$ for some $y^{*} \in \mathcal{B}_{0}$ such that $\left(\boldsymbol{\xi}^{*}, y^{*}\right)$ is a primal-dual solution to (1.2). Thus, $y^{*} \in \mathcal{K}^{\circ}$ and $\mathcal{L}\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)=\varphi\left(\boldsymbol{\xi}^{*}\right)$. Indeed, this implies $\left\langle\mathbf{y}^{*}, \mathbf{w}^{*}\right\rangle-\sigma_{\widetilde{\mathcal{C}}}\left(\mathbf{w}^{*}\right)=0$ which leads to $\sum_{i \in \mathcal{N}} w_{i}^{*}=\mathbf{0}$, i.e., $\mathbf{w}^{*} \in \mathcal{C}^{\circ}$. Hence, we have $0=\left\langle\mathbf{y}^{*}, \mathbf{w}^{*}\right\rangle=\sigma_{\widetilde{\mathcal{C}}}\left(\mathbf{w}^{*}\right)$, and it trivially follows that if $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ is a saddle point of $\mathcal{L}$ with $\mathbf{w}^{*} \neq \mathbf{0}$, then $\left(\boldsymbol{\xi}^{*}, \mathbf{0}, \mathbf{y}^{*}\right)$ is another saddle point of $\mathcal{L}$. Therefore, under Assumption 3, there is always a saddle point of the form $\left(\boldsymbol{\xi}^{*}, \mathbf{0}, \mathbf{y}^{*}\right)$, i.e., with $\mathbf{w}^{*}=\mathbf{0}$. In the rest, let $\mathbf{z}^{*}$ be a saddle point with components $\left(\boldsymbol{\xi}^{*}, \mathbf{0}, \mathbf{y}^{*}\right)$.

Next, we state few useful observations later used in the proof. Given some $\beta>0$, when primal-dual stepsizes are chosen as stated in (3.1), Lemma 3.4 implies that $\overline{\mathbf{Q}}(\beta) \succ 0$ and it follows from definitions of $D_{x}, D_{y}$ and $T$ that for all $\mathbf{z}, \mathbf{z}^{\prime} \in \mathcal{Z}$,

$$
\begin{align*}
D_{x}\left(\mathbf{x}, \mathbf{x}^{\prime}\right) & +D_{y}\left(\mathbf{y}, \mathbf{y}^{\prime}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{\prime}\right), \mathbf{y}-\mathbf{y}^{\prime}\right\rangle  \tag{3.12}\\
& \geq \sum_{i \in \mathcal{N}} \frac{1}{2} \max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}\left\|\xi_{i}-\xi_{i}^{\prime}\right\|^{2}+\frac{1}{4 \gamma}\left\|\mathbf{w}-\mathbf{w}^{\prime}\right\|^{2}+\frac{\gamma}{4}\left\|\mathbf{y}-\mathbf{y}^{\prime}\right\|^{2}
\end{align*}
$$

Moreover, the error term $E^{k+1}(\mathbf{z})$, defined in Lemma 3.3, trivially satisfies

$$
\begin{equation*}
E^{k+1}(\mathbf{z}) \leq \gamma\left(2\left\|\mathbf{e}^{k+1}\right\|+\left\|\mathbf{e}^{k}\right\|\right)\left(\frac{1}{\gamma}\left\|\mathbf{w}^{k+1}-\mathbf{w}\right\|+\left\|\mathbf{y}^{k+1}-\mathbf{y}\right\|\right), \quad \forall k \geq 0 \tag{3.13}
\end{equation*}
$$

3.2.1. Boundedness of dual iterate sequence. Next we show $\left\{\mathbf{y}^{k}\right\}_{k \geq 0}$ and $\left\{\mathbf{w}^{k}\right\}_{k \geq 0}$ are bounded. More specifically, our aim is to show that there exist $\bar{\beta}, \varsigma, \nu \in$ $\mathbb{R}_{+}$such that if we choose the step-sizes as in (3.1) for any $\gamma>0$ and $\beta \geq \bar{\beta}$, then

$$
\begin{equation*}
\max _{i \in \mathcal{N}}\left\{\left\|y_{i}^{k}\right\|\right\} \leq \beta, \quad\left\|\mathbf{w}^{k}\right\| \leq \varsigma, \quad\left\|\mathbf{e}^{k}\right\| \leq \nu \alpha^{q_{k-1}} k \tag{3.14}
\end{equation*}
$$

for all $k \geq 0$, where $q_{-1} \triangleq 0$ and $q_{0} \triangleq 0$. Below we provide the analysis for two sperate cases. We first define two quantities that are repeatedly used in the proof. Define $C_{0} \triangleq \sum_{k=1}^{\infty} \alpha^{q_{k-1}} k<+\infty-$ note $C_{0}>1$. Let $A_{0} \triangleq D_{x}\left(\mathbf{x}^{*}, \mathbf{x}^{0}\right)+D_{y}\left(\mathbf{y}^{*}, \mathbf{y}^{0}\right)-$ $\left\langle T\left(\mathbf{x}^{*}\right)-T\left(\mathbf{x}^{0}\right), \mathbf{y}^{*}-\mathbf{y}^{0}\right\rangle$. Since we initialize $\mathbf{w}^{0}=\mathbf{y}^{0}=\mathbf{0}$, the proof of Lemma 3.4 implies that $A_{0} \leq\left\|\boldsymbol{\xi}^{*}-\boldsymbol{\xi}^{0}\right\|_{\mathbf{D}_{\tau}}^{2}+\left\|\mathbf{y}^{*}\right\|_{\mathbf{D}_{\kappa}}^{2}$. Recall the definitions of $\bar{C}_{g}$ and $\bar{R}_{x}$ given in Section 3. Using (3.1), we get

$$
\begin{equation*}
A_{0} \leq\left\|\boldsymbol{\xi}^{*}-\boldsymbol{\xi}^{0}\right\|_{\mathbf{D}_{\tau}}^{2}+\left\|\mathbf{y}^{*}\right\|_{\mathbf{D}_{\kappa}}^{2} \leq(\beta+1) N \bar{R}_{x}^{2}+\left(\bar{C}_{g}+\frac{5}{2} \gamma\right) N\left\|y^{*}\right\|^{2} \triangleq \bar{A}_{0} \tag{3.15}
\end{equation*}
$$

In the rest we assume $\bar{C}_{g} \geq 1$.
CASE 1: Bound B on $\left\|y^{*}\right\|$ is available, i.e., $B \in(0, \infty)$. In this part, we assume that a nontrivial dual bound $B \in(0, \infty)$ is available. Suppose we set $\bar{\beta}=2 B$ and we choose the step-sizes as in (3.1) for some $\gamma>0$ and $\beta \geq \bar{\beta}$. Trivially, from (2.7d), we have $\max _{i \in \mathcal{N}}\left\|y_{i}^{k}\right\| \leq 2 B \leq \beta$ for $k \geq 0$. Hence, Lemma 3.3 shows that for all $k \geq 0$, (3.6) holds for $\beta_{k}=\beta$. Moreover, stepsize condition in (3.1) and Lemma 3.4 imply that $\overline{\mathbf{Q}}(\beta) \succ \mathbf{0}$. Therefore, for any $\ell \geq 0$, dropping the last term in (3.6), summing over $k \in\{0, \ldots, \ell\}$, and using Jensen's inequality, we get for all $\mathbf{z} \in \mathcal{Z}$,

$$
\begin{align*}
& (\ell+1)\left(\mathcal{L}\left(\overline{\mathbf{x}}^{\ell+1}, \mathbf{y}\right)-\mathcal{L}\left(\mathbf{x}, \overline{\mathbf{y}}^{\ell+1}\right)\right) \leq  \tag{3.16}\\
& {\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{0}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{0}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{0}\right), \mathbf{y}-\mathbf{y}^{0}\right\rangle\right]} \\
& -\left[D_{x}\left(\mathbf{x}, \mathbf{x}^{\ell+1}\right)+D_{y}\left(\mathbf{y}, \mathbf{y}^{\ell+1}\right)-\left\langle T(\mathbf{x})-T\left(\mathbf{x}^{\ell+1}\right), \mathbf{y}-\mathbf{y}^{\ell+1}\right\rangle\right]+\sum_{k=0}^{\ell} E^{k+1}(\mathbf{z})
\end{align*}
$$

where $\overline{\mathbf{x}}^{\ell+1} \triangleq \frac{1}{(\ell+1)} \sum_{k=1}^{\ell+1} \mathbf{x}^{k}$ and $\overline{\mathbf{y}}^{\ell+1} \triangleq \frac{1}{(\ell+1)} \sum_{k=1}^{\ell+1} \mathbf{y}^{k}$. For any $\ell \geq 0$, setting $\mathbf{z}=\mathbf{z}^{*}$ in (3.16), using $\mathcal{L}\left(\overline{\mathbf{x}}^{\ell+1}, \mathbf{y}^{*}\right)-\mathcal{L}\left(\mathbf{x}^{*}, \overline{\mathbf{y}}^{\ell+1}\right) \geq 0$, and (3.12) we obtain,

$$
\begin{equation*}
\frac{1}{4 \gamma}\left\|\mathbf{w}^{\ell+1}\right\|^{2}+\frac{\gamma}{4}\left\|\mathbf{y}^{*}-\mathbf{y}^{\ell+1}\right\|^{2} \leq A_{0}+\sum_{k=0}^{\ell} E^{k+1}\left(\mathbf{z}^{*}\right) \tag{3.17}
\end{equation*}
$$

Hence, using (3.13), (3.17) and the fact that $\mathbf{w}^{*}=0$, for all $\ell \geq 0$, we have

$$
\begin{align*}
\frac{\gamma}{8}\left(\frac{1}{\gamma}\left\|\mathbf{w}^{\ell+1}\right\|+\left\|\mathbf{y}^{*}-\mathbf{y}^{\ell+1}\right\|\right)^{2} & \leq \frac{1}{4 \gamma}\left\|\mathbf{w}^{\ell+1}\right\|^{2}+\frac{\gamma}{4}\left\|\mathbf{y}^{*}-\mathbf{y}^{\ell+1}\right\|^{2} \\
& \leq A_{0}+\sum_{k=1}^{\ell+1} \gamma\left(2\left\|\mathbf{e}^{k}\right\|+\left\|\mathbf{e}^{k-1}\right\|\right)\left(\frac{1}{\gamma}\left\|\mathbf{w}^{k}\right\|+\left\|\mathbf{y}^{*}-\mathbf{y}^{k}\right\|\right) \tag{3.18}
\end{align*}
$$

Next, we use Lemma 9.2 with $u_{k}=\frac{1}{\gamma}\left\|\mathbf{w}^{k}\right\|+\left\|\mathbf{y}^{*}-\mathbf{y}^{k}\right\|, S_{k}=\frac{8}{\gamma} A_{0}$ for $k \geq 0$, and $\lambda_{k}=8\left(2\left\|\mathbf{e}^{k}\right\|+\left\|\mathbf{e}^{k-1}\right\|\right)$ for $k \geq 1$. Note (3.12) and $\mathbf{w}^{0}=\mathbf{y}^{0}=\mathbf{0}$ imply that $A_{0} \geq \frac{\gamma}{4}\left\|\mathbf{y}^{*}\right\|^{2}$; hence, we have $u_{0}^{2} \leq S_{0}$. Thus, Lemma 9.2 implies that for all $\ell \geq 0$,

$$
\begin{equation*}
\frac{1}{\gamma}\left\|\mathbf{w}^{\ell+1}\right\|+\left\|\mathbf{y}^{*}-\mathbf{y}^{\ell+1}\right\| \leq \frac{1}{2} \sum_{k=1}^{\ell+1} \lambda_{k}+\sqrt{\frac{8 A_{0}}{\gamma}+\left(\frac{1}{2} \sum_{k=1}^{\ell+1} \lambda_{k}\right)^{2}} \leq 24 \sum_{k=1}^{\ell+1}\left\|\mathbf{e}^{k}\right\|+\sqrt{\frac{8 A_{0}}{\gamma}} \tag{3.19}
\end{equation*}
$$

For each $i \in \mathcal{N}$ and $k \geq 0$, the definition of $\mathcal{R}^{k}$ in (2.10) implies $\mathcal{R}_{i}^{k}(\mathbf{y}) \in \mathcal{B}_{0}$ for all $\mathbf{y}$; hence, from (2.11), $\left\|v_{i}^{k+1}\right\| \leq\left\|v_{i}^{k}+\gamma y_{i}^{k}\right\|+\gamma\left\|\mathcal{R}_{i}^{k}\left(\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)\right\| \leq\left\|v_{i}^{k}\right\|+4 \gamma B$; thus, $\max _{i \in \mathcal{N}}\left\|v_{i}^{k}\right\| \leq 4 \gamma B k$ for $k \geq 0$, and we trivially get the following bound:

$$
\begin{equation*}
\left\|\mathbf{v}^{k}\right\| \leq 4 \gamma \sqrt{N} B k, \quad \forall k \geq 0 \tag{3.20}
\end{equation*}
$$

Hence, for $k \geq 0$, since $\left\|\mathbf{y}^{k}\right\| \leq 2 \sqrt{N} B$, it follows from (2.10), (3.5) and (3.20) that

$$
\begin{equation*}
\left\|\mathbf{e}^{k+1}\right\| \leq N \Gamma \alpha^{q_{k}}\left\|\frac{1}{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right\| \leq 2 N^{\frac{3}{2}} B \Gamma \alpha^{q_{k}}(2 k+1) \Longrightarrow \quad \nu=4 N^{\frac{3}{2}} B \Gamma \tag{3.21}
\end{equation*}
$$

Therefore, $\left\|\mathbf{e}^{k}\right\|$ satisfies (3.14) for $\nu=4 N^{\frac{3}{2}} B \Gamma$. Using this result within (3.19), we obtain

$$
\begin{equation*}
\left\|\mathbf{w}^{\ell+1}\right\| \leq 24 \gamma \nu \sum_{k=1}^{\ell+1} \alpha^{q_{k-1}} k+\sqrt{8 A_{0} \gamma} \leq \varsigma \triangleq 24 \gamma \nu C_{0}+\sqrt{8 A_{0} \gamma}, \quad \forall \ell \geq 0 \tag{3.22}
\end{equation*}
$$

CASE 2: Bound $B$ on $\left\|y^{*}\right\|$ is not available, i.e., $B=\infty$. We set $B=+\infty$ in Algorithm 2.1. We prove the claim in (3.14) using induction; indeed, we construct $\bar{\beta}, \varsigma, \nu \in \mathbb{R}_{+}$and show for any $\gamma>0, \beta \geq \bar{\beta}$ and $K \geq 1$ that if (3.14) holds for all $k \in \mathcal{I} \triangleq\{0, \ldots, K-1\}$, then $\left\|\mathbf{e}^{K}\right\| \leq \nu \alpha^{q_{K-1}} K$ also holds and this implies $\left\|\mathbf{w}^{K}\right\| \leq \varsigma$ and $\max _{i \in \mathcal{N}}\left\{\left\|y_{i}^{K}\right\|\right\} \leq \beta$, which would complete the induction.

Since $\mathbf{v}^{0}=\mathbf{w}^{0}=\mathbf{0}$ and $\mathbf{y}^{0}=\mathbf{0},(3.14)$ trivially holds for $k=0$. Suppose for some $\beta>0$, (3.14) holds for $k \in \mathcal{I}$; hence, $\max _{i \in \mathcal{N}}\left\{\left\|y_{i}^{k}\right\|\right\} \leq \beta$ for $k \in \mathcal{I}$, and using the same arguments as in CASE 1, it can be shown that (3.16), (3.17), (3.18) and (3.19) hold for all $\ell \in \mathcal{I}$. Next, using (3.5), (3.19) implies that

$$
\begin{align*}
\left\|\mathbf{e}^{K}\right\| & =\left\|\mathcal{P}_{\tilde{\mathcal{C}}}\left(\frac{1}{\gamma} \mathbf{w}^{K-1}+\mathbf{e}^{K-1}+\mathbf{y}^{K-1}\right)-\mathcal{R}^{K-1}\left(\frac{1}{\gamma} \mathbf{w}^{K-1}+\mathbf{e}^{K-1}+\mathbf{y}^{K-1}\right)\right\| \\
& \leq N \Gamma \alpha^{q_{K-1}}\left(\left\|\mathbf{e}^{K-1}\right\|+24 \sum_{k=1}^{K-1}\left\|\mathbf{e}^{k}\right\|+\sqrt{\frac{8 A_{0}}{\gamma}}+\left\|\mathbf{y}^{*}\right\|\right) \\
& \leq N \Gamma \nu \alpha^{q_{K-1}}\left(\alpha^{q_{K-2}}(K-1)+24 \sum_{k=1}^{K-1} \alpha^{q_{k-1}} k+\left(\sqrt{\frac{8 A_{0}}{\gamma}}+\left\|\mathbf{y}^{*}\right\|\right) / \nu\right) \tag{3.23}
\end{align*}
$$

The assumption, $q_{k} \geq \log _{1 / \alpha}(24 N \Gamma(k+1))$ for $k \geq 0$, and $q_{-1}=0$ imply that $\alpha^{q_{k-1}} k \leq \frac{1}{24 N \Gamma}$ for $k \geq 0$. Thus, for $\nu \triangleq \frac{24}{23} N \Gamma\left(\left\|\mathbf{y}^{*}\right\|+\sqrt{\frac{8 A_{0}}{\gamma}}\right)$, (3.23) is indeed
bounded above by $\nu \alpha^{q_{K-1}} K$ which proves the induction on $\left\|\mathbf{e}^{K}\right\|$. Hence, using this result within (3.19) for $\ell=K-1$, we obtain

$$
\begin{equation*}
\frac{1}{\gamma}\left\|\mathbf{w}^{K}\right\|+\left\|\mathbf{y}^{K}\right\| \leq 24 \nu \sum_{k=1}^{K} \alpha^{q_{k-1}} k+\sqrt{\frac{8 A_{0}}{\gamma}}+\left\|\mathbf{y}^{*}\right\| \leq 24 \nu C_{0}+\sqrt{\frac{8 A_{0}}{\gamma}}+\left\|\mathbf{y}^{*}\right\| \tag{3.24}
\end{equation*}
$$

where $C_{0} \triangleq \sum_{k=1}^{\infty} \alpha^{q_{k-1}} k<+\infty$ and is independent of $\beta$. Thus, $\left\|\mathbf{w}^{K}\right\| \leq \gamma \beta$ and $\max _{i \in \mathcal{N}}\left\{\left\|y_{i}^{K}\right\|\right\} \leq \beta$ for all $\beta \geq\left(\frac{576}{23} N \Gamma C_{0}+1\right)\left(\sqrt{\frac{8 A_{0}}{\gamma}}+\sqrt{N}\left\|y^{*}\right\|\right)$. Hence, using the bound on $A_{0}$ in (3.15), we derive a sufficient condition on $\beta$ :

$$
\begin{equation*}
\beta \geq\left(\frac{576}{23} N \Gamma C_{0}+1\right) \sqrt{N}\left(\left\|y^{*}\right\|+\sqrt{\frac{8}{\gamma}\left((\beta+1) \bar{R}_{x}^{2}+\left(\bar{C}_{g}+\frac{5}{2} \gamma\right)\left\|y^{*}\right\|^{2}\right)}\right) . \tag{3.25}
\end{equation*}
$$

Note (3.25) implies that there exists $\bar{\beta} \in \mathbb{R}$ such that $\bar{\beta} \geq\left\|\mathbf{y}^{*}\right\|$ and for all $\beta \geq \bar{\beta}$ and $\varsigma=\gamma \beta$, (3.14) holds when the stepsizes are chosen as in (3.1) using $\beta$. Thus, when primal step-sizes $\left[\tau_{i}\right]_{i \in \mathcal{N}}$ chosen sufficiently small and $\left\{q_{k}\right\}$ chosen such that $q_{k} \geq \log _{1 / \alpha}(24 N \Gamma(k+1))$ and $\sum_{k=1}^{\infty} \alpha^{q_{k-1}} k<\infty$, both $\left\{\mathbf{y}^{k}\right\}$ and $\left\{\mathbf{w}^{k}\right\}$ are bounded. Moreover, solving the quadratic inequality in (3.25), we get

$$
\begin{equation*}
\beta=\mathcal{O}\left(\frac{1}{\gamma} N^{3} \Gamma^{2} C_{0}^{2} \bar{R}_{x}^{2}+N^{3 / 2} \Gamma C_{0}\left(\left\|y^{*}\right\|+\sqrt{\frac{1}{\gamma}\left(\bar{R}_{x}^{2}+\bar{C}_{g}\left\|y^{*}\right\|^{2}\right)}\right)\right) \tag{3.26}
\end{equation*}
$$

If $g_{i}$ is an affine function $\left(L_{g_{i}}=0\right)$ for all $i \in \mathcal{N}$, then choosing $q_{k}$ as before and setting $\tau_{i}=\left(\max \left\{1, L_{f_{i}}\right\}+C_{g_{i}}\right)^{-1}$ for $i \in \mathcal{N}$ guarantees that $\left\{\mathbf{y}^{k}\right\}_{k}$ and $\left\{\mathbf{w}^{k}\right\}_{k}$ are bounded. Moreover, since $\mathbf{D}_{\tau}$ does not depend on $\beta$, the term $(\beta+1) \bar{R}_{x}^{2}$ on the rhs of (3.25) becomes $\bar{R}_{x}^{2}$; thus, $\beta=\mathcal{O}\left(N^{3 / 2} \Gamma C_{0}\left(\left\|y^{*}\right\|+\sqrt{\frac{1}{\gamma}\left(\bar{R}_{x}^{2}+\bar{C}_{g}\left\|y^{*}\right\|^{2}\right)}\right)\right)$.
3.2.2. Convergence of iterates. In previous section, we showed that there exist $\bar{\beta}, \varsigma, \nu \in \mathbb{R}_{+}$such that if we choose the step-sizes as in (3.1) for any $\gamma>0$ and $\beta \geq \bar{\beta}$, then (3.14) holds for all $k \geq 0$. Consider a saddle point $\mathbf{z}^{*}=\left[\mathbf{x}^{* \top} \mathbf{y}^{* \top}\right]^{\top}$ of $\mathcal{L}$ in (2.5), where $\mathbf{x}^{*}=\left[\boldsymbol{\xi}^{* \top} \mathbf{w}^{* \top}\right]^{\top}$. Trivially, (3.13) and (3.14) imply that

$$
\begin{equation*}
\sum_{k=0}^{\infty} E^{k+1}\left(\mathbf{z}^{*}\right) \leq 3 \gamma \max _{k \geq 0}\left\{\frac{1}{\gamma}\left\|\mathbf{w}^{k+1}-\mathbf{w}^{*}\right\|+\left\|\mathbf{y}^{k+1}-\mathbf{y}^{*}\right\|\right\} \sum_{k=0}^{\infty}\left\|\mathbf{e}^{k+1}\right\|<\infty \tag{3.27}
\end{equation*}
$$

Evaluating (3.6) at $\mathbf{z}=\mathbf{z}^{*}$, we get

$$
\begin{equation*}
0 \leq \mathcal{L}\left(\mathbf{x}^{k+1}, \mathbf{y}^{*}\right)-\mathcal{L}\left(\mathbf{x}^{*}, \mathbf{y}^{k+1}\right) \leq \mathbf{a}^{k}-\mathbf{a}^{k+1}-\mathbf{b}^{k}+\mathbf{c}^{k} \tag{3.28}
\end{equation*}
$$

for $k \geq 0$, where $\mathbf{a}^{k} \triangleq D_{x}\left(\mathbf{x}^{*}, \mathbf{x}^{k}\right)+D_{y}\left(\mathbf{y}^{*}, \mathbf{y}^{k}\right)-\left\langle T\left(\mathbf{x}^{*}\right)-T\left(\mathbf{x}^{k}\right), \mathbf{y}^{*}-\mathbf{y}^{k}\right\rangle, \mathbf{b}^{k} \triangleq$ $\frac{1}{2}\left\|\mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right)\right\|_{\overline{\mathbf{Q}}(\beta)}^{2}$ and $\mathbf{c}^{k} \triangleq E^{k+1}\left(\mathbf{z}^{*}\right)$ for $k \geq 0$. Clearly, $\mathbf{b}^{k} \geq 0$ and $\mathbf{c}^{k} \geq 0$ for $k \geq 0$. Moreover, from (3.12), we get $\mathbf{a}^{k} \geq 0$ for $k \geq 0$. Since $\sum_{k=0}^{\infty} E^{k+1}\left(\mathbf{z}^{*}\right)<\infty$, Lemma 9.1 implies that $\lim _{k \rightarrow \infty} \mathbf{a}^{k}$ exists. Thus, $\left\{\mathbf{a}^{k}\right\}$ is a bounded sequence; and due to (3.12), $\left\{\mathbf{z}^{k}\right\}$ is bounded as well. Consequently, there exists a subsequence $\left\{\mathbf{z}^{k_{n}}\right\}_{n}$ such that $\mathbf{z}^{k_{n}} \rightarrow \mathbf{z}^{\#}$ as $n \rightarrow \infty$. Thus, there exists $N_{1}$ such that for all $n \geq N_{1}$, we have $\left\|\mathbf{z}^{k_{n}}-\mathbf{z}^{\#}\right\|<\frac{\epsilon}{2}$. Moreover, Lemma 9.1 also implies $\sum_{k=0}^{\infty}\left\|\mathbf{u}\left(\mathbf{z}^{k+1}, \mathbf{z}^{k}\right)\right\|_{\overline{\mathbf{Q}}(\beta)}^{2}<\infty$. Since $\overline{\mathbf{Q}}(\beta) \succ \mathbf{0}$, for any $\epsilon>0$, there exists $N_{2}$ such that for all $n \geq N_{2}$, we have $\left\|\mathbf{z}^{k_{n}+1}-\mathbf{z}^{k_{n}}\right\|<\frac{\epsilon}{2}$. Therefore, by letting $N=\max \left\{N_{1}, N_{2}\right\}$ we get $\left\|\mathbf{z}^{k_{n}+1}-\mathbf{z}^{\#}\right\|<$ $\epsilon$, i.e., $\mathbf{z}^{k_{n}+1} \rightarrow \mathbf{z}^{\#}$ as $n \rightarrow \infty$.

Note that (3.14) implies $\left\|\mathbf{e}^{k}\right\| \rightarrow 0$ as $k \rightarrow \infty$ for any $\left\{q_{k}\right\}$ such that $\sum_{k=1}^{\infty} \alpha^{q_{k}} k<$ $+\infty$. Recall that $\psi_{x}(\mathbf{x})=\frac{1}{2}\|\boldsymbol{\xi}\|_{\mathbf{D}_{\tau}}^{2}+\frac{1}{2}\|\mathbf{w}\|_{\mathbf{D}_{\gamma}}^{2}$, and $\psi_{y}(\mathbf{y})=\frac{1}{2}\|\mathbf{y}\|_{\mathbf{D}_{\kappa}}^{2}$ are the strongly
convex functions corresponding to Bregman distance functions $D_{x}$ and $D_{y}$, respectively. In particular, $D_{x}(\mathbf{x}, \overline{\mathbf{x}})=\psi_{x}(\mathbf{x})-\psi_{x}(\overline{\mathbf{x}})-\left\langle\nabla \psi_{x}(\overline{\mathbf{x}}), \mathbf{x}-\overline{\mathbf{x}}\right\rangle$, and $D_{y}$ is defined similarly. The optimality conditions for (3.7) imply that for all $n \in \mathbb{Z}_{+}, \mathbf{q}^{n} \in$ $\partial \rho\left(\mathbf{x}^{k_{n}+1}\right)$ and $\mathbf{p}^{n} \in \partial h\left(\mathbf{y}^{k_{n}+1}\right)$, where $\mathbf{q}^{n} \triangleq \nabla \psi_{x}\left(\mathbf{x}^{k_{n}}\right)-\nabla \psi_{x}\left(\mathbf{x}^{k_{n}+1}\right)-\left(\nabla f\left(\mathbf{x}^{k_{n}}\right)+\right.$ $\left.\mathbf{J} T\left(\mathbf{x}^{k_{n}}\right)^{\top} \mathbf{y}^{k_{n}}+U \mathbf{e}^{k_{n}}\right)$, and $\mathbf{p}^{n} \triangleq \nabla \psi_{y}\left(\mathbf{y}^{k_{n}}\right)-\nabla \psi_{y}\left(\mathbf{y}^{k_{n}+1}\right)+2 T\left(\mathbf{x}^{k_{n}+1}\right)-T\left(\mathbf{x}^{k_{n}}\right)+$ $\gamma\left(2 \mathbf{e}^{k_{n}+1}-\mathbf{e}^{k_{n}}\right)$. Since $\nabla \psi_{x}$ and $\nabla \psi_{y}$ are continuously differentiable on $\operatorname{dom} \rho$ and dom $h$, respectively, and since $\rho$ and $h$ are proper, closed convex functions, it follows from Theorem 24.4 in [39] that $\partial \rho\left(\mathbf{x}^{\#}\right) \ni \lim _{n} \mathbf{q}^{n}=-\nabla f\left(\mathbf{x}^{\#}\right)-\mathbf{J} T\left(\mathbf{x}^{\#}\right)^{\top} \mathbf{y}^{\#}$, and $\partial h\left(\mathbf{y}^{\#}\right) \ni \lim _{n} \mathbf{p}^{n}=T\left(\mathbf{x}^{\#}\right)$, which also implies that $\mathbf{z}^{\#}$ is a saddle point of (1.4).

Since (3.28) is true for any saddle point $\mathbf{z}^{*}$, by setting $\mathbf{z}^{*}=\mathbf{z}^{\#}$ in (3.28), one can conclude that $\mathbf{s}^{\#} \triangleq \lim _{k} \mathbf{s}^{k} \geq 0$ exists, where $\mathbf{s}^{k} \triangleq D_{x}\left(\mathbf{x}^{\#}, \mathbf{x}^{k}\right)+D_{y}\left(\mathbf{y}^{\#}, \mathbf{y}^{k}\right)-$ $\left\langle T\left(\mathbf{x}^{\#}\right)-T\left(\mathbf{x}^{k}\right), \mathbf{y}^{\#}-\mathbf{y}^{k}\right\rangle$ for $k \geq 0$. Since $\lim _{n}\left\langle T\left(\mathbf{x}^{\#}\right)-T\left(\mathbf{x}^{k_{n}}\right), \mathbf{y}^{\#}-\mathbf{y}^{k_{n}}\right\rangle=0$ (from $\mathbf{z}^{k_{n}} \rightarrow \mathbf{z}^{\#}$ ), clearly $\mathbf{s}^{\#}=\lim _{n \rightarrow \infty} \mathbf{s}^{k_{n}}=0$, which together with (3.12) implies that $\mathbf{z}^{k} \rightarrow \mathbf{z}^{\#}$.
3.2.3. Convergence rate. Recall that we initialize $\mathbf{v}^{0}=\mathbf{w}^{0}=\mathbf{0}$ and $\mathbf{y}^{0}=\mathbf{0}$; hence, the inequality in (3.16) can be written more explicitly as follows: let $\overline{\boldsymbol{\xi}}^{K} \triangleq$ $\frac{1}{K} \sum_{k=1}^{K} \boldsymbol{\xi}^{k}$, and $\overline{\mathbf{w}}^{K} \triangleq \frac{1}{K} \sum_{k=1}^{K} \mathbf{w}^{k}$, then for any $\boldsymbol{\xi}, \mathbf{w}$ and $\mathbf{y}$, and for all $K \geq 1$,

$$
\begin{equation*}
\mathcal{L}\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{w}}^{K}, \mathbf{y}\right)-\mathcal{L}\left(\boldsymbol{\xi}, \mathbf{w}, \overline{\mathbf{y}}^{K}\right) \leq \Theta(\mathbf{z}) / K \tag{3.29}
\end{equation*}
$$

where $\Theta(\mathbf{z}) \triangleq \frac{1}{2 \gamma}\|\mathbf{w}\|^{2}+\langle\mathbf{y}, \mathbf{w}\rangle+\sum_{i \in \mathcal{N}}\left[\frac{1}{2 \tau_{i}}\left\|\xi_{i}-\xi_{i}^{0}\right\|^{2}+\frac{1}{2 \kappa_{i}}\left\|y_{i}\right\|^{2}+\left\langle g_{i}\left(\xi_{i}\right)-g_{i}\left(\xi_{i}^{0}\right), y_{i}\right\rangle\right]$ $+\sum_{k=0}^{K-1} E^{k+1}(\mathbf{z})$. Given the step-size condition in (3.1), Schur complement condition guarantees that $\left[\begin{array}{cc}\frac{1}{\tau_{i}} & C_{g_{i}} \\ C_{g_{i}} & \frac{1}{\kappa_{i}}\end{array}\right] \preceq\left[\begin{array}{cc}\frac{2}{\tau_{i}} & 0 \\ 0 & \frac{2}{\kappa_{i}}\end{array}\right]$ for any $i \in \mathcal{N}$; therefore,

$$
\begin{equation*}
\Theta(\mathbf{z}) \leq \sum_{i \in \mathcal{N}}\left[\frac{1}{\tau_{i}}\left\|\xi_{i}-\xi_{i}^{0}\right\|^{2}+\frac{1}{\kappa_{i}}\left\|y_{i}\right\|^{2}\right]+\frac{1}{2 \gamma}\|\mathbf{w}\|^{2}+\langle\mathbf{y}, \mathbf{w}\rangle+\sum_{k=0}^{K-1} E^{k+1}(\mathbf{z}) \tag{3.30}
\end{equation*}
$$

In the rest, fix $K \geq 1$ and a saddle-point $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ of $\mathcal{L}$ in (2.5) such that $\mathbf{w}^{*}=\mathbf{0}$. Let $\hat{y}^{K} \triangleq 2\left\|y^{*}\right\| \mathcal{P}_{\mathcal{K}^{\circ}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) /\left\|\mathcal{P}_{\mathcal{K}^{\circ}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right)\right\| \in \mathcal{K}^{\circ}$, and define $\hat{\mathbf{y}}^{K}=\left[\hat{y}_{i}^{K}\right]_{i \in \mathcal{N}}$ such that $\hat{y}_{i}^{K}=\hat{y}^{K}$ for all $i \in \mathcal{N}$, i.e., $\hat{\mathbf{y}}^{K}=\mathbf{1} \otimes \hat{y}^{K} \in \widetilde{\mathcal{C}}$, and also define $\hat{\mathbf{w}}^{K} \triangleq$ $\left\|\mathcal{P}_{\mathcal{C}^{\circ}}\left(\overline{\mathbf{y}}^{K}\right)\right\|^{-1} \mathcal{P}_{\mathcal{C}^{\circ}}\left(\overline{\mathbf{y}}^{K}\right)$, where $\mathcal{C} \supset \widetilde{\mathcal{C}}$ defined in (2.2) is a closed convex cone and $\mathcal{C}^{\circ}$ denotes its polar cone. Note that $\hat{\mathbf{y}}^{K} \in \mathcal{C}$ and $\hat{\mathbf{w}}^{K} \in \mathcal{C}^{\circ}$ imply $\left\langle\hat{\mathbf{y}}^{K}, \hat{\mathbf{w}}^{K}\right\rangle \leq 0$. Recall that every closed convex cone $\mathcal{Q} \subset \mathbb{R}^{m}$ induces an orthogonal decomposition on $\mathbb{R}^{m}$, i.e., according to Moreau decomposition, for any $y \in \mathbb{R}^{m}$, there exist $y^{1} \in \mathcal{Q}$, and $y^{2} \in \mathcal{Q}^{\circ}$ such that $y=y^{1}+y^{2}$ and $y^{1} \perp y^{2}$; in particular, $y^{1}=\mathcal{P}_{\mathcal{Q}}(y)$ and $y^{2}=\mathcal{P}_{\mathcal{Q}} \circ(y)$. Thus, $\left\langle\hat{\mathbf{w}}^{K}, \overline{\mathbf{y}}^{K}\right\rangle=\left\langle\hat{\mathbf{w}}^{K}, \mathcal{P}_{\mathcal{C}}\left(\overline{\mathbf{y}}^{K}\right)+\mathcal{P}_{\mathcal{C}}{ }^{\circ}\left(\overline{\mathbf{y}}^{K}\right)\right\rangle=\left\|\mathcal{P}_{\mathcal{C}^{\circ}}\left(\overline{\mathbf{y}}^{K}\right)\right\|=d_{\mathcal{C}}\left(\overline{\mathbf{y}}^{K}\right)$. Note that for each $i \in \mathcal{N}$ we have $\bar{y}_{i}^{K} \in \mathcal{K}^{\circ}$ since $y_{i}^{k} \in \mathcal{K}^{\circ}$ for all $k=1, \ldots, K$ and $\mathcal{K}$ is convex; hence, $\sigma_{\mathcal{K}}\left({\underset{\sim}{\bar{y}}}_{i}^{K}\right)=0$ for $i \in \mathcal{N}$. Moreover, $\hat{\mathbf{w}}^{K} \in \mathcal{C}^{\circ}$ implies $\sigma_{\mathcal{C}}\left(\hat{\mathbf{w}}^{K}\right)=\mathbb{1}_{\mathcal{C}^{\circ}}\left(\hat{\mathbf{w}}^{K}\right)=0$; and since $\widetilde{\mathcal{C}} \subset \mathcal{C}$, we also have $\sigma_{\widetilde{\mathcal{C}}}\left(\hat{\mathbf{w}}^{K}\right) \leq \sigma_{\mathcal{C}}\left(\hat{\mathbf{w}}^{K}\right)=0$. Therefore, we can conclude that $\sigma_{\widetilde{\mathcal{C}}}\left(\hat{\mathbf{w}}^{K}\right)=0$ since $\mathbf{0} \in \widetilde{\mathcal{C}}$. These observations imply that

$$
\begin{equation*}
\mathcal{L}\left(\boldsymbol{\xi}^{*}, \hat{\mathbf{w}}^{K}, \overline{\mathbf{y}}^{K}\right)=\varphi\left(\boldsymbol{\xi}^{*}\right)-\sum_{i \in \mathcal{N}}\left\langle g_{i}\left(\xi_{i}^{*}\right), \bar{y}_{i}^{K}\right\rangle-d_{\mathcal{C}}\left(\overline{\mathbf{y}}^{K}\right) . \tag{3.31}
\end{equation*}
$$

Similarly, from the definition of $\hat{y}^{K} \in \mathcal{K}^{\circ},-\sum_{i \in \mathcal{N}}\left\langle g_{i}\left(\bar{\xi}_{i}^{K}\right), \hat{y}^{K}\right\rangle=2\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right)$, and since $\hat{\mathbf{y}}^{K} \in \widetilde{\mathcal{C}}$, we also have $\left\langle\overline{\mathbf{w}}^{K}, \hat{\mathbf{y}}^{K}\right\rangle-\sigma_{\widetilde{\mathcal{C}}}\left(\overline{\mathbf{w}}^{K}\right) \leq \sup _{\mathbf{w}}\left\langle\mathbf{w}, \hat{\mathbf{y}}^{K}\right\rangle-\sigma_{\widetilde{\mathcal{C}}}(\mathbf{w})=$ $\mathbb{1}_{\widetilde{\mathcal{C}}}\left(\hat{\mathbf{y}}^{K}\right)=0$. Note $\sigma_{\mathcal{K}}\left(\hat{y}^{K}\right)=0$ since $\hat{y}^{K} \in \mathcal{K}^{\circ}$. Thus, we conclude that $\mathcal{L}$ satisfies

$$
\begin{equation*}
\mathcal{L}\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{w}}^{K}, \hat{\mathbf{y}}^{K}\right) \geq \varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)+2\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) . \tag{3.32}
\end{equation*}
$$

Combining (3.31) and (3.32), we get

$$
\begin{align*}
& \mathcal{L}\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{w}}^{K}, \hat{\mathbf{y}}^{K}\right)-\mathcal{L}\left(\boldsymbol{\xi}^{*}, \hat{\mathbf{w}}^{K}, \overline{\mathbf{y}}^{K}\right)  \tag{3.33}\\
& \quad \geq \varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)+2\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right)+d_{\mathcal{C}}\left(\overline{\mathbf{y}}^{K}\right)+\sum_{i \in \mathcal{N}}\left\langle g_{i}\left(\xi_{i}^{*}\right), \bar{y}_{i}^{K}\right\rangle .
\end{align*}
$$

Moreover, $\left\langle\hat{\mathbf{y}}^{K}, \hat{\mathbf{w}}^{K}\right\rangle \leq 0,(3.29)$ and (3.30) imply that

$$
\begin{equation*}
\mathcal{L}\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{w}}^{K}, \hat{\mathbf{y}}^{K}\right)-\mathcal{L}\left(\boldsymbol{\xi}^{*}, \hat{\mathbf{w}}^{K}, \overline{\mathbf{y}}^{K}\right) \leq \Theta\left(\hat{\mathbf{z}}^{K}\right) / K \leq \frac{\Lambda_{1}+\sum_{k=0}^{K-1} E^{k+1}\left(\hat{\mathbf{z}}^{K}\right)}{K} \triangleq \frac{\Lambda(\gamma, \beta)}{K}, \tag{3.34}
\end{equation*}
$$

where $\hat{\mathbf{z}}^{K}=\left[\boldsymbol{\xi}^{* \top}\left(\hat{\mathbf{w}}^{K}\right)^{\top}\left(\hat{\mathbf{y}}^{K}\right)^{\top}\right]^{\top}$ and $\Lambda_{1} \triangleq \frac{1}{2 \gamma}+\sum_{i \in \mathcal{N}}\left[\frac{1}{\tau_{i}}\left\|\xi_{i}^{*}-\xi_{i}^{0}\right\|^{2}+\frac{4}{\kappa_{i}}\left\|y^{*}\right\|^{2}\right]$. Recall that we fixed a saddle point $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ such that $\mathbf{w}^{*}=\mathbf{0}$; hence, we have $\mathcal{L}\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)=\varphi\left(\boldsymbol{\xi}^{*}\right)$ and $\sigma_{\widetilde{\mathcal{C}}}\left(\mathbf{w}^{*}\right)=0$. Moreover, since $\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)$ is a saddle-point, we have $\mathcal{L}\left(\overline{\boldsymbol{\xi}}^{K}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)-\mathcal{L}\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right) \geq 0$ and $\mathcal{L}\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \mathbf{y}^{*}\right)-\mathcal{L}\left(\boldsymbol{\xi}^{*}, \mathbf{w}^{*}, \overline{\mathbf{y}}^{K}\right) \geq 0 ;$ therefore, these facts imply that

$$
\begin{equation*}
\sum_{i \in \mathcal{N}}\left\langle g_{i}\left(\xi_{i}^{*}\right), \bar{y}_{i}^{K}\right\rangle \geq 0, \quad \varphi\left(\bar{\xi}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)+\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\bar{\xi}^{K}\right)\right) \geq 0 \tag{3.35}
\end{equation*}
$$

where we used $y^{*} \in \mathcal{K}^{\circ}$, i.e., $\left\langle y^{*}, y\right\rangle \leq\left\langle y^{*}, \mathcal{P}_{\mathcal{K}^{\circ}}(y)\right\rangle \leq\left\|y^{*}\right\| d_{\mathcal{K}}(y)$ for all $y \in \mathbb{R}^{m}$. Therefore, combining (3.33), (3.34), and (3.35) gives us the infeasibility and consensus results in (3.2) and also the upper bound in (3.3); while the inequality on the left in (3.35) gives us the lower bound for the suboptimality.

To show that $\Lambda(\gamma, \beta)$ is finite and independent of $K$, we bound $\sum_{k=0}^{K-1} E^{k+1}\left(\hat{\mathbf{z}}^{K}\right)$. As in (3.27), using (3.13), (3.14) and (3.19), we get

$$
\begin{align*}
\sum_{k=0}^{K-1} E^{k+1}\left(\hat{\mathbf{z}}^{K}\right) & \leq 3 \gamma \max _{k=0, \ldots, K-1}\left\{\frac{1}{\gamma}\left\|\mathbf{w}^{k+1}-\hat{\mathbf{w}}^{K}\right\|+\left\|\mathbf{y}^{k+1}-\hat{\mathbf{y}}^{K}\right\|\right\} \sum_{k=0}^{K-1}\left\|\mathrm{e}^{k+1}\right\| \\
& \triangleq \Lambda_{2} \leq 3 \nu C_{0}\left(1+\sqrt{8 A_{0} \gamma}+\gamma\left(24 \nu C_{0}+3 \sqrt{N}\left\|y^{*}\right\|\right)\right) \tag{3.36}
\end{align*}
$$

- recall $\sum_{k=0}^{\infty}\left\|\mathbf{e}^{k+1}\right\|=\nu C_{0}$. Below we specify the bound in (3.36) for both cases.

For CASE 1 where $B$ is known, $\nu=4 N^{\frac{3}{2}} \Gamma B$ (see (3.21)) and $\left\|y^{*}\right\| \leq B$; hence, using these facts and the bound on $A_{0}$ given in (3.15) within (3.36), we get

$$
\Lambda_{2} \leq N^{\frac{3}{2}} \Gamma C_{0} B \mathcal{O}\left(1+\sqrt{\gamma N\left(B \bar{R}_{x}^{2}+B^{2} \bar{C}_{g}\right)}+\gamma N^{\frac{3}{2}} \Gamma C_{0} B\right)
$$

Moreover, the second inequality in (3.15) implies $\Lambda_{1}=\mathcal{O}\left(\frac{1}{\gamma}+N\left(B \bar{R}_{x}^{2}+B^{2} \bar{C}_{g}\right)+\right.$ $\gamma N B^{2}$ ). Our aim is to optimize the $\mathcal{O}(1)$ constant of $\Lambda_{1}+\Lambda_{2}$ via carefully selecting the free parameter $\gamma$. Setting $\gamma=\left(N^{3 / 2} \Gamma C_{0} B\right)^{-1}$ gives $\Lambda(\gamma, \beta)=\mathcal{O}\left(N B\left(\bar{R}_{x}^{2}+\right.\right.$ $\left.\left.\bar{C}_{g} B\right)+N^{3 / 2} \Gamma C_{0} B\left(1+\sqrt{\frac{\bar{R}_{x}^{2}+\bar{C}_{g} B}{N^{1 / 2} \Gamma C_{0}}}\right)\right)$ which implies the $N$ dependency in (3.4).

For CASE 2 where $B$ is not known, $\nu=\frac{24}{23} N \Gamma\left(\left\|\mathbf{y}^{*}\right\|+\sqrt{\frac{8 A_{0}}{\gamma}}\right)$ - see the discussion below (3.23); hence, from (3.36), we get $\Lambda_{2} \leq \mathcal{O}\left(N^{2} \Gamma^{2} C_{0}^{2}\left(A_{0}+\gamma N \max \left\{1,\left\|y^{*}\right\|^{2}\right\}\right)\right)$. For the sake of simplicity, suppose $\left\|y^{*}\right\| \geq 1$. Moreover, $\Lambda_{1}=\mathcal{O}\left(\frac{1}{\gamma}+\bar{A}_{0}\right)$, and since $A_{0} \leq \bar{A}_{0}-\operatorname{see}(3.15), \Lambda_{1}+\Lambda_{2}=\mathcal{O}\left(\frac{1}{\gamma}+N^{2} \Gamma^{2} C_{0}^{2} \bar{A}_{0}\right)$. Selecting $\gamma=N^{\frac{3}{2}} \Gamma C_{0} \bar{R}_{x}^{2}$, (3.15) and the bound on $\beta$ in (3.26) together imply that $\Lambda_{1}+\Lambda_{2}=\mathcal{O}\left(N^{\frac{9}{2}} \Gamma^{3} C_{0}^{3} \bar{R}_{x}^{2}\left\|y^{*}\right\|^{2}\right)$, assuming $N^{\frac{3}{2}} \Gamma>\bar{C}_{g} / \bar{R}_{x}^{2}$ and $N>1 / \bar{R}_{x}^{2}$ which are reasonable since we are interested in the bounds when $N$ is large. Moreover, when $g_{i}$ 's are linear functions $\left(L_{g_{i}}=0\right)$ the bound $\bar{A}_{0}$ can be simplified, i.e., $\bar{A}_{0}=N\left(\bar{R}_{x}^{2}+\left(\bar{C}_{g}+\frac{5 \gamma}{2}\right)\left\|y^{*}\right\|^{2}\right)$. Therefore, choosing $\gamma=\left(N^{\frac{3}{2}} \Gamma C_{0}\right)^{-1}$, we get $\Lambda_{1}+\Lambda_{2}=\mathcal{O}\left(N^{3} \Gamma^{2} C_{0}^{2}\left(\bar{R}_{x}^{2}+\bar{C}_{g}\left\|y^{*}\right\|^{2}\right)\right)$.

Remark 3.5. For CASE 1, assuming $\sum_{k=0}^{\infty} \alpha^{q_{k}}(k+1)^{2}<+\infty$ in addition to $C_{0}<+\infty$, one can observe that using (3.20) and (3.21), the bound $\mathcal{O}(1)$ bound takes a simpler form: $\Lambda(\gamma, \beta)=\Lambda_{1}+\sum_{k=0}^{K-1} E^{k+1}\left(\hat{\mathbf{z}}^{K}\right) \leq \frac{1}{\gamma}+N\left(B \bar{R}_{x}^{2}+B^{2} \bar{C}_{g}\right)+\gamma N B^{2}+$ $12 N^{\frac{3}{2}} \Gamma B\left[\sum_{k=1}^{K} \alpha^{q_{k-1}} k+4 \sqrt{N} B \gamma \sum_{k=1}^{K} \alpha^{q_{k-1}} k(k+1)\right]$.
4. Fully distributed step-size rule. Recall that step-size selection rule in (3.1) of Theorem 3.3 requires some sort of coordination among the nodes in $\mathcal{N}$ because there is a fixed $\gamma>0$ coupling and affecting all nodes' step-size choice. To overcome this issue, we will define $\gamma_{i}>0$ for each node, and let the nodes to choose this parameter independently. Let $\mathbf{D}_{\gamma} \triangleq \operatorname{diag}\left(\left[\frac{1}{\gamma_{i}} \mathbf{I}_{m}\right]_{i \in \mathcal{N}}\right) \succ 0$ and define $\gamma \triangleq\left[\gamma_{i}\right]_{i \in \mathcal{N}}$ and $\widehat{\mathcal{C}} \triangleq\left\{\mathbf{p} \in \mathcal{Y}: \quad \exists \bar{y} \in \mathbb{R}^{m}\right.$ s.t. $\left.\frac{1}{\sqrt{\gamma_{i}}} p_{i}=\bar{y} \quad \forall i \in \mathcal{N}, \quad\|\bar{y}\| \leq 2 B\right\}-$ here, $\mathbf{p}=\left[p_{i}\right]_{i \in \mathcal{N}}$. Recall the definition of Bregman distance function given in Definition 2: $D_{x}(\mathbf{x}, \overline{\mathbf{x}})=\frac{1}{2}\|\boldsymbol{\xi}-\overline{\boldsymbol{\xi}}\|_{\mathbf{D}_{\tau}}^{2}+\frac{1}{2}\|\mathbf{w}-\overline{\mathbf{w}}\|_{\mathbf{D}_{\gamma}}^{2}$. Switching to $\mathbf{D}_{\gamma}$ as defined above, (2.7a) should be replaced with $\mathbf{w}^{k+1} \leftarrow \operatorname{argmin}_{\mathbf{w}} \sigma_{\widetilde{\mathcal{C}}}(\mathbf{w})-\left\langle\mathbf{y}^{k}, \mathbf{w}\right\rangle+\frac{1}{2}\left\|\mathbf{w}-\mathbf{v}^{k}\right\|_{\mathbf{D}_{\gamma}}^{2}$. Using the change of variables $\hat{\mathbf{w}} \triangleq \mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{w}$, it can be rewritten as

$$
\begin{equation*}
\mathbf{w}^{k+1} \leftarrow \mathbf{D}_{\gamma}^{-\frac{1}{2}} \underset{\hat{\mathbf{w}}}{\operatorname{argmin}} \sigma_{\widehat{\mathcal{C}}}(\hat{\mathbf{w}})+\frac{1}{2}\left\|\hat{\mathbf{w}}-\left(\mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{v}^{k}+\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}^{k}\right)\right\|^{2} \tag{4.1}
\end{equation*}
$$

where we use the fact that $\sigma_{\widetilde{\mathcal{C}}}\left(\mathbf{D}_{\gamma}^{-\frac{1}{2}} \hat{\mathbf{w}}\right)=\sigma_{\widehat{\mathcal{C}}}(\hat{\mathbf{w}})$. Now, we can write (4.1) in a proximal form and using Moreau's decomposition, we get

$$
\mathbf{w}^{k+1}=\mathbf{D}_{\gamma}^{-\frac{1}{2}} \operatorname{prox}_{\sigma_{\widehat{\mathcal{C}}}}\left(\mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{v}^{k}+\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}^{k}\right)=\mathbf{D}_{\gamma}^{-\frac{1}{2}}\left(\mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{v}^{k}+\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}^{k}-\mathcal{P}_{\widehat{\mathcal{C}}}\left(\mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{v}^{k}+\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}^{k}\right)\right) .
$$

Note $\mathbf{p} \in \widehat{\mathcal{C}}$ implies that $\mathbf{D}_{\gamma}^{\frac{1}{2}} \mathbf{p}=\mathbf{1}_{N} \otimes \bar{y}$ for some $\bar{y} \in \mathbb{R}^{m}$ such that $\|\bar{y}\| \leq 2 B$. Therefore, for $\mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{n_{0}}$, the projection of $\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}$ onto $\widehat{\mathcal{C}}$ can be computed as

$$
\begin{align*}
\mathcal{P}_{\widehat{\mathcal{C}}}\left(\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}\right) & =\underset{\mathbf{p} \in \widehat{\mathcal{C}}}{\operatorname{argmin}} \frac{1}{2}\left\|\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}-\mathbf{p}\right\|^{2}=\mathbf{D}_{\gamma}^{-\frac{1}{2}}\left(\mathbf{1} \otimes \underset{\|\bar{y}\| \leq 2 B}{\operatorname{argmin}} \frac{1}{2}\left\|\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathbf{y}-\mathbf{D}_{\gamma}^{-\frac{1}{2}}(\mathbf{1} \otimes \bar{y})\right\|^{2}\right) \\
& =\mathbf{D}_{\gamma}^{-\frac{1}{2}} \mathcal{P}_{\mathcal{B}}\left(\frac{1}{\sum_{i \in \mathcal{N}} \gamma_{i}}\left(\mathbf{1 1}^{\top} \otimes \mathbf{I}_{m}\right) \mathbf{D}_{\gamma}^{-1} \mathbf{y}\right) . \tag{4.2}
\end{align*}
$$

Let $\mathcal{P}_{\gamma}(\mathbf{y}) \triangleq \mathbf{1}_{N} \otimes \mathcal{P}_{\mathcal{B}_{0}}\left(\frac{1}{\sum_{i \in \mathcal{N}} \gamma_{i}} \sum_{i \in \mathcal{N}} \gamma_{i} y_{i}\right)$; hence, we get that

$$
\begin{equation*}
\mathbf{w}^{k+1}=\mathbf{D}_{\gamma}^{-1}\left(\mathbf{D}_{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}-\mathcal{P}_{\gamma}\left(\mathbf{D}_{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)\right) \tag{4.3}
\end{equation*}
$$

Thus, we propose approximating $\mathcal{P}_{\gamma}(\cdot)$ using an approximate convex combination operator $\mathcal{R}_{\gamma}^{k}(\cdot)=\left[\mathcal{R}_{i}^{k}(\cdot)\right]_{i \in \mathcal{N}}$ such that it can be computed in a distributed way, i.e., $\mathcal{R}_{i}^{k}(\cdot)$ can be computed at $i \in \mathcal{N}$ using local communication. More precisely, suppose $\mathcal{R}_{\gamma}^{k}$ satisfies a slightly modified version of Assumption 4, where (2.10) is replaced with

$$
\begin{equation*}
\mathcal{R}_{\gamma}^{k}(\mathbf{w}) \in \mathcal{B}, \quad\left\|\mathcal{R}_{\gamma}^{k}(\mathbf{w})-\mathcal{P}_{\gamma}(\mathbf{w})\right\| \leq N \Gamma \alpha^{q_{k}}\|\mathbf{w}\|, \quad \forall \mathbf{w} \in \mathbb{R}^{n_{0}} \tag{4.4}
\end{equation*}
$$

Provided that such an operator exists, instead of (2.11), we set $\mathbf{v}^{k+1}$ as follows:

$$
\begin{equation*}
\mathbf{v}^{k+1} \leftarrow \mathbf{D}_{\gamma}^{-1}\left(\mathbf{D}_{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}-\mathcal{R}_{\gamma}^{k}\left(\mathbf{D}_{\gamma} \mathbf{v}^{k}+\mathbf{y}^{k}\right)\right) \tag{4.5}
\end{equation*}
$$

With this modification, we can still show that the iterate sequence converges to a primal-dual optimal solution with $\mathcal{O}(1 / K)$ ergodic rate provided that primal-dual
step-sizes $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ and $\left\{\gamma_{i}\right\}_{i \in \mathcal{N}}$ are chosen such that $\tau_{i}=\left(\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}+\right.$ $\left.C_{g_{i}}\right)^{-1}, \kappa_{i}=\left(C_{g_{i}}+\frac{5 \gamma_{i}}{2}\right)^{-1}$ for all $i \in \mathcal{N}$.

In the rest of this section, for both undirected and directed time-varying communication networks, we provide an operator $\mathcal{R}_{\gamma}^{k}$ satisfying (4.4). For $\mathbf{y}=\left[y_{i}\right]_{i \in \mathcal{N}} \in \mathcal{Y}$, define $p_{\gamma}(\mathbf{y}) \triangleq \frac{1}{\sum_{i \in \mathcal{N}} \gamma_{i}} \sum_{i \in \mathcal{N}} \gamma_{i} y_{i}$; hence, we have $\mathcal{P}_{\gamma}(\mathbf{y})=\mathbf{1}_{N} \otimes \mathcal{P}_{\mathcal{B}_{0}}\left(p_{\gamma}(\mathbf{y})\right)$. Therefore, we should consider distributed approximation of $p_{\gamma}(\mathbf{y})$. Given $y_{i} \in \mathbb{R}^{m}$ and $\gamma_{i}>0$, which are only known at node $i \in \mathcal{N}$, we next discuss extensions of techniques discussed in Section 2.1 to compute the convex combination $\sum_{i \in \mathcal{N}} \gamma_{i} y_{i} / \sum_{i \in \mathcal{N}} \gamma_{i}$.

First, suppose that $\left\{\mathcal{G}^{t}\right\}$ is a time-varying undirected graph and $\left\{V^{t}\right\}_{t \in \mathbb{Z}_{+}}$be a corresponding sequence of weight matrices satisfying Assumption 5. For $\mathbf{w}=$ $\left[w_{i}\right]_{i \in \mathcal{N}} \in \mathcal{Y}$ such that $w_{i} \in \mathbb{R}^{m}$ for $i \in \mathcal{N}$, define

$$
\begin{equation*}
\mathcal{R}_{\gamma}^{k}(\mathbf{w}) \triangleq \mathcal{P}_{\mathcal{B}}\left(\left(\operatorname{diag}\left(W^{t_{k}+q_{k}, t_{k}} \boldsymbol{\gamma}\right)^{-1} W^{t_{k}+q_{k}, t_{k}} \otimes \mathbf{I}_{m}\right) \mathbf{D}_{\gamma}^{-1} \mathbf{w}\right) \tag{4.6}
\end{equation*}
$$

to approximate $\mathcal{P}_{\gamma}(\cdot)$ in (4.3). Note that $\mathcal{R}_{\gamma}^{k}(\cdot)$ can be computed in a distributed fashion requiring $q_{k}$ communications with the neighbors for each node. Using Lemma 2.1, it is easy to show that $\mathcal{R}_{\gamma}^{k}$ given in (4.6) satisfies the condition in (4.4).

Second, suppose that $\left\{\mathcal{G}^{t}\right\}$ is a time-varying $M$-strongly-connected directed graph, and $\left\{V^{t}\right\}_{t \in \mathbb{Z}_{+}}$be the corresponding weight-matrix sequence as defined in (2.13) within Section 2.1 .2 - so that (4.6) can be computed over time-varying directed network. Given any $\mathbf{w}=\left[w_{i}\right]_{i \in \mathcal{N}}$ and $\left\{\gamma_{i}\right\}_{i \in \mathcal{N}}$, the results in [29] immediately imply that for any $s \in \mathbb{Z}_{+}$, the vector $\left(\operatorname{diag}\left(W^{t, s} \gamma\right)^{-1} W^{t, s} \otimes \mathbf{I}_{m}\right) \mathbf{D}_{\gamma}^{-1} \mathbf{w}$ converges to the consensus convex combination vector $\mathbf{1}_{N} \otimes p_{\gamma}(\mathbf{w})$ with a geometric rate as $t$ increases. Indeed, this can be trivially achieved by using a different initialization for the push-sum method. Next, we state a slightly modified version of the convergence result in Lemma 2.2.

Lemma 4.1. Suppose that the digraph sequence $\left\{\mathcal{G}^{t}\right\}_{t \geq 1}$ is uniformly strongly connected ( $M$-strongly connected), where $\mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$. Given node-specific data $\left\{w_{i}\right\}_{i \in \mathcal{N}}$ $\subset \mathbb{R}^{m}$ and $\left\{\gamma_{i}\right\}_{i \in \mathcal{N}} \subset \mathbb{R}_{++}$, for any fixed integer $s \geq 0$, the following bound holds for all integers $t>s$ :

$$
\left\|\operatorname{diag}\left(W^{t, s} \boldsymbol{\gamma} \otimes \mathbf{I}_{m}\right)^{-1}\left(W^{t, s} \otimes \mathbf{I}_{m}\right) \mathbf{D}_{\gamma}^{-1} \mathbf{w}-\mathbf{1}_{N} \otimes p_{\gamma}(\mathbf{w})\right\| \leq \frac{8 \sqrt{N}}{\gamma_{\min } \delta} \sum_{i \in \mathcal{N}} \gamma_{i}\left\|w_{i}\right\| \alpha^{t-s-1}
$$

for some $\delta \geq \frac{1}{N^{N M}}$ and $0<\alpha \leq\left(1-\frac{1}{N^{N M}}\right)^{\frac{1}{M}}$, where $N=|\mathcal{N}|$ and $\gamma_{\min }=\min _{i \in \mathcal{N}} \gamma_{i}$.
Proof. The proof follows from Corollary 2 and the proof of Lemma 1 in [29].
Thus, $\mathcal{R}_{\gamma}^{k}(\cdot)$ defined in (4.6) satisfies the requirement $\left\|\mathcal{R}_{\gamma}^{k}(\mathbf{w})-\mathcal{P}_{\gamma}(\mathbf{w})\right\| \leq N \Gamma \alpha^{q_{k}}\|\mathbf{w}\|$ in (4.4) for $\Gamma=\frac{\|\gamma\|}{\gamma_{\min } \sqrt{N}} \frac{8}{\delta \alpha}$ and for some $\alpha \in(0,1)$ and $\delta>0$ as stated in Lemma 4.1.
5. A Distributed Algorithm for Static Network Topology. We extend the results in [1] to nonlinear constraint functions $\left\{g_{i}\right\}_{i \in \mathcal{N}}$. Given an undirected, static communication network $\mathcal{G}=(\mathcal{N}, \mathcal{E})$, following the discussion in Section 1.2, the corresponding SP problem for the static network is given as $\min _{\boldsymbol{\xi}, \mathbf{w}} \max _{\mathbf{y}}\left\{\sum_{i \in \mathcal{N}} \varphi_{i}\left(\xi_{i}\right)-\right.$ $\left.\left\langle g_{i}\left(\xi_{i}\right), y_{i}\right\rangle-\langle\mathbf{w}, M \mathbf{y}\rangle: y_{i} \in \mathcal{K}^{\circ}, \forall i \in \mathcal{N}\right\}$. In Fig. 5.1, we propose a modified version of DPDA-S algorithm [1] (see Section 1.2) to solve (1.2) over $\mathcal{G}$.

DPDA-S needs only one communication round per iteration; moreover, since DPDA-S does not require inexact averaging (hence no error accumulation), its analysis is much simpler than and directly follows from the analysis of DPDA-D. The following theorem states the convergence rate for the iterates of DPDA-S.

Theorem 5.1. Suppose Assumptions 1, 2 with $\mathcal{G}^{t}=\mathcal{G}$ for $t \geq 0$ and 3 hold. For any $\gamma>0$, let the primal-dual step-sizes $\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}$ be chosen such that

$$
\begin{equation*}
\tau_{i}=\left(\max \left\{1, L_{f_{i}}+\beta L_{g_{i}}\right\}+C_{g_{i}}\right)^{-1}, \quad \kappa_{i}=\left(C_{g_{i}}+\gamma\left(4 d_{\max }+\frac{1}{2}\right)\right)^{-1}, \quad \forall i \in \mathcal{N} . \tag{5.1}
\end{equation*}
$$

```
Algorithm DPDA-S \(\left(\boldsymbol{\xi}^{0}, \gamma,\left\{\tau_{i}, \kappa_{i}\right\}_{i \in \mathcal{N}}\right)\)
Initialization: \(y_{i}^{0} \leftarrow 0, s_{i}^{0} \leftarrow 0, \quad i \in \mathcal{N}\)
Step \(k\) : \((k \geq 0)\)
    1. \(\xi_{i}^{k+1} \leftarrow \operatorname{prox}_{\tau_{i} \rho_{i}}\left(\xi_{i}^{k}-\tau_{i}\left(\nabla f_{i}\left(\xi_{i}^{k}\right)-\mathbf{J} g_{i}^{\top} y_{i}^{k}\right)\right), \quad p_{i}^{k+1} \leftarrow \sum_{j \in \mathcal{N}_{i}}\left(s_{i}^{k}-s_{j}^{k}\right), \quad i \in \mathcal{N}\)
    2. \(y_{i}^{k+1} \leftarrow \mathcal{P}_{\mathcal{K}^{\circ} \cap \mathcal{B}_{0}}\left[y_{i}^{k}-\kappa_{i}\left(2 g_{i}\left(\xi_{i}^{k+1}\right)-g_{i}\left(\xi_{i}^{k}\right)+\gamma p_{i}^{k+1}\right)\right], \quad i \in \mathcal{N}\)
    3. \(s_{i}^{k+1} \leftarrow y_{i}^{k+1}+\sum_{\ell=0}^{k+1} y_{i}^{\ell}, \quad i \in \mathcal{N}\)
```

Fig. 5.1. Distributed Primal Dual Algorithm for Static $\mathcal{G}$ (DPDA-S)
for some $\beta>0$. Given $B \in(0, \infty]$, let $\mathcal{B}_{0} \triangleq\left\{y \in \mathbb{R}^{m}:\|y\| \leq 2 B\right\}$. Starting from $\mathbf{s}^{0}=\mathbf{y}^{0}=\mathbf{0}$ and an arbitrary $\boldsymbol{\xi}^{0}$, let $\left\{\left(\boldsymbol{\xi}^{k}\right)\right\}_{k \geq 0}$ be the primal, and $\left\{\mathbf{y}^{k}\right\}_{k \geq 0}$ be the dual iterate sequence generated by DPDA-S, displayed in Fig. 5.1. For any $\gamma>0$, if $\beta>0$ is chosen as discussed below, then $\left\{\left(\boldsymbol{\xi}^{k}, \mathbf{y}^{k}\right)\right\}_{k \geq 0}$ converges to $\left(\boldsymbol{\xi}^{*}, \mathbf{y}^{*}\right)$ such that $\mathbf{y}^{*}=\mathbf{1} \otimes y^{*}$ and $\left(\boldsymbol{\xi}^{*}, y^{*}\right)$ is an optimal primal-dual solution to (1.2). Moreover, both infeasibility, $F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right)$, and suboptimality, $\left|\varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)\right|$ are $\mathcal{O}(1 / K)$, i.e.,

$$
\begin{align*}
& F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right) \triangleq\|M \overline{\mathbf{y}}\|+\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) \leq \frac{\Lambda(\gamma, \beta)}{K}  \tag{5.2}\\
& 0 \leq \varphi\left(\overline{\boldsymbol{\xi}}^{K}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)+\left\|y^{*}\right\| d_{\mathcal{K}}\left(-g\left(\overline{\boldsymbol{\xi}}^{K}\right)\right) \leq \frac{\Lambda(\gamma, \beta)}{K}-F\left(\overline{\boldsymbol{\xi}}^{K}, \overline{\mathbf{y}}^{K}\right), \tag{5.3}
\end{align*}
$$

for all $K \geq 1$, where $\Lambda(\gamma, \beta)=\frac{1}{2 \gamma}+\sum_{i \in \mathcal{N}} \frac{1}{\tau_{i}}\left\|\xi_{i}^{*}-\xi_{i}^{0}\right\|^{2}+\frac{4}{\kappa_{i}}\left\|y^{*}\right\|^{2}$.
(CASE 1): If a dual bound is known, i.e., $B<\infty$, then (5.2) and (5.3) hold for $\beta=2 B$; moreover, setting $\gamma=(N B)^{-1}$ gives $\Lambda(\gamma, \beta)=\mathcal{O}\left(N B\left(\bar{R}_{x}^{2}+\bar{C}_{g} B+1\right)\right)$.
(CASE 2): If the dual bound does not exist, then set $B=\infty$ within DPDAS. There exists $\bar{\beta}>0$ such that (5.2) and (5.3) hold for all $\beta \geq \bar{\beta}$; moreover, selecting $\gamma=\bar{R}_{x}^{2} \sqrt{N / d_{\max }}$ leads to $\Lambda(\gamma, \beta)=\mathcal{O}\left(N^{\frac{3}{2}} \sqrt{d_{\max }} \bar{R}_{x}^{2} \max \left\{1,\left\|y^{*}\right\|^{2}\right\}\right)$ and $\bar{\beta}=\mathcal{O}\left(\sqrt{N d_{\max }} \max \left\{1,\left\|y^{*}\right\|\right\}\right)$ for $N$ sufficiently large ${ }^{2}$.

Proof. The results follow from the analysis of DPDA-D in Section 3 and [1].
Remark 5.1. In Theorem 2 of [1], the rate result is provided for the case $g_{i}$ is affine for $i \in \mathcal{N}$. For this case, a dual bound is not needed; hence, the suboptimality and infeasibility rate is $\mathcal{O}(\Lambda / K)$ for some $\Lambda=\mathcal{O}\left(N\left(\bar{R}_{x}^{2}+\bar{C}_{g}\left\|y^{*}\right\|^{2}\right)\right)$ when $\gamma=1 / N$.
6. Computing a dual bound. Recall that the definition of $\widetilde{\mathcal{C}}$ in (2.3) involves a bound $B$ such that $\left\|y^{*}\right\| \leq B$ for some dual optimal solution $y^{*}$. In this section, we show that given a Slater point we can find a ball containing the optimal dual set for problem (1.2). To this end, we first derive some results without assuming convexity.

Let $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ and $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be arbitrary functions of $\boldsymbol{\xi}$, and $\mathcal{K} \subset \mathbb{R}^{m}$ be a cone. For now, we do not assume convexity for $\varphi, g$, and $\mathcal{K}$, which are the components of the following generic problem

$$
\begin{equation*}
\varphi^{*} \triangleq \min _{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi}) \quad \text { s.t. } \quad g(\boldsymbol{\xi}) \in-\mathcal{K}: y \in \mathcal{K}^{\circ} \tag{6.1}
\end{equation*}
$$

where $y \in \mathbb{R}^{m}$ denotes the dual vector. Let $q$ denote the dual function, i.e.,

$$
q(y) \triangleq \begin{cases}\inf _{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi})-y^{\top} g(\boldsymbol{\xi}), & \text { if } y \in \mathcal{K}^{\circ} ;  \tag{6.2}\\ -\infty, & \text { o.w. }\end{cases}
$$

We assume that there exists $\hat{y} \in \mathcal{K}^{\circ}$ such that $q(\hat{y})>-\infty$. Since $q$ is a closed concave function, this assumption implies that $-q$ is a proper closed convex function. Next

[^2]we show that for any $\bar{y} \in \operatorname{dom} q=\left\{y \in \mathbb{R}^{m}: q(y)>-\infty\right\}$, the superlevel set $Q_{\bar{y}} \triangleq\{y \in \operatorname{dom} q: q(y) \geq q(\bar{y})\} \subset \mathcal{K}^{\circ}$ is contained in a Euclidean ball centered at the origin, of which radius can be computed efficiently. A special case of this dual boundedness result is well known when $\mathcal{K}=\mathbb{R}_{+}^{m}[43]$ - see Lemma 1.1 in [35]; however, it is not trivial to extend this result to an arbitrary cone $\mathcal{K}$ with $\operatorname{int}(\mathcal{K}) \neq \emptyset$.

Lemma 6.1. Let $\overline{\boldsymbol{\xi}}$ be a Slater point for (6.1), i.e., $\overline{\boldsymbol{\xi}} \in \operatorname{relint}(\operatorname{dom} \varphi)$ such that $-g(\overline{\boldsymbol{\xi}}) \in \operatorname{int}(\mathcal{K})$. Then for all $\bar{y} \in \operatorname{dom} q$, the superlevel set $Q_{\bar{y}}$ is bounded as follows,

$$
\begin{equation*}
\|y\| \leq(\varphi(\overline{\boldsymbol{\xi}})-q(\bar{y})) / r^{*}, \quad \forall y \in Q_{\bar{y}} \tag{6.3}
\end{equation*}
$$

where $0<r^{*} \triangleq \min _{w}\left\{-w^{\top} g(\overline{\boldsymbol{\xi}}):\|w\|=1, w \in \mathcal{K}^{*}\right\}$. Note that this is not a convex problem due to the equality constraint; instead, one can upper bound (6.3) using $0<\tilde{r} \leq r^{*}$, which can be efficiently computed by solving a convex problem

$$
\begin{equation*}
\tilde{r} \triangleq \min _{w}\left\{-w^{\top} g(\overline{\boldsymbol{\xi}}):\|w\|_{1}=1, w \in \mathcal{K}^{*}\right\} . \tag{6.4}
\end{equation*}
$$

Proof. For any $y \in Q_{\bar{y}} \subset \mathcal{K}^{\circ}$, we have that

$$
\begin{equation*}
q(\bar{y}) \leq q(y)=\inf _{\boldsymbol{\xi}}\left\{\varphi(\boldsymbol{\xi})-y^{\top} g(\boldsymbol{\xi})\right\} \leq \varphi(\overline{\boldsymbol{\xi}})-y^{\top} g(\overline{\boldsymbol{\xi}}), \tag{6.5}
\end{equation*}
$$

which implies that $y^{\top} g(\overline{\boldsymbol{\xi}}) \leq \varphi(\overline{\boldsymbol{\xi}})-q(\bar{y})$. Since $-g(\overline{\boldsymbol{\xi}}) \in \operatorname{int}(\mathcal{K})$ and $y \in \mathcal{K}^{\circ}$, we clearly have $y^{\top} g(\overline{\boldsymbol{\xi}})>0$ whenever $y \neq \mathbf{0}$. Indeed, since $-g(\overline{\boldsymbol{\xi}}) \in \operatorname{int}(\mathcal{K})$, there exist $r>0$ such that $-g(\overline{\boldsymbol{\xi}})+r u \in \mathcal{K}$ for all $\|u\| \leq 1$. Hence, for $y \neq \mathbf{0}$, by choosing $u=y /\|y\|$ and using the fact that $y \in \mathcal{K}^{\circ}$, we get that $0 \geq(-g(\boldsymbol{\xi})+r y /\|y\|)^{\top} y$. Therefore, (6.5) implies that for all $y \in Q_{\bar{y}}, r\|y\| \leq y^{\top} g(\overline{\boldsymbol{\xi}}) \leq \varphi(\overline{\boldsymbol{\xi}})-q(\bar{y})$; hence, $\|y\| \leq \frac{\varphi(\overline{\boldsymbol{\xi}})-q(\bar{y})}{r}$. Now, we will characterize the largest radius $r^{*}>0$ such that $\mathcal{B}\left(-g(\overline{\boldsymbol{\xi}}), r^{*}\right) \subset \mathcal{K}$, where $\mathcal{B}(-g(\overline{\boldsymbol{\xi}}), r) \triangleq\{-g(\overline{\boldsymbol{\xi}})+r u:\|u\| \leq 1\}$. Note that $r^{*}>0$ can be written explicitly as follows: $r^{*}=\max \left\{r: d_{\mathcal{K}}(-g(\boldsymbol{\xi})+r u) \leq 0, \quad \forall u\right.$ s.t. $\left.\|u\| \leq 1\right\}$. Let $\gamma(r) \triangleq \sup \left\{d_{\mathcal{K}}(-g(\overline{\boldsymbol{\xi}})+r u):\|u\| \leq 1\right\}$; hence, $r^{*}=\max \{r: \gamma(r) \leq 0\}$. Note that for any fixed $u \in \mathbb{R}^{m}, d_{\mathcal{K}}(-g(\boldsymbol{\xi})+r u)$ as a function of $r$ is a composition of a convex function $d_{\mathcal{K}}(\cdot)$ with an affine function in $r$; hence, it is convex in $r \in \mathbb{R}$ for all $u \in \mathbb{R}^{m}$. Moreover, since supremum of convex functions is also convex, $\gamma(r)$ is convex in $r$. From the definition of $d_{\mathcal{K}}(\cdot)$, we have

$$
\begin{equation*}
\gamma(r)=\sup _{\|u\| \leq 1} \inf _{\boldsymbol{\xi} \in \mathcal{K}}\|\boldsymbol{\xi}+g(\overline{\boldsymbol{\xi}})-r u\|=\sup _{\|u\| \leq 1} \inf _{\boldsymbol{\xi} \in \mathcal{K}} \sup _{\|w\| \leq 1} w^{\top}(\boldsymbol{\xi}+g(\overline{\boldsymbol{\xi}})-r u) \tag{6.6}
\end{equation*}
$$

Since $\left\{w \in \mathbb{R}^{m}:\|w\| \leq 1\right\}$ is a compact set, and the function in (6.6) is a bilinear function of $w$ and $\boldsymbol{\xi}$ for each $u$, the inner $\inf _{\boldsymbol{\xi}}$ and $\sup _{w}$ can be interchanged to obtain,

$$
\gamma(r)=\sup _{\|u\| \leq 1} \sup _{\|w\| \leq 1} \inf _{\boldsymbol{\xi} \in \mathcal{K}} w^{\top}(\boldsymbol{\xi}+g(\overline{\boldsymbol{\xi}})-r u)=\sup _{\substack{\|u\| \leq 1 \\\|w\| \leq 1 \\ w \in \mathcal{K}^{*}}} w^{\top}(g(\overline{\boldsymbol{\xi}})-r u)=\sup _{\substack{\|w\| \leq 1 \\ w \in \mathcal{K}^{*}}} w^{\top} g(\overline{\boldsymbol{\xi}})+r\|w\| .
$$

Let $w^{*}(r)$ be one of the maximizers. It is easy to see that $\left\|w^{*}(r)\right\|=1$, since the supremum of a convex function over a convex set is attained on the boundary of the set. Therefore, $\gamma(r)=\sup _{\substack{\|w\|=1 \\ w \in \mathcal{K}^{*}}} w^{\top} g(\overline{\boldsymbol{\xi}})+r$. Since $r^{*}=\max \{r: \gamma(r) \leq 0\}$,

$$
\left(P_{1}\right): \quad r^{*}=\max \left\{r: r \leq-\sup \left\{w^{\top} g(\overline{\boldsymbol{\xi}}):\|w\|=1, w \in \mathcal{K}^{*}\right\}\right\}=\min _{\substack{\|w\|=1 \\ w \in \mathcal{K}^{*}}}-w^{\top} g(\overline{\boldsymbol{\xi}})
$$

Note that $\left(P_{1}\right)$ is not a convex problem due to boundary constraint, $\|w\|=1$. Next, we define a related convex problem: $\min _{\substack{\|w\|_{1}=1 \\ w \in \mathcal{K}^{*}}}-w^{\top} g(\overline{\boldsymbol{\xi}}) \leq r^{*}=\min _{\substack{\|w\|=1 \\ w \in \mathcal{K}^{*}}}-w^{\top} g(\overline{\boldsymbol{\xi}})$, to
lowerbound $r^{*}$ so that we can upper bound the right hand side of (6.3). Let $w^{*}$ be an optimal solution to $\left(P_{1}\right)$ and define $\bar{w}=w^{*} /\left\|w^{*}\right\|_{1}$. Clearly, $\|\bar{w}\|_{1}=1$ and $\bar{w} \in \mathcal{K}^{*}$. Moreover, since $\left\|w^{*}\right\|_{1} \geq\left\|w^{*}\right\|=1$ we have that

$$
0<\tilde{r}=\min _{\substack{\| w_{1}=1 \\ w \in \mathcal{K}^{*}}}-w^{\top} g(\overline{\boldsymbol{\xi}}) \leq-\bar{w}^{\top} g(\overline{\boldsymbol{\xi}})=-\frac{1}{\left\|w^{*}\right\|_{1}} w^{* \top} g(\overline{\boldsymbol{\xi}}) \leq-w^{* \top} g(\overline{\boldsymbol{\xi}})=r^{*} .
$$

Remark 6.1. Consider the problem in (1.2). Given a Slater point $\overline{\boldsymbol{\xi}}$, one needs to solve the minimization problem (6.4) in a distributed fashion, e.g., using the method in [2], to obtain a dual bound $B \in(0,+\infty)$. Suppose $\varphi_{i}(\cdot) \geq \underline{\varphi}$ for all $i \in \mathcal{N}$ and $N$ is known by all agents. Once $\tilde{r}$, the optimal value to (6.4) is computed, one can set $B=(\varphi(\overline{\boldsymbol{\xi}})-N \varphi) / \tilde{r}$, i.e., $\bar{y}=\mathbf{0}$. Moreover, if a Slater point exists but not available, one can solve the problem of $\overline{\boldsymbol{\xi}}=\operatorname{argmin}_{\boldsymbol{\xi}} \mathcal{F}\left(\sum_{i \in \mathcal{N}} g_{i}\left(\xi_{i}\right)\right)$ in a distributed fashion using methods proposed in [11] to obtain a Slater point where $\mathcal{F}: \mathbb{R}^{m} \rightarrow \mathbb{R}$ is a generalized logarithm function for the proper cone $\mathcal{K}$ (see Section 11.6.1 in [6] for the definition). Next, $B$ can be computed as discussed previously.

Remark 6.2. Let $g_{j}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be the components of $g: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ for $j=$ $1, \ldots, m$, i.e., $g(\boldsymbol{\xi})=\left[g_{j}(\boldsymbol{\xi})\right]_{j=1}^{m}$. When $\mathcal{K}=\mathbb{R}_{+}^{m}$, Lemma 1.1 in [35] implies that for any $\bar{y} \in \operatorname{dom} q$ and $\overline{\boldsymbol{\xi}}$ such that $g_{j}(\overline{\boldsymbol{\xi}})<0$ for all $j=1, \ldots, m$, every $y \in Q_{\bar{y}}$ satisfies $\|y\| \leq(\varphi(\overline{\boldsymbol{\xi}})-q(\bar{y})) / \bar{r}$, where $\bar{r} \triangleq \min \left\{-g_{j}(\overline{\boldsymbol{\xi}}): j=1, \ldots, m\right\}$. Note our result in Lemma 6.1 gives the same bound since $r^{*}=\min _{w}\left\{-w^{\top} g(\overline{\boldsymbol{\xi}}):\|w\|=1, w \in \mathbb{R}_{+}^{m}\right\}=\bar{r}$.
7. Numerical Experiments. We implemented the DPDA-D algorithm and tested its performance on two different set of problems.
7.1. Basis Pursuit Denoising (BPD) Problem. Let $\xi^{*} \in \mathbb{R}^{n}$ be an unknown sparse vector, i.e., most of its elements are zero. Suppose $r \in \mathbb{R}^{m}$ denotes a vector of $m \ll n$ noisy linear measurements of $\boldsymbol{\xi}^{*}$ using the measurement matrix $R \in \mathbb{R}^{m \times n}$, i.e., $\left\|R \xi^{*}-r\right\| \leq \epsilon$ for some $\epsilon \geq 0$. The BPD problem can be formulated as

$$
\begin{equation*}
\min _{\boldsymbol{\xi}}\|\boldsymbol{\xi}\|_{1} \quad \text { s.t. } \quad\|R \boldsymbol{\xi}-r\| \leq \epsilon . \tag{7.1}
\end{equation*}
$$

BPD appears in the context of compressed sensing [17] and the objective is to recover the unknown sparse $\boldsymbol{\xi}^{*}$ from a small set of measurement or transform values in $r$.

Given a set of computing nodes $\mathcal{N}$, suppose each node $i \in \mathcal{N}$ knows $r \in \mathbb{R}^{m}$ and stores only $n_{i}$ columns of $R$ corresponding to a submatrix $R_{i} \in \mathbb{R}^{m \times n_{i}}$ such that $n=\sum_{i \in \mathcal{N}} n_{i}$ and $R=\left[R_{i}\right]_{i \in \mathcal{N}}$. Partitioning the decision vector $\boldsymbol{\xi}=\left[\xi_{i}\right]_{i \in \mathcal{N}}$ accordingly, BPD problem in (7.1) can be rewritten as follows:

$$
\begin{equation*}
\min _{\xi_{i} \in \mathbb{R}^{i} i, i \in \mathcal{N}} \sum_{i \in \mathcal{N}}\left\|\xi_{i}\right\|_{1} \quad \text { s.t. }\left\|\sum_{i \in \mathcal{N}} R_{i} \xi_{i}-r\right\| \leq \epsilon . \tag{7.2}
\end{equation*}
$$

Note (7.2) can be cast into the form similar to (1.2). Indeed, let $\chi: \mathbb{R} \rightarrow \mathbb{R} \cup\{+\infty\}$ such that $\chi(t)=0$ if $t=\epsilon$, and $+\infty$ otherwise; and let $\mathcal{K}$ denote the second-order cone, i.e., $\mathcal{K}=\left\{(y, t) \in \mathbb{R}^{m} \times \mathbb{R}:\|y\| \leq t\right\}$. Hence, (7.2) can be written as

$$
\min _{t \in \mathbb{R}, \xi_{i} \mathbb{R}^{n_{i}},} \sum_{i \in \mathcal{N}} \sum_{i \in \mathcal{N}}\left\|\xi_{i}\right\|_{1}+\chi(t) \quad \text { s.t. } \quad\left(\sum_{i \in \mathcal{N}} R_{i} \xi_{i}-r, t\right) \in \mathcal{K} .
$$

First, we test the effect of network topology on the performance of the proposed algorithm, and then to benchmark this distributed algorithm, we also solve the same problem in a centralized way using Prox-JADMM algorithm proposed in [14]. Note that Prox-JADMM solves the problem in a centralized fashion which naturally has a faster convergence than a decentralized algorithm. The aim of this comparison is to show that the convergence of the proposed decentralized algorithm would be competitive with a centralized method when nature of the problem requires to store and access the data in a decentralized manner.
7.1.1. Problem generation. In the rest, we consider two different forms of the problem in (7.1): noisy, i.e., $\epsilon>0$ and noise free, i.e., $\epsilon=0$. In our experiments, we set $n=120$ and $m=20$. For the noisy case, as suggested in [3], the target signal $\boldsymbol{\xi}^{*}$ is generated by choosing $\kappa=20$ of its elements, uniformly at random, drawn from the standard Gaussian distribution and the rest of the elements are set to 0 . Moreover, each element of $R=\left[R_{i j}\right]$ is i.i.d with standard normal distribution, and the measurement $r=R \boldsymbol{\xi}^{*}+\eta$ where $\eta \in \mathbb{R}^{m}$ such that each of its elements is i.i.d. according to Gaussian distribution with mean 0 and variance $\sigma^{2}=\kappa 10^{-S / 10}$ - this would generate a measurement vector $r$ with the signal-to-noise ratio (SNR) equal to $S$ where $\operatorname{SNR}(r) \triangleq 10 \log _{10}\left(\mathbb{E}\left[\left\|R \xi^{*}\right\|^{2}\right] / \mathbb{E}\left[\|\eta\|^{2}\right]\right)$. In our experiments, we consider $S=30 \mathrm{~dB}$ or 40 dB . Finally, $\epsilon>0$ is chosen such that $\operatorname{Pr}\left(\|\eta\|^{2} \leq \epsilon^{2}\right)=1-\alpha$, and we let $\alpha=0.05$. For the noise-free case, the noise parameters, i.e., $\sigma^{2}$ and $\epsilon$ are set to 0 ; hence, the constraint for the noise-free case is a linear one, i.e., $\sum_{i \in \mathcal{N}} R_{i} \xi_{i}=r$ - the rest of the problem components are generated as in the noisy case.

Generating an undirected small-world network: Let $\mathcal{G}_{u}=\left(\mathcal{N}, \mathcal{E}_{u}\right)$ be generated as a random small-world network. Given $|\mathcal{N}|$ and the desired number of edges $\left|\mathcal{E}_{u}\right|$, we choose $|\mathcal{N}|$ edges creating a random cycle over nodes, and then the remaining $\left|\mathcal{E}_{u}\right|-|\mathcal{N}|$ edges are selected uniformly at random.

Generating a time-varying undirected network: We first generate a random small-world $\mathcal{G}_{u}=\left(\mathcal{N}, \mathcal{E}_{u}\right)$ as described above. Next, given $M \in \mathbb{Z}_{+}$, and $p \in(0,1)$, for each $k \in \mathbb{Z}_{+}$, we generate $\mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$, the communication network at time $t \in\{(k-1) M, \ldots, k M-2\}$ by sampling $\left\lceil p\left|\mathcal{E}_{u}\right|\right\rceil$ edges of $\mathcal{G}_{u}$ uniformly at random and we set $\mathcal{E}^{k M-1}=\mathcal{E}_{u} \backslash \bigcup_{t=(k-1) M}^{k M-2} \mathcal{E}^{t}$. In all experiments, we set $M=5, p=0.8$ and the number of communications per iteration is set to $q_{k}=10 \log (k+1)$.
7.1.2. Effect of Network Topology. In this section, we test the effect of network topology on the performance of DPDA-S and DPDA-D on undirected communication networks. We consider four scenarios in which the number of nodes $N \in\{10,40\}$ and the average number of edges per node, $\left|\mathcal{E}^{t}\right| / N$, is either 1.2 or $\approx 3.6$. For each scenario, we plot relative suboptimality, i.e., $\left|\varphi\left(\boldsymbol{\xi}^{k}\right)-\varphi\left(\boldsymbol{\xi}^{*}\right)\right| /\left|\varphi\left(\boldsymbol{\xi}^{*}\right)\right|$, infeasibility, i.e., $\left(\left\|\sum_{i \in \mathcal{N}} R_{i} \xi_{i}^{k}-r\right\|-\epsilon\right)_{+}$, and consensus violation, i.e., $\max _{i \in \mathcal{N}} \| y_{i}^{k}-$ $\frac{1}{|\mathcal{N}|} \sum_{j \in \mathcal{N}} y_{j}^{k} \|$ versus iteration number $k$. All the plots show the average statistics over 50 randomly generated replications. In each of these independent replications, both $R$ and $\boldsymbol{\xi}^{*}$ are also randomly generated in addition to random communication networks.

Testing DPDA-S on static undirected communication networks: We generated the static small-world networks $\mathcal{G}=(\mathcal{N}, \mathcal{E})$, as described above in Section 7.1.1, for $(|\mathcal{N}|,|\mathcal{E}|) \in\{(10,15),(10,45),(40,60),(40,180)\}$ and solved the BPD problem in (7.1) using DPDA-S described in Fig. 5.1 corresponding to SNR values $S \in\{30,+\infty\}$ - when $S=+\infty \mathrm{dB}$, we have $R \boldsymbol{\xi}^{*}=r$, i.e., noise-free case. The elements of the initial point $\boldsymbol{\xi}^{0}$ are i.i.d standard uniform distribution; and step-sizes are chosen as follows: $\gamma=2|\mathcal{N}| /|\mathcal{E}|, \tau_{i}=\frac{1}{\left\|R_{i}\right\|}$, and $\kappa_{i}=\frac{1}{2 \gamma d_{i}+\left\|R_{i}\right\|}$ for $i \in \mathcal{N}$. The performance of DPDA-S in terms of suboptimality, infeasibility and consensus violation is displayed in Fig. 7.1. It is clear that when compared to the effect of average edge density, the network size $|\mathcal{N}|$ has more influence on the convergence rate, i.e., the smaller the network faster the convergence is. On the other hand, for fixed size network, as expected, higher the density faster the convergence is, especially for consensus violation statistics.

Testing DPDA-D on time-varying undirected communication networks: We first generated an undirected small-world network $\mathcal{G}_{u}=\left(\mathcal{N}, \mathcal{E}_{u}\right)$ as described earlier. Next, we generated $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$ as described in Section 7.1.1. We chose the initial


Fig. 7.1. Effect of network topology on the convergence rate of $D P D A-S$ : top row corresponds to noise free and bottom row corresponds to noisy experiments with $S=30 \mathrm{~dB}$.
point $\boldsymbol{\xi}^{0}$ of DPDA-D such that the components are i.i.d with the standard uniform distribution, and set the step-sizes as follows: $\gamma=1, \tau_{i}=\frac{1}{|\mathcal{N}|\left\|R_{i}\right\|}$, and $\kappa_{i}=\frac{1}{\gamma+\left\|R_{i}\right\| /|\mathcal{N}|}$ for $i \in \mathcal{N}$. The performance of DPDA-D in terms of suboptimality, infeasibility and consensus violation is displayed in Fig. 7.2. It is clear that when compared to the effect of average edge density, the network size $|\mathcal{N}|$ has more influence on the convergence rate, i.e., the smaller the network faster the convergence is; however, the average edge density does not seem to have a significant impact on the convergence.


Fig. 7.2. Effect of network topology on the convergence of DPDA-D: top row corresponds to noise free and bottom row corresponds to noisy experiments with $S=30 \mathrm{~dB}$.
7.1.3. Benchmarking against a centralized algorithm. In this section, we benchmark DPDA-S on undirected network and DPDA-D on both undirected and directed networks against Prox-JADMM algorithm implemented on BPD problems under three different noise levels; $S=30 \mathrm{~dB}, S=40 \mathrm{~dB}$ and noise free, i.e., $S=$ $+\infty$ dB. Prox-JADMM is a multi-block ADMM using Jacobian type updates and
block- $i$ update has an additional proximal term $\frac{1}{2}\left\|\xi_{i}-\xi_{i}^{k}\right\|_{P_{i}}^{2}$ for each $i \in \mathcal{N}$, where $\left\{P_{i}\right\}_{i \in \mathcal{N}}$ are positive-definite matrices satisfying certain conditions. We choose the parameters for Prox-JADMM algorithm as suggested in Section 3.2. of [14], i.e., by setting the matrix $P_{i}$ in the proximal term to be $P_{i}=\left(N \mathbf{I}-10 R_{i}^{\top} R_{i}\right) /\|r\|_{1}$ for $i \in \mathcal{N}$ and $\left\{P_{i}\right\}_{i \in \mathcal{N}}$ are adaptively updated by the strategy discussed in Section 2.3. of [14].

Static undirected network: In each replication, we generate a random smallworld network $\mathcal{G}=(\mathcal{N}, \mathcal{E})$ and choose the algorithm parameters as in static network experiments of Section 7.1.2. The comparison between DPDA-S and Prox-JADMM in terms of suboptimality, infeasibility and consensus violation is displayed in Fig. 7.3 for different levels of noise. We observe that the lower signal-to-noise ratio leads to faster convergence, and the noise-free case has the slowest convergence. For all noise levels, DPDA-S is competitive with Prox-JADMM.




Fig. 7.3. Comparison of DPDA-S and Prox-JADMM over undirected static network for three different noise levels

Time-varying undirected network: For undirected time-varying networks we fix $N=10$ and $\left|\mathcal{E}^{t}\right| / N=1.2$, i.e., $\left|\mathcal{E}_{u}\right| / N=1.5$ - we observe the same convergence behavior for the other network scenarios discussed in Section 7.1.2. In each replication, we generate the network sequence $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$ and choose the parameters as in timevarying network experiments of Section 7.1.2. Fig. 7.4 shows the comparison between the two methods in terms of suboptimality, infeasibility and consensus violation. We observe that different noise-levels lead to similar convergence patterns; however, the lower signal-to-noise ratio leads to faster convergence, and the noise-free case has the slowest convergence. For all noise levels DPDA-D is competitive against ProxJADMM - slightly slower rate of DPDA-D is the price we pay for the decentralized setting to reach consensus on the dual price over the time-varying network.




Fig. 7.4. Comparison of $D P D A-D$ and Prox-JADMM over undirected time-varying network for three different noise levels

Time-varying directed network: In this scenario, similar to [36] we consider the strongly-connected directed graph $\mathcal{G}_{d}=\left(\mathcal{N}, \mathcal{E}_{d}\right)$ in Fig. 7.6 with $N=12$ nodes and $\left|\mathcal{E}_{d}\right|=24$ directed edges. We generated $\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$ as in the undirected case, but using $\mathcal{G}_{d}$ instead of $\mathcal{G}_{u}$, with parameters $M=5, p=0.8$, and $q_{k}=10 \log (k+1)$; hence,


Fig. 7.5. Comparison of DPDA-D and Prox-JADMM over directed time-varying network with three noise levels


FIG. 7.6. $\mathcal{G}_{d}=\left(\mathcal{N}, \mathcal{E}_{d}\right)$ directed strongly connected graph
$\left\{\mathcal{G}^{t}\right\}_{t \geq 0}$ is $M$-strongly-connected. Moreover, communication weight matrices $V^{t}$ are formed according to rule (2.13), and we used the approximate averaging operator $\mathcal{R}^{k}$ given in (2.14). We set the step-sizes as in the time-varying undirected case. Fig. 7.5 illustrates the comparison between DPDA-D and Prox-JADMM in terms of suboptimality, infeasibility and consensus violation when the network is both timevarying and directed. The results of this experiment are similar to those for the time-varying undirected case; hence, using unidirectional communications instead of bidirectional did not adversely affect the convergence of DPDA-D.
7.2. Multi-channel Power Allocation Problem. Multi-channel power allocation is a classic problem in information theory. Suppose there are a set of nodes connected to each other over a time-varying wireless communication network and all transmitting information to a receiver. Let the communication graph be $\mathcal{G}^{t}=\left(\mathcal{N}, \mathcal{E}^{t}\right)$ at time $t>0$. Each node $i \in \mathcal{N}$ transmits information over a different channel with bandwidth $b_{i}$ (given) with a signal transmission power $s_{i} \in\left[0, u_{i}\right]$ Watts and the signal is exposed to Gaussian White (uncorrelated) noise of additive nature, with power $w_{i}$ Watts (given). According to Shannon-Hartley equation the maximum capacity of the channel associated with node $i \in \mathcal{N}$ is $b_{i} \log _{2}\left(1+s_{i} / w_{i}\right)$. Suppose we want to minimize the total power of the system subject to certain capacity requirement $\delta>0$, i.e., $\min \left\{\sum_{i \in \mathcal{N}} s_{i}: \sum_{i \in \mathcal{N}} b_{i} \log _{2}\left(1+s_{i} / w_{i}\right) \geq \delta, 0 \leq s_{i} \leq u_{i}\right\}$.

For numerical experiments, we consider the particular setup described in [28]:

$$
\begin{equation*}
\min _{\boldsymbol{\xi}=\left[\xi_{i}\right]_{i \in \mathcal{N}}} \sum_{i \in \mathcal{N}} c_{i} \xi_{i} \quad \text { s.t. } \quad \sum_{i \in \mathcal{N}} b_{i} \log \left(1+\xi_{i}\right) \geq \delta, \quad \boldsymbol{\xi} \in[0,1]^{|\mathcal{N}|} . \tag{7.3}
\end{equation*}
$$

where $\mathbf{c}=\left[c_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$ and $\mathbf{b}=\left[b_{i}\right]_{i \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{N}|}$ are chosen uniformly at random between 0 and 1 . We consider both static and dynamic networks; dynamic ones are generated as in Section 7.1 .1 with $|\mathcal{N}|=50$ nodes and $\left|\mathcal{E}_{u}\right|=150$ edges, and the static one is set to $\mathcal{G}=\left(\mathcal{N}, \mathcal{E}_{u}\right)$. In the experiments we set $\delta=5$. For benchmarking, we compared our algorithm against Consensus-based Saddle-Point Subgradient (CoBa-

SPS) ${ }^{3}$ [28] and Mirror-prox [23] - the former one is a decentralized algorithm while the latter one is a centralized algorithm. Mirror-prox algorithm requires the global Lipschitz constant of $\nabla \mathcal{L}$, where $\mathcal{L}(\boldsymbol{\xi}, \mathbf{y})=\mathbf{c}^{\top} \boldsymbol{\xi}+\left\langle\sum_{i \in \mathcal{N}} b_{i} \log \left(1+\xi_{i}\right)-\delta, \mathbf{y}\right\rangle$, for $\boldsymbol{\xi} \in[0,1]^{|\mathcal{N}|}$, which is $\sqrt{2}\|\mathbf{b}\|$. Mirror-prox and CoBa-SPS, also, require a bound on the dual solutions. Similar to [28], for the Slater Point $\overline{\boldsymbol{\xi}}=\mathbf{1}_{|\mathcal{N}|}$, we have that $\left\|y^{*}\right\| \leq$ $\frac{N \max _{i \in \mathcal{N}} c_{i}}{\log (2)\left(\sum_{i \in \mathcal{N}} b_{i}\right)-\delta}$. We compare DPDA-S against CoBa-SPS and Mirror-prox. Since CoBa-SPS can only handle static network, when the network topology is time-varying, we compare DPDA-D only against Mirror-prox, where we set $q_{k}=10 \log (k+1)$ within DPDA-D. We choose our step-sizes according to (3.1) where $L_{f_{i}}=0, L_{g_{i}}=C_{g_{i}}=b_{i}$, and we set $\gamma=1 /|\mathcal{N}|$. Fig. 7.7 shows the performance of DPDA-D in terms of suboptimality and infeasibility as well as consensus violation. The performance of our method is comparable with the centralized Mirror-prox and slightly slower rate of DPDA-D is the price we pay for the decentralized setting. Fig. 7.8 compares the performance of DPDA-S against CoBa-SPS and Mirro-prox. Although CoBa-SPS finds a feasible solution and remains feasible, the iterates are far from optimality; and both suboptimality and consensus violations decrease with a slow rate. DPDA-S has a superior performance compared to CoBa-SPS. The discontinuity within infeasibility plots (middle figures) is due to achieving occasional feasibility when both primal and dual iterates are approaching their optimal solutions.


Fig. 7.7. Comparison between DPDA-D and Mirror-prox.


Fig. 7.8. Comparison among DPDA-S, Mirror-prox, and CoBa-SPS.
8. Conclusions. We propose a distributed primal-dual algorithm, DPDA-D, for solving cooperative multi-agent convex resource sharing problems over time-varying (un)directed communication networks, where only local communications are allowed. The objective is to minimize the sum of agent-specific composite convex functions subject to a conic constraint that couples agents' decisions. We show that the DPDAD iterate sequence converges to $\epsilon$-suboptimality/infeasibility within $\mathcal{O}(1 / \epsilon)$ number of iterations. To the best of our knowledge, this is the best rate result for our setting. Moreover, DPDA-D employs agent-specific constant step-sizes using local information. As a potential future work, we plan to analyze convergence rates of similar primal-dual algorithms under certain strong convexity assumptions.

[^3]
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## 9. Appendix.

Lemma 9.1. [38] Let $\left\{a^{k}\right\},\left\{b^{k}\right\},\left\{c^{k}\right\}$, and $\left\{d^{k}\right\}$ be non-negative real sequences such that $a^{k+1} \leq\left(1+d^{k}\right) a^{k}-b^{k}+c^{k}$ for all $k \geq 0, \sum_{k=0}^{\infty} c^{k}<\infty$, and $\sum_{k=0}^{\infty} d^{k}<\infty$. Then $a=\lim _{k \rightarrow \infty} a^{k}$ exists, and $\sum_{k=0}^{\infty} b^{k}<\infty$.

Lemma 9.2. Assume that $\left\{u_{k}\right\}_{k=0}^{K} \subset \mathbb{R}_{+}$satisfies $u_{0}^{2} \leq S_{0}$ and $u_{k}^{2} \leq S_{k}+$ $\sum_{i=1}^{k} \lambda_{i} u_{i}$ for all $k \in\{1, \ldots, K\}$ for some $\left\{S_{k}\right\}_{k=0}^{K}$ non-decreasing in $k$ and $\left\{\lambda_{k}\right\}_{k=1}^{K} \subset$ $\mathbb{R}_{+}$. Then, the following inequality holds for all $k \in\{1, \ldots, K\}$ :

$$
u_{k} \leq \frac{1}{2} \sum_{i=1}^{k} \lambda_{i}+\left(S_{k}+\left(\frac{1}{2} \sum_{i=1}^{k} \lambda_{i}\right)^{2}\right)^{1 / 2}
$$


[^0]:    ${ }^{*}$ In our prior work [1], published in the Proceedings of the $50^{t h}$ Asilomar Conference on Signals, Systems and Computer, we studied the resource sharing problem over static and undirected communication networks - see Section 1.2 for details.

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[^1]:    ${ }^{1} \mathrm{~A}$ directed graph $\mathcal{G}$ is balanced when each node has equal number of in-degree and out-degree.

[^2]:    ${ }^{2}$ For simple bounds, we assume $N>1 / \bar{R}_{x}^{2}$ and $\sqrt{N d_{\max }} \geq \bar{C}_{g} / \bar{R}_{x}^{2}$.

[^3]:    ${ }^{3}$ The code is available online and it is used to implement on problem (7.3).

