

# On the largest critical value of $T_n^{(k)}$

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## Abstract

We study the quantity

$$\tau_{n,k} := \frac{|T_n^{(k)}(\omega_{n,k})|}{T_n^{(k)}(1)},$$

where  $T_n$  is the Chebyshev polynomial of degree  $n$ , and  $\omega_{n,k}$  is the rightmost zero of  $T_n^{(k+1)}$ .

Since the absolute values of the local maxima of  $T_n^{(k)}$  increase monotonically towards the end-points of  $[-1, 1]$ , the value  $\tau_{n,k}$  shows how small is the largest critical value of  $T_n^{(k)}$  relative to its global maximum  $T_n^{(k)}(1)$ .

In this paper, we improve and extend earlier estimates by Erdős–Szegő, Eriksson and Nikolov in several directions.

Firstly, we show that the sequence  $\{\tau_{n,k}\}_{n=k+2}^\infty$  is monotonically decreasing in  $n$ , hence derive several sharp estimates, in particular

$$\tau_{n,k} \leq \begin{cases} \tau_{k+4,k} = \frac{1}{2k+1} \frac{3}{k+3}, & n \geq k+4, \\ \tau_{k+6,k} = \frac{1}{2k+1} \left(\frac{5}{k+5}\right)^2 \beta_k, & n \geq k+6, \end{cases}$$

where  $\beta_k < \frac{2+\sqrt{10}}{5} \approx 1.032$ .

We also obtain an upper bound which is uniform in  $n$  and  $k$ , and that implies in particular

$$\tau_{n,k} \approx \left(\frac{2}{e}\right)^k, \quad n \geq k^{3/2}; \quad \tau_{n,n-m} \approx \left(\frac{em}{2}\right)^{m/2} n^{-m/2}; \quad \tau_{n,n/2} \approx \left(\frac{4}{\sqrt{27}}\right)^{n/2}.$$

Finally, we derive the exact asymptotic formulae for the quantities

$$\tau_k^* := \lim_{n \rightarrow \infty} \tau_{n,k} \quad \text{and} \quad \tau_m^{**} := \lim_{n \rightarrow \infty} n^{m/2} \tau_{n,n-m},$$

which show that our upper bounds for  $\tau_{n,k}$  and  $\tau_{n,n-m}$  are asymptotically correct with respect to the exponential terms given above.

## 1 Introduction and statement of the results

We study the quantity

$$\tau_{n,k} := \frac{|T_n^{(k)}(\omega_k)|}{T_n^{(k)}(1)},$$

where  $T_n$  is the Chebyshev polynomial of degree  $n$ , and  $\omega_k$  is the rightmost zero of  $T_n^{(k+1)}$ .

Since the absolute values of the local maxima of  $T_n^{(k)}$  increase monotonically towards the end-points of  $[-1, 1]$ , the value  $\tau_{n,k}$  shows how small is the largest critical value of  $T_n^{(k)}$  relative to its global maximum  $T_n^{(k)}(1)$  (see Figure 1).

This value is useful in several applications which include some Markov-type inequalities [7], [14], [18], the Landau–Kolmogorov inequalities for intermediate derivatives [8], [18], where the

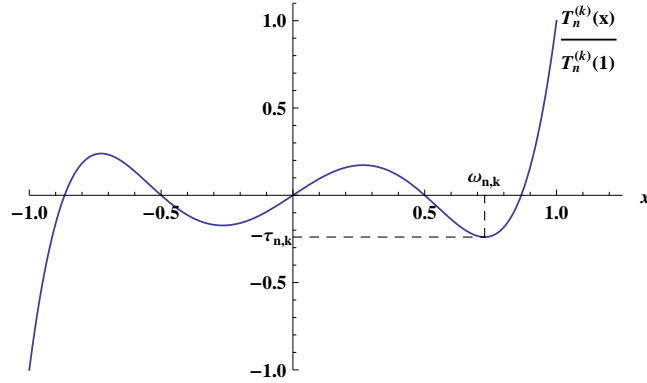


Figure 1: The last relative extremum  $\tau_{n,k}$  (here,  $k = 1$ ,  $n = 6$ ).

estimates of  $f^{(k)}$  on a subinterval slightly smaller than  $[-1, 1]$  are needed, and also in studying extreme zeros of ultraspherical polynomials.

Let us mention previous results. For the first derivative ( $k = 1$ ), Erdős–Szegő [7] showed that  $\tau_{3,1} = \frac{1}{3}$ ,  $\tau_{4,1} = \frac{1}{3}(\frac{2}{3})^{1/2}$  and proved that

$$\tau_{n,1} \leq \frac{1}{4}, \quad n \geq 5. \quad (1.1)$$

For arbitrary  $k \geq 1$ , Eriksson [8] and Nikolov [14] independently showed that

$$\tau_{n,k} \leq \frac{1}{2k+1}, \quad n \geq k+2, \quad (1.2)$$

with a better estimate when  $n$  is large relative to  $k$ ,

$$\tau_{n,k} \leq \frac{1}{2k+1} \frac{8}{2k+7}, \quad n \gtrsim k^{3/2}. \quad (1.3)$$

(The exact condition in [8], [14] was  $\omega_{n,k} \geq 1 - \frac{8}{2k+7}$  which implies the above inequality between  $n$  and  $k$  via the upper estimate  $\omega_{n,k} < 1 - \frac{k^2}{n^2}$ .)

In this paper, motivated by the applications mentioned above, we refine and extend inequalities (1.1)–(1.3) in several directions.

1) Our first observation is a monotone behaviour of the value  $\tau_{n,k}$  with respect to  $n$ .

**Theorem 1.1** *For a fixed  $k \in \mathbb{N}$ , the values  $\tau_{n,k}$  decrease monotonically in  $n$ , i.e.,*

$$\tau_{n+1,k} < \tau_{n,k} < \cdots < \tau_{k+3,k} < \tau_{k+2,k}. \quad (1.4)$$

*In particular, for any fixed  $k \in \mathbb{N}$  and any  $m \geq 2$ , we have*

$$\tau_{n,k} \leq \tau_{k+m,k}, \quad n \geq k+m. \quad (1.5)$$

In fact, such a monotone decrease of the relative values of the local extrema takes place for the ultraspherical polynomials  $P_n^{(\lambda)}$  with any parameter  $\lambda > 0$ . This remarkable result is due to Szász [19] and, for reader's convenience and to keep the paper self-contained, we state it as Theorem 2.1 and give a short proof.

2) Our next result is several sharp estimates for  $\tau_{n,k}$  which follow from (1.5). Namely, since  $T_{k+m}^{(k)}$  is a symmetric polynomial of degree  $m$ , for small  $m = 2..6$  we compute the value of its largest extremum, hence  $\tau_{k+m,k}$ , explicitly and then use (1.5).

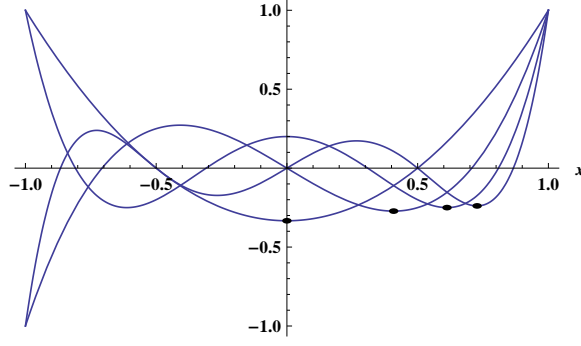


Figure 2: The last relative extrema  $\tau_{n,1}$ ,  $3 \leq n \leq 6$ .

**Theorem 1.2** *We have*

$$\tau_{n,k} \leq \begin{cases} \tau_{k+2,k} = \frac{1}{2k+1}, & n \geq k+2, \\ \tau_{k+3,k} = \frac{1}{2k+1} \left(\frac{2}{k+2}\right)^{1/2}, & n \geq k+3, \\ \tau_{k+4,k} = \frac{1}{2k+1} \frac{3}{k+3}, & n \geq k+4. \end{cases} \quad (1.6)$$

These estimates contain earlier results (1.1)-(1.2) as particular cases, and the last inequality in (1.6) improves (1.3) by the factor of  $\frac{3}{4}$  and removes the unnecessary restrictions on  $n$  and  $k$ .

The next pair of estimates strengthens (1.6). It also shows that, although the nice pattern for  $\tau_{k+m,k}$  in (1.6) is no longer true for  $m \geq 5$ , an approximate behaviour  $\tau_{k+m,k} \approx \left(\frac{m}{k+m}\right)^{m/2}$  is very much suggestive.

**Theorem 1.3** *We have*

$$\tau_{n,k} \leq \begin{cases} \tau_{k+5,k} = \frac{1}{2k+1} \left(\frac{4}{k+4}\right)^{3/2} \alpha_k, & n \geq k+5, \\ \tau_{k+6,k} = \frac{1}{2k+1} \left(\frac{5}{k+5}\right)^2 \beta_k, & n \geq k+6, \end{cases} \quad (1.7)$$

where the values  $\alpha_k, \beta_k$  increase monotonically to the following limits,

$$\alpha_k < \alpha_* = \frac{\sqrt{3(3+\sqrt{6})}}{4} = 1.0108\dots, \quad \beta_k < \beta_* = \frac{2+\sqrt{10}}{5} = 1.0325\dots$$

Let us note that, because of monotonicity of  $\tau_{n,k}$ , for any fixed moderate  $k$  and a moderate  $n_0$ , one can compute numerically the value  $\tau_{n_0,k}$ , thus getting for particular  $k$  the estimate

$$\tau_{n,k} \leq \tau_{n_0,k}, \quad n \geq n_0.$$

which would be better than those in (1.6) and (1.7).

3) An approximate behaviour  $\tau_{k+m,k} \approx \left(\frac{m}{k+m}\right)^{m/2}$  in (1.6)-(1.7) suggests that when  $m$  is fixed and  $k$  grows, then  $\tau_{n,n-m} = \tau_{k+m,k}$  is of a polynomial decay in  $n$ , i.e.,

$$\tau_{n,n-m} = \mathcal{O}(n^{-m/2}) \quad (n \rightarrow \infty),$$

while when  $k$  is fixed and  $n$  grows, we have an exponential estimate in  $k$ ,

$$\tau_{n,k} = \mathcal{O}(e^{-\gamma k}) \quad (n \rightarrow \infty).$$

We prove that such a behaviour is indeed the case by establishing first the upper bounds for  $\tau_{n,k}$  which are uniform in  $n$  and  $k$ , and then considering different relations between  $n$  and  $k$ .

**Theorem 1.4** For every  $n, k \in \mathbb{N}$  with  $n \geq k + 2$ , we have

$$\tau_{n,k}^2 \leq \frac{1}{2} \left(1 + \frac{k}{n}\right) \left(\frac{n}{k}\right)^{2k} \left(\frac{n+k}{n-k}\right)^{-1} \quad (1.8)$$

$$\leq c_1^2 k^{\frac{1}{2}} \left(1 - \frac{k^2}{n^2}\right)^{\frac{1}{2}} \frac{(2n)^{2k} (n-k)^{n-k}}{(n+k)^{n+k}}, \quad c_1^2 = \frac{e^2}{2\sqrt{\pi}}. \quad (1.9)$$

As a consequence of Theorem 1.4 we obtain the following statement.

**Theorem 1.5** We have the following estimates:

(i) if  $k \in \mathbb{N}$  is fixed and  $n$  grows, then

$$\tau_{n,k} \leq c_1 \left(\frac{2}{e}\right)^k \frac{k^{1/4}}{\left(1 - \frac{k^2}{n^2}\right)^{k/2}}, \quad (1.10)$$

in particular

$$\tau_{n,k} < c_2 \left(\frac{2}{e}\right)^k k^{1/4}, \quad n \geq k^{3/2}; \quad (1.11)$$

(ii) if  $n - k = m \in \mathbb{N}$  is fixed and  $n$  grows, then

$$\tau_{n,n-m} \leq c_3 m^{1/4} \left(\frac{me}{2}\right)^{m/2} n^{-m/2}; \quad (1.12)$$

(iii) if  $k = \lfloor \lambda n \rfloor \in \mathbb{N}$ , where  $\lambda \in (0, 1)$ , and  $n$  grows, then we have an exponential decay

$$\tau_{n,\lambda n} \leq c_4 n^{1/4} \rho_\lambda^{n/2}, \quad \rho_\lambda < 1,$$

in particular

$$\tau_{n,n/2} < c_1 n^{1/4} \left(\frac{4}{\sqrt{27}}\right)^{n/2}.$$

We can reformulate Theorem 1.5 in the form which shows, for a fixed  $k$  and growin  $n$ , the rate of decrease of the values  $\tau_{n,k}$  in (1.4).

**Corollary 1.6** We have

$$\tau_{n,k} \lesssim \begin{cases} k^{-m/2}, & n \geq k + m, \\ \left(\frac{4}{\sqrt{27}}\right)^k, & n \geq 2k, \\ \left(\frac{2}{e}\right)^k, & n \geq k^{3/2}. \end{cases} \quad (1.13)$$

**Remark 1.7** Exponential estimate (1.11) becomes superior to the polynomial estimates (1.7) only when  $k \geq 10$ .

4) Finally, we establish the asymptotics of the values of  $\lim_{n \rightarrow \infty} \tau_{n,k}$  and  $\lim_{n \rightarrow \infty} \tau_{n,n-m}$  which shows that the upper bounds in (1.11) and (1.12) are asymptotically correct with respect to the exponential terms therein.

**Theorem 1.8** We have

$$\tau_k^* := \lim_{n \rightarrow \infty} \tau_{n,k} = C_0 \left(\frac{2}{e}\right)^k e^{-a_0 k^{1/3}} k^{-1/6} (1 + \mathcal{O}(k^{-1/3})), \quad (1.14)$$

$$\tau_m^{**} := \lim_{n \rightarrow \infty} n^{m/2} \tau_{n,n-m} = C_1 \left(\frac{em}{2}\right)^{m/2} e^{-a_1 m^{1/3}} m^{-1/6} (1 + \mathcal{O}(m^{-1/3})), \quad (1.15)$$

where the pairs of constants

$$a_0 = 1.8557..., \quad C_0 = 1.1966.. \quad \text{and} \quad a_1 = 2.3381..., \quad C_1 = 1.0660...$$

can be explicitly represented in terms of the Airy function.

The rest of the paper is organised as follows. In Section 2, we present a proof of Theorem 1.1, which is deduced from a more general statement, Theorem 2.1, about monotonicity of the relative extrema of ultraspherical polynomials. In Section 3, we compute directly  $\tau_{k+m,k}$  for small  $m$ , thus proving Theorems 1.2-1.3. In Section 4, we adopt the majorant, originally introduced by Shaeffer and Duffin [16] for their alternative proof of the Markov inequality, and prove then Theorem 1.4. Theorem 1.5 is proved in Section 5. The proof of Theorem 1.8, which relies on some known asymptotic behaviour of orthogonal polynomials, is given in Section 6.

## 2 Monotonicity of the sequence $\{\tau_{n,k}\}_{n \geq k+2}$

Here, we prove that

$$\mu_{i,n}^{(\lambda)} = \frac{P_n^{(\lambda)}(y_{i,n}^{(\lambda)})}{P_n^{(\lambda)}(1)},$$

the relative values of the ordered local extrema of the ultraspherical polynomials  $P_n^{(\lambda)}$  with parameter  $\lambda$  decay monotonically with respect to  $n$  for any  $\lambda > 0$ . This includes Theorem 1.1 as a particular case since  $T_n^{(k)}$ , the  $k$ -th derivative of the Chebyshev polynomials of degree  $n$ , coincide up to a factor with  $P_{n-k}^{(\lambda)}$ , where  $\lambda = k$ .

We start with recalling some known facts about the ultraspherical polynomials (for more details, see [20, Chapter 4.7]).

For  $\lambda > -\frac{1}{2}$ ,  $\{P_n^{(\lambda)}\}_{n \in \mathbb{N}_0}$  stands for the sequence of ultraspherical polynomials, which are orthogonal on  $[-1, 1]$  with respect to the weight function  $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$ , with the standard normalization

$$P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}, \quad \lambda \neq 0.$$

The Chebyshev polynomials of the first and the second kind and the Legendre polynomials are particular cases of ultraspherical polynomials, they correspond up to a factor to the values  $\lambda = 0, 1$  and  $\frac{1}{2}$ , respectively. Moreover, due to the properties

$$\begin{aligned} T'_n(x) &= n P_{n-1}^{(1)}(x), \\ \frac{d}{dx} P_n^{(\lambda)}(x) &= 2\lambda P_{n-1}^{(\lambda+1)}(x), \quad \lambda \neq 0, \end{aligned}$$

the derivatives of the Chebyshev polynomials are ultraspherical polynomials, too,

$$T_n^{(k)}(x) = c_{n,\lambda} P_{n-k}^{(\lambda)}(x), \quad \lambda = k, \quad k = 1, \dots, n. \quad (2.1)$$

We will work with the re-normalised ultraspherical polynomials

$$p_n^{(\lambda)}(x) := P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1), \quad (2.2)$$

so that  $p_n^{(\lambda)}(1) = 1$ . It is clear that the absolute values of the local extrema of  $p_n^{(\lambda)}$  are equal to the relative values of the local extrema of  $P_n^{(\lambda)}$  compared to  $P_n^{(\lambda)}(1)$ .

Theorem 1.1 is a consequence of the following general statement.

**Theorem 2.1** *Let  $y_{1,n}^{(\lambda)} > y_{2,n}^{(\lambda)} > \dots > y_{n-1,n}^{(\lambda)}$  be the zeros of the ultraspherical polynomial  $p_{n-1}^{(\lambda+1)}$ , i.e., the abscissae of the local extrema of  $p_n^{(\lambda)}$ , in the reverse order. Set  $y_{n,n}^{(\lambda)} := -1$ , and denote*

$$\mu_{i,n}^{(\lambda)} := |p_n^{(\lambda)}(y_{i,n}^{(\lambda)})|, \quad i = 1, \dots, n.$$

1) If  $\lambda > 0$ , then

$$\mu_{i,n+1}^{(\lambda)} < \mu_{i,n}^{(\lambda)} \quad \text{for } i = 1, 2, \dots, n. \quad (2.3)$$

2) If  $-\frac{1}{2} < \lambda < 0$ , then inequalities (2.3) hold with the opposite sign.

3) (If  $\lambda = 0$ , then  $p_n^{(\lambda)} = T_n$  and we have equalities in (2.3) as all local extrema of  $T_n$  and  $T_{n+1}$  are of the absolute value 1.)

**Proof.** We omit index  $\lambda$ , so set  $p_n := p_n^{(\lambda)}$ , and we will use the next two identities which readily follow from [20, eqn.(4.7.28)]:

$$\begin{aligned} p_n(x) &= -\frac{1}{n+2\lambda} x p'_n(x) + \frac{1}{n+1} p'_{n+1}(x), \\ p_{n+1}(x) &= -\frac{1}{n+2\lambda} p'_n(x) + \frac{1}{n+1} x p'_{n+1}(x). \end{aligned}$$

From those we deduce that

$$p_n(x)^2 - p_{n+1}(x)^2 = (1-x^2) \left[ \frac{1}{(n+1)^2} p'_{n+1}(x)^2 - \frac{1}{(n+2\lambda)^2} p'_n(x)^2 \right], \quad (2.4)$$

and we rearrange his equality as follows,

$$f(x) := p_n(x)^2 + \frac{1-x^2}{(n+2\lambda)^2} p'_n(x)^2 \stackrel{(2.4)}{=} p_{n+1}(x)^2 + \frac{1-x^2}{(n+1)^2} p'_{n+1}(x)^2. \quad (2.5)$$

Clearly,  $f$  is a polynomial of degree  $2n$  which interpolates both  $p_n^2$  and  $p_{n+1}^2$  at the points of their local maxima in  $[-1, 1]$ . Moreover,  $f'$  vanishes at the zeros of both  $p'_n$  and  $p'_{n+1}$ , therefore, with some constant  $c_n^{(\lambda)}$ ,

$$f'(x) = c_n^{(\lambda)} p'_n(x) p'_{n+1}(x). \quad (2.6)$$

Next, we determine the sign of  $c_n^{(\lambda)}$ . Let  $a_n, a_{n+1}$  be the leading coefficients of  $p_n$  and  $p_{n+1}$ , respectively, and note that, since  $p_n(1) = p_{n+1}(1) = 1$ , we have  $a_n, a_{n+1} > 0$ . Then, equating the leading coefficients of  $f'$  in representations (2.5) and (2.6), respectively, we obtain

$$2na_n^2 \left( 1 - \frac{n^2}{(n+2\lambda)^2} \right) = c_n^{(\lambda)} n(n+1) a_n a_{n+1},$$

whence

$$c_n^{(\lambda)} = \frac{1}{n+1} \frac{a_n}{a_{n+1}} \frac{4\lambda(2n+2\lambda)}{(n+2\lambda)^2} \Rightarrow \text{sign } c_n^{(\lambda)} = \text{sign } \lambda$$

Thus, (2.6) becomes

$$f'(x) = c \lambda p'_n(x) p'_{n+1}(x), \quad c = c_{n,\lambda} > 0. \quad (2.7)$$

Now we can prove Theorem 2.1. Let  $\lambda > 0$ . Then from (2.7) and the interlacing of zeros of  $p'_n$  and  $p'_{n+1}$  we conclude that

$$f'(x) < 0, \quad x \in (y_{i,n}^{(\lambda)}, y_{i,n+1}^{(\lambda)}), \quad i = 1, \dots, n,$$

i.e.,  $f$  is monotonically decreasing on each interval  $(y_{i,n}^{(\lambda)}, y_{i,n+1}^{(\lambda)})$ . From (2.5), we have

$$\begin{aligned} f(y_{i,n}^{(\lambda)}) &= p_n(y_{i,n}^{(\lambda)})^2 = |\mu_{i,n}^{(\lambda)}|^2, \\ f(y_{i,n+1}^{(\lambda)}) &= p_{n+1}(y_{i,n+1}^{(\lambda)})^2 = |\mu_{i,n+1}^{(\lambda)}|^2, \end{aligned}$$

therefore

$$|\mu_{i,n+1}^{(\lambda)}| < |\mu_{i,n}^{(\lambda)}|, \quad i = 1, \dots, n.$$

Clearly, if  $\lambda < 0$ , then the sign is reversed. □

**Proof of Theorem 1.1.** By (2.1) and (2.2), we have

$$\frac{T_n^{(k)}(x)}{T_n^{(k)}(1)} = \frac{P_{n-k}^{(\lambda)}(x)}{P_{n-k}^{(\lambda)}(1)} = p_{n-k}^{(\lambda)}(x), \quad \lambda = k \geq 1.$$

Hence,  $\tau_{n,k} = \mu_{1,n-k}^{(\lambda)}$ , and then Theorem 2.1 yields

$$\tau_{n,k} = \mu_{1,n-k}^{(\lambda)} > \mu_{1,n+1-k}^{(k)} = \tau_{n+1,k}.$$

Theorem 1.1 is proved.  $\square$

### 3 Proof of Theorems 1.2-1.3

By Theorem 1.1, for any fixed  $m$ , the value  $\tau_{k+m,k}$  gives an upper bound for all  $\tau_{n,k}$ , namely

$$\tau_{n,k} \leq \tau_{k+m,k}, \quad n \geq k + m,$$

so here we determine the latter values directly for  $m = 2..6$ .

We will need the expansion formula for the  $n$ -th Chebyshev polynomial,

$$\begin{aligned} T_n(x) &= \frac{n}{2} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{(n-i-1)!}{i!(n-2i)!} (2x)^{n-2i} \\ &= 2^{n-1} x^n - 2^{n-3} n x^{n-2} \end{aligned} \quad (3.1)$$

$$+ 2^{n-6} n(n-3) x^{n-4} - \frac{1}{3} 2^{n-8} n(n-4)(n-5) x^{n-6} + \dots \quad (3.2)$$

From this we compute expression for  $T_n^{(n-6)}$  in (3.7), and then differentiate it to find all further derivatives  $T_n^{(n-m)}$  for  $m = 5..2$ .

We will denote the point of the rightmost local extrema of  $T_n^{(k)}$  by  $x_*$ , i.e.,  $x_* := \omega_{n,k}$ . Since  $T_n^{(k+1)}(x_*) = 0$  then, for the value of  $T_n^{(k)}(x_*)$  we will also use simplifications arising from the formula

$$T_n^{(k)}(x_*) = T_n^{(k)}(x_*) - c_{n,k} x_* T_n^{(k+1)}(x_*)$$

where we choose the constant  $c_{n,k}$  to cancel high degree monomials.

**1) The case  $k = n - 2$  (or equivalently  $n = k + 2$ ).** We have

$$T_n^{(n-2)}(x) = c^{-1} [2(n-1)x^2 - 1], \quad (3.3)$$

whence  $x_* = 0$  and

$$\tau_{n,n-2} = \frac{|T_n^{(n-2)}(x_*)|}{T_n^{(n-2)}(1)} = \frac{1}{2n-3} \Rightarrow \tau_{k+2,k} = \frac{1}{2k+1}.$$

**2) The case  $k = n - 3$  (or equivalently  $n = k + 3$ ).** We obtain

$$T_n^{(n-3)}(x) = c^{-1} [2(n-1)x^3 - 3x], \quad (3.4)$$

hence  $c T_n^{(n-3)}(1) = 2n - 5$ . From (3.3), we find  $x_*^2 = \frac{1}{2(n-1)}$  and

$$c T_n^{(n-3)}(x_*) = -2x_* = -\frac{2}{\sqrt{2(n-1)}}.$$

Respectively,

$$\tau_{n,n-3} = \frac{|T_n^{(n-3)}(x_*)|}{T_n^{(n-3)}(1)} = \frac{1}{2n-5} \sqrt{\frac{2}{n-1}} \Rightarrow \tau_{k+3,k} = \frac{1}{2k+1} \sqrt{\frac{2}{k+2}}.$$

**3) The case  $k = n - 4$  (or equivalently  $n = k + 4$ ). We have**

$$T_n^{(n-4)}(x) = c^{-1} [4(n-1)(n-2)x^4 - 12(n-2)x^2 + 3]. \quad (3.5)$$

hence

$$c T_n^{(n-4)}(1) = 4n^2 - 24n + 35 = (2n-5)(2n-7).$$

From (3.4), we find  $x_*^2 = \frac{3}{2(n-1)}$  and

$$c T_n^{(n-4)}(x_*) = -6(n-2)x_*^2 + 3 = \frac{3}{n-1} [-3(n-2) + (n-1)] = -\frac{3(2n-5)}{n-1}.$$

Respectively,

$$\tau_{n,n-4} = \frac{|T_n^{(n-4)}(x_*)|}{T_n^{(n-4)}(1)} = \frac{1}{2n-7} \frac{3}{n-1} \Rightarrow \tau_{k+4,k} = \frac{1}{2k+1} \frac{3}{k+3}.$$

The cases 1)-3) prove estimates (1.6), hence Theorem 1.2.  $\square$

**4) The case  $k = n - 5$  (or equivalently  $n = k + 5$ ). We have**

$$T_n^{(n-5)}(x) = c^{-1} [4(n-1)(n-2)x^5 - 20(n-2)x^3 + 15x], \quad (3.6)$$

hence

$$c T_n^{(n-5)}(1) = 4n^2 - 32n + 63 = (2n-9)(2n-7).$$

From (3.5), we find

$$\begin{aligned} x_*^2 &= \frac{3(n-2) + \sqrt{9(n-2)^2 - 3(n-1)(n-2)}}{2(n-1)(n-2)} \\ &= \frac{1}{n-1} \frac{3 + \sqrt{6-t}}{2}, \quad t := t_k = \frac{3}{n-2} = \frac{3}{k+3}. \end{aligned}$$

and

$$c T_n^{(n-5)}(x_*) = -4x_* [2(n-2)x_*^2 - 3].$$

After simplifications we obtain

$$\tau_{n,n-5} = \frac{|T_n^{(n-5)}(x_*)|}{T_n^{(n-5)}(1)} = \frac{1}{2n-9} \frac{8}{(n-1)^{3/2}} \alpha_k \Rightarrow \tau_{k+5,k} = \frac{1}{2k+1} \frac{4^{3/2}}{(k+4)^{3/2}} \alpha_k,$$

where

$$\alpha_k := \frac{1}{2\sqrt{2}} \frac{\sqrt{3 + \sqrt{6-t}}}{2-t} (\sqrt{6-t} - t) = \frac{1}{2\sqrt{2}} \frac{(y+3)^{3/2}}{y+2} =: f(y), \quad y := \sqrt{6-t}.$$

The function  $f$  is increasing for  $y > 0$ , hence

$$t_k > t_{k+1} \Rightarrow y_k < y_{k+1} \Rightarrow \alpha_k < \alpha_{k+1} < \alpha_*$$



where

$$\alpha_* = \lim_{t \rightarrow 0} \alpha_k = \frac{1}{2\sqrt{2}} \frac{\sqrt{3+\sqrt{6}}}{2} \sqrt{6} = \frac{\sqrt{3(3+\sqrt{6})}}{4}.$$

**5) The case  $k = n - 6$  (or equivalently  $n = k + 6$ ).** From (3.2), we have

$$T_n^{(n-6)}(x) = c^{-1} \left[ 8(n-1)(n-2)(n-3)x^6 - 60(n-2)(n-3)x^4 + 90(n-3)x^2 - 15 \right], \quad (3.7)$$

hence

$$cT_n^{(n-6)}(1) = 8n^3 - 108n^2 + 478n - 693 = (2n-11)(2n-9)(2n-7).$$

From (3.6), we find

$$\begin{aligned} x_*^2 &= \frac{5(n-2) + \sqrt{25(n-2)^2 - 15(n-1)(n-2)}}{2(n-1)(n-2)} \\ &= \frac{1}{n-1} \frac{5 + \sqrt{10-3t}}{2}, \quad t := t_k = \frac{5}{n-2} = \frac{5}{k+4} \end{aligned}$$

and

$$cT_n^{(n-6)}(x_*) = -4(2n-7)x_*^2 \left[ 2(n-2)x_*^2 - 5 \right].$$

After simplifications, we obtain

$$\tau_{n,n-6} = \frac{|T_n^{(n-6)}(x_*)|}{T_n^{(n-6)}(1)} = \frac{1}{2n-11} \frac{5^2}{(n-1)^2} \beta_k \Rightarrow \tau_{k+6,k} = \frac{1}{2k+1} \frac{5^2}{(k+5)^2} \beta_k,$$

where

$$\beta_k := \frac{2}{5^2} \frac{5 + \sqrt{10-3t}}{2-t} \left( \sqrt{10-3t} - t \right) = \frac{2}{5^2} \frac{(y+5)^2}{y+2} =: g(y), \quad y := \sqrt{10-3t}.$$

The function  $g$  is increasing for  $y > 1$ , hence

$$t_k > t_{k+1} \Rightarrow y_k < y_{k+1} \Rightarrow \beta_k < \beta_{k+1} < \beta_*,$$

where

$$\beta_* = \lim_{t \rightarrow 0} \beta_k = \frac{2}{5^2} \frac{5 + \sqrt{10}}{2} \sqrt{10} = \frac{2 + \sqrt{10}}{5}.$$

The cases 4)-5) prove estimates (1.7), hence Theorem 1.3.  $\square$

## 4 Estimates based on the Duffin–Schaeffer majorant

In this section, we prove Theorem 1.4. Our proof is based on the upper bound  $\tau_{n,k} < \delta_{n,k}$  which uses the so-called Duffin–Schaeffer majorant.

**Definition 4.1** With  $T_n$  the Chebyshev polynomial of degree  $n$ , and  $S_n(x) := \frac{1}{n} \sqrt{1-x^2} T'_n(x)$ , we define the *Duffin–Schaeffer majorant*  $D_{n,k}(\cdot)$  as

$$D_{n,k}(x) := \{[T_n^{(k)}(x)]^2 + [S_n^{(k)}(x)]^2\}^{1/2}, \quad x \in (-1, 1). \quad (4.1)$$

This majorant was introduced by Shaeffer–Duffin [16] who proved that, if  $p$  is a polynomial of degree not exceeding  $n$ , then

$$\|p\| \leq 1 \Rightarrow |p^{(k)}(x)| \leq D_{n,k}(x), \quad x \in (-1, 1). \quad (4.2)$$

which may be viewed as a generalization of the pointwise Bernstein inequality  $|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|$  to higher derivatives.

**Lemma 4.2** *The majorant  $D_{n,k}$  has the following properties.*

1. We have

$$|T_n^{(k)}(x)| \leq D_{n,k}(x) \quad \text{for all } x \in (-1, 1). \quad (4.3)$$

2.  $D_{n,k}(x) = |T_n^{(k)}(x)|$  at zeros of  $S_n^{(k)}$ , in particular,

$$D_{n,k}(0) = |T_n^{(k)}(0)| \quad \text{if } n - k \text{ is even.} \quad (4.4)$$

3. The majorant  $D_{n,k}(\cdot)$  is a strictly increasing function on  $[0, 1]$ .

4. We have the explicit formulae  $\frac{1}{n^2} [D_{n,1}(x)]^2 = \frac{1}{1-x^2}$  and

$$\frac{1}{n^2} [D_{n,k}(x)]^2 = \sum_{m=0}^{k-1} \frac{b_{m,n}}{(1-x^2)^{k+m}}, \quad k \geq 2, \quad (4.5)$$

where

$$b_{m,n} = c_{m,k}(n^2 - (m+1)^2) \cdots (n^2 - (k-1)^2), \quad (4.6)$$

$$c_{m,k} := \begin{cases} 1, & m = 0, \\ \binom{k-1+m}{2m} (2m-1)!!^2, & m \geq 1. \end{cases} \quad (4.7)$$

**Proof.** Claim 1 and the first half of Claim 2 follow directly from Definition 4.1. Equality (4.4) is due to the fact that  $T_n$  and  $S_n$  are of different parity, so if  $n-k$  is even, then  $T_n^{(k)}$  is an even function and  $S_n^{(k)}$  is an odd one, hence  $S_n^{(k)}(0) = 0$ . The third property was proved by Schaeffer–Duffin [16], and it also follows easily from the formulas (4.5)-(4.6) which were established by Shadrin [17].  $\square$

Here are few particular expressions for  $D_{n,k}(\cdot)$ .

$$\begin{aligned} \frac{1}{n^2} [D_{n,1}(x)]^2 &= \frac{1}{1-x^2}, \\ \frac{1}{n^2} [D_{n,2}(x)]^2 &= \frac{(n^2-1)}{(1-x^2)^2} + \frac{1}{(1-x^2)^3}, \\ \frac{1}{n^2} [D_{n,3}(x)]^2 &= \frac{(n^2-1)(n^2-4)}{(1-x^2)^3} + \frac{3(n^2-4)}{(1-x^2)^4} + \frac{9}{(1-x^2)^5}, \\ \frac{1}{n^2} [D_{n,4}(x)]^2 &= \frac{(n^2-1)(n^2-4)(n^2-9)}{(1-x^2)^4} + \frac{6(n^2-4)(n^2-9)}{(1-x^2)^5} \\ &\quad + \frac{45(n^2-9)}{(1-x^2)^6} + \frac{225}{(1-x^2)^7}. \end{aligned}$$

**Lemma 4.3** *Let  $\omega_k := \omega_{n,k}$  be the rightmost zero of  $T_n^{(k+1)}$ . Then*

$$\omega_k < x_k, \quad \text{where } x_k^2 := 1 - \frac{k^2}{n^2}. \quad (4.8)$$

**Proof.** The claim can be deduced from numerous upper bounds for the extreme zeros of ultraspherical polynomials. For instance, in [14] Nikolov proved that  $\omega_k^2 \leq \frac{n^2 - (k+2)^2}{n^2 + \alpha_{n,k}}$ , with some  $\alpha_{n,k} > 0$ , hence

$$\omega_k^2 \leq \frac{n^2 - (k+2)^2}{n^2} \leq \frac{n^2 - k^2}{n^2} = x_k^2. \quad (4.9)$$

□

From (4.3), monotonicity of  $D_{n,k}(\cdot)$  and inequality (4.8), it follows immediately that

$$|T_n^{(k)}(\omega_k)| \leq D_{n,k}(\omega_k) < D_{n,k}(x_k),$$

hence the following statement.

**Proposition 4.4** *We have*

$$\tau_{n,k} < \delta_{n,k}, \quad \delta_{n,k} := \frac{D_{n,k}(x_k)}{T_n^{(k)}(1)}.$$

We proceed with estimates of  $\delta_{n,k}$ , using the explicit expression (4.5) for  $D_{n,k}(\cdot)$ .

**Lemma 4.5** *We have*

$$\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k} B_{n,k}, \quad (4.10)$$

where

$$A_{n,k} = \frac{(2k-1)!!^2}{k^{2k}} \sum_{m=0}^{k-1} \frac{c_{m,k}}{k^{2m}} \frac{n^{2m}}{(n^2-1^2) \cdots (n^2-m^2)}, \quad (4.11)$$

$$B_{n,k} = \frac{n^{2k}}{n^2(n^2-1^2) \cdots (n^2-(k-1)^2)}. \quad (4.12)$$

**Proof.** From (4.5) – (4.6), we obtain

$$\begin{aligned} [\delta_{n,k}]^2 &= \frac{[D_{k,n}(x_k)]^2}{[T_n^{(k)}(1)]^2} \\ &= \frac{1}{[T_n^{(k)}(1)]^2} n^2 \sum_{m=0}^{k-1} \frac{c_{m,k}}{(1-x_k^2)^{k+m}} (n^2 - (m+1)^2) \cdots (n^2 - (k-1)^2) \\ &= \frac{n^2(n^2-1^2) \cdots (n^2-(k-1)^2)}{[T_n^{(k)}(1)]^2 (1-x_k^2)^k} \sum_{m=0}^{k-1} \frac{c_{m,k}}{(1-x_k^2)^m} \frac{1}{(n^2-1^2) \cdots (n^2-m^2)} \end{aligned}$$

and substitution

$$\frac{1}{[T_n^{(k)}(1)]^2} = \frac{(2k-1)!!^2}{[n^2(n^2-1^2) \cdots (n^2-(k-1)^2)]^2}, \quad 1-x_k^2 = \frac{k^2}{n^2},$$

gives (4.10) – (4.12) after a rearrangement. □

**Remark 4.6** Whereas the value  $\tau_{n,k}$  is defined only for  $n \geq k+2$ , the values of  $A_{n,k}$  and  $B_{n,k}$  in (4.11)-(4.12) are well-defined for  $n \geq k$ . We will use this fact in the next lemma where the values  $A_{k,k}$  and  $B_{k,k}$  will be considered.

**Lemma 4.7** *We have*

$$\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k} B_{n,k}, \quad (4.13)$$

where

$$A_{n,k} \leq \frac{1}{2} \frac{(2k)!}{k^{2k}}, \quad B_{n,k} = \frac{n+k}{n} \frac{n^{2k}(n-k)!}{(n+k)!}. \quad (4.14)$$

**Proof.** Expression for  $B_{n,k}$  in (4.14) is just a rearrangement of (4.12).

As to the inequality for  $A_{n,k}$  in (4.14), it is clear from (4.11) that  $A_{n,k}$  decreases when  $n$  grows, therefore

$$A_{n,k} \leq A_{k,k}, \quad n \geq k.$$

With  $n = k$ , we have  $x_k = 1 - \frac{k^2}{n^2} = 0$ , and also  $D_{k,k}(0) = T_k^{(k)}(0)$  by (4.4), therefore

$$A_{k,k}B_{k,k} = [\delta_{k,k}]^2 = \frac{[D_{k,k}(0)]^2}{[T_k^{(k)}(1)]^2} \stackrel{(4.4)}{=} \frac{[T_k^{(k)}(0)]^2}{[T_k^{(k)}(1)]^2} = 1,$$

hence,  $A_{k,k} = 1/B_{k,k}$ , and from formula (4.14), we find

$$A_{k,k} = \frac{1}{B_{k,k}} = \frac{k}{2k} \frac{(2k)!}{k^{2k}},$$

hence the result.  $\square$

**Remark 4.8** If we consider the first estimate in (4.9), namely

$$\omega_k \leq x'_k, \quad \text{where} \quad x_k'^2 = 1 - \frac{(k+2)^2}{n^2}, \quad n \geq k+2,$$

then we obtain

$$A_{n,k} \leq A'_{k+2,k} = \gamma_k^2 \frac{1}{2} \frac{(2k)!}{k^{2k}}, \quad \gamma_k^2 = \frac{(k+2)}{(2k+1)} \left( \frac{k}{k+2} \right)^{2k}$$

i.e., we can improve the estimate (4.14) (and all subsequent estimates) by the factor of  $\gamma_k$  (or  $\gamma_k^2$ ). Note that

$$\gamma_k \approx \frac{1}{\sqrt{2}} \frac{1}{e^2}.$$

Now, we prove Theorem 1.4 which is the following statement.

**Theorem 4.9** For every  $n, k \in \mathbb{N}$  with  $n \geq k+2$ , we have

$$\tau_{n,k}^2 \leq \frac{1}{2} \left( 1 + \frac{k}{n} \right) \left( \frac{n}{k} \right)^{2k} \left( \frac{n+k}{n-k} \right)^{-1} \quad (4.15)$$

$$\leq c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{\frac{1}{2}} \frac{(2n)^{2k} (n-k)^{n-k}}{(n+k)^{n+k}}, \quad c_1^2 = \frac{e^2}{2\sqrt{\pi}}. \quad (4.16)$$

**Proof.** The first part is just the estimate (4.13),

$$\tau_{n,k}^2 < \delta_{n,k}^2 = A_{n,k}B_{n,k} < \frac{1}{2} \frac{n+k}{n} \frac{n^{2k}}{k^{2k}} \frac{(n-k)!(2k)!}{(n+k)!}.$$

To prove the second inequality we use the following version of Stirling's formula

$$\sqrt{2\pi} \left( \frac{N}{e} \right)^N \sqrt{N} < N! < e \left( \frac{N}{e} \right)^N \sqrt{N}. \quad (4.17)$$

This gives

$$\begin{aligned} \frac{1}{2} \frac{n+k}{n} \frac{n^{2k}}{k^{2k}} \frac{(n-k)!(2k)!}{(n+k)!} &\leq \frac{1}{2} \frac{n+k}{n} \frac{n^{2k}}{k^{2k}} \frac{e^2}{\sqrt{2\pi}} \frac{\sqrt{n-k}\sqrt{2k}}{\sqrt{n+k}} \frac{(n-k)^{n-k}(2k)^{2k}}{(n+k)^{n+k}} \\ &= \frac{e^2}{2\sqrt{\pi}} \frac{n+k}{n} \frac{\sqrt{n-k}\sqrt{k}}{\sqrt{n+k}} \frac{(n-k)^{n-k}(2n)^{2k}}{(n+k)^{n+k}} \\ &= c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{\frac{1}{2}} \frac{(n-k)^{n-k}(2n)^{2k}}{(n+k)^{n+k}}, \end{aligned}$$

and that finishes the proof.  $\square$

## 5 Proof of Theorem 1.5

We rewrite inequality (4.16) in a more convenient form

$$\tau_{n,k} \leq c_1^2 k^{\frac{1}{2}} \left(1 - \frac{k^2}{n^2}\right)^{\frac{1}{2}} \left(\frac{2n}{n+k}\right)^{n+k} \left(\frac{n-k}{2n}\right)^{n-k}, \quad c_1^2 = \frac{e}{2\sqrt{\pi}}. \quad (5.1)$$

We will prove each part of Theorem 1.5 as a separate lemma.

**Lemma 5.1** *If  $k \in \mathbb{N}$  is fixed and  $n$  grows, then*

$$\tau_{n,k} \leq c_1 \left(\frac{2}{e}\right)^k \frac{k^{1/4}}{(1 - \frac{k^2}{n^2})^{k/4}}, \quad (5.2)$$

in particular

$$\tau_{n,k} < c_2 \left(\frac{2}{e}\right)^k k^{1/4}, \quad n \geq k^{3/2}. \quad (5.3)$$

**Proof.** We write (5.1) in the form

$$\tau_{n,k}^2 \leq c_1^2 L_1 L_2,$$

where

$$L_1 := k^{\frac{1}{2}} \left(1 - \frac{k^2}{n^2}\right)^{1/2} < k^{1/2}, \quad (5.4)$$

and

$$L_2 := \left(\frac{2n}{n+k}\right)^{n+k} \left(\frac{n-k}{2n}\right)^{n-k} = 2^{2k} \frac{(1 - \frac{k}{n})^{n-k}}{(1 + \frac{k}{n})^{n+k}}.$$

We use then the inequalities  $(1 - \frac{1}{x})^{x-1/2} < \frac{1}{e}$  and  $(1 + \frac{1}{x})^{x+1/2} > e$ , where  $x > 1$ , to derive

$$\begin{aligned} \left(1 - \frac{k}{n}\right)^{n-k} &= \left(1 - \frac{k}{n}\right)^{(\frac{n}{k} - \frac{1}{2})k} \left(1 - \frac{k}{n}\right)^{-k/2} < \frac{1}{e^k} \left(1 - \frac{k}{n}\right)^{-k/2}, \\ \left(1 + \frac{k}{n}\right)^{n+k} &= \left(1 + \frac{k}{n}\right)^{(\frac{n}{k} + \frac{1}{2})k} \left(1 + \frac{k}{n}\right)^{k/2} > e^k \left(1 + \frac{k}{n}\right)^{k/2}. \end{aligned}$$

Therefore

$$L_2 < 2^{2k} \frac{1}{e^{2k}} \frac{1}{(1 - \frac{k^2}{n^2})^{k/2}},$$

and that combined with (5.4) proves (5.2).

If  $n \geq k^{3/2}$  and  $k \geq 2$ , then  $(1 - \frac{k^2}{n^2})^{k/4} > (1 - \frac{1}{k})^{k/4} > 2^{-1/2}$ , so (5.3) is valid with  $c_2 = 2^{1/2} c_1$ . If  $k = 1$ , then  $n \geq 3$ , and  $(1 - \frac{k^2}{n^2})^{k/4} > 2^{-1/2}$  as well, and that proves (5.3) as well.  $\square$

**Lemma 5.2** *If  $n - k = m$  is fixed and  $n$  grows, then*

$$\tau_{n,n-m} \leq c_3 m^{1/4} \left(\frac{me}{2}\right)^{m/2} n^{-m/2}, \quad (5.5)$$

**Proof.** We consider the inequality (5.1)

$$\tau_{n,k}^2 \leq c_1^2 k^{\frac{1}{2}} \left(1 - \frac{k^2}{n^2}\right)^{1/2} \left(\frac{2n}{n+k}\right)^{n+k} \left(\frac{n-k}{2n}\right)^{n-k},$$

and then estimate the factors using substitution  $n - k = m$  where appropriate. We have

$$\begin{aligned} k^{\frac{1}{2}} \left(1 - \frac{k^2}{n^2}\right)^{1/2} &= (n-k)^{1/2} \left(\frac{k(n+k)}{n^2}\right)^{1/2} \leq 2^{1/2} m^{1/2}, \\ \left(\frac{2n}{n+k}\right)^{n+k} &= \left(1 + \frac{n-k}{n+k}\right)^{n+k} = \left(1 + \frac{m}{n+k}\right)^{n+k} < e^m, \\ \left(\frac{n-k}{2n}\right)^{n-k} &= \left(\frac{m}{2n}\right)^m. \end{aligned}$$

Thus, (5.5) follows with  $c_3 = 2^{1/4} c_1$ .  $\square$

**Lemma 5.3** *If  $k = \lfloor \lambda n \rfloor$ , where  $\lambda \in (0, 1)$ , then as  $n$  grows, we have an exponential decay*

$$\tau_{n,k} \leq c_4 n^{1/4} \rho_\lambda^{n/2}, \quad \rho_\lambda < 1, \quad (5.6)$$

*in particular*

$$\tau_{n,n/2} < c_1 n^{1/4} \left( \frac{4}{\sqrt{27}} \right)^{n/2}.$$

**Proof.** With  $k = \lfloor \lambda n \rfloor$ , set  $\lambda' := \frac{k}{n}$ , and note that

$$\lambda n - 1 \leq k \leq \lambda n \Rightarrow \lambda - \frac{1}{n} \leq \lambda' \leq \lambda. \quad (5.7)$$

Substitution  $k = \lambda' n$  in (5.1) gives

$$\begin{aligned} \tau_{n,k}^2 &\leq c_1^2 k^{\frac{1}{2}} \left( 1 - \frac{k^2}{n^2} \right)^{1/2} \left( \frac{2n}{n+k} \right)^{n+k} \left( \frac{n-k}{2n} \right)^{n-k} \\ &< c_1^2 n^{1/2} \rho_{\lambda'}^n, \end{aligned}$$

where

$$\rho_{\lambda'} = \left( \frac{2}{1+\lambda'} \right)^{1+\lambda'} \left( \frac{1-\lambda'}{2} \right)^{1-\lambda'} < 1, \quad \lambda' \in (0, 1).$$

On using that  $g(x) := \ln \rho_x$  satisfies  $g'(x) > -1$  for  $x \in (0, 1)$ , we derive from (5.7) that

$$\rho_{\lambda'} < e^{1/n} \rho_\lambda,$$

and that proves (5.6) with  $c_4 = e^{1/2n} c_1$ . If  $\lambda = \frac{1}{2}$  we obtain  $\rho_{1/2} = 2(\frac{1}{2})^{1/2} / (\frac{3}{2})^{3/2} = \frac{4}{\sqrt{27}}$ .  $\square$

## 6 The asymptotic formulas

In this section, we derive the asymptotic formulas (1.14)-(1.15) of Theorem 1.8.

1) We start with the asymptotic formula for

$$\tau_k^* := \lim_{n \rightarrow \infty} \tau_{n,k}.$$

For  $\alpha, \beta > -1$ , we denote by  $\{P_m^{(\alpha, \beta)}\}$  the sequence of Jacobi polynomials which are orthogonal with respect to the weight  $w_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta$ , with the standard normalization

$$P_m^{(\alpha, \beta)}(1) = \binom{m+\alpha}{m}. \quad (6.1)$$

Note that derivatives of the Chebyshev polynomials are related to Jacobi polynomials in the following way,

$$T_n^{(k)} = c_{n,k} P_m^{(\nu, \nu)}, \quad m = n - k, \quad \nu = k - \frac{1}{2}. \quad (6.2)$$

We will use the asymptotic property of Jacobi polynomials which is described in terms of Bessel functions (see [20, sect. 8.1]), namely the following equality from [21]

$$\lim_{m \rightarrow \infty} m^{-\alpha} P_m^{(\alpha, \beta)}(y_{m,r}) = \left( \frac{j_{\alpha+1,r}}{2} \right)^{-\alpha} J_\alpha(j_{\alpha+1,r}), \quad (6.3)$$

where  $y_{m,r}$  is the point of the  $r$ -th local extremum of  $P_m^{(\alpha, \beta)}$  counted in decreasing order and  $j_{\nu,r}$  is the  $r$ -th positive zero of the Bessel function  $J_\nu$ .

**Lemma 6.1** *We have*

$$\tau_k^* = \Gamma(\nu + 1) \left( \frac{j_{\nu+1,1}}{2} \right)^{-\nu} |J_\nu(j_{\nu+1,1})|, \quad \nu = k - \frac{1}{2}. \quad (6.4)$$

**Proof.** By (6.1)-(6.2), since  $\omega_{n,k} = y_{m,1}$ , we have

$$\tau_k^* = \lim_{m \rightarrow \infty} \frac{|P_m^{(\nu,\nu)}(y_{m,1})|}{P_m^{(\nu,\nu)}(1)} = \frac{\lim_{m \rightarrow \infty} m^{-\nu} |P_m^{(\nu,\nu)}(y_{m,1})|}{\lim_{m \rightarrow \infty} m^{-\nu} \binom{m+\nu}{m}} = \frac{L_1}{L_2}.$$

By (6.3),

$$L_1 = \left( \frac{j_{\nu+1,1}}{2} \right)^{-\nu} |J_\nu(j_{\nu+1,1})|,$$

while for the denominator we use

$$\binom{m+\nu}{m} = \frac{\Gamma(m+\nu+1)}{\Gamma(m+1)\Gamma(\nu+1)}, \quad \lim_{m \rightarrow \infty} m^{-\nu} \frac{\Gamma(m+\nu+1)}{\Gamma(m+1)} = 1,$$

to obtain

$$L_2 = 1/\Gamma(\nu+1)$$

and that proves the lemma.  $\square$

**Lemma 6.2 ([12])** *The first positive zero  $j_{\nu,1}$  of the Bessel function  $J_\nu$  obeys the following asymptotic expansion*

$$j_{\nu,1} = \nu + a\nu^{1/3} + \mathcal{O}(\nu^{-1/3}), \quad a = -i_1/2^{1/3} = 1.8557... \quad (6.5)$$

where  $i_1$  is the first negative zero of the Airy function  $\text{Ai}(x)$ .

**Lemma 6.3** *We have*

$$J_\nu(j_{\nu+1,1}) = -\left(\frac{2}{\nu}\right)^{2/3} \text{Ai}'(i_1) + \mathcal{O}(\nu^{-1}). \quad (6.6)$$

**Proof.** We will need the asymptotic behavior of  $J_\nu(\nu x)$  for large (fixed)  $\nu$  and  $x \geq 1$  (that is, around the first positive zero  $j_{\nu,1}$ ), which is given by the following formula (see [15, Chapter 11] or [12]),

$$J_\nu(\nu x) = \frac{\phi(z)}{\nu^{1/3}} \left[ \text{Ai}(\nu^{2/3} z) (1 + \mathcal{O}(\nu^{-2})) + \frac{\text{Ai}'(\nu^{2/3} z)}{\nu^{4/3}} (B_0(z) + \mathcal{O}(\nu^{-2})) \right], \quad (6.7)$$

where  $0 < B_0(z) \leq B_0(0)$  for  $z \leq 0$  and

$$z = -\left(\frac{3}{2}\sqrt{x^2-1} - \frac{3}{2}\sec^{-1}(x)\right)^{2/3}, \quad \phi(z) = \left(\frac{4z}{1-x^2}\right)^{1/4}, \quad x \geq 1.$$

Let  $x = 1 + \delta$ , where  $\delta = \mathcal{O}(\nu^{-2/3})$  and  $\delta > 0$ . Then

$$\sec^{-1}(x) = \arccos\left(\frac{1}{1+\delta}\right) = \sqrt{2\delta} \left(1 - \frac{5\delta}{12} + \mathcal{O}(\delta^2)\right),$$

whence we obtain for  $z$  and  $\phi(z)$

$$z = -2^{1/3}\delta \left(1 + \mathcal{O}(\delta)\right), \quad \phi(z) = 2^{1/3} + \mathcal{O}(\delta).$$

Substitution of these quantities in (6.7) yields

$$J_\nu(\nu(1+\delta)) = \left(\frac{2}{\nu}\right)^{1/3} \left( \text{Ai}(-\nu^{2/3}2^{1/3}\delta) + \mathcal{O}(\nu^{-2/3}) \right) + \mathcal{O}(\nu^{-1}) \quad (6.8)$$

From (6.5), we have

$$\begin{aligned} j_{\nu+1,1} &= \nu + 1 - \frac{i_1}{2^{1/3}}(\nu + 1)^{1/3} + \mathcal{O}(\nu^{-1/3}) \\ &= \nu(1 + \delta_0), \quad \delta_0 = -\frac{i_1}{2^{1/3}}\nu^{-2/3} + \nu^{-1} + \mathcal{O}(\nu^{-4/3}), \end{aligned}$$

so putting this into (6.8), we conclude

$$\begin{aligned} J_\nu(j_{\nu+1,1}) &= \left(\frac{2}{\nu}\right)^{1/3} \left( \text{Ai}(i_1 - (\frac{2}{\nu})^{1/3} + \mathcal{O}(\nu^{-2/3})) + \mathcal{O}(\nu^{-2/3}) \right) \\ &= \left(\frac{2}{\nu}\right)^{2/3} \text{Ai}'(i_1) + \mathcal{O}(\nu^{-1}), \end{aligned}$$

and that proves the lemma.  $\square$

**Proof of Theorem 1.8, part (1.14).** With the substitution  $\nu = k - \frac{1}{2}$ , we obtain

$$\begin{aligned} |J_{k-\frac{1}{2}}(j_{k+\frac{1}{2},1})| &\stackrel{(6.6)}{=} \left(\frac{2}{k}\right)^{2/3} \left( |\text{Ai}'(i_1)| + \mathcal{O}(k^{-1/3}) \right), \\ j_{k+\frac{1}{2},1} &\stackrel{(6.5)}{=} k + \frac{1}{2} + ak^{1/3} + \mathcal{O}(k^{-1/3}), \\ \left(\frac{j_{k+\frac{1}{2},1}}{2}\right)^{-(k-\frac{1}{2})} &= \left(\frac{2}{k}\right)^{k-1/2} e^{-1/2} e^{-ak^{1/3}} (1 + \mathcal{O}(k^{-1/3})), \\ \Gamma(k + \frac{1}{2}) &= \frac{(2k)!}{4^k k!} \sqrt{\pi} = \left(\frac{k}{e}\right)^k \sqrt{2\pi} (1 + \mathcal{O}(k^{-1})), \end{aligned}$$

and formula (6.4) gives

$$\tau_k^* = C_0 \left(\frac{2}{e}\right)^k e^{-a_0 k^{1/3}} k^{-1/6} (1 + \mathcal{O}(k^{-1/3})),$$

where

$$C_0 = 4^{1/3} \sqrt{\frac{\pi}{e}} |\text{Ai}'(i_1)|, \quad a_0 = -i_1/2^{1/3},$$

and that proves the first part of Theorem 1.8.  $\square$

2) Next, we will prove the asymptotic formula for

$$\tau_m^{**} := \lim_{n \rightarrow \infty} n^{m/2} \tau_{n,n-m}.$$

Note that, with  $m = n - k$  fixed, and provided that the limit exists, we have

$$\tau_m^{**} := \lim_{k \rightarrow \infty} (k + m)^{m/2} \tau_{k+m,k} = \lim_{k \rightarrow \infty} k^{m/2} \tau_{k+m,k}.$$

We will use the relation

$$T_{k+m}^{(k)} = c_{m,k} P_m^{(\lambda)}, \quad \lambda = k,$$

and the asymptotic properties of the ultraspherical polynomials  $P_m^{(\lambda)}$  expressed in terms of the Hermite polynomials  $H_m$  (see [20, eq. (5.6.3)])

$$\lim_{\lambda \rightarrow \infty} \lambda^{-m/2} P_m^{(\lambda)} \left( \frac{x}{\sqrt{\lambda}} \right) = \frac{H_m(x)}{m!}. \quad (6.9)$$

**Lemma 6.4** *We have*

$$\tau_m^{**} = 2^{-m} |H_m(x'_m)|,$$

where  $x'_m$  is the point of the rightmost extremum of  $H_m$ .



**Proof.** With  $\omega_{k+m,k}$  and  $x'_m$  being the points of the rightmost local extrema of  $T_{k+m}^{(k)} = c_{m,k} P_m^{(k)}$  and  $H_m$ , respectively, it follows from (6.9) that, for a fixed  $m$ , we have

$$\tau_{k+m,k} = \frac{|P_m^{(k)}(\omega_{k+m,k})|}{P_m^{(k)}(1)} \sim \frac{k^{m/2} |H_m(x'_m)|}{m! \binom{m+2k-1}{m}} \sim 2^{-m} k^{-m/2} |H_m(x'_m)|, \quad k \rightarrow \infty,$$

and this implies

$$\tau_m^{**} = \lim_{k \rightarrow \infty} k^{m/2} \tau_{k+m,k} = 2^{-m} |H_m(x'_m)|.$$

Lemma is proved.  $\square$

**Proof of Theorem 1.8, part (1.15).** For approximation of  $H_m(x'_m)$  we will use the formula of Plancherel - Rotach ([20, Theorem 8.22.9]). Actually, we need only the third part of this theorem, concerning the approximation of  $H_m$  around its turning point, where the behaviour of the polynomial changes from oscillatory to monotonically increasing. It states that if

$$x = (2m+1)^{\frac{1}{2}} - 2^{-\frac{1}{2}} 3^{-\frac{1}{3}} m^{-\frac{1}{6}} t, \quad t \in \mathbb{C}, \quad (6.10)$$

then

$$e^{-x^2/2} H_m(x) = 3^{\frac{1}{3}} \pi^{-\frac{3}{4}} 2^{\frac{m}{2} + \frac{1}{4}} (m!)^{\frac{1}{2}} m^{-\frac{1}{12}} \left\{ A(t) + O(m^{-\frac{2}{3}}) \right\}, \quad (6.11)$$

where  $A(z) = 3^{-\frac{1}{3}} \pi \text{Ai}(-3^{-\frac{1}{3}} z)$  is the normalized Airy function. Moreover, the asymptotic formula (6.11) holds uniformly when  $t \in \mathbb{C}$  is bounded.

Let  $x_m$  be the largest zero of  $H_m$ , then ([20, eq. (6.32.5)])

$$x_m = (2m+1)^{\frac{1}{2}} - 2^{-\frac{1}{2}} 3^{-\frac{1}{3}} m^{-\frac{1}{6}} i_1^* + O(m^{-5/6}), \quad m \geq 1,$$

where  $i_1^* = -3^{1/3} i_1$  is the first zero of  $A(z)$ . Since  $H'_m(x) = 2m H_{m-1}(x)$ , we have for  $m \geq 2$

$$\begin{aligned} x'_m &= x_{m-1} = (2m-1)^{\frac{1}{2}} - 2^{-\frac{1}{2}} 3^{-\frac{1}{3}} m^{-\frac{1}{6}} i_1^* + O(m^{-5/6}) \\ &= x_m - (2m)^{-\frac{1}{2}} + O(m^{-5/6}), \end{aligned}$$

and we can put  $x'_m$  in the form (6.10), with  $t = t'_m$  where

$$t'_m = i_1^* + 3^{\frac{1}{3}} m^{-\frac{1}{3}} + O(m^{-2/3}).$$

Then formula (6.11) gives

$$\begin{aligned} H_m(x'_m) &= e^{\frac{1}{2}(x'_m)^2} 3^{\frac{1}{3}} \pi^{-\frac{3}{4}} 2^{\frac{m}{2} + \frac{1}{4}} (m!)^{\frac{1}{2}} m^{-\frac{1}{12}} \left\{ A(t'_m) + O(m^{-\frac{2}{3}}) \right\} \\ &= -(2em)^{\frac{m}{2}} e^{-|i_1| m^{1/3}} m^{-1/6} \sqrt{2\pi/e} \text{Ai}'(i_1) \left( 1 + O(m^{-1/3}) \right). \end{aligned}$$

Finally, we obtain

$$\tau_m^{**} = 2^{-m} |H_m(x'_m)| = \left( \frac{em}{2} \right)^{m/2} e^{-a_1 m^{1/3}} m^{-1/6} \left( 1 + O(m^{-1/3}) \right),$$

where

$$C_1 = \sqrt{\frac{2\pi}{e}} \text{Ai}'(i_1), \quad a_1 = |i_1|.$$

Theorem 1.8 is proved.  $\square$

## 7 Remarks

1. As was mentioned in introduction, Theorem 2.1 is due to Szász [19]. The statement of Theorem 2.1 appears in [1, pp. 304–305] along with a proof of the Legendre case ( $\lambda = 1/2$ ). The proof of this case originally was given by Szegő, who confirmed a conjecture made by J. Todd. Our proof follows the same approach. An alternative proof of Theorem 1.1 can be obtained using some results of Bojanov and Naidenov in [2].

2. To obtain better approximation of  $\tau_{n,k}$  one needs more precise asymptotic formulae for ultraspherical and Hermite polynomials and bounds for their extreme zeros. In this connection we refer to [3, 4, 5, 6, 9, 11, 22].

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## References

- [1] G. E. ANDREWS, R. ASKEY AND R. ROY, “Special Functions”, Cambridge University Press, 2002.
- [2] B. BOJANOV AND N. NAIDENOV, On oscillating polynomials, *J. Approx. Theory* **162** (2010), 1766–1878.
- [3] K. DRIVER AND K. JORDAAN, Bounds for extreme zeros of some classical orthogonal polynomials, *J. Approx. Theory* **164** (2012), 1200–1204.
- [4] A. ELBERT, Some recent results on the zeros of Bessel functions and orthogonal polynomials, *J. Comp. Appl. Math.* **133** (2001), 65–83.
- [5] A. ELBERT AND A. LAFORGIA, Upper bounds for the zeros of ultraspherical polynomials, *J. Approx. Theory* **61** (1990), 88–97.
- [6] A. ELBERT AND A. LAFORGIA, Asymptotic formulas for ultraspherical polynomials  $P_n^{(\lambda)}(x)$  and their zeros for large values of  $\lambda$ , *Proc. Amer. Math. Soc.* **114** (1992), 371–377.
- [7] P. ERDŐS AND G. SZEGŐ, On a problem of I. Schur, *Ann. Math.* **43** (1942), no. 2, 451–470.
- [8] B.-O. ERIKSSON, Some best constants in the Landau inequality on a finite interval, *J. Approx. Theory* **94** (1998), no. 3, 420–454.
- [9] L. GATTESCHI, Asymptotics and bounds for the zeros of Laguerre polynomials, a survey, *J. Comp. Appl. Math.* **144** (2002), 7–27.
- [10] C. GIORDANO AND A. LAFORGIA, Elementary approximations for zeros of Bessel functions, *J. Comp. Appl. Math.* **9** (1983), 221–228.
- [11] I. KRASIKOV, Sharp Inequalities for Hermite Polynomials, in: “Constructive Theory of Functions, Varna 2005” (B. Bojanov, Ed.), pp. 176–182, Prof. Marin Drinov Academic Publishing House, Sofia, 2006.
- [12] T. LANG AND R. WONG, “Best possible” upper bounds for the first two positive zeros of the Bessel function  $J_\nu(x)$ : The infinite case, *J. Comp. Appl. Math.* **71** (1996), 311–329.
- [13] L. LORCH AND R. UBERTI, “Best possible” upper bounds for the first positive zeros of the Bessel function - the finite part, *J. Comp. Appl. Math.* **75** (1996), 249–258.

- [14] G. NIKOLOV, Inequalities of Duffin-Schaeffer type. II, *East J. Approx.* **11** (2005), no. 2, 147–168.
- [15] F. W. J. OLVER, “Asymptotics and special functions”, Academic Press, New York, 1974.
- [16] A. C. SCHAEFFER AND R. J. DUFFIN, On some inequalities of S. Bernstein and W. Markoff for derivatives of polynomials, *Bull. Amer. Math. Soc.* **44** (1938), no. 4, 289–297.
- [17] A. SHADRIN, Twelve proofs of the Markov inequality, in “Approximation theory: a volume dedicated to Borislav Bojanov”, pp. 233–298, Prof. M. Drinov Acad. Publ. House, Sofia, 2004.
- [18] A. SHADRIN, The Landau-Kolmogorov inequality revisited, *Discrete Contin. Dyn. Syst.* **34** (2014), no. 3, 1183–1210.
- [19] O. SZÁSZ, On the relative extrema of ultraspherical polynomials, *Boll. Un. Mat. Ital.* **5** (1950), no. 3, 125127.
- [20] G. SZEGŐ, “Orthogonal Polynomials”, Amer. Math. Soc. Colloq. Publ., v. 23, Providence, RI, 1975.
- [21] M. T. VACCA, Determinazione asintotica per  $n \rightarrow \infty$  degli estremi relativi dell’  $n$ -esimo polinomio di Jacobi, *Boll. Un. Mat. Ital.* **8** (1953), no. 3, 277–280.
- [22] R. WONG AND J.-M. ZHANG, Asymptotic monotonicity of the relative extrema of Jacobi polynomials, *Canad. J. Math.* **46** (1994), no. 6, 1318 - 1337.

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