

# EXISTENCE THEOREMS FOR A CRYSTAL SURFACE MODEL INVOLVING THE P-LAPLACE OPERATOR

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**ABSTRACT.** The manufacturing of crystal films lies at the heart of modern nanotechnology. How to accurately predict the motion of a crystal surface is of fundamental importance. Many continuum models have been developed for this purpose, including a number of PDE models, which are often obtained as the continuum limit of a family of kinetic Monte Carlo models of crystal surface relaxation that includes both the solid-on-solid and discrete Gaussian models. In this paper we offer an analytical perspective into some of these models. To be specific, we study the existence of a weak solution to the boundary value problem for the equation  $-\Delta e^{-\operatorname{div}(|\nabla u|^{p-2}\nabla u)} + au = f$ , where  $p > 1, a > 0$  are given numbers and  $f$  is a given function. This problem is derived from a crystal surface model proposed by J.L. Marzuola and J. Weare (2013 Physical Review, E 88, 032403). The mathematical challenge is due to the fact that the principal term in our equation is an exponential function of a p-Laplacian. Existence of a suitably-defined weak solution is established under the assumptions that  $p \in (1, 2]$ ,  $N \leq 4$ , and  $f \in W^{1,p}$ . Our investigations reveal that the key to our existence assertion is how to control the set where  $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$  is  $\pm\infty$ .

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$ . Given that  $p > 1$ ,  $a > 0$ , and a function  $f = f(x)$ , we consider the boundary value problem

$$(1.1) \quad -\Delta e^{-\Delta_p u} + au = f \quad \text{in } \Omega,$$

$$(1.2) \quad \nabla u \cdot \nu = \nabla e^{-\Delta_p u} \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where  $\Delta_p$  is the  $p$ -Laplacian, i.e.,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ , and  $\nu$  is the unit outward normal to  $\partial\Omega$ .

Our interest in the problem originates in the mathematical description of the evolution of a crystal surface. The surface of a crystal below the roughing temperature consists of steps and terraces. According to the Burton, Cabrera and Frank (BCF) model [3], atoms detach from the steps, diffuse across terraces, and reattach at new locations, inducing an overall evolution of the crystal surface. At the nanoscale, the motion of steps is described by large systems of ordinary differential equations for step positions ([1], [7]). At the macro-scale, this description is often reduced conveniently to nonlinear PDEs for macroscopic variables such as the surface height and slope profiles (see [14, 7] and the references therein).

To see the connection between our problem (1.1)-(1.2) and certain existing continuum models, we first observe from the conservation of mass that the dynamic equation for the surface height profile  $u(t, x)$  of a solid film is governed by

$$(1.3) \quad \partial_t u + \operatorname{div} J = 0,$$

where  $J$  is the adatom flux. By Fick's law [22],  $J$  can be written as

$$J = -M(\nabla u) \nabla \rho_s.$$

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1991 *Mathematics Subject Classification.* 35D30, 35Q82, 35A01.

*Key words and phrases.* Crystal surface model, existence, exponential function of a p-Laplacian, nonlinear fourth order equations.

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Here  $M(\nabla u)$  is the mobility and  $\rho_s$  is the local equilibrium density of adatoms. On account of the Gibbs-Thomson relation [16, 27, 22], which is connected to the theory of molecular capillarity, the corresponding local equilibrium density of adatoms is determined by

$$\rho_s = \rho_0 e^{\frac{\mu}{kT}},$$

where  $\mu$  is the chemical potential,  $\rho_0$  is a constant reference density,  $T$  is the temperature, and  $k$  is the Boltzmann constant. Denote by  $\Omega$  the “step locations area” of interest. Then we can take the general surface energy  $G(u)$  to be

$$(1.4) \quad G(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx, \quad p \geq 1.$$

The justification for this, as observed in [23], is that it can retain many of the interesting features of the microscopic system that are lost in the more standard scaling regime. The chemical potential  $\mu$  is defined as the change per atom in the surface energy. That is,

$$(1.5) \quad \mu = \frac{\delta G}{\delta u} = -\Delta_p u.$$

After incorporating those physical parameters into the scaling of the time and/or spatial variables [8, 19], we can rewrite the evolution equation for  $u$  as

$$(1.6) \quad \partial_t u = \operatorname{div} \left( M(\nabla u) \nabla e^{\frac{\delta G}{\delta u}} \right).$$

In the diffusion-limited (DL) regime, where the dynamics is dominated by the diffusion across the terraces and  $M \equiv 1$  the above equations reduces to

$$(1.7) \quad \partial_t u = \operatorname{div} \left( \nabla e^{\frac{\delta G}{\delta u}} \right) = \Delta e^{-\Delta_p u}.$$

This equation is assumed to hold in a space-time domain  $\Omega_T \equiv \Omega \times (0, T)$ ,  $T > 0$ , coupled with the following initial boundary conditions

$$(1.8) \quad \nabla u \cdot \nu = \nabla e^{-\Delta_p u} \cdot \nu = 0 \quad \text{on } \Sigma_T \equiv \partial\Omega \times (0, T),$$

$$(1.9) \quad u(x, 0) = u_0(x) \quad \text{on } \Omega.$$

As we shall see, a priori estimates for this problem are rather weak. As a result, an existence theorem seems to be hopeless. Instead, we focus on the associated stationary problem. That is, we discretize the time derivative in (1.7), thereby obtaining the following stationary equation

$$(1.10) \quad \frac{u - v}{\delta} - \Delta e^{-\Delta_p u} = 0 \quad \text{in } \Omega.$$

Here  $v$  is a given function. Initially,  $v = u_0(x)$ . The positive number  $\delta$  is the step size. Set  $a = \frac{1}{\delta}$  and  $f = \frac{1}{\delta}v$ . This leads to the boundary value problem (1.1)-(1.2).

The objective of this paper is to establish an existence assertion for the stationary problem (1.1)-(1.2), while the time-dependent problem (1.7)-(1.9) is left open.

If we linearize the exponential term

$$(1.11) \quad e^{-\Delta_p u} \approx 1 - \Delta_p u,$$

then (1.7) reduces to

$$(1.12) \quad \partial_t u + \Delta \Delta_p u = 0.$$

Giga-Kohn [11] proved that there is a finite time extinction for the equation when  $p > 1$ . For the difficult case of  $p = 1$ , Giga-Giga [10] developed an  $H^{-1}$  total variation gradient flow to analyze this equation and they showed that the solution may instantaneously develop jump discontinuity in the explicit example of important crystal facet dynamics. This explicit construction of the jump discontinuity solution for facet dynamics was extended to the exponential PDE in [19].

The time-dependent problem in the case where  $p = 2$  has been investigated in [20]. The mathematical novelty there is that the exponent  $-\Delta u$  is only a measure. But the singular part of the measure is such that the composition  $e^{-\Delta u}$  is still a well-defined function. A gradient flow approach to the problem can be found in [8]. We also would like to mention two other related articles [9, 21]. Note that if  $p = 2$  then the principal term in (1.1), i.e.,  $e^{-\Delta u}$ , can be viewed as a monotone operator in a suitable function space. This property is essential to the results in [20, 8]. If  $p \neq 2$ , this property is no longer true. Moreover, the exponent becomes nonlinear. Subsequently, we lose most of the a priori estimates in [20]. What remains is collected in the following

**Lemma 1.1.** *If  $u$  is a classical solution of (1.7)-(1.9), then we have*

$$(1.13) \quad \frac{1}{p} \int_{\Omega} |\nabla u(x, s)|^p dx + 4 \int_{\Omega_s} |\nabla \sqrt{\rho}|^2 dx dt = \frac{1}{p} \int_{\Omega} |\nabla u_0(x)|^p dx,$$

$$(1.14) \quad \int_{\Omega} \ln \rho dx = 0,$$

$$(1.15) \quad \int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx.$$

where  $s > 0$ ,  $\Omega_s = \Omega \times (0, s)$ , and

$$(1.16) \quad \rho = e^{-\Delta_p u}.$$

*Proof.* We calculate

$$(1.17) \quad \int_{\Omega} \Delta_p u \partial_t u dx = - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \partial_t u dx = - \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla u|^p dx,$$

$$(1.18) \quad \begin{aligned} \int_{\Omega} \Delta e^{-\Delta_p u} \cdot \Delta_p u dx &= - \int_{\Omega} \nabla e^{-\Delta_p u} \cdot \nabla \Delta_p u dx \\ &= \int_{\Omega} e^{-\Delta_p u} |\nabla \Delta_p u|^2 dx \\ &= 4 \int_{\Omega} \left| \nabla e^{-\frac{1}{2} \Delta_p u} \right|^2 dx. \end{aligned}$$

Multiply through (1.7) by  $\Delta_p u$  and integrate the resulting equation with respect to the space variables over  $\Omega$  to obtain

$$(1.19) \quad \frac{1}{p} \frac{d}{dt} \int_{\Omega} |\nabla u(x, t)|^p dx + 4 \int_{\Omega} \left| \nabla e^{-\frac{1}{2} \Delta_p u} \right|^2 dx = 0.$$

Integrate (1.19) with respect to  $t$  to arrive at (1.13). By (1.16), we have

$$(1.20) \quad -\Delta_p u = \ln \rho \quad \text{on } \Omega.$$

Integrate the above equation over  $\Omega$  to obtain (1.14). Similarly, we can integrate (1.7) over  $\Omega$  to get (1.15).  $\square$

Unfortunately, this lemma is not enough for an existence assertion for problem (1.7)-(1.9). To gain any further results, we are facing two main challenges. First, it does not seem possible to derive any meaningful estimates in the time variable such as estimates (6) and (9) in [20]. Second, do equations (1.13) and (1.14) really imply that  $\rho \in L^q(\Omega_T)$  for some  $q \geq 1$  in the context here? Obviously, the two are interconnected. In the stationary problem (1.1)-(1.2), of course, the first challenge mentioned earlier goes away, but the second one remains. Thus the main mathematical interest of problem (1.1)-(1.2) is how to suitably interpolate between  $\ln \rho$  and  $\nabla \sqrt{\rho}$ . We must point

out that condition (1.14) is rather weak. Indeed, we can easily construct a sequence  $\{f_j\}$  such that

$$(1.21) \quad f_j \rightarrow \infty \quad \text{a.e. on } \Omega \text{ as } j \rightarrow \infty, \text{ and}$$

$$(1.22) \quad \int_{\Omega} f_j dx = 0.$$

For example, take  $\Omega = (0, 1)$  and define

$$f_j(s) = \begin{cases} j & \text{if } 0 \leq s < \frac{1}{j}, \\ j - 4j^3(s - \frac{1}{j}) & \text{if } \frac{1}{j} \leq s < \frac{3}{2j}, \\ -2j^2 + j + 4j^3(s - \frac{3}{2j}) & \text{if } \frac{3}{2j} \leq s < \frac{2}{j}, \\ j & \text{if } \frac{2}{j} \leq s \leq 1. \end{cases}$$

Note that  $f_j$  is continuous and piecewise linear and satisfies the boundary condition (1.2). This means that equation (1.14) cannot be approximated. On the other hand, we obviously can not prevent a sequence with the boundary condition (1.2) from going to infinity if we only have some control on its partial derivatives. That is to say, neither (1.13) nor (1.14) alone is sufficient for our purpose, and we must find a right combination between the two and equation (1.1). This constitutes the core of our mathematical analysis. Our investigations reveal that the set where  $\Delta_p u$  is negative infinity and the set where it is positive infinity play two significantly different roles with the former commanding most of our attention, while the latter is similar to the case already considered in [20, 21].

In view of Lemma 1.1 and the analysis in [20], we can give the following definition of a weak solution.

**Definition 1.2.** *We say that a pair  $(u, \rho)$  is a weak solution to (1.1)-(1.2) if the following conditions hold:*

- (D1)  $\rho \in W^{2,p^*}(\Omega)$ ,  $u \in W^{1,p}(\Omega)$ ,  $\Delta_p u \in \mathcal{M}(\overline{\Omega}) \cap (W^{1,p}(\Omega))^*$ , where  $p^* = \frac{Np}{N-p}$ ,  $(W^{1,p}(\Omega))^*$  is the dual space of  $W^{1,p}(\Omega)$ , and  $\mathcal{M}(\overline{\Omega})$  is the space of bounded Radon measures on  $\overline{\Omega}$ ;  
 (D2) Let

$$-\Delta_p u = g_a + \nu_s$$

be the Lebesgue decomposition of  $-\Delta_p u$  ([4], p.42). That is,  $g_a \in L^1(\Omega)$  and the support of  $\nu_s \equiv A_0$  has Lebesgue measure 0. Then there holds

$$(1.23) \quad \rho = e^{g_a} \quad \text{a.e. on } \Omega;$$

(D3) We have

$$(1.24) \quad -\Delta \rho + a u = f \quad \text{a.e. on } \Omega,$$

$$(1.25) \quad \nabla \rho \cdot \nu = 0 \quad \text{a.e. on } \partial \Omega.$$

The boundary condition  $\nabla u \cdot \nu = 0$  on  $\partial \Omega$  is satisfied in the sense

$$\langle -\Delta_p u, \xi \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi dx \quad \text{for all } \xi \in W^{1,p}(\Omega),$$

where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $W^{1,p}(\Omega)$  and  $(W^{1,p}(\Omega))^*$ .

An example in [20] shows that the singular part in  $\Delta_p u$  is an intrinsic property of our solutions. Physically, the singularities represent rupture defects and pinning on the surface evolution due to the asymmetric in the exponential curvature dependent mobility. The pinning point ruptures in the epitaxial growth models was carefully studied numerically in [23]. An easy way to remove the

singular part in  $-\Delta_p u$  is by adding a lower order perturbation to the equation (1.1). To be precise, we consider the problem

$$(1.26) \quad -\Delta e^{-\Delta_p u} + \varepsilon \Delta_p u + au = f \quad \text{in } \Omega$$

$$(1.27) \quad \nabla u \cdot \nu = \nabla e^{-\Delta_p u} \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where  $\varepsilon > 0$  is a small perturbation parameter. In this case, we will have  $\Delta_p u \in L^2(\Omega)$ . Indeed, we use  $-\Delta_p u$  as a test function in (1.26) to obtain

$$(1.28) \quad \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |\nabla e^{-\frac{1}{2}\Delta_p u}|^2 dx + \varepsilon \int_{\Omega} (\Delta_p u)^2 dx \leq c \int_{\Omega} |\nabla f|^p dx.$$

Our main result is the following

**Theorem 1.3.** *Assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with  $C^{2,\alpha}$  boundary for some  $\alpha \in (0, 1)$ ,  $N \leq 4$ ,  $a > 0$ , and  $f \in W^{1,p}(\Omega)$  with  $1 < p \leq 2$ . Then there is a weak solution to (1.1)-(1.2).*

We have not considered the case where  $p = 1$ . The physical relevance of this case can be found in [17]. It is also related to the motion by surface curvature. Our key compactness result Claim 3.9 relies on (ii) in Lemma 2.3, which fails when  $p = 1$ . Thus it would be interesting to know if we can take the limit of our solutions as  $p \rightarrow 1$ . Theorem 1.3 should also hold for  $p > 2$ . In the remark following Claim 3.6 below we shall see why we have to require  $p \leq 2$ . Since we allow  $p$  to be arbitrarily close 1, the Sobolev embedding theorem forces us to impose the condition  $N \leq 4$ . The uniqueness assertion for problem (1.1)-(1.2) is still open. The difficulty here is due to the fact that the operator  $-\Delta e^{-\Delta_p u}$  does not seem to be monotone anymore for  $p \neq 2$ .

A solution to (1.1)-(1.2) will be constructed as the limit of a sequence of approximate solutions. The key is to design an approximation scheme that can generate sufficiently regular approximate solutions so that all the preceding formal calculations are made vigorous. Then we must be able to show that the sequence of approximate solutions does not converge to infinity a.e. on  $\Omega$ . This is accomplished in Section 3. In Section 2 we state a few preparatory lemmas, while in Section 4 we make some further remarks about the time-dependent problem.

Finally, we make some remarks about the notation. The letter  $c$  denotes a positive constant. In theory, its value can be computed from various given data. In the applications of the Sobolev embedding theorems, whenever the term  $N - 2$  appears in a denominator, it is understood that  $N > 2$  because the case where  $N = 2$  can always be handled separately.

## 2. PRELIMINARIES

In this section we state a few preparatory lemmas.

Relevant interpolation inequalities for Sobolev spaces are listed in the following lemma.

**Lemma 2.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ . Denote by  $\|\cdot\|_p$  the norm in the space  $L^p(\Omega)$ . Then we have:*

- (1)  $\|f\|_q \leq \varepsilon \|f\|_r + \varepsilon^{-\sigma} \|f\|_p$ , where  $\varepsilon > 0$ ,  $p \leq q \leq r$ , and  $\sigma = \left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{q} - \frac{1}{r}\right)$ ;
- (2) If  $\partial\Omega$  is Lipschitz, then for each  $\varepsilon > 0$  and each  $q \in (1, p^*)$ , where  $p^* = \frac{pN}{N-p}$  if  $N > p \geq 1$  and any number bigger than  $p$  if  $N = p$ , there is a positive number  $c = c(\varepsilon, p)$  such that

$$(2.1) \quad \|f\|_q \leq \varepsilon \|\nabla f\|_p + c \|f\|_1 \quad \text{for all } f \in W^{1,p}(\Omega).$$

- (3) If  $\partial\Omega$  is  $C^{2,\alpha}$  for some  $\alpha \in (0, 1)$  and  $q$  is given as in (2), then

$$(2.2) \quad \|\nabla g\|_q \leq \varepsilon \|\Delta g\|_p + c \|g\|_1 \quad \text{for all } g \in W^{2,p}(\Omega).$$

This lemma can be found in [12, 13, 26]. The next lemma collects a few frequently used elementary inequalities.

**Lemma 2.2.** *For  $x, y \in \mathbb{R}^N$  and  $a, b \in \mathbb{R}^+$ , we have:*

- (4)  $|x|^{p-2}x \cdot (x - y) \geq \frac{1}{p}(|x|^p - |y|^p)$ ;  
 (5)  $ab \leq \varepsilon a^p + \frac{1}{\varepsilon^{q/p}} b^q$  if  $\varepsilon > 0$ ,  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Lemma 2.3.** *Let  $x, y$  be any two vectors in  $\mathbb{R}^N$ . Then:*

(i) *For  $p \geq 2$ ,*

$$((|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y)) \geq \frac{1}{2^{p-1}}|x - y|^p;$$

(ii) *For  $1 < p \leq 2$ ,*

$$(1 + |x|^2 + |y|^2)^{\frac{2-p}{2}} ((|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y)) \geq (p-1)|x - y|^2.$$

The proof of this lemma is contained in ([25], p. 146-148).

**Lemma 2.4.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary  $\partial\Omega$ . Consider the problem*

$$(2.3) \quad -\Delta_p u + \tau|u|^{p-2}u = f \quad \text{in } \Omega,$$

$$(2.4) \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where  $\tau > 0$ ,  $p > 1$ ,  $f \in L^{\frac{p}{p-1}}(\Omega)$ . Without loss of generality, we also assume

$$(2.5) \quad p < N.$$

Then there is a unique weak solution  $u$  to the above problem in the space  $W^{1,p}(\Omega)$ . Furthermore, if  $f$  also lies in the space  $L^q(\Omega)$  with

$$(2.6) \quad q > \frac{N}{p},$$

$u$  is bounded and we have the estimate

$$(2.7) \quad \|u\|_\infty \leq c\|u\|_1 + c(\|f\|_q)^{\frac{1}{p-1}}.$$

*Proof.* We do not believe that the estimate (2.7) is new. Since we cannot find a good reference for it, we shall offer a proof here. We employ a technique of iteration of  $L^q$  norms originally due to Moser [24]. Without loss of generality, assume

$$(2.8) \quad \|u^+\|_\infty = \|u\|_\infty.$$

Set  $b = (\|f\|_q)^{\frac{1}{p-1}}$ . For each  $s > \frac{p-1}{p}$  we use  $\frac{s^p}{ps-p+1}(u^+ + b)^{ps-p+1}$  as a test function in (2.3) to obtain

$$(2.9) \quad \begin{aligned} & s^p \int_{\Omega} (u^+ + b)^{ps-p} |\nabla u^+|^p dx \\ & + \frac{\tau s^p}{ps-p+1} \int_{\Omega} |u|^{p-2} u (u^+ + b)^{ps-p+1} dx \\ & = \frac{s^p}{ps-p+1} \int_{\Omega} f (u^+ + b)^{ps-p+1} dx. \end{aligned}$$

Note that

$$(2.10) \quad \int_{\Omega} |u|^{p-2} u^- (u^+ + b)^{ps-p+1} dx = b^{ps-p+1} \int_{\Omega} |u|^{p-2} u^- dx.$$

Letting  $u = u^+ - u^-$  in (2.3) and integrating the resulting equation over  $\Omega$ , which amounts to using 1 as a test function in the equation, yield

$$(2.11) \quad \tau \int_{\Omega} |u|^{p-2} u^+ dx = \tau \int_{\Omega} |u|^{p-2} u^- dx + \int_{\Omega} f dx.$$

Substitute this into (2.10) to obtain

$$\begin{aligned}
 \int_{\Omega} |u|^{p-2} u^{-} (u^{+} + b)^{ps-p+1} dx &= b^{ps-p+1} \int_{\Omega} |u|^{p-2} u^{+} dx - \frac{b^{ps-p+1}}{\tau} \int_{\Omega} f dx \\
 (2.12) \qquad \qquad \qquad &\leq \int_{\Omega} (u^{+} + b)^{ps} dx + \frac{1}{\tau} \int_{\Omega} |f| (u^{+} + b)^{ps-p+1} dx.
 \end{aligned}$$

Keeping this in mind, we can derive from (2.9) that

$$\begin{aligned}
 &\int_{\Omega} |\nabla(u^{+} + b)^s|^p dx \\
 &\leq \frac{\tau s^p}{ps - p + 1} \int_{\Omega} (u^{+} + b)^{ps} dx + \frac{2s^p}{ps - p + 1} \int_{\Omega} |f| (u^{+} + b)^{ps-p+1} dx \\
 &\leq \frac{\tau s^p}{ps - p + 1} \int_{\Omega} (u^{+} + b)^{ps} dx \\
 &\quad + \frac{2s^p}{ps - p + 1} \|f(u^{+} + b)^{-p+1}\|_q \left( \int_{\Omega} (u^{+} + b)^{\frac{psq}{q-1}} dx \right)^{\frac{q-1}{q}} \\
 &\leq \frac{\tau s^p}{ps - p + 1} \int_{\Omega} (u^{+} + b)^{ps} dx + \frac{2s^p}{ps - p + 1} \left( \int_{\Omega} (u^{+} + b)^{\frac{psq}{q-1}} dx \right)^{\frac{q-1}{q}} \\
 (2.13) \qquad \qquad \qquad &\leq \frac{cs^p}{ps - p + 1} \left( \int_{\Omega} (u^{+} + b)^{\frac{psq}{q-1}} dx \right)^{\frac{q-1}{q}}, \quad c = c(\Omega, \tau, q).
 \end{aligned}$$

Here we have used the fact that

$$(2.14) \qquad \|f(u^{+} + b)^{-p+1}\|_q = \left( \int_{\Omega} \frac{|f|^q}{(u^{+} + b)^{(p-1)q}} dx \right)^{\frac{1}{q}} \leq \frac{\|f\|_q}{b^{p-1}} = 1.$$

With the aid of the Sobolev inequality, we obtain

$$\begin{aligned}
 &\left( \int_{\Omega} (u^{+} + b)^{\frac{sNp}{N-p}} dx \right)^{\frac{N-p}{N}} \\
 &\leq c \int_{\Omega} |\nabla(u^{+} + b)^s|^p dx + c \int_{\Omega} (u^{+} + b)^{sp} dx \\
 &\leq \frac{cs^p}{ps - p + 1} \left( \int_{\Omega} (u^{+} + b)^{\frac{psq}{q-1}} dx \right)^{\frac{q-1}{q}} + c \int_{\Omega} (u^{+} + b)^{sp} dx \\
 (2.15) \qquad \qquad \qquad &\leq \frac{cs^p}{ps - p + 1} \left( \int_{\Omega} (u^{+} + b)^{\frac{psq}{q-1}} dx \right)^{\frac{q-1}{q}}.
 \end{aligned}$$

The last step is due to the fact that

$$\frac{s^p}{ps - p + 1} \geq 1.$$

Set  $\chi = \frac{N}{N-p}/\frac{q}{q-1}$ . Our assumption (2.6) implies that  $\chi > 1$ . We can write (2.13) in the form

$$\begin{aligned}
 \|u^{+} + b\|_{\frac{psq\chi}{q-1}} &\leq \left( \frac{c}{ps - p + 1} \right)^{\frac{1}{ps}} s^{\frac{1}{s}} \|u^{+} + b\|_{\frac{psq}{q-1}} \\
 (2.16) \qquad \qquad \qquad &\leq c^{\frac{1}{s}} s^{\frac{1}{s}} \|u^{+} + b\|_{\frac{psq}{q-1}}, \quad \text{provided that } s \geq 1.
 \end{aligned}$$

In view of the proof in ([12], p. 190), we take  $s = \chi^m, m = 0, 1, 2, \dots$  in the above inequality. Iterating and taking the limit lead to

$$(2.17) \qquad \|u^{+} + b\|_{\infty} \leq c \|u^{+} + b\|_{\frac{pq}{q-1}},$$

whence by the interpolation inequality (1) in Lemma 2.1 we have

$$(2.18) \quad \|u^+ + b\|_\infty \leq c\|u^+ + b\|_1.$$

This implies the desired result.  $\square$

Further regularity results for solutions to equations of p-laplace type can be found in [2, 29] and the references therein.

Our existence theorem is based upon the following fixed point theorem, which is often called the Leray-Schauder Theorem ([12], p.280).

**Lemma 2.5.** *Let  $B$  be a map from a Banach space  $\mathcal{B}$  into itself. Assume:*

- (H1)  *$B$  is continuous;*
- (H2) *the images of bounded sets of  $B$  are precompact;*
- (H3) *there exists a constant  $c$  such that*

$$\|z\|_{\mathcal{B}} \leq c$$

*for all  $z \in \mathcal{B}$  and  $\sigma \in [0, 1]$  satisfying  $z = \sigma B(z)$ .*

*Then  $B$  has a fixed point.*

**Lemma 2.6.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with Lipschitz boundary and  $1 \leq p < N$ . Then there is a positive number  $c = c(N)$  such that*

$$(2.19) \quad \|u - u_S\|_{p^*} \leq \frac{cd^{N+1-\frac{p}{N}}}{|S|^{\frac{1}{p}}} \|\nabla u\|_p \quad \text{for each } u \in W^{1,p}(\Omega),$$

*where  $S$  is any measurable subset of  $\Omega$  with  $|S| > 0$ ,  $u_S = \frac{1}{|S|} \int_S u dx$ , and  $d$  is the diameter of  $\Omega$ .*

This lemma can be inferred from Lemma 7.16 in [12]. Also see [13, 26]. It is a version of the Poincaré inequality.

### 3. PROOF OF THEOREM 1.3

In this section we first design an approximation scheme for problem (1.1)-(1.2). Then we obtain a weak solution by passing to the limit in our approximate problems.

Following [20], we introduce a new unknown function

$$(3.1) \quad \psi = -\Delta_p u.$$

Then regularize this equation by adding the term  $\tau|u|^{p-2}u$ ,  $\tau > 0$ , to its right-hand side. This is due to the Neumann boundary condition in our problem. By the same reason, we add  $\tau\psi$  to (1.1). This leads to the study of the system

$$(3.2) \quad -\Delta e^\psi + \tau\psi + au = f \quad \text{in } \Omega,$$

$$(3.3) \quad -\Delta_p u + \tau|u|^{p-2}u = \psi \quad \text{in } \Omega$$

coupled with the boundary conditions

$$(3.4) \quad \nabla u \cdot \nu = \nabla e^\psi \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where we assume

$$(3.5) \quad f \in L^\infty(\Omega).$$

This is our approximating problem. Basically, we have transformed a fourth-order equation into a system of two second-order equations. A mathematical motivation behind this construction is given in [20].



**Theorem 3.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with  $C^{2,\alpha}$  boundary with some  $\alpha \in (0, 1)$ , and assume that  $1 < p < N$  and (3.5) hold. Then there is a weak solution  $(\psi, u)$  to (3.2)-(3.4) with*

$$(3.6) \quad \psi \in W^{2,q}(\Omega) \text{ for each } q > 1,$$

$$(3.7) \quad u \in C^{1,\lambda}(\overline{\Omega}), \quad \lambda \in (0, 1).$$

.

*Proof.* The existence assertion will be established via the Leray-Schauder Theorem. For this purpose, we define an operator  $B$  from  $L^\infty(\Omega)$  into itself as follows: for each  $g \in L^\infty(\Omega)$  we say  $B(g) = \psi$  if  $\psi$  is the unique solution of the linear boundary value problem

$$(3.8) \quad -\operatorname{div}(e^g \nabla \psi) + \tau \psi = f - au \quad \text{in } \Omega,$$

$$(3.9) \quad \nabla \psi \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where  $u$  solves the problem

$$(3.10) \quad -\Delta_p u + \tau |u|^{p-2} u = g \quad \text{in } \Omega,$$

$$(3.11) \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Concerning the preceding boundary value problem, a theorem in ([25], p.124) asserts that the problem has a weak solution  $u$  in the space  $W^{1,p}(\Omega)$ . Obviously, the uniqueness of such a solution is a consequence of Lemma 2.3. In fact, we can further conclude from [2, 18, 29] that  $u$  satisfies (3.7). Observe that since  $g \in L^\infty(\Omega)$  the equation (3.8) is uniformly elliptic. According to the classical regularity theory for linear elliptic equations, problem (3.8)-(3.9) has a unique solution  $\psi$  in the space  $W^{1,2}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$  for some  $\beta \in (0, 1)$  ([12], Chap. 8). Therefore, we can conclude that  $B$  is well-defined, continuous, and maps bounded sets into precompact ones. It remains to show that there is a positive number  $c$  such that

$$(3.12) \quad \|\psi\|_\infty \leq c$$

for all  $\psi \in L^\infty(\Omega)$  and  $\sigma \in [0, 1]$  satisfying

$$\psi = \sigma B(\psi).$$

This equation is equivalent to the boundary value problem

$$(3.13) \quad -\Delta e^\psi + \tau \psi = \sigma(f - au) \quad \text{in } \Omega,$$

$$(3.14) \quad -\Delta_p u + \tau |u|^{p-2} u = \psi \quad \text{in } \Omega,$$

$$(3.15) \quad \nabla u \cdot \nu = \nabla e^\psi \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

**Claim 3.2.** *We have*

$$(3.16) \quad \|\psi\|_q \leq \frac{1}{\tau} \|f - au\|_q \quad \text{for each } q > 2, \text{ and thus}$$

$$(3.17) \quad \|\psi\|_\infty \leq \frac{1}{\tau} \|f - au\|_\infty.$$

Furthermore,

$$(3.18) \quad \|\psi\|_2 \leq \frac{1}{\tau} \|f\|_2$$

*Proof.* We just need to slightly modify the proof of Claim 2.1 in [20]. Let  $q > 2$  be given. Then the function  $|\psi|^{q-2}\psi$  lies in  $W^{1,2}(\Omega)$  and  $\nabla(|\psi|^{q-2}\psi) = (q-1)|\psi|^{q-2}\nabla\psi$ . Multiply through (3.13)

by this function and integrate the resulting equation over  $\Omega$  to obtain

$$\begin{aligned} (q-1) \int_{\Omega} e^{\psi} |\psi|^{q-2} |\nabla \psi|^2 dx + \tau \int_{\Omega} |\psi|^q dx &= \sigma \int_{\Omega} (f - au) |\psi|^{q-2} \psi dx \\ &\leq \int_{\Omega} |f - au| |\psi|^{q-1} dx \\ &\leq \|f - au\|_q \|\psi\|_q^{q-1}. \end{aligned}$$

Dropping the first integral in the above inequality yields (3.16).

Multiplying  $\psi$  through (3.13), we obtain

$$(3.19) \quad \int_{\Omega} e^{\psi} |\nabla \psi|^2 dx + \tau \int_{\Omega} \psi^2 dx = -\sigma \int_{\Omega} au\psi dx + \sigma \int_{\Omega} f\psi dx.$$

Upon using  $u$  as a test function in (3.14), we can derive

$$\int_{\Omega} u\psi dx = \int_{\Omega} |\nabla u|^p dx + \tau \int_{\Omega} u^p dx \geq 0.$$

Keeping this in mind, we deduce from (3.19) that

$$\tau \int_{\Omega} \psi^2 dx \leq \sigma \int_{\Omega} f\psi dx \leq \int_{\Omega} |f| |\psi| dx \leq \|f\|_2 \|\psi\|_2.$$

Then (3.18) follows.  $\square$

To continue the proof of Theorem 3.1, multiply through (3.13) by  $e^{\psi} - 1$  and integrate the resulting equation over  $\Omega$  to obtain

$$\begin{aligned} \int_{\Omega} |\nabla e^{\psi}|^2 dx + \tau \int_{\Omega} \psi(e^{\psi} - 1) dx &= -\sigma \int_{\Omega} au(e^{\psi} - 1) dx + \sigma \int_{\Omega} f(e^{\psi} - 1) dx \\ (3.20) \quad &\leq \left| \int_{\Omega} au(e^{\psi} - 1) dx \right| + \|f\|_{\infty} \int_{\Omega} |(e^{\psi} - 1)| dx. \end{aligned}$$

In view of (2.1), the first integral on the right-hand side in the above equation can be estimated as follows:

$$\begin{aligned} \left| \int_{\Omega} u(e^{\psi} - 1) dx \right| &\leq \|u\|_2 \|e^{\psi} - 1\|_2 \\ (3.21) \quad &\leq c\varepsilon \|\nabla e^{\psi}\|_2 + c\|e^{\psi} - 1\|_1, \quad \varepsilon > 0. \end{aligned}$$

Here we have taken (3.18) into account. For each  $M > 0$  we have

$$\begin{aligned} \int_{\Omega} |(e^{\psi} - 1)| dx &= \int_{|\psi| > M} |e^{\psi} - 1| dx + \int_{|\psi| \leq M} |e^{\psi} - 1| dx \\ (3.22) \quad &\leq \frac{1}{M} \int_{\Omega} \psi (e^{\psi} - 1) dx + c(M). \end{aligned}$$

Plug (3.21) and then (3.22) into (3.20), choose  $\varepsilon$  suitably small and  $M$  suitably large in the resulting inequality, thereby derive

$$(3.23) \quad \int_{\Omega} |\nabla e^{\psi}|^2 dx + \tau \int_{\Omega} \psi(e^{\psi} - 1) dx \leq c.$$

This combined with (3.22) implies that  $\psi$  is bounded in  $L^q(\Omega)$  for each  $q \geq 1$ . Thus we apply Lemma 2.4 to obtain that  $u$  is bounded in  $L^{\infty}(\Omega)$ . Consequently, (3.12) follows from (3.17) and (3.5). As we mentioned earlier, we can infer (3.7) from [18, 2, 29]. This together with the classical Calderón-Zygmund estimate implies (3.6). The proof is complete.  $\square$

*Proof of Theorem 1.3.* Without loss of genearity, we may assume that

$$(3.24) \quad f \in L^\infty(\Omega) \cap W^{1,p}(\Omega).$$

Otherwise,  $f$  can be approximated by a sequence in the above space in  $W^{1,p}(\Omega)$ . We shall show that we can take  $\tau \rightarrow 0$  in (3.2)-(3.4). For this purpose we need to derive estimates that are uniform in  $\tau$ . We write

$$(3.25) \quad u = u_\tau, \quad \psi = \psi_\tau.$$

Then problem (3.2)-(3.4) becomes

$$(3.26) \quad -\Delta \rho_\tau + \tau \psi_\tau + a u_\tau = f \quad \text{in } \Omega,$$

$$(3.27) \quad e^{\psi_\tau} = \rho_\tau \quad \text{in } \Omega,$$

$$(3.28) \quad -\Delta_p u_\tau + \tau |u_\tau|^{p-2} u_\tau = \psi_\tau \quad \text{in } \Omega,$$

$$(3.29) \quad \nabla u_\tau \cdot \nu = \nabla \rho_\tau \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

We also view  $\{u_\tau, \rho_\tau, \psi_\tau\}$  as a sequence in the subsequent proof. Take  $\tau = \frac{1}{k}$ , where  $k$  is a positive integer, for example. The rest of the proof is divided into several claims.  $\square$

**Claim 3.3.** *We have*

$$(3.30) \quad \int_\Omega |\nabla \sqrt{\rho_\tau}|^2 dx + \tau \int_\Omega \psi_\tau^2 dx + \int_\Omega |\nabla u_\tau|^p dx + \tau \int_\Omega u_\tau^p dx \leq c,$$

$$(3.31) \quad \|u_\tau\|_{W^{1,p}(\Omega)} \leq c.$$

*Proof.* Use  $\psi_\tau = \ln \rho_\tau$  as a test function in (3.26) to obtain

$$(3.32) \quad 4 \int_\Omega |\nabla \sqrt{\rho_\tau}|^2 dx + \tau \int_\Omega \psi_\tau^2 dx + a \int_\Omega u_\tau \psi_\tau dx = \int_\Omega f \psi_\tau dx.$$

With the aid of (3.28), we evaluate the last two integrals in the above equation as follows:

$$(3.33) \quad \int_\Omega u_\tau \psi_\tau dx = \int_\Omega |\nabla u_\tau|^p dx + \tau \int_\Omega u_\tau^p dx,$$

$$(3.34) \quad \begin{aligned} \int_\Omega f \psi_\tau dx &= \int_\Omega |\nabla u_\tau|^{p-2} \nabla u_\tau \nabla f dx + \tau \int_\Omega |u_\tau|^{p-2} u_\tau f dx \\ &\leq \|\nabla f\|_p \|\nabla u_\tau\|_p^{p-1} + \tau \|f\|_p \|u_\tau\|_p^{p-1}. \end{aligned}$$

Plug (3.33) and (3.34) into (3.32), apply the interpolation inequality (5) in Lemma 2.2 in the resulting inequality, and thereby obtain

$$(3.35) \quad \begin{aligned} &\int_\Omega |\nabla \sqrt{\rho_\tau}|^2 dx + \tau \int_\Omega \psi_\tau^2 dx + \int_\Omega |\nabla u_\tau|^p dx + \tau \int_\Omega u_\tau^p dx \\ &\leq c \int_\Omega |\nabla f|^p dx + c\tau \int_\Omega |f|^p dx \\ &\leq c \int_\Omega |\nabla f|^p dx + c \int_\Omega |f|^p dx. \end{aligned}$$

From here on we assume that  $\tau \leq 1$ . The above estimate gives (3.30). Integrate (3.26) over  $\Omega$  to yield

$$(3.36) \quad \left| a \int_\Omega u_\tau dx \right| = \left| \int_\Omega f dx - \tau \int_\Omega \psi_\tau dx \right| \leq c.$$

Subsequently, we can apply the Poincaré inequality to get

$$\begin{aligned}
 \|u_\tau\|_{p^*} &\leq \|u_\tau - \frac{1}{|\Omega|} \int_\Omega u_\tau dx\|_{p^*} + \frac{1}{|\Omega|^{1-\frac{1}{p}}} \left| \int_\Omega u_\tau dx \right| \\
 (3.37) \quad &\leq c \|\nabla u_\tau\|_p + \frac{1}{|\Omega|^{1-\frac{1}{p}}} \left| \int_\Omega u_\tau dx \right| \leq c.
 \end{aligned}$$

Thus (3.31) follows. The proof is complete.  $\square$

**Claim 3.4.** *There exists a subsequence of  $\{\rho_\tau\}$ , still denoted by  $\{\rho_\tau\}$ , such that*

$$(3.38) \quad \rho_\tau \rightarrow \rho \text{ a.e. on } \Omega \text{ as } \tau \rightarrow 0.$$

*Proof.* We use  $\arctan \rho_\tau$  as a test function in (3.26) to obtain

$$\begin{aligned}
 \int_\Omega \frac{|\nabla \rho_\tau|^2}{1 + \rho_\tau^2} dx &= \int_\Omega (f - \tau \psi_\tau - a u_\tau) \arctan \rho_\tau dx \\
 (3.39) \quad &\leq \pi \int_\Omega |f - \tau \psi_\tau - a u_\tau| dx \leq c.
 \end{aligned}$$

Thus  $\{\arctan \rho_\tau\}$  is bounded in  $W^{1,2}(\Omega)$ . We can extract a subsequence of  $\{\arctan \rho_\tau\}$  which converges a.e. on  $\Omega$ . It follows that  $\rho_\tau = \tan(\arctan \rho_\tau)$  also converges a.e. along the subsequence. This completes the proof.  $\square$

It should be noted that at this point we cannot rule out the possibility that  $\{\arctan \rho_\tau\}$  goes to  $\frac{\pi}{2}$  on a large set. Thus the limit  $\rho$  may not be finite a.e. on  $\Omega$ .

**Claim 3.5.** *If  $\rho$  is finite on a set of positive measure, then there is a subsequence of  $\{\rho_\tau\}$  which is bounded in  $L^q(\Omega)$  for each  $1 \leq q < \frac{N}{N-2}$ .*

*Proof.* Our assumption implies that there is a positive number  $s_0$  such that the set

$$(3.40) \quad \Omega_{s_0} = \{x \in \Omega : \rho(x) \leq s_0\}$$

has positive measure. According to Claim 3.4 and Egoroff's theorem, for each  $\varepsilon > 0$  there is a closed set  $K \subset \Omega_{s_0}$  such that  $|\Omega_{s_0} \setminus K| < \varepsilon$  and  $\rho_\tau \rightarrow \rho$  uniformly on  $K$ . We take  $\varepsilon = \frac{1}{2}|\Omega_{s_0}|$ . Then the corresponding  $K$  has positive measure. We easily conclude from the uniform convergence that there is a positive number  $s_1 > s_0$  with the property

$$(3.41) \quad \rho_\tau \leq s_1 \text{ on } K.$$

For each  $s > s_1$  we use  $\left(\frac{1}{s} - \frac{1}{\rho_\tau}\right)^+$  as a test function in (3.26) to obtain

$$(3.42) \quad \int_\Omega \left| \nabla \ln^+ \frac{\rho_\tau}{s} \right|^2 dx = \int_\Omega (f - \tau \psi_\tau - a u_\tau) \left( \frac{1}{s} - \frac{1}{\rho_\tau} \right)^+ dx \leq \frac{c}{s}.$$

Denote by  $S_{\tau,s}$  the set where the function  $\left(\frac{1}{s} - \frac{1}{\rho_\tau}\right)^+$  is 0. It follows that

$$(3.43) \quad K \subset S_{\tau,s} \text{ for all } s > s_1 \text{ and sufficiently many } \tau > 0.$$

Thus we may apply Lemma 2.6 to obtain

$$\begin{aligned}
 \ln^2 2 |\{x \in \Omega : \rho_\tau(x) > 2s\}|^{\frac{N-2}{N}} &\leq \left[ \int_\Omega \left( \ln^+ \frac{\rho_\tau}{s} \right)^{\frac{2N}{N-2}} dx \right]^{\frac{N-2}{N}} \\
 (3.44) \quad &\leq \frac{c}{|K|} \int_\Omega \left| \nabla \ln^+ \frac{\rho_\tau}{s} \right|^2 dx \leq \frac{c}{s}.
 \end{aligned}$$

This implies the desired result.  $\square$

**Claim 3.6.** *The set where  $\rho$  is finite has positive measure.*

*Proof.* We argue by contradiction. Suppose that the claim is false. Then we have

$$(3.45) \quad \rho = \infty \quad \text{a.e. on } \Omega.$$

For each  $L > 0$  we define

$$(3.46) \quad \gamma_L(s) = \begin{cases} L & \text{if } s > L, \\ s & \text{if } -L \leq s \leq L, \\ -L & \text{if } s < -L. \end{cases}$$

Fix  $L > 1$ . Multiply through (3.26) by  $\gamma_L((\rho_\tau - 1)^+)$  and integrate to obtain

$$(3.47) \quad \int_{\Omega} |\nabla \gamma_L((\rho_\tau - 1)^+)|^2 dx = \int_{\Omega} (f - \tau \psi_\tau - a u_\tau) \gamma_L((\rho_\tau - 1)^+) dx \leq cL.$$

Here we have used the fact that

$$(3.48) \quad \nabla \gamma_L((\rho_\tau - 1)^+) = 0 \quad \text{on the set where either } \rho_\tau \leq 1 \text{ or } \rho_\tau > L + 1.$$

Now consider the sequence  $\{\ln \rho_\tau \gamma_L((\rho_\tau - 1)^+)\}$ . It is easy to see that

$$(3.49) \quad \ln \rho_\tau \gamma_L((\rho_\tau - 1)^+) \geq 0 \quad \text{a.e. on } \Omega,$$

$$(3.50) \quad \lim_{\tau \rightarrow 0} \ln \rho_\tau \gamma_L((\rho_\tau - 1)^+) = \infty \quad \text{a.e. on } \Omega.$$

By (3.27) and (3.28), we have

$$(3.51) \quad -\Delta_p u_\tau + \tau |u_\tau|^{p-2} u_\tau = \ln \rho_\tau \quad \text{in } \Omega.$$

Use  $\gamma_L((\rho_\tau - 1)^+)$  as a test function in the above equation, thereby deriving

$$(3.52) \quad \begin{aligned} & \int_{\Omega} \ln \rho_\tau \gamma_L((\rho_\tau - 1)^+) dx \\ &= \int_{\Omega} |\nabla u_\tau|^{p-2} \nabla u_\tau \nabla \gamma_L((\rho_\tau - 1)^+) dx \\ & \quad + \tau \int_{\Omega} |u_\tau|^{p-2} u_\tau \gamma_L((\rho_\tau - 1)^+) dx \\ &\leq \|\nabla \gamma_L((\rho_\tau - 1)^+)\|_p \|\nabla u_\tau\|_p^{p-1} + L\tau \int_{\Omega} |u_\tau|^{p-1} dx \leq c(L). \end{aligned}$$

The last step is due to the assumption  $p \leq 2$ . It follows from Fatou's lemma that the left-hand side of the above inequality goes to  $\infty$  as  $\tau \rightarrow 0$ . This gives us a contradiction. The proof is complete.  $\square$

We would like to make a remark about the condition  $p \leq 2$ . Note from (3.16) that

$$(3.53) \quad \tau \|\psi_\tau\|_{p^*} \leq \|f - a u_\tau\|_{p^*}.$$

Thus this condition could be avoided here if we had the estimate

$$(3.54) \quad \|\nabla \rho_\tau\|_p \leq c \|f - \tau \psi_\tau - a u_\tau\|_{\frac{Np}{N+p}}.$$

The above inequality is valid if  $\rho_\tau$  satisfies the Dirichlet boundary condition  $\rho_\tau|_{\partial\Omega} = 0$ . In our case, the right hand side of (3.54) seems to also depend on the  $L^1$ -norm of  $\rho_\tau$ .

**Claim 3.7.** *The sequence  $\{\ln \rho_\tau\}$  is bounded in  $L^1(\Omega)$ .*

*Proof.* Use the number 1 as a test function in (3.51) to get

$$(3.55) \quad \left| \int_{\Omega} \ln \rho_{\tau} dx \right| = \tau \left| \int_{\Omega} |u_{\tau}|^{p-2} u_{\tau} dx \right| \leq c\tau^{\frac{1}{p}}.$$

The last step is due to (3.35). By virtue of Claims 3.5 and 3.6,

$$(3.56) \quad \text{the sequence } \{\rho_{\tau}\} \text{ is bounded in } L^q(\Omega) \text{ for each } 1 \leq q < \frac{N}{N-2}.$$

We will use this for  $q = 1$ . Also keeping (3.55) in mind, we estimate

$$(3.57) \quad \begin{aligned} \int_{\Omega} |\ln \rho_{\tau}| dx &= \int_{\Omega} \ln^+ \rho_{\tau} dx + \int_{\Omega} \ln^- \rho_{\tau} dx \\ &= 2 \int_{\Omega} \ln^+ \rho_{\tau} dx - \int_{\Omega} \ln \rho_{\tau} dx \\ &\leq 2 \int_{\Omega} \rho_{\tau} dx + c\tau^{\frac{1}{p}} \leq c. \end{aligned}$$

□

**Claim 3.8.** *The sequence  $\{\rho_{\tau}\}$  is precompact in  $W^{1,2}(\Omega)$ .*

*Proof.* Note that our assumptions on  $N, p$  imply

$$(3.58) \quad \frac{2N}{N+2} < \frac{Np}{N-p} = p^*.$$

Use  $\rho_{\tau} - 1$  as a test function in (3.26) to obtain

$$(3.59) \quad \begin{aligned} \int_{\Omega} |\nabla \rho_{\tau}|^2 dx &\leq \int_{\Omega} (f - au_{\tau})(\rho_{\tau} - 1) dx \\ &\leq \|f - au_{\tau}\|_{\frac{2N}{N+2}} \|\rho_{\tau} - 1\|_{\frac{2N}{N-2}} \\ &\leq c \|\nabla \rho_{\tau}\|_2 + c \|\rho_{\tau} - 1\|_2. \end{aligned}$$

Here we have used the fact that  $\psi_{\tau}(\rho_{\tau} - 1) \geq 0$ . Use (3.56) and the interpolation inequality (2.1) in (3.59) to obtain

$$(3.60) \quad \int_{\Omega} |\nabla \rho_{\tau}|^2 dx \leq c.$$

This combined with (3.56) implies that

$$(3.61) \quad \{\rho_{\tau}\} \text{ is bounded in } W^{1,2}(\Omega).$$

Thus  $\{\rho_{\tau}\}$  is precompact in  $L^q(\Omega)$  for each  $q \in [0, 2^*)$ . Set  $g_{\tau} = f - au_{\tau} - \tau\psi_{\tau}$ . Then by (3.53), the sequence  $\{g_{\tau}\}$  is bounded in  $L^{p^*}(\Omega)$ . Let  $\tau_1, \tau_2 \in (0, 1)$ . We calculate from (3.26) that

$$(3.62) \quad \begin{aligned} \int_{\Omega} |\nabla \rho_{\tau_1} - \nabla \rho_{\tau_2}|^2 dx &\leq \int_{\Omega} (g_{\tau_1} - g_{\tau_2})(\rho_{\tau_1} - \rho_{\tau_2}) dx \\ &\leq c \|g_{\tau_1} - g_{\tau_2}\|_{p^*} \|\rho_{\tau_1} - \rho_{\tau_2}\|_{\frac{Np}{Np-N+p}} \\ &\leq c \|\rho_{\tau_1} - \rho_{\tau_2}\|_{\frac{Np}{Np-N+p}}. \end{aligned}$$

In view of our assumptions on  $N, p$ , we have that

$$(3.63) \quad \frac{Np}{Np - N + p} < 2^*.$$

The claim follows from the precompactness of  $\{\rho_{\tau}\}$  in  $L^q(\Omega)$  for each  $q \in [0, 2^*)$ . □

**Claim 3.9.** *At least a subsequence of  $\{\nabla u_{\tau}\}$  converges a.e. on  $\Omega$ .*

*Proof.* We will show that  $\{u_\tau\}$  is precompact in  $W^{1,q}(\Omega)$  for each  $q < p$ . The idea behind the proof has appeared elsewhere. See, for example, the proof of Lemma 2.2 in [31].

By (3.31) and Egoroff's theorem, for each  $\delta > 0$  there is a closed set  $E \subset \Omega$  with the properties

$$(3.64) \quad |\Omega \setminus E| \leq \delta,$$

$$(3.65) \quad u_\tau \rightarrow u \quad \text{uniformly on } E.$$

Subsequently, we can find a positive number  $K$  so that

$$(3.66) \quad |u_\tau| \leq K \quad \text{on } E.$$

For any  $\varepsilon > 0$  we have

$$(3.67) \quad |u_{\tau_1} - u_{\tau_2}| < \varepsilon \quad \text{on } E \text{ for sufficiently small } \tau_1, \tau_2.$$

We can derive from (3.28) and Lemma 2.3 that

$$\begin{aligned} & \int_E (|\nabla u_{\tau_1}|^{p-2} \nabla u_{\tau_1} - |\nabla u_{\tau_2}|^{p-2} \nabla u_{\tau_2}) \cdot \nabla (u_{\tau_1} - u_{\tau_2}) dx \\ & \leq \int_\Omega (|\nabla u_{\tau_1}|^{p-2} \nabla u_{\tau_1} - |\nabla u_{\tau_2}|^{p-2} \nabla u_{\tau_2}) \cdot \nabla \gamma_\varepsilon (u_{\tau_1} - u_{\tau_2}) dx \\ & = \int_\Omega (\ln \rho_{\tau_1} - \tau_1 |u_{\tau_1}|^{p-2} u_{\tau_1} - \ln \rho_{\tau_2} + \tau_2 |u_{\tau_2}|^{p-2} u_{\tau_2}) \gamma_\varepsilon (u_{\tau_1} - u_{\tau_2}) dx \\ (3.68) \quad & \leq c\varepsilon, \end{aligned}$$

where  $\gamma_\varepsilon$  is obtained by replacing  $L$  with  $\varepsilon$  in (3.46). Apply (ii) in Lemma 2.3 and (3.66) to deduce

$$(3.69) \quad \int_E |\nabla (u_{\tau_1} - u_{\tau_2})|^2 dx \leq c\varepsilon.$$

Thus  $\{\nabla u_\tau\}$  is precompact in  $(L^2(E))^N$ . Let  $q < p$  be given. We estimate

$$\begin{aligned} \int_\Omega |\nabla (u_\tau - u)|^q dx &= \int_E |\nabla (u_\tau - u)|^q dx + \int_{\Omega \setminus E} |\nabla (u_\tau - u)|^q dx \\ &\leq c|\Omega \setminus E|^{1-\frac{q}{p}} + \int_E |\nabla (u_\tau - u)|^q dx \\ (3.70) \quad &\leq c\delta^{1-\frac{q}{p}} + \int_E |\nabla (u_\tau - u)|^q dx. \end{aligned}$$

Therefore,

$$(3.71) \quad \limsup_{\tau \rightarrow 0} \leq c\delta^{1-\frac{q}{p}} \text{ at least along a subsequence.}$$

Since  $\delta$  is arbitrary, this implies the desired result.  $\square$

In view of Lemma 2.1, (3.56), the classical Calderón and Zygmund estimate,  $\{\rho_\tau\}$  is bounded in  $W^{2,q}(\Omega)$ , where  $q = \min\{2, \frac{Np}{N-p}\}$ . Passing to subsequences if necessary, we may assume

$$(3.72) \quad u_\tau \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega),$$

$$(3.73) \quad \rho_\tau \rightharpoonup \rho \text{ weakly in } W^{2,q}(\Omega) \text{ and a.e. on } \Omega.$$

By virtue of Claim 3.9,

$$(3.74) \quad |\nabla u_\tau|^{p-2} \nabla u_\tau \rightharpoonup |\nabla u|^{p-2} \nabla u \text{ weakly in } \left(L^{\frac{p}{p-1}}(\Omega)\right)^N.$$

With the aid of Fatou's Lemma, we deduce from Claim 3.7 that

$$\int_\Omega |\ln \rho| dx \leq \liminf_{\tau \rightarrow 0} \int_\Omega |\ln \rho_\tau| dx \leq c.$$

Therefore, the set

$$A_0 = \{(x) \in \Omega : \rho(x) = 0\}$$

has Lebesgue measure 0. This combined with (3.73) asserts that

$$(3.75) \quad \ln \rho_\tau \rightarrow \ln \rho \quad \text{a.e. on } \Omega.$$

Obviously, we have from (3.31) that

$$(3.76) \quad \tau |u_\tau|^{p-2} u_\tau \rightarrow 0 \text{ strongly in } L^{\frac{p}{p-1}}(\Omega), \text{ and thus a.e. on } \Omega$$

(passing to a subsequence if need be). Recall (3.51) to obtain

$$(3.77) \quad -\Delta_p u_\tau \rightarrow \ln \rho \quad \text{a.e. on } \Omega.$$

On the other hand, we conclude from Claim 3.7 and (3.31) that the sequence  $\{-\Delta_p u_\tau\}$  is bounded in both  $L^1(\Omega)$  and  $(W^{1,p}(\Omega))^*$ . Hence we have

$$(3.78) \quad -\Delta_p u_\tau \rightharpoonup -\Delta_p u \equiv \mu \quad \text{weakly in both } \mathcal{M}(\overline{\Omega}) \text{ and } (W^{1,p}(\Omega))^*.$$

The key issue is: do we have

$$-\Delta_p u = \mu = \ln \rho?$$

The following claim addresses this issue.

**Claim 3.10.** *The restriction of  $\mu$  to the set  $\overline{\Omega} \setminus A_0$  is a function. This function is exactly  $\ln \rho$ . That is, the Lebesgue decomposition of  $\mu$  with respect to the Lebesgue measure is  $\ln \rho + \nu_s$ , where  $\nu_s$  is a measure supported in  $A_0$ , and we have*

$$(3.79) \quad \rho = e^\mu \quad \text{on the set } \overline{\Omega} \setminus A_0.$$

*That is,  $\ln \rho$  is the function  $g_a$  in the definition of a weak solution.*

*Proof.* The proof is almost identical to the proof of Proposition 3.7 in [20]. For the reader's convenience, we shall reproduce it here. Keep in mind that since  $\mu \in (W^{1,p}(\Omega))^*$  each function in  $W^{1,p}(\Omega)$  is  $\mu$ -measurable, and thus it is well-defined except on a set of  $\mu$  measure 0. Furthermore,  $\langle \mu, v \rangle = \int_\Omega v d\mu$  for each  $v \in W^{1,p}(\Omega)$ . For  $\varepsilon > 0$  let  $\theta_\varepsilon$  be a smooth function on  $\mathbb{R}$  having the properties

$$\theta_\varepsilon(s) = \begin{cases} 1 & \text{if } s \geq 2\varepsilon, \\ 0 & \text{if } s \leq \varepsilon \end{cases} \quad \text{and} \\ 0 \leq \theta_\varepsilon \leq 1 \quad \text{on } \mathbb{R}.$$

Then it is easy to verify from Claim 3.8 that we still have

$$(3.80) \quad \theta_\varepsilon(\rho_\tau) \rightarrow \theta_\varepsilon(\rho) \quad \text{strongly in } W^{1,p}(\Omega) \text{ for each } p \leq 2.$$

Pick a function  $\xi$  from  $C^\infty(\overline{\Omega})$ . Multiply through (3.51) by  $\xi \theta_\varepsilon(\rho_\tau)$  and integrate the resulting equation over  $\Omega$  to obtain

$$(3.81) \quad - \int_\Omega \Delta_p u_\tau \theta_\varepsilon(\rho_\tau) \xi dx + \tau \int_\Omega |u_\tau|^{p-2} u_\tau \theta_\varepsilon(\rho_\tau) \xi dx = \int_\Omega \ln \rho_\tau \theta_\varepsilon(\rho_\tau) \xi dx.$$

For each fixed  $\varepsilon$  we can infer from (3.56) that the sequence  $\{\ln \rho_\tau \theta_\varepsilon(\rho_\tau)\}$  is bounded in  $L^p(\Omega)$  for any  $p > 1$ . This, along with (3.73), gives

$$\int_\Omega \ln \rho_\tau \theta_\varepsilon(\rho_\tau) \xi dx \rightarrow \int_\Omega \theta_\varepsilon(\rho) \ln \rho \xi dx.$$

Observe from (3.80) and (3.78) that

$$(3.82) \quad - \int_\Omega \Delta_p u_\tau \theta_\varepsilon(\rho_\tau) \xi dx = \langle -\Delta_p u_\tau, \theta_\varepsilon(\rho_\tau) \xi \rangle \rightarrow \int_\Omega \theta_\varepsilon(\rho) \xi d\mu.$$



Taking  $\tau \rightarrow 0$  in (3.81) yields

$$(3.83) \quad \int_{\Omega} \theta_{\varepsilon}(\rho) \xi d\mu = \int_{\Omega} \theta_{\varepsilon}(\rho) \ln \rho \xi dx.$$

Obviously,  $\rho \in W^{1,p}(\Omega)$ , and thus it is well-defined except on a set of  $\mu$  measure 0. We can easily conclude from the definition of  $\theta_{\varepsilon}$  that  $\{\theta_{\varepsilon}(\rho)\}$  converges everywhere on the set where  $\rho$  is defined as  $\varepsilon \rightarrow 0$ . With the aid of the Dominated Convergence Theorem, we can take  $\varepsilon \rightarrow 0$  in (3.83) to obtain

$$\int_{\Omega \setminus A_0} \xi d\mu = \int_{\Omega \setminus A_0} \ln \rho \xi dx.$$

This is true for every  $\xi \in C^{\infty}(\overline{\Omega})$ , which means

$$(3.84) \quad \mu = \ln \rho \quad \text{on } \Omega \setminus A_0.$$

This proves the claim.  $\square$

With this claim, the proof of Theorem 1.3 is now totally completed.

#### 4. REMARKS ABOUT THE TIME-DEPENDENT PROBLEM

In this section, we first fabricate an approximation scheme for the time-dependent problem (1.7)-(1.9). This is based upon Theorem 3.1. Then we show that estimate (1.13) is preserved for the approximate problems.

Let  $T > 0$  be given. For each  $j \in \{1, 2, \dots\}$ , we divide the time interval  $[0, T]$  into  $j$  equal subintervals. Set

$$\delta = \frac{T}{j}.$$

We discretize (1.7)-(1.9) as follows. For  $k = 1, \dots, j$ , we solve recursively the system

$$(4.1) \quad \frac{u_k - u_{k-1}}{\delta} - \Delta e^{\psi_k} + \delta \psi_k = 0 \quad \text{in } \Omega,$$

$$(4.2) \quad -\Delta_p u_k + \delta |u_k|^{p-2} u_k = \psi_k \quad \text{in } \Omega,$$

$$(4.3) \quad \nabla e^{\psi_k} \cdot \nu = \nabla u_k = 0 \quad \text{on } \partial\Omega.$$

Introduce the functions

$$(4.4) \quad \tilde{u}_j(x, t) = \frac{t - t_{k-1}}{\delta} u_k(x) + \left(1 - \frac{t - t_{k-1}}{\delta}\right) u_{k-1}(x), \quad x \in \Omega, \quad t \in (t_{k-1}, t_k],$$

$$(4.5) \quad \bar{u}_j(x, t) = u_k(x), \quad x \in \Omega, \quad t \in (t_{k-1}, t_k],$$

$$(4.6) \quad \bar{\psi}_j(x, t) = \psi_k(x), \quad x \in \Omega, \quad t \in (t_{k-1}, t_k],$$

where  $t_k = k\delta$ . We can rewrite (4.1)-(4.3) as

$$(4.7) \quad \frac{\partial \tilde{u}_j}{\partial t} - \Delta e^{\bar{\psi}_j} + \delta \bar{\psi}_j = 0 \quad \text{in } \Omega_T,$$

$$(4.8) \quad -\Delta_p \bar{u}_j + \delta |\bar{u}_j|^{p-2} \bar{u}_j = \bar{\psi}_j \quad \text{in } \Omega_T.$$

We proceed to derive a priori estimates for the sequence of approximate solutions  $\{\tilde{u}_j, \bar{u}_j, \bar{\psi}_j\}$ .

**Proposition 4.1.** *There holds*

$$(4.9) \quad \begin{aligned} & \frac{1}{p} \max_{0 \leq t \leq T} \int_{\Omega} |\nabla \bar{u}_j|^p dx + 4 \int_{\Omega_T} |\nabla e^{\frac{1}{2} \bar{\psi}_j}|^2 dx dt \\ & \quad + \frac{\delta}{p} \max_{0 \leq t \leq T} \int_{\Omega} |\bar{u}_j|^p dx + \delta \int_{\Omega_T} \bar{\psi}_j^2 dx dt \\ & \leq \frac{1}{p} \int_{\Omega} |\nabla u_0|^p dx + \frac{\delta}{p} \int_{\Omega} |u_0|^p dx. \end{aligned}$$

Obviously, this proposition is the discretized version of (1.13).

*Proof.* Multiply through (4.1) by  $\psi_k$  and integrate to obtain

$$(4.10) \quad \int_{\Omega} \frac{u_k - u_{k-1}}{\delta} \psi_k dx + \int_{\Omega} \nabla(e^{\psi_k}) \cdot \nabla \psi_k dx + \delta \int_{\Omega} \psi_k^2 dx = 0.$$

The second integral in the preceding equation is computed as follows:

$$(4.11) \quad \begin{aligned} \int_{\Omega} \nabla(e^{\psi_k}) \cdot \nabla \psi_k dx &= \int_{\Omega} e^{\psi_k} |\nabla \psi_k|^2 dx \\ &= 4 \int_{\Omega} |\nabla e^{\frac{1}{2}\psi_k}|^2 dx. \end{aligned}$$

Using  $u_k - u_{k-1}$  as a test function in (4.2) yields

$$(4.12) \quad \begin{aligned} \int_{\Omega} (u_k - u_{k-1}) \psi_k dx &= \int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k (\nabla u_k - \nabla u_{k-1}) dx \\ &\quad + \delta \int_{\Omega} |u_k|^{p-2} u_k (u_k - u_{k-1}) dx \\ &\geq \frac{1}{p} \int_{\Omega} (|\nabla u_k|^p - |\nabla u_{k-1}|^p) dx + \frac{\delta}{p} \int_{\Omega} (|u_k|^p - |u_{k-1}|^p) dx. \end{aligned}$$

The last step is due to (3) in Lemma 2.2. Substituting (4.11) and (4.12) into (4.10) yield

$$(4.13) \quad \begin{aligned} &\frac{1}{p\delta} \int_{\Omega} (|\nabla u_k|^p - |\nabla u_{k-1}|^p) dx + 4 \int_{\Omega} |\nabla e^{\frac{1}{2}\psi_k}|^2 dx \\ &+ \frac{1}{p} \int_{\Omega} (|u_k|^p - |u_{k-1}|^p) dx + \delta \int_{\Omega} \psi_k^2 dx \leq 0. \end{aligned}$$

Then the proposition follows from multiplying through the above inequality by  $\delta$  and summing up the resulting one over  $k$ .  $\square$

Obviously, this theorem is not enough to justify passing to the limit in (4.7). It remains open to find additional estimates to accomplish the feat.

**Acknowledgment.** The author is grateful to Prof. Jian-Guo Liu for originally bringing this problem to his attention.

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