Output Consensus of Networked Hammerstein and Wiener Systems *

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Abstract

In this paper we consider the output consensus problem of networked Hammerstein and Wiener systems in a noisy environment. The Hammerstein or Wiener system is assumed to be open-loop stable, and its static nonlinearity is allowed to grow up but not faster than a polynomial. A control algorithm based on the distributed stochastic approximation algorithm with expanding truncations is designed and it is shown that under the designed control the output consensus is achieved. The numerical simulation given in the paper justifies the theoretical assertions.

Key words: Output consensus, multi-agent system, Hammerstein system, Wiener system, distributed stochastic approximation.

1 Introduction

In past decades the consensus problem for multi-agent systems has drawn much attention from researchers for its close connection with problems arising from biological science, physical science, computer science, and other areas. Among the early theoretical studies the work Jadbabaie et al. (2003) gives the mathematical explanation for the physical phenomenon discovered by Vicsek et al. (1995), pointing out that the problem can be reduced to stability analysis for a class of first order integrator systems. This problem is then considered in Saber et al. (2004) for cases including the fixed directed graph, the switched directed graphs, and the undirected graph with time delay. Besides, the concept of algebraic connectivity is expanded in Saber et al. (2004) from the undirected graph to the balanced directed graph, which plays an important role in achieving averaged consensus. More general adjacency matrices and Laplace matrices in connection with the consensus of multi-agents

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are considered in Ren and Beard (2005), where it is pointed out that for time varying graphs the asymptotic consensus can be guaranteed if the union graph contains a spanning tree. All works mentioned above are mainly with the first-order linear systems in a noise-free communication environment.

Since the noise is unavoidable in practice, the noise environment has naturally been taken into account in later research. The fixed directed graph with white observation noise is dealt with in Li and Zhang (2009), where the sufficient and necessary conditions are obtained to guarantee the asymptotic unbiased mean square consensus. Further, for the time varying graph the sufficient conditions are given in Li and Zhang (2010) to ensure the mean square and almost sure consensus. The almost sure consensus for the first-order discrete-time systems on fixed graph with observation noise is transformed to the convergence analysis of some stochastic approximation algorithm in Huang and Manton (2009). The noise conditions used in Huang and Manton (2009) and Li and Zhang (2010) have been weakened in Fang et al. (2012). Since the second-order systems are of explicit physical meaning, consensus of the second-order multiagent systems has also attracted attention from many researchers Chen et al. (2013), Ren (2008), and Yu et al. (2017).

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The works concerned above are with linear systems, but in practice nonlinear systems are ubiquitous. The consensus problem for some nonlinear systems can be reduced to the consensus of linear multi-agent systems by local linearization of the nonlinearity if it satisfies the Lipschitz condition (Saber et al. (2007)). The linearization approach is hard to work when the Lipschitz condition does not take place. The consensus problem of multi-agent nonlinear systems is considered in many papers e.g., Hua et al. (2016), Li et al. (2013), Liu et al. (2013), Liu and Huang (2017), Munz et al. (2011), and Wang et al. (2017), where various types of consensus including the leader-following consensus, finite-time consensus, adaptive consensus etc. are discussed under different settings. For example, the tracking consensus for a class of high-order nonlinear systems with unknown parameters and external disturbances is considered in Wang et al. (2017), where the distributed adaptive control based on back-stepping method is given so that the boundedness of the closed-loop system and the output tracking consensus are achieved.

In general, for the consensus problem of nonlinear systems to guarantee stability of the closed-loop systems is of primary importance. For this the growth rate of system nonlinearity, the noise, and the step-size used in the algorithm play an important role. When the classical Lyapunov method is used in stability analysis, rather strong conditions are usually required. For example, in the leader-following case a stable leader is needed to serve as a reference signal in order to avoid divergence of agents, and thus the problem turns to tracking consensus for multi-agents (Hua et al. (2016), Liu and Huang (2017), and Wang et al. (2017)), while in other cases the stability of closed-loop systems is normally guaranteed by imposing the Lipschitz condition on nonlinearity (Li et al. (2013) and Liu et al. (2013)).

In this paper the nonlinear dynamical systems of agents are either the Hammerstein system or the Wiener system, which have the wide background in practice. In contrast to the conditions discussed above, here neither a stable leader nor the Lipschitz condition are needed. As mentioned in Fang et al. (2012), Huang and Manton (2009), and Li and Zhang (2010), there is a close relationship between the consensus problem of multi-agents and the root-seeking problem for an unknown function. We apply the distributed stochastic approximation algorithm with expanding truncations (DSAAWET) to solve the output consensus problem for networked Hammerstein and Wiener systems. DSAAWET used in the paper is not exactly the same as that proposed in Lei (2016) and Lei and Chen (2015), but the basic idea remains the same.

The rest of the paper is organized as follows. The problem for output consensus of networked Hammerstein and Wiener systems is described in Section 2. The control is defined by a DSAAWET in Section 3. The properties of noises appearing in DSAAWET are analyzed in Section 4. In Section 5, the auxiliary sequences are introduced. Convergence of the algorithm is proved in Section 6. Simulation and conclusions are given in Sections 7 and 8, respectively.

Notations

Let \mathbb{R} denote the real line, and \mathbb{R}^n the linear space of n-dimensional vectors. \mathbb{I}_n denotes the n-dimensional vector with all entries equal to one. For $\mathbf{x} = [x_1, \cdots, x_N]^T \in \mathbb{R}^N$, define the norm $\|\mathbf{x}\|_{\infty} = \max_{1 \leq i \leq N} \{|x_i|\}$. By $\mathbb{R}^{m \times n}$ we denote the linear space of $m \times n$ matrices, and by Null(A) the null space of matrix A. Let [s] denote the integer part of a nonnegative real number s, and $\mathrm{Span}\{e_1, \cdots, e_n\}$ be the linear subspace spanned by vectors e_1, \cdots, e_n . Set $p \wedge q \triangleq \min\{p, q\}$, and $p \vee q \triangleq \max\{p, q\}$.

2 Problem Description

Consider a network of N agents and the corresponding topology being an undirected graph $\mathcal{G}=(\mathcal{N},\mathcal{E})$, where $\mathcal{N}=\{1,2,\cdots,N\}$ is the node (agent) set and $\mathcal{E}\subset\mathcal{N}\times\mathcal{N}$ is the edge set. The neighbor set of agent i is denoted by $\mathcal{N}_i=\{j|(j,i)\in\mathcal{E}\}$ and it is assumed that $i\notin\mathcal{N}_i$, i.e., the graph \mathcal{G} does not contain any self-loops. A path from agent j to agent i is denoted by $j=i_0,i_1,\cdots,i_{d_{i_j}-1},i_{d_{i_j}}=i$, where $(i_s,i_{s+1})\in\mathcal{E},s=0,1,\cdots,d_{i_j}-1$ and d_{ij} is called the length of this path. The shortest length of these paths is called the distance from j to i, still denoted as d_{ij} . $d(\mathcal{G})\triangleq\max\{d_{i,j},i,j\in\mathcal{N}\}$ is the diameter of \mathcal{G} . $P=[p_{ij}]_{N\times N}$ is called the adjacency matrix, if $p_{ij}>0,j\in\mathcal{N}_i; p_{ij}=0,j\notin\mathcal{N}_i$. Set $p_i\triangleq\sum_{j\in\mathcal{N}_i}p_{i,j}$ and $D=\mathrm{diag}\{p_1,\cdots,p_N\}$ which is called the degree matrix. $L\triangleq D-P$ is the Laplace matrix of \mathcal{G} .

The dynamics of agent $i \in \mathcal{N}$ is the following SISO discrete-time Hammerstein or Wiener system.

Hammerstein system:

$$v_{i,k} = f_i(u_{i,k}), C_i(z)y_{i,k+1} = D_i(z)v_{i,k}$$
 (1)

Wiener system:

$$C_i(z)v_{i,k+1} = D_i(z)u_{i,k}, \ y_{i,k+1} = f_i(v_{i,k+1}),$$
 (2)

where $u_{i,k}, v_{i,k}, y_{i,k} \in \mathbb{R}$ are the control input, internal variable, and output, respectively. The first subscript i represents agent and the second subscript k represents the discrete-time. $f_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is an unknown function.

By z we denote the backward shift operator: $zy_{i,k+1} = y_{i,k}$. In (1) and (2)

$$C_i(z) = 1 + c_{i,1}z + \dots + c_{i,p}z^p$$

and

$$D_i(z) = 1 + d_{i,1}z + \dots + d_{i,q}z^q$$

are polynomials of z, where $c_{i,s}, d_{i,r} \in \mathbb{R}, s = 1, 2, \dots, p, r = 1, 2, \dots, q$ are unknown parameters, and p and q are also unknown and may depend on i but we still denote them as p and q for simplicity of writing.

The observation of neighbor $j \in \mathcal{N}_i$ at agent $i \in \mathcal{N}$ is

$$z_{ij,k+1} = y_{j,k+1} + \epsilon_{ij,k+1}, \tag{3}$$

where $\epsilon_{ij,k+1}$ is the observation noise.

Define the function

$$h_i(u) = \begin{cases} \frac{d_i}{c_i} f_i(u), & \text{if agent } i \text{ is the Hammerstein system,} \\ f_i(\frac{d_i}{c_i} u), & \text{if agent } i \text{ is the Wiener system,} \end{cases}$$

where $c_i \triangleq 1 + \sum_{s=1}^p c_{i,s}, d_i \triangleq 1 + \sum_{s=1}^q d_{i,s}$. By the following condition A1, $c_i = C_i(1) \neq 0$.

We list the conditions to be used.

A1: $C_i(z)$, $i \in \mathcal{N}$ are stable, i.e., the roots of $C_i(z)$ are outside the unit disk.

- A2:i) $f_i(\cdot)$, $i \in \mathcal{N}$ are continuous.
 - ii) There exists an unknown constant $\mu > 0$ such that

$$|f_i(u)| = O(|u|^{\mu}) \text{ as } |u| \to \infty \ \forall i \in \mathcal{N}.$$

iii) $h_i(u) : \mathbb{R} \to \mathbb{R}, i \in \mathcal{N}$ are strictly monotonically increasing and have range $(-\infty, +\infty)$.

A3: $\{\epsilon_{ij,k}\}_{k\geq 1}, i\in\mathcal{N}, j\in\mathcal{N}_i$ are mutually independent sequences with

$$\mathbb{E}\epsilon_{ij,k} = 0, \ \sup_{k} \mathbb{E}|\epsilon_{ij,k}|^2 < \infty. \tag{4}$$

A4: \mathcal{G} is a connected and undirected graph.

The output consensus problem is stated as follows: at agent $i \in \mathcal{N}$ the control input $u_{i,k+1}$ should be designed on the basis of its output $y_{i,k+1}$ and the observations on neighbors $z_{ij,k+1}, j \in \mathcal{N}_i$ so that the outputs of all agents converge to the same limit

$$\lim_{k \to \infty} y_k = y^0 \mathbb{1}_N \text{ a.s.}, \tag{5}$$

where $y_k = [y_{1,k}, \dots, y_{N,k}]^T$ and $y^0 \in \mathbb{R}$ may depend on samples $\omega \in \Omega$.

Under A1 and A2 i) by Lemma 1 in Chen (2007) it is known that for both (1) and (2), $\lim_{k\to\infty} y_{i,k} = h_i(u_i)$ as $\lim_{k\to\infty} u_{i,k} = u_i$.

Set

$$\mathbf{u}_k = [u_{1,k}, \cdots, u_{N,k}]^T$$

and

$$h(\mathbf{u}_k) = [h_1(u_{1,k}), \cdots, h_N(u_{N,k})]^T.$$

Then, the problem is restated as follows: at agent $i \in \mathcal{N}$ the control input $u_{i,k+1}$ should be designed on the basis of its output $y_{i,k+1}$ and the observations $z_{ij,k+1}, j \in \mathcal{N}_i$ on neighbors so that

$$\lim_{k \to \infty} \mathbf{u}_k = \mathbf{u}^0 \in J \triangleq \{\mathbf{u} \in \mathbb{R}^N : \mathbf{h}(\mathbf{u}) \in \text{Span}\{\mathbb{1}_N\}\},$$
(6)

where $\mathbf{u}^0 \triangleq [u_1^0, \cdots, u_N^0]^T$ may depend on samples.

For solvability of the consensus problem, J in (6) must be nonempty, i.e., the intersection of ranges of $h_i(\cdot)$, $i = 1, \dots, N$ should be nonempty. Besides, the steady output may depend on samples since communication between agents is corrupted by noises. In order to reach consensus for almost all samples, the strong condition A2 iii) is imposed on $h_i(\cdot)$.

Set $c_{i,s} = d_{i,r} = 0$ for s > p and r > q and define matrices

$$C_{i} \triangleq \begin{bmatrix} -c_{i,1} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & 1 \\ -c_{i} & (p)(q)+1 & 0 & \cdots & 0 \end{bmatrix},$$

$$D_i \triangleq \left[1 \ d_{i,1} \ \cdots \ d_{i,p \vee q}\right]^T,$$

$$G_1 \triangleq [1 \quad 0 \cdots 0]_{1 \times ((p \vee q) + 1)}.$$

The dynamical systems of (1) and (2) can be written in the state space form

$$Y_{i,k+1} = C_i Y_{i,k} + D_i v_{i,k}, \ y_{i,k} = G_1 Y_{i,k},$$

 $V_{i,k+1} = C_i V_{i,k} + D_i u_{i,k}, \ v_{i,k} = G_1 V_{i,k}.$

Then it follows that

$$Y_{i,s+l+1} = C_i^{l+1} Y_{i,s} + \sum_{k=s}^{s+l} C_i^{s+l-k} D_i v_{i,k}, \ \forall s, \ l \ge 0,$$
(7)

$$V_{i,s+l+1} = C_i^{l+1} V_{i,s} + \sum_{k=s}^{s+l} C_i^{s+l-k} D_i u_{i,k}, \ \forall s, \ l \ge 0.$$
(8)

It is well known that under A1, C_i are stable and there are r > 0 and $\delta > 0$ such that

$$||C_i^k|| \le re^{-\delta k} \ \forall k \ge 0 \ \forall i \in \mathcal{N}. \tag{9}$$

For Hammerstein systems (1) from (7) and (9) it follows that

$$||Y_{i,k}|| \le ||C_i^k|| ||Y_{i,0}|| + \sum_{s=0}^{k-1} ||C_i^{k-s-1}|| ||D_i|| |v_{i,s}|$$

$$\le re^{-\delta k} ||Y_{i,0}|| + r \sup_{0 < s < k-1} |v_{i,s}| ||D_i||. \tag{10}$$

Similarly, for Wiener systems (2), from (8) and (9) we have

$$||V_{i,k}|| \le ||C_i^k|| ||V_{i,0}|| + \sum_{s=0}^{k-1} ||C_i^{k-s-1}|| ||D_i|| |u_{i,s}|$$

$$\le re^{-\delta k} ||V_{i,0}|| + r \sup_{0 \le s \le k-1} |u_{i,s}| ||D_i||. \tag{11}$$

3 Control algorithm

Define $g(\mathbf{u}) \triangleq [g_1(\mathbf{u}), \cdots, g_N(\mathbf{u})]^T : \mathbb{R}^N \to \mathbb{R}^N$, where

$$g_i(\mathbf{u}) \triangleq \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(u_j) - h_i(u_i))$$
$$= \sum_{j \in \mathcal{N}_i} p_{ij} h_j(u_j) - p_i h_i(u_i).$$

By A4, it is noted in Saber et al. (2004) and Ren and Beard (2005) that $L=D-P\in\mathbb{R}^{N\times N}$ is nonnegative definite and

$$Null\{L\} = Span\{1_N\}. \tag{12}$$

Since g(u) = -Lh(u), by (12) it follows that

$$g(u) = 0 \Leftrightarrow h(u) \in \text{span}\{1_N\} \Leftrightarrow u \in J,$$
 (13)

i.e., the root set of g(u) coincides with J defined in (6). Therefore, the output consensus problem is transformed

to root-seeking for some function. The control algorithm will be constructed on the basis of DSAAWET. The regression function of agent i is $g_i(\mathfrak{u})$.

Assume the estimate for the roots of g(u) at time k is $u_k = [u_{1,k}, \dots, u_{N,k}]^T$. Then, as observation of $g_i(u_k)$ at i we may take

$$O_{i,k+1} = \sum_{j \in \mathcal{N}_i} p_{ij}(z_{ij,k+1} - y_{i,k+1})$$

$$= \sum_{j \in \mathcal{N}_i} p_{ij}y_{j,k+1} - p_iy_{i,k+1} + \sum_{j \in \mathcal{N}_i} p_{ij}\epsilon_{ij,k+1}$$

$$= g_i(\mathbf{u}_k) + \epsilon_{i,k+1}, \qquad (14)$$

where $\epsilon_{i,k+1} = \epsilon_{i,k+1}^{(1)} + \epsilon_{i,k+1}^{(2)} + \epsilon_{i,k+1}^{(3)}$ is the observation noise, where

$$\epsilon_{i,k+1}^{(1)} = \sum_{j \in \mathcal{N}_i} p_{ij} \epsilon_{ij,k+1}, \tag{15}$$

$$\epsilon_{i,k+1}^{(2)} = p_i(h_i(u_{i,k}) - y_{i,k+1}), \tag{16}$$

and

$$\epsilon_{i,k+1}^{(3)} = \sum_{j \in \mathcal{N}_i} p_{ij}(y_{j,k+1} - h_j(u_{j,k})). \tag{17}$$

Similar to DSAAWET proposed in Lei (2016), Lei and Chen (2015), and Fang and Chen (2001), we construct the distributed root-seeking algorithm:

$$\sigma'_{i,k} = \max\{\sigma_{j,k}, j \in \mathcal{N}_i, \sigma_{i,k}\}, \sigma_{i,1} = 0, k \ge 1, (18)$$

$$u'_{i,k} = u_{i,k} I_{\{\sigma'_{i,k} = \sigma_{i,k}\}} + u_i^* I_{\{\sigma'_{i,k} > \sigma_{i,k}\}},$$
(19)

$$O_{i,k+1} = \sum_{j \in \mathcal{N}_i} p_{ij} (z_{ij,k+1} - y_{i,k+1}), \tag{20}$$

$$u_{i,k+1} = \left(u'_{i,k} + a_k O_{i,k+1}\right) I_{\{|u'_{i,k} + a_k O_{i,k+1}| < M_{\sigma'_{i,k}}\}} + u_i^* I_{\{|u'_{i,k} + a_k O_{i,k+1}| \ge M_{\sigma'_{i,k}}\}}, \tag{21}$$

$$\sigma_{i,k+1} = \sigma'_{i,k} + I_{\{|u'_{i,k} + a_k O_{i,k+1}| \ge M_{\sigma'_{i,k}}\}}, \tag{22}$$

where $I_{\{\cdot\}}$ is the indicator function, $a_k = \frac{1}{k}$, $M_k = \ln(k + c_M)$, and $c_M > 0$ is a constant such that $|u_i^*| < M_0$, $i = 1, 2, \dots, N$. The new information obtained by agent i at k+1 is $\{z_{ij,k+1}, \sigma_{j,k}, j \in \mathcal{N}_i\}$ and the output $y_{i,k+1}$. The algorithm is distributed, since at each agent only the local information is used.

We explain the algorithm. The information obtained by agent $i \in \mathcal{N}$ at k+1 includes the observations $z_{ij,k+1}$ and truncation numbers $\sigma_{j,k}$ for neighbors in addition to itself's output and truncation number. In order to ensure

control algorithms of agents update gradually and simultaneously, we need the differences of truncation numbers at agents are not too large. In the ideal case the truncations at all agents occur at the same time. However, the truncation numbers at agents cannot be always the same, since only the local information can be used and no global information is available. To make sure the differences of truncation numbers among agents are not too big we first introduce $\sigma_{i,k}' = \max\{\sigma_{j,k}, j \in \mathcal{N}_i, \sigma_{i,k}\}$ and then force the truncation number of agent i at k+1 to catch up with its neighbors at k. In other words, $\sigma_{i,k+1}$ is set to equal to the largest one of $\sigma_{j,k}, j \in \mathcal{N}_i$ when agent i finds the truncation number $\sigma_{i,k}$ is smaller than $\sigma_{i,k}'$, even if the estimate is still within the truncation bound. This is what (22) means.

4 Properties of noises

Noticing the choice of step-size and truncation bound, similar to Lemma 2 in Chen (2007) we have the following lemma.

Lemma 1 Assume A1,A2 i), A2 ii), and A3. For any $l: l = 0, 1, 2, \dots, [\ln s] + m$ with m being a given positive integer, the following limits take place:

$$\sum_{k=s}^{s+l} a_k \xrightarrow[s \to \infty]{} 0, \tag{23}$$

$$\left|\sum_{k=1}^{s+l} a_k h_i(u_{i,k})\right| \xrightarrow[s \to \infty]{} 0 \ \forall i \in \mathcal{N},$$
 (24)

$$\left|\sum_{k=s}^{s+l} a_k y_{i,k+1}\right| \xrightarrow[s \to \infty]{} 0 \ \forall i \in \mathcal{N},$$
 (25)

$$\left|\sum_{k=s}^{s+t} a_k g_i(\mathbf{u}_k)\right| \xrightarrow[s \to \infty]{} 0 \ \forall i \in \mathcal{N}, \tag{26}$$

$$\left|\sum_{k=s}^{s+l} a_k \epsilon_{i,k+1}\right| \xrightarrow[s \to \infty]{} 0 \text{ a.s. } \forall i \in \mathcal{N},$$
 (27)

$$\left| \sum_{k=s}^{s+l} a_k O_{i,k+1} \right| \xrightarrow[s \to \infty]{} 0 \text{ a.s. } \forall i \in \mathcal{N}.$$
 (28)

PROOF. The proof of (23) follows from the following chain of inequalities and equality:

$$\sum_{k=s}^{s+l} a_k \le \sum_{k=s}^{s+\lceil \ln s \rceil + m} \frac{1}{k} \le \sum_{k=s}^{s+\lceil \ln s \rceil + m} \int_{k-1}^k \frac{\mathrm{d}x}{x}$$
$$= \ln \frac{s + \lceil \ln s \rceil + m}{s - 1} \xrightarrow[s \to \infty]{} 0.$$

Noticing $\sigma_{i,k} \leq k-1 \ \forall i \in \mathcal{N}$, we have

$$|u_{i,k}| \le M_{\sigma'_{i,k}} \le M_{k-1} = \ln(k-1+c_M),$$
 (29)

which by A2 ii) means that there exists $\alpha_1 > 0$ such that

$$|h_i(u_{i,k})| \le \alpha_1 (\ln k)^{\mu}. \tag{30}$$

As a result, for $v=0,1,\cdots,q$ and $l=0,1,\cdots,[\ln k]+m$ we have

$$|\sum_{k=s}^{s+l} a_k h_i(u_{i,k})| \le \alpha_1 \sum_{k=s}^{s+[\ln s]+m} \frac{(\ln k)^{\mu}}{k}$$

$$\le \alpha_1 \sum_{k=s}^{s+[\ln s]+m} \int_{k-1}^{k} \frac{(\ln x)^{\mu}}{x} dx$$

$$= \int_{s-1}^{s+[\ln s]+m} ((\ln x)^{\mu} d(\ln x))$$

$$= \alpha_1 \frac{1}{1+\mu} ((\ln (s+[\ln s]+N))^{1+\mu} - (\ln (s-1))^{1+\mu})$$

$$= \frac{\alpha_1}{1+\mu} (\ln (s-1))^{1+\mu} ((1+\frac{\ln (1+\frac{[\ln s]+m+1}{s-1})}{\ln (s-1)})^{1+\mu} - 1)$$

$$= (\ln (s-1))^{1+\mu} O(\frac{\ln (1+\frac{[\ln s]+m+1}{s-1})}{\ln (s-1)})$$

$$= (\ln (s-1))^{\mu} \cdot O(\frac{[\ln s]+m+1}{s-1})$$

$$= O(\frac{(\ln s)^{1+\mu} + (m+1)(\ln s)^{\mu}}{s-1}) \xrightarrow[s\to\infty]{} 0, \tag{31}$$

and thus (24) is proved.

In view of A2 ii), (10), (11), and (29), in both cases (1) and (2) there exists $\alpha_2 > 0$ such that

$$|y_{i,k+1}| < \alpha_2 (\ln k)^{\mu}$$
 (32)

for sufficiently large k. Then similar to (31), (25) can be proved.

By the definition of $g_i(\mathbf{u}_k)$, there exists $\alpha_3 > 0$ such that

$$|q_i(\mathbf{u}_k)| < \alpha_3 (\ln k)^{\mu}$$

for sufficiently large k. So, the proof for (26) can also be carried out in a way similar to that for (31).

From A3 and $a_k = \frac{1}{k}$ it is clear that

$$\left|\sum_{k=1}^{\infty} a_k \epsilon_{ij,k+1}\right| < \infty \text{ a.s. } \forall i, j \in \mathcal{N},$$
 (33)

which implies

$$|\sum_{k=1}^{\infty} a_k \epsilon_{i,k+1}^{(1)}| = |\sum_{k=1}^{\infty} a_k \sum_{j \in \mathcal{N}_i} p_{ij} \epsilon_{ij,k+1}| < \infty \text{ a.s.}$$

In view of (24) and (25), we have $\left|\sum_{k=s}^{s+l} a_k \epsilon_{i,k+1}^{(l)}\right| \xrightarrow[s \to \infty]{} 0, l = 2, 3$. Then (27) follows from the definition of $\epsilon_{i,k}$.

Finally, (28) is straightforwardly obtained from (26) and (27). $\ \square$

As in Lei (2016) and Lei and Chen (2015), for a positive integer $m \geq 0$, set $r(i,m) = \inf\{n \geq 1, \sigma_{i,n} \geq m\}$, the smallest time for agent *i*'s truncation number to reach m and $r(m) = \inf\{r(i,m), i \in \mathcal{N}\}$, the smallest time for some agent's truncation number to reach m. So r(i,0) = r(0) = 1. Set $\inf \emptyset = \infty$.

We now prove that

$$0 \le r(i, m) - r(m) \le d(\mathcal{G}) \ \forall i \in \mathcal{N} \text{ when } r(m) < \infty,$$
(34)

where $d(\mathcal{G})$ is the diameter of \mathcal{G} . The first inequality of (34) is seen by the definitions of r(i,m) and r(m). Assume r(j,m) = r(m), i.e., the truncation number of agent j first reaches m. The shortest path from j to i is denoted as $j = i_0, i_1, \cdots, i_{d_{ij}} = i$ where $(i_s, i_{s+1}) \in \mathcal{E}, s = 0, 1, \cdots, d_{ij}$. From (18), $\sigma'_{i_{s+1}, r(i_s, m)} \geq \sigma_{i_s, r(i_s, m)} = m$ and by (22), $\sigma_{i_{s+1}, r(i_s, m)+1} \geq \sigma'_{i_{s+1}, r(i_s, m)}$ which implies that $\sigma_{i_{s+1}, r(i_s, m)+1} \geq m$, i.e., the truncation number of agent i_{s+1} is bigger than or equal to m at $r(i_s, m) + 1$. Therefore,

$$r(i_{s+1}, m) \le r(i_s, m) + 1, \ s = 0, 1, \dots, d_{ij} - 1.$$
 (35)

Noticing the path from j to i, by (35) we have $r(i,m) = r(i_{d_{ij}}, m) \le r(i_{d_{ij}-1}, m) + 1 \le \cdots \le r(i_0, m) + d_{ij} = r(m) + d_{ij} \le r(m) + d(\mathcal{G})$, which implies the second inequality of (34).

Define the positive integer $m(k,T) \triangleq \max\{m: \sum_{s=k}^m a_s \leq T\}$ for a given T>0. It directly follows from definition that $\sum_{s=k}^{m(k,T)} a_s \leq T < \sum_{s=k}^{m(k,T)+1} a_s$. From the estimation $\int_s^{s+1} \frac{1}{t} \mathrm{d}t < \frac{1}{s} < \int_{s-1}^s \frac{1}{t} \mathrm{d}t, s > 1$, we have

$$T \ge \sum_{s=k}^{m(k,T)} a_s > \sum_{s=k}^{m(k,T)} \int_s^{s+1} \frac{1}{t} dt$$
$$= \int_t^{m(k,T)+1} \frac{1}{t} dt = \ln \frac{m(k,T)+1}{k}$$

and

$$T < \sum_{s=k}^{m(k,T)+1} a_s < \sum_{s=k}^{m(k,T)+1} \int_{s-1}^{s} \frac{1}{t} dt$$
$$= \int_{k-1}^{m(k,T)+1} \frac{1}{t} dt = \ln \frac{m(k,T)+1}{k-1}.$$

From here it follows that m(k,T) satisfies

$$(k-1)\exp(T) - 1 < m(k,T) < k\exp(T) - 1.$$
 (36)

Lemma 2 Assume that A1,A2i, A2i, and A3 hold. At the samples $\omega \in \Omega$ where (33) holds, for a given T > 0, for sufficiently large C > 0 and any $l : l = k, k + 1, \dots, \left(\left(r(i, m_{i,k} + 1) - 1 \right) \land m(k, T) \right)$, the sequence $\{ \epsilon_{i,k} \}$ satisfies

$$\limsup_{k \to \infty} \left| \sum_{s=k}^{l} a_s \epsilon_{i,s+1} I_{\{\|\mathbf{u}_s\| \le C\}} \right| = 0 \ \forall i \in \mathcal{N}, \quad (37)$$

where $m_{i,k} = \sup\{m : r(i,m) \le k\}$ is the biggest truncation number of agent i up to time k.

PROOF. It is noticed that the inequality $l \leq r(i, m_{i,k} + 1) - 1$ means that at time l the $(m_{i,k} + 1)$ th truncation has not happened yet for agent i, i.e., there is no truncation in (18)-(22) as $s = k, k + 1 \cdots, l$ for agent i. By $r(i, 0) = 1 \leq k$, the set $\{m : r(i, m) \leq k\}$ is nonempty, and then $m_{i,k}$ are well defined. For (37) it suffices to prove

$$\limsup_{k \to \infty} \left| \sum_{s=k}^{l} a_s \epsilon_{i,s+1}^{(h)} I_{\{\|\mathbf{u}_s\| \le C\}} \right| = 0, \ h = 1, 2, 3, \quad (38)$$

where $l = k, k+1, \dots, ((r(i, m_{i,k}+1)-1) \wedge m(k,T))$. From A3, (38) holds for h = 1.

By noticing

$$-\epsilon_{i,k+1}^{(2)} = p_i (y_{i,k+1} - h_i(u_{i,k})),$$

if $l \le k + [\ln k]$, (38) holds for h = 2 by (24) and (25). Assume $l > k + [\ln k]$ in what follows. Since the definitions of $h_i(\cdot)$ are different for the Hammerstein and Wiener systems, let us consider two cases separately.

1) The dynamics of agent i is a Hammerstein system (1).

In this case $\epsilon_{i,k+1}^{(2)}$ can be written as

$$-\epsilon_{i,k+1}^{(2)} = p_i \left(y_{i,k+1} - h_i(u_{i,k}) \right) \qquad \lim_{k \to \infty} |u_{i,s-h} - u_{i,s-h-1}| = 0, \tag{43}$$

$$= p_i \left(y_{i,k+1} - \frac{d_i}{c_i} f_i(u_{i,k}) \right)$$

$$= p_i \left(y_{i,k+1} - \frac{1}{c_i} C_i(z) y_{i,k+1} + \frac{1}{c_i} D_i(z) f_i(u_{i,k}) - \frac{d_i}{c_i} f_i(u_{i,k}) \right)$$
where $s = k + [\ln k] + 1, k + [\ln k] + 2, \dots, l ; h = 0, 1, \dots, v - 1 ; v = 1, 2, \dots, q.$ Since $s - h = 0, 1, \dots, v - 1 ; v = 1, 2, \dots, q.$ Since $s - h = 0, 1, \dots, v - 1 ; v = 1, 2, \dots, q.$ Since $s - h = 0, 1, \dots, v - 1 ; v = 1, 2, \dots, q.$ Since $s - h = 0, \dots, v = 1, \dots, q.$ Since $s - h = 0, \dots, q = 0, \dots, q.$ Since $s - h = 0, \dots, q = 0, \dots, q$

By taking notice of the discussion for the case of l < $k + [\ln k]$ given before, for (38) to hold for h = 2 it suffices to verify

$$\limsup_{k \to \infty} \left| \sum_{s=k+[\ln k]+1}^{l} a_s \sum_{v=1}^{q} d_{i,v} (f_i(u_{i,s-v}) -f_i(u_{i,s})) I_{\{\|\mathbf{u}_s\| \le C\}} \right| = 0$$
 (39)

and

$$\limsup_{k \to \infty} \left| \sum_{s=k+[\ln k]+1}^{l} a_s \sum_{v=1}^{p} c_{i,v}(y_{i,s+1} -y_{i,s+1-v}) I_{\{\|\mathbf{u}_s\| \le C\}} \right| = 0$$
 (40)

for $l > k + [\ln k]$.

We first prove (39). Noticing $l \leq m(k,T)$ we have

$$\left| \sum_{s=k+[\ln k]+1}^{l} a_s \sum_{v=1}^{q} d_{i,v} (f_i(u_{i,s-v}) - f_i(u_{i,s})) I_{\{\|\mathbf{u}_s\| \le C\}} \right|$$

$$\leq T \sup_{k+[\ln k]+1 \le s \le l} \{ \sum_{v=1}^{q} d_{i,v} |f_i(u_{i,s-v}) - f_i(u_{i,s})| I_{\{\|\mathbf{u}_s\| \le C\}} \}.$$

$$(41)$$

which means that we need only to prove

$$\lim_{k \to \infty} |f_i(u_{i,s-v}) - f_i(u_{i,s})| = 0,$$

where $s=k+[\ln k]+1, k+[\ln k]+2, \cdots, l; v=1,2,\cdots,q,$ when $|u_{i,s}|\leq \|\mathbf{u}_s\|\leq C.$ By continuity of $f_i(\cdot)$ and $|u_{i,s}| \leq C$, it is sufficient to show

$$\lim_{k \to \infty} |u_{i,s} - u_{i,s-v}| = 0, \tag{42}$$

where $s=k+[\ln k]+1, k+[\ln k]+2, \cdots, l; v=1,2,\cdots,q.$ By noticing $|u_{i,s}-u_{i,s-v}|\leq \sum_{h=0}^{v-1}|u_{i,s-h}-u_{i,s-h}|$

 $u_{i,s-h-1}|, (42)$ follows from

$$\lim_{k \to \infty} |u_{i,s-h} - u_{i,s-h-1}| = 0, \tag{43}$$

$$\lim_{k \to \infty} |u_{i,t+1} - u_{i,t}| = 0, \tag{44}$$

where
$$t = k + [\ln k] - q + 1, k + [\ln k] - q + 2, \dots, l - 1.$$

As noticed at the beginning of the proof, there is no truncation in (18)-(22) as $s = k, \dots, l \text{ and } \sigma'_{i,k} = \sigma_{i,k}, u'_{i,k} = \sigma_{i,k},$ $u_{i,k}$, i.e.,

$$u_{i,s+1} = u_{i,s} + a_s O_{i,s+1}, \ s = k, k+1, \cdots, l-1.$$

By (28) we have

$$|u_{i,s+1} - u_{i,s}| \xrightarrow[k \to \infty]{} 0, \ s = k, k+1, \cdots, l-1$$
 (45)

which imply (44) and hence (39) holds.

We continue to prove (40). Similar to (41) it is reduced to prove

$$\lim_{k \to \infty} |y_{i,s+1} - y_{i,s+1-v}| = 0, \tag{46}$$

where $s = k + [\ln k] + 1, k + [\ln k] + 2, \dots, l; v =$ $1, 2, \dots, p$, when $|u_{i,s}| \le ||\mathbf{u}_{s}|| \le C$. By noticing $(y_{i,s+1} - y_{i,s+1-v}) = \sum_{h=0}^{v-1} (y_{i,s-h+1} - y_{i,s-h})$, similar to the proof of (43) for (46) it suffices to prove

$$\lim_{k \to \infty} |y_{i,s-h+1} - y_{i,s-h}| = 0$$

for $|u_{i,s}| \le ||u_s|| \le C$, where $s = k + [\ln k] + 1, k + 1$ $[\ln k] + 2, \dots, l; h = 0, 1, \dots, v - 1; v = 1, 2, \dots, p.$ Since $s - h = k + [\ln k] - p + 2, k + [\ln k] - p + 3, \dots, l$, this is equivalent to verifying

$$\lim_{k \to \infty} |y_{i,k+[\ln k]+t+1} - y_{i,k+[\ln k]+t}| = 0, \qquad (47)$$

where $t = 2 - p, 3 - p, \dots, l - k - [\ln k]$, when $|u_{i,s}| \le$ $\|\mathbf{u}_s\| \leq C$. From (9), (30) and (32), we have the following chain of inequalities or equalities:

$$\begin{aligned} &|y_{i,k+[\ln k]+t+1} - y_{i,k+[\ln k]+t}| \\ &\leq \|Y_{i,k+[\ln k]+t+1} - Y_{i,n_k+[\ln k]+t}\| \\ &\leq \|C_i^{[\ln k]+t+1}Y_{i,k} - C_i^{[\ln k]+t}Y_{i,k}\| \\ &+ \|\sum_{g=k}^{k+[\ln k]+t} C_i^{k+[\ln k]+t-g} D_i f_i(u_{i,g}) \\ &- \sum_{g=k}^{k+[\ln k]+t-1} C_i^{k+[\ln k]+t-1-g} D_i f_i(u_{i,g})\| \\ &\leq re^{-\delta([\ln k]+t)} \alpha_2 (\ln k)^{\mu} |re^{-\delta} - 1| + \|C_i^{[\ln k]+t} D_i f_i(u_{i,k})\| \\ &+ \|\sum_{g=k}^{k+[\ln k]+t-1} C_i^{k+[\ln k]+t-1-g} D_i (f_i(u_{i,g+1}) - f_i(u_{i,g}))\| \\ &\leq r\alpha_2 e^{-\delta([\ln k]+t)} (\ln k)^{\mu} |re^{-\delta} - 1| \\ &+ r\alpha_1 \|D_i\| e^{-\delta([\ln k]+t)} (\ln k)^{\mu} \\ &+ \frac{r\|D_i\|}{1-e^{-\delta}} \max_{k \leq s \leq l-1} |f_i(u_{i,s+1}) - f(u_{i,s})|. \end{aligned} \tag{48}$$

The first and second terms at the right-hand side of (48) tend to zero as $k \to \infty$ and the third term also tends to zero by (45), $|u_{i,s}| \le ||u_s|| \le C$, and continuity of $f_i(\cdot)$.

2) The dynamics of agent i is a Wiener system (2).

In this case $\epsilon_{i,k+1}^{(2)}$ can be written as

$$-\epsilon_{i,k+1}^{(2)} = y_{i,k+1} - h_i(u_{i,k})$$

$$= f_i(v_{i,k+1}) - f_i(\frac{d_i}{c_i}u_{i,k})$$

$$= f_i(v_{i,k+1}) - f_i(\frac{1}{c_i}C_i(z)v_{i,k+1})$$

$$-\frac{1}{c_i}D_i(z)u_{i,k} + \frac{d_i}{c_i}u_{i,k}$$

$$= f_i(v_{i,k+1}) - f_i(v_{i,k+1} + \frac{1}{c_i}\sum_{v=1}^p c_{i,v}(v_{i,k+1-v} - v_{i,k+1}))$$

$$+\frac{1}{c_i}\sum_{v=1}^q d_{i,v}(u_{i,k} - u_{i,k-v}).$$
(49)

Similar to the preceding proof in 1) it suffices to prove

$$\limsup_{k \to \infty} \left| \sum_{s=k+[\ln k]+1}^{l} a_s \epsilon_{i,s+1}^{(2)} I_{\{\|\mathbf{u}_s\| \le C\}} \right| = 0$$
 (50)

when $l > k + [\ln k]$. From $l \le m(k,T)$ and $\|\mathbf{u}_s\| \le C$ it follows that $\|v_{i,s}\| \le C_1$ for some $C_1 > 0$ by (11).

Similar to (41), (50) follows from

$$\lim_{k \to \infty} \left| f_i(v_{i,s+1}) - f_i \left(v_{i,s+1} + \frac{1}{c_i} \sum_{v=1}^p c_{i,v}(v_{i,s+1-v} - v_{i,s+1}) + \frac{1}{c_i} \sum_{v=1}^q d_{i,v}(u_{i,s} - u_{i,s-v}) \right) \right| = 0,$$
(51)

where $s = k + [\ln k] + 1, k + [\ln k] + 2, \dots, l$. By continuity of $f_i(\cdot)$, (51) follows from

$$\lim_{k \to \infty} |u_{i,s} - u_{i,s-v}| = 0, \tag{52}$$

where $s = k + [\ln k] + 1, k + [\ln k] + 2, \dots, l; v = 1, 2, \dots, q$, and

$$\lim_{k \to \infty} |v_{i,s+1} - v_{i,s+1-v}| = 0, \tag{53}$$

where $s = k + [\ln k] + 1, k + [\ln k] + 2, \dots, l ; v = 1, 2, \dots, p$.

Similar to (42), we can verify (52), while from (48) with $y_{i,k}, Y_{i.k}, f_i(u_{i,k})$ replaced by $v_{i,k}, V_{i,k}, u_{i,k}$, respectively, we conclude

$$\lim_{k \to \infty} |v_{i,k+[\ln k]+t+1} - v_{i,k+[\ln k]+t}| = 0, \quad (54)$$

where $t = 2 - p, 3 - p, \dots, l - k - [\ln k]$. Therefore, (53) can be proved in a way similar to that for (46).

Similarly, it can be verified that (38) holds for h = 3. \Box

5 Auxiliary sequences

Since the growth rate of $f_i(\cdot)$ in (1) and (2) may be faster than linearly, the SA proposed by Robbins and Monro (1951) may diverge. Therefore, we apply the expanding truncation technique introduced in Chen (2002), and the resulting algorithm is called SAAWET. The distributed algorithm (18) - (22) is hard to be written as a centralized SAAWET, because the truncation numbers for different agents may be different. By introducing auxiliary sequences, we can transform the DSAAWET to a centralized algorithm by the treatment proposed in Lei (2016), Lei and Chen (2015), and Fang and Chen (2001).

The discrete-time axis is divided by the sequence $\{r(m): r(m) < \infty\}$ into $[r(m), r(m+1)), m=1, 2, \cdots$. Define $\overline{r}(i,m) = r(i,m) \wedge r(m+1)$. When $r(m) < \infty$, by (34) we have $0 \leq \overline{r}(i,m) - r(m) \leq d(\mathcal{G})$ which implies that $\overline{r}(i,m) - r(m)$ is bounded. Similar to Lei (2016), Lei

and Chen (2015), and Fang and Chen (2001), we define auxiliary sequences $\{\overline{u}_{i,k}\}$ and $\{\overline{\epsilon}_{i,k+1}\}$ as follows

$$\forall k : r(m) \leq k < \overline{r}(i, m),$$

$$\begin{cases} \overline{u}_{i,k} = u_i^*, \\ \overline{\epsilon}_{i,k+1} = -\sum_{j \in \mathcal{N}_i} p_{ij}(h_j(\overline{u}_{j,k}) - h_i(u_i^*)), \end{cases}$$

$$\forall k : \overline{r}(i, m) \leq k < r(m+1),$$

$$(55)$$

 $\begin{cases} \overline{u}_{i,k} = u_{i,k}, \\ \overline{\epsilon}_{i,k+1} = \epsilon_{i,k+1} + \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(u_{j,k}) - h_j(\overline{u}_{j,k})). \end{cases}$ (56)

For $\{\overline{u}_{i,k}\}$ and $\{\overline{\epsilon}_{i,k+1}\}$ we have the following lemma.

Lemma 3 The sequences $\{\overline{u}_{i,k}\}$ and $\{\overline{\epsilon}_{i,k+1}\}$ satisfy the following recursion:

$$\overline{u}_{i,k+1} = (\overline{u}_{i,k} + a_k(g_i(\overline{u}_k) + \overline{\epsilon}_{i,k+1}))
\cdot I_{\{\max_{j \in \mathcal{N}} | \overline{u}_{j,k} + a_k(g_j(\overline{u}_k) + \overline{\epsilon}_{j,k+1}) | < M_{\overline{\sigma}_k}\}}
+ u_i^* I_{\{\max_{j \in \mathcal{N}} | \overline{u}_{j,k} + a_k(g_j(\overline{u}_k) + \overline{\epsilon}_{j,k+1}) | \ge M_{\overline{\sigma}_k}\}}, (57)
\overline{\sigma}_{k+1} = \overline{\sigma}_k + I_{\{\max_{j \in \mathcal{N}} | \overline{u}_{j,k} + a_k(g_j(\overline{u}_k) + \overline{\epsilon}_{j,k+1}) | \ge M_{\overline{\sigma}_k}\}}, (58)$$

where $\overline{\sigma}_k = \max\{\sigma_{j,k}, j \in \mathcal{N}\}.$

PROOF. Define $\tilde{u}_{i,k}, \forall i \in \mathcal{N}$ and $\tilde{\sigma}_k$ as follows

$$\tilde{u}_{i,k+1} = (\tilde{u}_{i,k} + a_k(g_i(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{i,k+1}))
\cdot I_{\{\max_{j \in \mathcal{N}} |\tilde{u}_{j,k} + a_k(g_j(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{j,k+1}) | < M_{\tilde{\sigma}_k}\}}
+ u_i^* I_{\{\max_{j \in \mathcal{N}} |\tilde{u}_{j,k} + a_k(g_j(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{j,k+1}) | \ge M_{\tilde{\sigma}_k}\}}, (59)$$

$$\tilde{\sigma}_{k+1} = \tilde{\sigma}_k + I_{\{\max_{j \in \mathcal{N}} |\tilde{u}_{j,k} + a_k(g_j(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{j,k+1}) | \ge M_{\tilde{\sigma}_k}\}}, (60)$$

where $\tilde{u}_{i,1} = \overline{u}_{i,1}, \tilde{\sigma}_1 = \overline{\sigma}_1$. For (57) and (58), it suffices to prove $\tilde{u}_{i,k} = \overline{u}_{i,k}, \forall i \in \mathcal{N}, \tilde{\sigma}_k = \overline{\sigma}_k$, and $\tilde{\sigma}_k = \max\{\sigma_{j,k}, j \in \mathcal{N}\} \ \forall k \geq 1$.

Set $\overline{O}_{i,k+1} = g_i(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{i,k+1}$. Consider $k \in [r(m), r(m+1))$. When $k \in [r(m), \overline{r}(i,m))$ from (55) we have

$$\overline{O}_{i,k+1} = \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(\overline{u}_{j,k}) - h_i(\overline{u}_{i,k})) + \overline{\epsilon}_{i,k+1}$$

$$= \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(\overline{u}_{j,k}) - h_i(u_i^*)) + \overline{\epsilon}_{i,k+1}$$

$$= 0.$$
(61)

When $k \in [\overline{r}(i, m), r(m+1))$ from (56) we have

$$\overline{O}_{i,k+1} = \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(\overline{u}_{j,k}) - h_i(\overline{u}_{i,k})) + \overline{\epsilon}_{i,k+1}$$

$$= \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(\overline{u}_{j,k}) - h_i(u_{i,k})) + \overline{\epsilon}_{i,k+1}$$

$$= \sum_{j \in \mathcal{N}_i} p_{ij} (h_j(u_{j,k}) - h_i(u_{i,k})) + \epsilon_{i,k+1}$$

$$= g_i(\mathbf{u}_k) + \epsilon_{i,k+1}$$

$$= O_{i,k+1}, \tag{62}$$

where the last equality is from (14). Combining (61) and (62) leads to

$$\overline{O}_{i,k+1} = \begin{cases} 0, & k \in [r(m), \overline{r}(i,m)), \\ O_{i,k+1}, & k \in [\overline{r}(i,m), r(m+1)). \end{cases}$$
(63)

We prove the lemma by induction. Firstly, $\tilde{u}_{i,1} = \overline{u}_{i,1}, \forall i \in \mathcal{N}, \tilde{\sigma}_1 = \overline{\sigma}_1$ and $\tilde{\sigma}_1 = \max\{\sigma_{j,1}, j \in \mathcal{N}\}$ by definitions in (59) and (60). Assume that we have proved $\tilde{u}_{i,s} = \overline{u}_{i,s} \ \forall i \in \mathcal{N}, \tilde{\sigma}_s = \overline{\sigma}_s$, and $\tilde{\sigma}_s = \max\{\sigma_{j,s}, j \in \mathcal{N}\}$ for $1 \leq s \leq k$. Without loss of generality assume $k \in [r(m), r(m+1))$ for some $r(m) < \infty$. By definition of r(m), there exists j_0 such that $\sigma_{j_0,k} = m$ and $\sigma_{j,k} < m+1$ for any j. From the inductive assumption it follows that

$$\tilde{\sigma}_k = \overline{\sigma}_k = \max\{\sigma_{j,k}, j \in \mathcal{N}\} = m.$$
 (64)

Then, for k + 1 we proceed to prove

$$\tilde{u}_{i,k+1} = \overline{u}_{i,k+1}, \forall i \in \mathcal{N},$$
(65)

$$\tilde{\sigma}_{k+1} = \overline{\sigma}_{k+1} \quad \tilde{\sigma}_{k+1} = \max\{\sigma_{j,k+1}, j \in \mathcal{N}\}.$$
 (66)

The proof is carried out by three steps. In the first two steps, we prove (65) and (66) for $k \in [r(m), r(m+1)-1)$, respectively. In the third step, we discuss the case where k = r(m+1) - 1.

Set

$$\mathcal{N}^{(1)} = \{ j \in \mathcal{N} : r(j, m) = r(m) \},$$

$$\mathcal{N}^{(2)} = \{ j \in \mathcal{N} : r(m) < r(j, m) < r(m+1) \},$$

$$\mathcal{N}^{(3)} = \{ j \in \mathcal{N} : r(j, m) \ge r(m+1) \}.$$

It is clear that $\mathcal{N} = \mathcal{N}^{(1)} \cup \mathcal{N}^{(2)} \cup \mathcal{N}^{(3)}$, and $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}$, and $\mathcal{N}^{(3)}$ are disjoint.

Step 1: We prove that

$$|\tilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| = |\overline{u}_{j,k} + a_k \overline{O}_{j,k+1}| < M_{\tilde{\sigma}_k}, \forall j \in \mathcal{N},$$
(67)

i.e., $I_{\{\max_{j\in\mathcal{N}}|\tilde{u}_{j,k}+a_k\overline{O}_{j,k+1}|< M_{\tilde{\sigma}_k}\}}=1$ for $k\in[r(m),r(m+1)-1)$.

Noticing the inductive assumptions, $\tilde{u}_{j,k} = \overline{u}_{j,k} \ \forall j \in \mathcal{N}$, and $\tilde{\sigma}_k = \overline{\sigma}_k = m$, we verify (67) for agents in $\mathcal{N}^{(1)}$, $\mathcal{N}^{(2)}$ and $\mathcal{N}^{(3)}$, respectively.

1) For $j \in \mathcal{N}^{(1)}$, we have r(j,m) = r(m) by definition. Since $k \in [r(m), r(m+1)-1)$, the truncation number of agent j has reached m at time k and the next truncation will not occur at k+1. Then it follows that $\sigma_{j,k} = \sigma_{j,r(j,m)} = m = \tilde{\sigma}_k$, where the second equality is from the definition of r(j,m), and $\overline{r}(j,m) = r(j,m) \wedge r(m+1) = r(m)$, which means that $[r(m),\overline{r}(j,m))$ in (55) is empty. By (18) and (19), we have $\sigma_{j,k}' = \sigma_{j,k}$ and $u_{j,k}' = u_{j,k}$, respectively. From (56) it follows that $\overline{u}_{j,k} = u_{j,k}$ and from (63) that $\overline{O}_{j,k+1} = O_{j,k+1}$, which implies

$$\begin{aligned} |\widetilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| &= |\overline{u}_{j,k} + a_k \overline{O}_{j,k+1}| \\ &= |u_{j,k} + a_k O_{j,k+1}| < M_{\sigma_{j,k}} = M_m = M_{\tilde{\sigma}_k}, \end{aligned}$$

where the inequality holds because k + 1 < r(m + 1).

2) For $j \in \mathcal{N}^{(2)}$, we have r(m) < r(j,m) < r(m+1) by definition. Since $k \in [r(m), r(m+1)-1)$, the truncation number of j is smaller than m at k. Then it follows that $\sigma_{j,k} \leq m = \tilde{\sigma}_k$ and $\overline{r}(j,m) = r(j,m)$. By (55) and (56)

$$\overline{u}_{j,k} = \begin{cases} u_j^*, & k \in [r(m), r(j, m)), \\ u_{j,k}, & k \in [r(j, m), r(m+1) - 1). \end{cases}$$

From (63) it follows that

$$\overline{O}_{j,k+1} = \begin{cases} 0, & k \in [r(m), r(j,m)) \\ O_{j,k+1}, & k \in [r(j,m), r(m+1)-1). \end{cases}$$

Besides, by (18) and (19), we have $\sigma_{j,k}^{'}=\sigma_{j,k}$ and $u_{j,k}^{'}=u_{j,k}$ for $k\in[r(j,m),r(m+1)-1)$. Therefore,

$$|\tilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| = |\overline{u}_{j,k} + a_k \overline{O}_{j,k+1}|$$

$$= \begin{cases} |u_j^*|, & k \in [r(m), r(j, m)), \\ |u_{j,k} + a_k O_{j,k+1}|, & k \in [r(j, m), r(m+1) - 1). \end{cases}$$
(68)

Noticing $|u_j^*| < M_0 \le M_{\sigma_{j,k}}$ and k < r(m+1) - 1, for both cases at the right-hand side of (68) we always have

$$|\tilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| < M_{\sigma_{j,k}} \le M_m = M_{\tilde{\sigma}_k}.$$

3) For $j \in \mathcal{N}^{(3)}$, we have r(j,m) > r(m+1) by definition. Then we derive $\sigma_{j,k} < m = \tilde{\sigma}_k$ and $\bar{r}(j,m) = r(m+1)$ 1), which means $[\overline{r}(j,m),r(m+1))$ in (56) is empty. From (55) it follows that $\overline{u}_{j,k}=u_j^*$ and from (63) that $\overline{O}_{j,k+1}=0$. Therefore,

$$|\widetilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| = |\overline{u}_{j,k} + a_k \overline{O}_{j,k+1}| = |u_j^*|.$$

Noticing $|u_i^*| < M_0 \le M_{\sigma_{i,k}}$, we have

$$|\tilde{u}_{j,k} + a_k \overline{O}_{j,k+1}| < M_{\sigma_{j,k}} < M_m = M_{\tilde{\sigma}_k}.$$

Until now we have proved (67) for $k \in [r(m), r(m+1) - 1)$.

From (67) it follows that

$$\tilde{\sigma}_{k+1} = \tilde{\sigma}_k = \overline{\sigma}_k = \max{\{\sigma_{j,k}, j \in \mathcal{N}\}}$$

for $k \in [r(m), r(m+1) - 1)$. Since k + 1 < r(m+1), the possible (m+1)th truncation can happen only after time k + 1. Therefore,

$$\max\{\sigma_{i,k}, j \in \mathcal{N}\} = \max\{\sigma_{i,k+1}, j \in \mathcal{N}\} = \overline{\sigma}_{i,k+1}$$

Thus, (66) holds for k+1 when $k \in [r(m), r(m+1)-1)$.

Step 2: We proceed to prove that (65) holds for k + 1 when $k \in [r(m), r(m+1) - 1)$. Paying attention to (59),(67) and the inductive assumption, we have

$$\tilde{u}_{i,k+1} = \tilde{u}_{i,k} + a_k \overline{O}_{i,k+1},
= \overline{u}_{i,k} + a_k \overline{O}_{i,k+1}, k \in [r(m), r(m+1) - 1), \forall i \in \mathcal{N}.$$
(69)

Similar to **Step 1**, we can prove (65) for agents in $\mathcal{N}^{(1)}$, $\mathcal{N}^{(2)}$, and $\mathcal{N}^{(3)}$, respectively.

1) For $i \in \mathcal{N}^{(1)}$, r(i,m) = r(m) by definition, i.e., agent i earlier than other agents reaches m truncations and $\overline{r}(i,m) = r(i,m) = r(m)$. This means that $[r(m),\overline{r}(i,m))$ in (55) is empty. By (18), $\sigma_{i,k}^{'} = \sigma_{i,k}$ and by (19), $u_{i,k}^{'} = u_{i,k}$. From (56), we have $\overline{u}_{i,k} = u_{i,k}$, and also $\overline{u}_{i,k+1} = u_{i,k+1}$ since k+1 < r(m+1). From (63) it follows that $\overline{O}_{i,k+1} = O_{i,k+1}$. Since k+1 < r(m+1), i.e., there is no truncation for agent i at k+1, from (21) it follows that

$$u_{i,k+1} = u_{i,k} + a_k O_{i,k+1}, \ k \in [r(m), r(m+1) - 1),$$

which associated with (69) yields $\tilde{u}_{i,k+1} = \overline{u}_{i,k+1}$.

2) For $i \in \mathcal{N}^{(2)}$, r(m) < r(i,m) < r(m+1) by definition, and $\overline{r}(i,m) = r(i,m)$. From (55) and (56) it follows that

$$\overline{u}_{i,k} = \begin{cases} u_i^*, & k \in [r(m), r(i, m)), \\ u_{i,k}, & k \in [r(i, m), r(m+1) - 1), \end{cases}$$
(70)

and from (63) that

$$\overline{O}_{i,k+1} = \begin{cases} 0, & k \in [r(m), r(i,m)), \\ O_{i,k+1}, & k \in [r(i,m), r(m+1) - 1). \end{cases}$$
(71)

Combining (69), (70) and (71), we have

$$\tilde{u}_{i,k+1} = \begin{cases} u_i^*, & k \in [r(m), r(i,m)), \\ u_{i,k} + a_k O_{i,k+1} = u_{i,k+1}, & k \in [r(i,m), r(m+1) - 1). \end{cases}$$
(72)

We explain the last line of (72). Notice k+1 < r(m+1), i.e., there is no truncation at k+1. From (18) and (19), we have $\sigma_{j,k}^{'} = \sigma_{j,k}$ and $u_{j,k}^{'} = u_{j,k}$ for $k \in [r(j,m), r(m+1)-1)$. Then, the last line of (72) is derived from (21).

We now proceed to prove

$$\overline{u}_{i,k+1} = \begin{cases} u_i^*, & k \in [r(m), r(i, m)), \\ u_{i,k+1}, & k \in [r(i, m), r(m+1) - 1). \end{cases}$$
(73)

It is clear that for $k+1 \in [r(m)+1,r(m+1))$ there are three possible cases: $k+1 \in (r(m),r(i,m)), k+1 \in (r(i,m),r(m+1))$, and k+1=r(i,m). For the first two cases (73) is directly derived from (55) and (56). We need only to verify the case k+1=r(i,m). To this end, we have $\overline{u}_{i,k+1}=\overline{u}_{i,r(i,m)}=u_{i,r(i,m)}$ by (56) and $u_{i,r(i,m)}=u_i^*$ by the definition of r(i,m). Thus, (73) has been proved. This together with (72) implies that $\tilde{u}_{i,k+1}=\overline{u}_{i,k+1}$ for $k\in [r(m),r(m+1)-1)$.

3) For $i \in \mathcal{N}^{(3)}$, $r(i,m) \geq r(m+1)$ by definition and hence $\overline{r}(i,m) = r(m+1)$. This means that $[\overline{r}(j,m),r(m+1))$ in (56) is empty. By (55), $\overline{u}_{i,k} = u_i^*$ and $\overline{u}_{i,k+1} = u_i^*$ for $k+1 \in (r(m),r(m+1))$. From (63), we have $\overline{O}_{i,k+1} = 0$ which associated with (69) leads to $\tilde{u}_{i,k+1} = u_i^* = \overline{u}_{i,k+1}$.

Step 3: If $r(m+1) = \infty$, then we have completed the proof. This is because it is unnecessary to consider the case k = r(m+1) - 1. It remains to show (65) and (66) when $r(m+1) < \infty$ and k = r(m+1) - 1.

For k=r(m+1)-1, at k+1 the (m+1)th truncation occurs for some agent j_0 , i.e., there exists j_0 such that $|u_{j_0,k}+a_kO_{j_0,k+1}|\geq M_m$ and then $\sigma_{j_0,k+1}=m+1$. Since $r(j_0,m)< k+1=r(m+1)$, we have $\overline{r}(j_0,m)=r(j_0,m)$. From (56) it follows that $\overline{u}_{j_0,k}=u_{j_0,k}$ and from (63) that $\overline{O}_{j_0,k+1}=O_{j_0,k+1}$. Therefore, by the inductive assumption we have

$$\begin{split} \tilde{\sigma}_{k+1} &= \tilde{\sigma}_k + I_{\{\max_{j \in \mathcal{N}} | \tilde{u}_{j,k} + a_k \overline{O}_{j,k+1} | \geq M_{\tilde{\sigma}_k} \}} \\ &= \tilde{\sigma}_k + I_{\{|u_{j_0,k} + a_k O_{j_0,k+1}| \geq M_{\tilde{\sigma}_k} \}} \\ &= \overline{\sigma}_k + 1. \end{split}$$
 (74)

Further by (64), the chain of equalities (74) can be continued as follows

$$\widetilde{\sigma}_{k+1} = \overline{\sigma}_k + 1 = m + 1 = \sigma_{j_0, k+1}
= \overline{\sigma}_{r(m+1)} = \max\{\sigma_{j,k+1}, j \in \mathcal{N}\} = \overline{\sigma}_{i,k+1}.$$

Thus, (66) is proved for k = r(m+1) - 1. We then have $\tilde{u}_{i,k+1} = u_i^* \ \forall i \in \mathcal{N}$ from (59), and $\overline{u}_{i,k+1} = \overline{u}_{i,r(m+1)} = u_i^* \ \forall i \in \mathcal{N}$ from (55). This means that $\tilde{u}_{i,k+1} = \overline{u}_{i,k+1} \ \forall i \in \mathcal{N}, k = r(m+1) - 1$.

Thus, we have proved (65) and (66) for k+1 and completed the proof. \Box

Set $\overline{\mathbf{u}}_k = [\overline{u}_{1,k}, \cdots, \overline{u}_{N,k}]^T, \overline{\epsilon}_{k+1} = [\overline{\epsilon}_{1,k+1}, \cdots, \overline{\epsilon}_{N,k+1}]^T$ and $g(\overline{\mathbf{u}}) = [g_1(\overline{\mathbf{u}}), \cdots, g_N(\overline{\mathbf{u}})]^T$. Then (57) and (58) can be written as

$$\overline{\mathbf{u}}_{k+1} = (\overline{\mathbf{u}}_k + a_k(\mathbf{g}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1}))
\cdot I_{\{\|\overline{\mathbf{u}}_k + a_k(\mathbf{g}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} < M_{\overline{\sigma}_k}\}}
+ u^* I_{\{\|\overline{\mathbf{u}}_k + a_k(\mathbf{g}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} \ge M_{\overline{\sigma}_k}\}},$$

$$\overline{\sigma}_{k+1} = \overline{\sigma}_k + I_{\{\|\overline{\mathbf{u}}_k + a_k(\mathbf{g}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} > M_{\overline{\sigma}_k}\}}.$$
(75)

Lemma 4 Assume A1, A2 i), A2 ii), A3, and A4 hold.

Lemma 4 Assume A1, A2 i), A2 ii), A3, and A4 hold. At the samples $\omega \in \Omega$ where (33) holds, for a given T > 0 and for sufficiently large C > 0, the noise $\{\overline{\epsilon}_{i,k+1}\}$ defined in (55) and (56) satisfies

$$\lim_{k \to \infty} \left| \sum_{s=k}^{l} a_s \overline{\epsilon}_{i,s+1} I_{\{\|\overline{\mathbf{u}}_s\| \le C\}} \right| = 0, \ \forall i \in \mathcal{N},$$

$$l = k, k+1, \cdots, \left(\left(r(m_k + 1) - 1 \right) \wedge m(k, T) \right),$$

$$(77)$$

where $m_k = \sup\{m : r(m) \le k\}$ is the biggest number of truncations occurred at agents up to time k.

PROOF. It is worth noting that the inequality or equality $l \leq r(m_k + 1) - 1$ means that the $(m_k + 1)$ th truncation has not happened yet for any agent, i.e., there is no truncation in (18)-(22) as $s = k, k + 1, \dots, l$ for any agent.

Similar to Lemma 2, since $r(0) = 1 \le k$, the set $\{m : r(m) \le k\}$ is nonempty. This means that m_k is well defined. Besides, we have

$$\left| \sum_{s=k}^{l} a_{s} \overline{\epsilon}_{i,s+1} I_{\{\|\overline{\mathbf{u}}_{s}\| \leq C\}} \right| \leq \left| \sum_{s=k}^{l} a_{s} \overline{\epsilon}_{i,s+1} \right|$$

$$\cdot I_{\{r(m_{k}) \leq s < \overline{r}(i,m_{k})\}} I_{\{\|\overline{\mathbf{u}}_{s}\| \leq C\}}$$

$$+ \left| \sum_{s=k}^{l} a_{s} \overline{\epsilon}_{i,s+1} I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{\|\overline{\mathbf{u}}_{s}\| \leq C\}} \right|.$$
 (78)

First, let us consider the first term at the right-hand side of (78). By noticing (55) it follows that

$$\left| \sum_{s=k}^{l} a_{s} \overline{\epsilon}_{i,s+1} I_{\{r(m_{k}) \leq s < \overline{r}(i,m_{k})\}} I_{\{\|\overline{\mathbf{u}}_{s}\| \leq C\}} \right|$$

$$= \left| \sum_{s=k}^{l} a_{s} \sum_{j \in \mathcal{N}_{i}} p_{ij} (h_{j}(\overline{u}_{j,k}) - h_{i}(u_{i}^{*})) \right|$$

$$\cdot I_{\{r(m_{k}) \leq s < \overline{r}(i,m_{k})\}} I_{\{\|\mathbf{u}_{s}\| \leq C\}} \right|.$$

$$(79)$$

From (55) and (56) we have

$$\overline{u}_{j,k} = \begin{cases} u_j^*, & k \in [r(m), \overline{r}(j,m)), \\ u_{j,k}, & k \in [\overline{r}(j,m), r(m+1)). \end{cases}$$
(80)

Therefore, similar to (29), we see $|\overline{u}_{j,k}| \leq \ln(k-1+c_M)$ for sufficiently large k. This together with the boundedness of $\overline{r}(i, m_k) - r(m_k)$ as proved for (26) implies that the right-hand side of (79) tends to zero as $k \to \infty$.

We now analyze the second term at the right-hand side of (78). From (56) it follows that

$$\left| \sum_{s=k}^{l} a_{s} \overline{\epsilon}_{i,s+1} I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{\|\overline{\mathbf{u}}_{s}\| \leq C\}} \right| \\
\leq \left| \sum_{s=k}^{l} a_{s} \epsilon_{i,s+1} I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{\|\mathbf{u}_{s}\| \leq C\}} \right| \\
+ \left| \sum_{s=k}^{l} a_{s} \sum_{j \in \mathcal{N}_{i}} p_{ij} (h_{j}(u_{j,s}) - h_{j}(\overline{u}_{j,s})) \right. \\
\cdot I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{\|\mathbf{u}_{s}\| \leq C\}} \right|. \tag{81}$$

When $\overline{r}(i, m_k) = r(i, m_k)$, by Lemma 2 the first term at the right-hand side of (81) tends to zero as $k \to \infty$, while it is zero when $\overline{r}(i, m_k) = r(m_k + 1)$.

In view of (80), we have

$$\begin{split} &\sum_{s=k}^{l} a_{s} \sum_{j \in \mathcal{N}_{i}} p_{ij} (h_{j}(u_{j,s}) - h_{j}(\overline{u}_{j,s})) \\ &= \sum_{j \in \mathcal{N}_{i}} p_{ij} \sum_{s=k}^{l} a_{s} (h_{j}(u_{j,s}) - h_{j}(\overline{u}_{j,s})) \\ &= \sum_{j \in \mathcal{N}_{i}} p_{ij} \sum_{s=k}^{l} a_{s} (h_{j}(u_{j,s}) - h_{j}(\overline{u}_{j,s})) I_{\{r(m_{k}) \leq s < \overline{r}(j,m_{k})\}} \\ &= \sum_{j \in \mathcal{N}_{i}} p_{ij} \sum_{s=k}^{l} a_{s} (h_{j}(u_{j,s}) - h_{j}(u_{j}^{*})) I_{\{r(m_{k}) \leq s < \overline{r}(j,m_{k})\}}. \end{split}$$

Therefore, the second term at the right-hand side of (81)

can be rewritten as

$$|\sum_{s=k}^{l} a_{s} \sum_{j \in \mathcal{N}_{i}} p_{ij}(h_{j}(u_{j,s}) - h_{j}(\overline{u}_{j,s})) \cdot I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{\|\mathbf{u}_{s}\| \leq C\}}| = |\sum_{j \in \mathcal{N}_{i}} p_{ij} \sum_{s=k}^{l} a_{s}(h_{j}(u_{j,s}) - h_{j}(u_{j}^{*})) \cdot I_{\{\overline{r}(i,m_{k}) \leq s < r(m_{k}+1)\}} I_{\{r(m_{k}) \leq s < \overline{r}(j,m_{k})\}} I_{\{\|\mathbf{u}_{s}\| \leq C\}}|.$$
(82)

By the boundedness of $\overline{r}(j, m_k) - r(m_k)$, similar to (26), the right-hand side of (82) tends zero as $k \to \infty$.

Combining (78), (79), (81), and (82) we conclude the lemma. $\ \square$

The sequence $\{\overline{\mathbf{u}}_k\}$ has the following property along any its convergent subsequence.

Lemma 5 Assume A1, A2 i), A2 ii), A3, and A4 hold. Let $\{\overline{\mathbf{u}}_k\}$ be generated by (75) and (76) and let $\{\overline{\mathbf{u}}_{n_k}\} \subset \{\overline{\mathbf{u}}_k\}$ be a its convergent subsequence. For any sample $\omega \in \Omega$ where (33) holds, there exist $c_1 > 0$ and $T_1 > 0$ such that

$$\|\overline{\mathbf{u}}_{s+1} - \overline{\mathbf{u}}_{n_k}\| \le c_1 T, \forall s : n_k \le s \le m(n_k, T) \ \forall T \in [0, T_1]$$
(83)

for sufficiently large k.

PROOF. By definition $g(\cdot): \mathbb{R}^N \to \mathbb{R}^N$ is continuous. By the boundedness of $\{\overline{\mathbf{u}}_{n_k}\}$, take $M > \sup_k \{\|\mathbf{u}_{n_k}\|\}$ and set $G = \sup_{\|\mathbf{u}\| \le M+1} \|g(\mathbf{u})\|$. Take $c_1 = 2NG$ and $c_1 > 0$ such that $c_1 c_1 < 1$. We now show that $c_1 c_1 < 1$ satisfy (83) for sufficiently large k.

For $T \in [0, T_1]$, set

$$s_k \triangleq \min\{s > n_k, \|\overline{\mathbf{u}}_{s+1} - \overline{\mathbf{u}}_{n_k}\| > c_1 T\}. \tag{84}$$

We need only to prove $s_k > m(n_k, T)$ for sufficiently large k. If $s_k = \infty$, i.e., $\{s \ge n_k, \|\overline{\mathbf{u}}_{s+1} - \overline{\mathbf{u}}_{n_k}\| > c_1 T\}$ is empty, then we have completed the proof. In the following we assume $s_k < \infty$.

We first prove

$$s_k \le r(\overline{\sigma}_{n_k} + 1) - 1 \tag{85}$$

for sufficiently large k. If $\lim_{k\to\infty} \overline{\sigma}_k = \sigma < \infty$, i.e., the largest number of truncations the algorithm can reach is σ , then $r(\overline{\sigma}_{n_k} + 1) = r(\sigma + 1) = \inf \emptyset = \infty$ for sufficiently large k. This implies (85). On the other hand,

if $\lim_{k\to\infty} \overline{\sigma}_k = \infty$, then $M_{\overline{\sigma}_{n_k}} > M+1$ for sufficiently large k. Since

$$\|\overline{\mathbf{u}}_{s+1} - \overline{\mathbf{u}}_{n_k}\| \le c_1 T, \ \forall s : n_k \le s < s_k,$$

we have

$$\|\overline{\mathbf{u}}_{s+1}\| \le \|\overline{\mathbf{u}}_{n_k}\| + c_1 T < M + 1, \forall s : n_k \le s < s_k.$$
(86)

This means that there is no truncation in (75) and (76) from n_k to s_k and thus (85) is proved.

We now show

$$s_k > m(n_k, T). (87)$$

Assume the converse $s_k \leq m(n_k, T)$. By (86) and the definition of G, we have

$$\left|\sum_{s=n_k}^{s_k} a_s \overline{g}_i(\overline{\mathbf{u}}_s)\right| \le GT = \frac{c_1 T}{2N}.$$
 (88)

Moreover, from (85) and Lemma 4 (where C > M+1), it follows that

$$\left|\sum_{s=n_k}^{s_k} a_s \overline{\epsilon}_{i,s+1}\right| \le \frac{c_1 T}{2N} \tag{89}$$

for sufficiently large k. Thus, combining (88) and (89) we have

$$\begin{split} &\|\overline{\mathbf{u}}_{s_k+1} - \overline{\mathbf{u}}_{n_k}\| \leq \sum_{i=1}^N |\overline{u}_{i,s_k+1} - \overline{u}_{i,n_k}| \\ &\leq \sum_{i=1}^N (|\sum_{s=n_k}^{s_k} a_s \overline{g}_i(\overline{\mathbf{u}}_s)| + |\sum_{s=n_k}^{s_k} a_s \overline{\epsilon}_{i,s+1}|) \leq c_1 T, \end{split}$$

which contradicts with (84). So, the converse assumption is false, and $s_k > m(n_k, T)$ for sufficiently large k. Thus, the lemma is proved. \square

6 Convergence of algorithm

Noticing

$$g(\overline{u}_k) = (P - D)h(\overline{u}_k) = -Lh(\overline{u}_k),$$

we can rewrite (75) and (76) as

$$\overline{\mathbf{u}}_{k+1} = (\overline{\mathbf{u}}_k + a_k(-L\mathbb{h}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1}))
\cdot I_{\{\|\overline{\mathbf{u}}_k + a_k(-L\mathbb{h}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} \le M_{\overline{\sigma}_k}\}}
+ \mathbf{u}^* I_{\{\|\overline{\mathbf{u}}_k + a_k(-L\mathbb{h}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} > M_{\overline{\sigma}_k}\}},$$
(90)

$$\overline{\sigma}_{k+1} = \overline{\sigma}_k + I_{\{\|\overline{\mathbf{u}}_k + a_k(-L\mathbb{h}(\overline{\mathbf{u}}_k) + \overline{\epsilon}_{k+1})\|_{\infty} > M_{\overline{\sigma}_k}\}}. \tag{91}$$

Lemma 6 Assume A2 iii) holds. The set $J \cap \{u \in \mathbb{R}^N : \mathbb{1}^T u = c\}$ is a singleton, where c is a constant.

PROOF. In fact, under condition A2 iii) for any given $\alpha_i > 0, i = 1, 2, \dots, N$ and constant c, the system of equations with respect to u_1, \dots, u_N

$$\begin{cases}
h_1(u_1) = \dots = h_N(u_N), \\
\alpha_1 u_1 + \dots + \alpha_N u_N = c
\end{cases}$$
(92)

has a unique solution. Set $h_1(u_1) = \cdots = h_N(u_N) = b$. By A2 iii) $h_i(\cdot)$ has the inverse function $h_i^{-1}(\cdot)$, which is strictly monotonically increasing with range $(-\infty, +\infty)$. For any given constant c, consider the equation of b

$$\alpha_1 h_1^{-1}(b) + \dots + \alpha_N h_N^{-1}(b) = c.$$

Since $\alpha_i > 0$, $\alpha_1 h_1^{-1}(\cdot) + \cdots + \alpha_N h_N^{-1}(\cdot)$ is also strictly monotonically increasing with range $(-\infty, +\infty)$. Therefore, the above equation has a unique solution denoted as b_c , which means the original system of equations (92) has the unique solution $u_i = h_i^{-1}(b_c), i = 1, 2, \cdots, N$. Thus the lemma is proved, since $\mathbbm{1}^T \mathbbm{1}^T = u_1 + \cdots + u_N$ and the first N-1 equations in (92) are equivalent to $h(u) \in \mathrm{Span}\{\mathbbm{1}\}$. \square

Notice that A2 i) and iii) imply

$$0 \le \int_{u_i^{(0)}}^c h_i(t) dt \xrightarrow[|c| \to \infty]{} \infty, \tag{93}$$

where $u_i^{(0)}$ is the root of $h_i(\cdot)$. Define the continuously differentiable function $v(\cdot): \mathbb{R}^N \to \mathbb{R}$ as

$$v(\mathbf{u}) = \sum_{i \in \mathcal{N}} \int_{u_i^{(0)}}^{u_i} h_i(t) dt.$$
 (94)

It is clear that $\nabla_{\mathbf{u}} v(\mathbf{u}) = \mathbb{h}(\mathbf{u})$. In view of (93) and (94), there exists a constant $c_0 > 0$ such that $c_0 > \|\mathbf{u}^*\|_{\infty}$ and

$$\min\{\int_{u_i^{(0)}}^{c_0} h_i(t) dt, \int_{u_i^{(0)}}^{-c_0} h_i(t) dt\}$$

$$> \sum_{j \in \mathcal{N}} \int_{u_j^{(0)}}^{u_j^*} h_j(t) dt = v(\mathbf{u}^*).$$
(95)

By A4, the Laplace matrix L of the undirected graph has the following property

$$y^T L y = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} p_{ij} (y_i - y_j)^2 = \frac{1}{2} \sum_{i,j \in \mathcal{N}} p_{ij} (y_i - y_j)^2,$$

for $y = [y_1, \dots, y_N]^T \in \mathbb{R}^N$. Therefore, for any $0 < \delta < \Delta$ there exists $\zeta > 0$ such that

$$\sup_{\delta \le d(\mathbf{u}, J) \le \Delta} \nabla_{\mathbf{u}}^{T} v(\mathbf{u}) (-L) h(\mathbf{u})$$

$$= \frac{1}{2} \sup_{\delta \le d(\mathbf{u}, J) \le \Delta} - \sum_{(i,j) \in \mathcal{E}} p_{i,j} (h_i(u_i) - h_j(u_j))^2 \le -\zeta.$$
(96)

Theorem 2.2.1 in Chen (2002) is not applicable for proving the convergence $d(\overline{\mathfrak{u}}_k,J) \xrightarrow[k \to \infty]{} 0$ for the algorithm (90) and (91), since we cannot guarantee that $v(J) \triangleq \{v(\mathfrak{u}) | \mathfrak{u} \in J\}$ is nowhere dense. However, the idea of proof of that theorem can still be used.

Define $S^*(\epsilon) \triangleq \{\mathbf{u}: \mathbbm{1}_N^T \mathbf{u} \in [\mathbbm{1}_N^T \mathbf{u}^* - \epsilon, \mathbbm{1}_N^T \mathbf{u}^* + \epsilon]\}, \epsilon \geq 0$ and $S^*(0) \triangleq \{u: \mathbbm{1}_N^T \mathbf{u} = \mathbbm{1}_N^T \mathbf{u}^*\}$. To prove the convergence of SAAWET the key step is to show that the truncation ceases in a finite number of steps. This also is the key issue for convergence analysis of the algorithm (90) and (91), and its proof is motivated by that for SAAWET given in Chen (2002).

Lemma 7 Assume A1-A4 hold. For samples $\omega \in \Omega$ where (33) holds,

$$\lim_{k \to \infty} \overline{\sigma}_k \triangleq \overline{\sigma} < \infty. \tag{97}$$

PROOF. Assume the converse $\overline{\sigma}_k \xrightarrow[k \to \infty]{} \infty$, i.e., there exists a sequence $\{n_l\}_{l \geq 1}$ such that $\overline{\sigma}_{n_l} = \overline{\sigma}_{n_l-1} + 1$ and $\overline{\mathbf{u}}_{n_l} = \mathbf{u}^*$. To reach a contradiction we complete the proof by three steps.

Step 1: We first show

$$d(\overline{\mathbf{u}}_k, S^*(0)) \xrightarrow[k \to \infty]{} 0 \tag{98}$$

under the converse assumption.

The discrete-time axis can be divided into $[n_l, n_{l+1}), l = 1, 2, \cdots$ by $\{n_l\}$ and we investigate the limit of $\overline{\mathbf{u}}_k$ on these intervals. Since $\overline{\mathbf{u}}_{n_l} = \mathbf{u}^*$, (98) is satisfied at $\overline{\mathbf{u}}_{n_l}$. We need only to consider (n_l, n_{l+1}) . When T > 0 is sufficiently small, by Lemma 5 there is no truncation for (90) and (91) from n_l to $m(n_l, T)$ for sufficiently large l, i.e., $m(n_l, T) < r(\overline{\sigma}_{n_l} + 1) = n_{l+1}$. This means $r(\overline{\sigma}_{n_l}) = n_l \le m(n_l, T) < r(\overline{\sigma}_{n_l} + 1) = n_{l+1}$. From (36) it follows that $m(n_l, T) - n_l > (\exp(T) - 1)n_l - \exp(T) - 1 \xrightarrow[l \to \infty]{} \infty$. Let $i_l \in \arg\max_{j \in \mathcal{N}} \{r(j, \overline{\sigma}_{n_l})\}$. By $r(i_l, \overline{\sigma}_{n_l}) - r(\overline{\sigma}_{n_l}) < d(\mathcal{G})$, we have

$$n_l \le r(i_l, \overline{\sigma}_{n_l}) \le m(n_l, T) < r(\overline{\sigma}_{n_l} + 1) = n_{l+1}$$

for sufficiently large l. In the following, the limit of \overline{u}_k will be shown on $(n_l, r(i_l, \overline{\sigma}_{n_l})]$ and $(r(i_l, \overline{\sigma}_{n_l}), n_{l+1})$, respectively.

1) For $k \in (n_l, r(i_l, \overline{\sigma}_{n_l})]$

$$\overline{\mathbf{u}}_k = \overline{\mathbf{u}}_{k-1} + a_{k-1}(-L\mathbb{h}(\overline{\mathbf{u}}_{k-1}) + \overline{\epsilon}_k).$$

Multiplying this equation with $\mathbb{1}_N^T$ from the left and noticing that $\mathbb{1}_N$ is the left eigenvector of L corresponding to the eigenvalue 0 and that $\overline{\mathbb{u}}_{n_l} = \mathbb{u}^*$, we have

$$\mathbb{1}_{N}^{T}\overline{\mathbf{u}}_{k} = \mathbb{1}_{N}^{T}\overline{\mathbf{u}}_{k-1} + a_{k-1}\mathbb{1}_{N}^{T}\overline{\epsilon}_{k}
= \mathbb{1}_{N}^{T}\mathbf{u}^{*} + \sum_{s=n_{l}}^{k-1} a_{s}\mathbb{1}_{N}^{T}\overline{\epsilon}_{s+1}.$$
(99)

Since $k-1 < m(n_l, T)$, by (86), and Lemma 4 (where C > M+1) the second term at the right-hand side of (99) tends to zero as $n_l \to \infty$. Thus, $\mathbb{1}_N^T \overline{\mathbb{u}}_k \xrightarrow[k \to \infty]{} \mathbb{1}_N^T \mathbb{u}^*$ for $k \in (n_l, r(i_l, \overline{\sigma}_{n_l})]$.

2) For $k \in (r(i_l, \overline{\sigma}_{n_l}), n_{l+1})$, noticing $r(j, \overline{\sigma}_{n_l}) < k < r(\overline{\sigma}_{n_l} + 1), \forall j$, by (14) and (63), it follows that $\overline{\mathbb{O}}_k = \mathbb{O}_k = (P - D) \mathbb{y}_k + \epsilon_k^{(1)}$, where $\mathbb{O}_k = [O_{1,k}, \cdots, O_{N,k}]^T$ and $\epsilon_k^{(1)} = [\epsilon_{1,k}^{(1)}, \cdots, \epsilon_{N,k}^{(1)}]$. Then, we have

$$\overline{\mathbf{u}}_k = \overline{\mathbf{u}}_{k-1} + a_{k-1} \overline{\mathbb{O}}_k$$

$$= \overline{\mathbf{u}}_{k-1} + a_{k-1} \mathbb{O}_k = \overline{\mathbf{u}}_{k-1} + a_{k-1} (-L \mathbf{y}_k + \epsilon_k^{(1)}).$$

Multiplying this equation with $\mathbb{1}_N^T$ from left and noticing $\mathbb{1}_N$ is the left eigenvector of L corresponding to eigenvalue 0, we derive

$$\mathbb{1}_{N}^{T}\overline{\mathbf{u}}_{k} = \mathbb{1}_{N}^{T}\overline{\mathbf{u}}_{k-1} + a_{k-1}\mathbb{1}_{N}^{T}\epsilon_{k}^{(1)}
= \mathbb{1}_{N}^{T}\overline{\mathbf{u}}_{r(i_{l},\overline{\sigma}_{n_{l}})} + \sum_{s=r(i_{l},\overline{\sigma}_{n_{l}})}^{k-1} a_{s}\mathbb{1}_{N}^{T}\epsilon_{s+1}^{(1)}.$$
(100)

The second term at the right-hand side of (100) tends to zero as $k \to \infty$ since $\|\sum_{k=1}^{\infty} a_k \epsilon_k^{(1)}\| < \infty$ by A3. We now show that the first term at the right-hand side of (100) converges:

$$\mathbb{1}_{N}^{T}\overline{\mathbb{u}}_{r(i_{l},\overline{\sigma}_{n_{l}})} \xrightarrow[k \to \infty]{} \mathbb{1}_{N}^{T}\mathbb{u}^{*}. \tag{101}$$

Consider the components of $\overline{\mathbb{u}}_{r(i_l,\overline{\sigma}_{n_l})}$. First, by (56) it follows that

$$\overline{u}_{i_l,r(i_l,\overline{\sigma}_{n_l})} = u_{i_l,r(i_l,\overline{\sigma}_{n_l})} = u_{i_l}^*, \tag{102}$$

while for $j \neq i_l$ we have $r(\overline{\sigma}_{n_l}) \leq r(j, \overline{\sigma}_{n_l}) \leq r(i_l, \overline{\sigma}_{n_l}) < r(i_l, \overline{\sigma}_{n_l})$

 $r(\overline{\sigma}_{n_l}+1)$, and hence

$$\overline{u}_{j,r(i_{l},\overline{\sigma}_{n_{l}})} = u_{j,r(i_{l},\overline{\sigma}_{n_{l}})}$$

$$= u_{j,r(j,\overline{\sigma}_{n_{l}})} + \sum_{s=r(j,\overline{\sigma}_{n_{l}})}^{r(i_{l},\overline{\sigma}_{n_{l}})-1} a_{s}O_{j,s+1}$$

$$= u_{j}^{*} + \sum_{s=r(j,\overline{\sigma}_{n_{l}})}^{r(i_{l},\overline{\sigma}_{n_{l}})-1} a_{s}O_{j,s+1}. \tag{103}$$

Noticing $r(i_l, \overline{\sigma}_{n_l}) - r(j, \overline{\sigma}_{n_l}) \leq r(i_l, \overline{\sigma}_{n_l}) - r(\overline{\sigma}_{n_l}) \leq d(\mathcal{G})$ and (28), we conclude that the second term at the right-hand side of (103) tends to zero as $k \to \infty$. This combined with (102) implies (101). Therefore, by (100) we have $\mathbb{1}_N^T \overline{\mathbb{u}}_k \xrightarrow[k \to \infty]{} \mathbb{1}_N^T \mathbb{u}^*$ for $k \in (r(i_l, \overline{\sigma}_{n_l}), n_{l+1})$.

Combining 1) and 2) we derive (98).

Step 2: Consider $\omega \in \Omega$ where (33) holds. Let the sequence $\{\overline{\mathbf{u}}_k\}$ be generated by (90) and (91) and let S be a closed subset of \mathbb{R}^N such that $J \cap S \neq \emptyset$. Assume $\overline{\mathbf{u}}_k \in S$ for $k \geq k_0$. If $[\delta_1, \delta_2]$ is an interval such that $d([\delta_1, \delta_2], v(J \cap S)) > 0$, then for any bounded subsequence $\{\overline{\mathbf{u}}_{n_k}\}_{n_k \geq k_0} \subset \{\overline{\mathbf{u}}_k\}_{k \geq k_0}, \{v(\overline{\mathbf{u}}_n)\}_{n \geq k_0}$ cannot cross $[\delta_1, \delta_2]$ infinitely many times with starting points $\overline{\mathbf{u}}_{n_k}$ where " $\{v(\overline{\mathbf{u}}_n)\}_{n \geq k_0}$ crosses $[\delta_1, \delta_2]$ with starting points $\overline{\mathbf{u}}_{n_k}$ means that $v(\overline{\mathbf{u}}_{n_k}) \leq \delta_1$ and there exists $l_k > n_k$ such that $v(\overline{\mathbf{u}}_{l_k}) \geq \delta_2$ and $\delta_1 < v(\overline{\mathbf{u}}_n) < \delta_2$ for $n : n_k < n < l_k$.

In what follows n and n_l are always assumed to be equal to or greater than n_0 . Assume the converse that $v(\overline{\mathbb{u}}_n)$ crosses $[\delta_1, \delta_2]$ infinitely many times with starting points $\overline{\mathbb{u}}_{n_k}$. Without loss of generality, we may assume $\{\overline{\mathbb{u}}_{n_k}\}$ is convergent: $\lim_{k\to\infty} \overline{\mathbb{u}}_{n_k} \triangleq \overline{\mathbb{u}} \in S$ where $\overline{\mathbb{u}} \in S$ because S is closed. From (28) and (63) we have

$$\|\overline{\mathbf{u}}_{n_k+1} - \overline{\mathbf{u}}_{n_k}\| = \|a_{n_k}\overline{\mathbf{0}}_{n_k+1}\| \xrightarrow[k \to \infty]{} 0.$$
 (104)

By the definition of crossing, $v(\overline{\mathbf{u}}_{n_k}) \leq \delta_1 < v(\overline{\mathbf{u}}_{n_k+1})$ which associated with (104) implies $\lim_{k\to\infty} v(\overline{\mathbf{u}}_{n_k}) = \delta_1 = v(\overline{\mathbf{u}})$. In view of $d([\delta_1,\delta_2],v(J\cap S)) > 0$ and $\overline{\mathbf{u}} \in S$, we conclude that

$$d(\overline{\mathbf{u}}, J) > 0. \tag{105}$$

This is because if $d(\overline{\mathbb{u}},J)=0$, then $d(\overline{\mathbb{u}}_{n_k},J) \xrightarrow[k \to \infty]{} 0$. However, $d(\overline{\mathbb{u}}_{n_k},S) \xrightarrow[k \to \infty]{} 0$, so $d(\overline{\mathbb{u}}_{n_k},J\cap S) \xrightarrow[k \to \infty]{} 0$, which implies that $d(v(\overline{\mathbb{u}}_{n_k}),v(J\cap S)) \xrightarrow[k \to \infty]{} 0$. This contradicts to $\lim_{k \to \infty} v(\overline{\mathbb{u}}_{n_k}) = \delta_1$ and $d([\delta_1,\delta_2],v(J\cap S)) > 0$.

When T > 0 is sufficiently small, by Lemma 5 there is no truncation for (90) and (91) with time k running from

 n_k to $m(n_k,T)$ for sufficiently large n_k , i.e., $m(n_k,T) < r(\overline{\sigma}_{n_k}+1)$. Assume k is large enough so that (83) also holds. Then, we have

$$v(\overline{\mathbf{u}}_{m(n_k,T)+1}) - v(\overline{\mathbf{u}}_{n_k}) = \sum_{l=n_k}^{m(n_k,T)} a_l \overline{\mathbf{O}}_{l+1}^T \nabla_{\mathbf{u}} v(\overline{\mathbf{u}}) + o(T)$$

$$= \sum_{l=n_k}^{m(n_k,T)} a_l \mathbf{h}^T(\overline{\mathbf{u}}) (P - D) \nabla_{\mathbf{u}} v(\overline{\mathbf{u}}) + \sum_{l=n_k}^{m(n_k,T)} a_l \overline{\epsilon}_{l+1}$$

$$+ \sum_{l=n_k}^{m(n_k,T)} a_l (\mathbf{h}(\overline{\mathbf{u}}_l) - \mathbf{h}(\overline{\mathbf{u}})) (P - D) \nabla_{\mathbf{u}} v(\overline{\mathbf{u}}) + o(T).$$

$$(106)$$

By (86) and Lemma 4 (where C > M+1), it follows that

$$\sum_{l=n_k}^{m(n_k,T)} a_l \overline{\epsilon}_{l+1} = o(T)$$
 (107)

for sufficiently small T > 0 and large enough k. By Lemma 5, the continuity of $h(\cdot)$, and (86) we have

$$\sum_{l=n_k}^{m(n_k,T)} a_l(\mathbb{h}(\overline{\mathbb{u}}_l) - \mathbb{h}(\overline{\mathbb{u}}))(P-D)\nabla_{\mathbb{u}}v(\overline{\mathbb{u}}) = o(T).$$
(108)

This incorporating with (106), (107) and (108) leads to

$$v(\overline{\mathbf{u}}_{m(n_k,T)+1}) - v(\overline{\mathbf{u}}_{n_k})$$

$$= \sum_{l=n_k}^{m(n_k,T)} a_l \mathbf{h}^T(\overline{\mathbf{u}})(P-D)\nabla_{\mathbf{u}}v(\overline{\mathbf{u}}) + o(T). \quad (109)$$

Thus, by (96), (105), and (109) there exists a $\zeta > 0$ such that

$$v(\overline{\mathbf{u}}_{m(n_k,T)+1}) - v(\overline{\mathbf{u}}_{n_k}) < -\frac{\zeta}{2}T \tag{110}$$

for sufficiently large k. On the other hand, by Lemma $\,\,5\,$ we have

$$\max_{n_k \le s \le m(n_k, T)} |v(\overline{\mathbf{u}}_{s+1}) - v(\overline{\mathbf{u}}_{n_k})| \xrightarrow[T \to 0]{} 0$$

which means that $v(\overline{\mathbf{u}}_{m(n_k,T)+1}) \in [\delta_1, \delta_2]$ for sufficiently small T > 0. This is a contradiction to (110).

Step 3: By (95) and the definition of $v(\cdot)$ we have

$$\begin{split} & \inf_{\|\mathbf{u}\|_{\infty} = c_0} v(\mathbf{u}) = \min_{i \in \mathcal{N}} \Big\{ \min\{ \int_{u_i^0}^{c_0} h_i(t) \mathrm{d}t, \\ & \int_{u_i^0}^{-c_0} h_i(t) \mathrm{d}t \} \Big\} > \sum_{j \in \mathcal{N}} \int_{u_j^0}^{u_j^*} h_j(t) \mathrm{d}t \\ & = v(\mathbf{u}^*). \end{split}$$

By Lemma 6, $J \cap S^*(0)$ is a singleton. Thus, there exists a nonempty interval $[\delta_1, \delta_2]$ such that $d([\delta_1, \delta_2], v(J \cap S^*(0))) > 0$ and $[\delta_1, \delta_2] \subset (v(\mathfrak{u}^*), \inf_{\|\mathfrak{u}\|_{\infty} = c_0} v(\mathfrak{u}))$. So, there exists a sufficiently small $\epsilon > 0$ such that $d([\delta_1, \delta_2], v(J \cap S^*(\epsilon))) > 0$. By (98), there exists k_0 such that $\overline{\mathfrak{u}}_k \in S^*(\epsilon)$ for $k \geq k_0$. Since $M_k \xrightarrow[k \to \infty]{} \infty$, there exists k_1 such that $M_{\overline{\sigma}_{n_k}} > c_0$ for $k > k_1$. By the definition of $\{n_k\}$ we have

$$\|\overline{\mathbf{u}}_{n_{k+1}-1} + a_{n_{k+1}-1}\overline{\mathbb{O}}_{n_{k+1}}\|_{\infty} > M_{\overline{\sigma}_{n_k}}$$

for $k > k_0 \vee k_1$. This means that $\{v(\overline{\mathbf{u}}_n), n > k_0 \vee k_1\}$ crosses $[\delta_1, \delta_2]$ infinitely many times with starting points $\overline{\mathbf{u}}_{n_k} = \mathbf{u}^*$, but it is impossible by assertion in **Step 2**. Thus, the lemma is proved. \square

Set $\overline{\sigma} = \overline{\sigma}_K$ where K is the smallest time when the algorithm defined by (90) and (91) has no more truncations. For the samples $\omega \in \Omega$ where (33) holds, Lemma 7 means that there are only a finite number of truncations for (90) and (91). At the same time, noticing (55) and (56), we conclude that there are only finitely number of times for which $\{\overline{\mathbf{u}}_k\}$ may differ from $\{\mathbf{u}_k\}$, and $\overline{\mathbf{u}}_k = \mathbf{u}_k$ for $k \geq K$. For $\epsilon \geq 0$ set $S(\epsilon) \triangleq \{\mathbf{u} : \mathbbm{1}_N^T \mathbf{u} \in [\mathbbm{1}_N^T \mathbf{u}^* + \sum_{s=K}^\infty a_s \mathbbm{1}_N^T \overline{\epsilon}_{s+1} - \epsilon, \mathbbm{1}_N^T \mathbf{u}^* + \sum_{s=K}^\infty a_s \mathbbm{1}_N^T \overline{\epsilon}_{s+1} + \epsilon]\}$. After the truncation having ceased the algorithm becomes

$$\overline{\mathbf{u}}_{k+1} = \overline{\mathbf{u}}_k + a_k(-L\mathbf{y}_{k+1} + \epsilon_{k+1}^{(1)}), \ k \ge K,$$

where $\epsilon_{k+1}^{(1)}=[\epsilon_{1,k+1}^{(1)},\cdots,\epsilon_{N,k+1}^{(1)}].$ Therefore

$$\sum_{s=K}^{\infty} a_s \mathbb{1}_N^T \overline{\epsilon}_{s+1} = \sum_{s=K}^{\infty} a_s \mathbb{1}_N^T \epsilon_{s+1}^{(1)} < \infty,$$

and hence $S(\epsilon)$ is well defined. Similar to **Step 1** in the proof of Lemma 7, we have $d(\overline{u}_k, S(0)) \xrightarrow[k \to \infty]{} 0$.

Theorem 8 Assume A1-A4 hold. Then applying the algorithm (18)-(22) to the multi-agent systems composed of the Hammerstein systems (1) and the Wiener systems (2) leads to consensus:

$$y_{i,k} \xrightarrow[k \to \infty]{} y^0 \ a.s., \forall i \in \mathcal{N},$$
 (111)

where $y^0=y^0(\omega)$ may depend on samples ω and is such that $|y^0|<\infty$ a.s., and $y^0\mathbbm{1}_N=\mathbbm{h}(\mathbbm{u}^0)$, where

$$h_i(u) = \begin{cases} \frac{d_i}{c_i} f_i(u), & \text{if } i \text{ is the Hammerstein system,} \\ f_i(\frac{d_i}{c_i} u), & \text{if } i \text{ is the Wiener system.} \end{cases}$$

PROOF. We first note that $J \cap S(0) \triangleq \{u^0\}$ is a singleton by Lemma 6. By Lemma 1 in Chen (2007), if $\lim_{k\to\infty} u_{i,k} = u_i$, then $\lim_{k\to\infty} y_{i,k} = h_i(u_i)$. Therefore, to prove (111) it suffices to show that $d(u_k, J \cap S(0)) \xrightarrow[k\to\infty]{} 0$ or $u_k \xrightarrow[k\to\infty]{} u^0$. From Lemma 7 we know that $\{\overline{u}_k\}$ is bounded. Set

$$v_1 \triangleq \liminf_{k \to \infty} v(\overline{\mathbf{u}}_k) \leq \limsup_{k \to \infty} v(\overline{\mathbf{u}}_k) \triangleq v_2.$$

If $v_1 < v_2$, then there exist δ_1 and δ_2 such that $\delta_1 < \delta_2$, $d([\delta_1, \delta_2], v(J \cap S(0)) > 0$, and $[\delta_1, \delta_2] \subset (v_1, v_2)$. Thus, there exist $\epsilon > 0$ and k_0 such that $d([\delta_1, \delta_2], v(J \cap S(\epsilon))) > 0$ and $\overline{u}_k \in S(\epsilon), k > k_0$. This implies that $\{v(\overline{u}_k), k > k_0\}$ crosses $[\delta_1, \delta_2]$ infinitely many times and contradicts to what proved in **Step 2** in the proof of Lemma 7. So, $v_1 = v_2$, i.e., $\{v(\overline{u}_k)\}$ is convergent.

Assume the converse: there exists $\{\overline{\mathbf{u}}_{n_k}\}\subset\{\overline{\mathbf{u}}_k\}$ such that $\lim_{k\to\infty}\overline{\mathbf{u}}_{n_k}=\overline{\mathbf{u}}\neq\mathbf{u}^0$. Noticing $d(\overline{\mathbf{u}}_k,S(0))\xrightarrow[k\to\infty]{}0$, similar to (110) we have $v(\overline{\mathbf{u}}_{m(n_k,T)+1})-v(\overline{\mathbf{u}}_{n_k})\leq -\frac{\zeta}{2}T$ for sufficiently small T>0 and large enough k. This contradicts to the convergence of $v(\overline{\mathbf{u}}_k)$. Therefore, $d(\overline{\mathbf{u}}_k,J\cap S(0))\xrightarrow[k\to\infty]{}0$. By Lemma 7, $\{\mathbf{u}_k\}$ may differ from $\{\overline{\mathbf{u}}_k\}$ only by a finite number of terms, so $d(\mathbf{u}_k,J\cap S(0))\xrightarrow[k\to\infty]{}0$. \square

7 Numerical simulation

Consider the undirected communication graph with four agents presented in Fig. 1. The corresponding Laplace matrix is as follows:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$

Using the algorithm given by (18)-(22), we compute the output of the systems for three cases: 1. All four agents are Hammerstein systems (H) but with different parameters and static functions. 2. All four agents are Wiener systems (W) but with different parameters and static functions. 3. Two are Hammerstein systems and

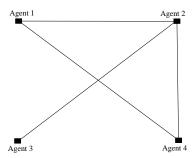


Fig. 1. Simulation graph

the other two are Wiener systems. To be precise, they are as follows.

Case 1.

$$\begin{split} H: \begin{cases} v_{1,k} &= f_1(u_{1,k}) = -u_{1,k}^3 - u_{1,k}, \\ y_{1,k+1} + 0.2y_{1,k} + 0y_{1,k-1} + 0.6y_{1,k-2} &= v_{1,k} \\ -0.3v_{1,k-1} - 1.2v_{1,k-2}; \end{cases} \\ H: \begin{cases} v_{2,k} &= f_2(u_{2,k}) = -2u_{2,k} + 1, \\ y_{2,k+1} + 0.6y_{2,k} + 0.5y_{2,k-1} + 0.4y_{2,k-2} &= v_{2,k} \\ -v_{2,k-1} - 2v_{2,k-2}; \end{cases} \\ H: \begin{cases} v_{3,k} &= f_3(u_{3,k}) = (u_{3,k} - 1)^3, \\ y_{3,k+1} - 0.15y_{3,k} + 0y_{3,k-1} + 0.5y_{3,k-2} &= v_{3,k} \\ +0.2v_{3,k-1} - 0.4v_{3,k-2}; \end{cases} \\ H: \begin{cases} v_{4,k} &= f_4(u_{4,k}) = u_{4,k}^3 + 1, \\ y_{4,k+1} + 0.76y_{4,k} + 0.5y_{4,k-1} + 0.6y_{4,k-2} &= v_{4,k} \\ +0.5v_{4,k-1}. \end{cases} \end{split}$$

Case 2.

$$\begin{split} W: \begin{cases} v_{1,k+1} + 0.2v_{1,k} + 0v_{1,k-1} + 0.6v_{1,k-2} &= u_{1,k} \\ -0.3u_{1,k-1} - 1.2u_{1,k-2}, \\ y_{1,k+1} &= f_1(v_{1,k+1}) = -v_{1,k+1}^3 - v_{1,k+1}; \\ W: \begin{cases} v_{2,k+1} + 0.6v_{2,k} + 0.5v_{2,k-1} + 0.4v_{2,k-2} &= u_{2,k} \\ -u_{2,k-1} - 2u_{2,k-2}, \\ y_{2,k+1} &= f_2(v_{2,k+1}) = -2v_{2,k+1} + 1; \\ \end{cases} \\ W: \begin{cases} v_{3,k+1} - 0.15v_{3,k} + 0v_{3,k-1} + 0.5v_{3,k-2} &= u_{3,k} \\ +0.2u_{3,k-1} - 0.4u_{3,k-2}, \\ y_{3,k+1} &= f_3(v_{3,k+1}) &= (v_{3,k+1} - 1)^3; \\ \end{cases} \\ W: \begin{cases} v_{4,k+1} + 0.76v_{4,k} + 0.5v_{4,k-1} + 0.6v_{4,k-2} &= u_{4,k} \\ +0.5u_{4,k-1}, \\ y_{4,k+1} &= f_4(v_{4,k+1}) &= v_{4,k+1}^3 + 1. \end{cases} \end{split}$$

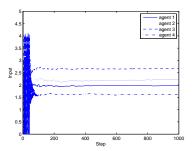


Fig. 2. Inputs for Case 1

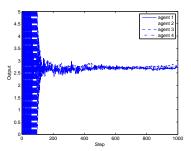


Fig. 3. Outputs for Case 1

Case 3.

$$\begin{split} W: \begin{cases} v_{1,k+1} + 0.2v_{1,k} + 0v_{1,k-1} + 0.6v_{1,k-2} &= u_{1,k} \\ -0.3u_{1,k-1} - 1.2u_{1,k-2}, \\ y_{1,k+1} &= f_1(v_{1,k+1}) = -v_{1,k+1}^3 - v_{1,k+1}; \end{cases} \\ W: \begin{cases} v_{2,k+1} + 0.6v_{2,k} + 0.5v_{2,k-1} + 0.4v_{2,k-2} &= u_{2,k} \\ -u_{2,k-1} - 2u_{2,k-2}, \\ y_{2,k+1} &= f_2(v_{2,k+1}) = -2v_{2,k+1} + 1; \end{cases} \\ H: \begin{cases} v_{3,k} = f_3(u_{3,k}) = (u_{3,k} - 1)^3, \\ y_{3,k+1} - 0.15y_{3,k} + 0y_{3,k-1} + 0.5y_{3,k-2} &= v_{3,k} \\ +0.2v_{3,k-1} - 0.4v_{3,k-2}; \end{cases} \\ H: \begin{cases} v_{4,k} = f_4(u_{4,k}) = u_{4,k}^3 + 1, \\ y_{4,k+1} + 0.76y_{4,k} + 0.5y_{4,k-1} + 0.6y_{4,k-2} &= v_{4,k} \\ +0.5v_{4,k-1}. \end{cases} \end{split}$$

All observation noises $\{\epsilon_{12,k}\}, \{\epsilon_{14,k}\}, \{\epsilon_{21,k}\}, \{\epsilon_{23,k}\}, \{\epsilon_{24,k}\}, \{\epsilon_{32,k}\}, \{\epsilon_{41,k}\}, \text{ and } \{\epsilon_{42,k}\} \text{ are mutually independent and normally distributed with with zero mean and variance 1. It can straightforwardly be verified that A1-A4 hold. We take all initial values to be 0 and <math>u_1^* = 1, u_2^* = 2, u_3^* = 3, u_4^* = 4, c_M = 55$. The algorithm (18) - (22) is applied. Simulation results are presented in Fig. 2 - Fig. 7 for cases 1-3, respectively. From the figures it is seen that the output consensus is achieved and the input at all agents are convergent for all these cases.

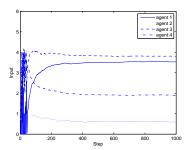


Fig. 4. Inputs for Case 2

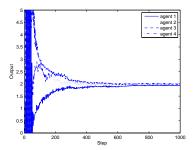


Fig. 5. Outputs for Case 2

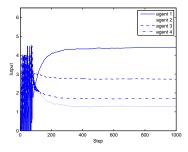


Fig. 6. Inputs for Case 3

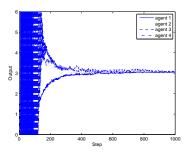


Fig. 7. Outputs for Case 3

8 Conclusions

In this paper the output consensus problem of networked Hammerstein and Wiener systems is studied in a noisy communication environment. Each agent being a Hammerstein or Wiener system is assumed to be open-loop stable, and its static nonlinearity is allowed to grow up but not faster than a polynomial. A control algorithm based on DSAAWET is proposed. The algorithm is transformed to a centralized SAAWET by introducing auxiliary sequences, but its convergence is established by a treatment different from a direct application of the convergence theorem presented in Chen (2002). It is shown that the proposed algorithm leads to the output consensus with probability one. For further research it is of interest to consider the output consensus problem for other subsystems, for example, the nonlinear ARX systems, the multi-input multi-output systems, etc. It is also of interest to consider other kind of communication graphs.

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References

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Control* 48(6), 988-1001.

Vicsek, T., Czirk, Benjacob, A. E., Cohen, I. I. & Shochet, O. (1995). Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6), 1226.

Olfati-Saber, R., Fax, J. A. & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Autom. Control*, 49(9), 1520-1533.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control*, 50, 655-661.

Li, T., & Zhang, J. F. (2009). Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions. *Automatica*, 45(8), 1929-1936.

Li, T., & Zhang, J. F. (2010). Consensus conditions of multiagent systems with time-varying topologies and stochastic communication noises. *IEEE Trans. Autom. Control*, 55(9), 2043-2057.

Huang, M. Y., & Manton, J. H. (2009). Coordination and consensus of networked agents with noisy measurements: Stochastic algorithms and asymptotic behavior. SIAM Journal on Control and Optimization, 48(1), 134-161.

Fang, H. T., Chen, H.-F., & Wen, L. (2012). On control of strong consensus for networked agents with noisy observations. *Journal of Systems Science Complexity*, 25(1), 1-12.

Ren, W. (2008). On consensus algorithms for double-integrator dynamics. *IEEE Trans. on Autom. Control*, 53(6), 1503-1509.

Chen, Y., Lv, J. H., Yu, X. H., & Lin, Z. L. (2013). Consensus of discrete-time second-order multiagent systems based on infinite products of general stochastic matrices. *SIAM Journal on Control and Optimization*, 51(4), 3274-3301.

Yu, W. W., Li, Y., Wen, G. H., Yu, X. H., & Cao, J. D. (2017). Observer design for tracking consensus in secondorder multi-agent systems: Fractional order less than two. *IEEE Trans. on Autom. Control*, 62(2), 894-900.

- Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1), 215-233.
- Hua, C.-C., You, X., & Guan, X.-P. (2016). Leader-following consensus for a class of high-order nonlinear multi-agent systems. Automatica, 73, 138-144.
- Li, Z. K., Ren, W., Liu, X. D., & Fu, M. Y. (2013). Consensus of multi-agent systems with general linear and lipschitz nonlinear dynamics using distributed adaptive protocols. *IEEE Trans. Autom. Control*, 58(7), 1786-1791.
- Liu, K. E., Xie, G. M., Ren, W., & Wang, L. (2013). Consensus for multi-agent systems with inherent nonlinear dynamics under directed topologies. Systems Control Letters, 62(2), 152-162.
- Liu, W., & Huang, J. (2017). Adaptive leader-following consensus for a class of higher-order nonlinear multi-agent systems with directed switching networks. *Automatica*, 79, 84-92.
- Wang, W., Wen, C. Y., & Huang, J. S. (2017). Distributed adaptive asymptotically consensus tracking control of nonlinear multi-agent systems with unknown parameters and uncertain disturbances. *Automatica*, 77, 133-142.
- Munz, U., Papachristodoulou, A., & Allgower, F. (2011). Robust consensus controller design for nonlinear relative degree two multi-agent systems with communication constraints. *IEEE Trans. Autom. Control*, 56(1), 145-151.
- Lei, J. L. (2016). Cooperative Estimation and Optimization over Multi-Agent Networks with Uncertainties. *PhD The*sis, The University of Chineses Academy of Sciences.
- Lei, J. L., & Chen, H.-F. (2015). Distributed stochastic approximation algorithm with expanding truncations: Algorithm and applications. arXiv:1410.7180v4.
- Chen, H.-F. (2002). Stochastic Approximation and Its Applications. Kluwer Academic Publisher, *Dordrecht*, The Netherlands.
- Chen, H.-F. (2007). Adaptive regulator for Hammerstein and Wiener systems with noisy observations. *IEEE Trans. Au*tom. Control, 52, 703-709.
- Fang, H. T., & Chen, H.-F. (2001). Asymptotic behavior of asynchronous stochastic approximation. *Science in China*, 44(4), 249-258.
- Robbins, H., & Monro, S. (1951). A stochastic approximation method. *Ann. Math. Statist.*, 22, 400-407.