# SECOND-ORDER KKT OPTIMALITY CONDITIONS FOR MULTI-OBJECTIVE OPTIMAL CONTROL PROBLEMS 

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#### Abstract

In this paper, we study second-order necessary and sufficient optimality conditions of Karush-Kuhn-Tucker-type for locally optimal solutions in the sense of Pareto to a class of multi-objective optimal control problems with mixed pointwise constraint. To deal with the problems, we first derive second-order optimality conditions for abstract multiobjective optimal control problems which satisfy the Robinson constraint qualification. We then apply the obtained results to our concrete problems. The proofs of obtained results are direct, self-contained without using scalarization techniques.


## 1. Introduction

Let $L_{j}:[0,1] \times \mathbb{R}^{n} \times \mathbb{R}^{l} \rightarrow \mathbb{R}$ with $j=1,2, \ldots, m, \varphi:[0,1] \times \mathbb{R}^{n} \times \mathbb{R}^{l} \rightarrow \mathbb{R}^{n}$, and $g:[0,1] \times \mathbb{R}^{n} \times \mathbb{R}^{l} \rightarrow \mathbb{R}$ be given functions. We consider the multi-objective optimal control problem of finding a control vector $u \in L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$ and the corresponding state $x \in$ $C\left([0,1], \mathbb{R}^{n}\right)$ which solve

$$
\begin{array}{ll}
\operatorname{Min}_{\mathbb{R}_{+}^{m}} & I(x, u) \\
\text { s.t. } & x(t)=x_{0}+\int_{0}^{t} \varphi(s, x(s), u(s)) d s \text { for a.e. } t \in[0,1], \\
& g(t, x(t), u(t)) \leq 0 \text { for a.e. } t \in[0,1] . \tag{3}
\end{array}
$$

Here $x_{0}$ is a given vector in $\mathbb{R}^{n}$ and the multi-objective function $I$ is given by

$$
I(x, u)=\left(I_{1}(x, u), I_{2}(x, u), \ldots, I_{m}(x, u)\right)
$$

where

$$
I_{j}(x, u):=\int_{0}^{1} L_{j}(t, x(t), u(t)) d t
$$

We denote by (MCP) the problem (11)-(3) and by $\Phi$ its feasible set, that is, $\Phi$ consists of couples $(x, u) \in C\left([0,1], \mathbb{R}^{n}\right) \times L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$ which satisfy constraints (2)-(3)).

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The multi-objective optimal control problems are important in mechanics and economy. For example, when we want to minimize energy and time of a system, we need to use twoobjective optimal control which has a form like (11)-(3) (see for instance [14]). Recently, problem (MCP) has been studied by several mathematicians. For papers which have a close connection to the present work, we refer the readers to [2, 3, 7, 14, 15, 17, 19, 20, 21, 27] and the references therein. In these papers, the authors mainly studied numerical methods and first-order necessary optimality conditions for multi-objective optimal control problems. However, to the best of our knowledge, so far there have been no papers investigating second-order optimality conditions for multi-objective optimal control problems. The study of second-order optimality conditions for optimization problems as well as for multi-objective optimal control problems is a fundamental topic in optimization theory. The second-order optimality conditions play an important role in solution stability and numerical methods of finding optimal solutions.

In this paper, we will focus on deriving second-order necessary optimality conditions and second-order sufficient optimality conditions of Karush-Kuhn-Tucker (KKT) type for the multi-objective optimal control problem (MCP). In order to establish second-order KKT optimality conditions for the (MCP), we first derive second-order optimality conditions for abstract multi-objective optimal control problems which satisfy the Robinson constraint qualification. We then apply the obtained results to our concrete problem.

In contrast with multi-objective optimal control problems, there have been some papers dealing with second-order KKT optimality conditions for vector optimization problems recently. For papers of this topic, we refer the reader to [6, 10, 11, 18] and references given therein. In [6, 18, the second-order KKT optimality conditions were derived by scalarization method via the so-called oriented distance function which was used by Ginchev et al. in [8] for the first time. However, this approach has also some certain limits because the oriented distance function is often nonsmooth. In [10, 11], by using Motzkin's theorem of the alternative, Jiménez et al. presented some second-order KKT optimality conditions for vector optimization problems under suitable constraint qualification conditions. Although the constraint qualification conditions used in [10, 11] are weaker than the Robinson constraint qualification, the "sigma" terms in the obtained second-order conditions do not vanish. In addition, those results can not apply to the (MCP) directly as the Robinson constraint qualification does not hold for the (MCP).

In the present paper, we derive second-order KKT optimality conditions for vector optimization directly via separation theorems. We then establish second-order KKT conditions without sigma terms for an abstract multi-objective optimal control problem under the Robinson constraint qualification. It is worth pointing out that our method is natural
and intrinsic. The obtained results approach a theory of no-gap second-order optimality conditions for multi-objective optimal control problems.

The paper is organized as follows. In Section 2, we set up notation and terminology, and state main results. Section 3 is intended to derive second-order KKT optimality conditions for a class of vector optimization problems. In Section 4, we establish second-order KKT necessary optimality conditions for an abstract multi-objective optimal control problem, which is based on the obtained result of Section 3. The proofs of the main results will be provided in Section 5. In Section 6, we give some examples to illustrate the main results.

## 2. Assumptions and statements of the main results

In this section, $C\left([0,1], \mathbb{R}^{n}\right)$ is the Banach space of continuous vector-valued functions $x:[0,1] \rightarrow \mathbb{R}^{n}$ with the norm $\|x\|_{0}=\max _{t \in[0,1]}|x(t)|$ and $\mathbb{R}^{n}$ is the Euclidean space of $n$-tuples $\xi=\left(\xi_{1}, \ldots, \xi_{n}\right)$ with the norm $|\xi|=\left(\sum_{i=1}^{n} \xi_{i}^{2}\right)^{\frac{1}{2}}$. For each $1 \leq p \leq \infty, L^{p}\left([0,1], \mathbb{R}^{l}\right)$ stands for the Lebesgue spaces with the norms $\|\cdot\|_{p}$. For convenience, we put $X=C\left([0,1], \mathbb{R}^{n}\right)$ and $U=L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$. In the sequel, $L$ and $\phi$ stand for $\left(L_{1}, \ldots, L_{m}\right)$ and $(\varphi, g)$, respectively. Define

$$
Q=\left\{v \in L^{\infty}([0,1], \mathbb{R}) \mid v(t) \leq 0, \text { a.e. } t \in[0,1]\right\}
$$

Let us impose the following assumptions on $L$ and $\phi$.
(H1) The function $L$ is a Carathéodory function and $\phi$ is a continuous mapping. For a.e. $t \in[0,1], L(t, \cdot \cdot \cdot)$ is of class $C^{2}$ while $\phi(t, \cdot, \cdot)$ is of class $C^{2}$ for all $t \in[0,1]$. Besides, for each $M>0$, there exist numbers $k_{L M}>0$ and $k_{\phi M}>0$ such that

$$
\begin{aligned}
\left|L\left(t, x_{1}, u_{1}\right)-L\left(t, x_{2}, u_{2}\right)\right| & +\left|\nabla L\left(t, x_{1}, u_{1}\right)-\nabla L\left(t, x_{2}, u_{2}\right)\right|+ \\
& +\left|\nabla^{2} L\left(t, x_{1}, u_{1}\right)-\nabla^{2} L\left(t, x_{2}, u_{2}\right)\right| \leq k_{L M}\left(\left|x_{1}-x_{2}\right|+\left|u_{1}-u_{2}\right|\right)
\end{aligned}
$$

for a.e. $t \in[0,1]$ and for all $x_{i} \in \mathbb{R}^{n}, u_{i} \in \mathbb{R}^{l}$ satisfying $\left|x_{i}\right| \leq M,\left|u_{i}\right| \leq M$ with $i=1,2$, and

$$
\begin{aligned}
\left|\phi\left(t, x_{1}, u_{1}\right)-\phi\left(t, x_{2}, u_{2}\right)\right| & +\left|\nabla \phi\left(t, x_{1}, u_{1}\right)-\nabla \phi\left(t, x_{2}, u_{2}\right)\right|+ \\
& +\left|\nabla^{2} \phi\left(t, x_{1}, u_{1}\right)-\nabla^{2} \phi\left(t, x_{2}, u_{2}\right)\right| \leq k_{\phi M}\left(\left|x_{1}-x_{2}\right|+\left|u_{1}-u_{2}\right|\right)
\end{aligned}
$$

for all $t \in[0,1], x_{i} \in \mathbb{R}^{n}, u_{i} \in \mathbb{R}^{l}$ satisfying $\left|x_{i}\right| \leq M,\left|u_{i}\right| \leq M$ with $i=1,2$. Moreover, we require that the functions

$$
L(t, 0,0),|\nabla L(t, 0,0)|,\left|\nabla^{2} L(t, 0,0)\right|, \phi(t, 0,0),|\nabla \phi(t, 0,0)|,\left|\nabla^{2} \phi(t, 0,0)\right|
$$

belong to $L^{\infty}([0,1], \mathbb{R})$.
(H2) Given a couple $(\bar{x}, \bar{u}) \in \Phi$, there exist $i_{0} \in\{1,2, \ldots, l\}$ and $\alpha>0$ such that

$$
\left|g_{u_{i_{0}}}(t, \bar{x}(t), \bar{u}(t))\right| \geq \alpha \text { for a.e. } t \in[0,1] .
$$

Hereafter, $L[t], \varphi[t], g[t], L_{x}[t], \varphi_{u}[t], g_{u}[t]$ and so on, stand for

$$
\begin{aligned}
& L(t, \bar{x}(t), \bar{u}(t)), \varphi(t, \bar{x}(t), \bar{u}(t)), g(t, \bar{x}(t), \bar{u}(t)), \\
& L_{x}(t, \bar{x}(t), \bar{u}(t)), \varphi_{u}(t, \bar{x}(t), \bar{u}(t)), g_{u}(t, \bar{x}(t), \bar{u}(t)), \ldots
\end{aligned}
$$

We denote by $\mathbb{R}_{+}^{m}$ the nonnegative orthant of $\mathbb{R}^{m}$, where $m \in \mathbb{N}:=\{1,2, \ldots\}$. The interior of $\mathbb{R}_{+}^{m}$ is denoted by int $\mathbb{R}_{+}^{m}$.

Definition 2.1. Assume that $\bar{z}=(\bar{x}, \bar{u})$ is a feasible point of the (MCP). We say that:
(i) $\bar{z}$ is a locally weak Pareto solution of the (MCP) if there exists $\epsilon>0$ such that for all $(x, u) \in(B(\bar{x}, \epsilon) \times B(\bar{u}, \epsilon)) \cap \Phi$, one has

$$
I(x, u)-I(\bar{x}, \bar{u}) \notin-\operatorname{int} \mathbb{R}_{+}^{m} .
$$

(ii) $\bar{z}$ is a locally Pareto solution of the (MCP) if there exists $\epsilon>0$ such that for all $(x, u) \in(B(\bar{x}, \epsilon) \times B(\bar{u}, \epsilon)) \cap \Phi$, one has

$$
I(x, u)-I(\bar{x}, \bar{u}) \notin-\mathbb{R}_{+}^{m} \backslash\{0\} .
$$

Let us denote by $\mathcal{C}_{0}(\bar{z})$ the set of vectors $z=(x, u) \in C\left([0,1], \mathbb{R}^{n}\right) \times L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$ such that the following conditions hold:

$$
\begin{aligned}
& \left(c_{1}\right) \int_{0}^{1}\left(L_{x}[t] x(t)+L_{u}(t) u(t)\right) d t \in-\mathbb{R}_{+}^{m} ; \\
& \left(c_{2}\right) x(\cdot)=\int_{0}^{(\cdot)}\left(\varphi_{x}[s] x(s)+\varphi_{u}[s] u(s)\right) d s ; \\
& \left(c_{3}\right) g_{x}[\cdot] x+g_{u}[\cdot] u \in \operatorname{cone}(Q-g(\cdot, \bar{x}, \bar{u})) .
\end{aligned}
$$

Let $\mathcal{C}(\bar{z})$ be the closure of $\mathcal{C}_{0}(\bar{z})$ in $C\left([0,1], \mathbb{R}^{n}\right) \times L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$. We call $\mathcal{C}(\bar{z})$ the critical cone of the (MCP) at $\bar{z}$. Each vector $z \in \mathcal{C}(\bar{z})$ is called a critical direction to the (MCP) at $\bar{z}$. It is easily seen that $\mathcal{C}(\bar{z})$ is a closed convex cone containing 0 .

The following theorem gives necessary optimality conditions for the (MCP).
Theorem 2.1. Suppose that assumptions (H1) and (H2) are valid and $\bar{z}$ is a locally weak Pareto solution of the (MCP). Then, for each $z \in \mathcal{C}(\bar{z})$, there exist a vector $\lambda \in \mathbb{R}_{+}^{m}$ with $|\lambda|=1$, an absolutely continuous function $\bar{p}:[0,1] \rightarrow \mathbb{R}^{n}$ and a function $\theta \in L^{1}([0,1], \mathbb{R})$ such that the following conditions are fulfilled:
(i) $\theta \in N(Q, g[\cdot])$;
(ii) (the adjoint equation)

$$
\left\{\begin{array}{l}
\dot{\bar{p}}(t)=-\lambda^{T} L_{x}[t]-\varphi_{x}[t] \bar{p}(t)-\theta(t) g_{x}[t] \quad \text { a.e. } t \in[0,1], \\
\bar{p}(1)=0 ;
\end{array}\right.
$$

(iii) (the stationary condition in u)

$$
\lambda^{T} L_{u}[t]+\bar{p}^{T} \varphi_{u}[t]+\theta(t) g_{u}[t]=0 \quad \text { a.e. } t \in[0,1] ;
$$

(iv) (the non-negative condition)

$$
\begin{aligned}
\int_{0}^{1}\left(\sum_{j=1}^{m} \lambda_{j} \nabla^{2} L_{j}[t] z(t), z(t)\right) d t & +\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] z(t), z(t)\right) d t \\
& +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] z(t), z(t)\right) d t \geq 0
\end{aligned}
$$

A point $\bar{z} \in \Phi$ satisfying conditions (i)-(iii) of Theorem 2.1 w.r.t $(\lambda, \bar{p}, \theta)$ is called a KKT point of the (MCP).

In multi-objective optimization problems, the critical cone for second-order sufficient conditions is often required bigger than the one for second-order necessary conditions. Therefore, we need to enlarge $\mathcal{C}(\bar{z})$ to deal with second-order sufficient conditions. We denote by $\mathcal{C}^{\prime}(\bar{z})$ the set of vectors $(x, u) \in C\left([0,1], \mathbb{R}^{n}\right) \times L^{2}\left([0,1], \mathbb{R}^{l}\right)$ which satisfy the following conditions:

$$
\begin{array}{ll}
\left(c_{1}^{\prime}\right) & \int_{0}^{1}\left(L_{x}[t] x(t)+L_{u}[t] u(t)\right) d t \in-\mathbb{R}_{+}^{m} \\
\left(c_{2}^{\prime}\right) & x(\cdot)=\int_{0}^{(\cdot)}\left(\varphi_{x}[s] x(s)+\varphi_{u}[s] u(s)\right) d s ; \\
\left(c_{3}^{\prime}\right) & g_{x}[t] x(t)+g_{u}[t] u(t) \in T((-\infty, 0] ; g[t]) \text { for a.e. } t \in[0,1] .
\end{array}
$$

Obviously, $\mathcal{C}^{\prime}(\bar{z})$ is a closed convex cone and $\mathcal{C}(\bar{z}) \subset \mathcal{C}^{\prime}(\bar{z})$.
We now introduce the concept of locally strong Pareto solution for the multi-objective optimal control problem (MCP).

Definition 2.2. Let $\bar{z}=(\bar{x}, \bar{u}) \in \Phi$ be a feasible point of the (MCP). We say that $\bar{z}$ is a locally strong Pareto solution of the (MCP) if there exist a number $\epsilon>0$ and a vector $c \in \operatorname{int} \mathbb{R}_{+}^{m}$ such that for all $(x, u) \in\left(B_{X}(\bar{x}, \epsilon) \times B_{U}(\bar{u}, \epsilon)\right) \cap \Phi$, one has

$$
I(x, u)-I(\bar{x}, \bar{u})-c\|u-\bar{u}\|_{2}^{2} \notin-\mathbb{R}_{+}^{m} \backslash\{0\}
$$

Clearly, every locally strong Pareto solution of the (MCP) is also a locally Pareto solution of this problem. Note that in Definition 2.2 we use two norms to define locally strong Pareto solutions. Here $\left.B_{U}(\bar{u}, \epsilon)\right)$ is a ball in $L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$ while $\|u-\bar{u}\|_{2}$ is the norm in $L^{2}\left([0,1], \mathbb{R}^{l}\right)$.

The following theorem provides sufficient conditions for locally strong Pareto solutions.
Theorem 2.2. Suppose that assumptions (H1) and (H2) are valid and $\bar{z} \in \Phi$. Suppose that there exist a vector $\lambda \in \mathbb{R}_{+}^{m}$ with $|\lambda|=1$, an absolutely continuous function $\bar{p}:[0,1] \rightarrow \mathbb{R}^{n}$ and a function $\theta \in L^{2}([0,1], \mathbb{R})$ satisfy conditions $(i),(i i)$ and (iii) of Theorem 2.1 and the
strict second-order condition

$$
\begin{align*}
\int_{0}^{1}\left(\lambda^{T} \nabla^{2} L_{j}[t] z(t), z(t)\right) d t & +\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] z(t), z(t)\right) d t \\
& +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] z(t), z(t)\right) d t>0 \tag{4}
\end{align*}
$$

for all $z=(x, u) \in \mathcal{C}^{\prime}(\bar{z}) \backslash\{0\}$. Furthermore, there is a number $\gamma_{0}>0$ such that

$$
\begin{equation*}
\lambda^{T} L_{u u}[t](\xi, \xi) \geq \gamma_{0}|\xi|^{2} \quad \forall \xi \in \mathbb{R}^{l} \tag{5}
\end{equation*}
$$

Then, $(\bar{x}, \bar{u})$ is a locally strong Pareto solution of the (MCP).

## 3. Abstract multi-objective optimization

Assume that $Z$ and $E$ are Banach spaces with the dual spaces $Z^{*}$ and $E^{*}$, respectively. We consider the following multi-objective optimization problem:

$$
\begin{array}{ll}
\operatorname{Min}_{\mathbb{R}_{+}^{m}} & f(z)=\left(f_{1}(z), \ldots, f_{m}(z)\right)  \tag{MP1}\\
\text { s.t. } & G(z) \in Q
\end{array}
$$

where $f: Z \rightarrow \mathbb{R}^{m}$ and $G: Z \rightarrow E$ are of class $C^{2}$, and $Q$ is a nonempty closed convex subset in $E$. We denote by $\Sigma$ the feasible set of the (MP1), that is,

$$
\Sigma=\{z \in Z \mid G(z) \in Q\}
$$

To derive optimality conditions for the (MP1) we need some concepts of variational analysis. Let $X$ be a Banach space with the dual $X^{*}, B_{X}$ and $B_{X}(x, r)$ stand for the closed unit ball and the closed ball with center $x$ and radius $r$, respectively. Given a subset $A$ of $X$, we denote the interior and the closure of $A$ respectively by $\operatorname{int} A$ and $\bar{A}$.

Let $\Omega$ be a nonempty and closed subset in $X$ and $\bar{x} \in \Omega$. The sets

$$
\begin{aligned}
T^{b}(\Omega ; \bar{x}) & :=\left\{h \in X \mid \forall t_{k} \rightarrow 0^{+}, \exists h_{k} \rightarrow h, \bar{x}+t_{k} h_{k} \in \Omega \quad \forall k \in \mathbb{N}\right\} \\
T(\Omega ; \bar{x}) & :=\left\{h \in X \mid \exists t_{k} \rightarrow 0^{+}, \exists h_{k} \rightarrow h, \bar{x}+t_{k} h_{k} \in \Omega \quad \forall k \in \mathbb{N}\right\}
\end{aligned}
$$

are called the adjacent tangent cone and the contingent cone to $\Omega$ at $\bar{x}$, respectively. It is well-known that when $\Omega$ is convex, then

$$
T^{b}(\Omega ; \bar{x})=T(\Omega ; \bar{x})=\overline{\Omega(\bar{x})},
$$

where $\Omega(\bar{z})$ is defined as follows

$$
\Omega(\bar{x}):=\operatorname{cone}(\Omega-\bar{x})=\{\lambda(h-\bar{x}) \mid h \in \Omega, \lambda>0\} .
$$

Let $\bar{x} \in \Omega$ and $h \in X$. The sets

$$
\begin{aligned}
T^{2 b}(\Omega ; \bar{x}, h) & :=\left\{w \in X \mid \forall t_{k} \rightarrow 0^{+}, \exists w_{k} \rightarrow w, \bar{x}+t_{k} h+\frac{1}{2} t_{k}^{2} w_{k} \in \Omega \quad \forall k \in \mathbb{N}\right\}, \\
T^{2}(\Omega ; \bar{x}, h) & :=\left\{w \in X \mid \exists t_{k} \rightarrow 0^{+}, \exists w_{k} \rightarrow w, \bar{x}+t_{k} h+\frac{1}{2} t_{k}^{2} w_{k} \in \Omega \quad \forall k \in \mathbb{N}\right\},
\end{aligned}
$$

are called the second-order adjacent tangent set and the second-order contingent tangent set to $\Omega$ at $\bar{x}$ in the direction $h$, respectively. Clearly, $T^{2 b}(\Omega ; \bar{x}, h)$ and $T^{2}(\Omega ; \bar{x}, h)$ are closed sets and

$$
T^{2 b}(\Omega ; \bar{x}, h) \subset T^{2}(\Omega ; \bar{x}, h), T^{2 b}(\Omega ; \bar{x}, 0)=T^{b}(\Omega ; \bar{x}), T^{2}(\Omega ; \bar{x}, 0)=T(\Omega ; x)
$$

It is noted that if $\Omega$ is convex, then so is $T^{2 b}(\Omega ; \bar{x}, h)$. However, $T^{2}(\Omega ; \bar{x}, h)$ may not be convex when $\Omega$ is convex (see, for example, [1]). In the case that $\Omega$ is a convex set, the normal cone to $\Omega$ at $\bar{x}$ is defined by

$$
N(\Omega ; \bar{x}):=\left\{x^{*} \in X^{*} \mid\left\langle x^{*}, x-\bar{x}\right\rangle \leq 0 \quad \forall x \in \Omega\right\},
$$

or, equivalently,

$$
N(\Omega ; \bar{x})=\left\{x^{*} \in X^{*} \mid\left\langle x^{*}, h\right\rangle \leq 0 \quad \forall h \in T(\Omega ; \bar{x})\right\} .
$$

Definition 3.1. We say that $\bar{z} \in \Sigma$ is a locally weak Pareto solution of the (MP1) if there exists $\epsilon>0$ such that for all $z \in B_{Z}(\bar{z}, \epsilon) \cap \Sigma$, one has $f(z)-f(\bar{z}) \notin-\operatorname{int} \mathbb{R}_{+}^{m}$.

We say that the Robinson constraint qualification holds at $\bar{z} \in \Sigma$ if the following condition is verified

$$
0 \in \operatorname{int}\left[\nabla G(\bar{z})\left(B_{Z}\right)-(Q-G(\bar{z})) \cap B_{E}\right] .
$$

According to [25, Theorem 2.1], the Robinson constraint qualification is equivalent to the following condition:

$$
E=\nabla G(\bar{z}) Z-\operatorname{cone}(Q-G(\bar{z}))
$$

When the Robinson constraint qualification holds at $\bar{z}$, we say $\bar{z}$ is a regular point of the (MP1). Hereafter we always assume that $\bar{z}$ is a feasible regular point of the (MP1).

Let us define the following critical cones

$$
\begin{aligned}
\mathcal{C}_{1}(\bar{z}) & =\left\{d \in Z \mid \nabla f(\bar{z}) d \in-\mathbb{R}_{+}^{m}, \nabla G(\bar{z}) d \in T(Q ; G(\bar{z}))\right\} \\
\mathcal{C}_{01}(\bar{z}) & =\left\{d \in Z \mid \nabla f(\bar{z}) d \in-\mathbb{R}_{+}^{m}, \nabla G(\bar{z}) d \in \operatorname{cone}(Q-G(\bar{z}))\right\} \\
\mathcal{C}_{1 *}(\bar{z}) & =\overline{\mathcal{C}_{01}(\bar{z})}
\end{aligned}
$$

For each $d \in \mathcal{C}(\bar{z})$, put

$$
I(\bar{z}, d)=\left\{i \in I \mid \nabla f_{i}(\bar{z}) d=0\right\}
$$

where $I:=\{1, \ldots, m\}$.

Lemma 3.1. If $\bar{z}$ is a locally weak Pareto solution of the (MP1), then

$$
\mathcal{C}_{1}(\bar{z})=\left\{d \in Z \mid \nabla f(\bar{z}) d \in-\mathbb{R}_{+}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right), \nabla G(\bar{z}) d \in T(Q ; G(\bar{z}))\right\}
$$

Consequently, the following set

$$
\left\{d \in Z \mid \nabla f(\bar{z}) d \in-\operatorname{int} \mathbb{R}_{+}^{m}, \nabla G(\bar{z}) d \in T(Q ; G(\bar{z}))\right\}
$$

is empty.
Proof. Thanks to [12, Theorem 3.1], we have

$$
\nabla G(\bar{z})^{-1}(T(Q ; G(\bar{z})))=T\left(G^{-1}(Q) ; \bar{z}\right)=T(\Sigma ; \bar{z})
$$

Take $d \in \mathcal{C}_{1}(\bar{z})$, then $d \in T(\Sigma ; \bar{z})$. Hence there exists a sequence $\left\{\left(t_{k}, d_{k}\right)\right\}$ converging to $\left(0^{+}, d\right)$ such that $\bar{z}+t_{k} d_{k} \in \Sigma$ for all $k \in \mathbb{N}$. Since $\bar{z}$ is a locally weak Pareto solution of the (MP1), we may assume that

$$
f\left(\bar{z}+t_{k} d_{k}\right)-f(\bar{z}) \in \mathbb{R}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right)
$$

for all $k \in \mathbb{N}$. From Mean Value Theorem for differentiable functions, we have

$$
t_{k} \nabla f(\bar{z}) d_{k}+o\left(t_{k}\right) \in \mathbb{R}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right)
$$

or, equivalently,

$$
\nabla f(\bar{z}) d_{k}+\frac{o\left(t_{k}\right)}{t_{k}} \in \mathbb{R}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right)
$$

for all $k \in \mathbb{N}$. Letting $k \rightarrow \infty$, by the closedness of $\mathbb{R}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right)$, we get

$$
\nabla f(\bar{z}) d \in \mathbb{R}^{m} \backslash\left(-\operatorname{int} \mathbb{R}_{+}^{m}\right)
$$

The proof is complete.
Lemma 3.2. Let $\bar{z}$ be a locally weak Pareto solution of the (MP1) and $\Lambda_{1}(\bar{z})$ be the set of normalized Karush-Kuhn-Tucker multipliers of the (MP1) at $\bar{z}$, that is,

$$
\Lambda_{1}(\bar{z}):=\left\{\left(\lambda, e^{*}\right) \in \mathbb{R}_{+}^{m} \times E^{*}\left|\langle\lambda, \nabla f(\bar{z})\rangle+\nabla G(\bar{z})^{*} e^{*}=0,|\lambda|=1, e^{*} \in N(Q ; G(\bar{z}))\right\} .\right.
$$

Then $\Lambda_{1}(\bar{z})$ is a nonempty bounded and compact set in $\mathbb{R}^{m} \times E^{*}$ with respect to topology $\tau_{\mathbb{R}^{m}} \times \tau\left(E^{*}, E\right)$, where $\tau\left(E^{*}, E\right)$ is the weakly star topology in $E^{*}$.

Proof. We first claim that $\Lambda_{1}(\bar{z})$ is nonempty. Indeed, put

$$
\Psi=\left\{\left(\nabla f_{1}(\bar{z}) d+r_{1}, \ldots, \nabla f_{m}(\bar{z}) d+r_{m}, \nabla G(\bar{z}) d-v\right) \mid d \in Z, v \in T(Q ; G(\bar{z})), r_{i} \geq 0, i \in I\right\}
$$

Then, $\Psi$ is a convex subset in $\mathbb{R}^{m} \times E$. By the Robinson constraint qualification, there exists $\rho>0$ such that

$$
\begin{equation*}
B_{E}(0, \rho) \subset \nabla G(\bar{z})\left(B_{Z}\right)-(Q-G(\bar{z})) \cap B_{E} \tag{6}
\end{equation*}
$$

This implies that

$$
B_{E}(0, \rho) \subset \nabla G(\bar{z})\left(B_{Z}\right)-T(Q ; G(\bar{z})) \cap B_{E}
$$

For each $i \in I$, put

$$
\alpha_{i}=\left\|\nabla f_{i}(\bar{z})\right\|=\sup _{d \in B_{Z}}\left|\nabla f_{i}(\bar{z}) d\right|
$$

It is easily seen that

$$
\left(\alpha_{1},+\infty\right) \times \ldots \times\left(\alpha_{m},+\infty\right) \times B(0, \rho) \subset \Psi
$$

Thus, $\Psi$ has a nonempty interior. We show that $(0,0) \notin \operatorname{int} \Psi$. If otherwise, there exist $\epsilon_{1}>0, \ldots, \epsilon_{m}>0$ such that

$$
\left(-\epsilon_{1}, \epsilon_{1}\right) \times \ldots \times\left(-\epsilon_{m}, \epsilon_{m}\right) \times\{0\} \subset \Psi
$$

This implies that there exist $d \in Z, v \in T(Q ; G(\bar{z}))$, and $r_{i} \geq 0, i \in I$ such that

$$
\left\{\begin{array}{l}
\nabla f_{i}(\bar{z}) d+r_{i}<0, \quad i \in I \\
\nabla G(\bar{z}) d-v=0
\end{array}\right.
$$

Consequently, the system

$$
\left\{\begin{array}{l}
\nabla f_{i}(\bar{z}) d<0, \quad i \in I \\
\nabla G(\bar{z}) d \in T(Q ; G(\bar{z}))
\end{array}\right.
$$

has at least one solution $d \in Z$, which contradicts the conclusion of Lemma 3.1. We now can separate $(0,0)$ from $\Psi$ by a hyperplane, i.e., there exists a functional $\left(\lambda, e^{*}\right) \in\left(\mathbb{R}^{m} \times\right.$ $\left.E^{*}\right) \backslash\{(0,0)\}$ such that

$$
\begin{equation*}
\sum_{i=1}^{m} \lambda_{i}\left(\nabla f_{i}(\bar{z}) d+r_{i}\right)+\left\langle e^{*}, \nabla G(\bar{z}) d-v\right\rangle \geq 0 \tag{7}
\end{equation*}
$$

for all $d \in Z, v \in T(Q ; G(\bar{z})), r_{i} \geq 0, i \in I$. From (7) it follows that $\lambda_{i} \geq 0$ for all $i \in I$, that is, $\lambda \in \mathbb{R}_{+}^{m}$. Putting $v=0$ and $r_{i}=0, i \in I$ into (7), we get

$$
\left(\langle\lambda, \nabla f(\bar{z})\rangle+\nabla G(\bar{z})^{*} e^{*}\right) d \geq 0 \quad \forall d \in Z
$$

Thus, $\langle\lambda, \nabla f(\bar{z})\rangle+\nabla G(\bar{z})^{*} e^{*}=0$. Putting this equation and $r_{i}=0, i \in I$, into (7), one has

$$
e^{*}(v) \leq 0 \quad \forall v \in T(Q ; G(\bar{z}))
$$

Hence, $e^{*} \in N(Q ; G(\bar{z}))$. We now show that $\lambda \neq 0$. Indeed, if otherwise, then we have

$$
\begin{equation*}
\left\langle e^{*}, \nabla G(\bar{z}) d-v\right\rangle \geq 0 \tag{8}
\end{equation*}
$$

for all $d \in Z, v \in T(Q ; G(\bar{z}))$. Again by the Robinson constraint qualification, one has

$$
E=\nabla G(\bar{z})(Z)-Q(G(\bar{z}))
$$

This and (8) imply that $e^{*}=0$, a contradiction. $\operatorname{Put}\left(\bar{\lambda}, \bar{e}^{*}\right)=\left(\frac{\lambda}{|\lambda|}, \frac{e^{*}}{|\lambda|}\right)$. Then we have $\left(\bar{\lambda}, \bar{e}^{*}\right) \in \Lambda_{1}(\bar{z})$, as required.

We now claim that $\Lambda_{1}(\bar{z})$ is bounded. Indeed, fix $\left(\lambda_{0}, e_{0}^{*}\right) \in \Lambda_{1}(\bar{z})$. Then, for any $\left(\lambda, e^{*}\right)$ belonging to $\Lambda_{1}(\bar{z})$, we have

$$
\begin{cases}\left\langle\lambda_{0}, \nabla f(\bar{z})\right\rangle+\nabla G(\bar{z})^{*} e_{0}^{*}=0, & e_{0}^{*} \in N(Q ; G(\bar{z})), \\ \langle\lambda, \nabla f(\bar{z})\rangle+\nabla G(\bar{z})^{*} e^{*}=0, \quad e^{*} \in N(Q ; G(\bar{z}))\end{cases}
$$

Since (6), for each $y \in B_{E}(0, \rho)$, there exist $z \in B_{Z}$ and $w \in(Q-G(\bar{z})) \cap B_{E}$ such that $y=\nabla G(\bar{z}) z-w$. It follows that

$$
\begin{aligned}
\left\langle e_{0}^{*}-e^{*}, y\right\rangle & =\left\langle e_{0}^{*}-e^{*}, \nabla G(\bar{z}) z-w\right\rangle \\
& =\left\langle\nabla G(\bar{z})^{*}\left(e_{0}^{*}-e^{*}\right), z\right\rangle-\left\langle e_{0}^{*}-e^{*}, w\right\rangle \\
& =\left\langle\left(\lambda-\lambda_{0}\right) \nabla f(\bar{z}), z\right\rangle-\left\langle e_{0}^{*}, w\right\rangle+\left\langle e^{*}, w\right\rangle \\
& \leq 2\|\nabla f(\bar{z})\|+\left\|e_{0}^{*}\right\| .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\langle-e^{*}, y\right\rangle & \leq 2\|\nabla f(\bar{z})\|+\left\|e_{0}^{*}\right\|+\left\|e_{0}^{*}\right\|\|y\| \\
& \leq 2\|\nabla f(\bar{z})\|+\left\|e_{0}^{*}\right\|+\left\|e_{0}^{*}\right\| \rho .
\end{aligned}
$$

Replacing $y$ by $\rho y$ with $\|y\| \leq 1$, we get

$$
\left\|e^{*}\right\| \rho \leq 2\|\nabla f(\bar{z})\|+\left\|e_{0}^{*}\right\|(1+\rho)
$$

Consequently,

$$
\left\|e^{*}\right\| \leq \frac{2}{\rho}\|\nabla f(\bar{z})\|+\frac{1+\rho}{\rho}\left\|e_{0}^{*}\right\|
$$

Thus, $\Lambda_{1}(\bar{z})$ is bounded. It is easy to check that the set $\Lambda_{1}(\bar{z})$ is closed with respect to topology $\tau_{\mathbb{R}^{m}} \times \tau\left(E^{*}, E\right)$. Thanks to [4, Theorem 3.16], $\Lambda_{1}(\bar{z})$ is compact.

To derive second-order necessary conditions for the (MP1), we need the following result.
Lemma 3.3. Let $\bar{z} \in \Sigma$ and $d \in \mathcal{C}_{1}(\bar{z})$. If $\bar{z}$ is a locally weak Pareto solution of the (MP1), then the following system

$$
\begin{align*}
& \nabla f_{i}(\bar{z}) z+\nabla^{2} f_{i}(\bar{z}) d^{2}<0, \quad \forall i \in I(\bar{z}, d)  \tag{9}\\
& \nabla G(\bar{z}) z+\nabla^{2} G(\bar{z}) d^{2} \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d) \tag{10}
\end{align*}
$$

has no solution $z \in Z$.

Proof. Arguing by contradiction, assume that the system (19)-(10) admits a solution, say $z$. From (10) it follows that

$$
z \in \nabla G(\bar{z})^{-1}\left[T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)-\nabla^{2} G(\bar{z}) d^{2}\right] .
$$

By the Robinson constraint qualification and [12, Theorem 3.1], we have

$$
T^{2 b}(\Sigma ; \bar{z}, d)=\nabla G(\bar{z})^{-1}\left[T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)-\nabla^{2} G(\bar{z}) d^{2}\right]
$$

Thus, $z \in T^{2 b}(\Sigma ; \bar{z}, d)$. Let $\left\{t_{k}\right\}$ be an arbitrary sequence converging to $0^{+}$. Then, there exists a sequence $\left\{z_{k}\right\}$ tending to $z$ such that

$$
w_{k}:=\bar{z}+t_{k} d+\frac{1}{2} t_{k}^{2} z_{k} \in \Sigma \quad \forall k \in \mathbb{N} .
$$

For each $i \in I(\bar{z}, d)$ and $k \in \mathbb{N}$, we have

$$
\begin{aligned}
f_{i}\left(w_{k}\right)-f_{i}(\bar{z}) & =\left[f_{i}\left(w_{k}\right)-f_{i}\left(\bar{z}+t_{k} d\right)\right]+\left[f_{i}\left(\bar{z}+t_{k} d\right)-f_{i}(\bar{z})-t_{k}\left\langle\nabla f_{i}(\bar{z}), d\right\rangle\right] \\
& =\frac{1}{2} t_{k}^{2}\left\langle\nabla f_{i}\left(\bar{z}+t_{k} d\right), z_{k}\right\rangle+\frac{1}{2} t_{k}^{2} \nabla^{2} f_{i}(\bar{z}) d^{2}+o\left(t_{k}^{2}\right)
\end{aligned}
$$

Therefore,

$$
\lim _{k \rightarrow \infty} \frac{f_{i}\left(w_{k}\right)-f_{i}(\bar{z})}{\frac{1}{2} t_{k}^{2}}=\left\langle\nabla f_{i}(\bar{z}), z\right\rangle+\nabla^{2} f_{i}(\bar{z}) d^{2}
$$

This and (9) imply that

$$
f_{i}\left(w_{k}\right)<f_{i}(\bar{z})
$$

for all $i \in I(\bar{z}, d)$ and $k$ large enough. For each $i \in I \backslash I(\bar{z}, d)$, we have $\left\langle\nabla f_{i}(\bar{z}), d\right\rangle<0$. From this and the fact that

$$
\lim _{k \rightarrow \infty} \frac{f_{i}\left(w_{k}\right)-f_{i}(\bar{z})}{t_{k}}=\left\langle\nabla f_{i}(\bar{z}), d\right\rangle
$$

it follows that

$$
f_{i}\left(w_{k}\right)<f_{i}(\bar{z})
$$

for all $k$ large enough. Thus there exists $k$ large enough such that

$$
f_{i}\left(w_{k}\right)<f_{i}(\bar{z}) \quad \forall i \in I,
$$

which contradicts the fact that $\bar{z}$ is a locally weak Pareto solution of the (MP1).
Problem (MP1) is associated with the Lagrangian $\mathcal{L}_{1}\left(z, \lambda, e^{*}\right)=\lambda^{T} f(z)+e^{*} G(z)$. The following theorem gives some second-order necessary optimality conditions for the (MP1).

Theorem 3.1. Suppose that $\bar{z}$ is a locally weak Pareto solution of the (MP1). Then, for each $d \in \mathcal{C}_{1 *}(\bar{z})$, there exists $\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})$ such that the following non-negative second-order condition is valid:

$$
\nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq 0
$$

Proof. We first prove the theorem for $d \in \mathcal{C}_{01}(\bar{z})$. We claim that $0 \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$. Indeed, since $\nabla G(\bar{z}) d \in$ cone $(Q-G(\bar{z}))$, there exists $\mu>0$ such that $\mu \nabla G(\bar{z}) d \in Q-G(\bar{z})$. From $0 \in Q-G(\bar{z})$ and the convexity of $Q-G(\bar{z})$, for any $0<\alpha<\mu$, we get

$$
\frac{\alpha}{\mu} \mu \nabla G(\bar{z}) d+\left(1-\frac{\alpha}{\mu}\right) .0 \in Q-G(\bar{z}) .
$$

This implies that $G(\bar{z})+\alpha \nabla G(\bar{z}) d \in Q$ for all $\alpha \in(0, \mu)$. Hence, $0 \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$ as required. We consider the following set

$$
\begin{aligned}
\Pi:= & \left\{\left(\nabla^{2} f_{1}(\bar{z}) d^{2}+\nabla f_{1}(\bar{z}) z+r_{1}, \ldots, \nabla^{2} f_{m}(\bar{z}) d^{2}+\nabla f_{m}(\bar{z}) z+r_{m}, \nabla^{2} G(\bar{z}) d^{2}+\nabla G(\bar{z}) z-v\right) \mid\right. \\
& \left.z \in Z, v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d), r_{i} \geq 0, i \in I\right\} .
\end{aligned}
$$

Thanks to Lemma 3.1, the set $I(\bar{z}, d)$ is nonempty. Thus, by choosing $\lambda_{i}=0$ and removing the component $i$-th of $\Pi$ for $i \in I \backslash I(\bar{z}, d)$, we may assume that $I(\bar{z}, d)=I$. We claim that $\Pi$ is a convex set with a nonempty interior and $(0,0) \notin \operatorname{int} \Pi$. The convexity of $\Pi$ follows directly from the convexity of $T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$ and $\mathbb{R}_{+}^{m}$. By the Robinson constraint qualification, there exists $\rho>0$ such that

$$
B_{E}(0, \rho) \subset \nabla G(\bar{z})\left[B_{Z}\right]-T^{b}(Q ; G(\bar{z})) \cap B_{E}
$$

Since $0 \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$ and [12, Proposition 3.1], we have

$$
T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}))=T^{b}\left(T^{b}(Q ; G(\bar{z})), \nabla G(\bar{z})\right)
$$

Putting $V=\nabla G(\bar{z})+B_{E}(0, \rho)$ and $\rho_{1}=1+\|\nabla G(\bar{z}) d\|$, we have

$$
\begin{aligned}
V & \subset \nabla G(\bar{z})\left[B_{Z}\right]-\left[T^{b}(Q ; G(\bar{z})) \cap B_{E}-\nabla G(\bar{z})\right] \\
& \subset \nabla G(\bar{z})\left[B_{Z}\right]-\left[T^{b}(Q ; G(\bar{z}))-\nabla G(\bar{z})\right] \cap B_{E}\left(0, \rho_{1}\right) \\
& \subset \nabla G(\bar{z})\left[B_{Z}\right]-T^{b}\left(T^{b}(Q ; G(\bar{z})), \nabla G(\bar{z})\right) \cap B_{E}\left(0, \rho_{1}\right) \\
& =\nabla G(\bar{z})\left[B_{Z}\right]-T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z})) \cap B_{E}\left(0, \rho_{1}\right) .
\end{aligned}
$$

By the Robinson constraint qualification,

$$
\begin{equation*}
E=\nabla G(\bar{z})[Z]-Q(G(\bar{z}))=\nabla G(\bar{z})[Z]-T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d) \tag{11}
\end{equation*}
$$

For each $i \in I$, we put

$$
\alpha_{i}=\nabla^{2} f_{i}(\bar{z}) d^{2}+\sup \left\{\nabla f_{i}(\bar{z}) z \mid z \in B_{Z}\left(0, \rho_{1}\right)\right\}<+\infty
$$

We then have

$$
\left(\alpha_{1},+\infty\right) \times\left(\alpha_{2},+\infty\right) \times \ldots \times\left(\alpha_{m},+\infty\right) \times \hat{V} \subset \Pi
$$

where $\hat{V}:=\nabla^{2} G(\bar{z}) d^{2}+V$. This implies that the interior of $\Pi$ is nonempty. We now show that $(0,0) \notin \operatorname{int} \Pi$. If otherwise, there exist $\epsilon_{1}>0, \epsilon_{2}>0, \ldots, \epsilon_{m}>0$ such that

$$
\left(-\epsilon_{1}, \epsilon_{1}\right) \times\left(-\epsilon_{2}, \epsilon_{2}\right) \times \ldots \times\left(-\epsilon_{m}, \epsilon_{m}\right) \times\{0\} \subset \Pi
$$

This implies that there exist $z \in Z, v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d), r_{i} \geq 0, i \in I$, satisfying

$$
\left\{\begin{array}{l}
\nabla^{2} f_{i}(\bar{z}) d^{2}+\nabla f_{i}(\bar{z}) z+r_{i}<0, \quad i \in I \\
\nabla^{2} G(\bar{z}) d^{2}+\nabla G(\bar{z}) z-v=0
\end{array}\right.
$$

Consequently, $z$ is a solution of the following system

$$
\left\{\begin{array}{l}
\nabla^{2} f_{i}(\bar{z}) d^{2}+\nabla f_{i}(\bar{z}) z<0, \quad i \in I \\
\nabla^{2} G(\bar{z}) d^{2}+\nabla G(\bar{z}) z \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)
\end{array}\right.
$$

contrary to Lemma 3.3, Since $(0,0) \notin \operatorname{int} \Pi$, we can separate $(0,0)$ from $\Pi$ by a hyperplane, i.e., there exists a functional $\left(\lambda, e^{*}\right) \in\left(\mathbb{R}^{m} \times E^{*}\right) \backslash\{(0,0)\}$ such that

$$
\begin{equation*}
\sum_{i=1}^{m} \lambda_{i}\left(\nabla^{2} f_{i}(\bar{z}) d^{2}+\nabla f_{i}(\bar{z}) z+r_{i}\right)+\left\langle e^{*}, \nabla^{2} G(\bar{z}) d^{2}+\nabla G(\bar{z}) z-v\right\rangle \geq 0 \tag{12}
\end{equation*}
$$

for all $r_{i} \geq 0, i \in I, z \in Z$, and $v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$. By (12), we have $\lambda_{i} \geq 0$ for all $i \in I$, i.e., $\lambda \in \mathbb{R}_{+}^{m}$. We claim that $\lambda$ is a nonzero vector. If otherwise, then we have

$$
\left\langle e^{*}, \nabla^{2} G(\bar{z}) d^{2}+\nabla G(\bar{z}) z-v\right\rangle \geq 0
$$

for all $z \in Z$, and $v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$, or, equivalently,

$$
\left\langle e^{*}, \nabla^{2} G(\bar{z}) d^{2}+w\right\rangle \geq 0
$$

for all $w \in \nabla G(\bar{z})(Z)-T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$. This and (11) imply that $e^{*}=0$, contrary to the fact that $\left(\lambda, e^{*}\right) \neq(0,0)$. We now rewrite (12) as follows

$$
\begin{equation*}
\left\langle\nabla_{z} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right), z\right\rangle+\nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d)+\sum_{i=1}^{m} \lambda_{i} r_{i}-\left\langle e^{*}, v\right\rangle \geq 0 \tag{13}
\end{equation*}
$$

for all $r_{i} \geq 0, i \in I, z \in Z$, and $v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$. It follows that $\nabla_{z} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)=0$. Putting $\nabla_{z} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)=0$ and $r_{i}=0, i \in I$, into (13), we get

$$
\nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq\left\langle e^{*}, v\right\rangle
$$

for all $v \in T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)$. Thus,

$$
\begin{equation*}
\nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq \sigma\left(e^{*}, T^{2 b}(Q ; G(\bar{z}), \nabla G(\bar{z}) d)\right) \geq 0 \tag{14}
\end{equation*}
$$

By dividing both sides of (14) by $|\lambda|$, we obtain that

$$
\sup _{\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})} \nabla_{z z}^{2} \mathcal{L}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq 0
$$

We now take any $d \in \mathcal{C}_{1 *}(\bar{z})$. Then, there exists a sequence $\left\{d_{k}\right\} \subset \mathcal{C}_{01}(\bar{z})$ converging to d. From what has already been proved, we have

$$
\begin{equation*}
\sup _{\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})} \nabla_{z z}^{2} \mathcal{L}\left(\bar{z}, \lambda, e^{*}\right)\left(d_{k}, d_{k}\right) \geq 0 \tag{15}
\end{equation*}
$$

Since the set $\Lambda_{1}(\bar{z})$ is compact in topology $\tau_{\mathbb{R}^{m}} \times \tau\left(E^{*}, E\right)$, the function

$$
\psi(d):=\sup _{\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})} \nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d)
$$

is continuous. Letting $k \rightarrow \infty$ in (15), we get

$$
\sup _{\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})} \nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq 0
$$

Again, by the compactness of $\Lambda_{1}(\bar{z})$, there exists $\left(\lambda, e^{*}\right) \in \Lambda_{1}(\bar{z})$ such that

$$
\nabla_{z z}^{2} \mathcal{L}_{1}\left(\bar{z}, \lambda, e^{*}\right)(d, d) \geq 0
$$

The proof is complete.

From Theorem 3.1 we want to ask whether the conclusion is still true if $\mathcal{C}_{1 *}(\bar{z})$ is replaced by $\mathcal{C}_{1}(\bar{z})$. Clearly, $\mathcal{C}_{1 *}(\bar{z}) \subseteq \mathcal{C}_{1}(\bar{z})$. In the case of single-objective ( $m=1$ ) under assumptions that $Q$ is polyhedric at $G(\bar{z})$ and $\nabla G(\bar{z})$ is surjective, [1, Proposition 3.54] showed that $\mathcal{C}_{1}(\bar{z})=\mathcal{C}_{1 *}(\bar{z})$. However, when $m>1$, the proof of Proposition 3.54 in [1] is collapsed. The reason is that the condition $\langle\lambda, \nabla f(\bar{z})\rangle=0$ with $\lambda \neq 0$ does not imply $\nabla f(\bar{z})=0$. We do not know whether the equality $\mathcal{C}_{1}(\bar{z})=\mathcal{C}_{1 *}(\bar{z})$ is valid. Therefore, we leave here the following conjecture.

- Conjecture: Suppose that $\nabla G(\bar{z}): Z \rightarrow E$ is surjective and $Q$ is polyhedric at $G(\bar{z})$. If $\bar{z}$ is a locally weak Pareto solution of the (MP1), then $\mathcal{C}_{1}(\bar{z})=\mathcal{C}_{1 *}(\bar{z})$.


## 4. Abstract multi-objective optimal control problems

Let $E_{0}, E, X$ and $U$ be Banach spaces and $Q$ be a nonempty closed convex set in $E$. Define $Z=X \times U$ and assume that

$$
\begin{aligned}
& I: X \times U \rightarrow \mathbb{R}^{m} \\
& F: X \times U \rightarrow E_{0}, \\
& G: X \times U \rightarrow E
\end{aligned}
$$

are given mappings. We consider the following multi-objective optimal control problem of finding a control $u \in U$ and the corresponding state $x \in X$ which solve

$$
\begin{array}{ll}
\operatorname{Min}_{\mathbb{R}_{+}^{m}} & I(x, u), \\
\text { s.t. } & F(x, u)=0,  \tag{MP2}\\
& G(x, u) \in Q .
\end{array}
$$

We denote by $\Phi$ the feasible of the (MP2) and put

$$
D=\{(x, u) \in Z \mid F(x, u)=0\} .
$$

Fix $z_{0}=\left(y_{0}, u_{0}\right) \in \Phi$. We denoted by $\Lambda_{2}\left(z_{0}\right)$ the set of multipliers $\left(\lambda, v^{*}, e^{*}\right) \in \mathbb{R}_{+}^{m} \times E_{0}^{*} \times E^{*}$ with $|\lambda|=1$, which satisfies the following conditions

$$
\nabla_{z} \mathcal{L}_{2}\left(z, \lambda, v^{*}, e^{*}\right)=0, e^{*} \in N\left(Q, G\left(z, w_{0}\right)\right)
$$

where $\mathcal{L}_{2}\left(z, \lambda, v^{*}, e^{*}\right)$ is the Lagrangian which is given by

$$
\mathcal{L}_{2}\left(z, v^{*}, e^{*}\right)=\langle\lambda, I(z)\rangle+\left\langle v^{*}, F(z)\right\rangle+\left\langle e^{*}, G(z)\right\rangle .
$$

We also denote by $\mathcal{C}_{2}\left(z_{0}\right)$ the closure of $\mathcal{C}_{02}\left(z_{0}\right)$ in $Z$, where

$$
\mathcal{C}_{02}\left(z_{0}\right):=\left\{d \in Z \mid \nabla I\left(z_{0}\right) d \in-\mathbb{R}_{+}^{m}, \nabla F\left(z_{0}\right) d=0, \nabla G\left(z_{0}\right) d \in \operatorname{cone}\left(Q-G\left(z_{0}\right)\right\} .\right.
$$

The set $\mathcal{C}_{2}\left(z_{0}\right)$ is called the critical cone of the (MP2) at $z_{0}$.
Let us introduce the following assumptions:
(A1) There exist positive numbers $r_{1}, r_{1}^{\prime}$ such that the mapping $I(\cdot, \cdot), F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are twice continuously Fréchet differentiable on $B_{X}\left(x_{0}, r_{1}\right) \times B_{U}\left(u_{0}, r_{1}^{\prime}\right)$;
(A2) The mapping $F_{x}\left(z_{0}\right)$ is bijective;
$(A 3) \nabla G\left(z_{0}\right)\left(T\left(D ; z_{0}\right)\right)=E$.
From assumptions (A1) and (A3), we have that $F(\cdot, \cdot)$ is continuously differentiable on $B_{X}\left(x_{0}, r_{1}\right) \times B_{U}\left(u_{0}, r_{1}^{\prime}\right)$ and $F_{x}\left(z_{0}\right)$ is bijective. By the implicit function theorem (see [26, Theorem 4.E]), there exist balls $B_{X}\left(x_{0}, r_{2}\right), B_{U}\left(u_{0}, r_{2}^{\prime}\right)$ with $r_{2}<r_{1}, r_{2}^{\prime}<r_{1}^{\prime}$ such that for each $u \in B_{U}\left(u_{0}, r_{2}^{\prime}\right)$, the equation

$$
F(x, u)=0
$$

has a unique solution $x=\zeta(u) \in B_{X}\left(x_{0}, r_{2}\right)$. Moreover, the mapping

$$
\zeta: B_{U}\left(u_{0}, r_{2}^{\prime}\right) \rightarrow B_{X}\left(x_{0}, r_{2}\right)
$$

is of class $C^{2}$ and $\zeta\left(u_{0}\right)=x_{0}$. Thus,

$$
\begin{equation*}
F(\zeta(u), u)=0 \quad \forall u \in B_{U}\left(u_{0}, r_{2}^{\prime}\right) \tag{16}
\end{equation*}
$$

We now define the following mappings:

$$
\begin{array}{ll}
J: U \rightarrow \mathbb{R}^{m}, & J(u):=J(\zeta(u), u) \\
H: U \rightarrow E, & H(u):=G(\zeta(u), u) \tag{17}
\end{array}
$$

Then we can show that $\left(x_{0}, u_{0}\right)$ is a locally weak Pareto solution of the (MP2) if and only if $u_{0}$ is a locally weak Pareto solution of the following problem:

$$
\begin{align*}
& \operatorname{Min}_{\mathbb{R}_{+}^{m}} J(u)  \tag{MP3}\\
& \text { s.t. } H(u) \in Q
\end{align*}
$$

Problem (MP3) is associated with the Lagrangian

$$
\mathcal{L}_{3}\left(u, \lambda, e^{*}\right)=\lambda J(u)+e^{*} H(u)
$$

Given a feasible point $u_{0}$ of the (MP3), we define

$$
\mathcal{C}_{03}\left(u_{0}\right)=\left\{u \in U \mid \nabla J\left(u_{0}\right) u \in-\mathbb{R}_{+}^{m}, \nabla H\left(u_{0}\right) u \in \operatorname{cone}\left(Q-H\left(u_{0}\right)\right)\right\}
$$

and $\mathcal{C}_{3}\left(u_{0}\right)=\overline{\mathcal{C}_{03}\left(u_{0}\right)}$ the interior critical cone and the critical cone at $u_{0}$, respectively.
The following theorem provides second-order necessary optimality conditions for the (MP2).
Theorem 4.1. Suppose that $z_{0}$ is a feasible point of the (MP2) and assumptions $(A 1)-(A 3)$ are satisfied. If $z_{0}$ is a locally weak Pareto solution of the (MP2), then, for each $d \in \mathcal{C}_{2}\left(z_{0}\right)$, there exists a nonzero triple $\left(\lambda, v^{*}, e^{*}\right) \in \Lambda_{2}\left(z_{0}\right)$ such that

$$
\nabla_{z}^{2} \mathcal{L}_{2}\left(z_{0}, e^{*}, v^{*}\right)(d, d)=\left\langle\lambda J_{z z}\left(z_{0}\right) d, d\right\rangle+\left\langle v^{*} F_{z z}\left(z_{0}\right) d, d\right\rangle+\left\langle e^{*} G_{z z}\left(z_{0}\right) d, d\right\rangle \geq 0
$$

Proof. Since $z_{0}=\left(x_{0}, u_{0}\right)$ is a locally weak Pareto solution to the (MP2), $u_{0}$ is a locally weak Pareto solution of the (MP3). By assumption $(A 2), \nabla F\left(z_{0}\right)$ is surjective. Indeed, for any $v \in E_{0}$, there exists $x \in X$ such that $F_{x}(\bar{z}) x=v$. Hence, $(x, 0) \in X \times U$ and $\nabla F(\bar{z})(x, 0)=v$. This means that $\nabla F(\bar{z})$ is surjective. From this and [16, Lemma 2.2], it follows that

$$
T\left(D ; z_{0}\right)=\left\{(x, u) \in Z \mid F_{x}\left(z_{0}\right) x+F_{u}\left(z_{0}\right) u=0\right\}=\left\{\left(\zeta^{\prime}\left(u_{0}\right) u, u\right) \mid u \in U\right\}
$$

Combining this with $(A 3)$, we get

$$
\begin{aligned}
E & \subseteq\left\{\nabla_{x} G\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u+\nabla_{u} G\left(z_{0}\right) u \mid u \in U\right\} \\
& \subseteq \nabla H\left(u_{0}\right)(U)-\operatorname{cone}\left(Q-H\left(u_{0}\right)\right)
\end{aligned}
$$

where $H$ is defined by (17). Hence the Robinson constraint qualification for the (MP3) is satisfied at $u_{0}$.

Fix any $d=(x, u) \in \mathcal{C}_{2}\left(z_{0}\right)$. Then there exists a sequence $\left\{d_{k}\right\}=\left\{\left(x_{k}, u_{k}\right)\right\} \subset \mathcal{C}_{02}\left(z_{0}\right)$ such that $d_{k} \rightarrow d$. Since $F_{x}\left(z_{0}\right) x_{k}+F_{u}\left(z_{0}\right) u_{k}=0$, we have $x_{k}=\zeta^{\prime}\left(u_{0}\right) u_{k}$ and so $u_{k} \in \mathcal{C}_{03}\left(u_{0}\right)$.

Consequently, $u \in \mathcal{C}_{3}\left(u_{0}\right)$. By Theorem [3.1, there exists a multipliers $\left(\lambda, e^{*}\right) \in \Lambda_{*}\left(u_{0}\right)$ such that the following conditions hold:
(a) $\lambda \nabla J\left(u_{0}\right)+\nabla e^{*} H\left(u_{0}\right)=0, e^{*} \in N\left(Q, H\left(u_{0}\right)\right)$,
(b) $\left\langle\lambda \nabla^{2} J\left(u_{0}\right) u, u\right\rangle+\left\langle e^{*} \nabla^{2} H\left(u_{0}\right) u, u\right\rangle \geq 0$.

Note that from (16), we have

$$
\begin{equation*}
F(\zeta(v), v)=0 \quad \forall v \in B_{U}\left(u_{0}, r_{3}^{\prime}\right) \tag{18}
\end{equation*}
$$

Taking first-order derivative on both sides, we get

$$
F_{x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right)+F_{u}\left(z_{0}\right)=0
$$

and so

$$
\begin{equation*}
\zeta^{\prime}\left(u_{0}\right)^{*}=-F_{u}\left(z_{0}\right)^{*}\left(F_{x}^{*}\left(z_{0}\right)\right)^{-1} . \tag{19}
\end{equation*}
$$

From (a), we have

$$
\lambda I_{x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right)+\lambda I_{u}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right)+e^{*} G_{u}\left(z_{0}\right)=0
$$

or, equivalently,

$$
\begin{equation*}
\zeta^{\prime}\left(u_{0}\right)^{*}\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right)=-\left(\lambda I_{u}\left(z_{0}\right)+e^{*} G_{u}\left(z_{0}\right)\right) \tag{20}
\end{equation*}
$$

Let us put

$$
\begin{equation*}
\phi=\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right), v^{*}=-\left(F_{x}^{*}\left(z_{0}\right)\right)^{-1} \phi \tag{21}
\end{equation*}
$$

Then, from (19) and (20), we have

$$
-F_{u}\left(z_{0}\right)^{*}\left(F_{x}^{*}\left(z_{0}\right)\right)^{-1} \phi=-\left(\lambda I_{u}\left(z_{0}\right)+e^{*} G_{u}\left(z_{0}\right)\right) .
$$

Consequently,

$$
\left\{\begin{array}{l}
F_{u}\left(z_{0}\right)^{*} v^{*}+\lambda I_{u}\left(z_{0}\right)+e^{*} G_{u}\left(z_{0}\right)=0 \\
F_{x}\left(z_{0}\right)^{*} v^{*}=-\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right)
\end{array}\right.
$$

This is equivalent to

$$
\nabla I\left(z_{0}\right)^{*} \lambda^{T}+\nabla F\left(z_{0}\right)^{*} v^{*}+\nabla G\left(z_{0}\right)^{*} e^{*}=0 .
$$

Hence we have $\left(\lambda, v^{*}, e^{*}\right) \in \Lambda_{2}\left(z_{0}\right)$.
Let us define the following function

$$
\psi(t):=\mathcal{L}_{3}\left(u_{0}+t u, \lambda, e^{*}\right)=\lambda J\left(u_{0}+t u\right)+e^{*} H\left(u_{0}+t u\right),-1<t<1, u \in \mathcal{C}_{3}\left(u_{0}\right) .
$$

Then, by (b), we have

$$
\psi^{\prime \prime}(0)=\nabla_{u}^{2} \mathcal{L}_{3}\left(u_{0}\right)(u, u)=\left\langle\lambda \nabla^{2} J\left(u_{0}\right) u, u\right\rangle+\left\langle e^{*} \nabla^{2} H\left(u_{0}\right) u, u\right\rangle \geq 0
$$

On the other hand, by simple calculation, we get

$$
\begin{align*}
\psi^{\prime \prime}(0)= & \left\langle\lambda I_{x x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, \zeta^{\prime}\left(u_{0}\right) u\right\rangle+\left\langle\lambda I_{x u}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, u\right\rangle+\left\langle\lambda I_{u x}\left(z_{0}\right) u, \zeta^{\prime}\left(u_{0}\right) u\right\rangle \\
& +\left\langle\lambda I_{u u}\left(z_{0}\right) u, u\right\rangle+\left\langle e^{*} G_{x x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, \zeta^{\prime}\left(u_{0}\right) u\right\rangle+\left\langle e^{*} G_{x u}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, u\right\rangle \\
& +\left\langle e^{*} G_{u x}\left(z_{0}\right) u, \zeta^{\prime}\left(u_{0}\right) u\right\rangle+\left\langle e^{*} G_{u u}\left(z_{0}\right) u, u\right\rangle+\left\langle\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right) \zeta^{\prime \prime}\left(u_{0}\right) u, u\right\rangle \\
= & \left\langle\lambda I_{x x}\left(z_{0}\right) x, x\right\rangle+\left\langle\lambda I_{x u}\left(z_{0}\right) x, u\right\rangle+\left\langle\lambda I_{u x}\left(z_{0}\right) u, x\right\rangle+\left\langle\lambda I_{u u}\left(z_{0}\right) u, u\right\rangle \\
& +\left\langle e^{*} G_{x x}\left(z_{0}\right) x, x\right\rangle+\left\langle e^{*} G_{x u}\left(z_{0}\right) x, u\right\rangle+\left\langle e^{*} G_{u x}\left(z_{0}\right) u, x\right\rangle+\left\langle e^{*} G_{u u}\left(z_{0}\right) u, u\right\rangle \\
& +\left\langle\left(\zeta^{\prime \prime}\left(u_{0}\right) u\right)^{*}\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right), u\right\rangle . \tag{22}
\end{align*}
$$

Taking second-order derivatives on both sides of (18) at $u_{0}$ and acting on $u \in \mathcal{C}^{\prime \prime}\left(u_{0}\right)$ and $v \in U$, we obtain

$$
\begin{aligned}
\left\langle F_{x}\left(z_{0}\right) \zeta^{\prime \prime}\left(u_{0}\right) u, v\right\rangle+\left\langle F_{x x}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, \zeta^{\prime}\left(u_{0}\right) v\right\rangle & +\left\langle F_{x u}\left(z_{0}\right) \zeta^{\prime}\left(u_{0}\right) u, v\right\rangle+ \\
& +F_{u x}\left(z_{0}\right)\left(u, \zeta^{\prime}\left(z_{0}\right) v\right)+F_{u u}\left(z_{0}\right)(u, v)=0
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\left\langle F_{x}\left(z_{0}\right) \zeta^{\prime \prime}\left(u_{0}\right) u, v\right\rangle & = \\
- & {\left[\left\langle F_{x x}\left(z_{0}\right) x, \zeta^{\prime}\left(u_{0}\right) v\right\rangle+\left\langle F_{x u}\left(z_{0}\right) x, v\right\rangle+\left\langle F_{u x}\left(z_{0}\right) u, \zeta^{\prime}\left(z_{0}\right) v\right\rangle+\left\langle F_{u u}\left(z_{0}\right) u, v\right\rangle\right] . }
\end{aligned}
$$

It follows that

$$
\left(\zeta^{\prime \prime}\left(u_{0}\right) u\right)^{*}=-\left[F_{x x}\left(z_{0}\right) x \zeta^{\prime}\left(u_{0}\right)+F_{x u}\left(z_{0}\right) x+F_{u x}\left(z_{0}\right) u \zeta^{\prime}\left(z_{0}\right)+F_{u u}\left(z_{0}\right) u\right]^{*}\left(F_{x}\left(z_{0}\right)^{*}\right)^{-1}
$$

Combining this with formula (21), we have

$$
\begin{aligned}
& \left(\zeta^{\prime \prime}\left(u_{0}\right) u\right)^{*}\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right)=-\left(\zeta^{\prime \prime}\left(u_{0}\right) u\right)^{*} \phi \\
& \quad=-\left[F_{x x}\left(z_{0}\right) x \zeta^{\prime}\left(u_{0}\right)+F_{x u}\left(z_{0}\right) x+F_{u x}\left(z_{0}\right) u \zeta^{\prime}\left(z_{0}\right)+F_{u u}\left(z_{0}\right) u\right]^{*}\left(F_{x}\left(z_{0}\right)^{*}\right)^{-1} \phi \\
& \quad=-\left[F_{x x}\left(z_{0}\right) x \zeta^{\prime}\left(u_{0}\right)+F_{x u}\left(z_{0}\right) x+F_{u x}\left(z_{0}\right) u \zeta^{\prime}\left(z_{0}\right)+F_{u u}\left(z_{0}\right) u\right]^{*} v^{*}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left\langle\left(\zeta^{\prime \prime}\left(u_{0}\right) u\right)^{*}\right. & \left.\left(\lambda I_{x}\left(z_{0}\right)+e^{*} G_{x}\left(z_{0}\right)\right), u\right\rangle= \\
& =\left\langle v^{*} F_{x x}\left(z_{0}\right) x, \zeta^{\prime}\left(u_{0}\right) u\right\rangle+\left\langle v^{*} F_{x u}\left(z_{0}\right) x, u\right\rangle+\left\langle F_{u x}\left(z_{0}\right) u, \zeta^{\prime}\left(z_{0}\right) u\right\rangle+\left\langle v^{*} F_{u u}\left(z_{0}\right) u, u\right\rangle \\
& =\left\langle v^{*} F_{x x}\left(z_{0}\right) x, x\right\rangle+\left\langle v^{*} F_{x u}\left(z_{0}\right) x, u\right\rangle+\left\langle F_{u x}\left(z_{0}\right) u, x\right\rangle+\left\langle v^{*} F_{u u}\left(z_{0}\right) u, u\right\rangle .
\end{aligned}
$$

Inserting this term into (22), we obtain

$$
\begin{aligned}
\left\langle\lambda \nabla^{2} J\left(u_{0}\right) u, u\right\rangle & +\left\langle e^{*} \nabla^{2} H\left(u_{0}\right) u, u\right\rangle= \\
& =\left\langle\lambda \nabla^{2} I\left(z_{0}\right) d, d\right\rangle+\left\langle v^{*} \nabla^{2} F\left(z_{0}\right) d, d\right\rangle+\left\langle e^{*} \nabla^{2} G\left(z_{0}\right) d, d\right\rangle \geq 0 .
\end{aligned}
$$

The proof is complete.

## 5. Proofs of main results

5.1. Proof of Theorem 2.1. For the proof, we first put

$$
\begin{aligned}
& X=C\left([0,1], \mathbb{R}^{n}\right), U=L^{\infty}\left([0,1], \mathbb{R}^{l}\right), \\
& E_{0}=C\left([0,1], \mathbb{R}^{n}\right), E=L^{\infty}([0,1], \mathbb{R}),
\end{aligned}
$$

and define the following mappings

$$
\begin{aligned}
& F: X \times U \rightarrow E_{0}, F(x, u):=x-x_{0}-\int_{0}^{(\cdot)} \varphi(s, x(s), u(s)) d s \\
& G: X \times U \rightarrow E, G(x, u):=g(\cdot, x(\cdot), u(\cdot))
\end{aligned}
$$

The problem (MCP) can be formulated in the form of the problem (MP3). Therefore, we can apply Theorem 4.1 for the (MCP) in order to derive necessary optimality conditions.

Step 1. Verification of assumptions ( $A 1$ )-(A3).

- Verification of $(A 1)$. From $(H 1)$ we see that the mapping $I, F$ and $G$ are of class $C^{2}$ around $\bar{z}$. Hence, $(A 1)$ is valid. Here $\nabla I_{j}(\bar{z}), \nabla^{2} I(\bar{z}), \nabla F(\bar{z}), \nabla^{2} F(\bar{z}), \nabla G(\bar{z})$ and $\nabla^{2} G(\bar{z})$ are defined by:

$$
\begin{aligned}
& I_{j x}(\bar{z}) x=\int_{0}^{1} L_{j x}[s] x(s) d s, I_{j u}(\bar{z}) u=\int_{0}^{1} L_{j u}[s] u(s) d s \\
& \left.F_{x}(\bar{z}) x=x-\int_{0}^{(\cdot)} \varphi_{x}[s] x(s) d s, F_{u}(\bar{z}) u=-\int_{0}^{(\cdot)} \varphi_{u}[s] u(s) d s\right), \\
& \nabla G(\bar{z})=\left(G_{x}(\bar{z}), G_{u}(\bar{z})\right)=\left(g_{x}[\cdot], g_{u}[\cdot]\right), \\
& \left\langle\nabla^{2} I_{j}(\bar{z}) z, z\right\rangle=\int_{0}^{1}\left(\nabla^{2} L_{j}[s] z(s), z(s)\right) d s \\
& \left\langle\nabla^{2} F(\bar{z}) z, z\right\rangle=-\int_{0}^{(\cdot)}\left(\nabla^{2} \varphi[s] z(s), z(s)\right) d s
\end{aligned}
$$

for all $z=(x, u) \in Z$, and

$$
\nabla^{2} G(\bar{z})=\left[\begin{array}{ll}
g_{x x}[\cdot] & g_{x u}[\cdot] \\
g_{u x}[\cdot] & g_{u u}[\cdot]
\end{array}\right]
$$

- Verification of $(A 2)$. Taking any $v \in E_{0}$, we consider equation $F_{x}(\bar{z}) x=v$. This equation is equivalent to

$$
x=\int_{0}^{(\cdot)} \varphi_{x}[s] x(s) d s+v
$$

By assumption $(H 1)$, we have $\varphi_{x}[\cdot] \in L^{\infty}\left([0,1], \mathbb{R}^{n}\right)$. By [9, Lemma 1, p. 51], the equation has a unique solution $x \in X$. Hence ( $A 2$ ) is valid.

- Verification of $(A 3)$. Let $D:=\{(x, u) \in Z \mid F(z)=0\}$. Under assumption (A2), the mapping $\nabla F(\bar{z}): X \times U \rightarrow E_{0}$ is surjective. This implies that

$$
T(D ; \bar{z})=\left\{(x, u) \in Z \mid F_{x}(\bar{z}) x+F_{u}(\bar{z}) u=0\right\}
$$

Therefore, assumption $(A 3)$ is amount to saying that for each $v \in E$, there exists $(x, u) \in Z$ satisfying

$$
\begin{align*}
& F_{x}(\bar{z}) x+F_{u}(\bar{z}) u=0,  \tag{23}\\
& G_{x}(\bar{z}) x+G_{u}(\bar{z}) u=v . \tag{24}
\end{align*}
$$

We will find $u$ in the form $u=\left(0,0, \ldots, u_{i_{0}}, 0, \ldots, 0\right)$. Consider the following equation

$$
F_{x}(\bar{z}) x+F_{u}(\bar{z}) \frac{v-G_{x}(\bar{z}) x}{g_{i_{0} u}[\cdot]}=0
$$

This equation is equivalent to

$$
x=\int_{0}^{(\cdot)}\left(\varphi_{x}[s]+\varphi_{u}[s] \frac{g_{x}[s]}{g_{i_{0} u}[\cdot]}\right) x(s) d s+\int_{0}^{(\cdot)} \varphi_{u}[s] \frac{v}{g_{i_{0} u}[s]} d s .
$$

By (H2), $\varphi_{u}[\cdot] \cdot \frac{g_{x}[\cdot]}{\left.g_{i_{0} u} \cdot\right]}$ and $\varphi_{u}[\cdot] \frac{v}{g_{i_{0} u}[\cdot]}$ belong to $L^{\infty}\left([0,1], \mathbb{R}^{n}\right)$. Thanks to [9, Lemma 1, p. 51], the above equation has a unique solution $x \in X$. Choosing $u=\left(0,0, \ldots, u_{i_{0}}, 0, \ldots, 0\right)$ with

$$
u_{i_{0}}=\frac{v-G_{x}(\bar{z}) x}{g_{i_{0} u}[\cdot]} .
$$

We see that $(x, u)$ satisfies equations (23)-(24). Hence assumption $(A 3)$ is fulfilled.
Step 2. Deriving optimality conditions.
Let $\mathcal{L}\left(z, \lambda, v^{*}, e^{*}\right)=\lambda I(x, u)+v^{*} F(x, u)+e^{*} G(x, u)$ be the Lagrangian associated with the (MCP). According to Theorem 4.1, for each $z=(\tilde{x}, \tilde{u}) \in \mathcal{C}(\bar{z})$, there exist multipliers $\lambda \in \mathbb{R}_{+}^{m}$ with $|\lambda|=1, v^{*} \in E_{0}^{*}$ and $e^{*} \in E^{*}$ such that the following conditions are valid:

$$
\begin{align*}
& e^{*} \in N(Q, G(\bar{z})),  \tag{25}\\
& \lambda I_{x}(\bar{z})+v^{*} F_{x}(\bar{z})+e^{*} G_{x}(\bar{z})=0  \tag{26}\\
& \lambda I_{u}(\bar{z})+v^{*} F_{u}(\bar{z})+e^{*} G_{u}(\bar{z})=0,  \tag{27}\\
& \left\langle\lambda \nabla^{2} I(\bar{z}) z, z\right\rangle+\left\langle v^{*} \nabla^{2} F(\bar{z}) z, z\right\rangle+\left\langle e^{*} \nabla^{2} G(\bar{z}) z, z\right\rangle \geq 0 . \tag{28}
\end{align*}
$$

Here $v^{*}$ is a signed Radon measure and $e^{*}$ is a signed and finite additive measure on $[0,1]$ which is absolutely continuous w.r.t the Lebesgue measure $|\cdot|$ on $[0,1]$. By Riesz's Representation (see [9, Chapter 01, p. 19] and [13, Theorem 3.8, p. 73]), there exists a vector function of bounded variation $\nu$, which is continuous from the right and vanish at zero such that

$$
\left\langle v^{*}, y\right\rangle=\int_{0}^{1} y(t) d \nu(t) \quad \forall y \in E_{0}
$$

where $\int_{0}^{1} y(t) d \nu(t)$ is the Riemann-Stieltjes integral.
Define $\bar{p}:[0,1] \rightarrow \mathbb{R}^{n}$ by setting

$$
\bar{p}(t)=\nu((t, 1])=\nu(t)-\nu(1)
$$

Clearly, $\bar{p}(1)=0$ and the function $\bar{p}$ is of bounded variation. By the Fubini Theorem, for each $x \in C\left([0,1], \mathbb{R}^{n}\right)$, we have

$$
\begin{align*}
\left\langle v^{*} F_{x}(\bar{z}), x\right\rangle & =\left\langle v^{*}, x-\int_{0}^{(\cdot)} \varphi_{x}[s] x(s) d s\right\rangle \\
& =\int_{0}^{1} x^{T}(t) d \nu(t)-\int_{0}^{1} \int_{0}^{t} \varphi_{x}[s] x(s) d s d \nu(t) \\
& =\int_{0}^{1} x^{T}(t) d \nu(t)-\int_{0}^{1} \varphi_{x}[s] x(s) d s \int_{s}^{1} d \nu(t) \\
& =\int_{0}^{1} x^{T}(t) d \nu(t)+\int_{0}^{1} \varphi_{x}[s] x(s) \bar{p}(s) d s \tag{29}
\end{align*}
$$

Similarly, for any $u \in L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$, we get

$$
\begin{equation*}
\left\langle v^{*} F_{u}(\bar{z}), u\right\rangle=\int_{0}^{1} \int_{0}^{t} \varphi_{u}[s] u(s) d s d \nu(t)=\int_{0}^{1} \bar{p}(s)^{T} \varphi_{u}[s] u(s) d s \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle v^{*} \nabla^{2} F(\bar{z}) z, z\right\rangle=\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[s] z(s), z(s)\right) d s \tag{31}
\end{equation*}
$$

From (27) and (30), we have

$$
\begin{equation*}
\int_{0}^{1} \lambda L_{u}[s] u(s) d s+\int_{0}^{1} \bar{p}(s)^{T} \varphi_{u}[s] u(s) d s+\left\langle e^{*}, G_{u}(\bar{z}) u\right\rangle=0 \quad \forall u \in U \tag{32}
\end{equation*}
$$

Let us claim that $e^{*}$ can be represented by a density in $L^{1}([0,1], \mathbb{R})$. Indeed, let $\bar{d}$ be an arbitrary element of $T(Q ; G(\bar{z}))$. Then, by assumption (H2), we have

$$
\begin{aligned}
\left|\omega g_{u}[t]\right|^{2}+\left[(\omega \bar{d}(t))^{+}\right]^{2} & \geq\left|\omega g_{u}[t]\right|^{2} \\
& \geq \omega^{2}\left|g_{u_{i_{0}}}[t]\right|^{2} \\
& \geq \alpha^{2} \omega^{2}
\end{aligned}
$$

for all $\omega \in \mathbb{R}$ and for a.e. $t \in[0,1]$. Thanks to [24, Theorem 3.2], there exist measurable mappings $a:[0,1] \times \mathbb{R} \rightarrow \mathbb{R}^{l}, c:[0,1] \times \mathbb{R} \rightarrow[0, \infty)$ and a constant $R>0$ such that

$$
G_{u}(\bar{z}) a(t, \omega)=g_{u}[t] a(t, \omega)=\omega+c(t, \omega) \bar{d}(t)
$$

and

$$
|a(t, \omega)| \leq R|\omega|,|c(t, \omega)| \leq R|\omega|
$$

for all $\omega \in \mathbb{R}$. We now take any $v \in E$ and put $u(t)=a(t, v(t))$. Then $u \in U$ and we have

$$
\left\langle e^{*}, G_{u}(\bar{z}) a(\cdot, v(\cdot))\right\rangle=\left\langle e^{*}, v\right\rangle+\left\langle e^{*}, c(\cdot, v) \bar{d}\right\rangle \leq\left\langle e^{*}, v\right\rangle
$$

because $e^{*} \in N(Q, G(\bar{z}))$ and $c(\cdot, v) \bar{d} \in T(Q ; G(\bar{z}))$. Inserting $u(t)=a(t, v(t))$ into (32), we get

$$
\int_{0}^{1} \lambda L_{u}[s] a(s, v(s)) d s+\int_{0}^{1} \bar{p}(s)^{T} \varphi_{u}[s] a(s, v(s)) d s+\left\langle e^{*}, v\right\rangle \geq 0 \quad \forall v \in E .
$$

This implies that

$$
\left|\left\langle e^{*}, v\right\rangle\right| \leq R \int_{0}^{1}\left|\lambda L_{u}[s]\right||v(s)| d s+R \int_{0}^{1}\left|\bar{p}(s)^{T} \varphi_{u}[s] \| v(s)\right| d s
$$

From this and [9, Proposition 5, p. 348], there is a function $\theta \in L^{1}([0,1], \mathbb{R})$ such that

$$
\begin{equation*}
\left\langle e^{*}, v\right\rangle=\int_{0}^{1} \theta(t) v(t) d t \quad \forall v \in E \tag{33}
\end{equation*}
$$

Therefore the claim is justified.
Based on the representation of $e^{*}$, (25), and (32), we obtain assertions (i) and (iii). Also, from (33), (29) and (26), we get

$$
\int_{0}^{1} \lambda^{T} L_{x}[s] x(s) d s+\int_{0}^{1} x^{T}(t) d \nu(t)+\int_{0}^{1} \bar{p}(t)^{T} \varphi_{x}[s] x(s) d s+\int_{0}^{1} \theta(s)^{T} g_{x}[s] x(s) d s=0
$$

for all $x \in X$. This is equivalent to

$$
\begin{equation*}
-\int_{0}^{1} x^{T}(t) d \nu(t)=\int_{0}^{1}\left(\lambda^{T} L_{x}[s]+\bar{p}(s)^{T} \varphi_{x}[s]+\theta(s)^{T} g_{x}[s]\right) x(s) d s \quad \forall x \in X \tag{34}
\end{equation*}
$$

We now fix any vector $\xi \in \mathbb{R}^{n}$ and $t \in[0,1]$. Define $x_{t}(s)=\xi \chi_{(t, 1]}(s)$, where $\chi_{(t, 1]}(\cdot)$ is the indicator function of $(t, 1]$. Let us define

$$
\vartheta(s)=\lambda^{T} L_{x}[s]+\bar{p}(t)^{T} \varphi_{x}[s]+\theta(s)^{T} g_{x}[s] .
$$

Then, $\vartheta(\cdot) \in L^{1}\left([0,1], \mathbb{R}^{n}\right)$ and so are the functions $\vartheta(s)$ and $s \vartheta(s)$. By the Lebesgue Differentiation Theorem (see [23, Theorem 7.15]), these functions have Lebesgue points for a.e. on $[0,1]$. Let us denote by $P$ and $P^{\prime}$ the sets of Lebesgue points of $\vartheta(s)$ and $s \vartheta(s)$, respectively. Then $\left|P \cap P^{\prime}\right|=1$ and we have the following key lemma.

Lemma 5.1. For each $t \in P \cap P^{\prime}$, the following equality is valid:

$$
-\int_{0}^{1} x_{t}^{T}(s) d \nu(s)=\int_{0}^{1}\left(\lambda^{T} L_{x}[s]+\bar{p}(s)^{T} \varphi_{x}[s]+\theta(s)^{T} g_{x}[s]\right) x_{t}(s) d s
$$

Proof. Note that any function with bounded variation as well as any signed Radon measure can be represented as the difference of two increasing functions, and the difference of two positive Radon measures, respectively (see [23, Corollary 2.7] and [22, Lemma 13.6]). Therefore, we can assume that $\nu$ is increasing, right continuous and of bounded variation.

For each $\epsilon$ with $t<\epsilon<1$, we define a function $x_{\epsilon}$ as follows.

$$
x_{\epsilon}(s)= \begin{cases}\xi & \text { if } s \in[\epsilon, 1] \\ \frac{\xi(s-t)}{\epsilon-t} & \text { if } s \in[t, \epsilon] \\ 0 & \text { if } s \in[0, t]\end{cases}
$$

Then, $x_{\epsilon} \in C\left([0,1], \mathbb{R}^{n}\right)$. By (34), we have

$$
-\int_{0}^{1} x_{\epsilon}^{T}(s) d \nu(s)=\int_{0}^{1}\left(\lambda^{T} L_{x}[s]+\bar{p}(t)^{T} \varphi_{x}[s]+\theta(s)^{T} g_{x}[s]\right) x_{\epsilon}(s) d s
$$

or, equivalently,

$$
\begin{equation*}
-\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} d \nu(s)-\int_{\epsilon}^{1} \xi d \nu(s)=\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} \vartheta(s) d s+\int_{\epsilon}^{1} \xi \vartheta(s) d s \tag{35}
\end{equation*}
$$

By Mean Value Theorem (see [23, Theorem 2.27, p. 33]), there is a point $t^{\prime} \in[t, \epsilon]$ such that

$$
\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} d \nu(s)=\frac{\xi\left(t^{\prime}-t\right)}{\epsilon-t}(\nu(\epsilon)-\nu(t))
$$

Hence

$$
\begin{aligned}
\left|\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} d \nu(s)\right| & =\frac{|\xi|\left|\left(t^{\prime}-t\right)\right|}{\epsilon-t}(\nu(\epsilon)-\nu(t)) \\
& \leq \frac{|\xi| \mid \epsilon-t) \mid}{\epsilon-t}(\nu(\epsilon)-\nu(t)) \\
& \leq|\xi|(\nu(\epsilon)-\nu(t))
\end{aligned}
$$

By letting $\epsilon \rightarrow t^{+}$and using the right continuity of $\nu$, we see that

$$
\begin{equation*}
\left|\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} d \nu(s)\right| \rightarrow 0 \text { as } \epsilon \rightarrow t^{+} \tag{36}
\end{equation*}
$$

Also, we have

$$
\left|\int_{t}^{1} \xi d \nu(s) d s-\int_{\epsilon}^{1} \xi d \nu(s)\right|=\left|\int_{t}^{\epsilon} \xi d \nu(s)\right| \leq|\xi|(\nu(\epsilon)-\nu(t)) \rightarrow 0 \text { as } \epsilon \rightarrow t^{+}
$$

Consequently,

$$
\int_{\epsilon}^{1} \xi d \nu(s) d s \rightarrow \int_{t}^{1} \xi d \nu(s) \text { as } \epsilon \rightarrow t^{+}
$$

For the first term of (35), we have from the Lebesgue Differentiation Theorem (see [23, Theorem 7.16]) that

$$
\begin{aligned}
\left|\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} \vartheta(s) d s\right| & \leq|\xi| \frac{1}{\epsilon-t} \int_{t}^{\epsilon}|(s-t) \vartheta(s)| d s \\
& \leq|\xi| \frac{1}{\epsilon-t} \int_{t}^{\epsilon}|s \vartheta(s)-t \vartheta(t)| d s+|\xi| \frac{t}{\epsilon-t} \int_{t}^{\epsilon}|\vartheta(t)-\vartheta(s)| d s \rightarrow 0
\end{aligned}
$$

as $\epsilon \rightarrow t^{+}$. Hence

$$
\left|\int_{t}^{\epsilon} \frac{\xi(s-t)}{\epsilon-t} \vartheta(s) d s\right| \rightarrow 0 \quad \text { as } \epsilon \rightarrow t^{+}
$$

The convergence

$$
\begin{equation*}
\int_{\epsilon}^{1} \xi^{T} \vartheta(s) d s \rightarrow \int_{t}^{1} \xi^{T} \vartheta(s) d s \text { as } \epsilon \rightarrow t^{+} \tag{37}
\end{equation*}
$$

is straightforward. Passing the limit both sides of (35) and using (36)-(37), we obtain

$$
-\int_{t}^{1} \xi^{T} d \nu(s)=\int_{t}^{1} \xi^{T} \vartheta(s) d s
$$

The proof of the lemma is complete.
From Lemma 5.1, we have for a.e. $t \in[0,1]$ that

$$
-\int_{t}^{1} \xi^{T} d \nu(s)=\int_{t}^{1}\left(\lambda^{T} L_{x}[t]+\bar{p}(t)^{T} \varphi_{x}[t]+\theta(t) g_{x}[t]\right) \xi d s
$$

or, equivalently,

$$
\xi^{T} \bar{p}(t)=\int_{t}^{1} \xi^{T}\left(\lambda^{T} L_{x}[t]+\bar{p}(t)^{T} \varphi_{x}[t]+\theta(t) g_{x}[t]\right) d s
$$

Since $\xi$ is arbitrary, we obtain

$$
\begin{aligned}
& \dot{\bar{p}}(t)=-\lambda^{T} L_{x}[t]-\bar{p}(t)^{T} \varphi_{x}[t]-\theta(t) g_{x}[t] \quad \text { a.e. } \quad t \in[0,1] \\
& \bar{p}(1)=0
\end{aligned}
$$

which is assertion (ii) of Theorem 2.1. Finally, from (31) and (28), we have

$$
\begin{aligned}
\int_{0}^{1}\left(\sum_{j=1}^{m} \lambda_{j} \nabla^{2} L_{j}[t] z(t), z(t)\right) d t & +\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] z(t), z(t)\right) d t \\
& +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] z(t), z(t)\right) d t \geq 0
\end{aligned}
$$

which is assertion (iv) of the theorem. The proof of Theorem 2.1 is complete.
5.2. Proof of Theorem [2.2. In this proof, we will use the Sobolev space $W^{1,2}\left([0,1], \mathbb{R}^{n}\right)$ which consists of absolutely continuous functions $x$ with $\dot{x} \in L^{2}\left([0,1], \mathbb{R}^{n}\right)$.

Let us define the Lagrangian $\mathcal{L}(z, \lambda, \bar{p}, \theta)$ associated with the (MCP) by setting

$$
\mathcal{L}(z, \lambda, \bar{p}, \theta):=\lambda^{T} I(z)+\bar{p}^{T} F(z)+\theta G(z)
$$

where

$$
\begin{aligned}
& \lambda^{T} I(z)=\int_{0}^{1} \lambda^{T} L(s, x(s), u(s)) d s \\
& \bar{p}^{T} F(z)=\int_{0}^{1} \dot{\bar{p}}(s) x(s) d s+\int_{0}^{1} \bar{p}(s) \varphi(s, x(s), u(s)) d s, \\
& \theta G(z)=\int_{0}^{1} \theta(s) g(s, x(s), u(s)) d s
\end{aligned}
$$

Then, from conditions (ii) and (iii) of Theorem [2.1, we can show that

$$
\begin{equation*}
\nabla_{z} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)=0 \tag{38}
\end{equation*}
$$

We now return to the proof of the theorem. Suppose the the theorem was false. Then, we could find sequences $\left\{\left(x_{k}, u_{k}\right)\right\} \subset \Phi$ and $\left\{c_{k}\right\} \subset \operatorname{int} \mathbb{R}_{+}^{m}$ such that $\left(x_{k}, u_{k}\right) \rightarrow(\bar{x}, \bar{u}), c_{k} \rightarrow 0$ and

$$
\begin{equation*}
I\left(x_{k}, u_{k}\right)-I(\bar{x}, \bar{u})-c_{k}\left\|u_{k}-\bar{u}\right\|_{2}^{2} \in-\mathbb{R}_{+}^{m} \backslash\{0\} . \tag{39}
\end{equation*}
$$

Clearly, $\left(x_{k}, u_{k}\right) \neq(\bar{x}, \bar{u})$ for all $k \in \mathbb{N}$. By replacing the sequence $\left\{\left(x_{k}, u_{k}\right)\right\}$ by a subsequence we may assume that $u_{k}=\bar{u}$ or $u_{k} \neq \bar{u}$ for all $k \in \mathbb{N}$. If $u_{k}=\bar{u}$ for all $k \in \mathbb{N}$, then we have

$$
x_{k}(t)=x_{0}+\int_{0}^{t} \varphi\left(s, x_{k}(s), \bar{u}(s)\right) d s
$$

and

$$
\bar{x}(t)=x_{0}+\int_{0}^{t} \varphi(s, \bar{x}(s), \bar{u}(s)) d s
$$

Hence,

$$
x_{k}(t)-\bar{x}(t)=\int_{0}^{t}\left(\varphi\left(s, x_{k}(s), \bar{u}(s)\right)-\varphi(s, \bar{x}(s), \bar{u}(s))\right) d s
$$

From this and (H1), there exist numbers $M>0$ and $k_{\varphi M}>0$ such that for $k$ large enough, we have

$$
\left|\varphi\left(s, x_{k}(s), \bar{u}(s)\right)-\varphi(s, \bar{x}(s), \bar{u}(s))\right| \leq k_{\varphi M}\left|x_{k}(t)-\bar{x}(t)\right|
$$

Hence,

$$
\left|x_{k}(t)-\bar{x}(t)\right| \leq \int_{0}^{t} k_{\varphi M}\left|x_{k}(s)-\bar{x}(s)\right| d s
$$

Using the Gronwall Inequality (see [5, 18.1.i, p. 503]), we get $x_{k}=\bar{x}$, a contradiction. Therefore, we have that $u_{k} \neq \bar{u}$ for all $k \in \mathbb{N}$.

Define $t_{k}=\left\|u_{k}-\bar{u}\right\|_{2}, \hat{x}_{k}=\frac{x_{k}-\bar{x}}{t_{k}}$ and $\hat{u}_{k}=\frac{u_{k}-\bar{u}}{t_{k}}$. Then $t_{k} \rightarrow 0^{+}$and $\left\|\hat{u}_{k}\right\|_{2}=1$. Since $L^{2}\left([0,1], \mathbb{R}^{l}\right)$ is reflexive, we may assume that $\hat{u}_{k} \rightharpoonup \hat{u}$. From the above, we have

$$
\begin{equation*}
\lambda^{T} I\left(z_{k}\right)-\lambda^{T} I(\bar{z}) \leq t_{k}^{2} \lambda^{T} c_{k} \leq t_{k}^{2}|\lambda|\left|c_{k}\right| \leq o\left(t_{k}^{2}\right) \tag{40}
\end{equation*}
$$

We claim that $\hat{x}_{k}$ converges uniformly to some $\hat{x}$ in $C\left([0,1], \mathbb{R}^{n}\right)$. In fact, since $\left(x_{k}, u_{k}\right) \in \Phi$, we have

$$
x_{k}(t)=x_{0}+\int_{0}^{t} \varphi\left(s, x_{k}(s), u_{k}(s)\right) d s .
$$

Since $x_{k}=\bar{x}+t_{k} \hat{x}_{k}$, we have

$$
\begin{equation*}
t_{k} \hat{x}_{k}(t)=\int_{0}^{t}\left(\varphi\left(s, x_{k}(s), u_{k}(s)\right)-\varphi(s, \bar{x}(s), \bar{u}(s)) d s\right. \tag{41}
\end{equation*}
$$

Since $x_{k} \rightarrow \bar{x}$ uniformly and $u_{k} \rightarrow \bar{u}$ in $L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$, there exists a constant $\varrho>0$ such that $\left\|x_{k}\right\|_{0} \leq \varrho,\left\|u_{k}\right\|_{\infty} \leq \varrho$. By assumption (H1), there exists $k_{\varphi, \varrho}>0$ such that

$$
\left|\varphi\left(s, x_{k}(s), u_{k}(s)\right)-\varphi(s, \bar{x}(s), \bar{u}(s))\right| \leq k_{\varphi, \varrho}\left(\left|x_{k}(s)-\bar{x}(s)\right|+\left|u_{k}(s)-\bar{u}(s)\right|\right)
$$

for a.e. $s \in[0,1]$. Hence we have from (41) that

$$
\left|\hat{x}_{k}(t)\right| \leq \int_{0}^{t} k_{\varphi, \varrho}\left(\left|\hat{x}_{k}(s)\right|+\left|\hat{u}_{k}(s)\right|\right) d s
$$

and

$$
\begin{equation*}
\left|\dot{\hat{x}}_{k}(t)\right| \leq k_{\varphi, \varrho}\left(\left|\hat{x}_{k}(t)\right|+\left|\hat{u}_{k}(t)\right|\right) . \tag{42}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
\left|\hat{x}_{k}(t)\right| & \leq \int_{0}^{t} k_{\varphi, \varrho}\left|\hat{x}_{k}(s)\right| d s+\int_{0}^{1} k_{\varphi, \varrho}\left|\hat{u}_{k}(s)\right| d s \\
& \leq \int_{0}^{t} k_{\varphi, \varrho}\left|\hat{x}_{k}(s)\right| d s+k_{\varphi, \varrho}\left(\int_{0}^{1}\left|\hat{u}_{k}(s)\right|^{2} d s\right)^{1 / 2} \\
& \leq \int_{0}^{t} k_{\varphi, \varrho}\left|\hat{x}_{k}(s)\right| d s+k_{\varphi, \varrho}, \quad\left(\left\|\hat{u}_{k}\right\|_{2}=1\right)
\end{aligned}
$$

Using the Gronwall Inequality, we have

$$
\left|\hat{x}_{k}(t)\right| \leq k_{\varphi, \varrho} \exp \left(k_{\varphi, \varrho}\right)
$$

From this and (42), we see that

$$
\left|\dot{\hat{x}}_{k}(t)\right|^{2} \leq 2 k_{\varphi, \varrho}^{2}\left(\left|\hat{x}_{k}(t)\right|^{2}+\left|\hat{u}_{k}(t)\right|^{2}\right) \leq 2 k_{\varphi, \varrho}^{2}\left(k_{\varphi, \varrho}^{2} \exp \left(2 k_{\varphi, \varrho}\right)+\left|\hat{u}_{k}\right|^{2}\right) .
$$

Hence,

$$
\int_{0}^{1}\left|\dot{\hat{x}}_{k}(t)\right|^{2} d t \leq 2 k_{\varphi, \varrho}^{2}\left(k_{\varphi, \varrho}^{2} \exp \left(2 k_{\varphi, \varrho}\right)+1\right)
$$

Consequently, $\left\{\hat{x}_{k}\right\}$ is bounded in $W^{1,2}\left([0,1], \mathbb{R}^{n}\right)$. By passing subsequence, we can assume that $\hat{x}_{k} \rightharpoonup \hat{x}$ weakly in $W^{1,2}\left([0,1], \mathbb{R}^{n}\right)$. Thanks to [4, Theorem 8.8], the embedding $W^{1,2}\left([0,1], \mathbb{R}^{n}\right) \hookrightarrow C\left([0,1], \mathbb{R}^{n}\right)$ is compact. Hence we have that $\hat{x}_{k} \rightarrow \hat{x}$ uniformly on $[0,1]$. The claim is justified. The remains of the proof is divided into some steps.

Step 1. Showing that $(\hat{x}, \hat{u}) \in \mathcal{C}^{\prime}(\bar{z})$.

By a Taylor expansion, we have from (39) that

$$
\begin{equation*}
I_{x}(\bar{z}) \hat{x}_{k}+I_{u}(\bar{z}) \hat{u}_{k}+\frac{o\left(t_{k}\right)}{t_{k}} \in-\mathbb{R}_{+}^{m} \tag{43}
\end{equation*}
$$

Note that $L_{j u}[\cdot] \in L^{\infty}\left([0,1], \mathbb{R}^{l}\right)$ and $I_{j u}(\bar{z}): L^{2}\left([0,1], \mathbb{R}^{l}\right) \rightarrow \mathbb{R}$ is a continuous linear mapping, where

$$
\left\langle I_{j u}(\bar{z}), u\right\rangle:=\int_{0}^{1} L_{j u}[s] u(s) d s \quad \forall u \in L^{2}\left([0,1], \mathbb{R}^{l}\right)
$$

By [4, Theorem 3.10], $I_{j u}(\bar{z})$ is weakly continuous on $L^{2}\left([0,1], \mathbb{R}^{l}\right)$. By letting $k \rightarrow \infty$ in (43), we get

$$
\begin{equation*}
I_{x}(\bar{z}) \hat{x}+I_{u}(\bar{z}) \hat{u} \in-\mathbb{R}_{+}^{m} \tag{44}
\end{equation*}
$$

Since $F(\bar{z})=0, F\left(x_{k}, u_{k}\right)=0$ and by a Taylor expansion, we have

$$
F_{x}(\bar{z}) \hat{x}_{k}+F_{u}(\bar{z}) \hat{u}_{k}+\frac{o\left(t_{k}\right)}{t_{k}}=0
$$

By the same arguments as the above and letting $k \rightarrow \infty$, we obtain

$$
\begin{equation*}
F_{x}(\bar{z}) \hat{x}+F_{u}(\bar{z}) \hat{u}=0 \tag{45}
\end{equation*}
$$

Since $G\left(x_{k}, u_{k}\right)-G(\bar{x}, \bar{u}) \in Q-G(\bar{x}, \bar{u})$ and by a Taylor expansion, we have

$$
G_{x}(\bar{z}) \hat{x}_{k}+G_{u}(\bar{z}) \hat{u}_{k}+\frac{o\left(t_{k}\right)}{t_{k}} \in \operatorname{cone}(Q-G(\bar{x}, \bar{u})) \subset T(Q ; G(\bar{x}, \bar{u}))
$$

where $T(Q ; G(\bar{x}, \bar{u}))$ is the tangent cone to $Q$ at $G(\bar{x}, \bar{u})$ in $L^{\infty}([0,1], \mathbb{R})$. It is easily seen that

$$
\begin{aligned}
T(Q ; G(\bar{x}, \bar{u})) & \subseteq\left\{v \in L^{\infty}([0,1], \mathbb{R}) \mid v(t) \in T((-\infty, 0] ; g[t]) \quad \text { a.e. }\right\} \\
& \subseteq\left\{v \in L^{2}([0,1], \mathbb{R}) \mid v(t) \in T((-\infty, 0] ; g[t]) \quad \text { a.e. }\right\}
\end{aligned}
$$

Hence,

$$
\begin{equation*}
G_{x}(\bar{z}) \hat{x}_{k}+G_{u}(\bar{z}) \hat{u}_{k}+\frac{o\left(t_{k}\right)}{t_{k}} \in\left\{v \in L^{2}([0,1], \mathbb{R}) \mid v(t) \in T((-\infty, 0] ; g[t]) \quad \text { a.e. }\right\} . \tag{46}
\end{equation*}
$$

Note that

$$
\left\{v \in L^{2}([0,1], \mathbb{R}) \mid v(t) \in T((-\infty, 0] ; g[t]) \text { a.e. }\right\}=T_{L^{2}}(Q ; G(\bar{x}, \bar{u}))
$$

where $T_{L^{2}}(Q ; G(\bar{x}, \bar{u}))$ is the tangent cone to the set $Q$ at $G(\bar{x}, \bar{u})$ in $L^{2}([0,1], \mathbb{R})$. Since $T_{L^{2}}(Q ; G(\bar{x}, \bar{u}))$ is a closed convex set in $L^{2}([0,1], \mathbb{R})$, it is also a weakly closed set in $L^{2}([0,1], \mathbb{R})$. Since

$$
G_{u}(\bar{z}): L^{2}\left([0,1], \mathbb{R}^{l}\right) \rightarrow L^{2}([0,1], \mathbb{R})
$$

is a continuous linear mapping, [4, Theorem 3.10] implies that it is continuous from weakly topology of $L^{2}\left([0,1], \mathbb{R}^{l}\right)$ to weakly topology of $L^{2}([0,1], \mathbb{R})$. By passing the limit in (46) when $k \rightarrow \infty$, we obtain

$$
G_{x}(\bar{z}) \hat{x}+G_{u}(\bar{u}) \hat{u} \in\left\{v \in L^{2}([0,1], \mathbb{R}) \mid v(t) \in T((-\infty, 0] ; g[t]) \text { a.e. }\right\}
$$

Combining this with (44) and (45), we get $(\hat{x}, \hat{u}) \in \mathcal{C}^{\prime}(\bar{z})$.

Step 2. Showing that $(\hat{x}, \hat{u})=0$.
By a second-order Taylor expansion for $\mathcal{L}$ and (38), we get

$$
\mathcal{L}\left(z_{k}, \lambda, \bar{p}, \theta\right)-\mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)=\frac{t_{k}^{2}}{2} \nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)\left(\hat{z}_{k}, \hat{z}_{k}\right)+o\left(t_{k}^{2}\right), \quad\left(\hat{z}_{k}=\left(\hat{x}_{k}, \hat{u}_{k}\right)\right)
$$

On the other hand from (40), we have

$$
\mathcal{L}\left(z_{k}, \lambda, \bar{p}, \theta\right)-\mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)=\lambda^{T}\left(I\left(z_{k}\right)-I(\bar{z})\right)+\left\langle\theta, G\left(z_{k}\right)-G(\bar{z})\right\rangle \leq o\left(t_{k}^{2}\right)
$$

Here we used the fact that $\theta \in N(Q, G(\bar{z}))$ and $F\left(z_{k}\right)=F(\bar{z})=0$. Therefore, we have

$$
\frac{t_{k}^{2}}{2} \nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)\left(\hat{z}_{k}, \hat{z}_{k}\right)+o\left(t_{k}^{2}\right) \leq o\left(t_{k}^{2}\right)
$$

or, equivalently,

$$
\begin{equation*}
\nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)\left(\hat{z}_{k}, \hat{z}_{k}\right) \leq \frac{o\left(t_{k}^{2}\right)}{t_{k}^{2}} \tag{47}
\end{equation*}
$$

By letting $k \rightarrow \infty$, we obtain

$$
\nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)(\hat{z}, \hat{z}) \leq 0
$$

By a simple calculation, we have

$$
\begin{aligned}
\int_{0}^{1}\left(\lambda^{T} \nabla^{2} L[t] \hat{z}(t), \hat{z}(t)\right) d t+\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] \hat{z}(t), \hat{z}(t)\right) d t & +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] \hat{z}(t), \hat{z}(t)\right) d t \\
& =\nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)(\hat{z}, \hat{z}) \leq 0
\end{aligned}
$$

Combining this with (4), we must have $\hat{z}=0$.

Step 3. Showing a contradiction.

From (5) and (47), we have

$$
\begin{aligned}
\frac{o\left(t_{k}^{2}\right)}{t_{k}^{2}} \geq & \nabla_{z z}^{2} \mathcal{L}(\bar{z}, \lambda, \bar{p}, \theta)\left(\hat{z}_{k}, \hat{z}_{k}\right) \\
= & \int_{0}^{1}\left(\lambda^{T} \nabla^{2} L[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t+\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t \\
& +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t \\
= & \int_{0}^{1} \lambda^{T} L_{u u}[t] \hat{u}_{k}^{2}(t) d t+2 \int_{0}^{1} \lambda^{T} L_{x u}[t] \hat{x}_{k}(t) \hat{u}_{k}(t) d t+\int_{0}^{1} \lambda^{T} L_{x x}[t] \hat{x}_{k}^{2}(t) d t+ \\
& +\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t+\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t \\
\geq & \gamma_{0}+2 \int_{0}^{1} \lambda^{T} L_{x u}[t] \hat{x}_{k}(t) \hat{u}_{k}(t) d t+\int_{0}^{1} \lambda^{T} L_{x x}[t] \hat{x}_{k}^{2}(t) d t+ \\
& +\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t+\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] \hat{z}_{k}(t), \hat{z}_{k}(t)\right) d t .
\end{aligned}
$$

By letting $k \rightarrow \infty$ and using the fact $\hat{z}=0$, we obtain $0 \geq \gamma_{0}$, which is impossible. The proof of Theorem 2.2 is complete.

## 6. Examples

In this section, we give some examples to illustrate the main results. The first example shows us how to use Theorem 2.1 and Theorem 2.2 to obtain solutions of the (MCP). The second one indicates the important role of the second-order necessary optimality conditions in checking optimal solutions.

Example 6.1. Consider the problem (MCP), where

$$
\begin{aligned}
L(t, x(t), u(t)) & =\left(x_{1}^{2}(t)+u_{1}^{2}(t), x_{2}^{2}(t)+u_{2}^{2}(t)\right) \\
\varphi(t, x(t), u(t)) & =\left(u_{1}(t), u_{2}(t)\right) \\
x_{0} & =(0,0) \\
g(t, x(t), u(t)) & =x_{1}(t)+x_{2}(t)-u_{1}(t)-u_{2}(t)
\end{aligned}
$$

for all $x(t)=\left(x_{1}(t), x_{2}(t)\right), u(t)=\left(u_{1}(t), u_{2}(t)\right)$ and $t \in[0,1]$. Then, the feasible solution set of the (MCP) is

$$
\begin{aligned}
\Phi=\left\{(x, u) \in C\left([0,1], \mathbb{R}^{2}\right) \times L^{\infty}\left([0,1], \mathbb{R}^{2}\right)\right. & \mid x(t)=\int_{0}^{t} u(s) d s \\
& \left.x_{1}(t)+x_{2}(t)-u_{1}(t)-u_{2}(t) \leq 0 \text { a.e. } t \in[0,1]\right\} .
\end{aligned}
$$

It is easy to see that conditions (H1) and (H2) are valid. We will use conditions (i)-(iii) of Theorem 2.1 to find out KKT points of the (MCP) which are good candidates for optimal solutions. Assume that $\bar{z}=(\bar{x}, \bar{u})$ is a feasible solution of the (MCP) and satisfies conditions (i)-(iii) of Theorem 2.1 with respect to $(\lambda, \bar{p}, \theta)$. By simple computations, we have

$$
\begin{aligned}
& L_{1 x}[t]=\left(2 \bar{x}_{1}, 0\right)^{T}, L_{2 x}[t]=\left(0,2 \bar{x}_{2}\right)^{T}, \varphi_{1 x}[t]=\varphi_{2 x}[t]=(0,0)^{T}, g_{x}[t]=(1,1)^{T}, \\
& L_{1 u}[t]=\left(2 \bar{u}_{1}, 0\right)^{T}, L_{2 u}[t]=\left(0,2 \bar{u}_{2}\right)^{T}, \varphi_{1 u}[t]=(1,0)^{T}, \varphi_{2 u}[t]=(0,1)^{T}, g_{u}[t]=(-1,-1)^{T} .
\end{aligned}
$$

From conditions (ii) and (iii), we get

$$
\left\{\begin{array}{l}
\dot{\bar{p}}_{1}=-2 \lambda_{1} \bar{x}_{1}-\theta  \tag{48}\\
\dot{\bar{p}}_{2}=-2 \lambda_{2} \bar{x}_{2}-\theta \\
\bar{p}_{1}(1)=\bar{p}_{2}(1)=0
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
2 \lambda_{1} \bar{u}_{1}+\bar{p}_{1}-\theta=0  \tag{49}\\
2 \lambda_{2} \bar{u}_{2}+\bar{p}_{2}-\theta=0
\end{array}\right.
$$

Combining (48) and (49) yields

$$
\left\{\begin{array}{l}
\dot{\bar{p}}_{1}=-2 \lambda_{1} \bar{x}_{1}-2 \lambda_{1} \bar{u}_{1}-\bar{p}_{1}  \tag{50}\\
\dot{\bar{p}}_{2}=-2 \lambda_{2} \bar{x}_{2}-2 \lambda_{2} \bar{u}_{2}-\bar{p}_{2}
\end{array}\right.
$$

By condition (2), one has

$$
\left\{\begin{array}{l}
\dot{\bar{x}}_{1}=\bar{u}_{1}  \tag{51}\\
\dot{\bar{x}}_{2}=\bar{u}_{2}
\end{array}\right.
$$

Then inserting these equations into (50), we obtain

$$
\left\{\begin{array} { l } 
{ \dot { \overline { p } } _ { 1 } = - 2 \lambda _ { 1 } \overline { x } _ { 1 } - 2 \lambda _ { 1 } \dot { \overline { x } } _ { 1 } - \overline { p } _ { 1 } , } \\
{ \dot { \overline { p } } _ { 2 } = - 2 \lambda _ { 2 } \overline { x } _ { 2 } - 2 \lambda _ { 2 } \dot { \overline { x } } _ { 2 } - \overline { p } _ { 2 } , }
\end{array} \Leftrightarrow \left\{\begin{array}{l}
\dot{p}_{1}+2 \lambda_{1} \dot{\bar{x}}_{1}=-\bar{p}_{1}-2 \lambda_{1} \bar{x}_{1} \\
\dot{\bar{p}}_{2}+2 \lambda_{2} \dot{\bar{x}}_{2}=-\bar{p}_{2}-2 \lambda_{2} \bar{x}_{2}
\end{array}\right.\right.
$$

This implies that

$$
\left\{\begin{array}{l}
\bar{p}_{1}+2 \lambda_{1} \bar{x}_{1}=c_{1} \exp (-t),  \tag{52}\\
\bar{p}_{2}+2 \lambda_{2} \bar{x}_{2}=c_{2} \exp (-t),
\end{array}\right.
$$

where $c_{1}, c_{2} \in \mathbb{R}$ are constants. Hence,

$$
\begin{equation*}
\bar{p}_{1}-\bar{p}_{2}+2 \lambda_{1} \bar{x}_{1}-2 \lambda_{2} \bar{x}_{2}=c_{3} \exp (-t) \tag{53}
\end{equation*}
$$

where $c_{3}:=c_{1}-c_{2}$. From (49) and (51), we have

$$
\bar{p}_{1}-\bar{p}_{2}=-2 \lambda_{1} \dot{\bar{x}}_{1}+2 \lambda_{2} \dot{\bar{x}}_{2}
$$

Inserting this equation into (53), we get

$$
\begin{equation*}
-2 \lambda_{1} \dot{\bar{x}}_{1}+2 \lambda_{2} \dot{\bar{x}}_{2}+2 \lambda_{1} \bar{x}_{1}-2 \lambda_{2} \bar{x}_{2}=c_{3} \exp (-t) \tag{54}
\end{equation*}
$$

Let $\alpha=2 \lambda_{1} \bar{x}_{1}-2 \lambda_{2} \bar{x}_{2}$. Then equation (54) becomes

$$
-\dot{\alpha}+\alpha=c_{3} \exp (-t)
$$

From this equation and $\alpha(0)=0$, it is easily seen that

$$
\alpha(t)=\frac{1}{2} c_{3} \exp (-t)-\frac{1}{2} c_{3} \exp (t) \quad \forall t \in[0,1]
$$

This and (51) imply that

$$
\left\{\begin{array}{l}
2 \lambda_{1} \bar{x}_{1}-2 \lambda_{2} \bar{x}_{2}=\frac{1}{2} c_{3} \exp (-t)-\frac{1}{2} c_{3} \exp (t),  \tag{55}\\
2 \lambda_{1} \bar{u}_{1}-2 \lambda_{2} \bar{u}_{2}=-\frac{1}{2} c_{3} \exp (-t)-\frac{1}{2} c_{3} \exp (t)
\end{array}\right.
$$

We see that, for each $\lambda \in \mathbb{R}_{+}^{2} \backslash\{0\}$, every solution $(\bar{x}, \bar{u})$ of system (55) is a KKT point of the (MCP). To illustrate Theorem [2.1, let us verify condition (iv) at a solution of the system (55). Let $\bar{x}=(0,0), \bar{u}=(0,0)^{T}$. By (52) and $\bar{p}(1)=(0,0)$, we have $c_{1}=0, c_{2}=0$, and $\bar{p}(t)=(0,0)$ for all $t \in[0,1]$. Consequently, $c_{3}=0$ and $\theta(t)=0$ for all $t \in[0,1]$. Thus, $(\bar{x}, \bar{u})$ is a solution of the system (55) for every $\lambda \in \mathbb{R}_{+}^{2} \backslash\{0\}$. By simple calculation, we get

$$
\int_{0}^{1}\left(\sum_{j=1}^{m} \lambda_{j} \nabla^{2} L_{j}[t] z(t), z(t)\right) d t=2 \lambda_{1} \int_{0}^{1}\left(x_{1}^{2}(t)+u_{1}^{2}(t)\right) d t+2 \lambda_{2} \int_{0}^{1}\left(x_{2}^{2}(t)+u_{2}^{2}(t)\right) d t \geq 0
$$

for all $z=(x, u) \in X \times U$ and $\lambda \in \mathbb{R}_{+}^{2} \backslash\{0\}$. Hence, condition (iv) is satisfied.
We now use Theorem 2.2 to show that $(\bar{x}, \bar{u})$ is a locally strong Pareto solution of the (MCP). Let $\lambda=\left(\frac{1}{2}, \frac{1}{2}\right), \bar{p}=(0,0)$ and $\theta=0$. Then, we have

$$
\int_{0}^{1}\left(\sum_{j=1}^{m} \lambda_{j} \nabla^{2} L_{j}[t] z(t), z(t)\right) d t=\int_{0}^{1}\left(x_{1}^{2}(t)+u_{1}^{2}(t)\right) d t+\int_{0}^{1}\left(x_{2}^{2}(t)+u_{2}^{2}(t)\right) d t>0
$$

for all $z=(x, u) \in X \times U \backslash\{(0,0)\}$. Hence condition (4) is satisfied. Furthermore, we see that

$$
\lambda^{T} L_{u u}[t](\xi, \xi)=|\xi|^{2}
$$

for all $\xi \in \mathbb{R}^{2}$. This implies that condition (5) holds at $\bar{z}$ with respect to $\gamma_{0}=1$. Thanks to Theorem 2.2, $\bar{z}$ is a locally strong Pareto solution of the (MCP).

Example 6.2. Let $\varphi$ and $g$ be as in Example 6.1 and $L$ be defined by

$$
L(t, x(t), u(t))=\left(x_{1}^{2}(t)-u_{1}^{2}(t), x_{2}^{2}(t)-u_{2}^{2}(t)\right)
$$

for all $x(t)=\left(x_{1}(t), x_{2}(t)\right), u(t)=\left(u_{1}(t), u_{2}(t)\right)$ and $t \in[0,1]$. Clearly, $(\bar{x}, \bar{u})=((0,0),(0,0))$ is a feasible point of the (MCP). We claim that $(\bar{x}, \bar{u})$ is not a locally weak Pareto solution
of the (MCP). Indeed, if otherwise, then due to Theorem 2.1, for each critical direction $z \in \mathcal{C}(\bar{z})$, there exist multipliers $\lambda, \bar{p}$ and $\theta$ such that conditions (i)-(iv) are fulfilled. Let $\tilde{x}(t)=(t, t)^{T}$ and $\tilde{u}(t)=(1,1)^{T}$ for all $t \in[0,1]$. It is easy to check that $z:=(\tilde{x}, \tilde{u})$ is a critical direction of the (MCP) at $\bar{z}$. By conditions (ii) and (iii) of Theorem 2.1, we get

$$
\left\{\begin{array} { l } 
{ \dot { \overline { p } } = - \theta ( 1 , 1 ) ^ { T } , } \\
{ p ( 1 ) = 0 , } \\
{ \overline { p } + \theta ( - 1 , - 1 ) ^ { T } = 0 , }
\end{array} \Leftrightarrow \left\{\begin{array}{l}
\dot{\bar{p}}_{1}=\dot{\bar{p}}_{2}=-\theta, \\
p(1)=0, \\
\bar{p}_{1}=\bar{p}_{2}=\theta .
\end{array}\right.\right.
$$

This implies that $\bar{p}_{1}(t)=\bar{p}_{2}(t)=\theta(t)=0$ for all $t \in[0,1]$. Since $\lambda \in \mathbb{R}_{+}^{2} \backslash\{0\}, \bar{p}=(0,0)^{T}$ and $\theta=0$, we have

$$
\begin{aligned}
& \int_{0}^{1}\left(\lambda_{1} \nabla^{2} L_{1}[t] z(t)+\lambda_{2} \nabla^{2} L_{2}[t] z(t), z(t)\right) d t+\int_{0}^{1}\left(\bar{p}(t)^{T} \nabla^{2} \varphi[t] z(t), z(t)\right) d t \\
& +\int_{0}^{1}\left(\theta(t) \nabla^{2} g[t] z(t), z(t)\right) d t \\
= & \int_{0}^{1}\left(\lambda_{1} \nabla^{2} L_{1}[t] z(t)+\lambda_{2} \nabla^{2} L_{2}[t] z(t), z(t)\right) d t \\
= & \int_{0}^{1}\left(\lambda_{1}+\lambda_{2}\right)\left(2 t^{2}-2\right) d t=-\frac{4}{3}\left(\lambda_{1}+\lambda_{2}\right)<0,
\end{aligned}
$$

which does not satisfy condition (iv) of Theorem 2.1. Hence, $(\bar{x}, \bar{u})$ is not a locally weak Pareto solution of the (MCP).

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## References

[1] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.
[2] S. Bellaassali and A. Jourani, Necessary optimality conditions in dynamic optimization, SIAM J. Control Optim., 42 (2004), pp. 2043-261.
[3] H. Bonnel and N. S. Pham, Nonsmooth optimization over the weakly or properly Pareto set of a linear-quadratic multi-objective control problem: Explicit optimality conditions, J. Indust. Manag. Optim., 7 (2011), pp. 789-809.
[4] H. Brézis, Functional Analysis, Sobolev spaces and Partial Differential Equations, Springer, 2010.
[5] L. Cesari, Optimization Theory and Applications, Problems with Ordinary Differential Equations, Springer-Verlage, New York Inc., 1983.
[6] H. GFrerer, Second-order optimality conditions for scalar and vector optimization problems in Banach spaces, SIAM J. Control Optim., 45 (2006), pp. 972-997.
[7] W. Grecksch, F. Heyde, G. Isac and Chr. Tammer, A characterization of approximate solutions of multiobjective stochatics optimal control problems, Optimization, 52 (2003), pp. 153-170.
[8] I. Ginchev, A. Guerraggio and M. Rocca, Second-order condition in $C^{1,1}$ constrained vector optimization, Math. Progam. Ser. B., 104 (2005), pp. 389-405.
[9] A. D. Ioffe and V. M. Tihomirov, Theory of Extremal Problems, North-Holland Publishing Company, 1979.
[10] B. Jiménez and V. Novo, Second order necessary conditions in set constrained differentiable vector optimization, Math. Methods Oper. Res., 58 (2003), pp. 299-317.
[11] B. Jiménez and V. Novo, Optimality conditions in differentiable vector optimization via second-order tangent sets, Appl. Math. Optim., 49 (2004), pp. 123-144.
[12] R. Cominetti, Metric regularity, tangent sets, and second-order optimality conditions, Appl. Math. Optim., 21 (1990), pp. 265-287.
[13] F. Hirsch and G. Lacombe, Elements of Functional Analysis, Springer-Verlage, 1999.
[14] C. Y. Kaya and H. Maurer, A numerical method for nonconvex multi-objective optimal control problems, Comput. Optim. Appl., 57 (2014), pp. 685-702.
[15] B. T. Kien, N.-C. Wong and J.-C. Yao, Necessary conditions for multi-objective optimal control problem with free end-time, SIAM J. Control Optim., 47 (2010), pp. 2251-2274.
[16] B. T. Kien and V. H. Nhu, Second-order necessary optimality conditions for a class of semilinear elliptic optimal control problems with mixed pointwise constraints, SIAM J. Control Optim., 52 (2014), pp. 1166-1202.
[17] T.-N. NGO AND N. HAYEK, Necessary conditions of Pareto optimality for multiobjective optimal control problems under constraints, Optimization, 66 (2017), pp. 149-177.
[18] E. Ning, W. Song and Y. Zhang, Second-order sufficient optimality conditions in vector optimization, J. Glob. Optim., 54 (2012), pp. 537-549.
[19] V. A. de Oliveira, G. N. Silva and M. A. Rojas-Medar, A class of multi-objective control problems, Optim. Control Appl. Meth., 30 (2009), pp. 77-86.
[20] V. A. de Oliveira and G. N. Silva, On sufficient optimality condition for multiobjective control problems, J. Glob. Optim., 64 (2016), pp. 721-744.
[21] Y.-H. Shao and K. Tsujioka, On proper-efficiency for nonsmooth multiobjective optimal control problems, Bull. Infor. Cyber., 32 (2000), pp. 139-155.
[22] M. Taylor, Measure Theory and Integration, American Mathematical Society, 2006.
[23] R. L. Wheeden and A. Zygmund, Measure and Integral, An Introduction to Real Analysis, Taylor and Francis Group, 2015.
[24] Z. Páles and V. Zeidan, Optimal control problems with set-valued control and state constraints, SIAM J. Optim., 14 (2003), pp. 334-358.
[25] J. Zowe and S. Kurcyusz, Regularity and stability for the mathematical programming problem in Banach spaces, Appl. Math. Optim., 5 (1979), pp. 49-62.
[26] E. Zeidler, Applied Functional Analysis, Main Principles and Their Applications, Springer-Verlag, 1995.
[27] Q. J. ZHU, Hamiltonian necessary conditions for a multiobjective optimal control problems with endpoint constraints, SIAM J. Control Optim., 39 (2000), pp. 97-112.
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