# A PROBABILISTIC CHARACTERIZATION OF RANDOM AND MALICIOUS COMMUNICATION FAILURES IN MULTI-HOP NETWORKED CONTROL

A. CETINKAYA, H. ISHII, AND T. HAYAKAWA

ABSTRACT. The control problem of a linear discrete-time dynamical system over a multi-hop network is explored. The network is assumed to be subject to packet drops by malicious and nonmalicious nodes as well as random and malicious data corruption issues. We utilize asymptotic tail-probability bounds of transmission failure ratios to characterize the links and paths of a network as well as the network itself. This probabilistic characterization allows us to take into account multiple failures that depend on each other, and coordinated malicious attacks on the network. We obtain a sufficient condition for the stability of the networked control system by utilizing our probabilistic approach. We then demonstrate the efficacy of our results in different scenarios concerning transmission failures on a multi-hop network.

#### 1. INTRODUCTION

Networked control systems incorporate communication networks to facilitate the exchange of measurement and control data between the plant and the controller [1]. Recently, multi-hop networks have been utilized in networked control operations [2, 3, 4, 5, 6, 7]. A multi-hop network such as a wireless ad hoc network consists of a number of nodes that are connected with a number of communication links. Ensuring orderly operation of a multi-hop networked control system can be challenging, as packets may sometimes fail to be transmitted at different parts of the network due to various reasons.

One of the reasons for transmission failures in a multi-hop network is channel noise in the communication links, which may corrupt the contents of state and control input data packets. The occurrence of data corruption in a communication network may be modeled using random processes [8]. In addition to channel noise, network congestion may also cause packet transmission failures. Routers may drop some packets to mitigate congestion [9].

Furthermore, it has become apparent that malicious attacks may also hamper transmissions in a networked control system. For instance, jamming attacks by

<sup>(</sup>A. Cetinkaya, H. Ishii) DEPARTMENT OF COMPUTER SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY, YOKOHAMA, 226-8502, JAPAN.

<sup>(</sup>T. Hayakawa) Department of Systems and Control Engineering, Tokyo Institute of Technology, Tokyo 152-8552, Japan.

*E-mail addresses*: ahmet@sc.dis.titech.ac.jp, ishii@c.titech.ac.jp, hayakawa@sc.e.titech.ac.jp.

Key words and phrases. Networked Control, Multi-hop Networks, Malicious Attacks, Random Failures.

This work was supported in part by the JST CREST Grant No. JPMJCR15K3 and by JSPS under Grant-in-Aid for Scientific Research Grant No. 15H04020.

an adversary may interfere with the communication on links and effectively prevent transmission of packets. This issue was investigated in several works from the viewpoints of wireless communications [10, 11], as well as control and game theory [12, 13, 14, 15, 16]. Transmissions of state and control input information between the plant and the controller may also fail due to malicious activities of routers. Malicious routers may intentionally drop some of the packets coming from and/or headed to certain nodes of the network [6, 17, 18]. The detection of such routing attacks can be challenging especially when the malicious nodes act normal for certain periods of time (see grayhole attacks in [19]). Furthermore, networks may face both malicious routing and random packet losses due to link errors [20, 21]. Understanding the effects of malicious attacks on networks is important from the viewpoint of cyber security of networked control systems [22, 23].

Our goal in this paper is to explore the effects of random and malicious transmission failures in a general multi-hop communication network and develop a network characterization to be used in the analysis of networked control systems. The key problem here is to characterize the failures for the overall multi-hop network in a nonconservative way while still taking into account mutually-dependent packet failures and coordinated malicious attacks on the network.

In the literature, researchers have proposed different characterizations of packet failures in a networked control system. Specifically, [2] explored control over a network with multiple links that introduce random packet drops. The results obtained in [2] utilize packet drop probabilities on the edges that constitute cut-sets of the network graph. This approach is also utilized in network characterization by [24]. Furthermore, [3] discussed almost sure networked control system stability, and [25] studied networked state estimation problem. Recently, [7] investigated mean square stability and robustness under delays and packet losses through a Markov jump linear system framework. Prior to these works, random packet losses have been studied for the simpler single-hop case. There, Bernoulli processes as well as time-homogeneous and time-inhomogeneous Markov chains are utilized for modeling purposes. In particular, [26] investigated the single-hop networked Kalman filtering problem under Bernoulli-type packet losses. Furthermore, for single-hop networked stabilization problems, packet losses have been modeled with Bernoulli processes in [27, 28] and finite-state time-homogeneous Markov chains in [29, 30]. In [31, 32, 33], where both random packet losses and malicious attacks in a single-hop networked control problem is considered, we modeled random packet losses through time-inhomogeneous Markov chains. In addition, stability as well as robust and optimal control problems concerned with general time-inhomogeneous Markov jump linear systems with polytopic uncertainties in transition probability matrices are investigated in [34, 35, 36], where the obtained results are applicable to multi-hop networked control systems with random packet losses.

In this paper, we consider a network with not only random failures but also malicious activities on nodes or links. As a result, failures may not always be modeled by Markov processes and packet losses may not always have well-defined probabilities. In [6], researchers explored a multi-hop network model with malicious nodes. There, the fault detection and isolation problem is explored for the case where the nodes induce delay in transmissions. Here, we consider a different setup and a different modeling approach. In particular, we do not investigate a detection problem, and we do not model the effect of delays. In this paper, we would like to characterize the overall packet exchange failures on a network between the plant and the controller by using the properties of the paths of the network and the communication links on those paths. We also note that our network characterization provides a high level model and it is tailored to be utilized for stability analysis in networked control as in [2]. Instead of specifying underlying physical channel models and routing protocols, we take a probabilistic approach to characterize transmission failure ratios on the links, paths of a network, as well as the network itself.

Our approach for modeling the overall packet failures in a network is built upon tail-probability bounds for the binary-valued processes that describe the occurrences of failures on the network. Specifically, each link on a multi-hop network is described through an asymptotic tail-probability bound of the transmission failure ratio of that link. This tail-probability-based approach is different from the typical random packet loss modeling approach of assigning probabilities to failures. Our approach can capture failures that occur due to both malicious and nonmalicious reasons. In fact, we utilized the tail-probability-based approach in [33] to study the combined effects of malicious-attacks following the discrete-time version of the attack model in [13, 16] and random packet losses modeled as Markov chains. Through this modeling approach, [33] provides a Lyapunov-based stability analysis, which is further enhanced in [37] by using linear programming techniques. Differently from [33], we show in this paper that when tail-probability bounds for the links are available, then we can obtain tail-probability bounds describing the overall failures on individual paths of a network. Then those bounds are used for deriving tail-probability bounds describing the overall failures on the network itself. Using our proposed characterizations, we obtain a probabilistic upper-bound for the average number of packet exchange failures between the plant and the controller, which we use in almost sure stability analysis of a discrete-time linear networked control system.

In the multi-hop setting, the location of failures and whether multiple failures depend on each other or not critically affect the quality of communication and hence the stabilization performance of the controller. Especially, the situation becomes more serious when the network is targeted by a number of adversaries that launch coordinated attacks on different locations in the communication network. Our tail-probability bound approach can handle such worst-case situations in a unified manner. In addition to our investigation on those situations, we also explore the case where one or more paths/links are known to be associated with random failures and the corresponding indicator processes are mutually independent. For such cases, we show that tighter results can be obtained. This is done by exploiting certain properties of the hidden Markov models that characterize the random failures.

The organization of the rest of the paper is as follows. In Section 2, we explain the networked control problem over a multi-hop network. We then present our multi-hop network characterization in Sections 3 and 4. Specifically, in Section 3, we give a characterization of the overall transmission failures in a multi-hop network based on the failures on individual paths of that network. Furthermore, in Section 4, we investigate failures on the paths that occur due to data-corruption and packet-dropping issues at nodes and communication links. We demonstrate the utility of our results in Section 5, and conclude the paper in Section 6.

4

TABLE 1. Notation concerning the links and the paths of a graph, binary-valued failure indicator processes concerning those links and paths, and the classes of failure indicator processes.

${\mathcal{P}}_i$	ith path on a graph representing a communication network
${\mathcal P}_{i,j}$	jth communication link on path $\mathcal{P}_i$
$l_{\mathcal{P}_i}$	Binary-valued process indicating overall transmission failures on path $\mathcal{P}_i$
$l_{\mathcal{P}_i} l_{\mathcal{P}_i}^{\mathcal{P}_i, j}$	Binary-valued process indicating transmission failures on $j \mathrm{th}$ link of path $\mathcal{P}_i$
$\Lambda_{ ho}$	General class of failure indicator processes
$\Gamma_{p_0,p_1}$	Class of indicator processes that characterize random failures
$\Pi_{\kappa,w}$	Class of failure indicator processes that characterize malicious attacks
$\theta^l$	Markov chain of a hidden Markov model with the output process $l\in \Gamma_{p_0,p_1}$
$h^l$	Output function of a hidden Markov model with the output process $l\in \Gamma_{p_0,p_1}$
$f(\mathcal{P})$	First link of path $\mathcal{P}$
$\mathcal{R}(\mathcal{P})$	Subpath obtained from path ${\mathcal P}$ by removing its first link

We note that the conference version of this paper appeared in [38]. In this paper, we provide more detailed discussions throughout the paper. Furthermore, we present new results in Sections 3, 4 and new examples in Section 5.

In this paper, we use  $\mathbb{N}_0$  and  $\mathbb{N}$  to denote nonnegative and positive integers, respectively. The notation  $\mathbb{P}[\cdot]$  denotes the probability on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with filtration  $\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$ . For binary numbers, the notation  $\vee$  represents the or-operation; moreover,  $\wedge$  represents the and-operation. Furthermore, Table 1 provides notation concerning graphs, binary-valued processes, and classes of such processes.

# 2. Control over Multi-hop Networks

In this section, we investigate the control of a linear plant over multi-hop networks depicted in Figure 1. On these networks, the plant and the controller exchange state measurement and control input packets. The transmissions are not subject to delay; however, there may be failures in packet exchange attempts between the plant and the controller.

We describe the linear dynamics of the plant by

(1) 
$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \in \mathbb{N}_0,$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  respectively denote the state and input matrices; furthermore,  $x(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^m$  are the state and the control input vectors, respectively.

The plant and the controller attempt to exchange state measurement and control input packets at each time instant t. Packet exchanges are attempted over multihop networks G and  $\tilde{G}$  as shown in Figure 1. The detailed models of these networks will be given in Section 3.1. We denote success or failure states of packet exchange attempts by using the binary-valued process  $\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ . In the case of a successful packet exchange (l(t) = 0), the plant transmits the state measurement to the controller; the controller uses the received state information together with a linear static feedback control law to compute the control input, which is then sent back to the plant. The transmitted control input is applied at the plant side. At certain time instants, either the state packet or the control input packet cannot be delivered due to network issues such as packet drops, jamming attacks, and other

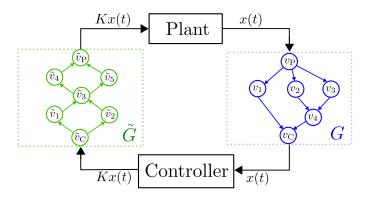


FIGURE 1. Multi-hop networked control system

communication errors. In such cases packet exchange attempts fail (l(t) = 1), and the control input at the plant side is set to 0, which is one of the common approaches in the literature (see [1] and the references therein).

Under this characterization, the control input u(t) applied at the plant side is given by

(2) 
$$u(t) \triangleq (1 - l(t)) K x(t), \quad t \in \mathbb{N}_0,$$

where  $K \in \mathbb{R}^{m \times n}$  represents the feedback gain.

Although we consider a static state-feedback control setup here, the techniques that we develop in this paper can also be used in conjunction with other control approaches. In particular, the predictive control approach of [39] and the event-triggered output-feedback control approach from our earlier work [32] can be studied within the context of multi-hop networked control by using the network characterizations that we develop here.

We assume that the information packets between the plant and the controller propagate with no delay, although there may be transmission failures due to:

1) packet drops by malicious nodes to prevent communication and/or by nonmalicious nodes to avoid congestion;

2) data corruption on communication links because of random channel errors and/or malicious jamming attacks.

We now introduce a class of processes that is useful in describing packet failure indicators in the paper.

**Definition 2.1**  $(\Lambda_{\rho})$ . Given a scalar  $\rho \in [0,1]$  we define the class of binary-valued processes  $\Lambda_{\rho}$  by

$$\Lambda_{\rho} \triangleq \Big\{l \colon l(t) \in \{0,1\}, t \in \mathbb{N}_0; \sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} l(t) > \rho k] < \infty \Big\}.$$

By this definition, we have  $\Lambda_{\rho_1} \subseteq \Lambda_{\rho_2}$  for  $\rho_1 \leq \rho_2$ . Furthermore, any binaryvalued process l satisfies  $l \in \Lambda_1$ .

In [33], we explored a problem similar to the one that we discuss in this paper. There, we considered a single direct communication channel between the plant and the controller. To describe the packet losses on this channel, we proposed a probabilistic characterization that is based on the following assumption.

5

**Assumption 2.2.** For the packet exchange failure indicator  $l(\cdot)$ , we have  $l \in \Lambda_{\rho}$  with  $\rho \in [0, 1]$ .

This assumption allows us to characterize a range of scenarios in a unified manner through the scalar  $\rho \in [0, 1]$ . For instance, the case where all packet exchange attempts fail  $(l(t) = 1, t \in \mathbb{N}_0)$  can be described by setting  $\rho = 1$ . Moreover, in the case where all packet exchange attempts are successful  $(l(t) = 0, t \in \mathbb{N}_0)$ , then  $l \in \Lambda_0$ . In addition to these two extreme cases, as we illustrate throughout the paper, Assumption 2.2 can also be used to describe random failures and malicious attacks.

When the packet exchange failures in a networked control system satisfy Assumption 2.2, then the scalar  $\rho \in [0, 1]$  also represents a bound on the asymptotic packet exchange failure ratio. Specifically, it follows from Borel-Cantelli Lemma that  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq \rho$ , almost surely. When this inequality holds with a small  $\rho$ , it means that the packet exchanges fail statistically rarely. We showed in [33] that the plant (1) can be stabilized over a network, if  $\rho$  is sufficiently small.

The stability analysis method developed in [33] allows us to obtain the following result, which presents sufficient stability conditions for the closed-loop networked control system (1), (2).

**Theorem 2.3.** Consider the dynamical system (1), (2). Suppose Assumption 2.2 holds with scalar  $\rho \in [0, 1]$ . If there exist a positive-definite matrix  $P \in \mathbb{R}^{n \times n}$  and scalars  $\beta \in (0, 1), \varphi \in [1, \infty)$  such that

- (3)  $(A+BK)^{\mathrm{T}} P (A+BK) \beta P \leq 0,$
- (4)  $A^{\mathrm{T}}PA \varphi P \le 0,$

 $\mathbf{6}$ 

(5)  $(1-\rho)\ln\beta + \rho\ln\varphi < 0,$ 

then the zero solution  $x(t) \equiv 0$  of the closed-loop system (1), (2) is asymptotically stable almost surely.

In Theorem 2.3, the conditions (3) and (4) characterize stability and instability of the closed-loop and the open-loop dynamics through scalars  $\beta$  and  $\varphi$ . These scalars also appear in condition (5). When  $\rho$  (and hence the packet exchange failure ratio) is sufficiently small so that (5) is satisfied, then we have almost sure asymptotic stability, which implies  $\mathbb{P}[\lim_{t\to\infty} ||x(t)|| = 0] = 1$ .

In Sections 3 and 4 below, we will present some key methods for obtaining  $\rho$  of a *multi-hop networked control system* (1), (2) to facilitate its stability analysis with Theorem 2.3. Specifically, we will consider the setting where the state measurement and control input packets are attempted to be transmitted over *multi-hop networks* instead of a single direct channel, which was considered in [33]. We will use assumptions similar to Assumption 2.2 to characterize packet transmission failures on the paths between the plant and the controller as well as the individual links on those paths. We will then show that the packet exchange failures (represented by  $l(\cdot)$ ) in the overall networked control system satisfy Assumption 2.2 with a scalar  $\rho \in [0, 1]$  that depends on the network structure as well as failure models of links.

To facilitate the analysis in the following sections, we now define two more classes of binary-valued processes that are distinct from  $\Lambda_{\rho}$ . Our goal is to characterize more specific models for random and malicious failures.

In our earlier work [33], we utilized time-inhomogeneous Markov chains for characterizing random failures in a single communication channel. When we consider a multi-hop network composed of a number of channels that face random transmission failures, we are required to introduce a different characterization. This is because the overall failures in the network depends on the failures on each individual channel. Hence, the overall failures cannot always be described as a Markov chain. In order to describe random failures in multi-hop networks, we utilize time-inhomogeneous hidden Markov models (see [40, 41, 42]). The binary-valued process  $\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$  has a time-inhomogeneous hidden Markov model if

(6) 
$$l(t) = h^l(\theta^l(t)), \quad t \in \mathbb{N}_0,$$

where  $h^l: \Theta^l \to \{0, 1\}$  is a binary-valued function on a set  $\Theta^l$  of finite number of elements, and moreover,  $\{\theta^l(t) \in \Theta^l\}_{t \in \mathbb{N}_0}$  is an  $\mathcal{F}_t$ -adapted, finite-state, and time-inhomogeneous Markov chain with initial distributions  $\vartheta^l_q \in [0, 1], q \in \Theta^l$ , and time-varying transition probability functions  $p^l_{q,r}: \mathbb{N}_0 \to [0, 1], q, r \in \Theta^l$ , satisfying

$$\mathbb{P}[\theta^l(0) = q] = \vartheta^l_q,$$
$$\mathbb{P}[\theta^l(t+1) = r|\theta^l(t) = q] = p^l_{q,r}(t), \quad t \in \mathbb{N}_0.$$

The process  $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  is also called the output process of a hidden Markov model. Notice that  $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  depends on the Markov chain  $\{\theta^l(t) \in \Theta^l\}_{t \in \mathbb{N}_0}$  through function  $h^l$ , but it is not necessarily a Markov chain itself. Specifically, in certain cases, we may have  $\mathbb{P}[l(t+1) = 1|\mathcal{F}_t] \neq \mathbb{P}[l(t+1) = 1|l(t)]$ , which shows that Markov property (see [43]) does not hold. This is for example the case where  $l(\cdot)$  represents the failures on a Gilbert-Elliott channel (see [44, 45]). We also note that hidden Markov models naturally arise in the description of multi-hop networks. For instance,  $l(\cdot)$  may be the failure indicator of a path with multiple links. Even if the failures on individual links may satisfy the Markov property, the overall failure indicator  $l(\cdot)$  does not satisfy it due to dependence on the failure/success states of all individual links. In such cases,  $l(\cdot)$  follows a hidden Markov model, where the Markov chain  $\{\theta^l(t) \in \Theta^l\}_{t \in \mathbb{N}_0}$  represents the combined states of all individual links.

For a given binary-valued output process  $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  associated with a time-inhomogeneous hidden Markov model, let  $\Theta_0^l, \Theta_1^l \subset \Theta^l$  be given by

$$\Theta_0^l \triangleq \{r \in \Theta^l \colon h^l(r) = 0\}, \quad \Theta_1^l \triangleq \{r \in \Theta^l \colon h^l(r) = 1\}$$

In the next definition we introduce a class of binary-valued output processes associated with time-inhomogeneous hidden Markov models.

**Definition 2.4**  $(\Gamma_{p_0,p_1})$ . Given scalars  $p_0, p_1 \in [0,1]$  we define the class  $\Gamma_{p_0,p_1}$  of binary-valued output processes of time-inhomogeneous hidden Markov models by

$$\Gamma_{p_0,p_1} \triangleq \left\{ l \colon \sum_{r \in \Theta_0^l} p_{q,r}^l(t) \le p_0, q \in \Theta^l, t \in \mathbb{N}_0; \sum_{r \in \Theta_1^l} p_{q,r}^l(t) \le p_1, q \in \Theta^l, t \in \mathbb{N}_0 \right\}.$$

The advantage of the class  $\Gamma_{p_0,p_1}$  is that by utilizing the values  $p_0$  and  $p_1$  associated with different hidden Markov models, we can characterize the combined effects of those models even if detailed information on the Markov chain  $\theta^l$  and the output function  $h^l$  of those models are not available.

The following result establishes the relation between the classes  $\Gamma_{p_0,p_1}$  and  $\Lambda_{\rho}$ .

**Proposition 2.5.** Consider the binary-valued output process  $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  of a time-inhomogeneous hidden Markov model. If  $l \in \Gamma_{p_0,p_1}$  with  $p_1 \in (0,1)$ , then we have  $l \in \Lambda_{\rho}$  for any  $\rho \in (p_1,1]$ .

Before, we give the proof of Proposition 2.5, we first present a technical result that provides upper bounds on the tail probabilities of sums involving a binaryvalued output process associated with a time-inhomogeneous hidden Markov model.

**Lemma 2.6.** Let  $\{\xi(t) \in \Xi\}_{t \in \mathbb{N}_0}$  be a finite-state time-inhomogeneous Markov chain with transition probabilities  $p_{q,r} \colon \mathbb{N}_0 \to [0,1], q,r \in \Xi$ , and let  $\Xi_1 \subset \Xi$  be given by  $\Xi_1 \triangleq \{r \in \Xi : h(r) = 1\}$ , where  $h: \Xi \to \{0, 1\}$  is a binary-valued function. Furthermore, let  $\{\chi(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  be a binary-valued process that is independent of  $\{\xi(t) \in \Xi\}_{t \in \mathbb{N}_0}$ . Assume

(7) 
$$\sum_{r\in\Xi_1} p_{q,r}(t) \le \tilde{p}, \quad q\in\Xi, \quad t\in\mathbb{N}_0,$$

(8) 
$$\sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} \chi(t) > \tilde{w}k] < \infty$$

where  $\tilde{p} \in (0,1)$ ,  $\tilde{w} \in (0,1]$ . We then have for  $\rho \in (\tilde{p}\tilde{w},\tilde{w})$ ,

(9) 
$$\mathbb{P}[\sum_{t=0}^{k-1} h(\xi(t))\chi(t) > \rho k] \le \psi_k, \quad k \in \mathbb{N},$$

where  $\psi_k \triangleq \tilde{\sigma}_k + \phi^{-\rho k+1} \frac{((\phi-1)\tilde{p}+1)^{\tilde{w}k}-1}{(\phi-1)\tilde{p}}$ ,  $\phi \triangleq \frac{\frac{\rho}{\tilde{w}}(1-\tilde{p})}{\tilde{p}(1-\frac{\rho}{\tilde{w}})}$ ,  $\tilde{\sigma}_k \triangleq \mathbb{P}[\sum_{t=0}^{k-1} \chi(t) > \tilde{w}k], k \in \mathbb{N}$ . Moreover,  $\sum_{k=1}^{\infty} \psi_k < \infty$ .

Lemma 2.6 is an essential tool for dealing with different failure scenarios specific to multi-hop networks, and it generalizes a result for the fully observable Markov chains from our previous work [33]. In particular, in the case where  $\Xi = \{0, 1\}$  and  $h(r) = r, r \in \Xi$ , Lemma 2.6 recovers Lemma A.1 of [33]. The proof Lemma 2.6 is given in the Appendix.

We are now ready to prove Proposition 2.5.

Proof of Proposition 2.5. Notice that  $l \in \Lambda_{\rho}$  for  $\rho = 1$ , since  $l(\cdot)$  is binary-valued. For the case  $\rho \in (p_1, 1)$ , we show  $l \in \Lambda_{\rho}$  by employing Lemma 2.6. Specifically, let  $\tilde{p} = p_1, \, \tilde{w} = 1$ , and define the processes  $\{\xi(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$  and  $\{\chi(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ with  $\xi(t) = \theta_1^l(t)$  and  $\chi(t) = 1, t \in \mathbb{N}_0$ . Since the conditions in (7) and (8) are satisfied with  $\tilde{p} = p_1$ ,  $h = h^l$ ,  $\Xi = \Theta^l$ , and  $\Xi_1 = \Theta^l_1$ , it follows from Lemma 2.6 that

$$\sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} l(t) > \rho k] = \sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} h^l(\theta_1^l(t)) > \rho k] = \sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} h(\xi(t))\chi(t) > \rho k] < \infty,$$
which completes the proof.

which completes the proof.

Next, we introduce a class for binary-valued processes that we employ in characterizing the timing of malicious attacks.

**Definition 2.7**  $(\Pi_{\kappa,w})$ . Given scalars  $\kappa \geq 0$ ,  $w \in [0,1]$  we define the class of binary-valued processes  $\Pi_{\kappa,w}$  by

$$\Pi_{\kappa,w} \triangleq \big\{l \colon l(t) \in \{0,1\}, t \in \mathbb{N}_0; \ \mathbb{P}\big[\sum_{t=0}^{k-1} l(t) \le \kappa + wk\big] = 1, \ k \in \mathbb{N}\big\}.$$

The characterization for the class  $\Pi_{\kappa,w}$  is based on a discrete-time version of the malicious attack model used by [13, 16]. In this model, occurrences of malicious attacks are described by a process  $l(\cdot)$  such that  $\mathbb{P}\left[\sum_{t=0}^{k-1} l(t) \leq \kappa + wk\right] = 1$ ,  $k \in \mathbb{N}$ , where  $\kappa \geq 0$  represents an upper-bound for a number of initial attacks, and  $w \in (0, 1)$  represents a bound on the average attack rate. Lemma 2.3 in [33] shows that if  $l \in \Pi_{\kappa,w}$  with  $w \in (0, 1)$ , then  $l \in \Lambda_{\rho}$  for any  $\rho \in (w, 1]$ .

Note that the malicious attack characterization through the class  $\Pi_{\kappa,w}$  does not require the process  $l \in \Pi_{\kappa,w}$  to follow a particular distribution at each time. This is the key difference of the class  $\Pi_{\kappa,w}$  from the class  $\Gamma_{p_0,p_1}$  that represents random failures.

There are several ways an attacker can strategize when to cause transmission failures. For instance, game-theoretic [14, 15] and optimization-based methods can be used by the attacker to decide the timing of attacks. An important property of the class  $\Pi_{\kappa,w}$  is that it characterizes attacks by their maximum average attack rate but not by the specific timing strategy they follow. Thus by using  $\Pi_{\kappa,w}$ , we can capture the uncertainty in the generation of attacks, which may follow a deterministic strategy, or may involve randomness. Interestingly, an attacker can also make use of system dynamics as well as past/present state information to decide the timing of attacks to cause more damage to the system. In the following example, we discuss such an attack scenario.

**Example 2.8.** Consider the scenario where the plant's communication node  $v_{\rm P}$  in Figure 1 is compromised by an attacker. The attacker is assumed to utilize the knowledge of system dynamics and the state information for deciding whether to transmit packets or not. We represent the attacker's actions with a binary-valued process  $\{l_{\rm A}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ , where  $l_{\rm A}(t) = 0$  indicates that the attacker transmits the state packet at time t to nodes  $v_1, v_2, v_3$  and  $l_{\rm A}(t) = 1$  indicates no transmission. The attacker decides the values of the binary-valued process  $l_{\rm A}(t)$  according to

(10) 
$$l_{\mathcal{A}}(t) = a_t^*$$

where  $(a_t^*, a_{t+1}^*, \cdots, a_{t+N-1}^*) \in \{0, 1\}^N$  is a solution to the optimization problem (11)

$$\max_{\substack{(a_t, \cdots, a_{i+N-1})\\ \text{subject to } \sum_{i=0}^{t-1} l_{\mathcal{A}}(i) + \sum_{i=t}^{t+j-1} a_i \le \kappa_{\mathcal{A}} + w_{\mathcal{A}}(t+j), \quad j \in \{1, \dots, N\},$$

with  $\hat{x}(t+i) = (A+(1-a_{t+i-1})BK)(A+(1-a_{t+i-2})BK)\cdots(A+(1-a_t)BK)x(t),$   $\kappa_A \geq 0, w_A \in (0,1), and N \in \mathbb{N}.$  Here,  $\hat{x}(t+i)$  denotes the attacker's prediction of the future state at time t + i. In this strategy, the attacker decides to attack  $(l_A(t) = 1)$  or not  $(l_A(t) = 0)$  at time t, based on the solution of the optimization problem where the goal is to maximize the sum of squared predicted future state norms over the interval [t+1,t+N], while keeping the attack rate below a certain value  $w_A \in (0,1)$ . The positive integer  $N \in \mathbb{N}$  is the horizon in the optimization problem, and it can be large if the attacker has sufficient computational resources. The optimization problem in (11) is solved at each time step and the updated state information is used by the attacker for decision making.

The process  $\{l_A(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  under the attack strategy (10), (11) is not a Markov process, since the value of  $l_A(t)$  depends on not just the action  $l_A(t-1)$  but all previous actions  $l_A(0), l_A(1), \ldots, l_A(t-1)$ . Moreover,  $l_A(t)$  depends also on the state value x(t).

Notice that under this attack strategy, we have  $l_A \in \Pi_{\kappa_A, w_A}$ , and hence,  $l_A \in \Lambda_{\rho}$ for any  $\rho \in (w_A, 1]$ . This illustrates the generality of the class  $\Lambda_{\rho}$  characterized in Definition 2.1. Not only the output processes of hidden Markov models from the class  $\Gamma$ , but also the non-Markovian, state-dependent, and optimization-based attacks from the class  $\Pi$  belong to the class  $\Lambda_{\rho}$  for certain values of  $\rho$ . In Section 5.3, we further illustrate the effects of such attacks.

As we discussed above, a process l that belongs to either of the classes  $\Pi_{\kappa,w}$ and  $\Gamma_{p_0,p_1}$  also belongs to the class  $\Lambda_{\rho}$  for a suitable value of  $\rho$ . This observation suggests us that both random failures and malicious attacks can be characterized by utilizing the class  $\Lambda_{\rho}$ . We note also that there are cases where a process may belong to  $\Lambda_{\rho}$ , even though it does not belong to  $\Pi_{\kappa,w}$  or  $\Gamma_{p_0,p_1}$ . The following example discusses such a case.

**Example 2.9.** In our recent work [46], we investigated the effects of channel noise and jamming attacks on a wireless communication channel. There, we considered a physical channel model to determine the wireless transmission failure probabilities. In particular, we used the binary-valued process  $\{l_J(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  to indicate the transmission failures and the process  $\{v_J(t) \in [0,\infty)\}_{t\in\mathbb{N}_0}$  to denote the jamming interference power. These two processes are related to each other through the equality

(12) 
$$\mathbb{P}[l_{\mathcal{J}}(t) = 1 | v_{\mathcal{J}}(t) = v^*] = p(v^*), \quad v^* \ge 0,$$

where  $p: [0, \infty) \to [0, 1]$  is a function determined by the properties (such as the constant channel noise power) associated with the underlying channel. We showed that  $l_{\rm J} \in \Lambda_{\rho}$  holds for certain  $\rho$  values. Specifically, if there exist  $\kappa_{\rm J} \ge 0, \overline{v}_{\rm J} \ge 0$  such that  $\sum_{i=0}^{t-1} v(i) \le \kappa_{\rm J} + \overline{v}_{\rm J} t$  for  $t \in \mathbb{N}$ , then we have  $l_{\rm J} \in \Lambda_{\rho}$  for  $\rho \in (\hat{p}(\overline{v}), 1]$ , where  $\hat{p}: [0, \infty) \to [0, 1]$  is a concave function that upper-bounds p. Notice that in this case,  $l_{\rm J}$  depends on the uncertain power process  $v_{\rm J}$ . Moreover,  $l_{\rm J}$  does not necessarily belong to the class  $\Pi$  or  $\Gamma$ , even though it belongs to  $\Lambda_{\rho}$  for  $\rho \in (\hat{p}(\overline{v}), 1]$ .

As we make it clear in the following sections, the class  $\Lambda_{\rho}$  has useful properties. For instance, if two processes  $l_1$  and  $l_2$  belong to classes  $\Lambda_{\rho_1}$  and  $\Lambda_{\rho_2}$ , then the processes  $l_{\wedge}$  and  $l_{\vee}$  defined by setting  $l_{\wedge}(t) = l_1(t) \wedge l_2(t)$  and  $l_{\vee} = l_1(t) \vee l_2(t)$ belong to classes  $\Lambda_{\rho_{\wedge}}$  and  $\Lambda_{\rho_{\vee}}$ , respectively, where  $\rho_{\wedge}$  and  $\rho_{\vee}$  depend on  $\rho_1$  and  $\rho_2$ . Such properties enable us to model the failures on both the links and the paths of a network by using processes that belong to the class  $\Lambda_{\rho}$  for certain values of  $\rho$ .

# 3. RANDOM AND MALICIOUS PACKET FAILURES IN MULTI-HOP NETWORKS

In this section, we present a framework for modeling random and malicious packet transmission failures in the multi-hop networks that are used for exchanging state and control input packets between the plant and the controller.

3.1. Multi-Hop Network Model. We follow the approach of [24] and represent the networks between the plant and the controller by using directed acyclic graphs. To model the network, over which the *state packets* are transmitted from the plant to the controller, we consider the directed acyclic graph  $G \triangleq (V, E)$ , where Vdenotes the set of nodes, and  $E \subset V \times V$  denotes the set of edges. Here the nodes and the edges correspond respectively to communication devices and links. We represent the nodes at the plant and the controller with  $v_{\rm P} \in V$  and  $v_{\rm C} \in V$ , respectively. A path  $\mathcal{P}$  from a node  $v_1 \in V$  to another node  $v_h \in V$  is identified as a sequence of nonrepeating edges  $\mathcal{P} = ((v_1, v_2), (v_2, v_3), \dots, (v_{h-1}, v_h))$ . We write  $|\mathcal{P}|$  to denote the number of edges on the path  $\mathcal{P}$ .

11

Similarly, the network used for transmission of the *control input packets* from the controller to the plant is represented by graph  $\tilde{G} \triangleq (\tilde{V}, \tilde{E})$  with plant and controller nodes  $\tilde{v}_{\rm P} \in \tilde{V}$  and  $\tilde{v}_{\rm C} \in \tilde{V}$ . In practice, the same physical network may be used for transmission of both the state and the control input packets. For those cases, the nodes in V and  $\tilde{V}$  would correspond to the same physical devices.

We assume that there exists at least one directed path from node  $v_{\rm P}$  to node  $v_{\rm C}$ in G, and at least one directed path from node  $\tilde{v}_{\rm C}$  to node  $\tilde{v}_{\rm P}$  in  $\tilde{G}$ . This ensures that the underlying communication network topology allows packet transmissions between the plant and the controller. Note that each path represents a possible transmission route. When multiple routes are utilized, the same packet is attempted to be transmitted on all those routes. In the ideal case where packet drops and data corruption do not occur, packets can be delivered in either one of these routes at all times. In this paper, we are interested in the nonideal case, where at certain times, transmissions on these routes may fail.

**Example 3.1.** We show an example of G and  $\tilde{G}$  in Fig. 1, where the nodes corresponding to the plant and the controller are not directly connected, but state and control input packets can still be transmitted with the help of the intermediate nodes  $v_1, \ldots, v_4$  in G, and  $\tilde{v}_1, \ldots, \tilde{v}_5$  in  $\tilde{G}$ .

Intermediate nodes in networks forward data packets that they receive from their incoming edges to their outgoing edges. Depending on the communication protocol, the forwarding method may differ. For instance, in the *broadcast* method, intermediate nodes forward all data packets that they receive from their incoming edges to all the nodes that they are connected with their outgoing edges. On the other hand, it may also be the case that intermediate nodes follow a specific routing scheme, where a packet coming from a certain incoming edge is forwarded through a certain outgoing edge [47].

A packet exchange between the plant and the controller may fail if the state or the control input packets are dropped or get corrupted. Here note that corrupted data packets are allowed to be transmitted over intermediate nodes, but they are detected and discarded at the plant/controller nodes. Error-detecting codes can be used for this purpose. Note also that if the controller only receives corrupted versions of a state packet, the control input is not computed.

In the following sections, we present some key results for the analysis of packet failures on network G, which are directly applicable for analyzing  $\tilde{G}$ . In particular, we characterize failures on G in terms of the failures on different paths between the plant and the controller. Then we present a set of results that relate data corruption and packet dropping issues of nodes and links to the failures on each individual path of G. These results enable us to obtain  $\rho \in [0, 1]$  in Assumption 2.2, which is essential for analyzing the system in Fig. 1 with Theorem 2.3. We emphasize that the central problem here is to find  $\rho$  for the overall multi-hop network in Fig. 1 in a nonconservative way while still taking into account mutually-dependent packet failures and coordinated attacks on the network.

3.2. Packet Transmission Failures on Networks. We use the binary-valued process  $\{l_G(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  to indicate transmission failures on G. Specifically,  $l_G(t) = 0$  means that the state packet x(t) sent from the plant node  $v_P$  is successfully received at the controller node  $v_C$ . On the other hand,  $l_G(t) = 1$  indicates a failure, that is, the controller does not receive the state x(t).

Let  $c \in \mathbb{N}$  denote the number of paths on graph G from the node  $v_{\mathrm{P}}$  to the node  $v_{\mathrm{C}}$ , and let  $\mathcal{P}_i, i \in \{1, \ldots, c\}$ , denote these paths. In the example network G in Fig. 1, there are c = 3 paths

(13) 
$$\mathcal{P}_1 = ((v_{\rm P}, v_1), (v_1, v_{\rm C})), \ \mathcal{P}_2 = ((v_{\rm P}, v_2), (v_2, v_4), (v_4, v_{\rm C})), \ \mathcal{P}_3 = ((v_{\rm P}, v_3), (v_3, v_4), (v_4, v_{\rm C})).$$

Note that different paths may include the same link. Hence, when packet transmission is attempted on multiple paths, a link that is shared on those paths may be used multiple times. For instance,  $(v_4, v_C)$  is on both  $\mathcal{P}_2$  and  $\mathcal{P}_3$ . Hence,  $(v_4, v_C)$ may be utilized twice to forward the packets coming from  $v_2$  and  $v_3$ . On the other hand, the framework that we describe below also allows modeling the case where one of the packets is dropped at node  $v_4$  and not transmitted further. Furthermore, the packet drop can be random or malicious.

We use  $\{l_{\mathcal{P}_i}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  to indicate whether the state packet x(t) is successfully transmitted to the controller through path  $\mathcal{P}_i$  or not. Specifically,  $l_{\mathcal{P}_i}(t) = 0$  represents a successful transmission. On the other hand  $l_{\mathcal{P}_i}(t) = 1$  may indicate that the path is not utilized for transmission due to the particular routing scheme, or it may indicate a failure. Failures occur if packets get dropped on the path or if they get corrupted.

Thus, in network G, the transmission of the state packet x(t) from node  $v_{\rm P}$  to node  $v_{\rm C}$  results in failure if  $l_{\mathcal{P}_i}(t) = 1$  for all paths  $\mathcal{P}_i, i \in \{1, \ldots, c\}$ . Therefore,  $l_G(\cdot)$  is given by

(14) 
$$l_G(t) = l_{\mathcal{P}_1}(t) \wedge l_{\mathcal{P}_2}(t) \wedge \dots \wedge l_{\mathcal{P}_c}(t)$$

The following result presents a probabilistic and asymptotic bound for the packet transmission failure ratio of G as a function of the bounds for the individual paths  $\mathcal{P}_i$ .

**Proposition 3.2.** Assume for each path  $\mathcal{P}_i$  that we have  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$ , where  $\rho_{\mathcal{P}_i} \in [0,1]$ . Then  $l_G \in \Lambda_{\rho_G}$  with  $\rho_G \triangleq \min_{i \in \{1,...,c\}} \rho_{\mathcal{P}_i}$ .

*Proof.* First let  $i^* \in \arg\min_{i \in \{1,...,c\}} \rho_{\mathcal{P}_i}$ . It follows that  $\rho_G = \min_{i \in \{1,...,c\}} \rho_{\mathcal{P}_i} = \rho_{\mathcal{P}_i^*}$ . Now, by (14),

$$\sum_{t=0}^{k-1} l_G(t) \le \sum_{t=0}^{k-1} l_{\mathcal{P}_i}(t), \quad i \in \{1, \dots, c\},$$

and hence  $\sum_{t=0}^{k-1} l_G(t) \leq \sum_{t=0}^{k-1} l_{\mathcal{P}_{i^*}}(t)$ . Therefore,

$$\mathbb{P}[\sum_{t=0}^{k-1} l_G(t) > \rho_G k] \le \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}_{i^*}}(t) > \rho_{\mathcal{P}_{i^*}} k].$$

The result then follows, since  $l_{\mathcal{P}_{i^*}} \in \Lambda_{\rho_{\mathcal{P}_{i^*}}}$ .

The scalars  $\rho_{\mathcal{P}_i}$ ,  $i \in \{1, \ldots, c\}$ , in Proposition 3.2 represent bounds for asymptotic packet failure ratios on different paths of network G. Proposition 3.2 indicates that the minimum of these scalars is also a bound for the packet failure ratio of the whole network. Observe that if  $\rho_{\mathcal{P}_i} = 0$  for some path  $\mathcal{P}_i$ , then we have  $\rho_G = 0$ , which means that the state can be securely and reliably transmitted to the controller at all time instants. This is because the transmission on path  $\mathcal{P}_i$  never fails.

13

If, on the other hand,  $\rho_{\mathcal{P}_i} = 1$  for all paths  $\mathcal{P}_i$ , then  $\rho_G = 1$ , indicating all packet transmission attempts fail, almost surely.

Note that in Proposition 3.2, we do not assume that  $\{l_{\mathcal{P}_i}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  are mutually-independent processes. This allows us to deal with the scenarios where transmission failures on different paths may depend on each other. In particular, we can consider coordinated packet dropout attacks of several malicious routers on different paths. For instance, two malicious routers  $v_2$  and  $v_3$  in Fig. 1 may skip forwarding packets at the same time. Then transmissions on paths  $\mathcal{P}_2$  and  $\mathcal{P}_3$ given in (13) would both fail. Similarly, Proposition 3.2 is also useful when links on different paths are attacked at the same time by coordinated jamming attackers.

Another scenario that can be explored through Proposition 3.2 is related to packet drops by nonmalicious routers to prevent congestion [9]. For example, a nonmalicious router  $v_4$  in Fig. 1 may choose to forward only one of the packets coming from  $v_2$  and  $v_3$ . Then,  $l_{\mathcal{P}_2}(\cdot)$  and  $l_{\mathcal{P}_3}(\cdot)$  would be dependent processes. In particular, if there are no other failures in the network, then we have  $l_{\mathcal{P}_2}(t) = 1 - l_{\mathcal{P}_3}(t)$ .

We remark that by utilizing additional properties of the indicator processes  $l_{\mathcal{P}_i}(\cdot)$  for paths, we can obtain a better asymptotic failure bound  $\rho_G$  than the one provided in Proposition 3.2. In particular, if one or more paths are known to be associated with random failures and the corresponding indicator processes are mutually independent, we can obtain tighter results than Proposition 3.2. To this end, we first present the following result on the properties of a process that is obtained by using  $\wedge$  operation on the output processes of two mutually-independent hidden Markov models.

**Theorem 3.3.** Consider the binary-valued output processes  $\{l^{(1)} \in \{0,1\}\}_{t\in\mathbb{N}_0}$ and  $\{l^{(2)} \in \{0,1\}\}_{t\in\mathbb{N}_0}$  of hidden Markov models such that  $l^{(1)} \in \Gamma_{p_0^{(1)},p_1^{(1)}}, l^{(2)} \in \Gamma_{p_0^{(2)},p_1^{(2)}}$  with  $p_0^{(1)}, p_1^{(1)}, p_0^{(2)}, p_1^{(2)} \in [0,1]$ . Suppose that the Markov chains  $\{\theta^{l^{(1)}}(t) \in \Theta^{l^{(1)}}\}_{t\in\mathbb{N}_0}$  and  $\{\theta^{l^{(2)}}(t) \in \Theta^{l^{(2)}}\}_{t\in\mathbb{N}_0}$  associated with the processes  $l^{(1)}$  and  $l^{(2)}$  are mutually independent. Let  $\tilde{p}_0 \triangleq \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, p_0^{(2)} + p_1^{(2)}p_0^{(1)}, 1\}$  and  $\tilde{p}_1 \triangleq p_1^{(1)}p_1^{(2)}$ . Then the process  $\{\tilde{l}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  defined by

(15) 
$$l(t) = l^{(1)}(t) \wedge l^{(2)}(t), \quad t \in \mathbb{N}_0,$$

is the output process of a time-inhomogeneous hidden Markov model, and moreover,  $\tilde{l} \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$ .

*Proof.* Let  $\Theta^{\tilde{l}} \triangleq \{(q^{(1)}, q^{(2)}) : q^{(1)} \in \Theta^{l^{(1)}}, q^{(2)} \in \Theta^{l^{(2)}}\}$  and define the bivariate process  $\{\theta^{\tilde{l}}(t) \in \Theta^{\tilde{l}}\}_{t \in \mathbb{N}_0}$  by

$$\theta^{\tilde{l}}(t) = (\theta^{l^{(1)}}(t), \theta^{l^{(2)}}(t)), \quad t \in \mathbb{N}_0.$$

It follows that  $\{\theta^{\tilde{l}}(t) \in \Theta^{\tilde{l}}\}_{t \in \mathbb{N}_{0}}$  is a time-inhomogeneous Markov chain with initial distribution  $\vartheta^{\tilde{l}}_{(q^{(1)},q^{(2)})} = \vartheta^{l^{(1)}}_{q^{(1)}} \vartheta^{l^{(2)}}_{q^{(2)}}, q^{(1)} \in \Theta^{l^{(1)}}, q^{(2)} \in \Theta^{l^{(2)}}$ , and time-varying transition probabilities  $p^{\tilde{l}}_{(q^{(1)},q^{(2)}),(r^{(1)},r^{(2)})}(t) = p^{l^{(1)}}_{q^{(1)},r^{(1)}}(t)p^{l^{(2)}}_{q^{(2)},r^{(2)}}(t), t \in \mathbb{N}_{0}$ . Here, for  $j \in \{1,2\}, \vartheta^{l^{(j)}}$  and  $p^{l^{(j)}}(\cdot)$  respectively denote the initial distribution and the transition probability function for the Markov chain  $\{\theta^{l^{(j)}}(t) \in \Theta^{l^{(j)}}\}_{t \in \mathbb{N}_{0}}$  associated with the output process  $\{l^{(j)} \in \{0,1\}\}_{t \in \mathbb{N}_{0}}$ . Furthermore, it follows from (15)

that

$$\tilde{l}(t) = l^{(1)}(t) \wedge l^{(2)}(t) = l^{(1)}(t)l^{(2)}(t) = h^{l^{(1)}}(\theta^{l^{(1)}}(t))h^{l^{(2)}}(\theta^{l^{(2)}}(t)), \quad t \in \mathbb{N}_0.$$

Now let  $h^{\tilde{l}} \colon \Theta^{\tilde{l}} \to \{0,1\}$  be given by

$$h^{\tilde{l}}((q,r)) = h^{l^{(1)}}(q)h^{l^{(2)}}(r), \quad (q,r) \in \Theta^{\tilde{l}}.$$

It follows that (6) holds with l replaced with  $\tilde{l}$ . Thus,  $\{\tilde{l}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  is the output process of a time-inhomogeneous hidden Markov model.

Our next goal is to prove  $\tilde{l} \in \Gamma_{\tilde{p}_0,\tilde{p}_1}$  by showing

(16) 
$$\sum_{(r^{(1)}, r^{(2)}) \in \Theta_0^{\tilde{l}}} p_{(q^{(1)}, q^{(2)}), (r^{(1)}, r^{(2)})}^{\tilde{l}}(t) \le \tilde{p}_0,$$

(17) 
$$\sum_{(r^{(1)}, r^{(2)}) \in \Theta_1^{\tilde{l}}} p_{(q^{(1)}, q^{(2)}), (r^{(1)}, r^{(2)})}^{\tilde{l}}(t) \le \tilde{p}_1, \quad t \in \mathbb{N}_0$$

First, we show (16). Observe that

(18)  

$$\Theta_{0}^{\tilde{l}} = \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} \colon h^{\tilde{l}}((r^{(1)}, r^{(2)})) = 0 \}$$

$$= \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} \colon h^{l^{(1)}}(r^{(1)})h^{l^{(2)}}(r^{(2)}) = 0 \}.$$

Now let  $j_1, j_2 \in \{1, 2\}$  be such that  $j_1 \neq j_2$ . It follows from (18) that

(19)  

$$\Theta_{0}^{\tilde{l}} = \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} \colon h^{l^{(j_{1})}}(r^{(j_{1})}) = 1, h^{l^{(j_{2})}}(r^{(j_{2})}) = 0 \}$$

$$\cup \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} \colon h^{l^{(j_{1})}}(r^{(j_{1})}) = 0 \}.$$

Hence, by (19), we obtain

Now, since  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}, \ l^{(2)} \in \Gamma_{p_0^{(2)}, p_1^{(2)}},$  we have  $\sum_{r^{(j_1)} \in \Theta_0^{l}} p_{q^{(j_1)}, r^{(j_1)}}^{l^{(j_1)}}(t) \leq p_0^{(j_1)}, \sum_{r^{(j_1)} \in \Theta_1^{l^{(j_1)}}} p_{q^{(j_1)}, r^{(j_1)}}^{l^{(j_1)}}(t) \leq p_1^{(j_1)},$  and  $\sum_{r^{(j_2)} \in \Theta_0^{l^{(j_2)}}} p_{q^{(j_2)}, r^{(j_2)}}^{l^{(j_2)}}(t) \leq p_0^{(j_2)}, \ t \in \mathbb{R}$ 

N<sub>0</sub>. Furthermore, we have  $\sum_{r^{(j_2)} \in \Theta^{l^{(j_2)}}} p_{q^{(j_2)}, r^{(j_2)}}^{l^{(j_2)}}(t) = 1$ , since the summation is over all possible states  $r^{(j_2)} \in \Theta^{l^{(j_2)}}$ . Using these inequalities in (20), we obtain

(21) 
$$\sum_{(r^{(1)},r^{(2)})\in\Theta_0^{\tilde{l}}} p_{(q^{(1)},q^{(2)}),(r^{(1)},r^{(2)})}^{\tilde{l}}(t) \le p_0^{(j_1)} + p_1^{(j_1)} p_0^{(j_2)},$$

for  $t \in \mathbb{N}_0$ . Since (21) holds for all  $j_1, j_2 \in \{1, 2\}$  such that  $j_1 \neq j_2$ , we have (22)

$$\sum_{(r^{(1)},r^{(2)})\in\Theta_0^{\tilde{l}}} p_{(q^{(1)},q^{(2)}),(r^{(1)},r^{(2)})}^{\tilde{l}}(t) \le \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, p_0^{(2)} + p_1^{(2)}p_0^{(1)}\}, \quad t \in \mathbb{N}_0.$$

Furthermore, noting that  $\Theta_{0}^{\tilde{l}} \subset \Theta^{\tilde{l}}$ , we obtain  $\sum_{(r^{(1)}, r^{(2)}) \in \Theta_{0}^{\tilde{l}}} p_{(q^{(1)}, q^{(2)}), (r^{(1)}, r^{(2)})}^{\tilde{l}}(t) \leq \sum_{(r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}}} p_{(q^{(1)}, q^{(2)}), (r^{(1)}, r^{(2)})}^{\tilde{l}}(t) = 1, t \in \mathbb{N}_{0}$ . By using this inequality, it follows from (22) that (16) holds.

Next, we show (17). Notice that

(23)  

$$\Theta_{1}^{\tilde{l}} = \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} : h^{\tilde{l}}((r^{(1)}, r^{(2)})) = 1 \} \\
= \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} : h^{l^{(1)}}(r^{(1)})h^{l^{(2)}}(r^{(2)}) = 1 \} \\
= \{ (r^{(1)}, r^{(2)}) \in \Theta^{\tilde{l}} : h^{l^{(1)}}(r^{(1)}) = 1, h^{l^{(2)}}(r^{(2)}) = 1 \}$$

Noting that  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}, l^{(2)} \in \Gamma_{p_0^{(2)}, p_1^{(2)}}$ , we use (23) to obtain for  $t \in \mathbb{N}_0$ ,

$$\begin{split} &\sum_{(r^{(1)},r^{(2)})\in\Theta_{1}^{\tilde{l}}}p_{\left(q^{(1)},q^{(2)}\right),\left(r^{(1)},r^{(2)}\right)}^{\tilde{l}}(t) = \sum_{r^{(1)}\in\Theta_{1}^{l^{(1)}},r^{(2)}\in\Theta_{1}^{l^{(2)}}}p_{\left(q^{(1)},q^{(2)}\right),\left(r^{(1)},r^{(2)}\right)}^{\tilde{l}}(t) \\ &= \sum_{r^{(1)}\in\Theta_{1}^{l^{(1)}}}\sum_{r^{(2)}\in\Theta_{1}^{l^{(2)}}}p_{q^{(1)},r^{(1)}}^{l^{(1)}}(t)p_{q^{(2)},r^{(2)}}^{l^{(2)}}(t) \\ &= \sum_{r^{(1)}\in\Theta_{1}^{l^{(1)}}}p_{q^{(1)},r^{(1)}}^{l^{(1)}}(t)\sum_{r^{(2)}\in\Theta_{1}^{l^{(2)}}}p_{q^{(2)},r^{(2)}}^{l^{(2)}}(t) \le p_{1}^{(1)}p_{1}^{(2)}, \end{split}$$

which implies (17). Now since (16) and (17) hold, we have  $\tilde{l} \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$ .

Theorem 3.3 shows that when two hidden Markov output processes  $l^{(1)}$  and  $l^{(2)}$  are combined with  $\wedge$  operation, the resulting process  $\tilde{l}$  is also a hidden Markov output process. Furthermore, Theorem 3.3 provides the values of  $\tilde{p}_0, \tilde{p}_1$  for which  $\tilde{l} \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$ .

This result can be applied to obtain  $\rho_G$ . For instance, consider the case c = 2, where  $l_{\mathcal{P}_1}(\cdot)$  and  $l_{\mathcal{P}_2}(\cdot)$  are the output processes of hidden Markov models such that  $l_{\mathcal{P}_1} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}, l_{\mathcal{P}_2} \in \Gamma_{p_0^{(2)}, p_1^{(2)}}$ . It follows from Theorem 3.3 with  $l^{(1)} = l_{\mathcal{P}_1}$ ,  $l^{(2)} = l_{\mathcal{P}_2}$ , and  $\tilde{l} = l_G$  that  $l_G \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$  with  $\tilde{p}_1 \triangleq p_1^{(1)} p_1^{(2)}$ . Now, suppose that  $p_1^{(1)} p_1^{(2)} < 1$ . Notice that since  $l_{\mathcal{P}_1} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$  and  $l_{\mathcal{P}_2} \in \Gamma_{p_0^{(2)}, p_1^{(2)}}$ , we have  $l_{\mathcal{P}_1} \in \Lambda_{\rho_{\mathcal{P}_2}}$  with  $\rho_{\mathcal{P}_1} \in (p_1^{(1)}, 1]$  and  $\rho_{\mathcal{P}_2} \in (p_1^{(2)}, 1]$ . The direct application of Proposition 3.2 gives  $l_G \in \Lambda_{\rho_G}$  with  $\rho_G = \min\{\rho_{\mathcal{P}_1}, \rho_{\mathcal{P}_2}\}$ . However, by applying Proposition 2.5, we can obtain a smaller value for  $\rho_G$ . In fact by Proposition 2.5, we obtain  $l_G \in \Lambda_{\rho_G}$  for any  $\rho_G \in (p_1^{(1)} p_1^{(2)}, 1]$ . Notice that in the case where c > 2, Theorem 3.3 can be applied repeatedly. For instance, when c = 3, we can use Theorem 3.3 first for  $l_{\mathcal{P}_2}(t) \wedge l_{\mathcal{P}_3}(t)$  and then for  $l_G(t) = l_{\mathcal{P}_1}(t) \wedge (l_{\mathcal{P}_2}(t) \wedge l_{\mathcal{P}_3}(t))$ .

#### 16 CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL

Now consider the case where the graph G possesses some paths with indicator processes that are mutually independent but not all of them are associated with random failures. Even for this case, we can obtain results that are tighter than Proposition 3.2. To this end, we first provide the following result where we derive properties of a process that is obtained by using  $\wedge$  operation on a hidden Markov output process  $l^{(1)} \in \Gamma_{n^{(1)}n^{(1)}}$  and a binary-valued process  $l^{(2)} \in \Lambda_{\rho^{(2)}}$ .

**Theorem 3.4.** Consider the binary-valued processes  $\{l^{(1)} \in \{0,1\}\}_{t \in \mathbb{N}_0}$  and  $\{l^{(2)} \in \{0,1\}\}_{t \in \mathbb{N}_0}$  that satisfy  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$  and  $l^{(2)} \in \Lambda_{\rho^{(2)}}$  with  $p_1^{(1)}\rho^{(2)} < 1$ . Then the process  $\{\tilde{l}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  defined by

(24) 
$$\tilde{l}(t) = l^{(1)}(t) \wedge l^{(2)}(t), \quad t \in \mathbb{N}_0,$$

satisfies  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)}, 1]$ .

Proof. For the case where  $p_1^{(1)}\rho^{(2)} = 0$  and the case where  $p_1^{(1)} = 1$ , the result follows from Proposition 3.2. Now consider the case where  $p_1^{(1)} \in (0,1), \rho^{(2)} \in (0,1]$ . We will first show that  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for  $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)}, \rho^{(2)})$ . The process  $\{\chi(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ defined by  $\chi(t) = l^{(2)}(t), t \in \mathbb{N}_0$ , satisfies (8) with  $\tilde{w} = \rho^{(2)}$ . Furthermore,  $\{\xi(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  defined by  $\xi(t) = \theta^{l^{(1)}}(t), t \in \mathbb{N}_0$ , satisfies (7) with  $\tilde{p} = p_1^{(1)}, h = h^{l^{(1)}}, \Xi = \Theta^{l^{(1)}}$ , and  $\Xi_1 = \Theta_1^{l^{(1)}}$ . It then follows from Lemma 2.6 that for all  $\rho \in (\tilde{p}\tilde{w}, \tilde{w})$ , we have

$$\sum_{k \in \mathbb{N}} \mathbb{P}[\sum_{t=0}^{k-1} h(\xi(t))\chi(t) > \rho k] = \sum_{k \in \mathbb{N}} \mathbb{P}[\sum_{t=0}^{k-1} l^{(1)}(t) \wedge l^{(2)}(t) > \rho k] < \infty.$$

Now since  $\tilde{p}\tilde{w} = p_1^{(1)}\rho^{(2)}$  and  $\tilde{w} = \rho^{(2)}$ , it follows that  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  holds for  $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)},\rho^{(2)})$ . This implies  $\tilde{l} \in \Lambda_{\tilde{\rho}}$ ,  $\tilde{\rho} \in [\rho^{(2)},1]$ , since  $\Lambda_{\rho_1} \subseteq \Lambda_{\rho_2}$  for any  $\rho_1,\rho_2 \in [0,1]$  such that  $\rho_1 \leq \rho_2$ . Consequently, we have  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for  $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)},1] = (p_1^{(1)}\rho^{(2)},\rho^{(2)}) \cup [\rho^{(2)},1]$ .

Theorem 3.4 is concerned with  $\wedge$  operation applied to a process  $l^{(1)}(\cdot)$  from the hidden Markov model class  $\Gamma_{p_0^{(1)}, p_1^{(1)}}$  and another process  $l^{(2)}(\cdot)$  from the class  $\Lambda_{\rho^{(2)}}$ . It is shown that if  $p_1^{(1)}\rho^{(2)} < 1$ , then this operation results in a process  $\tilde{l}$  that satisfies  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)}, 1]$ . We note that the application of Proposition 3.2 to this situation would allow us to show  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (\min\{p_1^{(1)}, \rho^{(2)}\}, 1]$ . Notice that Proposition 3.2 in this case is conservative since  $\min\{p_1^{(1)}, \rho^{(2)}\} > p_1^{(1)}\rho^{(2)}$ . On the other hand, we note that Proposition 3.2 allows us to deal with processes that are not mutually independent.

**Remark 3.5.** Theorem 3.3 explains the joint effects of random transmission failures happening on two different paths. On the other hand, Theorem 3.4 is concerned with the case where one path is associated with random packet losses and the other path may be subject to more general types of failures (random, malicious, or a combination of both). In this sense, Theorem 3.4 considers a more general situation than that considered in Theorem 3.3. In fact Theorem 3.4 can also be utilized for two processes that are both associated with random failures. The difference between Theorem 3.3 and Theorem 3.4 is that they describe the joint effects of the processes through different classes ( $\Gamma$  and  $\Lambda$ , respectively).

17

TABLE 2. Comparison of the classes of processes obtained by combining processes  $l^{(1)}$  and  $l^{(2)}$  of different classes through  $\wedge$  operation. For the case where  $l^{(1)} \in \Lambda_{\rho^{(1)}}$  and  $l^{(2)} \in \Lambda_{\rho^{(2)}}$ , the processes  $l^{(1)}$  and  $l^{(2)}$  can be dependent; for other cases,  $l^{(1)}$  and  $l^{(2)}$  are assumed to be mutually independent.

$l^{(1)}$	$l^{(2)}$	$l^{(1)} \wedge l^{(2)}$	
	$\Gamma_{p_0^{(2)},p_1^{(2)}}$	$\Gamma_{\tilde{p}_0,\tilde{p}_1} \text{ with } \tilde{p}_0 \triangleq \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, p_0^{(2)} + p_1^{(2)}p_0^{(1)}, 1\} \text{ and } \tilde{p}_1 \triangleq p_1^{(1)}p_1^{(2)}$	
$\Gamma_{p_0^{(1)},p_1^{(1)}}$		(Theorem 3.3)	
	$\Lambda_{\rho^{(2)}}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in (p_1^{(1)}\rho^{(2)}, 1]$ (Theorem 3.4)	
$\Lambda_{\rho^{(1)}}$	$\Gamma_{p_0^{(2)},p_1^{(2)}}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in (p_1^{(2)}\rho^{(1)}, 1]$ (Theorem 3.4)	
$\rho^{(1)}$	$\Lambda_{\rho(2)}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in [\min\{\rho^{(1)}\rho^{(2)}\}, 1]$ (Proposition 3.2)	

We also note that when two processes associated with random failures are involved, Theorem 3.3 is more advantageous than Theorem 3.4, since Theorem 3.4 may result in conservatism in certain situations. Consider for example  $l^{(1)} \in \Gamma_{p_0^{(1)},p_1^{(1)}}$  and  $l^{(2)} \in \Gamma_{p_0^{(2)},p_1^{(2)}}$  associated both with random failures and  $l^{(3)} \in \Lambda_{\rho^{(3)}}$  associated with malicious packet losses. Since  $l^{(2)} \in \Lambda_{\rho^{(2)}}$  with  $\rho^{(2)} \in (p_1^{(2)}, 1]$ , we can apply Theorem 3.4 to obtain  $l^{(1)} \wedge l^{(2)} \in \Lambda_{\bar{\rho}}$  for all  $\bar{\rho} \in (p_1^{(1)}, \rho^{(2)}, 1]$ . Here, by Theorem 3.4, the process associated with joint failures  $(l^{(1)} \wedge l^{(2)})$  belongs to the class  $\Lambda_{\bar{\rho}}$ . Now, in order to explain the joint effects of  $l^{(1)} \wedge l^{(2)}$  together with yet another process  $l^{(3)} \in \Lambda_{\rho^{(3)}}$ , we would be required to use Proposition 3.2, which would result in conservatism as explained after the proof of Theorem 3.4. In order not to introduce unnecessary conservatism, we can first apply Theorem 3.3 to  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$  and  $l^{(2)} \in \Gamma_{p_0^{(2)}, p_1^{(2)}}$  to obtain  $l^{(1)} \wedge l^{(2)} \in \Gamma_{\bar{\rho}_0, \bar{\rho}_1}$  with  $\tilde{p}_0 \triangleq \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, p_0^{(2)} + p_1^{(2)}p_0^{(1)}, 1\}$  and  $\tilde{p}_1 \triangleq p_1^{(1)}p_1^{(2)}$ . Notice now that the process associated with joint failures  $(l^{(1)} \wedge l^{(2)})$  belongs to the class  $\Gamma_{\bar{p}_0, \bar{p}_1}$ . Then, we can apply Theorem 3.4 for processes  $l^{(1)} \wedge l^{(2)} \in \Gamma_{\bar{p}_0, \bar{p}_1}$  and  $l^{(3)} \in \Lambda_{\rho^{(3)}}$ , obtaining a less conservative result.

We summarize the results presented in this section in Table 2, where we indicate the classes obtained through the  $\land$  operation.

**Remark 3.6.** Although in this paper we assume delay-free packet transmissions, in practice delays are inevitable. Moreover, different routes represented by different paths may induce different amounts of delay for the transmission of the same packet. This is because the numbers of links on different paths  $\mathcal{P}_i$  may be different, and furthermore, links on those paths may have different transmission properties. As a result, the receiver node (e.g., the controller node  $v_{\rm C}$  in G) may obtain the same packet from different paths at different times. It is clear that the performance can be improved if the controller acts immediately upon receiving the first uncorrupted state packet from the path with the shortest delay. Furthermore, we can introduce setups that utilize a delay threshold: a path  $\mathcal{P}_i$  that faces delay beyond the threshold can be considered to have faced a failure  $(l_{\mathcal{P}_i}(t) = 0)$  in transmission of the state packet x(t). In such setups, the delay properties of the links on different paths have to be taken into account to fully model the processes  $\{l_{\mathcal{P}_i}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ . This requires further analysis that is different from what we provide in this paper. In particular, the effects of different delay models for links as well as the combined effects of delays and failures (due to data corruption and packet dropping) need to be investigated.

## 4. PACKET TRANSMISSION FAILURES ON PATHS OF A NETWORK

So far, in the previous section, we have looked at how packet failures on the paths of a network affect the overall packet transmission rate. In this section, our goal is to explore the effect of the failures at individual nodes and links of a path. To this end, we first consider the scenario where packet transmission failures occur due to only data corruption. We then explore the case where data corruption and packet drops may occur on the same path.

4.1. Characterization for Data Corrupting Paths. Let  $\mathcal{P}_{i,j}$  denote the *j*th edge on path  $\mathcal{P}_i$ . We use the binary-valued process  $\{l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  to denote the data corruption indicator for this link. For example, in Fig. 1, consider the second edge  $\mathcal{P}_{1,2} = (v_1, v_C)$  of path  $\mathcal{P}_1$  in (13). The state  $l_{\mathcal{P}_1}^{(v_1, v_C)}(t) = 1$  indicates that at time *t*, the packet flowing on path  $\mathcal{P}_1$  faces data corruption on the link  $(v_1, v_C)$ . This may be due to a jamming attack on this link, or due to channel noise, and moreover, it may also be the case that the node  $v_1$  maliciously corrupts the packet.

The notation for the data corruption indicator allows us to distinguish data corruption issues when we consider the same communication link on different paths. For instance, communication link  $(v_4, v_{\rm C})$  is on both  $\mathcal{P}_2$  and  $\mathcal{P}_3$ . It may be the case that node  $v_4$  corrupts all packets transmitted along  $\mathcal{P}_2$ , but none of the packets transmitted along  $\mathcal{P}_3$ . This situation can be described by setting  $l_{\mathcal{P}_2}^{(v_4,v_{\rm C})}(t) = 1$ ,  $t \in \mathbb{N}_0$ , and  $l_{\mathcal{P}_3}^{(v_4,v_{\rm C})}(t) = 0$ ,  $t \in \mathbb{N}_0$ .

State packet transmitted through path  $\mathcal{P}_i$  is subject to data corruption if there is data corruption on one (or more) of the edges in this path. Hence, for each  $i \in \{1, \ldots, c\}$ ,

(25) 
$$l_{\mathcal{P}_i}(t) = l_{\mathcal{P}_i}^{\mathcal{P}_{i,1}}(t) \lor l_{\mathcal{P}_i}^{\mathcal{P}_{i,2}}(t) \lor \cdots \lor l_{\mathcal{P}_i}^{\mathcal{P}_{i,|\mathcal{P}_i|}}(t).$$

The next result shows that an upper-bound for the asymptotic transmission failure rate of a path can be given as the sum of the failure rate bounds of the links on the path.

**Proposition 4.1.** Consider  $\{l_{\mathcal{P}_i}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  given by (25). Assume  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in \Lambda_{\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}}, j \in \{1,\ldots,|\mathcal{P}_i|\}$ , where  $\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in [0,1], j \in \{1,\ldots,|\mathcal{P}_i|\}$ , satisfy  $\sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \leq 1$ . Then  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$  with  $\rho_{\mathcal{P}_i} \triangleq \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}$ .

*Proof.* By (25),  $\sum_{t=0}^{k-1} l_{\mathcal{P}_i}(t) \leq \sum_{j=1}^{|\mathcal{P}_i|} \sum_{t=0}^{k-1} l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t)$ . Hence,

(26)  

$$\mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}_i}(t) > \rho_{\mathcal{P}_i}k] \leq \mathbb{P}[\sum_{j=1}^{|\mathcal{P}_i|} \sum_{t=0}^{k-1} l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t) > \rho_{\mathcal{P}_i}k] \leq \sum_{j=1}^{|\mathcal{P}_i|} \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t) > \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}k],$$

for  $k \in \mathbb{N}$ , where to obtain the last inequality, we used

$$\mathbb{P}[\sum_{j=1}^{|\mathcal{P}_i|} \gamma_j > 0] \leq \mathbb{P}[\bigcup_{j=1}^{|\mathcal{P}_i|} \{\gamma_j > 0\}] \leq \sum_{j=1}^{|\mathcal{P}_i|} \mathbb{P}[\gamma_j > 0]$$
  
with  $\gamma_j \triangleq \sum_{t=0}^{k-1} l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t) - \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}k$ . The result then follows from (26) since  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in \Lambda_{\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}}, j \in \{1, \dots, |\mathcal{P}_i|\}.$ 

Proposition 4.1 can be used to characterize overall failures on a path. Note that in Proposition 4.1, the indicators for the links are not necessarily mutuallyindependent processes. This allows us to model failures on different links that depend on each other. In particular, we can explore the effect of interference between links as well as coordinated jamming attacks targeting several links at the same time.

Note that in certain cases the result provided in Proposition 4.1 can be improved in terms of conservativeness. In particular, if one or more links in a path are known to be associated with random failures and the corresponding indicator processes are mutually independent, we can obtain less conservative results in comparison to Proposition 4.1. The following result is the counterpart of Theorem 3.3 for  $\lor$ operation and it is concerned with output processes of two mutually-independent hidden Markov models.

**Theorem 4.2.** Consider the binary-valued output processes  $\{l^{(1)} \in \{0,1\}\}_{t\in\mathbb{N}_0}$ and  $\{l^{(2)} \in \{0,1\}\}_{t\in\mathbb{N}_0}$  of hidden Markov models such that  $l^{(1)} \in \Gamma_{p_0^{(1)},p_1^{(1)}}, l^{(2)} \in \Gamma_{p_0^{(2)},p_1^{(2)}}$  with  $p_0^{(1)}, p_1^{(1)}, p_0^{(2)}, p_1^{(2)} \in [0,1]$ . Suppose that the Markov chains  $\{\theta^{l^{(1)}}(t) \in \Theta^{l^{(1)}}\}_{t\in\mathbb{N}_0}$  and  $\{\theta^{l^{(2)}}(t) \in \Theta^{l^{(2)}}\}_{t\in\mathbb{N}_0}$  associated with the processes  $l^{(1)}$  and  $l^{(2)}$ are mutually independent. Let  $\tilde{p}_0 \triangleq p_0^{(1)} p_0^{(2)}$  and  $\tilde{p}_1 \triangleq \min\{p_1^{(1)} + p_0^{(1)} p_1^{(2)}, p_1^{(2)} + p_0^{(2)} p_1^{(1)}, 1\}$ . Then the process  $\{\tilde{l}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  defined by

(27) 
$$\tilde{l}(t) = l^{(1)}(t) \vee l^{(2)}(t), \quad t \in \mathbb{N}_0$$

is the output process of a time-inhomogeneous hidden Markov model, and moreover,  $\tilde{l} \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$ .

*Proof.* Let  $\Theta^{\tilde{l}} \triangleq \{(q^{(1)}, q^{(2)}) : q^{(1)} \in \Theta^{l^{(1)}}, q^{(2)} \in \Theta^{l^{(2)}}\}$  and define the bivariate process  $\{\theta^{\tilde{l}}(t) \in \Theta^{\tilde{l}}\}_{t \in \mathbb{N}_0}$  by

$$\theta^{\tilde{l}}(t) = (\theta^{l^{(1)}}(t), \theta^{l^{(2)}}(t)), \quad t \in \mathbb{N}_0.$$

It follows that  $\{\theta^{\tilde{l}}(t) \in \Theta^{\tilde{l}}\}_{t \in \mathbb{N}_0}$  is a time-inhomogeneous Markov chain with initial distribution  $\vartheta^{\tilde{l}}_{(q^{(1)},q^{(2)})} = \vartheta^{l^{(1)}}_{(q^{(1)})} \vartheta^{l^{(2)}}_{q^{(2)}}, q^{(1)} \in \Theta^{l^{(1)}}, q^{(2)} \in \Theta^{l^{(2)}}$ , and time-varying transition probabilities  $p^{\tilde{l}}_{(q^{(1)},q^{(2)}),(r^{(1)},r^{(2)})}(t) = p^{l^{(1)}}_{q^{(1)},r^{(1)}}(t)p^{l^{(2)}}_{q^{(2)},r^{(2)}}(t), t \in \mathbb{N}_0$ . Here, for  $j \in \{1,2\}, \vartheta^{l^{(j)}}$  and  $p^{l^{(j)}}(\cdot)$  respectively denote the initial distribution and the transition probability function for the Markov chain  $\{\theta^{l^{(j)}}(t) \in \Theta^{l^{(j)}}\}_{t \in \mathbb{N}_0}$  associated with the output process  $\{l^{(j)} \in \{0,1\}\}_{t \in \mathbb{N}_0}$ . Furthermore, it follows from (15) that

$$\tilde{l}(t) = l^{(1)}(t) \vee l^{(2)}(t) = l^{(1)}(t) + (1 - l^{(1)}(t))l^{(2)}(t)$$
  
=  $h^{l^{(1)}}(\theta^{l^{(1)}}(t)) + (1 - h^{l^{(1)}}(\theta^{l^{(1)}}(t)))h^{l^{(2)}}(\theta^{l^{(2)}}(t)),$ 

19

#### 20 CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL

for  $t \in \mathbb{N}_0$ . Now let  $h^{\tilde{l}} : \Theta^{\tilde{l}} \to \{0, 1\}$  be given by

$$h^{\tilde{l}}((q,r)) = h^{l^{(1)}}(q) + (1 - h^{l^{(1)}}(q))h^{l^{(2)}}(r), \quad (q,r) \in \Theta^{\tilde{l}}.$$

It follows that (6) holds with l replaced with  $\tilde{l}$ . Thus,  $\{\tilde{l}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  is the output process of a time-inhomogeneous hidden Markov model.

To show  $l \in \Gamma_{\tilde{p}_0,\tilde{p}_1}$ , we can apply Theorem 3.3. To this end, we first note that for an output process  $l \in \Gamma_{p_0,p_1}$ , the complementing process  $\{l_c(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ given by  $l_c(t) = 1 - l(t), t \in \mathbb{N}_0$ , is the output process of a time-inhomogeneous hidden Markov model, for which  $\Theta^{l_c} = \Theta^l, \theta^{l_c}(t) = \theta^l(t), t \in \mathbb{N}_0$ . Furthermore, we have  $\Theta_0^{l_c} = \Theta_1^l$  and  $\Theta_1^{l_c} = \Theta_0^l$ , since  $h^{l_c}((q, r)) = 1 - h^l((q, r)), (q, r) \in \Theta^{l_c} = \Theta^l$ . As a consequence, we have  $l \in \Gamma_{p_0,p_1}$  if and only if  $l_c \in \Gamma_{p_1,p_0}$ . In the following, we show  $\tilde{l} \in \Gamma_{\tilde{p}_0,\tilde{p}_1}$  by proving that  $\{\tilde{l}_c(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  given by  $\tilde{l}_c(t) = 1 - \tilde{l}(t), t \in \mathbb{N}_0$ , satisfies  $\tilde{l}_c \in \Gamma_{\tilde{p}_1,\tilde{p}_0}$ .

First, observe that  $\tilde{l}_{c}(t) = 1 - \tilde{l}(t) = 1 - l^{(1)}(t) \lor l^{(2)}(t) = (1 - l^{(1)}(t)) \land (1 - l^{(2)}(t)) = l_{c}^{(1)}(t) \land l_{c}^{(2)}(t)$ , where  $l_{c}^{(i)}(t) = 1 - l^{(i)}(t)$ ,  $i \in \{1, 2\}$ . Since  $l^{(1)} \in \Gamma_{p_{0}^{(1)}, p_{1}^{(1)}}$ ,  $l^{(2)} \in \Gamma_{p_{0}^{(2)}, p_{0}^{(2)}}$ , we have  $l_{c}^{(1)} \in \Gamma_{p_{1}^{(1)}, p_{0}^{(1)}}$ ,  $l_{c}^{(2)} \in \Gamma_{p_{1}^{(2)}, p_{0}^{(2)}}$ . Finally, noting that  $\tilde{l}_{c}(\cdot)$  is obtained by using  $\land$  operation on processes  $l_{c}^{(1)}(\cdot)$ ,  $l_{c}^{(2)}(\cdot)$ , we can use Theorem 3.3, to obtain  $\tilde{l}_{c} \in \Gamma_{\tilde{p}_{1}, \tilde{p}_{0}}$ , which implies  $\tilde{l} \in \Gamma_{\tilde{p}_{0}, \tilde{p}_{1}}$ .

Theorem 4.2 shows that when two hidden Markov output processes  $l^{(1)}$  and  $l^{(2)}$  are combined with  $\vee$  operation, the resulting process  $\tilde{l}$  is also a hidden Markov output process. Furthermore, it provides the values of  $\tilde{p}_0, \tilde{p}_1$  for which  $\tilde{l} \in \Gamma_{\tilde{p}_0, \tilde{p}_1}$ .

Theorem 4.2 can be applied in obtaining  $\rho_{\mathcal{P}_i}$  for a given path  $\mathcal{P}_i$ . Consider for example a path  $\mathcal{P}_i$  with  $|\mathcal{P}_i| = 2$  links. Assume that the failure indicator processes  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,1}}(\cdot)$ ,  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,2}}(\cdot)$  associated with the links are mutually independent and  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,1}} \in \Gamma_{p_0^{(1)},p_1^{(1)}}$ ,  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,2}} \in \Gamma_{p_0^{(2)},p_1^{(2)}}$ . It follows from Theorem 4.2 with  $l^{(1)}(t) = l_{\mathcal{P}_i}^{\mathcal{P}_{i,1}}(t)$ and  $l^{(2)}(t) = l_{\mathcal{P}_i}^{\mathcal{P}_{i,2}}(t)$  that  $l_{\mathcal{P}_i} \in \Gamma_{\tilde{p}_0,\tilde{p}_1}$  with  $\tilde{p}_0 = p_0^{(1)}p_0^{(2)}$  and  $\tilde{p}_1 = \min\{p_1^{(1)} + p_0^{(1)}p_1^{(2)}, p_1^{(2)} + p_0^{(2)}p_1^{(1)}\} < 1$ , then by Proposition 2.5, we have  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$  with  $\rho_{\mathcal{P}_i} \in (\min\{p_1^{(1)} + p_0^{(1)}p_1^{(2)}, p_1^{(2)}, p_1^{(2)} + p_0^{(2)}p_1^{(2)}\}, 1]$ .

On the other hand, the direct application of Proposition 4.1 provides a conservative result. To apply Proposition 4.1, notice first that  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,1}} \in \Lambda_{\rho_{\mathcal{P}_{i,1}}}$ ,  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,2}} \in \Lambda_{\rho_{\mathcal{P}_{i,2}}}$ with  $\rho_{\mathcal{P}_{i,1}} \in (p_1^{(1)}, 1]$  and  $\rho_{\mathcal{P}_{i,2}} \in (p_1^{(2)}, 1]$ . Hence, by Proposition 4.1, we obtain the value  $\rho_{\mathcal{P}_i} = \rho_{\mathcal{P}_{i,1}} + \rho_{\mathcal{P}_{i,2}}$ , which implies that  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$  with  $\rho_{\mathcal{P}_i} \in (p_1^{(1)} + p_1^{(2)}, 1]$ . The inequality  $\min\{p_1^{(1)} + p_0^{(1)}p_1^{(2)}, p_1^{(2)} + p_0^{(2)}p_1^{(1)}\} \leq p_1^{(1)} + p_1^{(2)}$  allows us to conclude that Theorem 4.2 provides a less conservative range for  $\rho_{\mathcal{P}_i}$  compared to Proposition 4.1.

In certain scenarios a path  $\mathcal{P}_i$  may be composed of communication links with mutually-independent indicator processes some of which are not associated with random failures. In such scenarios, it is again possible to obtain results that are less conservative than those in Proposition 4.1. Specifically, in the following result we derive properties of a process that is obtained by using  $\vee$  operation on a hidden Markov output process  $l^{(1)} \in \Gamma_{p_0^{(1)}p_1^{(1)}}$  and a binary-valued process  $l^{(2)} \in \Lambda_{\rho^{(2)}}$ . **Theorem 4.3.** Consider the binary-valued processes  $\{l^{(1)} \in \{0,1\}\}_{t \in \mathbb{N}_0}$  and  $\{l^{(2)} \in \{0,1\}\}_{t \in \mathbb{N}_0}$  that satisfy  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$  and  $l^{(2)} \in \Lambda_{\rho^{(2)}}$  with  $p_1^{(1)} + p_0^{(1)} \rho^{(2)} < 1$ . Then the process  $\{\tilde{l}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  defined by

(28) 
$$\tilde{l}(t) = l^{(1)}(t) \vee l^{(2)}(t), \quad t \in \mathbb{N}_0,$$

satisfies  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)} + p_0^{(1)}\rho^{(2)}, 1]$ .

*Proof.* For the case where  $p_0^{(1)} = 1$  and the case where  $\rho^{(2)} = 0$ , the result follows from Proposition 4.1. Here, we consider the case where  $p_0^{(1)} \in (0,1), \rho^{(2)} \in (0,1]$ . Notice that since  $\tilde{l}(\cdot)$  is a binary-valued process, we have  $\Lambda_{\tilde{\rho}}$  for  $\tilde{\rho} = 1$ . Next, we show  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for  $\tilde{\rho} \in (p_1^{(1)} + p_0^{(1)}\rho^{(2)}, 1)$ . First, (28) implies

(29) 
$$\tilde{l}(t) = l^{(1)}(t) + (1 - l^{(1)}(t))l^{(2)}(t), \quad t \in \mathbb{N}_0.$$

Let  $\tilde{L}(k) \triangleq \sum_{t=0}^{k-1} \tilde{l}(t)$ . It follows from (29) that

(30) 
$$\tilde{L}(k) = \sum_{t=0}^{k-1} l^{(1)}(t) + \sum_{t=0}^{k-1} (1 - l^{(1)}(t)) l^{(2)}(t), \quad k \in \mathbb{N}.$$

Now, let  $\epsilon \triangleq \tilde{\rho} - p_1^{(1)} - p_0^{(1)} \rho^{(2)}$ ,  $\epsilon_2 \triangleq \min\{\frac{\epsilon}{2}, \frac{\rho^{(2)} - p_0^{(1)} \rho^{(2)}}{2}\}$ ,  $\epsilon_1 \triangleq \epsilon - \epsilon_2$ , and define  $\tilde{\rho}_1 \triangleq p_1^{(1)} + \epsilon_1$ ,  $\tilde{\rho}_2 \triangleq p_0^{(1)} \rho^{(2)} + \epsilon_2$ . Furthermore, let  $\tilde{L}_1(k) \triangleq \sum_{t=0}^{k-1} l^{(1)}(t)$  and  $\tilde{L}_2(k) \triangleq \sum_{t=0}^{k-1} (1 - l^{(1)}(t)) l^{(2)}(t)$ . We then have

(31)  

$$\mathbb{P}[\tilde{L}(k) > \tilde{\rho}k] = \mathbb{P}[\tilde{L}_1(k) + \tilde{L}_2(k) > \tilde{\rho}_1k + \tilde{\rho}_2k] \\
\leq \mathbb{P}\left[\left\{\tilde{L}_1(k) > \tilde{\rho}_1k\right\} \cup \left\{\tilde{L}_2(k) > \tilde{\rho}_2k\right\}\right] \\
\leq \mathbb{P}[\tilde{L}_1(k) > \tilde{\rho}_1k] + \mathbb{P}[\tilde{L}_2(k) > \tilde{\rho}_2k].$$

In the following we show  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_1(k) > \tilde{\rho}_1 k] < \infty$  and  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_2(k) > \tilde{\rho}_2 k] < \infty$ . First, note that

(32)  

$$\tilde{\rho}_{1} = p_{1}^{(1)} + \epsilon - \epsilon_{2} = \max\{p_{1}^{(1)} + \frac{\epsilon}{2}, p_{1}^{(1)} + \epsilon - \frac{\rho^{(2)} - p_{0}^{(1)}\rho^{(2)}}{2}\}$$

$$= \max\{\frac{p_{1}^{(1)} + \tilde{\rho} - p_{0}^{(1)}\rho^{(2)}}{2}, \tilde{\rho} - p_{0}^{(1)}\rho^{(2)} - \frac{\rho^{(2)} - p_{0}^{(1)}\rho^{(2)}}{2}\}\}$$

$$= \max\{\frac{p_{1}^{(1)} + \tilde{\rho} - p_{0}^{(1)}\rho^{(2)}}{2}, \frac{2\tilde{\rho} - \rho^{(2)}(1 + p_{0}^{(1)})}{2}\}.$$

As  $\frac{p_1^{(1)} + \tilde{\rho} - p_0^{(1)} \rho^{(2)}}{2} < 1$  and  $\frac{2\tilde{\rho} - \rho^{(2)}(1 + p_0^{(1)})}{2} < 1$ , it holds from (32) that  $\tilde{\rho}_1 \in (p_1^{(1)}, 1)$ . Since  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$ , we can use Proposition 2.5 with  $\rho$  replaced with  $\tilde{\rho}$  and l replaced with  $l^{(1)}$  to obtain  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_1(k) > \tilde{\rho}_1 k] < \infty$ .

replaced with  $l^{(1)}$  to obtain  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_1(k) > \tilde{\rho}_1 k] < \infty$ . Next, we use Lemma 2.6 to show that  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_2(k) > \tilde{\rho}_2 k] < \infty$ . To obtain this result, we first observe that  $\tilde{\rho}_2 > p_0^{(1)} \rho^{(2)}$ , since  $\epsilon_2 > 0$ . Moreover,

$$\tilde{\rho}_{2} = p_{0}^{(1)}\rho^{(2)} + \min\{\frac{\epsilon}{2}, \frac{\rho^{(2)} - p_{0}^{(1)}\rho^{(2)}}{2}\} \le p_{0}^{(1)}\rho^{(2)} + \frac{\rho^{(2)} - p_{0}^{(1)}\rho^{(2)}}{2} < p_{0}^{(1)}\rho^{(2)} + \rho^{(2)} - p_{0}^{(1)}\rho^{(2)} = \rho^{(2)},$$

#### 22 CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL

TABLE 3. Comparison of the classes of processes obtained by combining processes  $l^{(1)}$  and  $l^{(2)}$  of different classes through  $\lor$  operation. For the case where  $l^{(1)} \in \Lambda_{\rho^{(1)}}$  and  $l^{(2)} \in \Lambda_{\rho^{(2)}}$ , the processes  $l^{(1)}$  and  $l^{(2)}$  can be dependent; for other cases,  $l^{(1)}$  and  $l^{(2)}$  are assumed to be mutually independent.

$l^{(1)}$	$l^{(2)}$	$l^{(1)} \lor l^{(2)}$	
	$\Gamma_{p_0^{(2)},p_1^{(2)}}$	$\Gamma_{\tilde{p}_0,\tilde{p}_1} \text{ with } \tilde{p}_0 \triangleq p_0^{(1)} p_0^{(2)} \text{ and } \tilde{p}_1 \triangleq \min\{p_1^{(1)} + p_0^{(1)} p_1^{(2)}, p_1^{(2)} + p_0^{(2)} p_1^{(1)}, 1\}$	
$\Gamma_{p_0^{(1)},p_1^{(1)}}$		(Theorem 4.2)	
	$\Lambda_{\rho^{(2)}}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in (p_1^{(1)} + p_0^{(1)}\rho^{(2)}, 1]$ (Theorem 4.3)	
$\Lambda_{\rho(1)}$	$\Gamma_{p_0^{(2)},p_1^{(2)}}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in (p_1^{(2)} + p_0^{(2)}\rho^{(1)}, 1]$ (Theorem 4.3)	
$\rho^{(1)}$	$\Lambda_{\rho(2)}$	$\Lambda_{\tilde{\rho}}$ for $\tilde{\rho} \in [\rho^{(1)} + \rho^{(2)}, 1]$ (Proposition 4.1)	

and hence, we have  $\tilde{\rho}_2 \in (p_0^{(1)}\rho^{(2)}, \rho^{(2)})$ . Let  $\{l_c^{(1)}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  be defined by  $l_c^{(1)}(t) = 1 - l^{(1)}(t), t \in \mathbb{N}_0$ . Since  $l^{(1)} \in \Gamma_{p_0^{(1)}, p_1^{(1)}}$ , we have  $l_c^{(1)} \in \Gamma_{p_1^{(1)}, p_0^{(1)}}$ . Furthermore, since  $l^{(2)} \in \Lambda_{\rho^{(2)}}$ , conditions (7), (8) in the Lemma 2.6 hold with  $\tilde{p} = p_0^{(1)}$  and  $\tilde{w} = \rho^{(2)}$ , together with processes  $\{\xi(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  and  $\{\chi(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  defined by setting  $\xi(t) = \theta^{l_c}(t), \ \chi(t) = l_{\mathcal{M}}(t), \ t \in \mathbb{N}_0$ . Now, we have  $\tilde{L}_2(k) = \sum_{t=0}^{k-1} \xi(t)\chi(t)$  and hence, Lemma 2.6 implies  $\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_2(k) > \tilde{\rho}_2 k] < \infty$ . Finally, by (31), we arrive at

$$\sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}(k) > \tilde{\rho}_k] \le \sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_1(k) > \tilde{\rho}_1 k] + \sum_{k=1}^{\infty} \mathbb{P}[\tilde{L}_2(k) > \tilde{\rho}_2 k] < \infty,$$

which shows that  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)} + p_0^{(1)}\rho^{(2)}, 1]$ .

Theorem 4.3 is concerned with  $\lor$  operation applied to a process  $l^{(1)}(\cdot)$  from the hidden Markov model class  $\Gamma_{p_0^{(1)}, p_1^{(1)}}$  and another process  $l^{(2)}(\cdot)$  from the class  $\Lambda_{\rho^{(2)}}$ . It is shown that the  $\lor$  operation results in a process  $\tilde{l}$  that satisfies  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)} + p_0^{(1)}\rho^{(2)}, 1]$ . Notice that the application of Proposition 3.2 to this situation would allow us to show  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  for all  $\tilde{\rho} \in (p_1^{(1)} + \rho^{(2)}, 1]$ . Proposition 4.1 in this case is conservative since  $p_1^{(1)} + \rho^{(2)} \ge p_1^{(1)} + p_0^{(1)}\rho^{(2)}$  (and  $p_1^{(1)} + \rho^{(2)} > p_1^{(1)} + p_0^{(1)}\rho^{(2)}$  if  $p_0^{(1)} < 1$ ). We remark that the advantage of Proposition 4.1 may be that it allows us to deal with processes that are not mutually independent.

The results presented so far in this section are summarized in Table 3. There we show the classes of processes obtained by using the  $\lor$  operation.

**Remark 4.4.** By utilizing (14) and (25) together with the results so far presented, we can obtain  $\rho$  values for which the overall packet exchange failure indicator l satisfies  $l \in \Lambda_{\rho}$ . As discussed in Section 2,  $l \in \Lambda_{\rho}$  implies that the average number of packet exchange failures is upper-bounded by  $\rho$  (i.e.,  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq \rho$ , almost surely). We would like  $\rho$  to be a tight upper bound so that we can avoid additional conservatism in checking the closed-loop stability with Theorem 2.3. As discussed in Section 4 of [33], conditions (3)–(5) of Theorem 2.3 are tight for scalar systems when  $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) = \rho$ . However, notice that in certain cases, TABLE 4. Comparison of the conservativeness of our results with respect to the range of  $\tilde{\rho}$  of the process  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  obtained by combining mutually independent processes  $l^{(1)} \in \Gamma_{p_0^{(1)},p_1^{(1)}}$  and  $l^{(2)} \in \Gamma_{p_0^{(2)},p_1^{(2)}}$  through  $\wedge$  or  $\vee$  operations (i.e.,  $\tilde{l}(t) = l^{(1)}(t) \wedge l^{(2)}(t)$ or  $\tilde{l}(t) = l^{(1)}(t) \vee l^{(2)}(t)$ ). For each operation, less conservative results are marked with  $\star$  and they provide larger ranges of  $\tilde{\rho}$  (indicating that  $\tilde{l} \in \Lambda_{\tilde{\rho}}$  holds with smaller  $\tilde{\rho}$  values close to the lower bound of the range). However, these less conservative results are applicable only when  $l^{(1)}$  and  $l^{(2)}$  are mutually independent. The sequences of results that are not marked with  $\star$  are more conservative, but they are also applicable to scenarios where  $l^{(1)}$  and  $l^{(2)}$ are not necessarily mutually independent.

Operation	Applied Results	$\tilde{ ho}$ range
	$\star$ First Theorem 3.3, then Proposition 2.5	$(p_1^{(1)}p_1^{(2)},1]$
~	First Proposition 2.5, then Proposition 3.2	$(\min\{p_1^{(1)}, p_1^{(2)}\}, 1]$
	$\star$ First Theorem 4.2, then Proposition 2.5	$(\min\{p_1^{(1)} + p_0^{(1)}p_1^{(2)}, p_1^{(2)} + p_0^{(2)}p_1^{(1)}\}, 1]$
· ·	First Proposition 2.5, then Proposition 4.1	$(p_1^{(1)} + p_1^{(2)}, 1]$

*l* may be a nonergodic process for which  $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  takes different values for different outcomes  $\omega \in \Omega$  (see Remark 3.4 in [33]); moreover, in certain cases  $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  may not exist. In such cases  $\rho$  (as an upper bound of  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$ ) is still useful in stability analysis through Theorem 2.3. In some cases,  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  may be strictly smaller than  $\rho$  that we ob-

tain by using our results in this paper. The gap between  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  and  $\rho$  can be identified if all link failure model parameters and independence/dependence relations between communication links are known. Note that it may be difficult to obtain an analytical expression for  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  even if the properties of the failure processes are known exactly. This is because such properties may be time-dependent and may have complicated inter-dependence relations. We compare  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  and  $\rho$  numerically through repeated simulations for an example case in Section 5.3. To guarantee a small gap between  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$ and  $\rho$ , failure processes need to be placed in adequate classes  $\Gamma_{p_0,p_1}$  and  $\Pi_{\kappa,w}$  so that the bounds utilized in Definitions 2.4 and 2.7 are sufficiently tight. Furthermore, to keep the gap small, it is also essential to utilize Theorems 3.3, 3.4, 4.2, and 4.3 when the processes involved in  $\wedge$  and  $\vee$  operations are known to be mutually independent and at least one of them is from the class  $\Gamma$ . If, instead, Propositions 3.2 and 4.1 are used, this may introduce additional gap between  $\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)$  and  $\rho$ . This is because, in scenarios where random failures on links/paths occur independently, Propositions 3.2 and 4.1 are more conservative. In Table 4, we compare these results with Theorems 3.3 and 4.2 in terms of the conservatism that they may introduce.

4.2. Characterization of Paths with Packet Dropping Links. In the previous section, we provided a characterization for paths with data-corrupting links. In this section we look at packet drops.

A packet dropout on a communication link (v, w) may occur if v is a malicious router that intentionally skips forwarding packets (see blackhole and grayhole attacks in [19]). A nonmalicious router may also drop packets to avoid congestion. In addition to these two issues, a packet may also be dropped if the header part of the packet, which includes information on the destination of the packet, is corrupted. Furthermore, in scenarios where error-detection is implemented at intermediate nodes, corruption on the data part of a packet can be detected. As a result, a corrupted packet needs not be further transmitted. Such a scenario can also be studied within the packet drop framework.

Consider a link  $\mathcal{P}_{i,j} = (v, w)$  on path  $\mathcal{P}_i$ . Let  $t = 0, 1, \ldots$ , denote the indices of packets that node v possesses (or receives from previous nodes on path  $\mathcal{P}_i$ ). Observe that if there are links before  $\mathcal{P}_{i,j}$  that are packet-dropping, then the first packet (with index t = 0) that v receives may be different from x(0), and it may be the state at a later time. This is because x(0) may have been dropped before reaching node v. We use  $l_{\mathcal{P}_{i,j}}(t) \in \{0, 1\}$  to indicate the status of transmission of the (t+1)th packet (with index t) that node v possesses to node w.

The following result is concerned with the characterization of the failures on a path with only packet dropping links.

**Proposition 4.5.** Let  $\{l_{\mathcal{P}_i}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  denote the failure indicator of a path  $\mathcal{P}_i$ composed only of packet dropping links with failure indicators denoted by  $\{l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ . Assume  $l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in \Lambda_{\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}}$  with  $\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in [0,1]$ ,  $j \in \{1,\ldots,|\mathcal{P}_i|\}$ , that satisfy  $\sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \leq 1$ . Then  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$  with  $\rho_{\mathcal{P}_i} \triangleq \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}$ .

The proof of this result is skipped since it is similar to that of Theorem 4.8 provided in the next section, where we consider paths that include data corrupting and packet dropping communication links.

4.3. A General Characterization of Paths with Data Corruption and Packet Drops. In this section we investigate the effects of both data corruption and packet dropouts. Without loss of generality, we assume that links on a path are either *data-corrupting* or *packet-dropping*, but not both. Note that if in the original network, a link (v, w) is subject to both of the issues, we can artificially add a node v' and edges (v, v'), (v', w) to the graph, and consider (v, v') as a packet-dropping link and (v', w) as a data-corrupting link.

For a data-corrupting link (v, w), packets available at v are always transmitted to w, but their content may be externally manipulated or damaged during the transmission over this link. On the other hand, if (v, w) is a packet-dropping link, packets available at v may or may not be received by w, but never get corrupted on the communication link (v, w).

Our goal here is to obtain a relation between the asymptotic packet failure ratio of path  $\mathcal{P}_i$  and the failure ratios of the links on that path. To this end, we will use a recursive characterization for describing packet failures on paths. Specifically, consider a path

(33) 
$$\mathcal{P} \triangleq ((v_1, v_2), (v_2, v_3), \dots, (v_h, v_{h+1}))$$

of  $h \geq 1$  links, and consider the associated process  $\{l_{\mathcal{P}}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ . The state  $l_{\mathcal{P}}(t) = 0$  indicates that the (t+1)th packet that the first node  $v_1$  possesses can be successfully transmitted to the last node  $v_{h+1}$ , whereas  $l_{\mathcal{P}}(t) = 1$  indicates a failure.

If h = 1 in (33), then we have  $l_{\mathcal{P}}(t) = l_{\mathcal{P}}^{(v_1, v_2)}(t), t \in \mathbb{N}_0$ . Now consider the case  $h \geq 2$ , and let  $f(\mathcal{P})$  and  $\mathcal{R}(\mathcal{P})$  respectively denote the first link on  $\mathcal{P}$ , and the *subpath* composed of the rest of the links, that is,

$$(34) f(\mathcal{P}) = (v_1, v_2),$$

(35) 
$$\mathcal{R}(\mathcal{P}) = ((v_2, v_3), \dots, (v_h, v_{h+1})).$$

We illustrate  $f(\mathcal{P})$  and  $\mathcal{R}(\mathcal{P})$  on the left side of Fig. 2.

Next, we show that transmission failures on a path  $\mathcal{P}$  can be characterized through transmission failures on the link  $f(\mathcal{P})$  and the subpath  $\mathcal{R}(\mathcal{P})$ . Let  $\{l_{\mathcal{P}}^{f(\mathcal{P})}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  and  $\{l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  denote indicators of transmission failures on the link  $f(\mathcal{P})$  and the subpath  $\mathcal{R}(\mathcal{P})$ . If the link  $f(\mathcal{P}) = (v_1, v_2)$  is a datacorrupting link, then we would have  $l_{\mathcal{P}}(t) = l_{\mathcal{P}}^{f(\mathcal{P})}(t) \vee l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$ . But this relation does not hold if  $f(\mathcal{P})$  is a packet-dropping link, because the index t for  $l_{\mathcal{P}}^{f(\mathcal{P})}(t)$  and  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$  represent different packets.

Now, we will introduce a new process  $\{\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$  for which

(36) 
$$l_{\mathcal{P}}(t) = l_{\mathcal{P}}^{f(\mathcal{P})}(t) \vee \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t), \quad t \in \mathbb{N}_0$$

If  $f(\mathcal{P})$  is a data-corrupting link, then we define  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\cdot)$  by setting  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) = l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t), t \in \mathbb{N}_0$ . On the other hand, if  $f(\mathcal{P})$  is a packet-dropping link, then for  $t \in \mathbb{N}_0$ ,

(37) 
$$\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \triangleq \begin{cases} 0, & \text{if } l_{\mathcal{P}}^{f(\mathcal{P})}(t) = 1, \\ l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\hat{k}(t+1)-1), & \text{if } l_{\mathcal{P}}^{f(\mathcal{P})}(t) = 0, \end{cases}$$

where  $\hat{k}(t) \triangleq \sum_{i=0}^{t-1} (1 - l_{\mathcal{P}}^{f(\mathcal{P})}(i)), t \in \mathbb{N}$ . In this definition,  $\hat{k}(t+1)$  denotes the number of packets that are successfully transmitted from node  $v_1$  to node  $v_2$  among the first t+1 packets that node  $v_1$  possesses. Hence, the scalar  $\hat{k}(t+1)-1$  represents the index of the  $\hat{k}(t+1)$ th packet received by  $v_2$ . Moreover,  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\hat{k}(t+1)-1)$  indicates whether this packet is successfully transmitted from  $v_2$  over  $\mathcal{R}(\mathcal{P})$  to  $v_{h+1}$ .

Observe that by (37),  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$  is set to 0, if  $v_1$  drops the (t+1)th packet that it possesses. On the other hand, if  $v_1$  transmits this packet to  $v_2$ , the state  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) = 1$  indicates that further transmission of this packet on subpath  $\mathcal{R}(\mathcal{P})$  has failed, whereas  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) = 0$  indicates success. As a result, we have  $l_{\mathcal{P}}(t) = l_{\mathcal{P}}^{f(\mathcal{P})}(t) + (1 - l_{\mathcal{P}}^{f(\mathcal{P})}(t))\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$ , and hence, (36) holds.

The characterization above is recursive in the sense that it is recursively applied to describe failures on  $\mathcal{R}(\mathcal{P})$  by means of failures on the first link  $f(\mathcal{R}(\mathcal{P}))$  and the subpath  $\mathcal{R}(\mathcal{R}(\mathcal{P}))$ .

**Example 4.6.** In Fig. 2, we see a path  $\mathcal{P}$ , its packet-dropping first link  $f(\mathcal{P}) = (v_1, v_2)$ , and its subpath  $\mathcal{R}(\mathcal{P})$ , together with sample trajectories of  $l_{\mathcal{P}}^{f(\mathcal{P})}(\cdot)$ ,  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\cdot)$ ,  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\cdot)$  and  $l_{\mathcal{P}}(\cdot)$ . Note that the link  $f(\mathcal{P}) = (v_1, v_2)$  drops the first two packets, that is,  $l_{\mathcal{P}}^{f(\mathcal{P})}(t) = 1$  for  $t \in \{0, 1\}$ . By (37), we have  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(0) = \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(1) = 0$ , and as

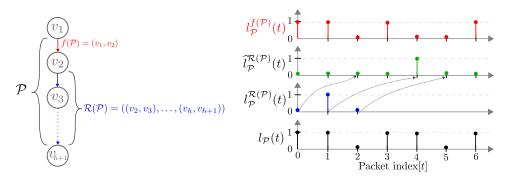


FIGURE 2. [Left] Paths  $\mathcal{P}$ ,  $\mathcal{R}(\mathcal{P})$ , and packet-dropping link  $f(\mathcal{P})$ . [Right] Trajectories of  $l_{\mathcal{P}}^{f(\mathcal{P})}(\cdot)$ ,  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\cdot)$ ,  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(\cdot)$  and  $l_{\mathcal{P}}(\cdot)$ .

a consequence of (36),  $l_{\mathcal{P}}(0) = l_{\mathcal{P}}(1) = 1$ . Furthermore, the link  $f(\mathcal{P})$  successfully transmits the 3rd packet, i.e,  $l_{\mathcal{P}}^{f(\mathcal{P})}(2) = 0$ . The value of  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(2)$  represents whether this packet is successfully transmitted over the subpath  $\mathcal{R}(\mathcal{P})$  or not. Since this packet is the first packet that  $v_2$  receives, its transmission state is represented by  $l_{\mathcal{R}(\mathcal{P})}(0)$ , and as a result,  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(2) = l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(0)$ . We have  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(2) = l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(0) = 0$ , indicating successful transmission over  $\mathcal{R}(\mathcal{P})$ . Now, by (36),  $l_{\mathcal{P}}(2) = 0$ , which indicates that the 3rd packet possessed by the first node  $v_1$  is successfully transmitted to the last node  $v_{h+1}$ .

The following result is essential for characterizing the failures on path  $\mathcal{P}$  through those on the first link  $f(\mathcal{P})$  and the subpath  $\mathcal{R}(\mathcal{P})$ . We obtain the result by first establishing the key inequality  $\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \leq \sum_{t=0}^{k-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$ , and then applying some of the arguments that we employed for proving Proposition 4.1.

**Lemma 4.7.** Consider path  $\mathcal{P}$  given in (33) for  $h \geq 1$ . Assume  $l_{\mathcal{P}}^{f(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{f(\mathcal{P})}}$  and  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}}$  with scalars  $\rho_{\mathcal{P}}^{f(\mathcal{P})}, \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in [0,1]$ , such that  $\rho_{\mathcal{P}}^{f(\mathcal{P})} + \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \leq 1$ . It follows that  $l_{\mathcal{P}} \in \Lambda_{\rho_{\mathcal{P}}}$  with  $\rho_{\mathcal{P}} = \rho_{\mathcal{P}}^{f(\mathcal{P})} + \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}$ .

*Proof.* We will first show that  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}}$ . If  $f(\mathcal{P})$  is a data-corrupting link, we have  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) = l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t), t \in \mathbb{N}_0$ , and therefore  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}}$ . Now consider the situation where  $f(\mathcal{P})$  is a packet-dropping link. For this case,

$$\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) = \begin{cases} 0, & \hat{k}(k) = 0, \\ \sum_{t=0}^{\hat{k}(k)-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t), & \hat{k}(k) \neq 0, \end{cases} \quad k \in \mathbb{N}.$$

Since  $\hat{k}(k) \leq k$  and  $l_{\mathcal{R}(\mathcal{P})}(t) \geq 0$ , we have  $\sum_{t=0}^{\hat{k}(k)-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \leq \sum_{t=0}^{k-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$ , and hence,  $\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) \leq \sum_{t=0}^{k-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t)$ . It follows that

$$\mathbb{P}[\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}k] \le \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}k]$$

Now, as  $l_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}}$ , we also have  $\hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})} \in \Lambda_{\rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}}$ .

Finally, since (36) holds, we have  $l_{\mathcal{P}}(t) \leq l_{\mathcal{P}}^{f(\mathcal{P})}(t) + \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t), t \in \mathbb{N}_0$ . Hence,

$$\mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}(t) > \rho_{\mathcal{P}} k] \leq \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}^{f(\mathcal{P})}(t) + \sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > \rho_{\mathcal{P}} k]$$

$$= \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}^{f(\mathcal{P})}(t) + \sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > (\rho_{\mathcal{P}}^{f(\mathcal{P})} + \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})})k]$$

$$\leq \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}^{f(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{f(\mathcal{P})}k] + \mathbb{P}[\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}k].$$
(38)

It then follows from (38) together with  $\sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}^{f(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{f(\mathcal{P})}k] < \infty$  and  $\sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} \hat{l}_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}(t) > \rho_{\mathcal{P}}^{\mathcal{R}(\mathcal{P})}k] < \infty$  that  $\sum_{k=1}^{\infty} \mathbb{P}[\sum_{t=0}^{k-1} l_{\mathcal{P}}(t) > \rho_{\mathcal{P}}k] < \infty$ , which completes the proof.

Now, for a given path  $\mathcal{P}_i$ , repeated application of Lemma 4.7 to  $\mathcal{P}_i$ ,  $\mathcal{R}(\mathcal{P}_i)$ ,  $\mathcal{R}(\mathcal{R}(\mathcal{P}_i))$ , ..., leads us to the following result.

**Theorem 4.8.** Assume 
$$l_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in \Lambda_{\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}}$$
 with  $\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \in [0,1], j \in \{1,\ldots,|\mathcal{P}_i|\}$ , that satisfy  $\sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} \leq 1$ . Then  $l_{\mathcal{P}_i} \in \Lambda_{\rho_{\mathcal{P}_i}}$  with  $\rho_{\mathcal{P}_i} \triangleq \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}$ .

Note that Theorem 4.8 allows us to consider both data-corruption and packetdropouts on links, and hence it generalizes Proposition 4.1.

**Remark 4.9.** By utilizing Theorem 4.8 together with Proposition 3.2, we can obtain  $\rho_G, \rho_{\tilde{G}} \in [0, 1]$  as upper-bounds for the average number of packet exchange failures on networks G and  $\tilde{G}$  such that  $l_G \in \Lambda_{\rho_G}$  and  $l_{\tilde{G}} \in \Lambda_{\rho_{\tilde{G}}}$ . Note that when  $l_G(t) = 1$ , then the controller either does not receive the state packet or receives corrupted versions, which are discarded. Hence, when  $l_G(t) = 1$ , no control input packet is attempted to be transmitted on  $\tilde{G}$ . This setting is similar to the situation that we discussed above for packet dropping links. Here we can consider the whole networked system as a path  $\mathcal{P}$  from node  $v_{\mathrm{P}}$  to node  $\tilde{v}_{\mathrm{P}}$ , where  $l_G(\cdot)$  corresponds to the indicator for the first packet dropping link  $(f(\mathcal{P}))$  and  $l_{\tilde{G}}(\cdot)$  corresponds to packet transmission failure indicator for the rest of the path  $(\mathcal{R}(\mathcal{P}))$ . Hence, by arguments similar to the ones used above, we can show that if  $\rho_G + \rho_{\tilde{G}} \leq 1$ , then  $l \in \Lambda_{\rho}$  with  $\rho = \rho_G + \rho_{\tilde{G}}$ . This  $\rho$  value is utilized for stability analysis with Theorem 2.3.

### 5. Illustrative Numerical Examples

In this section, we present illustrative examples to demonstrate the utility of our results in the characterization of communication failures on multi-hop networks. We also investigate the effects of those failures on the stability of a multi-hop networked control system.

Consider the networked control system (1), (2) with

(39) 
$$A = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}, \quad K = \begin{bmatrix} -2.9012 & -0.9411 \end{bmatrix}$$

together with the networks G and  $\tilde{G}$  in Fig. 1. This system was explored previously in [33] with a single channel network model. Here, differently from [33], we consider networks G and  $\tilde{G}$  that incorporate multiple paths and multiple links for packet transmissions. In what follows, we investigate various scenarios where we demonstrate the utility of our results in Sections 3 and 4 for characterizing overall network failures. For each different scenario, our goal is to find out the level of transmission failures that can be tolerated on the communication links so that the stability of the system (1), (2) is guaranteed.

5.1. Data corruption/packet dropout issues on both networks G and  $\tilde{G}$ . Consider the scenario where all links on both networks G and  $\tilde{G}$  are subject to malicious/nonmalicious data corruption or packet dropout issues. We explore the general situation where the failures may depend on each other. For this general setup, we can use Proposition 3.2 and Theorem 4.8 for the characterization of the overall transmission failures between the plant and the controller. As explained in Remark 4.9, the overall packet exchange failure process  $l(\cdot)$  satisfies Assumption 2.2 with  $\rho = \rho_G + \rho_{\tilde{G}}$ . Here,  $\rho_G$  and  $\rho_{\tilde{G}}$  are asymptotic failure ratios for the networks G and  $\tilde{G}$ , that is,  $l_G \in \Lambda_{\rho_G}$ ,  $l_{\tilde{G}} \in \Lambda_{\rho_{\tilde{G}}}$ .

To find the values of  $\rho_G$  and  $\rho_{\tilde{G}}$ , we can use Proposition 3.2 and Theorem 4.8. In particular, by Proposition 3.2 and Theorem 4.8, we obtain  $\rho_G = \min_{i \in \{1,...,c=3\}} \rho_{\mathcal{P}_i}$ where  $\rho_{\mathcal{P}_i} = \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}$ . Similarly, we have  $\rho_{\tilde{G}} = \min_{i \in \{1,...,\tilde{c}\}} \rho_{\tilde{\mathcal{P}}_i}$  and  $\rho_{\tilde{\mathcal{P}}_i} = \sum_{j=1}^{|\tilde{\mathcal{P}}_i|} \rho_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,j}}$ , where  $\tilde{c} = 4$  is the number of paths (denoted by  $\tilde{\mathcal{P}}_i$ ) from the controller node  $\tilde{v}_{\rm C}$  to the plant node  $\tilde{v}_{\rm P}$  on the network  $\tilde{G}$ . Consequently, Assumption 2.2 holds with

$$\rho = \min_{i \in \{1,...,c\}} \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} + \min_{i \in \{1,...,\tilde{c}\}} \sum_{j=1}^{|\tilde{\mathcal{P}}_i|} \rho_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,j}}$$

The effect of the asymptotic packet failure ratios  $\rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}}$ ,  $\rho_{\bar{\mathcal{P}}_i}^{\bar{\mathcal{P}}_{i,j}}$  on the stability of the networked system can be analyzed by using Theorem 2.3. First, we identify the values of asymptotic packet exchange failure ratio  $\rho$  in Assumption 2.2, for which the stability conditions (3)–(5) hold. For this numerical example, there exist a positive-definite matrix P and scalars  $\beta \in (0, 1), \varphi \in [1, \infty)$  that satisfy (3)–(5), when  $\rho$  is less than 0.411. Hence, Theorem 2.3 guarantees that the zero solution of the closed-loop system (1), (2) is asymptotically stable almost surely if the asymptotic packet transmission failure ratios satisfy

(40) 
$$\min_{i \in \{1,...,c\}} \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} + \min_{i \in \{1,...,\tilde{c}\}} \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,j}} \le 0.411.$$

The system operator can guarantee (40) by ensuring that at least one path in each network is sufficiently secure and reliable. In particular, if there exist a path  $\mathcal{P}_{i^*}$  in network G and a path  $\tilde{\mathcal{P}}_{i^*}$  in network  $\tilde{G}$  such that  $\sum_{j=1}^{|\mathcal{P}_{i^*}|} \rho_{\mathcal{P}_{i^*}}^{\mathcal{P}_{i^*,j}} +$  $\sum_{j=1}^{|\tilde{\mathcal{P}}_{i^*}|} \rho_{\tilde{\mathcal{P}}_{i^*}}^{\tilde{\mathcal{P}}_{i^*,j}} \leq 0.411$ , then (40) holds and the stability is guaranteed regardless of the security/reliability of all other paths. Another approach to guarantee (40) is to ensure a certain level of security/reliability for all links. For this example, if all links are sufficiently secure and reliable so that  $\rho_{\mathcal{P}_{i}}^{\mathcal{P}_{i,j}}, \rho_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,j}} \leq \bar{\rho} \triangleq \frac{0.411}{6}$ , then (40) holds and the stability is guaranteed. To see this, first note that the network G contains 3 paths, and the number of links on these paths are given by 2,3,3.

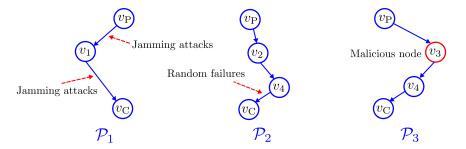
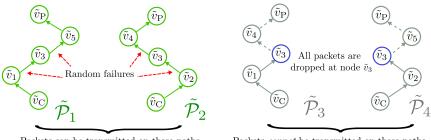


FIGURE 3. Paths from the plant node  $v_{\rm P}$  to the controller node  $v_{\rm C}$  in network G



Packets can be transmitted on these paths

Packets *cannot* be transmitted on these paths

FIGURE 4. Paths from the controller node  $\tilde{v}_{\rm C}$  to the plant node  $\tilde{v}_{\rm P}$  in network  $\tilde{G}$ 

Furthermore, the network  $\bar{G}$  contains 4 paths, each of which contains 4 links. It follows that

$$\min_{i \in \{1,...,c\}} \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\mathcal{P}_i}^{\mathcal{P}_{i,j}} + \min_{i \in \{1,...,\bar{c}\}} \sum_{j=1}^{|\mathcal{P}_i|} \rho_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,j}} \le \min_{i \in \{1,...,c\}} \sum_{j=1}^{|\mathcal{P}_i|} \overline{\rho} + \min_{i \in \{1,...,\bar{c}\}} \sum_{j=1}^{|\mathcal{P}_i|} \overline{\rho} \\
= \min\{2\overline{\rho}, 3\overline{\rho}, 3\overline{\rho}\} + \min\{4\overline{\rho}, 4\overline{\rho}, 4\overline{\rho}, 4\overline{\rho}, 4\overline{\rho}\} = 6\overline{\rho} \le 0.411,$$

which implies that (40) holds, and hence stability is guaranteed.

Notice that in this example, we have not made any particular assumption on the independence or randomness of the failures on the links. In fact, all links may be subject to failures caused by actions of coordinated adversaries. In such a case, the occurrence of the failures may be nonrandom, and moreover, the binary-valued processes that characterize failures on different links would depend on each other. For example, in the case of data-corruption attacks, the worst-case scenario would be that the failures on the paths are synchronized so that packet transmissions necessarily fail in all parallel paths (such as paths  $\mathcal{P}_1$ ,  $\mathcal{P}_2$ , and  $\mathcal{P}_3$  of the graph G) at the same time. Notice that the condition  $\rho_{\mathcal{P}_i,j}^{\mathcal{P}_{i,j}}, \rho_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,j}} \leq \bar{\rho}$  guarantees that such failures happen sufficiently rarely in average in the long run. Thus, networked stabilization can be achieved through the successful exchanges of measurement and control data.

In the following examples, we will illustrate how our results in Sections 3 and 4 can be used for scenarios where some information on the properties of the communication links are available.

5.2. Jamming Attacks and Random Transmission Failures on Multiple Links. Consider the network G and its paths shown in Fig. 3. We assume that the paths  $\mathcal{P}_1$  and  $\mathcal{P}_3$  are subject to malicious attacks. In particular, the node  $v_3$ on  $\mathcal{P}_3$  is assumed to be controlled by a malicious agent and all packets arriving at node  $v_3$  are dropped. Hence, we have  $l_{\mathcal{P}_3}^{\mathcal{P}_{3,2}}(t) = 1, t \in \mathbb{N}_0$ , which implies that  $l_{\mathcal{P}_3}(t) = 1, t \in \mathbb{N}_0$ . Moreover, the first and the second links on path  $\mathcal{P}_1$  are assumed to be subject to jamming attacks that cause data corruption. Here the worst-case scenario happens when the attackers coordinate and jam only one of the links at once. This would maximize the number of packet losses within the total energy constraint of the attackers. Our results take this worst-case into account.

In particular, suppose that the attacked links  $\mathcal{P}_{1,1}$  and  $\mathcal{P}_{1,2}$  satisfy  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}} \in \Pi_{\kappa,w}$ ,  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,2}} \in \Pi_{\kappa,w}$  with  $\kappa \geq 0$  and  $w \in (0,1)$ . Here, w provides a bound on average failures on each link and it is related to the energy available to each attacker. The fact that  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}} \in \Pi_{\kappa,w}$ ,  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,2}} \in \Pi_{\kappa,w}$  implies  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}} \in \Lambda_{\rho_{\mathcal{P}_1}}^{\mathcal{P}_{1,1}}$ ,  $l_{\mathcal{P}_1}^{\mathcal{P}_{1,2}} \in \Lambda_{\rho_{\mathcal{P}_1}}^{\mathcal{P}_{1,2}}$  for all  $\rho_{\mathcal{P}_1}^{\mathcal{P}_{1,1}}, \rho_{\mathcal{P}_1}^{\mathcal{P}_{1,1}} \in (w, 1]$ . It then follows from Proposition 4.1 that  $\rho_{\mathcal{P}_1} = \rho_{\mathcal{P}_1}^{\mathcal{P}_{1,1}} + \rho_{\mathcal{P}_1}^{\mathcal{P}_{1,1}}$ , and hence  $l_{\mathcal{P}_1} \in \Lambda_{\rho_{\mathcal{P}_1}}$  for  $\rho_{\mathcal{P}_1} \in (2w, 1]$ . Notice that  $l_{\mathcal{P}_1} \in \Lambda_{\rho_{\mathcal{P}_1}}$  holds even in the worst-case scenario mentioned above. Here  $\rho_{\mathcal{P}_1} \in (2w, 1]$  provides an upper-bound on the average number of overall failures on path  $\mathcal{P}_1$  in the worst-case scenario.

We further assume that the link  $(v_4, v_{\rm C})$  on path  $\mathcal{P}_2$  is subject to random data corruption and the associated failure indicator process  $\{l_{\mathcal{P}_2}^{\mathcal{P}_{2,3}}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$  satisfies  $l_{\mathcal{P}_2}^{\mathcal{P}_{2,3}} \in \Gamma_{p_0,p_1}$  with  $p_0, p_1 \in (0,1)$ . We consider ideal communication on the other links on path  $\mathcal{P}_2$ . Thus, by using (25), we obtain

$$l_{\mathcal{P}_2}(t) = l_{\mathcal{P}_2}^{\mathcal{P}_{2,1}}(t) \vee l_{\mathcal{P}_2}^{\mathcal{P}_{2,2}}(t) \vee l_{\mathcal{P}_2}^{\mathcal{P}_{2,3}}(t) = 0 \vee 0 \vee l_{\mathcal{P}_2}^{\mathcal{P}_{2,3}}(t) = l_{\mathcal{P}_2}^{\mathcal{P}_{2,3}}(t), \quad t \in \mathbb{N}_0,$$

and hence we also have  $l_{\mathcal{P}_2} \in \Gamma_{p_0,p_1}$ .

To characterize overall failures on network G, we use (14) and obtain

$$l_{G}(t) = l_{\mathcal{P}_{1}}(t) \wedge l_{\mathcal{P}_{2}}(t) \wedge l_{\mathcal{P}_{3}}(t) = l_{\mathcal{P}_{1}}(t) \wedge l_{\mathcal{P}_{2}}(t) \wedge 1 = l_{\mathcal{P}_{1}}(t) \wedge l_{\mathcal{P}_{2}}(t), \quad t \in \mathbb{N}_{0}.$$

Now, assuming that the failures on paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are mutually-independent, it follows from Theorem 3.4 with  $l^{(1)}(t) = l_{\mathcal{P}_2}(t)$ ,  $l^{(2)}(t) = l_{\mathcal{P}_1}(t)$ ,  $p_0^{(1)} = p_0$ ,  $p_1^{(1)} = p_1$ , and  $\rho^{(2)} = \rho_{\mathcal{P}_1}$  that if  $p_1\rho_{\mathcal{P}_1} < 1$ , then  $l_G \in \Lambda_{\rho_G}$  with  $\rho_G \in (p_1\rho_{\mathcal{P}_1}, 1]$ . Noting that  $\rho_{\mathcal{P}_1} \in (2w, 1]$ , we see that if  $2p_1w < 1$ , then  $l_G \in \Lambda_{\rho_G}$  for all  $\rho_G \in (2p_1w, 1]$ .

Next, consider the network  $\hat{G}$ . Suppose that  $\hat{G}$  is secure against attacks, but it is unreliable and subject to random transmission failures. To characterize the overall transmission failures on the network  $\tilde{G}$ , we utilize Theorems 3.3 and 4.2. Before we apply these results, we first describe the routing scheme on this network. Specifically, in this network  $\tilde{v}_3$  is assumed to be a router that forwards all incoming packets from node  $\tilde{v}_1$  only to node  $\tilde{v}_5$ , and all incoming packets from node  $\tilde{v}_2$  only to node  $\tilde{v}_4$ . Among the paths shown in Fig. 4, packets may be transmitted on  $\tilde{\mathcal{P}}_1$ and  $\tilde{\mathcal{P}}_2$ , but are never transmitted on  $\tilde{\mathcal{P}}_3$  and  $\tilde{\mathcal{P}}_4$  due to the routing scheme. Hence  $l_{\tilde{\mathcal{P}}_3}(t) = 1, l_{\tilde{\mathcal{P}}_4}(t) = 1, t \in \mathbb{N}_0.$ 

We assume that the links  $(\tilde{v}_1, \tilde{v}_3)$  and  $(\tilde{v}_3, \tilde{v}_5)$  on path  $\hat{\mathcal{P}}_1$  and the links  $(\tilde{v}_2, \tilde{v}_3)$ and  $(\tilde{v}_3, \tilde{v}_4)$  on  $\tilde{\mathcal{P}}_2$  face random data corruption issues. Furthermore, the failure indicator processes  $\{l_{\tilde{\mathcal{P}}_1}^{(\tilde{v}_1, \tilde{v}_3)}(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}, \{l_{\tilde{\mathcal{P}}_1}^{(\tilde{v}_3, \tilde{v}_5)}(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}, \{l_{\tilde{\mathcal{P}}_2}^{(\tilde{v}_2, \tilde{v}_3)}(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}, \{l_{\tilde{\mathcal{P}}_2}^{(\tilde{v}_3, \tilde{v}_4)}(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$  are assumed to be mutually independent processes that belong to the hidden Markov model class  $\Gamma_{p_0, p_1}$  with  $p_0, p_1 \in (0, 1)$  (see

Definition 2.4). The links that are connected directly to the plant and the controller nodes ( $\tilde{v}_{\rm P}$  and  $\tilde{v}_{\rm C}$ ) are considered to be ideal communication links. In other words,  $l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,1}}(t) = 0, \, l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,4}}(t) = 0, \, t \in \mathbb{N}_{0}, \, i \in \{1,2\}.$ 

Now, observe that the failure indicators for paths  $\tilde{\mathcal{P}}_1$  and  $\tilde{\mathcal{P}}_2$  satisfy

$$\begin{split} l_{\tilde{\mathcal{P}}_{i}}(t) &= l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,1}}(t) \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,2}}(t) \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,3}}(t) \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,4}}(t) = 0 \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,2}}(t) \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,3}}(t) \vee 0 \\ &= l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,2}}(t) \vee l_{\tilde{\mathcal{P}}_{i}}^{\tilde{\mathcal{P}}_{i,3}}(t), \quad t \in \mathbb{N}_{0}, \ i \in \{1,2\}. \end{split}$$

By applying Theorem 4.2 with  $l^{(1)}(t) = l_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,2}}(t), \ l^{(2)}(t) = l_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,3}}(t), \ p_0^{(1)} = p_0^{(2)} = p_0,$ and  $p_1^{(1)} = p_1^{(2)} = p_1$ , we obtain  $l_{\tilde{\mathcal{P}}_i} \in \Gamma_{\tilde{p}_0,\tilde{p}_1}$ ,  $i \in \{1,2\}$ , where  $\tilde{p}_0 = p_0^2$  and  $\tilde{p}_1 = \min\{p_1 + p_0p_1, p_1 + p_0p_1, 1\} = \min\{p_1 + p_0p_1, 1\}$ . Next, since  $l_{\tilde{P}_3}(t) = 1$ ,  $l_{\tilde{P}_4}(t) = 1$ ,  $t \in \mathbb{N}_0$ , we have

$$l_{\tilde{G}}(t) = l_{\tilde{\mathcal{P}}_{1}}(t) \wedge l_{\tilde{\mathcal{P}}_{2}}(t) \wedge l_{\tilde{\mathcal{P}}_{3}}(t) \wedge l_{\tilde{\mathcal{P}}_{4}}(t) = l_{\tilde{\mathcal{P}}_{1}}(t) \wedge l_{\tilde{\mathcal{P}}_{2}}(t) \wedge 1 \wedge 1 = l_{\tilde{\mathcal{P}}_{1}}(t) \wedge l_{\tilde{\mathcal{P}}_{2}}(t),$$

for  $t \in \mathbb{N}_0$ . It then follows from Theorem 3.3 with  $l^{(1)}(t) = l_{\tilde{\mathcal{P}}_1}(t), \ l^{(2)}(t) = l_{\tilde{\mathcal{P}}_2}(t), \ p_0^{(1)} = p_0^{(2)} = p_0^2$ , and  $p_1^{(1)} = p_1^{(2)} = \min\{p_1 + p_0p_1, 1\}$ , that we have  $l_{\tilde{G}} \in \Gamma_{p_0^{\tilde{G}}, p_1^{\tilde{G}}}$ with

$$\begin{split} p_0^{\tilde{G}} &= \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, p_0^{(2)} + p_1^{(2)}p_0^{(1)}, 1\} = \min\{p_0^{(1)} + p_1^{(1)}p_0^{(2)}, 1\} \\ &= \min\{p_0^2 + \min\{p_1 + p_0p_1, 1\}p_0^2, 1\} = \min\{p_0^2 + \min\{p_0^2p_1 + p_0^3p_1, p_0^2\}, 1\} \\ &= \min\{\min\{p_0^2 + p_0^2p_1 + p_0^3p_1, 2p_0^2\}, 1\} = \min\{p_0^2 + p_0^2p_1 + p_0^3p_1, 2p_0^2, 1\} \end{split}$$

and  $p_1^{\tilde{G}} = p_1^{(1)} p_1^{(2)} = (\min\{p_1 + p_0 p_1, 1\})^2 = \min\{(p_1 + p_0 p_1)^2, 1\}$ . Finally, in the case where  $(p_1 + p_0 p_1)^2 < 1$ , as a consequence of Proposition 2.5, we obtain  $l_{\tilde{G}} \in \Lambda_{\tilde{\rho}_{\tilde{G}}}$ for  $\rho_{\tilde{G}} \in (p_1^{\tilde{G}}, 1] = ((p_1 + p_0 p_1)^2, 1].$ 

Now, we note once again that the overall packet exchange failure indicator  $\{l(t) \in$  $\{0,1\}\}_{t\in\mathbb{N}_0}$  satisfies  $l\in\Lambda_{\rho}$  with  $\rho=\rho_G+\rho_{\tilde{G}}$ . Since  $\rho_G\in(2p_1w,1]$  and  $\rho_{\tilde{G}}\in((p_1+p_0p_1)^2,1]$ , it follows that if  $2p_1w+(p_1+p_0p_1)^2<1$ , then  $l\in\Lambda_{\rho}$  with  $\rho \in (2p_1w + (p_1 + p_0p_1)^2, 1]$ . As we discussed in Section 5.1, stability of the networked control system can be ensured when  $\rho \leq 0.411$ . It follows that if

(41) 
$$2p_1w + (p_1 + p_0p_1)^2 < 0.411,$$

then  $l \in \Lambda_{\rho}$  holds with  $\rho = 0.411$ , implying almost sure asymptotic stability of the networked control system. Observe that by utilizing the results in Sections 3 and 4, we are able to derive the sufficient stability condition (41) in terms of the attack rate  $w \in (0,1)$  associated with the links on path  $\mathcal{P}_1$  that are attacked by jamming attackers as well as random failure parameters  $p_0, p_1 \in (0, 1)$ .

5.3. State-dependent Attacks by a Malicious Node. In the scenarios discussed in Sections 5.1 and 5.2, the strategies of the attackers are not specified. However, in certain cases an attacker may have access to the state or control input information and be able to directly cause transmission failures between the plant and the controller. In such scenarios, the goal of the attacker might be to increase state norm with small amount of attacks. In this section, our goal is to illustrate such an attack strategy. In particular, we consider the case where the plant node  $v_{\rm P}$  is compromised by an attacker. The attacker is assumed to have access to the state information.

#### 32 CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL

We consider an attack strategy that is based on an optimization problem. In particular, the attacker at node  $v_{\rm P}$  decides whether to transmit the state information on links  $\mathcal{P}_{1,1} = (v_{\rm P}, v_1)$ ,  $\mathcal{P}_{2,1} = (v_{\rm P}, v_2)$ , and  $\mathcal{P}_{3,1} = (v_{\rm P}, v_3)$  after solving an optimization problem for maximizing the norm of the state at a future time. In particular, we consider the attack strategy (10), (11) discussed in Example 2.8, where we represent the attackers actions with a binary-valued process  $\{l_A(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ . In the case where  $l_A(t) = 0$ , the attacker transmits the state packet on  $\mathcal{P}_{1,1} = (v_{\rm P}, v_1)$ ,  $\mathcal{P}_{2,1} = (v_{\rm P}, v_2)$ , and  $\mathcal{P}_{3,1} = (v_{\rm P}, v_3)$ ; moreover,  $l_A(t) = 1$  indicates no transmission.

Observe that in this scenario, we have

$$l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}}(t) = l_{\mathcal{P}_2}^{\mathcal{P}_{2,1}}(t) = l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}}(t) = l_{\mathcal{A}}(t), \quad t \in \mathbb{N}_0$$

Since  $l_{A} \in \Pi_{\kappa,w}$ , we have  $l_{A} \in \Lambda_{\rho_{A}}$  with  $\rho_{A} \in (w, 1]$ . As a consequence,  $l_{\mathcal{P}_{i}}^{\mathcal{P}_{i,1}} \in \Lambda_{\rho_{\mathcal{P}_{i}}}^{\mathcal{P}_{i,1}}$  with  $\rho_{\mathcal{P}_{i}}^{\mathcal{P}_{i,1}} \in (w_{A}, 1]$  for all  $i \in \{1, 2, 3\}$ . Notice that  $l_{A}(t) = 1$  implies l(t) = 1, since the attacker can completely prevent the packet exchange between the plant and the controller. Observe also that if there are other sources of transmission failures on the network, then there may be times when l(t) = 1 even if  $l_{A}(t) = 0$ . As a result, if  $l_{A}(t) = 0$  then the attacker may not be able to correctly predict the state x(t+1) at time t, as there may or may not be a failure in the network that prevents control input to reach the plant. However, as the optimization problem in (11) is solved at each time step, the updated state information is used for decision.

Notice that in the case where all links other than  $\mathcal{P}_{1,1} = (v_{\mathrm{P}}, v_1), \mathcal{P}_{2,1} = (v_{\mathrm{P}}, v_2),$ and  $\mathcal{P}_{3,1} = (v_{\mathrm{P}}, v_3)$  are secure and reliable, we have  $l_G(t) = l_{\mathrm{A}}(t)$ , since

$$l_G(t) = l_{\mathcal{P}_1}(t) \wedge l_{\mathcal{P}_2}(t) \wedge l_{\mathcal{P}_3}(t) = l_{\mathcal{P}_1}^{\mathcal{P}_{1,1}}(t) \wedge l_{\mathcal{P}_2}^{\mathcal{P}_{2,1}}(t) \wedge l_{\mathcal{P}_3}^{\mathcal{P}_{3,1}}(t) = l_{\mathcal{A}}(t) \wedge l_{\mathcal{A}}(t) \wedge l_{\mathcal{A}}(t) = l_{\mathcal{A}}(t), \quad t \in \mathbb{N}_0.$$

Hence,  $l_G \in \Lambda_{\rho_G}$  with  $\rho_G \in (w, 1]$ .

Now suppose that the network  $\tilde{G}$  also faces failures. In particular, consider the setup in Section 5.2, where  $l_{\tilde{G}} \in \Lambda_{\tilde{\rho}_G}$  for  $\rho_{\tilde{G}} \in (p_1^{\tilde{G}}, 1] = ((p_1 + p_0 p_1)^2, 1]$ . Since  $l \in \Lambda_{\rho}$  with  $\rho = \rho_G + \rho_{\tilde{G}}$ , we have  $l \in \Lambda_{\rho}$  for all  $\rho \in (w + (p_1 + p_0 p_1)^2, 1]$ . Noting that the stability of the networked control system can be ensured when  $\rho \leq 0.411$ , we can impose a sufficient condition on the attack rate w. Specifically, for the scenario of this section, if

(42) 
$$w + (p_1 + p_0 p_1)^2 < 0.411,$$

then  $l \in \Lambda_{\rho}$  holds with  $\rho = 0.411$ , and thus the networked control system is almost surely asymptotically stable.

To illustrate the effect of different parameters in the attack strategy (11), we conduct simulations. First, we generate 50 sample trajectories of the process  $l_{\tilde{G}}$  by setting the failure processes  $l_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,2}}, l_{\tilde{\mathcal{P}}_i}^{\tilde{\mathcal{P}}_{i,3}}, i \in \{1, 2\}$ , as outputs of time-inhomogeneous hidden Markov models such that

(43) 
$$l_{\tilde{\mathcal{P}}_{i,j}}^{\tilde{\mathcal{P}}_{i,j}}(t) = \theta_{i,j}(t), \quad t \in \mathbb{N}_0, \quad i \in \{1,2\}, \quad j \in \{2,3\},$$

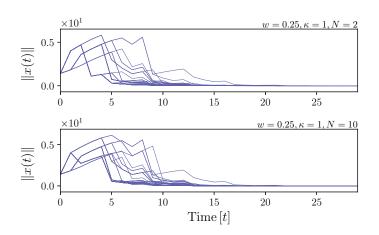


FIGURE 5. Comparison of state trajectories for different values of N in (11)

where  $\{\theta_{i,j}(t) \in \Theta^l = \{0,1\}\}_{t \in \mathbb{N}_0}$  is an  $\mathcal{F}_t$ -adapted finite-state time-inhomogeneous Markov chain with initial distributions and time-varying transition probability functions satisfying  $\mathbb{P}[\theta_{i,j}(0) = 0] = 0.6$  and

$$\mathbb{P}[\theta_{i,j}(t+1) = 1 | \theta_{i,j}(t) = 0] = 0.2 + 0.02 \cos^2(0.1t),$$
  
$$\mathbb{P}[\theta_{i,j}(t+1) = 1 | \theta_{i,j}(t) = 1] = 0.2 + 0.02 \sin^2(0.2t),$$

for  $t \in \mathbb{N}_0$ ,  $i \in \{1, 2\}$ ,  $j \in \{2, 3\}$ . Notice that  $l_{\tilde{p}_i}^{\tilde{p}_{i,2}}$ ,  $l_{\tilde{p}_i}^{\tilde{p}_{i,3}} \in \Gamma_{p_0,p_1}$ ,  $i \in \{1, 2\}$ , with  $p_0 = 0.8$ ,  $p_1 = 0.22$ . Next, for each sample trajectory of  $l_{\tilde{G}}$ , we simulate the networked control system (1), (2) under the attack strategy (11). In Fig. 5, we show state trajectories when attack parameters are selected as w = 0.25 and  $\kappa = 1$ . In top part of Fig. 5, the horizon parameter N = 2, and in the bottom part N = 10. Observe that with larger horizon variable, the attacker following the strategy (11) can utilize the resources more efficiently so that in certain sample state trajectories the state norm is set to larger values for longer durations even though in both cases N = 2 and N = 10, the attack belongs to the same class  $\Pi_{\kappa,w}$ . Notice that with  $p_0 = 0.8$ ,  $p_1 = 0.22$ , and w = 0.25, the inequality (42) holds. Therefore, the zero solution of the closed-loop system is almost surely asymptotically stable by Theorem 2.3.

For w = 0.25,  $\kappa = 1$ , and N = 2, we show in Fig. 6 the average number of failures on the networks  $\tilde{G}$  and G as well as the average number of total packet exchange failures between the plant and the controller. These averages are upper bounded in the long run by certain scalars. In particular, for a process l that satisfies  $l \in \Lambda_{\rho}$  with  $\rho \in (\underline{\rho}, 1]$ , we have  $\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq \underline{\rho}$  (see Lemma 3.3 in [33]). As a result, we have  $\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_{\tilde{G}}(t) \leq (p_1 + p_0 p_1)^2$ ,  $\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_G(t) \leq w$ , and  $\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq w + (p_1 + p_0 p_1)^2$ . Next, we consider the case where  $\kappa = 10$ , which corresponds to the case where

Next, we consider the case where  $\kappa = 10$ , which corresponds to the case where the attacker has more initial resources. In this case, we consider two scenarios w = 0.25 and w = 0.75, for which we obtain the sample state trajectories shown in Fig. 7. First, notice that in the case with  $\kappa = 10, w = 0.25$  (top plot in Fig. 7),

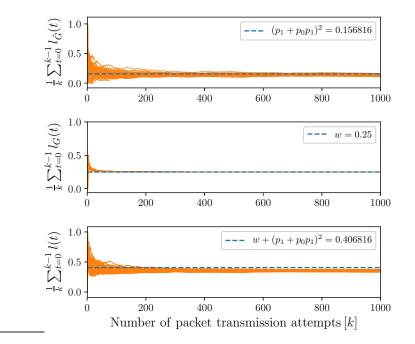


FIGURE 6. Average number of failures on networks  $\tilde{G}$  and G together with average number of overall packet exchange failures

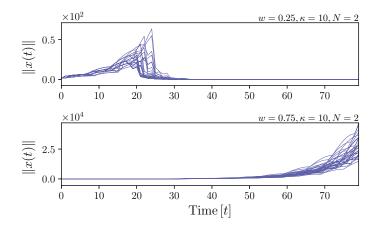


FIGURE 7. Comparison of state trajectories for different values of w in (10)

the state grows to larger values for longer durations in comparison to the case with  $\kappa = 1, w = 0.25$  (top plot in Fig. 5). In both cases, the stability conditions hold with w = 0.25, and therefore the state eventually converges to zero as guaranteed by Theorem 2.3. Note that, when w is set to larger values, the attacker can attack at a higher rate. For this example, we set w = 0.75 and observe that for such a high attack rate all sample state trajectories diverge (see bottom plot in Fig. 7).

# 6. CONCLUSION

In this paper, we explored state feedback control of a linear plant over an unreliable network that may also face malicious attacks. We developed a probabilistic approach to characterize the failures on the network in terms of the failures on different paths between the plant and the controller. We showed that the failures on each path can be described as a combination of data-corruption and packetdropout failures on the communication links of that particular path. We obtained sufficient conditions for almost sure asymptotic stability of the overall networked control system, which allow us to check stability by using a probabilistic upperbound obtained for the average number of packet exchange failures between the plant and the controller.

Our framework can take into account mutual independence/dependence relationships between the failures on different links and paths of a network, and as a result, it can provide relatively tight upper bounds for the long run average number of overall transmission failures on a network. This is achieved by utilizing upper bounds on the tail probabilities of sums involving a binary-valued process from a general class together with a binary-valued output process associated with a time-inhomogeneous hidden Markov model.

# Appendix A. Proof of Lemma 2.6

In this section, we prove Lemma 2.6. As in the proof of Lemma A.1 of [33], for proving Lemma 2.6, we use Markov's inequality and follow the approaches used for obtaining Chernoff-type tail distribution inequalities for sums of random variables (see Appendix B of [48] and Section 1.9 of [43]). In the proof Lemma 2.6, the following result also plays a key role.

**Lemma A.1.** Let  $\{\xi(t) \in \Xi\}_{t \in \mathbb{N}_0}$  be a finite-state time-inhomogeneous Markov chain with transition probabilities  $p_{q,r} \colon \mathbb{N}_0 \to [0,1]$ ,  $q,r \in \Xi$ . Furthermore, let  $\Xi_1 \subset \Xi$  be given by  $\Xi_1 \triangleq \{r \in \Xi \colon h(r) = 1\}$ , where  $h \colon \Xi \to \{0,1\}$  is a binary-valued function. Then for all  $\phi > 1$ ,  $s \in \mathbb{N}$ , and  $\tilde{p} \in [0,1]$  such that

(44) 
$$\sum_{r \in \Xi_1} p_{q,r}(t) \le \tilde{p}, \quad q \in \Xi, \quad t \in \mathbb{N}_0,$$

we have

(45) 
$$\mathbb{E}[\phi^{\sum_{j=1}^{s} h(\xi(t_j))}] \le \phi \left((\phi - 1)\tilde{p} + 1\right)^{s-1},$$

where  $t_1, t_2, \ldots, t_s \in \mathbb{N}_0$  denote indices such that  $0 \leq t_1 < t_2 < \ldots < t_s$ .

*Proof.* We use induction to show (45). First, we consider the case where s = 1. In this case, we have

(46) 
$$\mathbb{E}[\phi^{\sum_{j=1}^{s} h(\xi(t_j))}] = \mathbb{E}[\phi^{h(\xi(t_1))}] \le \phi.$$

Next, consider the case where s = 2. Observe that  $t_1 \leq t_2 - 1$ , thus the random variable  $\xi(t_1)$  is  $\mathcal{F}_{t_2-1}$ -measurable. Consequently,

(47) 
$$\mathbb{E}[\phi^{\sum_{j=1}^{s}h(\xi(t_{j}))}] = \mathbb{E}[\phi^{h(\xi(t_{1}))}\phi^{h(\xi(t_{2}))}] = \mathbb{E}[\mathbb{E}[\phi^{h(\xi(t_{1}))}\phi^{h(\xi(t_{2}))} | \mathcal{F}_{t_{2}-1}]] = \mathbb{E}[\phi^{h(\xi(t_{1}))}\mathbb{E}[\phi^{h(\xi(t_{2}))} | \mathcal{F}_{t_{2}-1}]],$$

35

where the last equality follows from the fact that  $\phi^{h(\xi(t_1))}$  is a measurable function of  $\xi(t_1)$ , and hence, it is  $\mathcal{F}_{t_2-1}$ -measurable. Now, since  $\{\xi(t) \in \Xi\}_{t \in \mathbb{N}_0}$  is a Markov chain, we have  $\mathbb{E}[\phi^{h(\xi(t_2))} | \mathcal{F}_{t_2-1}] = \mathbb{E}[\phi^{h(\xi(t_2))} | \xi(t_2-1)]$ . Therefore,

$$\mathbb{E}[\phi^{\sum_{j=1}^{s}h(\xi(t_{j}))}] = \mathbb{E}[\phi^{h(\xi(t_{1}))}\mathbb{E}[\phi^{h(\xi(t_{2}))} | \xi(t_{2}-1)]]$$

$$= \mathbb{E}\Big[\phi^{h(\xi(t_{1}))}\Big(\phi\mathbb{P}[h(\xi(t_{2})) = 1 | \xi(t_{2}-1)] + \mathbb{P}[h(\xi(t_{2})) = 0 | \xi(t_{2}-1)]\Big)\Big]$$

$$= \mathbb{E}\Big[\phi^{h(\xi(t_{1}))}\Big(\phi\mathbb{P}[h(\xi(t_{2})) = 1 | \xi(t_{2}-1)] + 1 - \mathbb{P}[h(\xi(t_{2})) = 1 | \xi(t_{2}-1)]\Big)\Big]$$
(48)

$$= \mathbb{E}\Big[\phi^{h(\xi(t_1))}\Big((\phi-1)\mathbb{P}[h(\xi(t_2))=1 \mid \xi(t_2-1)]+1\Big)\Big].$$

Here we obtain  $\mathbb{P}[h(\xi(t_2)) = 1 | \xi(t_2-1)] = \mathbb{P}[\xi(t_2) \in \Xi_1 | \xi(t_2-1)] = \sum_{r \in \Xi_1} \mathbb{P}[\xi(t_2) = r | \xi(t_2-1)]$ . Thus, by using (44) and (46), we arrive at

$$\mathbb{E}[\phi^{\sum_{j=1}^{s}h(\xi(t_j))}]\Big)\Big] \leq \mathbb{E}\Big[\phi^{h(\xi(t_1))}\Big((\phi-1)\tilde{p}+1\Big)\Big] = \mathbb{E}[\phi^{h(\xi(t_1))}]((\phi-1)\tilde{p}+1)$$

$$(49) \qquad \leq \phi((\phi-1)\tilde{p}+1).$$

Hence, we have that (45) is satisfied for  $s \in \{1, 2\}$ .

Now, assume that (45) holds for  $s = \tilde{s} > 2$ , i.e.,

(50) 
$$\mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}}h(\xi(t_j))}] \le \phi\left((\phi-1)\tilde{p}+1\right)^{\tilde{s}-1}$$

We will to prove that (45) holds for  $s = \tilde{s} + 1$ . By employing arguments similar to those used for obtaining (47)–(49), we get

$$\mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}+1}h(\xi(t_j))}] = \mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}}h(\xi(t_j))}\phi^{h(\xi(t_{\tilde{s}+1}))}]$$
  
$$= \mathbb{E}[\mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}}h(\xi(t_j))}\phi^{h(\xi(t_{\tilde{s}+1}))} \mid \mathcal{F}_{t_{\tilde{s}+1}-1}]]$$
  
$$= \mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}}h(\xi(t_j))}\mathbb{E}[\phi^{h(\xi(t_{\tilde{s}+1}))} \mid \mathcal{F}_{t_{\tilde{s}+1}-1}]]$$
  
(51)

$$= \mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}} h(\xi(t_j))} \mathbb{E}[\phi^{h(\xi(t_{\tilde{s}+1}))} | \xi(t_{\tilde{s}+1}-1)]] \le \mathbb{E}[\phi^{\sum_{j=1}^{\tilde{s}} h(\xi(t_j))}]((\phi-1)\tilde{p}+1).$$
  
Finally, by (50) and (51), we obtain (45) with  $s = \tilde{s} + 1.$ 

Next, by utilizing Lemma A.1, we prove Lemma 2.6.

Proof of Lemma 2.6. First, we define

$$\overline{h}(k) \triangleq [h(\xi(0)), h(\xi(1)), \dots, h(\xi(k-1))]^{\mathrm{T}},$$
  
$$\overline{\chi}(k) \triangleq [\chi(0), \chi(1), \dots, \chi(k-1)]^{\mathrm{T}}, \quad k \in \mathbb{N}.$$

Next, let  $F_{s,k} \triangleq \{\overline{\chi} \in \{0,1\}^k : \overline{\chi}^T \overline{\chi} = s\}, s \in \{0,1,\ldots,k\}, k \in \mathbb{N}$ . Here, we have  $F_{s_1,k} \cap F_{s_2,k} = \emptyset, s_1 \neq s_2$ , and moreover,  $\mathbb{P}[\overline{\chi}(k) \in \bigcup_{s=0}^k F_{s,k}] = 1, k \in \mathbb{N}$ . By utilizing these definitions we obtain for all  $\rho \in (\tilde{p}\tilde{w}, 1)$  and  $k \in \mathbb{N}$ ,

Since  $\xi(\cdot)$  and  $\chi(\cdot)$  are mutually-independent,

(53) 
$$\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi}(k) > \rho k \mid \overline{\chi}(k) = \overline{\chi}] = \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k].$$

Therefore, it follows from (52) and (53) that for  $k \in \mathbb{N}$ ,

$$\mathbb{P}[\sum_{t=0}^{k-1} h(\xi(t))\chi(t) > \rho k] = \sum_{s=0}^{k} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]\mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$
$$= \sum_{s=0}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]\mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$
$$+ \sum_{s=\lfloor \tilde{w}k \rfloor + 1}^{k} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]\mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$
(54)

In what follows, our goal is to find upper bounds for the two summation terms in (54). We start with the second term. Since  $\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \leq 1, \ k \in \mathbb{N}$ , we obtain

(55) 
$$\sum_{s=\lfloor \tilde{w}k \rfloor+1}^{k} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]\mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$
$$\leq \sum_{s=\lfloor \tilde{w}k \rfloor+1}^{k} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{\chi}(k) = \overline{\chi}] = \mathbb{P}[\sum_{t=0}^{k-1} \chi(t) > \tilde{w}k] = \tilde{\sigma}_{k},$$

for  $k \in \mathbb{N}$ . Next, we obtain an upper bound for the first term in (54). Observe that  $\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] = 0$  for  $\overline{\chi} \in F_{0,k}$ . It then follows that, for all  $k \in \mathbb{N}$  such that  $\lfloor \tilde{w}k \rfloor = 0$ , we have

(56) 
$$\sum_{s=0}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \mathbb{P}[\overline{\chi}(k) = \overline{\chi}] = 0.$$

Moreover, for all  $k \in \mathbb{N}$  such that  $\lfloor \tilde{w}k \rfloor \geq 1$ , we obtain

(57)  

$$\sum_{s=0}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \mathbb{P}[\overline{\chi}(k) = \overline{\chi}] = \sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$

To obtain an upper bound for the term  $\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]$ , we will utilize Markov's inequality and Lemma A.1. First, for  $s \in \{1, 2, \ldots, \lfloor \tilde{w}k \rfloor\}$ , let  $t_1(\overline{\chi}), t_2(\overline{\chi}), \ldots, t_s(\overline{\chi})$  denote the indices of the nonzero entries of  $\overline{\chi} \in F_{s,k}$  such that  $t_1(\overline{\chi}) < t_2(\overline{\chi}) < \cdots < t_s(\overline{\chi})$ . Consequently,

(58) 
$$\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] = \mathbb{P}[\sum_{j=1}^{s} \overline{h}_{t_{j}(\overline{\chi})}(k) > \rho k] = \mathbb{P}[\sum_{j=1}^{s} h(\xi(t_{j}(\overline{\chi}) - 1)) > \rho k],$$

for  $\overline{\chi} \in F_{s,k}$ ,  $s \in \{1, 2, \dots, \lfloor \tilde{w}k \rfloor\}$ , and  $k \in \mathbb{N}$  such that  $\lfloor \tilde{w}k \rfloor \geq 1$ .

#### CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL 38

Now observe that  $\phi > 1$ , since  $\rho \in (\tilde{p}\tilde{w}, \tilde{w})$ . By using Markov's inequality, we obtain

$$\mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \leq \mathbb{P}[\sum_{j=1}^{s} h(\xi(t_{j}(\overline{\chi}) - 1)) \geq \rho k] = \mathbb{P}[\phi^{\sum_{j=1}^{s} h(\xi(t_{j}(\overline{\chi}) - 1))} \geq \phi^{\rho k}]$$

$$\leq \phi^{-\rho k} \mathbb{E}[\phi^{\sum_{j=1}^{s} h(\xi(t_{j}(\overline{\chi}) - 1))}].$$
(59)

It follows from Lemma A.1 that  $\mathbb{E}[\phi^{\sum_{j=1}^{s}h(\xi(t_j(\overline{\chi})-1))}] \leq \phi((\phi-1)\tilde{p}+1)^{s-1}$ . This inequality together with (57) and (59) imply that for all  $k \in \mathbb{N}$  such that  $\lfloor \tilde{w}k \rfloor \geq 1$ ,

$$\sum_{s=0}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k]\mathbb{P}[\overline{\chi}(k) = \overline{\chi}]$$

$$\leq \sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \phi^{-\rho k} \phi \left((\phi - 1)\tilde{p} + 1\right)^{s-1} \mathbb{P}[\overline{\chi}_{k} = \overline{\chi}]$$

$$= \phi^{-\rho k+1} \sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \left((\phi - 1)\tilde{p} + 1\right)^{s-1} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{\chi}_{k} = \overline{\chi}]$$

$$= \phi^{-\rho k+1} \sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \left((\phi - 1)\tilde{p} + 1\right)^{s-1} \mathbb{P}[\overline{\chi}_{k} \in F_{s,k}]$$

$$\leq \phi^{-\rho k+1} \sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \left((\phi - 1)\tilde{p} + 1\right)^{s-1}.$$
(60)

Notice that to obtain the last inequality in (60), we employed the fact that  $\mathbb{P}[\overline{\chi}_k \in$  $[F_{s,k}] \leq 1$ . The summation term on the far right-hand side of (60) satisfies

(61) 
$$\sum_{s=1}^{\lfloor \tilde{w}k \rfloor} \left( (\phi-1)\tilde{p}+1 \right)^{s-1} = \frac{\left( (\phi-1)\tilde{p}+1 \right)^{\lfloor \tilde{w}k \rfloor} - 1}{\left( (\phi-1)\tilde{p}+1 \right) - 1} \le \frac{\left( (\phi-1)\tilde{p}+1 \right)^{\tilde{w}k} - 1}{(\phi-1)\tilde{p}}.$$

Therefore, from (60) and (61), we obtain

(62) 
$$\sum_{s=0}^{\lfloor \tilde{w}k \rfloor} \sum_{\overline{\chi} \in F_{s,k}} \mathbb{P}[\overline{h}^{\mathrm{T}}(k)\overline{\chi} > \rho k] \mathbb{P}[\overline{\chi}(k) = \overline{\chi}] \le \phi^{-\rho k+1} \frac{\left((\phi-1)\tilde{p}+1\right)^{\tilde{w}k}-1}{(\phi-1)\tilde{p}},$$

for all  $k \in \mathbb{N}$  such that  $|\tilde{w}k| \geq 1$ . The right-hand side of this inequality is zero if  $\lfloor \tilde{w}k \rfloor = 0$ . Therefore, (62) holds also for all  $k \in \mathbb{N}$ . By using this fact together with (54), (55), we arrive at (9). Next, we show  $\sum_{k=1}^{\infty} \psi_k < \infty$ . First of all, we have

(63) 
$$\sum_{k=1}^{\infty} \phi^{-\rho k+1} \frac{((\phi-1)\tilde{p}+1)^{\tilde{w}k}-1}{(\phi-1)\tilde{p}} = \frac{\phi}{(\phi-1)\tilde{p}} \sum_{k=1}^{\infty} \phi^{-\rho k} \left((\phi-1)\tilde{p}+1\right)^{\tilde{w}k} - \frac{\phi}{(\phi-1)\tilde{p}} \sum_{k=1}^{\infty} \phi^{-\rho k}.$$

We will prove that the summation terms on the right-hand side of (63) are both convergent. First, we have  $\phi^{-\rho} < 1$ , because  $\phi > 1$ . Therefore, the geometric series  $\sum_{k=1}^{\infty}\phi^{-\rho k}$  converges, that is,  $\sum_{k=1}^{\infty}\phi^{-\rho k}<\infty$ . Next, we show  $\phi^{-\rho}\left((\phi-1)\tilde{p}+1\right)^{\tilde{w}}<1$ . Observe that

(64) 
$$\phi^{-\rho} \left( (\phi - 1)\tilde{p} + 1 \right)^{\tilde{w}} = \left( \phi^{-\frac{\rho}{\tilde{w}}} \left( (\phi - 1)\tilde{p} + 1 \right) \right)^{\tilde{w}}.$$

Moreover,

$$\begin{split} \phi^{-\frac{\rho}{\bar{w}}}\left((\phi-1)\tilde{p}+1\right) &= \left(\frac{\frac{\rho}{\bar{w}}(1-\tilde{p})}{\tilde{p}(1-\frac{\rho}{\bar{w}})}\right)^{-\frac{\rho}{\bar{w}}} \left(\left(\frac{\frac{\rho}{\bar{w}}(1-\tilde{p})}{\tilde{p}(1-\frac{\rho}{\bar{w}})}-1\right)\tilde{p}+1\right) \\ &= \left(\frac{\tilde{p}\tilde{w}}{\rho}\right)^{\frac{\rho}{\bar{w}}} \left(\frac{1-\tilde{p}}{1-\frac{\rho}{\bar{w}}}\right)^{1-\frac{\rho}{\bar{w}}}. \end{split}$$

Here, we have  $\frac{\tilde{p}\tilde{w}}{\rho}, \frac{1-\tilde{p}}{1-\frac{p}{\tilde{w}}} \in (0,1) \cup (1,\infty)$ . As a result, since  $\ln v < v-1$  for any  $v \in (0,1) \cup (1,\infty)$ , we obtain

$$\ln\left(\phi^{-\frac{\rho}{\tilde{w}}}\left((\phi-1)\tilde{p}+1\right)\right) = \frac{\rho}{\tilde{w}}\ln\left(\frac{\tilde{p}\tilde{w}}{\rho}\right) + \left(1-\frac{\rho}{\tilde{w}}\right)\ln\left(\frac{1-\tilde{p}}{1-\frac{\rho}{\tilde{w}}}\right)$$
$$< \frac{\rho}{\tilde{w}}\left(\frac{\tilde{p}\tilde{w}}{\rho}-1\right) + \left(1-\frac{\rho}{\tilde{w}}\right)\left(\frac{1-\tilde{p}}{1-\frac{\rho}{\tilde{w}}}-1\right) = \tilde{p} - \frac{\rho}{\tilde{w}} + \frac{p}{\tilde{w}} - \tilde{p} = 0,$$

which implies  $\phi^{-\frac{\rho}{\tilde{w}}}((\phi-1)\tilde{p}+1) < 1$ . Thus, by (64),  $\phi^{-\rho}((\phi-1)\tilde{p}+1)^{\tilde{w}} < 1$ . It then follows that

(65) 
$$\sum_{k=1}^{\infty} \phi^{-\rho k} \left( (\phi - 1) \tilde{p} + 1 \right)^{\tilde{w}k} < \infty.$$

Finally, since  $\sum_{k=1}^{\infty} \phi^{-\rho k} < \infty$ , we obtain  $\sum_{k=1}^{\infty} \psi_k < \infty$  from (63) and (65).  $\Box$ 

# References

- J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," Proc. IEEE, vol. 95, no. 1, pp. 138–172, 2007.
- [2] V. Gupta, A. F. Dana, J. P. Hespanha, R. M. Murray, and B. Hassibi, "Data transmission over networks for estimation and control," *IEEE Trans. Automat. Control*, vol. 54, no. 8, pp. 1807–1819, 2009.
- [3] R. Alur, A. D'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss, "Compositional modeling and analysis of multi-hop control networks," *IEEE Trans. Automat. Control*, vol. 56, no. 10, pp. 2345–2357, 2011.
- [4] F. Smarra, A. D'Innocenzo, and M. D. Di Benedetto, "Optimal co-design of control, scheduling and routing in multi-hop control networks," in *Proc. IEEE Conf. Dec. Contr.*, pp. 1960–1965, 2012.
- [5] A. D'Innocenzo, M. D. Di Benedetto, and E. Serra, "Fault tolerant control of multi-hop control networks," *IEEE Trans. Automat. Control*, vol. 58, no. 6, pp. 1377–1389, 2013.
- [6] A. D'Innocenzo, F. Smarra, and M. D. Di Benedetto, "Resilient stabilization of multi-hop control networks subject to malicious attacks," *Automatica*, vol. 71, pp. 1–9, 2016.
- [7] F. Smarra, M. D. D. Benedetto, and A. D'Innocenzo, "A sub-optimal method for routing redundancy design over lossy networks," in *Proc. 20th IFAC World Congress*, pp. 2604–2609, 2017.
- [8] S. A. Khayam and H. Radha, "Markov-based modeling of wireless local area networks," in Proc. ACM MSWiM, pp. 100–107, 2003.
- S. Floyd and V. Jacobson, "Random early detection gateways for congestion avoidance," IEEE/ACM Trans. Networking, vol. 1, no. 4, pp. 397–413, 1993.
- [10] W. Xu, W. Trappe, Y. Zhang, and T. Wood, "The feasibility of launching and detecting jamming attacks in wireless networks," in *Proc. 6th ACM Int. Symp. Mobile Ad Hoc Network. Comput.*, pp. 46–57, 2005.

39

#### 40 CHARACTERIZATION OF FAILURES IN MULTI-HOP NETWORKED CONTROL

- [11] K. Pelechrinis, M. Iliofotou, and S. V. Krishnamurty, "Denial of service attacks in wireless networks: The case of jammers," *IEEE Commun. Surveys Tuts.*, vol. 13, no. 2, pp. 245–257, 2011.
- [12] S. Amin, A. A. Cárdenas, and S. S. Sastry, "Safe and secure networked control systems under Denial-of-Service attacks," in *Proc. 12th HSCC*, pp. 31–45, 2009.
- [13] C. De Persis and P. Tesi, "Input-to-state stabilizing control under denial-of-service," *IEEE Trans. Automat. Control*, vol. 60, no. 11, pp. 2930–2944, 2015.
- [14] S. Liu, P. X. Liu, and A. El Saddik, "A stochastic game approach to the security issue of networked control systems under jamming attacks," J. Franklin Inst., vol. 351, no. 9, pp. 4570–4583, 2014.
- [15] Y. Li, L. Shi, P. Cheng, J. Chen, and D. E. Quevedo, "Jamming attacks on remote state estimation in cyber-physical systems: A game-theoretic approach," *IEEE. Trans. Automat. Control*, vol. 60, no. 10, pp. 2831–2836, 2015.
- [16] C. De Persis and P. Tesi, "Networked control of nonlinear systems under Denial-of-Service," Syst. Control Lett., vol. 96, pp. 124–131, 2016.
- [17] B. Awerbuch, R. Curtmola, D. Holmer, C. Nita-Rotaru, and H. Rubens, "ODSBR: An ondemand secure Byzantine resilient routing protocol for wireless ad hoc networks," ACM Trans. Inf. and System Security, vol. 10, no. 4, p. 6, 2008.
- [18] M. M. E. A. Mahmoud and X. S. Shen, Security for Multi-hop Wireless Networks. Springer, 2014.
- [19] R. H. Jhaveri, S. J. Patel, and D. C. Jinwala, "DoS attacks in mobile ad hoc networks: A survey," in *Proc. ACTT*, pp. 535–541, 2012.
- [20] A. T. Mizrak, S. Savage, and K. Marzullo, "Detecting malicious packet losses," *IEEE Trans. Parallel Distrib. Syst.*, vol. 20, no. 2, pp. 191–206, 2009.
- [21] T. Shu and M. Krunz, "Privacy-preserving and truthful detection of packet dropping attacks in wireless ad hoc networks," *IEEE Trans. Mobile Computing*, vol. 14, no. 4, pp. 813–828, 2015.
- [22] A. A. Cárdenas, S. Amin, and S. Sastry, "Research challenges for the security of control systems," in *Proc. Conf. Hot Topics in Security*, 2008.
- [23] H. Sandberg, S. Amin, and K. H. Johansson, "Special issue on cyberphysical security in networked control systems," *IEEE Control Syst. Mag.*, vol. 35, no. 1, 2015.
- [24] A. F. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 789–804, 2006.
- [25] D. E. Quevedo, A. Ahlén, and K. H. Johansson, "State estimation over sensor networks with correlated wireless fading channels," *IEEE Trans. Automat. Control*, vol. 58, no. 3, pp. 581– 593, 2013.
- [26] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations," *IEEE Trans. Automat. Control*, vol. 49, no. 9, pp. 1453–1464, 2004.
- [27] H. Ishii, "Limitations in remote stabilization over unreliable channels without acknowledgements," Automatica, vol. 45, no. 10, pp. 2278–2285, 2009.
- [28] M. Lemmon and X. S. Hu, "Almost sure stability of networked control systems under exponentially bounded bursts of dropouts," in *Proc. 14th HSCC*, pp. 301–310, 2011.
- [29] V. Gupta, N. C. Martins, and J. S. Baras, "Optimal output feedback control using two remote sensors over erasure channels," *IEEE Trans. Automat. Control*, vol. 54, no. 7, pp. 1463–1476, 2009.
- [30] K. Okano and H. Ishii, "Stabilization of uncertain systems with finite data rates and Markovian packet losses," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 298–307, 2014.
- [31] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered control over unreliable networks subject to jamming attacks," in *Proc. IEEE Conf. Dec. Contr.*, pp. 4818–4823, 2015.
- [32] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered output feedback control resilient against jamming attacks and random packet losses," in *Proc. IFAC NecSys*, pp. 270–275, 2015.
- [33] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Networked control under random and malicious packet losses," *IEEE Trans. Automat. Control*, vol. 62, no. 5, pp. 2434–2449, 2017.
- [34] Y. Z. Lun, A. D'Innocenzo, and D. Di Beneditto, M., "On stability of time-inhomogeneous markov jump linear systems," in *Proc. IEEE Conf. Dec. Contr.*, pp. 5527–5532, 2016.

41

- [35] Y. Z. Lun, A. D'Innocenzo, and D. Di Beneditto, M., "Robust LQR for time-inhomogeneous Markov jump switched linear systems," in *Proc. 24th IFAC World Congress*, pp. 2199–2204, 2017.
- [36] Y. Z. Lun, A. D'Innocenzo, and D. Di Beneditto, M., "Optimal robust control and a separation principle for polytopic time-inhomogeneous Markov jump linear systems," in *Proc. IEEE Conf. Dec. Contr.*, pp. 6525–6530, 2017.
- [37] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Analysis of stochastic switched systems with application to networked control under jamming attacks," *IEEE Trans. Automat. Control*, to appear, 2018.
- [38] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Random and malicious packet transmission failures on multi-hop channels in networked control systems," in *Proc. IFAC NecSys*, pp. 49–54, 2016.
- [39] D. E. Quevedo and D. Nesic, "Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts," *IEEE Trans. Automat. Control*, vol. 56, no. 2, pp. 370–375, 2011.
- [40] B. D. O. Anderson, "From Wiener to hidden Markov models," IEEE Control Syst. Mag., vol. 19, no. 3, pp. 41–51, 1999.
- [41] O. Cappé, E. Moulines, and T. Rydén, Inference in Hidden Markov Models. Springer, 2009.
- [42] M. Vidyasagar, Hidden Markov Processes: Theory and Applications to Biology. Princeton University Press, 2014.
- [43] P. Billingsley, Probability and Measure. Wiley, 2012.
- [44] P. Sadeghi, R. A. Kennedy, P. B. Rapajic, and R. Shams, "Finite-state Markov modeling of fading channels," *IEEE Signal Process. Mag.*, vol. 25, no. 5, pp. 57–80, 2008.
- [45] M. Ellis, D. P. Pezaros, T. Kypraios, and C. Perkins, "A two-level Markov model for packet loss in UDP/IP-based real-time video applications targeting residential users," *Comput. Netw.*, vol. 70, no. 9, pp. 384–399, 2014.
- [46] A. Cetinkaya, H. Ishii, and T. Hayakawa, "The effect of time-varying jamming interference on networked stabilization," SIAM J. Control Optim., vol. 56, pp. 2398–2435, 2018.
- [47] D. Medhi and K. Ramasamy, Network Routing: Algorithms, Protocols, and Architectures. Morgan Kaufmann, 2007.
- [48] U. Madhow, Fundamentals of Digital Communication. Cambridge University Press, 2008.