# Multi-level Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities 

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#### Abstract

Parabolic partial differential equations (PDEs) and backward stochastic differential equations (BSDEs) have a wide range of applications. In particular, high-dimensional PDEs with gradient-dependent nonlinearities appear often in the state-of-the-art pricing and hedging of financial derivatives. In this article we prove that semilinear heat equations with gradient-dependent nonlinearities can be approximated under suitable assumptions with computational complexity that grows polynomially both in the dimension and the reciprocal of the accuracy.


## 1 Introduction

Parabolic partial differential equations (PDEs) and backward stochastic differential equations (BSDEs) are key ingredients in a number of models in physics and financial engineering; see, e.g., the references in [4]. These applications often lead to stochastic optimization problems which result in a semilinear or quasilinear PDE with a nonlinearity depending on the gradient of the solution. Moreover these PDEs are high-dimensional if the financial derivative depends on a whole basket of underlyings. So it is important to approximate the solutions of such PDEs approximately at single space-time points (the full solution function is presumeably hard to approximate in high dimensions; cf. Theorem 1 in Heinrich [6] for the elliptic case). The numerical analysis literature contains a multitude of approximation methods for parabolic PDEs and BSDEs; see the review in (4) and the recent article [2]. However, to the best of our knowledge, none of these methods except for the branching diffusion method fulfills the requirement that the computational complexity grows at most polynomially both in the dimension and in the reciprocal of the accuracy; see Section 6 in [4] for a detailed discussion. The branching diffusion method proposed in [7, 9, 8] meets this requirement. However, not only is this method only applicable to a special class of PDEs, it also requires the terminal/initial condition to be quite small (see Subsection 6.7 in (4) for a detailed discussion).

The recent article [3] proposes a family of approximation methods based on Picard approximations and multilevel Monte Carlo methods; see also (5) below. The simulation results in 4 suggest that these methods work satisfactory for 100-dimensional semilinear PDEs from applications. In addition Corollary 3.18 in [3] shows under suitable regularity assumptions on the exact solution for semilinear heat equations with gradient-independent nonlinearities that the computational complexity is bounded by $O\left(d \varepsilon^{-(4+\delta)}\right)$ for any $\delta \in(0, \infty)$, where $d$ is the dimensionality of the problem and $\varepsilon \in(0, \infty)$ is the prescribed accuracy. Generalizing the proof of Corollary 3.18 in [3] to the gradient-dependent case is nontrivial. In particular, we were not able to derive an inequality analogous to (56) in [3] involving a family of suitable seminorms to which one could apply a discrete Gronwall inequality.

So it remained an open problem to prove mathematically that semilinear PDEs with gradient-dependent nonlinearity and general terminal/initial condition can be approximated with a computational effort which grows at most polynomially both in the dimension and in the reciprocal of the prescribed accuracy. In this article we solve this problem for the first time. More precisely, Corollary 4.8 below shows under suitable regularity assumptions on the exact solution for semilinear heat equations with gradient-dependent nonlinearities that the computational complexity of the multi-level Picard approximations (5) is bounded by $O\left(d \varepsilon^{-(4+\delta)}\right)$ for any $\delta \in(0, \infty)$, where $d$ is the dimensionality of the problem and $\varepsilon \in(0, \infty)$ is the prescribed accuracy.

The structure of this article is as follows. Subsection 1.1 gathers notation that we frequently use. In Section 2 we introduce the setting which we consider throughout this article and, in particular, the multilevel

[^0]Picard approximations (5) with Gauß-Legendre quadrature rules given by (4). The reason for choosing GaußLegendre quadrature rules is the very fast convergence in case of sufficiently smooth integrands; cf. Lemma 4.5 below. Fast readers can then jump to Corollary 4.8, which is the main result of this article. For the proof of Corollary 4.8, we first derive the (recursive) bound (54) for the global error and then iterate this inequality to obtain the (non-recursive) bound (65) for the global error. Finally Lemma 3.3 provides an upper bound for the iterated Gauß-Legendre integrals over inverse square roots appearing in (65).

### 1.1 Notation

We denote by $\langle\cdot, \cdot\rangle:\left(\cup_{n \in \mathbb{N}}\left(\mathbb{R}^{n} \times \mathbb{R}^{n}\right)\right) \rightarrow[0, \infty)$ the function that satisfies for all $n \in \mathbb{N}, v=\left(v_{1}, \ldots, v_{n}\right)$, $w=\left(w_{1}, \ldots, w_{n}\right) \in \mathbb{R}^{n}$ that $\langle v, w\rangle=\sum_{i=1}^{n} v_{i} w_{i}$. For every $p \in \mathbb{N}$ we denote by $\|\cdot\|_{p}:\left(\cup_{n \in \mathbb{N}} \mathbb{R}^{n}\right) \rightarrow[0, \infty)$ and $\|\cdot\|_{\infty}:\left(\cup_{n \in \mathbb{N}} \mathbb{R}^{n}\right) \rightarrow[0, \infty)$ the functions that satisfy for all $n \in \mathbb{N}, v=\left(v_{1}, \ldots, v_{n}\right)$ that $\|v\|_{p}=\left[\sum_{i=1}^{n}\left|v_{i}\right|^{p}\right]^{1 / p}$ and $\|v\|_{\infty}=\max _{i=1, \ldots, n}\left|v_{i}\right|$. For every topological space $(E, \mathcal{E})$ we denote by $\mathcal{B}(E)$ the Borel-sigma-algebra on $(E, \mathcal{E})$. For all measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of $\mathcal{A} / \mathcal{B}$-measurable functions from $A$ to $B$. For every probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we denote by $\|\cdot\|_{L^{2}(\mathbb{P} ; \mathbb{R})}: \mathcal{M}(\mathcal{A}, \mathcal{B}(\mathbb{R})) \rightarrow[0, \infty]$ the function that satisfies for all $X \in \mathcal{M}(\mathcal{A}, \mathcal{B}(\mathbb{R}))$ that $\|X\|_{L^{2}(\mathbb{P} ; \mathbb{R})}=\sqrt{\mathbb{E}\left[|X|^{2}\right]}$. We denote by $\frac{0}{0}, 0 \cdot \infty, 0^{0}$, and $\sqrt{\infty}$ the extended real numbers given by $\frac{0}{0}=0,0 \cdot \infty=0,0^{0}=1$, and $\sqrt{\infty}=\infty$. For every $a \in(0, \infty)$ and every $b \in \mathbb{R}$ we denote by $\frac{a}{0}, \frac{-a}{0}, 0^{-a}, \frac{1}{0^{a}}, \frac{b}{\infty}$, and $0^{a}$ the extended real numbers given by $\frac{a}{0}=\infty, \frac{-a}{0}=-\infty, 0^{-a}=\infty$, $\frac{1}{0^{a}}=\infty, \frac{b}{\infty}=0$, and $0^{a}=0$. For every $A \subseteq \mathbb{Z}, a: A \rightarrow \mathbb{R}$, and $k \in \mathbb{Z}$ we denote by $\prod_{l=k}^{k-1} a(l)$ and $\sum_{l=k}^{k-1} a(l)$ the real numbers given by $\prod_{l=k}^{k-1} a(l)=1$ and $\sum_{l=k}^{k-1} a(l)=0$.

## 2 Multi-level Picard approximations

Let $T \in(0, \infty), d \in \mathbb{N}, g \in C^{2}\left(\mathbb{R}^{d}, \mathbb{R}\right), \Theta=\cup_{n \in \mathbb{N}} \mathbb{R}^{n}, L \in \mathbb{R}^{d+1}, K \in \mathbb{R}^{d}$, let $\left(\Omega, \mathcal{F}, \mathbb{P},\left(\mathbb{F}_{t}\right)_{t \in[0, T]}\right)$ be a stochastic basis, let $W^{\theta}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, \theta \in \Theta$, be independent standard $\left(\mathbb{F}_{t}\right)_{t \in[0, T]}$-Brownian motions with continuous sample paths, let $F: \mathcal{M}\left(\mathcal{B}\left([0, T] \times \mathbb{R}^{d}\right), \mathcal{B}\left(\mathbb{R}^{d+1}\right)\right) \rightarrow \mathcal{M}\left(\mathcal{B}\left([0, T] \times \mathbb{R}^{d}\right), \mathcal{B}(\mathbb{R})\right)$ satisfy for all $u_{1}, u_{2} \in \mathcal{M}\left(\mathcal{B}\left([0, T] \times \mathbb{R}^{d}\right), \mathcal{B}\left(\mathbb{R}^{d+1}\right)\right), r \in[0, T], y \in \mathbb{R}^{d}$ that

$$
\begin{equation*}
\left|\left(F\left(u_{1}\right)-F\left(u_{2}\right)\right)(r, y)\right| \leq \sum_{\nu=1}^{d+1} L_{\nu}\left|\left(u_{1}(r, y)-u_{2}(r, y)\right)_{\nu}\right| \tag{1}
\end{equation*}
$$

let $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ satisfy for all $x, y \in \mathbb{R}^{d}$ that

$$
\begin{equation*}
|g(x)-g(y)| \leq \sum_{\alpha=1}^{d} K_{\alpha}\left|(x-y)_{\alpha}\right| \tag{2}
\end{equation*}
$$

let $u^{\infty}=\left(u^{\infty}(r, y)\right)_{(r, y) \in[0, T] \times \mathbb{R}^{d}} \in C^{1,2}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$ satisfy for all $r \in(0, T), y \in \mathbb{R}^{d}$ that $u^{\infty}(T, y)=g(y)$ and

$$
\begin{equation*}
\left(\frac{\partial}{\partial r} u^{\infty}\right)(r, y)+\frac{1}{2}\left(\Delta_{y} u^{\infty}\right)(r, y)+\left(F\left(\left(u^{\infty}, \nabla_{y} u^{\infty}\right)\right)\right)(r, y)=0 \tag{3}
\end{equation*}
$$

let $\mathbf{u}^{\infty} \in C\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}^{d+1}\right)$ satisfy for all $r \in[0, T], y \in \mathbb{R}^{d}$ that $\mathbf{u}^{\infty}(r, y)=\left(u^{\infty}(r, y), \nabla_{y} u^{\infty}(r, y)\right)$, for every $n \in \mathbb{N}$ let $\left(c_{i}^{n}\right)_{i \in\{1, \ldots, n\}} \subseteq[-1,1]$ be the $n$ distinct roots of the Legendre polynomial $[-1,1] \ni x \mapsto$ $\frac{1}{2^{n} n!} \frac{d^{n}}{d x^{n}}\left[\left(x^{2}-1\right)^{n}\right] \in \mathbb{R}, q^{n,[a, b]}:[a, b] \rightarrow \mathbb{R}$ be the function which satisfies for all $t \in[a, b]$ that

$$
q^{n,[a, b]}(t)= \begin{cases}\int_{a}^{b}\left[\prod_{\substack{i \in\{1, \ldots, n\}, c_{i}^{n} \neq \frac{2 t-a+b)}{b-a}}} \frac{2 x-(b-a) c_{i}^{n}-(a+b)}{2 t-(b-a) c_{i}^{n}-(a+b)}\right] & :(a<b) \text { and }\left(\frac{2 t-(a+b)}{b-a} \in\left\{c_{1}^{n}, \ldots, c_{n}^{n}\right\}\right)  \tag{4}\\ 0 & : \text { else }\end{cases}
$$

let $\left(\mathbf{U}_{n, M, Q}^{\theta}\right)_{n, M, Q \in \mathbb{Z}, \theta \in \Theta} \subseteq \mathcal{M}\left(\mathcal{B}\left([0, T] \times \mathbb{R}^{d}\right) \otimes \mathcal{F}, \mathcal{B}\left(\mathbb{R} \times \mathbb{R}^{d}\right)\right)$ satisfy for all $n, M, Q \in \mathbb{N}, \theta \in \Theta,(s, x) \in$ $[0, T) \times \mathbb{R}^{d}$ that $\mathbf{U}_{0, M, Q}^{\theta}(s, x)=0$ and

$$
\begin{align*}
& \mathbf{U}_{n, M, Q}^{\theta}(s, x)=(g(x), 0)+\frac{1}{M^{n}} \sum_{i=1}^{M^{n}}\left(g\left(x+W_{T}^{(\theta, 0,-i)}-W_{s}^{(\theta, 0,-i)}\right)-g(x)\right)\left(1, \frac{W_{T}^{(\theta, 0,-i)}-W_{s}^{(\theta, 0,-i)}}{T-s}\right) \\
& +\sum_{l=0}^{n-1} \sum_{t \in(s, T)} \frac{q^{Q,[s, T]}(t)}{M^{n-l}} \sum_{i=1}^{M^{n-l}}\left(F\left(\mathbf{U}_{l, M, Q}^{(\theta, l, t, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(\theta,-l, i, t)}\right)\right)\left(t, x+W_{t}^{(\theta, l, i)}-W_{s}^{(\theta, l, i)}\right)\left(1, \frac{W_{t}^{(\theta, l, i)}-W_{s}^{(\theta, l, i)}}{t-s}\right) . \tag{5}
\end{align*}
$$

## 3 Preliminary results for Gauß－Legendre quadrature rules

Lemma 3.1 （Iterated Gauß－Legendre integration）．Assume the setting in Section $⿴ 囗 ⿱ 一 𧰨 殳$ and let $Q \in \mathbb{N}$ ．Then it holds for all $k \in \mathbb{N}, t_{0} \in[0, T)$ that

$$
\begin{equation*}
\sum_{\substack{t_{1}, \ldots, t_{k-1}, t_{k} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k-1}<t_{k}<T}}\left[\prod_{i=0}^{k-1} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right]=\left(T-t_{0}\right)^{k / 2} \prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right] . \tag{6}
\end{equation*}
$$

Proof of Lemma 3．1．First observe that for all $t_{0} \in[0, T)$ and $s \in[0,1]$ with $2 s-1 \in\left\{c_{1}^{Q}, c_{2}^{Q}, \ldots c_{Q}^{Q}\right\}$ the definition（4）and the integral transformation theorem with the substitution $\left[t_{0}, T\right] \ni x \mapsto \frac{x-t_{0}}{T-t_{0}} \in[0,1]$ prove that

$$
\begin{align*}
q^{Q,\left[t_{0}, T\right]}\left(s\left(T-t_{0}\right)+t_{0}\right) & =\int_{t_{0}}^{T}\left[\prod_{\substack{i \in\{1, \ldots, n\}, c_{i}^{Q} \neq \frac{2 s\left(T-t_{0}\right)+2 t_{0}-\left(t_{0}+T\right)}{T-t_{0}}}} \frac{2 x-\left(T-t_{0}\right) c_{i}^{Q}-\left(t_{0}+T\right)}{2 s\left(T-t_{0}\right)+2 t_{0}-\left(T-t_{0}\right) c_{i}^{Q}-\left(t_{0}+T\right)}\right] d x \\
& =\int_{t_{0}}^{T}\left[\prod_{\substack{i \in\{1, \ldots, n\}, c_{i}^{Q} \neq 2 s-1}} \frac{2\left(x-t_{0}\right)-\left(T-t_{0}\right) c_{i}^{Q}-\left(T-t_{0}\right)}{\left(T-t_{0}\right)\left(2 s-c_{i}^{Q}-1\right)}\right] d x  \tag{7}\\
& =\left(T-t_{0}\right) \int_{0}^{1}\left[\prod_{\substack{i \in\{1, \ldots, n\}, c_{i}^{Q} \neq 2 s-1}} \frac{2 y-c_{i}^{Q}-1}{2 s-c_{i}^{Q}-1}\right] d y
\end{align*}
$$

This and（4）show that for all $t_{0} \in[0, T)$ and $s \in[0,1]$ it holds that

$$
\begin{equation*}
q^{Q,\left[t_{0}, T\right]}\left(s\left(T-t_{0}\right)+t_{0}\right)=\left(T-t_{0}\right) q^{Q,[0,1]}(s) \tag{8}
\end{equation*}
$$

We prove（6）by induction on $k \in \mathbb{N}$ ．For the base case $k=1$ observe that（8）ensures that for all $t_{0} \in[0, T)$ it holds that

$$
\begin{equation*}
\sum_{t_{1} \in\left(t_{0}, T\right)} \frac{q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)}{\sqrt{t_{1}-t_{0}}}=\sum_{s \in(0,1)} \frac{q^{Q,\left[t_{0}, T\right]}\left(s\left(T-t_{0}\right)+t_{0}\right)}{\sqrt{s\left(T-t_{0}\right)}}=\left(T-t_{0}\right)^{1 / 2} \sum_{s \in(0,1)} \frac{q^{Q,[0,1]}(s)}{\sqrt{s}} \tag{9}
\end{equation*}
$$

This establishes（6）in the base case $k=1$ ．For the induction step $\mathbb{N} \ni k \rightarrow k+1 \in \mathbb{N}$ observe that the induction hypothesis implies that for all $t_{0} \in[0, T)$ it holds that

$$
\left.\left.\begin{array}{rl} 
& \sum_{\substack{t_{1}, \ldots, t_{k}, t_{k+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k}<t_{k+1}<T}}\left[\prod_{i=0}^{k} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right]=\sum_{t_{1} \in\left(t_{0}, T\right)} \frac{q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)}{\sqrt{t_{1}-t_{0}}}\left\{\sum_{\substack{t_{2}, \ldots, t_{k}, t_{k+1} \in \mathbb{R},}}\left[\prod_{i=1}^{k} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right]\right\} \\
= & \sum_{t_{1}<t_{2}<\ldots<t_{k}<t_{k+1}<T} \\
t_{1} \in\left(t_{0}, T\right)  \tag{10}\\
= & q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right) \\
\sqrt{t_{1}-t_{0}}
\end{array}\left(T-t_{1}\right)^{k / 2} \prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right]\right\}, \prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right]\right\}\left\{\sum_{t_{1} \in\left(t_{0}, T\right)} q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right) \frac{\left(T-t_{1}\right)^{k / 2}}{\sqrt{t_{1}-t_{0}}}\right\} .
$$

This together with（8）ensures that for all $t_{0} \in[0, T)$ it holds that

$$
\begin{align*}
& \sum_{\substack{t_{1}, \ldots, t_{k}, t_{k+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k}<t_{k+1}<T}}\left[\prod_{i=0}^{k} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right] \\
&=\left\{\prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right]\right\}\left\{\sum_{s \in(0,1)} q^{Q,\left[t_{0}, T\right]}\left(s\left(T-t_{0}\right)+t_{0}\right) \frac{\left(T-s\left(T-t_{0}\right)-t_{0}\right)^{k / 2}}{\sqrt{s\left(T-t_{0}\right)}}\right\}  \tag{11}\\
&=\left(T-t_{0}\right)^{(k+1) / 2} \prod_{i=0}^{k}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right] .
\end{align*}
$$

This finishes the induction step $\mathbb{N}_{0} \ni k \rightarrow k+1 \in \mathbb{N}$. Induction hence establishes (6). The proof of Lemma 3.1] is thus completed.

Lemma 3.2. Assume the setting in Section 2 and let $Q \in \mathbb{N}, j \in \mathbb{N}_{0}$. Then it holds that

$$
\begin{equation*}
\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{j}}{\sqrt{s}} \leq \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(j+1)}{\Gamma\left(j+\frac{3}{2}\right)} \tag{12}
\end{equation*}
$$

Proof of Lemma 3.2. The Leibniz formula ensures that for all $\varepsilon \in(0, \infty), s \in(0,1)$ it holds that

$$
\begin{align*}
\frac{d^{2 Q}}{d s^{2 Q}} \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}} & =\sum_{k=0}^{2 Q}\binom{2 Q}{k}\left[\frac{d^{2 Q-k}}{d s^{2 Q-k}} \frac{1}{\sqrt{s+\varepsilon}}\right]\left[\frac{d^{k}}{d s^{k}}(1-s)^{j}\right] \\
& =\sum_{k=0}^{2 Q}\binom{2 Q}{k}\left[(s+\varepsilon)^{-(2 Q-k+1 / 2)} \prod_{l=0}^{2 Q-k-1}\left(-\frac{1}{2}-l\right)\right]\left[(-1)^{k}(1-s)^{j-k} \prod_{l=0}^{k-1}(j-l)\right]  \tag{13}\\
& =\sum_{k=0}^{\min \{j, 2 Q\}}\binom{2 Q}{k}\left[(s+\varepsilon)^{-(2 Q-k+1 / 2)} \prod_{l=0}^{2 Q-k-1}\left(\frac{1}{2}+l\right)\right]\left[(1-s)^{j-k} \prod_{l=0}^{k-1}(j-l)\right] \\
& \geq 0 .
\end{align*}
$$

The error representation for the Gauß-Legendre quadrature rule (see, e.g., [1, Display (2.7.12)]) implies that for every $\varepsilon \in(0, \infty)$ there exists $\xi \in(0,1)$ such that it holds that

$$
\begin{equation*}
\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}}=\int_{0}^{1} \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}} d s-\left.\frac{(Q!)^{4}}{(2 Q+1)[(2 Q)!]^{3}} \frac{d^{2 Q}}{d s^{2 Q}}\right|_{s=\xi} \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}} \tag{14}
\end{equation*}
$$

This and (13) prove that for all $\varepsilon \in(0, \infty)$ it holds that

$$
\begin{equation*}
\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}} \leq \int_{0}^{1} \frac{(1-s)^{j}}{\sqrt{s+\varepsilon}} d s \leq \int_{0}^{1} \frac{(1-s)^{j}}{\sqrt{s}} d s=\frac{\Gamma\left(\frac{1}{2}\right) \Gamma(j+1)}{\Gamma\left(j+\frac{3}{2}\right)} \tag{15}
\end{equation*}
$$

Letting $\varepsilon \rightarrow 0$ in (15) completes the proof of Lemma 3.2,
Lemma 3.3 (Upper bound for iterated Gauß-Legendre integration). Assume the setting in Section $\mathbf{Q}^{2}$ and let $Q \in \mathbb{N}$. Then it holds for all $k \in \mathbb{N}, t_{0} \in[0, T)$ that

$$
\begin{equation*}
\sum_{\substack{t_{1}, \ldots, t_{k-1}, t_{k} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k-1}<t_{k}<T}}\left[\prod_{i=0}^{k-1} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right] \leq \frac{2\left(\left(T-t_{0}\right) \pi\right)^{k / 2}}{\Gamma\left(\frac{k}{2}\right)} \tag{16}
\end{equation*}
$$

Proof of Lemma 3.3. Throughout this proof let $w: \mathbb{N} \rightarrow \mathbb{R}$ be the function that satisfies for all $k \in \mathbb{N}$ that $w(k)=\prod_{i=0}^{k-1} \frac{\Gamma\left(\left\lfloor\frac{i}{2}\right\rfloor+1\right)}{\Gamma\left(\left\lfloor\frac{2}{2}\right\rfloor+\frac{3}{2}\right)}$. First observe that for all $k \in\{2 n: n \in \mathbb{N}\}$ it holds that

$$
\begin{equation*}
\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}+\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right) \Gamma\left(\frac{k}{2}\right) \frac{k}{2}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\left(\frac{k}{2}+\frac{1}{2}\right)}=\frac{\frac{k}{2}}{\frac{k}{2}+\frac{1}{2}} \leq 1 . \tag{17}
\end{equation*}
$$

Moreover, the fact that $\Gamma:(0, \infty) \rightarrow(0, \infty)$ is logarithmically convex ensures that for all $k \in\{2 n-1: n \in \mathbb{N}\}$ it holds that

$$
\begin{equation*}
\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)^{2}}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}+1\right)} \leq 1 \tag{18}
\end{equation*}
$$

This and (17) prove that for all $k \in \mathbb{N}$ it holds that

$$
\begin{equation*}
\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)} \leq 1 \tag{19}
\end{equation*}
$$

Next we show that for all $k \in \mathbb{N}$ it holds that

$$
\begin{equation*}
w(k) \leq \frac{2}{\Gamma\left(\frac{k}{2}\right)} \tag{20}
\end{equation*}
$$

We prove (20) by induction on $k \in \mathbb{N}$. For the base case $k=1$ we note that it holds that

$$
\begin{equation*}
w(1)=\frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}=\frac{2}{\Gamma\left(\frac{1}{2}\right)} \tag{21}
\end{equation*}
$$

This establishes (20) in the base case $k=1$. For the induction step $\mathbb{N} \ni k \rightarrow k+1 \in \mathbb{N}$ observe that the induction hypothesis and (19) show that

$$
\begin{equation*}
w(k+1)=w(k) \frac{\Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)} \leq \frac{2 \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\left\lfloor\frac{k}{2}\right\rfloor+\frac{3}{2}\right)} \frac{2}{\Gamma\left(\frac{k+1}{2}\right)} \leq \frac{2}{\Gamma\left(\frac{k+1}{2}\right)} \tag{22}
\end{equation*}
$$

This finishes the induction step $\mathbb{N} \ni k \rightarrow k+1 \in \mathbb{N}$. Induction hence establishes (20). Lemma 3.1, Lemma 3.2, the facts that $\forall s \in(0,1): q^{Q,[0,1]}(s) \geq 0$ (see, e.g., [1, Section 2.7]) and $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, and (20) show that for all $k \in \mathbb{N}, t_{0} \in[0, T)$ it holds that

$$
\begin{align*}
\sum_{\substack{t_{1}, \ldots, t_{k-1}, t_{k} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k}<1<t_{k}<T}}\left[\prod_{i=0}^{k-1} \frac{q^{Q,\left[t_{i}, T\right]}\left(t_{i+1}\right)}{\sqrt{t_{i+1}-t_{i}}}\right] & =\left(T-t_{0}\right)^{k / 2} \prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{i / 2}}{\sqrt{s}}\right] \\
& \leq\left(T-t_{0}\right)^{k / 2} \prod_{i=0}^{k-1}\left[\sum_{s \in(0,1)} q^{Q,[0,1]}(s) \frac{(1-s)^{\lfloor i / 2\rfloor}}{\sqrt{s}}\right]  \tag{23}\\
& \leq\left(T-t_{0}\right)^{k / 2} \prod_{i=0}^{k-1}\left[\frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\left\lfloor\frac{i}{2}\right\rfloor+1\right)}{\Gamma\left(\left\lfloor\frac{i}{2}\right\rfloor+\frac{3}{2}\right)}\right] \\
& =\left(T-t_{0}\right)^{k / 2} \Gamma\left(\frac{1}{2}\right)^{k} w(k) \\
& \leq \frac{2\left(\left(T-t_{0}\right) \pi\right)^{k / 2}}{\Gamma\left(\frac{k}{2}\right)}
\end{align*}
$$

This completes the proof of Lemma 3.3
Lemma 3.4 (Iterated sums). Let $n \in \mathbb{N}, l_{0} \in\{0, \ldots, n-1\}$, and $j \in\left\{1, \ldots, n-l_{0}-1\right\}$. Then it holds that

$$
\begin{equation*}
\sum_{\substack{l_{1}, \ldots, l_{j} \in \mathbb{N}, l_{0}<l_{1}<\ldots<l_{j}<n}} 1=\binom{n-l_{0}-1}{j} . \tag{24}
\end{equation*}
$$

Proof of Lemma 3.4. The natural number $\sum_{\substack{l_{1}, \ldots, l_{j} \in \mathbb{N}, l_{0}<l_{1}<\ldots<l_{j}<n}} 1$ is the number of ways to choose a subset of size $j$ elements from a set of $n-l_{0}-1$ elements. This completes the proof of Lemma 3.4,

Lemma 3.5 (Log-subadditivity). Let $d, p \in \mathbb{N}, x, y \in \mathbb{R}^{d}$, and let $\|\cdot\|: \mathbb{R}^{d} \rightarrow[0, \infty)$ be a norm. Then $1+\|x+y\|^{p} \leq(1+\|y\|)^{p}\left(1+\|x\|^{p}\right)$.
Proof of Lemma 3.5. It holds that

$$
\begin{align*}
1+\|x+y\|^{p} & \leq 1+(\|x\|+\|y\|)^{p}=1+\sum_{k=0}^{p}\binom{p}{k}\|x\|^{p-k}\|y\|^{k}=\left(1+\|x\|^{p}\right)\left(1+\sum_{k=1}^{p}\binom{p}{k} \frac{\|x\|^{p-k}}{1+\|x\|^{p}}\|y\|^{k}\right)  \tag{25}\\
& \leq\left(1+\|x\|^{p}\right)\left(1+\sum_{k=1}^{p}\binom{p}{k}\|y\|^{k}\right)=(1+\|y\|)^{p}\left(1+\|x\|^{p}\right) .
\end{align*}
$$

This completes the proof of Lemma 3.5

## 4 Error analysis for multi-level Picard approximations with GaußLegendre quadrature rules

Lemma 4.1 (Approximations are integrable). Assume the setting in Section $\mathbf{Q}^{2}$, let $p, M, Q \in \mathbb{N}$ and assume for all $t \in[0, T]$ that

$$
\begin{equation*}
\sup _{x \in \mathbb{R}^{d}} \frac{|g(x)|}{1+\|x\|_{1}^{p}}+\sup _{x \in \mathbb{R}^{d}} \frac{|(F(0))(t, x)|}{1+\|x\|_{1}^{p}}<\infty \tag{26}
\end{equation*}
$$

Then
(i) for all $n \in \mathbb{N}_{0}, \theta \in \Theta, s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{equation*}
\mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{n, M, Q}^{\theta}(s, x)\right)_{\nu}\right|}{1+\|x\|_{1}^{p}}\right]<\infty \tag{27}
\end{equation*}
$$

(ii) for all $n \in \mathbb{N}, \theta \in \Theta, s \in[0, T), t \in(s, T], x \in \mathbb{R}^{d}, \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{equation*}
\mathbb{E}\left[\left|\left(F\left(\mathbf{U}_{n, M, Q}^{\theta}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right]<\infty \tag{28}
\end{equation*}
$$

and
(iii) for all $n \in \mathbb{N}, \theta \in \Theta, s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{align*}
\mathbb{E}\left[\mathbf{U}_{n, M, Q}^{\theta}(s, x)\right] & =\mathbb{E}\left[g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right] \\
& +\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{n-1, M, Q}^{\theta}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)\right] \tag{29}
\end{align*}
$$

Proof of Lemma 4.1. We prove (iil) by induction on $n \in \mathbb{N}_{0}$. The induction base $n=0$ is clear. For the induction step $\mathbb{N}_{0} \ni n \rightarrow n+1 \in \mathbb{N}$, let $n \in \mathbb{N}_{0}$ and assume that (ii) holds for $n=0, n=1, \ldots, n=n$. The triangle inequality, Lemma 3.5. (22), and (11) ensure that for all $\theta \in \Theta, s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{n+1, M, Q}^{\theta}(s, x)\right)_{\nu}\right|}{1+\|x\|_{1}^{p}}\right] \leq \sup _{x \in \mathbb{R}^{d}} \frac{\left|(g(x), 0)_{\nu}\right|}{1+\|x\|_{1}^{p}}+\mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\right|}{1+\|x\|_{1}^{p}}\left|\left(1, \frac{W_{T}^{(\theta, 0,-i)}-W_{s}^{(\theta, 0,-i)}}{T-s}\right)_{\nu}\right|\right] \\
& +\sum_{l=0}^{n} \sum_{t \in(s, T)} q^{Q,[s, T]}(t) \mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(F\left(\mathbf{U}_{l, M, Q}^{(\theta, l, l, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(\theta, l, l, t)}\right)\right)\left(t, x+W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right)\right|}{1+\|x\|_{1}^{p}}\left|\left(1, \frac{W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}}{t-s}\right)_{\nu}\right|\right] \\
& \leq \sup _{x \in \mathbb{R}^{d}} \frac{|g(x)|}{1+\|x\|_{1}^{p}}+\sum_{\alpha=1}^{d} K_{\alpha} \mathbb{E}\left[\left|\left(W_{T}^{0}-W_{s}^{0}\right)_{\alpha}\left(1, \frac{W_{T}^{(\theta, 0,-i)}-W_{s}^{(\theta, 0,-i)}}{T-s}\right)_{\nu}\right|\right] \\
& +\sum_{l=1}^{n} \sum_{t \in(s, T)} q^{Q,[s, T]}(t) \sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{l, M, Q}^{(\theta, l, l, t)}-\mathbf{U}_{l-1, M, Q}^{(\theta,-l, 1, t)}\right)_{\nu_{1}}\left(t, x+W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right)\right|}{1+\left\|x+W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right\|_{1}^{p}} \cdot \frac{1+\left\|x+W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right\|_{1}^{p}}{1+\|x\|_{1}^{p}}\right. \\
& \left.\cdot\left|\left(1, \frac{W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}}{t-s}\right)_{\nu}\right|\right] \\
& +\sum_{t \in(s, T)} q^{Q,[s, T]}(t) \mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|(F(0))\left(t, x+W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}\right)\right|}{1+\left\|x+W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}\right\|_{1}^{p}} \cdot \frac{1+\left\|x+W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}\right\|_{1}^{p}}{1+\|x\|_{1}^{p}}\left|\left(1, \frac{W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}}{t-s}\right)_{\nu}\right|\right] \\
& \leq \sup _{x \in \mathbb{R}^{d}} \frac{|g(x)|}{1+\|x\|_{1}^{p}}+\sum_{\alpha=1}^{d} K_{\alpha} \mathbb{E}\left[\left|\left(W_{T}^{0}-W_{s}^{0}\right)_{\alpha}\left(1, \frac{W_{T}^{(\theta, 0,-i)}-W_{s}^{(\theta, 0,-i)}}{T-s}\right)_{\nu}\right|\right] \\
& +\sum_{l=1}^{n} \sum_{t \in(s, T)} q^{Q,[s, T]}(t) \sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \mathbb{E}\left[\sup _{y \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{l, M, Q}^{(\theta, l, 1, t)}(t, y)-\mathbf{U}_{l-1, M, Q}^{(\theta,-l, t)}(t, y)\right)_{\nu_{1}}\right|\left(1+\left\|W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right\|_{1}\right)^{p}}{1+\|y\|_{1}^{p}}\left|\left(1, \frac{W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}}{t-s}\right)_{\nu}\right|\right] \\
& +\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left[\sup _{y \in \mathbb{R}^{d}} \frac{|(F(0))(t, y)|}{1+\|y\|_{1}^{p}}\right] \mathbb{E}\left[\left(1+\left\|W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}\right\|_{1}\right)^{p}\left|\left(1, \frac{W_{t}^{(\theta, 0,1)}-W_{s}^{(\theta, 0,1)}}{t-s}\right)_{\nu}\right|\right] . \tag{30}
\end{align*}
$$

The fact that for all $l \in \mathbb{N}, \theta \in \Theta, s, t \in[0, T)$ the random variables $\mathbf{U}_{l, M, Q}^{(\theta, l, 1, t)}(t, \cdot)-\mathbf{U}_{l-1, M, Q}^{(\theta,-l, 1, t)}(t, \cdot)$ and $W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}$ are independent proves that for all $\theta \in \Theta, \nu \in\{1, \ldots, d+1\}, l \in \mathbb{N}, s \in[0, T], t \in(s, T]$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{l, M, Q}^{(\theta, l, l, t)}(t, x)-\mathbf{U}_{l-1, M, Q}^{(\theta,-l, l, t)}(t, x)\right)_{\nu_{1}}\right|\left(1+\left\|W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right\|_{1}\right)^{p}}{1+\|x\|_{1}^{p}}\left|\left(1, \frac{W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}}{t-s}\right)_{\nu}\right|\right] \\
& \quad=\mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{l, M, Q}^{(\theta, l, t, t)}(t, x)-\mathbf{U}_{l-1, M, Q}^{(\theta,-l, 1, t)}(t, x)\right)_{\nu_{1}}\right|}{1+\|x\|_{1}^{p}}\right] \mathbb{E}\left[\left(1+\left\|W_{t}^{(\theta, l, 1)}-W_{s}^{(\theta, l, 1)}\right\|_{1}\right)^{p}\left|\left(1, \frac{W_{t}^{(\theta, l, l)}-W_{s}^{(\theta, l, 1)}}{t-s}\right)_{\nu}\right|\right] \tag{31}
\end{align*}
$$

Combining (41), (30), (31), the assumption (26), and the induction hypothesis demonstrates that for all $\theta \in \Theta$, $s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{equation*}
\mathbb{E}\left[\sup _{x \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{n+1, M, Q}^{\theta}(s, x)\right)_{\nu}\right|}{1+\|x\|_{1}^{p}}\right]<\infty . \tag{32}
\end{equation*}
$$

This finishes the induction step $\mathbb{N}_{0} \ni n \rightarrow n+1 \in \mathbb{N}$. Induction hence establishes (ili). Next we note that the triangle inequality and (11) imply that for all $\theta \in \Theta, n \in \mathbb{N}, s \in[0, T), t \in(s, T], x \in \mathbb{R}^{d}, \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\left|\left(F\left(\mathbf{U}_{n, M, Q}^{\theta}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right] \\
& \leq \mathbb{E}\left[\left|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right]+\sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \mathbb{E}\left[\left|\left(\mathbf{U}_{n, M, Q}^{\theta}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right] \\
& \leq\left(\left[\sup _{y \in \mathbb{R}^{d}} \frac{|(F(0))(t, y)|}{1+\|y\|_{1}^{p}}\right]+\sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \mathbb{E}\left[\sup _{y \in \mathbb{R}^{d}} \frac{\left|\left(\mathbf{U}_{n, M, Q}^{\theta}(s, y)\right)_{\nu}\right|}{1+\|y\|_{1}^{p}}\right]\right) \mathbb{E}\left[\left(1+\left\|x+W_{t}^{0}-W_{s}^{0}\right\|_{1}^{p}\right)\left|\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right] \tag{33}
\end{align*}
$$

This, (26), and (ii) prove (iii). Next we note that (5), (iii), the fact that $\left(\mathbf{U}_{n, M, Q}^{\theta}\right)_{n \in \mathbb{N}_{0}}, \theta \in \Theta$, are identically distributed, and a telescope argument yield that for all $n \in \mathbb{N}_{0}, \theta \in \Theta, s \in[0, T)$ it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
& \mathbb{E}\left[\mathbf{U}_{n, M, Q}^{\theta}(s, x)\right]-\mathbb{E}\left[g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right] \\
& =\sum_{l=0}^{n-1} \sum_{t \in(s, T)} q^{Q,[s, T]}(t) \mathbb{E}\left[\left(F\left(\mathbf{U}_{l, M, Q}^{(\theta, l, 0, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(\theta,-, 0, t)}\right)\right)\left(t, x+W_{t}^{(\theta, l, 0)}-W_{s}^{(\theta, l, 0)}\right)\left(1, \frac{W_{t}^{(\theta, l, 0)}-W_{s}^{(\theta, l, 0)}}{t-s}\right)\right] \\
& =\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{n-1, M, Q}^{\theta}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)\right] . \tag{34}
\end{align*}
$$

This establishes (iiii). The proof of Lemma 4.1 is thus completed.
Lemma 4.2 (Nonlinear Feynman-Kac formula \& Bismut-Elworthy-Li formula). Assume the setting in Section囼, let $p \in \mathbb{N}$ and assume that

$$
\begin{equation*}
\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{\left\|\mathbf{u}^{\infty}(t, x)\right\|_{1}}{1+\|x\|_{1}^{p}}+\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{|F(0)(t, x)|}{1+\|x\|_{1}^{p}}<\infty . \tag{35}
\end{equation*}
$$

Then
(i) for all $s \in[0, T], x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
u^{\infty}(s, x)-\mathbb{E}\left[g\left(x+W_{T-s}^{0}\right)\right]=\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right) d t\right] \tag{36}
\end{equation*}
$$

and
(ii) for all $s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\mathbf{u}^{\infty}(s, x)-\mathbb{E}\left[g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right]=\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right) d t\right] \tag{37}
\end{equation*}
$$

Proof of Lemma 4.2. First note that the triangle inequality, (1), and (35) ensure that

$$
\begin{equation*}
\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{\left|\left(F\left(\mathbf{u}^{\infty}\right)\right)(t, x)\right|}{1+\|x\|_{1}^{p}} \leq \sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{|(F(0))(t, x)|}{1+\|x\|_{1}^{p}}+\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{\sum_{\nu=1}^{d+1} L_{\nu}\left|\left(\mathbf{u}^{\infty}(t, x)\right)_{\nu}\right|}{1+\|x\|_{1}^{p}}<\infty . \tag{38}
\end{equation*}
$$

Itô's formula and the PDE (3) imply that for all $s \in[0, T], t \in[s, T], x \in \mathbb{R}^{d}$ it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
& u^{\infty}\left(t, x+W_{t}^{0}-W_{s}^{0}\right)-u^{\infty}(s, x) \\
& =\int_{s}^{t}\left(\frac{\partial}{\partial r} u^{\infty}+\frac{1}{2} \Delta_{y} u^{\infty}\right)\left(r, x+W_{r}^{0}-W_{s}^{0}\right) d r+\int_{s}^{t}\left\langle\left(\nabla_{y} u^{\infty}\right)\left(r, x+W_{r}^{0}-W_{s}^{0}\right), d W_{r}^{0}\right\rangle  \tag{39}\\
& =-\int_{s}^{t}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(r, x+W_{r}^{0}-W_{s}^{0}\right) d r+\int_{s}^{t}\left\langle\left(\nabla_{y} u^{\infty}\right)\left(r, x+W_{r}^{0}-W_{s}^{0}\right), d W_{r}^{0}\right\rangle .
\end{align*}
$$

This, (35), and (38) show that for all $s \in[0, T], x \in \mathbb{R}^{d}$ it holds that $\mathbb{E}\left[\sup _{t \in[s, T]} \mid \int_{s}^{t}\left\langle\left(\nabla_{y} u^{\infty}\right)\left(r, x+W_{r}^{0}-\right.\right.\right.$ $\left.\left.\left.W_{s}^{0}\right), d W_{r}^{0}\right\rangle \mid\right]<\infty$. This ensures that $\mathbb{E}\left[\int_{s}^{T}\left\langle\left(\nabla_{y} u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right), d W_{t}^{0}\right\rangle\right]=0$. This and (39) prove for all $s \in[0, T], x \in \mathbb{R}^{d}$ that

$$
\begin{equation*}
u^{\infty}(s, x)-\mathbb{E}\left[g\left(x+W_{T-s}^{0}\right)\right]=u^{\infty}(s, x)-\mathbb{E}\left[u^{\infty}\left(T, x+W_{T}^{0}-W_{s}^{0}\right)\right]=\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right) d t\right] \tag{40}
\end{equation*}
$$

This proves (il). Next, the Bismut-Elworthy-Li formula (see, e.g., [5, Proposition 3.2]) together with (35) show that for all $i \in\{1, \ldots, d\}, s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\left.\frac{\partial}{\partial x_{i}} \mathbb{E}\left[g\left(x+W_{T-s}^{0}\right)\right]\right]=\mathbb{E}\left[g\left(x+W_{T-s}^{0}\right) \frac{\left(W_{T-s}^{0}\right)_{i}}{T-s}\right] . \tag{41}
\end{equation*}
$$

Moreover, the Bismut-Elworthy-Li formula (see, e.g., 5, Proposition 3.2]) together with (38) demonstrate that for all $i \in\{1, \ldots, d\}, s \in[0, T], t \in(s, T], x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\right]=\mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right) \frac{\left(W_{t-s}^{0}\right)_{i}}{t-s}\right] \tag{42}
\end{equation*}
$$

This and (38) ensure that for all $i \in\{1, \ldots, d\}, s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\frac{\partial}{\partial x_{i}} \int_{s}^{T} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\right] d t=\int_{s}^{T} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right) \frac{\left(W_{t-s}^{0}\right)_{i}}{t-s}\right] d t \tag{43}
\end{equation*}
$$

Combining this, Fubini's theorem, (36), and (41) shows that for all $s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\mathbf{u}^{\infty}(s, x)-\mathbb{E}\left[g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right]=\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right) d t\right] \tag{44}
\end{equation*}
$$

This proves (iii). The proof of Lemma 4.2 is thus completed.
Lemma 4.3 (Recursive bound for global error). Assume the setting in Section 圆, let $p, M, Q \in \mathbb{N}$, assume that

$$
\begin{equation*}
\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{\left\|\mathbf{u}^{\infty}(t, x)\right\|_{1}}{1+\|x\|_{1}^{p}}+\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d}} \frac{|F(0)(t, x)|}{1+\|x\|_{1}^{p}}<\infty \tag{45}
\end{equation*}
$$

and let $\varepsilon:[0, T] \times \mathbb{R}^{d} \rightarrow[0, \infty]^{d+1}$ be the function that satisfies for all $s \in[0, T], x \in \mathbb{R}^{d}, \nu \in\{1, \ldots, d+1\}$ that

$$
\begin{equation*}
(\varepsilon(s, x))_{\nu}=\left\lvert\, \mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{u}^{\infty}\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu}-\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu} d t\right] .\right.\right. \tag{46}
\end{equation*}
$$

Then for all $n, k \in \mathbb{N},\left(t_{0}, x\right) \in[0, T) \times \mathbb{R}^{d}, \nu_{0} \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \sum_{j=0}^{k-1} \sum_{\substack{ }} \sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, l_{1}, \ldots, l_{j+1} \in \mathbb{N}, l_{1}<\ldots<l_{j+1}=n}} \sum_{t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T} \sum_{1, \ldots, \nu_{j+1} \in\{1, \ldots, d+1\}} \frac{2^{j}}{\sqrt{M^{n-j-l_{1}}}}\left[\prod_{i=1}^{j} L_{\nu_{i}} q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)\right] \\
& \cdot\left\{\mathbb { 1 } _ { \{ 1 \} } ( \nu _ { j + 1 } ) \left(\mathbb { 1 } _ { \{ T \} } ( t _ { j + 1 } ) \left(\left\|\left(\varepsilon\left(t_{j}, x+W_{t_{j}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{j}} \prod_{i=1}^{j}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right.\right.\right. \\
& \left.+\frac{1}{\sqrt{M^{l_{1}}}}\left\|\left(g\left(x+W_{T}^{0}-W_{t_{0}}^{0}\right)-g\left(x+W_{t_{j}}^{0}-W_{t_{0}}^{0}\right)\right) \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.+\frac{q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)}{\sqrt{M^{l_{1}}}}\left\|(F(0))\left(t_{j+1}, x+W_{t_{j+1}}^{0}-W_{t_{0}}^{0}\right) \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right)  \tag{47}\\
& \left.+\frac{L_{\nu_{j+1}} q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)}{\sqrt{M^{l_{1}-1}}}\left\|\left(\mathbf{u}^{\infty}\left(t_{j+1}, x+W_{t_{j+1}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{j+1}} \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right\} \\
& +\sum_{\substack{l_{1}, \ldots, l_{k} \in \mathbb{N}, l_{1}<\ldots<l_{k}<n}} \sum_{\substack{t_{1}, \ldots, t_{k} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{k}<T}} \sum_{\nu_{1}, \ldots, \nu_{k} \in\{1, \ldots, d+1\}} \frac{2^{k}}{\sqrt{M^{n-k-l_{1}}}}\left[\prod_{i=1}^{k} L_{\nu_{i}} q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)\right] \\
& \cdot\left\|\left(\left(\mathbf{U}_{l_{1}, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t_{k}, x+W_{t_{k}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{k}} \prod_{i=1}^{k}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} .
\end{align*}
$$

Proof of Lemma 4.3. We note that (45) and (11) ensure that the function $\varepsilon$ is well-defined. First, we analyze the Monte Carlo error. Independence, Items (ii) and (iii) of Lemma 4.1, and (5) imply that for all $m \in \mathbb{N}, x \in \mathbb{R}^{d}$,
$s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \operatorname{Var}\left(\left(\mathbf{U}_{m, M, Q}^{0}(s, x)\right)_{\nu}\right)=\frac{1}{M^{m}} \operatorname{Var}\left(\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right) \\
& +\sum_{l=0}^{m-1} \frac{1}{M^{m-l}} \operatorname{Var}\left(\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{l, M, Q}^{(0, l, 1, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(0,-l, 1, t)}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right) \\
& \leq \frac{1}{M^{m}} \mathbb{E}\left[\left|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right|^{2}\right] \\
& +\sum_{l=0}^{m-1} \frac{1}{M^{m-l}} \mathbb{E}\left[\left|\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{l, M, Q}^{(0, l, 1, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(0,-l, 1, t)}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|^{2}\right] \tag{48}
\end{align*}
$$

Combining this, the triangle inequality, and (11) yields that for all $m \in \mathbb{N}, x \in \mathbb{R}^{d}, s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{m, M, Q}^{0}(s, x)-\mathbb{E}\left[\mathbf{U}_{m, M, Q}^{0}(s, x)\right]\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}=\left(\operatorname{Var}\left(\left(\mathbf{U}_{m, M, Q}^{0}(s, x)\right)_{\nu}\right)\right)^{1 / 2} \\
& \leq \frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=0}^{m-1}\left[\sum_{t \in(s, T)} \frac{q^{Q,[s, T]}(t)}{\sqrt{M^{m-l}}}\left\|\left(F\left(\mathbf{U}_{l, M, Q}^{(0, l, 1, t)}\right)-\mathbb{1}_{\mathbb{N}}(l) F\left(\mathbf{U}_{l-1, M, Q}^{(0,-l, 1, t)}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right] \\
& \leq \frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{m}}} \sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left\|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=1}^{m-1}\left[\sum_{t \in(s, T)} \frac{q^{Q,[s, T]}(t)}{\sqrt{M^{m-l}}}\left\|\sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \mid\left(\left(\mathbf{U}_{l, M, Q}^{(0, l, 1, t)}-\mathbf{U}_{l-1, M, Q}^{(0,-l, 1, t)}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\right\|\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\| \|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right] \tag{49}
\end{align*}
$$

This and the triangle inequality ensure that for all $m \in \mathbb{N}, x \in \mathbb{R}^{d}, s \in[0, T), \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{m, M, Q}^{0}(s, x)-\mathbb{E}\left[\mathbf{U}_{m, M, Q}^{0}(s, x)\right]\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{m}}} \sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left\|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=1}^{m-1} \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{L_{\nu_{1}} q^{Q,[s, T]}(t)}{\sqrt{M^{m-l}}}\left\|\left(\left(\mathbf{U}_{l, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=1}^{m-1} \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{L_{\nu_{1}} q^{Q,[s, T]}(t)}{\sqrt{M^{m-l}}}\left\|\left(\left(\mathbf{U}_{l-1, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& =\frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{m}}} \sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left\|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=0}^{m-1} \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{L_{\nu_{1}} q^{Q,[s, T]}}{\sqrt{M^{m-l-1}}(t)}\left(\frac{\mathbb{1}_{(0, m)}(l)}{\sqrt{M}}+\mathbb{1}_{(-1, m-1)}(l)\right)\left\|\left(\left(\mathbf{U}_{l, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \tag{50}
\end{align*}
$$

Next we analyze the time discretization error. Item (iiii) of Lemma 4.1 ensures that for all $m \in \mathbb{N}, s \in[0, T)$, $x \in \mathbb{R}^{d}$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\mathbf{U}_{m, M, Q}^{0}(s, x)-g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right] \\
& =\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{m-1, M, Q}^{0}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)\right] . \tag{51}
\end{align*}
$$

Item (iii) of Lemma 4.2 proves that for all $s \in[0, T), x \in \mathbb{R}^{d}$ it holds that

$$
\begin{equation*}
\mathbf{u}^{\infty}(s, x)-\mathbb{E}\left[g\left(x+W_{T}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)\right]=\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right) d t\right] \tag{52}
\end{equation*}
$$

This, (51), the triangle inequality, and Jensen's inequality show for all $m \in \mathbb{N}, s \in[0, T), x \in \mathbb{R}^{d}, \nu \in$ $\{1, \ldots, d+1\}$ that

$$
\begin{align*}
& \mid(\mathbb{E} {\left.\left[\mathbf{U}_{m, M, Q}^{0}(s, x)\right]-\mathbf{u}^{\infty}(s, x)\right)_{\nu} \mid } \\
&=\left\lvert\, \mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{m-1, M, Q}^{0}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right]\right. \\
& \left.-\mathbb{E}\left[\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu} d t\right] \right\rvert\, \\
& \leq\left|\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{m-1, M, Q}^{0}\right)-F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right]\right| \\
&+\left|\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu}-\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu} d t\right]\right|  \tag{53}\\
&=\left|\mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{U}_{m-1, M, Q}^{0}\right)-F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right]\right|+(\varepsilon(s, x))_{\nu} \\
& \leq \mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left[\sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}}\left|\left(\left(\mathbf{U}_{m-1, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\right|\right]\left|\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right|\right]+(\varepsilon(s, x))_{\nu} \\
& \leq \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} q^{Q,[s, T]}(t)\left\|\left(\left(\mathbf{U}_{m-1, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}+(\varepsilon(s, x))_{\nu} .
\end{align*}
$$

In the next step we combine the established bounds for the Monte Carlo error and for the time discretization error to obtain a bound for the global error. More formally, observe that (50) and (53) ensure that for all $m \in \mathbb{N}$, $s \in[0, T), x \in \mathbb{R}^{d}, \nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{m, M, Q}^{0}(s, x)-\mathbf{u}^{\infty}(s, x)\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq\left\|\left(\mathbf{U}_{m, M, Q}^{0}(s, x)-\mathbb{E}\left[\mathbf{U}_{m, M, Q}^{0}(s, x)\right]\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}+\left|\left(\mathbb{E}\left[\mathbf{U}_{m, M, Q}^{0}(s, x)\right]-\mathbf{u}^{\infty}(s, x)\right)_{\nu}\right|^{\leq} \frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{m}}} \sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left\|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}+(\varepsilon(s, x))_{\nu} \\
& +\sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} q^{Q,[s, T]}(t)\left\|\left(\left(\mathbf{U}_{m-1, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=0}^{m-1} \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{L_{\nu_{1}} q^{Q,[s, T]}(t)}{\sqrt{M^{m-l-1}}}\left(\frac{\mathbb{1}_{(0, m)}(l)}{\sqrt{M}}+\mathbb{1}_{(-1, m-1)}(l)\right)\left\|\left(\left(\mathbf{U}_{l, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq(\varepsilon(s, x))_{\nu}+\frac{1}{\sqrt{M^{m}}}\left\|\left(g\left(x+W_{T}^{0}-W_{s}^{0}\right)-g(x)\right)\left(1, \frac{W_{T}^{0}-W_{s}^{0}}{T-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{m}}} \sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left\|(F(0))\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{L_{\nu_{1}} q^{Q,[s, T]}(t)}{\sqrt{M^{m-1}}}\left\|\left(\mathbf{u}^{\infty}\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l=1}^{m-1} \sum_{t \in(s, T)} \sum_{\nu_{1}=1}^{d+1} \frac{2 L_{\nu_{1}} q^{Q,[s, T]}(t)}{\sqrt{M^{m-l-1}}}\left\|\left(\left(\mathbf{U}_{l, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right)_{\nu_{1}}\left(1, \frac{W_{t}^{0}-W_{s}^{0}}{t-s}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}
\end{align*}
$$

We prove (47) by induction on $k \in \mathbb{N}$. The base case $k=1$ follows immediately from (54). For the induction step $\mathbb{N} \ni k \mapsto k+1 \in \mathbb{N}$ let $k \in \mathbb{N}$ and assume that (47) holds for $k$. Inequality (54) and independence of $\left(\mathbf{U}_{m, M, Q}^{0}\right)_{m \in \mathbb{N}_{0}}$ and $W^{0}$ yield that for all $m \in \mathbb{N}, t_{1}, \ldots, t_{k} \in \mathbb{R}, x \in \mathbb{R}^{d}, \nu_{0}, \ldots, \nu_{k} \in\{1, \ldots, d+1\}$ with $t_{0}<t_{1}<\ldots<t_{k}<T$ it holds that

$$
\begin{align*}
& \left\|\left(\left(\mathbf{U}_{l_{1}, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t_{k}, x+W_{t_{k}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{k}} \prod_{i=1}^{k}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& =\left(\mathbb{E}\left[\left(\|\left(\mathbf{U}_{l_{1}, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t_{k}, z\right)\right)_{\nu_{k}} \|\left._{L^{2}(\mathbb{P} ; \mathbb{R})}^{2}\right|_{z=x+W_{t_{k}}^{0}-W_{t_{0}}^{0}} \prod_{i=1}^{k}\left(\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right)^{2}\right]\right)^{\frac{1}{2}} \\
& \leq\left\|\left(\varepsilon\left(t_{k}, x+W_{t_{k}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{k}} \prod_{i=1}^{k}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{l_{1}}}}\left\|\left(g\left(x+W_{T}^{0}-W_{t_{0}}^{0}\right)-g\left(x+W_{t_{k}}^{0}-W_{t_{0}}^{0}\right)\right)\left(1, \frac{W_{T}^{0}-W_{t_{k}}^{0}}{T-t_{k}}\right)_{\nu_{k}} \prod_{i=1}^{k}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{1}{\sqrt{M^{l_{1}}}} \sum_{t_{k+1} \in\left(t_{k}, T\right)} q^{Q,\left[t_{k}, T\right]}\left(t_{k+1}\right)\left\|(F(0))\left(t_{k+1}, x+W_{t_{k+1}}^{0}-W_{t_{0}}^{0}\right) \prod_{i=1}^{k+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{t_{k+1} \in\left(t_{k}, T\right)} \sum_{\nu_{k+1}=1}^{d+1} \frac{L_{\nu_{k+1}} q^{Q,\left[t_{k}, T\right]}\left(t_{k+1}\right)}{\sqrt{M^{l_{1}-1}}}\left\|\left(\mathbf{u}^{\infty}\left(t_{k+1}, x+W_{t_{k+1}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{k+1}} \prod_{i=1}^{k+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{l_{0}=1}^{m-1} \sum_{t_{k+1} \in\left(t_{k}, T\right)} \sum_{\nu_{k+1}=1}^{d+1} \frac{2 L_{\nu_{k+1}} q^{Q,\left[t_{k}, T\right]}\left(t_{k+1}\right)}{\sqrt{M^{l_{1}-1-l_{0}}}} \\
& \left\|\left(\left(\mathbf{U}_{l_{0}, M, Q}^{0}-\mathbf{u}^{\infty}\right)\left(t_{k+1}, x+W_{t_{k+1}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{k+1}} \prod_{i=1}^{k+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} . \tag{55}
\end{align*}
$$

This and the induction hypothesis complete the induction step $\mathbb{N} \ni k \rightarrow k+1 \in \mathbb{N}$. Induction hence establishes (47). This finishes the proof of Lemma 4.3

Theorem 4.4 (Global approximation error). Assume the setting in Section 园, let $p, n, Q \in \mathbb{N}, M \in \mathbb{N} \cap[2, \infty)$, $\nu_{0} \in\{1, \ldots, d+1\},\left(t_{0}, x\right) \in[0, T) \times \mathbb{R}^{d}$, assume that

$$
\begin{equation*}
\sup _{(t, z) \in[0, T] \times \mathbb{R}^{d}} \frac{\left\|\mathbf{u}^{\infty}(t, z)\right\|_{1}}{1+\|z\|_{1}^{p}}+\sup _{(t, z) \in[0, T] \times \mathbb{R}^{d}} \frac{|F(0)(t, z)|}{1+\|z\|_{1}^{p}}<\infty, \tag{56}
\end{equation*}
$$

let $C \in[0, \infty)$ be the real number given by

$$
\begin{equation*}
C=2\left(\sqrt{T-t_{0}}+1\right) \sqrt{\left(T-t_{0}\right) \pi}\left(\|L\|_{1}+1\right)+1 \tag{57}
\end{equation*}
$$

and let $\varepsilon:[0, T] \times \mathbb{R}^{d} \rightarrow[0, \infty]^{d+1}$ be the function that satisfies for all $s \in[0, T], y \in \mathbb{R}^{d}, \nu \in\{1, \ldots, d+1\}$ that

$$
\begin{equation*}
(\varepsilon(s, y))_{\nu}=\left\lvert\, \mathbb{E}\left[\left.\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{u}^{\infty}\right)\left(t, y+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu}-\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, y+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu} d t\right] \right\rvert\,\right.\right. \tag{58}
\end{equation*}
$$

Then it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\left(14(4 C)^{n-1}+1\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right] . \tag{59}
\end{align*}
$$

Proof of Theorem 4.4. Lemma 4.3 implies that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \sum_{j=0}^{n-1} \sum_{\substack{l_{1}, \ldots, l_{j+1} \in \mathbb{N}, l_{1}<\ldots<l_{j+1}=n}} \sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T}} \sum_{\nu_{1}, \ldots, \nu_{j+1} \in\{1, \ldots, d+1\}} \frac{2^{j}}{\sqrt{M^{n-j-l_{1}}}}\left[\prod_{i=1}^{j} L_{\nu_{i}} q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)\right] \\
& \quad \cdot\left\{\mathbb { 1 } _ { \{ 1 \} } ( \nu _ { j + 1 } ) \left(\mathbb { 1 } _ { \{ T \} } ( t _ { j + 1 } ) \left(\left\|\left(\varepsilon\left(t_{j}, x+W_{t_{j}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{j}} \prod_{i=1}^{j}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right.\right.\right. \\
& \left.\quad+\frac{1}{\sqrt{M^{l_{1}}}}\left\|\left(g\left(x+W_{T}^{0}-W_{t_{0}}^{0}\right)-g\left(x+W_{t_{j}}^{0}-W_{t_{0}}^{0}\right)\right) \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right)  \tag{60}\\
& \left.\quad+\frac{q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)}{\sqrt{M^{l_{1}}}}\left\|(F(0))\left(t_{j+1}, x+W_{t_{j+1}}^{0}-W_{t_{0}}^{0}\right) \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.\quad+\frac{L_{\nu_{j+1}} q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)}{\sqrt{M^{l_{1}-1}}}\left\|\left(\mathbf{u}^{\infty}\left(t_{j+1}, x+W_{t_{j+1}}^{0}-W_{t_{0}}^{0}\right)\right)_{\nu_{j+1}}^{j+1} \prod_{i=1}^{j+1}\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right\}
\end{align*}
$$

This, (22) and independence of Brownian increments prove that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \sum_{j=0}^{n-1} \sum_{\substack{ \\
l_{1}, \ldots, l_{j+1} \in \mathbb{N}, l_{1}<\ldots<l_{j+1}=n}} \sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T}} \sum_{\nu_{1}, \ldots, \nu_{j+1} \in\{1, \ldots, d+1\}} \frac{2^{j}}{\sqrt{M^{n-j-l_{1}}}}\left[\prod_{i=1}^{j} L_{\nu_{i}} q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)\right] \\
& \left\{\mathbb { 1 } _ { \{ 1 \} } ( \nu _ { j + 1 } ) \left(\mathbb { 1 } _ { \{ T \} } ( t _ { j + 1 } ) \left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{j}}\right] \prod_{i=1}^{j}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right.\right.\right.  \tag{61}\\
& \left.+\sum_{\alpha=1}^{d} \frac{K_{\alpha}}{\sqrt{M^{l_{1}}}}\left\|\left(W_{T}^{0}-W_{t_{j}}^{0}\right)_{\alpha}\left(1, \frac{W_{T}^{0}-W_{t_{j}}^{0}}{T-t_{j}}\right)_{\nu_{j}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \prod_{i=1}^{j}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.+\frac{q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M^{l_{1}}}} \prod_{i=1}^{j+1}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.+\frac{L_{\nu_{j+1}} q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left|\left(\mathbf{u}^{\infty}(t, z)\right)_{\nu_{j+1}}\right|\right]}{\sqrt{M^{l_{1}-1}}} \prod_{i=1}^{j+1}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right\} .
\end{align*}
$$

It holds for all $\nu \in\{1, \ldots, d+1\}, t \in[0, T)$ that
$\sum_{\alpha=1}^{d} K_{\alpha}\left\|\left(W_{T}^{0}-W_{t}^{0}\right)_{\alpha}\left(1, \frac{W_{T}^{0}-W_{t}^{0}}{T-t}\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}$
$=\sum_{\alpha=1}^{d} K_{\alpha}\left(\sqrt{T-t} \mathbb{1}_{\{1\}}(\nu)+\frac{\mathbb{1}_{[2, \infty)}(\nu)}{T-t}\left\|\left(W_{T}^{0}-W_{t}^{0}\right)_{\alpha}\left(W_{T}^{0}-W_{t}^{0}\right)_{\nu-1}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right)$
$=\sqrt{T-t}\|K\|_{1} \mathbb{1}_{\{1\}}(\nu)+\frac{\mathbb{1}_{[2, \infty)}(\nu)}{T-t}\left(K_{\nu-1}\left\|\left(W_{T}^{0}-W_{t}^{0}\right)_{\nu-1}^{2}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}+\sum_{\alpha \in\{1, \ldots, d\} \backslash\{\nu-1\}} K_{\alpha}\left\|\left(W_{T}^{0}-W_{t}^{0}\right)_{\alpha}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}^{2}\right)$
$=\sqrt{T-t}\|K\|_{1} \mathbb{1}_{\{1\}}(\nu)+\mathbb{1}_{[2, \infty)}(\nu)\left(\sqrt{3} K_{\nu-1}+\sum_{\alpha \in\{1, \ldots, d\} \backslash\{\nu-1\}} K_{\alpha}\right)$
$\leq \max \{\sqrt{T-t}, \sqrt{3}\}\|K\|_{1}$.

This and (61) show that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}} \\
& +\frac{\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M^{n}}} \sum_{t_{1} \in\left(t_{0}, T\right]} q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)\left\|\left(1, \frac{W_{t_{1}}^{0}-W_{t_{0}}^{0}}{t_{1}-t_{0}}\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\frac{\left[\sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}} \sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left|\left(\mathbf{u}^{\infty}(t, z)\right)_{\nu_{1}}\right|\right]}{\sqrt{M^{n-1}}} \sum_{t_{1} \in\left(t_{0}, T\right]} q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)\left\|\left(1, \frac{W_{t_{1}}^{0}-W_{t_{0}}^{0}}{t_{1}-t_{0}}\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& +\sum_{j=1}^{n-1} \sum_{\substack{l_{1}, \ldots, l_{j} \in \mathbb{N}, l_{1}<\ldots<l_{j}<n}} \sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T}} \sum_{\nu_{1}, \ldots, \nu_{j+1} \in\{1, \ldots, d+1\}} \frac{2^{j}}{\sqrt{M^{n-j-l_{1}}}}\left[\prod_{i=1}^{j} L_{\nu_{i}} q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)\right]  \tag{63}\\
& \cdot\left\{\mathbb { 1 } _ { \{ 1 \} } ( \nu _ { j + 1 } ) \left(\mathbb { 1 } _ { \{ T \} } ( t _ { j + 1 } ) \left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{j}}\right] \prod_{i=1}^{j}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right.\right.\right. \\
& \left.+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{l_{1}}}} \prod_{i=1}^{j}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.+\frac{q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M^{l_{1}}}} \prod_{i=1}^{j+1}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right) \\
& \left.+\frac{L_{\nu_{j+1}} q^{Q,\left[t_{j}, T\right]}\left(t_{j+1}\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left|\left(\mathbf{u}^{\infty}(t, z)\right)_{\nu_{j+1}}\right|\right]}{\sqrt{M^{1_{1}-1}}} \prod_{i=1}^{j+1}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right\} .
\end{align*}
$$

For all $j \in \mathbb{N}, \nu_{0}, \ldots, \nu_{j-1} \in\{1, \ldots, d+1\}$, and $t_{1}, \ldots, t_{j} \in \mathbb{R}$ satisfying $t_{0}<t_{1}<\ldots<t_{j}<T$ it holds that

$$
\begin{equation*}
\prod_{i=1}^{j}\left\|\left(1, \frac{W_{t_{i}}^{0}-W_{t_{i-1}}^{0}}{t_{i}-t_{i-1}}\right)_{\nu_{i-1}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}=\prod_{i=1}^{j}\left[\mathbb{1}_{\{1\}}\left(\nu_{i-1}\right)+\frac{\mathbb{1}_{[2, \infty)}\left(\nu_{i-1}\right)}{\sqrt{t_{i}-t_{i-1}}}\right] \leq\left(\sqrt{T-t_{0}}+1\right)^{j} \prod_{i=1}^{j} \frac{1}{\sqrt{t_{i}-t_{i-1}}} \tag{64}
\end{equation*}
$$

This and (63) ensure that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}}+\frac{\left(\sqrt{T-t_{0}}+1\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M^{n}}} \sum_{t_{1} \in\left(t_{0}, T\right]} \frac{q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)}{\sqrt{t_{1}-t_{0}}} \\
& +\frac{\left(\sqrt{T-t_{0}}+1\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right] \sum_{\nu_{1}=1}^{d+1} L_{\nu_{1}}}{\sqrt{M^{n-1}}} \sum_{t_{1} \in\left(t_{0}, T\right]} \frac{q^{Q,\left[t_{0}, T\right]}\left(t_{1}\right)}{\sqrt{t_{1}-t_{0}}} \\
& +\sum_{j=1}^{n-1} \sum_{\substack{l_{1}, \ldots, l_{j} \in \mathbb{N}, l_{1}<\ldots<l_{j}<n}}\left\{\frac{2^{j}\left(\sqrt{T-t_{0}}+1\right)^{j}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right]}{\sqrt{M^{n-j-l_{1}}}}\left[\sum_{\substack{t_{1}, \ldots, t_{j} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<T}} \prod_{i=1}^{j} \frac{q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)}{\sqrt{t_{i}-t_{i-1}}}\right]\left[\sum_{\nu_{1}, \ldots, \nu_{j} \in\{1, \ldots, d+1\}} \prod_{i=1}^{j} L_{\nu_{i}}\right]\right. \\
& +\frac{2^{j}\left(\sqrt{T-t_{0}}+1\right)^{j} \max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n-j}}}\left[\sum_{\substack{t_{1}, \ldots, t_{j} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<T}} \prod_{i=1}^{j} \frac{q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)}{\sqrt{t_{i}-t_{i-1}}}\right]\left[\sum_{\nu_{1}, \ldots, \nu_{j} \in\{1, \ldots, d+1\}} \prod_{i=1}^{j} L_{\nu_{i}}\right] \\
& +\frac{2^{j}\left(\sqrt{T-t_{0}}+1\right)^{j+1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M^{n-j}}}\left[\sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T}} \prod_{i=1}^{j+1} \frac{q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)}{\sqrt{t_{i}-t_{i-1}}}\right]\left[\sum_{\nu_{1}, \ldots, \nu_{j} \in\{1, \ldots, d+1\}} \prod_{i=1}^{j} L_{\nu_{i}}\right] \\
& +\frac{2^{j}\left(\sqrt{T-t_{0}}+1\right)^{j+1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]}{\sqrt{M^{n-j-1}}}\left[\sum_{\substack{t_{1}, \ldots, t_{j}, t_{j+1} \in \mathbb{R}, t_{0}<t_{1}<\ldots<t_{j}<t_{j+1} \leq T}} \prod_{i=1}^{j+1} \frac{q^{Q,\left[t_{i-1}, T\right]}\left(t_{i}\right)}{\sqrt{t_{i}-t_{i-1}}}\right]\left[\sum_{\nu_{1}, \ldots, \nu_{j+1} \in\{1, \ldots, d+1\}} \prod_{i=1}^{j+1} L_{\nu_{i}}\right] . \tag{65}
\end{align*}
$$

Observe that for all $j \in \mathbb{N}$ it holds that

$$
\begin{equation*}
\left[\sum_{\nu_{1}, \ldots, \nu_{j} \in\{1, \ldots, d+1\}} \prod_{i=1}^{j} L_{\nu_{i}}\right]=\|L\|_{1}^{j} . \tag{66}
\end{equation*}
$$

This, Lemma 3.3. (65), and the fact that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ imply that

$$
\left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}
$$

$$
\leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}}
$$

$$
+\frac{2 \sqrt{T-t_{0}}\left(\sqrt{T-t_{0}}+1\right)}{\sqrt{M^{n-1}}}\left(\frac{\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]}{\sqrt{M}}+\|L\|_{1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]\right)
$$

$$
+2 \sum_{j=1}^{n-1} \sum_{\substack{l_{1}, \ldots, l_{j} \in \mathbb{N}, l_{1}<\ldots<l_{j}<n}}\left\{\frac{2^{j}\|L\|_{1}^{j}\left(\sqrt{T-t_{0}}+1\right)^{j}\left(\sqrt{\left(T-t_{0}\right) \pi}\right)^{j}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right]}{\sqrt{M^{n-j-l_{1}}} \Gamma\left(\frac{j}{2}\right)}\right.
$$

$$
+\frac{2^{j}\|L\|_{1}^{j}\left(\sqrt{T-t_{0}}+1\right)^{j}\left(\sqrt{\left(T-t_{0}\right) \pi}\right)^{j} \max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n-j}} \Gamma\left(\frac{j}{2}\right)}+\frac{2^{j}\|L\|_{1}^{j}\left(\sqrt{T-t_{0}}+1\right)^{j+1}\left(\sqrt{\left(T-t_{0}\right) \pi}\right)^{j+1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \mid(F(0))(t, z) \|\right]}{\sqrt{M^{n-j}} \Gamma\left(\frac{j+1}{2}\right)}
$$

$$
\begin{equation*}
\left.+\frac{2^{j}\|L\|_{1}^{j+1}\left(\sqrt{T-t_{0}}+1\right)^{j+1}\left(\sqrt{\left(T-t_{0}\right) \pi}\right)^{j+1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]}{\sqrt{M^{n-j-1}} \Gamma\left(\frac{(+1}{2}\right)}\right\} . \tag{67}
\end{equation*}
$$

This, Lemma 3.4 and the definition (57) of $C$ show that

It holds for all $r \in[0, \infty)$ that

$$
\begin{align*}
\sum_{j=0}^{n-1} \frac{r^{j}}{\Gamma\left(\frac{j+1}{2}\right)} & \leq \frac{1}{\sqrt{\pi}}+\sum_{j=1}^{n-1} \frac{r^{j}}{\Gamma\left(\left\lfloor\frac{j+1}{2}\right\rfloor\right)}=\frac{1}{\sqrt{\pi}}+\sum_{l=1}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{r^{2 l-1}}{\Gamma(l)}+\sum_{l=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{r^{2 l}}{\Gamma(l)}  \tag{69}\\
& =\frac{1}{\sqrt{\pi}}+\sum_{l=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1} \frac{r^{2 l+1}}{l!}+\sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor-1} \frac{r^{2 l+2}}{l!} \leq \frac{1}{\sqrt{\pi}}+r(r+1) e^{r^{2}}
\end{align*}
$$

$$
\begin{aligned}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}}+\frac{C\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]\right)}{\sqrt{\pi} \sqrt{M^{n-1}}} \\
& +\frac{2\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right]}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)} \sum_{l_{1}=1}^{n-j} \sqrt{M}^{l_{1}}\binom{n-l_{1}-1}{j-1} \\
& +\frac{2 \max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)}\binom{n-1}{j}+\frac{\left.C\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \mid(F(0))(t, z)\right]\right]}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j+1}{2}\right)}\binom{n-1}{j} \\
& +\frac{\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j+1}}{\Gamma\left(\frac{+1}{2}\right)}\binom{n-1}{j} \\
& \leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{2\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right]}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)} \sum_{l=1}^{n-j} \sqrt{M}^{l}\binom{n-l-1}{j-1} \\
& +\frac{2 \max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}} \sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)}\binom{n-1}{j} . \tag{68}
\end{align*}
$$

Note that it holds for all $j \in\{0, \ldots, n-1\}$ that $\binom{n-1}{j} \leq \sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$. This and (69) ensure that

$$
\begin{align*}
\sum_{j=0}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j+1}{2}\right)}\binom{n-1}{j} & \leq(2 C)^{n-1} \sum_{j=0}^{n-1} \frac{\sqrt{M}^{j}}{\Gamma\left(\frac{j+1}{2}\right)} \leq(2 C)^{n-1}\left(\frac{1}{\sqrt{\pi}}+\sqrt{M}(\sqrt{M}+1) e^{M}\right)  \tag{70}\\
& \leq 3(2 C)^{n-1} M e^{M}
\end{align*}
$$

and that

$$
\begin{equation*}
\sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)}\binom{n-1}{j} \leq(2 C)^{n-1} \sqrt{M} \sum_{j=0}^{n-1} \frac{\sqrt{M}^{j}}{\Gamma\left(\frac{j+1}{2}\right)} \leq 3(2 C)^{n-1} \sqrt{M}^{3} e^{M} \tag{71}
\end{equation*}
$$

For all $j \in\{1, \ldots, n-1\}$ it holds that

$$
\begin{align*}
\sum_{l=1}^{n-j} \sqrt{M}^{l}\binom{n-l-1}{j-1} & =\sum_{l=j-1}^{n-2} \sqrt{M}^{n-l-1}\binom{l}{j-1} \leq \sqrt{M}^{n-1} \sum_{l=j-1}^{\infty}\left(\frac{1}{\sqrt{M}}\right)^{l}\binom{l}{j-1} \\
& =\frac{\sqrt{M}^{n-1}\left(\frac{1}{\sqrt{M}}\right)^{j-1}}{\left(1-\frac{1}{\sqrt{M}}\right)^{j}}=\frac{\sqrt{M}^{n-j}}{\left(1-\frac{1}{\sqrt{M}}\right)^{j}} . \tag{72}
\end{align*}
$$

This together with (69) ensures that

$$
\begin{align*}
\sum_{j=1}^{n-1} \frac{(C \sqrt{M})^{j}}{\Gamma\left(\frac{j}{2}\right)} \sum_{l=1}^{n-j} \sqrt{M}^{l}\binom{n-l-1}{j-1} & \leq \sqrt{M}^{n} \sum_{j=1}^{n-1} \frac{C^{j}}{\Gamma\left(\frac{j}{2}\right)\left(1-\frac{1}{\sqrt{M}}\right)^{j}} \leq \sqrt{M}^{n} \frac{C^{n-1}}{\left(1-\frac{1}{\sqrt{2}}\right)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{\Gamma\left(\frac{j}{2}\right)}  \tag{73}\\
& \leq(4 C)^{n-1} \sqrt{M}^{n}\left(\frac{1}{\sqrt{\pi}}+2 e\right) \leq 7(4 C)^{n-1} \sqrt{M}^{n} .
\end{align*}
$$

Combining (68), (70), (71), and (73) proves that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}(\varepsilon(t, z))_{\nu_{0}}\right]+\frac{\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n}}} \\
& +\frac{3 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{\left((t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}\right.}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]\right) \\
& +14(4 C)^{n-1}\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right]+\frac{6(2 C)^{n-1} e^{M} \max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}}{\sqrt{M^{n-3}}} \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\left(14(4 C)^{n-1}+1\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\|\varepsilon(t, z)\|_{\infty}\right] . \tag{74}
\end{align*}
$$

This completes the proof of Theorem 4.4
Lemma 4.5 (Quadrature error). Assume the setting in Section 圆, let $p, Q \in \mathbb{N}, x \in \mathbb{R}^{d}, s \in[0, T)$, and assume that $u^{\infty} \in C^{\infty}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$ and for all $k \in \mathbb{N}_{0}$ that

$$
\begin{equation*}
\sup _{(t, y) \in[0, T] \times \mathbb{R}^{d}} \frac{\left|\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, y)\right|}{1+\|y\|_{1}^{p}}<\infty . \tag{75}
\end{equation*}
$$

Then there exists $\xi \in[s, T]^{d+1}$ such that for all $\nu \in\{1, \ldots, d+1\}$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\sum_{t \in(s, T)} q^{Q,[s, T]}(t)\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu}-\int_{s}^{T}\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t-s}^{0}\right)\left(1, \frac{W_{t-s}^{0}}{t-s}\right)_{\nu} d t\right]  \tag{76}\\
& =\left(1, \nabla_{x}\right)_{\nu} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{2 Q+1} u^{\infty}\right)\left(\xi_{\nu}, x+W_{\xi_{\nu}}^{0}-W_{s}^{0}\right)\right] \frac{[Q!]^{4}(T-s)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}} .
\end{align*}
$$

Proof of Lemma 4.5. Observe that (75) and the dominated convergence theorem ensure that for every $k \in \mathbb{N}_{0}$ it holds that the function

$$
\begin{equation*}
[s, T] \ni t \mapsto \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(t, x+W_{t-s}^{0}\right)\right] \in \mathbb{R} \tag{77}
\end{equation*}
$$

is continuous. The assumption that $u^{\infty} \in C^{\infty}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$ and Itô's formula imply that for all $t \in[s, T]$, $k \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
& \left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)-\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(s, x)  \tag{78}\\
& =\int_{s}^{t}\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k+1} u^{\infty}\right)\left(v, x+W_{v}^{0}-W_{s}^{0}\right) d v+\int_{s}^{t}\left\langle\left(\nabla_{y}\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(v, x+W_{v}^{0}-W_{s}^{0}\right), d W_{v}^{0}\right\rangle .
\end{align*}
$$

This and (75) show that for all $k \in \mathbb{N}$ it holds that $\mathbb{E}\left[\sup _{t \in[s, T]}\left|\int_{s}^{t}\left\langle\left(\nabla_{y}\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(v, x+W_{v}^{0}-W_{s}^{0}\right), d W_{v}^{0}\right\rangle\right|\right]<$ $\infty$. This implies that for all $t \in[s, T], k \in \mathbb{N}$ it holds that $\mathbb{E}\left[\int_{s}^{t}\left\langle\left(\nabla_{y}\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(v, x+W_{v}^{0}-W_{s}^{0}\right), d W_{v}^{0}\right\rangle\right]=$ 0 . This, (78), and Fubini's theorem show that for all $t \in[s, T], k \in \mathbb{N}$ it holds that

$$
\begin{align*}
& \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]-\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(s, x) \\
& =\int_{s}^{t} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k+1} u^{\infty}\right)\left(v, x+W_{v}^{0}-W_{s}^{0}\right)\right] d v \tag{79}
\end{align*}
$$

Equation (79) (with $k=1$ ) together with (77) (with $k=2$ ) implies that the function $[s, T] \ni t \mapsto \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}\right.\right.$ $\mathbb{R}$ is continuously differentiable. Induction, (77), and (79) prove that it holds that the function $[s, T] \ni t \mapsto$ $\mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] \in \mathbb{R}$ is infinitely often differentiable. This, induction, and (79) demonstrate that for all $k \in \mathbb{N}, t \in[s, T]$ it holds that

$$
\begin{equation*}
\frac{\partial^{k}}{\partial t^{k}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]=\mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k+1} u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] \tag{80}
\end{equation*}
$$

Equation (3) and the error representation for the Gauß-Legendre quadrature rule (see, e.g., [1, Display (2.7.12)]) imply that there exists a real number $\xi_{1} \in[s, T]$ such that

$$
\begin{align*}
& \sum_{t \in[s, T]} q^{Q,[s, T]}(t) \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]-\int_{s}^{T} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] d t  \tag{81}\\
& =\int_{s}^{T} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] d t-\sum_{t \in[s, T]} q^{Q,[s, T]}(t) \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] \\
& =\left.\left(\frac{\partial^{2 Q}}{\partial t^{2 Q}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]\right)\right|_{t=\xi_{1}} \frac{[Q!]^{4}(T-s)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}} \\
& =\mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{2 Q+1} u^{\infty}\right)\left(\xi_{1}, x+W_{\xi_{1}}^{0}-W_{s}^{0}\right)\right] \frac{[Q!]^{4}(T-s)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}}
\end{align*}
$$

Equation (3), the Bismut-Elworthy-Li formula (see, e.g., [5, Proposition 3.2]) and the error representation for the Gauß-Legendre quadrature rule (see, e.g., [1, Display (2.7.12)]) imply for all $i \in\{1, \ldots, d\}$ that there exists a real number $\xi_{i+1} \in[s, T]$ such that

$$
\begin{align*}
& \sum_{t \in[s, T]} q^{Q,[s, T]}(t) \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(\frac{W_{s}^{0}-W_{t}^{0}}{s-t}\right)_{i}\right]-\int_{s}^{T} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\left(\frac{W_{s}^{0}-W_{t}^{0}}{s-t}\right)_{i}\right] d s \\
& =\sum_{t \in[s, T]} q^{Q,[s, T]}(t) \frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]-\int_{s}^{T} \frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(F\left(\mathbf{u}^{\infty}\right)\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] d t \\
& =\int_{s}^{T} \frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] d t \\
& \quad-\sum_{t \in[s, T]} q^{Q,[s, T]}(t) \frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right] \\
& =\left.\left(\frac{\partial^{2 Q}}{\partial t^{2 Q}} \frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right) u^{\infty}\right)\left(t, x+W_{t}^{0}-W_{s}^{0}\right)\right]\right)\right|_{t=\xi_{i}} \frac{[Q!]^{4}(T-s)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}} \\
& =\frac{\partial}{\partial x_{i}} \mathbb{E}\left[\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{2 Q+1} u^{\infty}\right)\left(\xi_{i}, x+W_{\xi_{i}}^{0}-W_{s}^{0}\right)\right] \frac{[Q!]^{4}(T-s)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}} . \tag{82}
\end{align*}
$$

This and (81) prove (76). This completes the proof of Lemma 4.5,

Corollary 4.6. Assume the setting in Section 图, assume that $u^{\infty} \in C^{\infty}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$, let $n, Q \in \mathbb{N}, M \in$ $\mathbb{N} \cap[2, \infty)$, $\nu_{0} \in\{1, \ldots, d+1\},\left(t_{0}, x\right) \in[0, T) \times \mathbb{R}^{d}, \alpha \in[0,1]$ and let $C \in[0, \infty)$ be the real number given by

$$
\begin{equation*}
C=2\left(\sqrt{T-t_{0}}+1\right) \sqrt{\left(T-t_{0}\right) \pi}\left(\|L\|_{1}+1\right)+1 \tag{83}
\end{equation*}
$$

Then it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\frac{\left(14(4 C)^{n-1}+1\right) T^{2 Q+1}}{Q^{2 \alpha Q}}\left[\sup _{k \in \mathbb{N}(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \frac{\left\|\left(1, \nabla_{y}\right)\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, z)\right\|_{\infty}}{(k!)^{1-\alpha}} \sup \right] . \tag{84}
\end{align*}
$$

Proof of Corollary 4.6. To prove (84) we assume w.l.o.g. that the right-hand side of (84) is finite. Observe that the Stirling-type formula in Robbins [10. Displays (1)-(2)] proves for all $k \in \mathbb{N}$ that

$$
\begin{equation*}
\sqrt{2 \pi k}\left[\frac{k}{e}\right]^{k} \leq k!\leq \sqrt{2 \pi k}\left[\frac{k}{e}\right]^{k} e^{\frac{1}{12}} \tag{85}
\end{equation*}
$$

This together with the fact that $e^{2} \leq 8$ and the fact that $\forall k \in \mathbb{N}: \pi e^{\frac{1}{3}} k \leq 8^{k}$ shows for all $k \in \mathbb{N}$ that

$$
\begin{align*}
\frac{k^{2 \alpha k}((2 k+1)!)^{1-\alpha}[k!]^{4}}{(2 k+1)[(2 k)!]^{3}} & \leq \frac{k^{2 \alpha k}[k!]^{4}}{[(2 k)!]^{2+\alpha}} \leq \frac{k^{2 \alpha k}\left[\sqrt{2 \pi} k^{k+\frac{1}{2}} e^{-k+\frac{1}{12}}\right]^{4}}{\left[\sqrt{2 \pi}(2 k)^{2 k+\frac{1}{2}} e^{-2 k}\right]^{2+\alpha}}=(\sqrt{2 \pi})^{2-\alpha} k^{1-\frac{\alpha}{2}} e^{\frac{1}{3}} 2^{-\left(2 k+\frac{1}{2}\right) 2}\left(\frac{e^{2 k}}{2^{2 k+\frac{1}{2}}}\right)^{\alpha}  \tag{86}\\
& \leq 2 \pi k e^{\frac{1}{3}} 2^{-4 k-1} e^{2 k} 2^{-2 k}=\pi e^{\frac{1}{3}} k\left(\frac{e^{2}}{64}\right)^{k} \leq \pi e^{\frac{1}{3}} k 8^{-k} \leq 1
\end{align*}
$$

Theorem 4.4 and Lemma 4.5 ensure that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\left(14(4 C)^{n-1}+1\right)\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\left(1, \nabla_{y}\right)\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{2 Q+1} u^{\infty}\right)(t, z)\right\|_{\infty} \frac{[Q!]^{4}(T-t)^{2 Q+1}}{(2 Q+1)[(2 Q)!]^{3}}\right] \tag{87}
\end{align*}
$$

It follows that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, M, Q}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})} \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\frac{\left(14(4 C)^{n-1}+1\right) T^{2 Q+1}}{Q^{2 \alpha Q}}\left[\sup _{l \in \mathbb{N}} \frac{2^{2 \alpha l}((2 l+1)!)^{1-\alpha}[l!]^{4}}{(2 l+1)[(2 l)!]^{3}}\right]\left[\sup _{k \in \mathbb{N}(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \frac{\left\|\left(1, \nabla_{y}\right)\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, z)\right\|_{\infty}}{(k!)^{1-\alpha}}\right] \\
& \leq \frac{7 C^{n} 2^{n-1} e^{M}}{\sqrt{M^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\frac{\left(14(4 C)^{n-1}+1\right) T^{2 Q+1}}{Q^{2 \alpha Q}}\left[\sup _{k \in \mathbb{N}} \sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \frac{\|\left(1, \nabla_{y}\right)\left(\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, z) \|_{\infty}\right.}{(k!)^{1-\alpha}}\right] . \tag{88}
\end{align*}
$$

This proves (84). The proof of Corollary 4.6 is thus completed.
The following corollary (Corollary 4.7) specializes Corollary 4.6 to the special case $n=M=Q$ and $\alpha=\frac{1}{4}$. For the choice of $\alpha$ note that the terms $\sqrt{M}^{-n}$ and $Q^{-2 \alpha Q}$ in the case $n=M=Q \in \mathbb{N} \cap[2, \infty)$ are equal if and only if $\alpha=\frac{1}{4}$.

Corollary 4.7. Assume the setting in Section 圆, assume that $u^{\infty} \in C^{\infty}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$, let $n \in \mathbb{N} \cap[2, \infty)$, $\nu_{0} \in\{1, \ldots, d+1\},\left(t_{0}, x\right) \in[0, T) \times \mathbb{R}^{d}$, and let $C \in[0, \infty)$ be the real number given by

$$
\begin{equation*}
C=2\left(\sqrt{T-t_{0}}+1\right) \sqrt{\left(T-t_{0}\right) \pi}\left(\|L\|_{1}+1\right)+1 \tag{89}
\end{equation*}
$$

Then it holds that

$$
\begin{align*}
& \left\|\left(\mathbf{U}_{n, n, n}^{0}\left(t_{0}, x\right)-\mathbf{u}^{\infty}\left(t_{0}, x\right)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}  \tag{90}\\
& \leq \frac{7 C^{n} 2^{n-1} e^{n}}{\sqrt{n^{n-3}}}\left(\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}|(F(0))(t, z)|\right]+\left[\sup _{(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}}\left\|\mathbf{u}^{\infty}(t, z)\right\|_{\infty}\right]+\max \left\{\sqrt{T-t_{0}}, \sqrt{3}\right\}\|K\|_{1}\right) \\
& +\frac{\left(14(4 C)^{n-1}+1\right) T^{2 n+1}}{\sqrt{n^{n}}}\left[\sup _{k \in \mathbb{N}(t, z) \in\left[t_{0}, T\right] \times \mathbb{R}^{d}} \frac{\|\left(1, \nabla_{y}\right)\left(\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, z) \|_{\infty}\right.}{(k!)^{3 / 4}}\right] .
\end{align*}
$$

The following main result of this article (Corollary 4.8) proves that if the constant (91) is finite, then the computational complexity (here measured in terms of the number of scalar normal random variables and in terms of function evaluations of $f$ and $g$ ) is bounded by $O\left(d \varepsilon^{-(4+\delta)}\right)$ for any $\delta \in(0, \infty)$ where $d$ is the dimensionality of the problem and $\varepsilon \in(0, \infty)$ is the prescribed accuracy.
Corollary 4.8 (Computational complexity in terms of global error). Assume the setting in Subsection圆, assume that $u^{\infty} \in C^{\infty}\left([0, T] \times \mathbb{R}^{d}, \mathbb{R}\right)$, let $\delta \in(0, \infty)$, let $C \in[0, \infty]$ be the extended real number given by

$$
\begin{equation*}
C=\left(\sup _{(t, z) \in[0, T] \times \mathbb{R}^{d}}|(F(0))(t, z)|+\sqrt{T+3}\|K\|_{1}+\sup _{k \in \mathbb{N}_{0}} \frac{\sup _{(t, z) \in[0, T] \times \mathbb{R}^{d}}\left\|\left(1, \nabla_{y}\right)\left(\left(\frac{\partial}{\partial r}+\frac{1}{2} \Delta_{y}\right)^{k} u^{\infty}\right)(t, z)\right\|_{\infty}}{(k!)^{3 / 4}}\right)^{(4+\delta)} \tag{91}
\end{equation*}
$$

assume that $C<\infty$, let $\left(\operatorname{RN}_{n, M, Q}\right)_{n, M, Q \in \mathbb{Z}} \subseteq \mathbb{N}_{0}$ be natural numbers which satisfy for all $n, M, Q \in \mathbb{N}$ that $\mathrm{RN}_{0, M, Q}=0$ and

$$
\begin{equation*}
\mathrm{RN}_{n, M, Q} \leq d M^{n}+\sum_{l=0}^{n-1}\left[Q M^{n-l}\left(d+\mathrm{RN}_{l, M, Q}+\mathbb{1}_{\mathbb{N}}(l) \cdot \mathrm{RN}_{l-1, M, Q}\right)\right] \tag{92}
\end{equation*}
$$

(for every $N \in \mathbb{N}$ we think of $\mathrm{RN}_{N, N, N}$ as the number of realizations of a scalar standard normal random variable required to compute one realization of the random variable $\left.U_{N, N, N}^{0}(0,0): \Omega \rightarrow \mathbb{R}\right)$, and let $\left(\mathrm{FE}_{n, M, Q}\right)_{n, M, Q \in \mathbb{Z}} \subseteq$ $\mathbb{N}_{0}$ be natural numbers which satisfy for all $n, M, Q \in \mathbb{N}$ that $\mathrm{FE}_{0, M, Q}=$ and

$$
\begin{equation*}
\mathrm{FE}_{n, M, Q} \leq M^{n}+\sum_{l=0}^{n-1}\left[Q M^{n-l}\left(1+\mathrm{FE}_{l, M, Q}+\mathbb{1}_{\mathbb{N}}(l)+\mathbb{1}_{\mathbb{N}}(l) \cdot \mathrm{FE}_{l-1, M, Q}\right)\right] \tag{93}
\end{equation*}
$$

(for every $N \in \mathbb{N}$ we think of $\mathrm{FE}_{N, N, N}$ as the number of function evaluations of $f$ and $g$ required to compute one realization of the random variable $\left.U_{N, N, N}^{0}(0,0): \Omega \rightarrow \mathbb{R}\right)$. Then it holds for all $N \in \mathbb{N}$ that

$$
\begin{align*}
& \mathrm{RN}_{N, N, N}+\mathrm{FE}_{N, N, N} \leq d\left[\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d} \nu \in\{1, \ldots, d+1\}}\left\|\left(\mathbf{U}_{N, N, N}^{0}(t, x)-\mathbf{u}^{\infty}(t, x)\right)_{\nu}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right]^{-(4+\delta)}  \tag{94}\\
& \cdot 16 C \sum_{n \in \mathbb{N}}(24(T+1))^{3(4+\delta) n}\left(\|L\|_{1}+1\right)^{(4+\delta) n} \sqrt{n}^{-\delta n}<\infty
\end{align*}
$$

Proof of Corollary 4.8. Lemma 3.15 and Lemma 3.16 in [3 imply that for all $N \in \mathbb{N}$ it holds that $\mathrm{RN}_{N, N, N} \leq$ $8 d N^{2 N}$ and $\mathrm{FE}_{N, N, N} \leq 8 N^{2 N}$. This and Corollary 4.7 yield for all $N \in \mathbb{N}$ that

$$
\begin{align*}
& \left(\mathrm{RN}_{N, N, N}+\mathrm{FE}_{N, N, N}\right)\left[\sup _{(t, x) \in[0, T] \times \mathbb{R}^{d} \nu \in\{1, \ldots, d+1\}}\left\|\left(\mathbf{U}_{n, n, n}^{0}(t, x)-\mathbf{u}^{\infty}(t, x)\right)_{\nu_{0}}\right\|_{L^{2}(\mathbb{P} ; \mathbb{R})}\right]^{(4+\delta)} \\
& \leq 8(d+1) N^{2 N} \cdot\left(\frac{7\left(2(\sqrt{T}+1) \sqrt{T \pi}\left(\|L\|_{1}+1\right)+1\right)^{N} 2^{N-1} e^{N}}{\sqrt{N^{N-3}}}+\frac{\left(14\left(8(\sqrt{T}+1) \sqrt{T \pi}\left(\|L\|_{1}+1\right)+4\right)^{N-1}+1\right) T^{2 N+1}}{\sqrt{N^{N}}}\right)^{(4+\delta)} C  \tag{95}\\
& \leq 8(d+1) N^{2 N} \cdot\left((24(T+1))^{3 N}\left(\|L\|_{1}+1\right)^{N} \sqrt{N}^{-N}\right)^{(4+\delta)} C \\
& \leq 16 d C \sum_{n \in \mathbb{N}}(24(T+1))^{3(4+\delta) n}\left(\|L\|_{1}+1\right)^{(4+\delta) n} \sqrt{n}^{-\delta n}
\end{align*}
$$

The right-hand side of (95) is clearly finite. This finishes the proof of Corollary 4.8

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