

SUBDIFFERENTIAL FORMULAE FOR THE SUPREMUM OF AN ARBITRARY FAMILY OF FUNCTIONS*

PEDRO PÉREZ-AROS[†]

Abstract. This work provides calculus for the Fréchet and limiting subdifferential of the pointwise supremum given by an arbitrary family of lower semicontinuous functions. We start our study showing fuzzy results about the Fréchet subdifferential of the supremum function. Posteriorly, we study in finite- and infinite-dimensional settings the limiting subdifferential of the supremum function. Finally, we apply our results to the study of the convex subdifferential; here we recover general formulae for the subdifferential of an arbitrary family of convex functions.

Key words. variational analysis and optimization, supremum functions, calculus rules, subdifferentials.

AMS subject classifications. 49J52, 49J53, 49Q10

1. Introduction. Many mathematical models concern the study of a constraint minimization problem represented by

$$(1) \quad \begin{array}{ll} \text{minimize } g & \text{subject to} \\ f_t(x) \leq 0, & \text{for all } t \in T \text{ and } x \in X, \end{array}$$

where T is an index set and the function g and f_t are defined in some space X . In these applications the (possibly nonsmooth) pointwise supremum $f := \sup f_t$ plays a crucial role in solving this optimization problem, because the constraint $f_t(x) \leq 0$ for all $t \in T$ can be recast as one single inequality constraint passing to the supremum function $f := \sup_T f_t$. For that reason, understanding the subdifferential of the function f is decisive in computing necessary optimality conditions. Problem (1) has been widely studied when the index set T is finite, and nowadays these results are available in numerous monographs of optimization and variational analysis (see for instance [2, 3, 6, 7, 25–27, 36]).

When the set T is infinite (1) is understood to be a problem of *infinite programming*, and when the space X is finite-dimensional the more precise terminology of *semi-infinite programming* appears due to the finite-dimensionality of the variable $x \in X$ and the infinitude of T . These classes of problems have been studied over the last sixty years by many researchers for the reason that several models in science can be represented as a constraint of the state or the control of a system during a period of time or in a region of the space. Within this framework, a classical assumption is the compactness of the set T together with some hypothesis about the continuity of the function $(t, x) \rightarrow f_t(x)$ and its gradient; in this context the set of active indices $T(x) := \{t \in T : f_t(x) = f(x)\}$ performs an important part in the study (see, e.g., [24]).

More recent papers have studied the convex subdifferential of the supremum function when T is an arbitrary index set and $\{f_t : t \in T\}$ is an arbitrary family of (possibly non-smooth) convex functions (see, for example, [8, 12–14, 23, 37] and the reference therein). Due to the possible emptiness of the set of active indices at a given point x , the authors have considered the ε -active index set $T_\varepsilon(x) := \{t \in T : f_t(x) \geq f(x) - \varepsilon\}$. In these works researchers have successfully calculated the convex subdifferential of

*Submitted to the editors DATE.

Funding: CONICYT-PCHA/doctorado nacional/ 2014-21140621

[†]Instituto de Ciencias de la Ingeniera, Universidad de O'Higgins, Chile (pedro.perez@uoh.cl)

the supremum function without any qualification about the data functions f'_t s, using the set of ε -active indices, the ε -subdifferential of the data and the *normal cone* of the domain of the function f , all of which are well-known concepts in convex analysis.

When the data functions $\{f_t\}_{t \in T}$ are non-convex and non-smooth, but *uniformly locally Lipschitz at point \bar{x}* , which means, there are constants $k, \varepsilon > 0$ such that

$$(2) \quad |f_t(x) - f_t(y)| \leq k\|x - y\|, \forall x \in \mathbb{B}(\bar{x}, \varepsilon), \forall t \in T,$$

we can refer to the classical result about the upper-estimate of the Clarke subdifferential of the function f at the point \bar{x} (see [6, Theorem 2.8.2]). It is important to recall that in this result the set T is compact and the function $t \rightarrow f_t(x)$ is upper-semi continuous for each $x \in \mathbb{B}(\bar{x}, \varepsilon)$. Recently, in [28] (see also [29]) the authors studied the limiting subdifferential of the function f at \bar{x} ; they assumed that T is an arbitrary index and the functions $\{f_t\}_{t \in T}$ satisfy (2). They provided new upper-estimates and improvements of the mentioned result relative to the Clarke subdifferential. Using these calculus rules they derived optimality conditions for infinite and semi-infinite programming.

However, as far as we know, the literature does not provide an upper-estimate for the subdifferential of an arbitrary family of functions $\{f_t : t \in T\}$. This observation motivates our research to derive general upper estimations for the subdifferential of the supremum function under an arbitrary index set T and without the uniform locally Lipschitz condition. The aim of this work is to extend the results of [28] and give general formulae for the subdifferential of the supremum function, in order to apply them to derive necessary optimality conditions for general problems in the framework of infinite programming. The main motivation for considering an arbitrary family of functions comes from the fact that indicators of sets are commonly used in variational analysis to study constraints and set-valued maps related with optimization problems (for example, stability of optimization problems and differentiability of set-valued maps) and they cannot, at least directly, be assumed to be locally Lipschitz. Furthermore, this approach allows us to also study the convex case, and recover general formulae in the convex case, which in particular shows a unifying approach to the study of the subdifferential of the supremum function. For the sake of brevity, we will confine ourselves to extending the results of [28], keeping in mind our applications for a future work.

The rest of the paper is organized as follows: In [Section 2](#) we summarize the notation that we use in this paper, which is classical in variation analysis. In [Subsection 3.1](#) we establish basic properties about the Fréchet subdifferential. We begin [Subsection 3.2](#) giving the definition of *robust infimum* (see [Definition 3.3](#)), this notion fits perfectly with our purpose. It can be understood as a bridge, which allows us to express the subgradient of the supremum function as *robust minimum* of perturbed functions, when the family $\{f_t : t \in T\}$ is an *increasing family of functions*. Nevertheless, the increasing property of the functions can be obtained considering the max functions over all finite sets of T (see [Theorem 3.8](#)). In [Section 4](#), where the main results are established, we study the limiting subdifferential, this section is divided into two subsections. First, we consider a finite-dimensional space; in this framework we establish a technical result (see [Lemma 4.1](#)), which can be applied to several results, but for simplicity we choose only one setting (see [Theorem 4.2](#)), where we provide a convex upper-estimation of the subdifferential. Second, we consider an infinite-dimensional Asplund space. This subsection starts with a result concerning a fuzzy calculus rule for the normal cone of an intersection of an arbitrary family

of sets (see [Theorem 4.5](#)). Later, we use the definition of *sequential normal epi-compactness* together with some results of *separable reduction* to get [Theorem 4.8](#); this gives as a consequence a generalization of [28, Theorem 3.2] (see [Theorem 4.9](#)), for non-necessarily uniformly Lipschitz functions. Finally, in [Section 5](#) we apply our results to the convex subdifferential, that is, when the functions f_t are convex. In this section we get new results and also we recover the general formula of Hantoute-López-Zălinescu [14, Theorem 4].

2. Notation. Throughout the paper and unless we stipulate to the contrary, we adopt the following notation, $(X, \|\cdot\|)$ will be an Asplund space (i.e., every separable subspace of X has separable dual) and X^* its topological dual, with its norm denoted by $\|\cdot\|_*$. The bilinear form $\langle \cdot, \cdot \rangle : X^* \times X \rightarrow \mathbb{R}$ is given by $\langle x^*, x \rangle := x^*(x)$. The weak*-topology on X^* is denoted by $w(X^*, X)$ (w^* , for short). The set of all convex, balanced and closed neighborhoods of a point x with respect to the topology τ is denoted by $\mathcal{N}_x(\tau)$ (\mathcal{N}_x for short). We will write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and we adopt the conventions $1/\infty = 0$, $0 \cdot \infty = 0 = 0 \cdot (-\infty)$ and $\infty + (-\infty) = (-\infty) + \infty = \infty$.

The closed unit ball in X and X^* are denoted by \mathbb{B} and \mathbb{B}^* respectively. For a point $x \in X$ (resp. $x^* \in X^*$) and a number $r \geq 0$ we set $\mathbb{B}(x, r) := x + r\mathbb{B}$ (resp. $\mathbb{B}^*(x^*, r) = x^* + r\mathbb{B}^*$). For a function $f : X \rightarrow \overline{\mathbb{R}}$ the set $\mathbb{B}(x, f, r)$ is defined as the set of all $x' \in \mathbb{B}(x, r)$ such that $|f(x) - f(x')| \leq r$. The symbol $x' \xrightarrow{f} x$ means $x' \rightarrow x$ and $f(x') \rightarrow f(x)$; we avoid some misunderstandings about the topology τ considered in the last convergence using the notation $x' \xrightarrow{\tau} x$ which emphasizes that the convergence $x' \rightarrow x$ is with respect to the topology τ .

We denote by $\text{int}(A)$, \overline{A} , $\text{co}(A)$ and $\overline{\text{co}}(A)$, the interior, the closure, the *convex hull* and the *closed convex hull* of A , respectively. The *affine subspace generated by* A is denoted by $\text{aff}(A)$. The *polar set* and *annihilator* of A are defined by

$$\begin{aligned} A^\circ &:= \{x^* \in X^* \mid \langle x^*, x \rangle \leq 1, \forall x \in A\}, \\ A^\perp &:= \{x^* \in X^* \mid \langle x^*, x \rangle = 0, \forall x \in A\}, \end{aligned}$$

respectively. The *indicator function* of A is defined as $\delta_A(x) := 0$, if $x \in A$ and $\delta_A(x) = +\infty$, if $x \notin A$.

Let $f : X \rightarrow \overline{\mathbb{R}}$ be a lower semicontinuous (lsc) function finite at x . Then

$$\hat{\partial}f(x) := \{x^* \in X^* \mid \liminf_{h \rightarrow 0} \frac{f(x+h) - f(x) - \langle x^*, h \rangle}{\|h\|} \geq 0\},$$

is called the *Fréchet (or regular) subdifferential* of f at x .

The *limiting (or Mordukhovich, or basic) subdifferential* and the *singular subdifferential* can be defined as

$$\begin{aligned} \partial f(x) &:= \{w^*\text{-}\lim x_n^* : x_n^* \in \hat{\partial}f(x_n), \text{ and } x_n \xrightarrow{f} x\}, \\ \partial^\infty f(x) &:= \{w^*\text{-}\lim \lambda_n x_n^* : x_n^* \in \hat{\partial}f(x_n), x_n \xrightarrow{f} x \text{ and } \lambda_n \rightarrow 0^+\}, \end{aligned}$$

respectively (see, e.g., [2, 3, 25, 27] for more details).

If $|f(x)| = +\infty$, we set $\partial f(x) := \emptyset$ for any of the previous subdifferentials. It is important to recall that when f is convex proper and lsc all of these subdifferentials coincide with the classical subdifferential of convex analysis

$$\partial f(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq f(y) - f(x), \forall y \in X\}.$$

For any set A , the Fréchet (or Regular) and the limiting (or Mordukhovich, or basic) normal cone of A at x are given by $\hat{N}(x, A) = \hat{\partial}\delta_A(x)$ and $N(x, A) = \partial\delta_A(x)$, respectively.

Consider a set T and a family of functions $\{f_t\}_{t \in T} \subseteq \overline{\mathbb{R}}^T$, we define the supremum function $f : X \rightarrow \overline{\mathbb{R}}$ by

$$(3) \quad f(x) := \sup_{t \in T} f_t(x), \quad \forall x \in X$$

The symbol $\mathcal{P}_f(T)$ denotes the set of all $F \subseteq T$ such that F is finite. For $F \in \mathcal{P}_f(T)$ we denote $f_F(x) := \max_{s \in F} f_s(x)$.

Following the notation of [28], \mathbb{R}^T is defined as the space of all multipliers $\lambda = (\lambda_t)$ and $\tilde{\mathbb{R}}^T$ denotes the set of all $\lambda \in \mathbb{R}^T$ such that $\lambda_t \neq 0$ for finitely many $t \in T$; by the symbol $\#\lambda$ we denote the cardinal number of $\text{supp } \lambda$. The *generalized simplex on T* is the set $\Delta(T) := \{\lambda \in \tilde{\mathbb{R}}^T : (\lambda_t) \geq 0 \text{ and } \sum_{t \in T} \lambda_t = 1\}$. For a point \bar{x} and $\varepsilon \geq 0$, the set of ε -active indices at \bar{x} is denoted by $T_\varepsilon(\{f_t\}_{t \in T}, \bar{x}) := \{t \in T : f_t(\bar{x}) \leq f_t(\bar{x}) + \varepsilon\}$ ($T_\varepsilon(\bar{x})$ for short), meanwhile the set of all ε -active sets at \bar{x} is denoted by $\mathcal{T}_\varepsilon(\{f_t\}_{t \in T}, \bar{x}) := \{F \in \mathcal{P}_f(T) : f(\bar{x}) \leq f_F(\bar{x}) + \varepsilon\}$ ($\mathcal{T}_\varepsilon(\bar{x})$ for short) and finally, we define

$$\Delta(T, \{f_t\}_{t \in T}, \bar{x}, \varepsilon) := \left\{ (\lambda_t) \in \tilde{\mathbb{R}}^T : \begin{array}{l} \lambda_t \geq 0 \text{ for all } t \in T, \\ \lambda_t \leq \varepsilon, \forall t \in T \setminus T_\varepsilon(\bar{x}) \\ \text{and } |\sum_{t \in T} \lambda_t - 1| \leq \varepsilon \end{array} \right\}$$

($\Delta(T, \bar{x}, \varepsilon)$ for short). When T is a directed set ordered by \preceq , which means (T, \preceq) is an ordered set and for every $t_1, t_2 \in T$ there exists $t_3 \in T$ such that $t_1 \preceq t_3$ and $t_2 \preceq t_3$, we say that the family of functions is increasing provided that for all $t_1, t_2 \in T$

$$t_1 \preceq t_2 \implies f_{t_1}(x) \leq f_{t_2}(x), \quad \forall x \in X.$$

3. Subdifferential of supremum function. In this section we establish some fuzzy calculus rules for the Fréchet subdifferential of the supremum function. First we start [subsection 3.1](#) recalling some basic properties of this subdifferential. Posteriorly, we use the aforementioned properties to get fuzzy calculus rules for the supremum function of an arbitrary family of lower-semicontinuous functions.

3.1. Basic properties of the Fréchet subdifferential. This section is devoted to stipulating some simple properties of the Fréchet subdifferentials. First, let us recall the following relation between the subdifferential and the normal cone to the epigraph of the function; a point x^* belongs to $\hat{\partial}f(x)$ if and only if $(x^*, -1) \in \hat{N}((x, f(x)), \text{epi } f)$.

Now we write the next result, which is useful to understand Fréchet normal vectors to the epigraph of a function in terms of subgradients in the Fréchet subdifferential, this result is well-known and we refer to [3, 20, 25, 27, 31, 34] for the proof.

PROPOSITION 3.1. *Let $f : X \rightarrow \overline{\mathbb{R}}$ be a proper lsc function and consider a point $(x^*, 0) \in \hat{N}(\text{epi } f, (x, f(x)))$. Hence for any $\varepsilon > 0$ there are points $y \in X$ and $(y^*, \lambda) \in \hat{N}(\text{epi } f, (y, f(y)))$ such that $\lambda \in (-\varepsilon, 0)$, $\|y - x\| \leq \varepsilon$, $|f(y) - f(x)| < \varepsilon$ and $y^* \in x^* + \varepsilon\mathbb{B}^*$.*

Next, we give some basic properties of the Fréchet subdifferentials. The first four properties are classical in the literature, the final one can be proved using [35, Theorem 3.1] by rewriting a Fréchet subgradient satisfying an optimization problem as in [28, Equation (3.8)]. Nevertheless, we provide a proof for completeness.

PROPOSITION 3.2. *The Fréchet subdifferential satisfies the following properties:*

P(i) *Consider an lsc function $f : X \rightarrow \overline{\mathbb{R}}$ and $x^* \in \hat{\partial}f(\bar{x})$. Then, for every $\varepsilon > 0$ there exists $\gamma > 0$ such that the function*

$$x \rightarrow f(x) - \langle x^*, x - \bar{x} \rangle + \varepsilon \|x - \bar{x}\| + \delta_{\mathbb{B}(\bar{x}, \gamma)}$$

attains its minimum at \bar{x} .

P(ii) *(Calculus estimation) For every $\varepsilon > 0$, any point $x \in X$ and every finite-dimensional subspace L of X , we have*

$$\hat{\partial}\delta_{\mathbb{B}(x, \varepsilon) \cap L}(x') \subseteq L^\perp, \forall x' \in \text{int } \mathbb{B}(x, \varepsilon).$$

P(iii) *(Enhanced Fuzzy Sum Rule) Consider an lsc function f , a convex Lipschitz function g and a point $x \in X$. If x is a local minimum of $f + g$ with $f(x) \in \mathbb{R}$, there are sequences $(x_n, x_n^*)_{n \in \mathbb{N}}$ such that $x_n^* \in \hat{\partial}f(x_n)$, $x_n \xrightarrow{f} x_0$, $x_n^* \xrightarrow{\|\cdot\|} x_0^*$ with $-x_0^* \in \hat{\partial}g(x)$.*

P(iv) *(Fuzzy Sum Rule) Consider a finite family of lsc functions $f_j : X \rightarrow \overline{\mathbb{R}}$ with $j \in J$ and $x^* \in \hat{\partial}(\sum_{j \in J} f_j)(x)$. Then, there are nets $(x_{\alpha, j}, x_{\alpha, j}^*)_{\alpha \in \mathbb{D}}$ such that $x_{\alpha, j}^* \in \hat{\partial}f_j(x_{\alpha, j})$, $x_{\alpha, j} \xrightarrow{f} x$ and $\sum_{j \in J_1} x_{\alpha, j}^* \xrightarrow{w^*} x^*$.*

P(v) *For every finite family of lsc functions $f_j : X \rightarrow \overline{\mathbb{R}}$ with $j \in J$ we have that for all $x \in X$*

$$(4) \quad \hat{\partial}f_j(x) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \sum \lambda_i \hat{\partial}f_j(x_j) : \begin{array}{l} x_j \in \mathbb{B}(x, f_j, \varepsilon), \lambda \in \Delta(J, x, \varepsilon) \\ \text{and } \#\lambda \leq \dim(X) + 1 \end{array} \right\}.$$

Proof. Items P(i) and P(ii) follow from definition. Item P(iii) is the well-known *Enhanced Fuzzy Sum Rule* (see, e.g., [7, 20, 25, 42, 43]). Item P(iv) is an equivalence of the *Enhanced Fuzzy Sum Rule* (see, e.g., [21]). Finally, we must prove Item P(v); to complete this task, it is enough to consider the pointwise maximum of two functions $g := \max\{f_1, f_2\}$. Let $x^* \in \hat{\partial}g(x)$, $\varepsilon \in (0, 1)$, $V \in \mathcal{N}_0(w^*)$, so by Item P(i) there exist $\gamma \in (0, \varepsilon)$ such that the function

$$y \rightarrow g(y) - \langle x^*, x - \bar{x} \rangle + \varepsilon \|y - x\| + \delta_{\mathbb{B}(x, \gamma)}(y)$$

attains its minimum at x . Hence, assuming that $\gamma > 0$ is small enough, one can suppose that

$$(5) \quad f_i(u) > f_i(x) - \varepsilon, \text{ for all } u \in \mathbb{B}(x, \gamma), i = 1, 2.$$

Now consider the function

$$X \times \mathbb{R}^2 \ni (w, \alpha_1, \alpha_2) \rightarrow m(\alpha_1, \alpha_2) + \delta_{\text{epi } f_1}(w, \alpha_1) + \delta_{\text{epi } f_2}(w, \alpha_2) - \phi(w) + \delta_{F \cap \mathbb{B}(x, \gamma)}(w),$$

where $m(\alpha_1, \alpha_2) := \max\{\alpha_1, \alpha_2\}$ and $\phi(y) := \langle x^*, x - \bar{x} \rangle - \varepsilon \|y - x\|$. This function has a local minimum at the point $(x, f_1(x), f_2(x))$, so by Item P(iv) we can choose

(i) $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ with $|f_i(x) - \alpha_i| \leq \gamma/2$ and $(q_1, q_2) \in \hat{\partial}m(\alpha_1, \alpha_2) = \{(p_1, p_2) \in \Delta(\{1, 2\}) : p_i = 0 \text{ if } \alpha_i < m(\alpha_1, \alpha_2)\}$.

(ii) $(w_i, \beta_i) \in \mathbb{B}(x, f_i(x), \gamma/2)$ and $(w_i^*, \lambda_i) \in \hat{\partial}\delta_{\text{epi } f_i}(w_i, \beta_i)$

such that $w_1^* + w_2^* \in x^* + V + V$, $|q_1 + \lambda_1| < \gamma/2$ and $|q_2 + \lambda_2| < \gamma/2$. Consequently, by (5) and Item (ii) we have that $(w_i, f_i(w_i)) \in \mathbb{B}(x, f_i(x), \varepsilon)$ by classical argumentation we have that $(w_i^*, \lambda_i) \in \hat{\partial}\delta_{\text{epi } f_i}(w_i, f_i(w_i))$ and $\lambda_i \leq 0$ (see, e.g., [7, 20, 25, 27]). Now, we

check that $(-\lambda_1, -\lambda_2) \in \Delta(\{1, 2\}, x, \varepsilon)$, indeed $|\lambda_1 + \lambda_2 - 1| = |\lambda_1 + \lambda_2 - q_1 + q_2| \leq \varepsilon$; moreover if $f_i(x) < g(x)$ (for small enough ε) we can assume (by [Item \(i\)](#)) that $\alpha_i < m(\alpha_1, \alpha_2)$, so $q_i = 0$ and consequently $|\lambda_i| \leq \varepsilon$. Now, if $\lambda_i^* \neq 0$ for $i = 1, 2$, we define $x_i^* := -\lambda_i^{-1} w_i^* \in \hat{\partial}f(w_i)$; otherwise if there exists some $\lambda_i = 0$, then one can approximate this element using [Proposition 3.1](#). Therefore, we have proved that

$$\hat{\partial}f_J(x) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \sum \lambda_t \hat{\partial}f_j(x_j) : \begin{array}{l} x_j \in \mathbb{B}(x, f_j, \varepsilon), \\ \lambda \in \Delta(J, x, \varepsilon) \end{array} \right\}.$$

Now assume that X is finite-dimensional. Consider $x^* = \sum_{i=1}^k \lambda_i x_i^*$ for some $k > \dim(X) + 1$ with $\lambda_i > 0$, $x_i^* \in \hat{\partial}f_{t_i}(x_i)$, $x_i \in \mathbb{B}(x, f_{t_i}, \varepsilon)$ and $\lambda \in \Delta(J, x, \varepsilon)$. Hence, $\{(x_i^*, 1)\}_{i=1}^k \subseteq X \times \mathbb{R}$ must be linearly dependent in $X \times \mathbb{R}$, and there are numbers $(\alpha_i)_{i=1}^k \subseteq \mathbb{R}$ not all equal to zero such that $\sum_{i=1}^k \alpha_i x_i^* = 0$ and $\sum \alpha_i = 0$. Now consider

$$(6) \quad \beta := \min \left\{ \frac{\lambda}{|\alpha_i|} : i \in I^+ \cup I^- \right\}, \text{ where } I^+ := \{i : \alpha_i > 0\} \text{ and } I^- := \{i : \alpha_i < 0\}.$$

Then,

1) If $\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}$ for some $i_0 \in I^+$, we notice that

$$x^* = \sum_{i=1}^k (\lambda_i - \beta \alpha_i) x_i^* = \sum_{\substack{i=1 \\ i \neq i_0}}^k (\lambda_i - \beta \alpha_i) x_i^*,$$

moreover $|\sum_{i=1}^k (\lambda_i - \beta \alpha_i) - 1| = |\sum_{i=1}^k \lambda_i - 1| \leq \varepsilon$ and for all $t_i \notin T_\varepsilon(x)$

1.1) If $i \in I^+$, $0 \leq \lambda_i - \beta \alpha_i \leq \lambda_i \leq \varepsilon$.

1.1) If $i \in I^-$, $0 \leq \lambda_i - \beta \alpha_i = \lambda_i + \beta |\alpha_i| \leq 2\lambda_i \leq 2\varepsilon$ (recall [\(6\)](#)).

2) If $\beta = \frac{\lambda_{i_0}}{\alpha_{i_0}}$ for some $i_0 \in I^-$, we notice that

$$x^* = \sum_{i=1}^k (\lambda_i + \beta \alpha_i) x_i^* = \sum_{\substack{i=1 \\ i \neq i_0}}^k (\lambda_i + \beta \alpha_i) x_i^*,$$

moreover $|\sum_{i=1}^k (\lambda_i + \beta \alpha_i) - 1| = |\sum_{i=1}^k \lambda_i - 1| \leq \varepsilon$ and for all $t_i \notin T_\varepsilon(x)$

2.1) If $i \in I^-$, $0 \leq \lambda_i + \beta \alpha_i \leq \lambda_i \leq \varepsilon$.

2.1) If $i \in I^+$, $0 \leq \lambda_i + \beta \alpha_i = \lambda_i + \beta |\alpha_i| \leq 2\lambda_i \leq 2\varepsilon$ (recall [\(6\)](#)).

Therefore,

$$x^* \in \left\{ \sum_{t \in J} \lambda_t \hat{\partial}f_t(x_t) : \begin{array}{l} x_t \in \mathbb{B}(x, f_t, 2\varepsilon), (\lambda_t) \in \Delta(J, x, 2\varepsilon) \\ \text{and } \#(\lambda_t) \leq k - 1 \end{array} \right\}.$$

Repeating the processes (if $k - 1 > \dim(X) + 1$) one gets that

$$x^* \in \left\{ \sum_{t \in J} \lambda_t \hat{\partial}f_t(x_t) : \begin{array}{l} x_t \in \mathbb{B}(x, f_t, 2^p \varepsilon), (\lambda_t) \in \Delta(J, x, 2^p \varepsilon) \\ \text{and } \#(\lambda_t) \leq \dim(X) + 1 \end{array} \right\}$$

with $p = \#J - \dim(X) - 1$. □

3.2. Fuzzy calculus rules for the subdifferential of the supremum function. In this section T will be an arbitrary index set and $f_t : X \rightarrow \overline{\mathbb{R}}$ will be a family of lsc functions. We recall that f is defined as the supremum function of the family (3).

The next definition is an adaptation of the notion of the *robust infimum* or the *decoupled infimum* used in subdifferential theory to get *fuzzy calculus rules* (see, e.g., [3, 19, 25, 27, 36, 37]).

DEFINITION 3.3 (robust infimum). *We will say that the family $\{f_t : t \in T\}$ has a robust infimum on $B \subseteq X$ provided that*

$$(7) \quad \inf_{x \in B} f(x) = \sup_{t \in T} \inf_{x \in B} f_t(x).$$

In addition, if there exists some $\bar{x} \in B$ such that $\sup_{t \in T} \inf_{x \in B} f_t(x) = f(\bar{x})$, then we will say that $\{f_t : t \in T\}$ has a robust minimum on $B \subseteq X$. Finally, we say that the family $\{f_t : t \in T\}$ has a robust local minimum at \bar{x} if $\{f_t : t \in T\}$ has a robust minimum on some neighborhood B of \bar{x} .

The next lemma shows a sufficient condition for the existence of a robust minimum. We recall that a function $g : X \rightarrow \overline{\mathbb{R}}$, where (X, τ) is a topological space, is called τ -infcompact provided that for every $\alpha \in \mathbb{R}$ the sublevel set $\{x \in X : g(x) \leq \alpha\}$ is τ -compact.

LEMMA 3.4. [Sufficient condition for robust minimum] *Let X be a Banach space and $B \subseteq X$. Suppose that $\{f_t : t \in T\}$ is an increasing family of τ -lsc, B is τ -closed and there exists some t_0 such that f_{t_0} is τ -infcompact on B , with τ some topology coarser (weaker or smaller) than the norm topology. Then the family $\{f_t : t \in T\}$ has a robust minimum on B .*

Proof. [37, Lemma 3.5] □

It is worth mentioning that in the above result the interchange between minimax in (7) is given without any convex-concave assumptions as in classical results (see, e.g., [3, 4, 11, 40, 41, 44]). This follows from the fact that in our result these assumptions are replaced by the increasing property of the family of functions.

REMARK 3.5. *it has not escaped our notice that the hypothesis of infcompactness of some f_t is necessary, even if the supremum function f is infcompact. Indeed, consider $f_n(x) = n^2x^2 - x^4$, then it is easy to see that $f_n \leq f_{n+1}$ and $f = \delta_{\{0\}}$; moreover $\inf_{\mathbb{R}} f_n = -\infty$ and $\inf_{\mathbb{R}} f = 0$.*

The next results give us a necessary condition for the existence of robust minimum in terms of an approximate Fermat's rule. More precisely, we have the following results

PROPOSITION 3.6. *Let $\{f_t : t \in T\}$ be an increasing family of lsc functions. If $\{f_t : t \in T\}$ has a robust local minimum at \bar{x} , then*

$$(8) \quad 0 \in \bigcap_{\varepsilon > 0} \text{cl}^{\|\cdot\|} \left\{ \bigcup \{ \hat{\partial} f_t(x) : x \in \mathbb{B}(\bar{x}, f_t, \varepsilon), t \in T_\varepsilon(\bar{x}) \} \right\}.$$

Proof. Assume that $\{f_t : t \in T\}$ has a robust minimum at \bar{x} on $B := \mathbb{B}(\bar{x}, \eta)$. Pick $\varepsilon \in (0, 1)$ and $\gamma \in (0, \min\{\eta/2, \varepsilon/2\})$, since \bar{x} is a robust minimum there exists some $t \in T$ such that $\inf_B f_t \geq f(\bar{x}) - \gamma^2 \geq f_t(\bar{x}) - \gamma^2$, so $|f_t(\bar{x}) - f(\bar{x})| \leq \gamma^2$ and \bar{x} is a γ^2 -minimum of $f_t + \delta_B$. Hence, by *Ekeland's Variational Principle* (see, e.g., [3]) there

exists $x_\gamma \in \mathbb{B}(\bar{x}, \gamma)$ such that $|f_t(x_\gamma) - f_t(\bar{x})| \leq \gamma^2$ and x_γ is a minimum of the function $f_t(\cdot) + \delta_B(\cdot) + \gamma \|\cdot - x_\gamma\|$, which implies that $f_t(\cdot) + \gamma \|\cdot - x_\gamma\|$ attains a local minimum at x_γ . By [Proposition 3.2 Item P\(iii\)](#) there exist sequences $(x_n, x_n^*) \in X \times X^*$ such that $x_n^* \in \hat{\partial}f_t(x_n)$, $x_n \xrightarrow{f_t} x_\gamma$, $x_n^* \xrightarrow{\|\cdot\|} \bar{x}^*$ with $\bar{x}^* \in \gamma\mathbb{B}^*$. Then, take $n \in \mathbb{N}$ such that $|f_t(x_n) - f_t(x_\gamma)| \leq \gamma$, $\|x_n - x_\gamma\| \leq \gamma$ and $0 \in \hat{\partial}f_t(x_n) + 2\gamma\mathbb{B}^*$. Therefore, $x_n \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$, $|f_t(\bar{x}) - f_t(x_n)| \leq \varepsilon$, $|f(\bar{x}) - f_t(x_n)| \leq \varepsilon$ and $0 \in \hat{\partial}f_t(x_n) + \varepsilon\mathbb{B}^*$; to that end $0 \in \bigcup\{\hat{\partial}f_t(x) : x \in \mathbb{B}(\bar{x}, f_t(\bar{x}), \varepsilon), t \in T_\varepsilon(\bar{x})\} + \varepsilon\mathbb{B}^*$. \square

Now, we notice that, in particular, [Lemma 3.4](#) shows that every minimum over a closed bounded set in a finite-dimensional space is necessarily a *robust local minimum*. This fact, together with the representation of [Item P\(i\)](#), helps us to understand the subgradients in terms of the definition of a *robust local minimum*. Also in an infinite-dimensional space, this compactness property can be forced using the w^* -topology. Consequently, we use [Proposition 3.6](#) to give an upper-estimation of the subdifferential of the supremum function of an increasing family of functions.

PROPOSITION 3.7. *Let $\{f_t : t \in T\}$ be an increasing family of lsc functions. Then for all $\bar{x} \in X$*

$$(9) \quad \hat{\partial}f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \bigcup \left\{ \hat{\partial}f_t(x) : x \in \mathbb{B}(\bar{x}, f_t(\bar{x}), \varepsilon), t \in T_\varepsilon(\bar{x}) \right\}.$$

Proof. Fix $x^* \in \hat{\partial}f(\bar{x})$, $V \in \mathcal{N}_0(w^*)$, $\varepsilon > 0$ and L a finite-dimensional subspace of X such that $L^\perp \subseteq V$, so by [Item P\(i\)](#) there exist a ball $B := \mathbb{B}(\bar{x}, \eta)$ such that the function $\tilde{f} := f - \langle x^*, \cdot - \bar{x} \rangle + \varepsilon \|\cdot - \bar{x}\| + \delta_{L \cap B}$ attains its minimum at \bar{x} .

Hence, consider the family of functions $\tilde{f}_t := f_t - \langle x^*, \cdot - \bar{x} \rangle + \varepsilon \|\cdot - \bar{x}\| + \delta_{L \cap B}$. It is easy to see that the family is increasing, $\tilde{f} = \sup_T \tilde{f}_t$ and there exists some $t \in T$ such that \tilde{f}_t is infcompact. Whence, [Lemma 3.4](#) shows that the family $\{\tilde{f}_t : t \in T\}$ has a robust local minimum at \bar{x} , and [Proposition 3.6](#) implies

$$(10) \quad 0 \in \bigcap_{\gamma > 0} \text{cl}^{w^*} \left\{ \bigcup \{ \hat{\partial}\tilde{f}_t(x) : x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \gamma), t \in T_\gamma(\{\tilde{f}_t\}_{t \in T}, \bar{x}) \} \right\}.$$

Now take $\nu \in (0, \min\{\varepsilon/3, \eta/3\})$ small enough such that $|\phi(w) - \phi(\bar{x})| \leq \varepsilon/3$ for all $w \in \mathbb{B}(\bar{x}, \nu)$, so by [\(10\)](#) there exist $t \in T_\nu(\{\tilde{f}_t\}_{t \in T}, \bar{x})$, $x \in \mathbb{B}(\bar{x}, \tilde{f}_t, \nu)$ and $w^* \in \hat{\partial}\tilde{f}_t(x) = \hat{\partial}(f - \phi + \delta_{B \cap L})(x)$ such that $w^* \in x^* + V$. This implies that $x \in \mathbb{B}(\bar{x}, f_t, \nu + \varepsilon/3)$ and $t \in T_{\nu + \varepsilon/3}(\{f_t\}_{t \in T}, \bar{x})$.

Now applying [Proposition 3.2 Items P\(ii\)](#) and [P\(iv\)](#) to \tilde{f}_t we get the existence of points $u \in X$ and $u^* \in X^*$ such that $u^* \in \hat{\partial}f_t(u)$, $u \in \mathbb{B}(x, f_t, \nu)$ and $u^* \in w^* + L^\perp + V = w^* + V$. Therefore $t \in T_\varepsilon(\{f_t\}_{t \in T}, \bar{x})$, $u \in \mathbb{B}(\bar{x}, f_t, \varepsilon)$ and $x^* \in u^* + V + V$. \square

Now we present a fuzzy calculus rule for a not necessarily increasing family of functions; we bypass this assumption using the family of finite sets of the index set T , which is always ordered by inclusion.

THEOREM 3.8. *Let $\{f_t : t \in T\}$ be an arbitrary family of lsc functions. Then for every $\bar{x} \in X$*

$$(11) \quad \hat{\partial}f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \bigcup_{\substack{F \in \mathcal{T}_\varepsilon(\bar{x}) \\ x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon)}} \bigcap_{\gamma > 0} \text{cl}^{w^*} \left\{ \sum_{t \in F} \lambda_t \hat{\partial}f_t(x_t) : \begin{array}{l} x_t \in \mathbb{B}(x', f_t, \gamma), \\ \lambda \in \Delta(F, x', \gamma) \text{ and} \\ \#\lambda \leq \dim(X) + 1 \end{array} \right\} \right\}$$

Proof. Consider the set $\tilde{T} := \mathcal{P}_f(T)$, ordered by $F_1 \preceq F_2$ if and only if $F_1 \subseteq F_2$, and the family of functions $\{f_F : F \in \tilde{T}\}$ (recall that $f_F = \max_{s \in F} f_s$), then it is easy to see that the family $\{f_F : F \in \tilde{T}\}$ is an increasing family of functions and $\sup_{F \in \tilde{T}} f_F = f$. Let $x^* \in \hat{\partial}f(\bar{x})$, thus by [Proposition 3.7](#)

$$x^* \in \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \bigcup \{ \hat{\partial}f_F(x') : x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon), F \in \tilde{T}_\varepsilon(\bar{x}) \} \right\}.$$

Now, if $w^* \in \hat{\partial}f_F(x')$ for some $x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon)$ and $F \in \tilde{T}_\varepsilon(\bar{x})$, we get $x' \in \mathbb{B}(\bar{x}, \varepsilon)$ and $F \in \mathcal{T}_\varepsilon(\bar{x})$, so using [Proposition 3.2 Item P\(v\)](#) we get

$$w^* \in \bigcap_{\gamma > 0} \text{cl}^{w^*} \left\{ \sum \lambda_t \hat{\partial}f_t(x_t) : \begin{array}{l} x_t \in \mathbb{B}(x', f_t, \gamma), \lambda \in \Delta(F, x', \gamma) \\ \text{and } \#\lambda \leq \dim(X) + 1 \end{array} \right\},$$

then (11) holds. \square

Here, it is important to compare the above result with [28, Theorem 3.1 part ii)]. In the mentioned result, only uniform Lipschitz continuous data was considered. Here, we extend this fuzzy calculus to arbitrary lsc data functions. Since the comparison between both results involves some technical estimations, we prefer to write this as a corollary.

COROLLARY 3.9. *Under the hypothesis of [Theorem 3.8](#) assume that the data function f_t is uniformly locally Lipschitz at \bar{x} . Then, for each $x^* \in \hat{\partial}f(\bar{x})$, $V \in \mathcal{N}_0(w^*)$ and $\varepsilon > 0$ there exist $\lambda \in \Delta(T_\varepsilon(\bar{x}))$ and $x_t \in \mathbb{B}(\bar{x}, \varepsilon)$ for all $t \in T_\varepsilon(\bar{x})$ such that*

$$(12) \quad x^* \in \sum_{t \in T_\varepsilon(\bar{x})} \lambda_t \hat{\partial}f_t(x_t) + V$$

Proof. Consider K as the constant of uniform Lipschitz continuity. Pick $x^* \in \hat{\partial}f(\bar{x})$, and by [Theorem 3.8](#) we have that

$$(13) \quad x^* \in \sum_{t \in F} \lambda_t \hat{\partial}f_t(x_t) + V$$

for some $F \in \mathcal{T}_\varepsilon(\bar{x})$, a point $x' \in \mathbb{B}(\bar{x}, f_F, \varepsilon)$, points $x_t \in \mathbb{B}(x', f_t, \gamma)$ and $\lambda \in \Delta(F, x', \gamma)$, we can assume that $\gamma \cdot \#F \leq \varepsilon$. First $\|x_t - \bar{x}\| \leq \|x_t - x'\| + \|\bar{x} - x'\| \leq \varepsilon + \gamma$. Second $F_\varepsilon(x') \subseteq T_{\varepsilon(K+3)}(\bar{x})$, this is because

$$\begin{aligned} f_t(\bar{x}) &\geq f_t(x') - \varepsilon K \geq f_F(x') - \varepsilon(K+1) \geq f_F(\bar{x}) - \varepsilon(K+2) \\ &\geq f(\bar{x}) - \varepsilon(K+3). \end{aligned}$$

Then, let us define $\tilde{\lambda} : T \rightarrow \mathbb{R}$ by

$$\tilde{\lambda}_t := \begin{cases} \frac{\lambda_t}{\sum_{t \in F_\gamma(x')} \lambda_t} & \text{if } t \in F_\gamma(x'), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{\lambda} \in \Delta(T_{\varepsilon(K+3)}(\bar{x}))$. Furthermore, we claim that

$$(14) \quad x^* \in \sum_{t \in T} \tilde{\lambda}_t \hat{\partial}f_t(x_t) + 3K\varepsilon\mathbb{B} + V.$$

Indeed, by (13) there are $x_t^* \in \hat{\partial}f_t(x_t)$ and $v^* \in V$ such that $x^* = \sum \lambda_t x_t^* + v^*$, then

$$\begin{aligned} \left\| \sum_{t \in T} \lambda_t x_t^* - \sum_{t \in T} \tilde{\lambda}_t x_t^* \right\| &= \left\| \sum_{t \in F_\gamma(x')} (\lambda_t - \tilde{\lambda}_t) x_t^* + \sum_{F \setminus F_\gamma(x')} \lambda_t x_t^* \right\| \\ &\leq \left| \sum_{t \in F_\gamma(x')} \lambda_t - 1 \right| K + K\varepsilon \leq \left| \sum_{t \in F} \lambda_t - 1 \right| K + 2\varepsilon K \\ &\leq 3K\varepsilon. \end{aligned}$$

Consequently, (14). Finally, taking ε small enough we have that (14) implies (13). \square

4. Limiting subdifferential of pointwise supremum. This section is divided into two subsections. The first one concerns the study of the notion of the limiting subdifferential in finite-dimensional Banach spaces. This setting is obviously motivated by the theory of *semi-infinite programming*; in this scenario we can obtain a better estimation of the limiting sequences obtained in Theorem 3.8. This result is given in Lemma 4.1; using this technical lemma, we focus on the particular case when the set T is a subset of a compact metric space (see Theorem 4.2). The second one corresponds to the infinite-dimensional setting; this subsection begins with a result concerning a *fuzzy intersection rule for the normal cone of an arbitrary intersection of sets* (see Theorem 4.5), which generalizes [30, Theorem 5.2]. Later the main result of this subsection is given in Theorem 4.8, where we explore the definition of *sequential normal epi-compactness* (see, e.g., [25]) and with this we extend [28, Theorem 3.2] (see Theorem 4.9).

4.1. Finite-dimensional spaces. In this subsection $\hat{\partial}$, ∂ and ∂^∞ mean the Fréchet subdifferential, the limiting subdifferential and the singular limiting subdifferential, respectively.

LEMMA 4.1. *Consider $\gamma_k \rightarrow 0$ and $x^* \in \partial f(x)$ and $y^* \in \partial^\infty f(x)$. Then there are sequences $\eta_k \rightarrow 0^+$, $\{t_{i,k}\} = F_k \in \mathcal{P}_f(T)$, $\{t_{i,k}^\infty\} = F_k^\infty \in \mathcal{P}_f(T)$ with $\#F_k \leq \dim(X) + 1$, $\#F_k^\infty \leq \dim(X) + 1$, $x'_k \rightarrow x$, $y'_k \rightarrow x$, $x_{i,k} \rightarrow x$, $y_{i,k} \rightarrow x$, $\lambda_{i,k} \in \Delta(F_k, x'_k, \gamma_k)$, $\lambda_{i,k}^\infty \in \Delta(F_k^\infty, y'_k, \gamma_k)$ such that:*

- i) $x^* = \lim_{k \rightarrow \infty} \sum_{i \in F_k} \lambda_{i,k} \cdot x_{i,k}^*$, $y^* = \lim_{k \rightarrow \infty} \eta_k \sum_{i \in F_k} \lambda_{i,k}^\infty \cdot y_{i,k}^*$,
- ii) $\lim_{k \rightarrow \infty} f_{F_k}(x'_k) = f(x)$, $\lim_{k \rightarrow \infty} f_{F_k^\infty}(y'_k) = f(x)$,
- iii) $\lim |f_{t_{i,k}}(x_{i,k}) - f_{t_{i,k}}(x'_k)| = 0$ and $\lim |f_{t_{i,k}^\infty}(y_{i,k}) - f_{t_{i,k}^\infty}(y'_k)| = 0$ for all i .

Moreover (by passing to a subsequence) one of the following conditions holds.

- (A) There exists $n_1 \in \mathbb{N}$ with $n_1 \leq \dim(X) + 1$ such that $\lambda_{i,k} \xrightarrow{k \rightarrow \infty} \lambda_i > 0$, $x_{i,k}^* \xrightarrow{k \rightarrow \infty} x_i^*$, $\lim f_{t_{i,k}}(x_{i,k}) = f(x)$ for $i \leq n_1$ and $\lambda_{i,k} \xrightarrow{k \rightarrow \infty} 0$, $\lambda_{i,k} \cdot x_{i,k}^* \xrightarrow{k \rightarrow \infty} x_i^*$ for $n_1 < i \leq n$, and $x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i>n_1} x_i^*$, or

- (B) There are $\nu_k \rightarrow 0$ such that $\nu_k \cdot \lambda_{i,k} \cdot x_{i,k}^* \xrightarrow{k \rightarrow \infty} x_i^*$ and $\sum_{i=1}^{n_1} x_i^* = 0$ with not all x_i^* equal to zero.

and (up to a subsequence) one of the following conditions holds.

- (A $^\infty$) There exists $n_2 \in \mathbb{N}$ with $n_2 \leq \dim(X) + 1$ such that $\lambda_{i,k}^\infty \xrightarrow{k \rightarrow \infty} \lambda_i^\infty > 0$, $y_{i,k}^* \xrightarrow{k \rightarrow \infty} y_i^*$, $\lim f_{t_{i,k}^\infty}(y_{i,k}) = f(x)$ for $i \leq n_2$ and $\lambda_{i,k}^\infty \xrightarrow{k \rightarrow \infty} 0$, $\lambda_{i,k}^\infty \cdot y_{i,k}^* \xrightarrow{k \rightarrow \infty} y_i^*$ for $n_2 < i \leq n$,

$$\text{and } y^* = \sum_{i=1}^{n_2} \lambda_i^\infty y_i^* + \sum_{i>n_2}^n y_i^*, \text{ or}$$

(B^∞) There are $\nu_k \rightarrow 0$ such that $\nu_k \cdot \eta_k \cdot \lambda_{i,k}^\infty \cdot y_{i,k}^* \xrightarrow{k \rightarrow \infty} y_i^*$ and $\sum_{i=1}^{n_1} y_i^* = 0$ with not all x_i^* equal to zero.

Proof. Define $N := \dim(X) + 1$ and consider $x^* \in \partial f(x)$ ($y^* \in \partial^\infty f(x)$, resp.), so (by definition) there exist $x_k \xrightarrow{f} x$ and $x_k^* \in \hat{\partial} f(x_k)$ ($y_k \xrightarrow{f} x$, η_k and $y_k^* \in \hat{\partial} f(y_k)$, resp.) such that $x_k^* \rightarrow x^*$ ($\eta_k y_k^* \rightarrow y^*$, resp.). Whence, by [Theorem 3.8](#), there exist $x'_k \in \mathbb{B}(x_k, \gamma_k)$ and $F_k = \{t_{i,k}\}_{k=1}^N \subseteq T$, with $|f_{F_k}(x'_k) - f(x_k)| \leq \gamma_k$ along with elements $x_{t_{i,k}} \in \mathbb{B}(x'_k, f_{t_{i,k}}, \gamma_k)$ and $z_k^* = \sum_{i=1}^N \lambda_{t_{i,k}} x_{t_{i,k}}^*$ with $\|z_k^* - x_k^*\|_* \leq \gamma_k$, $(\lambda_{k,i}) \in \Delta(F_k, x'_k, \gamma_k)$ and $x_{t_{i,k}}^* \in \hat{\partial} f_{t_{i,k}}(x_{i,k})$. Hence, $x^* = \lim_{k \rightarrow \infty} \sum_{i \in F_k} \lambda_{i,k} \cdot x_{i,k}^*$, $\lim_{k \rightarrow \infty} f_{F_k}(x'_k) = f(x)$ and $\lim_{k \rightarrow \infty} (f_{t_{i,k}}(x_{i,k}) - f_{t_{i,k}}(x'_k)) = 0$. Similarly, for the case $y^* \in \partial^\infty f(x)$, there exist $y'_k \in \mathbb{B}(y_k, \gamma_k)$ and $F_k^\infty = \{t_{i,k}^\infty\}_{k=1}^N \subseteq T$, with $|f_{F_k^\infty}(y'_k) - f(y_k)| \leq \gamma_k$ along with elements $y_{t_{i,k}^\infty} \in \mathbb{B}(y'_k, f_{t_{i,k}^\infty}, \gamma_k)$ and $w_k^* = \sum_{i=1}^N \lambda_{t_{i,k}^\infty}^\infty \eta_k y_{t_{i,k}^\infty}^*$ with $\|w_k^* - y_k^*\|_* \leq \gamma_k$, $(\lambda_{k,i}^\infty) \in \Delta(F_k^\infty, y'_k, \gamma_k)$ and $y_{t_{i,k}^\infty}^* \in \hat{\partial} f_{t_{i,k}^\infty}(y_{i,k})$.

Now, we focus on the case $x^* \in \partial f(x)$; by passing to a subsequence, we have that $\lambda_{i,k} \rightarrow \lambda_i$ with $(\lambda_i) \in \Delta(\{1, \dots, N\})$ and (relabeling it if necessary) we may assume that $\lambda_k \neq 0$ for all $i = 1, \dots, n_1$ and $\lambda_k = 0$ for all $i = n_1 + 1, \dots, N$.

On the one hand if $\sup\{\|\lambda_{i,k} x_{i,k}^*\|_* : i = 1, \dots, N; k \in \mathbb{N}\} < +\infty$ (up to a subsequence) we can assume that $\lambda_{i,k} x_{i,k}^* \rightarrow \lambda_i x_i^*$ for all $i = 1, \dots, n_1$ and $\lambda_{i,k} x_{i,k}^* \rightarrow x_i^*$ for all $i = n_1 + 1, \dots, N$, therefore $x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i>n_1}^n x_i^*$. Next, we claim that $\lim_{k \rightarrow \infty} f_{t_{i,k}}(x_{i,k}) = f(x)$ for all $i = 1, \dots, n_1$. Indeed, define $\gamma := \min\{\lambda_i/2 : i = 1, \dots, n_1\}$, then for all k (large enough) such that $\gamma_k \leq \gamma$ and $\lambda_k > \gamma$ (recall $t_{k,i} \in \Delta(F_k, x'_k, \gamma_n)$) we have that

$$f_{t_{i,k}}(x'_k) + \gamma_k \geq \max_{s \in F_k} f_s(x'_k) \geq f_{t_{i,k}}(x'_k),$$

so, taking the limits we obtain that

$$\lim_{k \rightarrow \infty} f_{t_{i,k}}(x'_k) \geq \lim_{k \rightarrow \infty} \max_{s \in F_k} f_s(x'_k) = f(x) \geq \lim_{k \rightarrow \infty} f_{t_{i,k}}(x'_k),$$

which implies the desired conclusion.

On the other hand, if $\sup\{\|\lambda_{i,k} x_{i,k}^*\|_* : i = 1, \dots, N; k \in \mathbb{N}\} = +\infty$ (by passing to a subsequence) $\eta_k := \left(\max_{i=1, \dots, k} \|\lambda_{i,k} x_{i,k}^*\|_* \right)^{-1} \rightarrow 0$ and (w.l.o.g.) $\eta_k \lambda_{i,k} x_{i,k}^* \rightarrow x_i^*$ for all $i = 1, \dots, N$, which implies that $\sum_{i=1}^{n_1} x_i^* = 0$ with not all x_i^* equal to zero.

The case $y^* \in \partial^\infty f(x)$ follows similar arguments, so we omit the proof. \square

Now we are going to apply the above result to a framework, where the functions f_t 's represent a control in a region. We assume that T is contained in a metric space and \bar{T} is compact. For this reason we introduce the following definitions.

A family of lsc functions $\{f_t : t \in T\}$ is said to be *continuously subdifferentiable* at x with respect to $\hat{\partial}$ provided that for every sequence $T \times X \times [0, +\infty) \ni (t_n, x_n, \lambda_n) \rightarrow (t, x, \lambda) \in T \times X \times [0, +\infty)$ and points $w_n^* \in \hat{\partial} f_{t_n}(x_n)$ with $\lambda_n w_n^* \rightarrow w^*$ one has

$$w^* \in \lambda \circ \partial f_t(x) := \begin{cases} \lambda \partial f_t(x) & \text{if } \lambda > 0, \\ \partial^\infty f_t(x) & \text{if } \lambda = 0, \end{cases}$$

To our knowledge, the next definition was introduced in [32], where the authors studied generalized notions of differentiation for parameter-dependent set valued maps and mappings. For a point $x \in X$ and $t \in \overline{T} \setminus T$ we define the *extended subdifferential* and the *extended singular subdifferential* at (t, x) as

$$\begin{aligned} \partial f_t(x) &:= \left\{ x^* \in X^* : \begin{array}{l} \exists t_k \in T, t_k \rightarrow t, x_k \rightarrow x, x_k^* \in \hat{\partial} f_{t_k}(x_k) \\ \text{s.t. } f_{t_k}(x_k) \rightarrow f(x), \text{ and } x_k^* \rightarrow x^* \end{array} \right\}, \\ \partial^\infty f_t(x) &:= \left\{ x^* \in X^* : \begin{array}{l} \exists t_k \in T, t_k \rightarrow t, \eta_k \rightarrow 0^+, x_k \rightarrow x, x_k^* \in \hat{\partial} f_{t_k}(x_k) \\ \text{s.t. } \limsup f_{t_k}(x_k) \leq f(x), \text{ and } \eta_k x_k^* \rightarrow x^* \end{array} \right\}, \end{aligned}$$

respectively. Finally, we denote the *extended active index set* at x by $\overline{T}(x) = T(x) \cup (\overline{T} \setminus T)$.

THEOREM 4.2. *Consider a family of lsc functions $\{f_t : t \in T\}$ where T is a subset of a metric space and \overline{T} is compact. Assume that the following conditions hold at a point \bar{x}*

- (a) *For every $\bar{t} \in T$, $\limsup_{(t,x) \rightarrow (\bar{t}, \bar{x})} f_t(x) \leq f_{\bar{t}}(\bar{x})$.*
- (b) *The family is $\{f_t : t \in T\}$ continuously subdifferentiable at \bar{x} .*
- (c) *The set $\text{co}(\bigcup_{t \in \overline{T}} \partial^\infty f_t(\bar{x}))$ does not contain lines.*

Then

$$\begin{aligned} \partial f(\bar{x}) &\subseteq \text{co} \left(\bigcup_{t \in \overline{T}(\bar{x})} \partial f_t(\bar{x}) \right) + \text{co} \left(\bigcup_{t \in \overline{T}} \partial^\infty f_t(\bar{x}) \right), \text{ and} \\ \partial^\infty f(\bar{x}) &\subseteq \text{co} \left(\bigcup_{t \in \overline{T}} \partial^\infty f_t(\bar{x}) \right). \end{aligned}$$

Proof. Consider $x^* \in \partial f(\bar{x})$. Now, using the notation of Lemma 4.1 and by the compactness of \overline{T} we can assume that $t_{k,i} \rightarrow t_i \in \overline{T}$. Moreover, Item (c) contradicts Lemma 4.1 Items (B) and (B $^\infty$), which means, Lemma 4.1 Items (A) and (A $^\infty$) must hold. Hence we can write $x^* = \sum_{i=1}^{n_1} \lambda_i x_i^* + \sum_{i > n_1}^n x_i^*$.

- If $i \leq n_1$ and $t_i \in T$: By assumption Item (a) and Lemma 4.1 Item (A) necessarily $f(\bar{x}) = f_{t_i}(\bar{x})$, i.e., $t \in T(\bar{x})$. Also, Item (b) implies $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i \leq n_1$ and $t_i \in \overline{T} \setminus T$: By Lemma 4.1 Item (A) we get that $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i > n_1$ and $t_i \in T$: By assumption Item (b) we get $x_i^* \in \partial f_{t_i}(\bar{x})$.
- If $i > n_1$ and $t_i \in \overline{T} \setminus T$: By Lemma 4.1 Item (A) implies that $x_i^* \in \partial f_{t_i}(\bar{x})$.

This completes the first part. The case $y^* \in \partial^\infty f(\bar{x})$ follows similar arguments so we omit the proof. \square

It is important to mention that similar results have been shown in the literature; we refer to [6, 29, 32] for some examples. In the above result we did not go for the greater stage of generality, and we established the result only to show one possible application of Lemma 4.1.

REMARK 4.3. *It has not escaped our notice that the convex envelope appears in Theorem 4.2 due to the fact that at the moment of taking the convergent subsequence in the index $t_{k,i} \rightarrow t_i$ we cannot ensure, in a general framework, that there could exist two limit points $t_i = t_j$ for $i \neq j$. Nevertheless, the reader can force this condition imposing some assumptions over the index set, the simplest example is when the index set is finite.*

Now let us finish this subsection with an example which shows an application of [Theorem 4.2](#) for a countable number of functions.

EXAMPLE 4.4. Consider $T = \mathbb{N}$ and the sequence of functions

$$f_n(x, y) = \begin{cases} nx^2 + \frac{n}{n-1} \log(|y| + 1) - \frac{1}{n} & \text{if } x \geq 0, \\ \frac{n}{n-1} \log(|y| + 1) - \frac{1}{n} & \text{if } x < 0. \end{cases}$$

Here, it is worth noting that all functions f_n are locally Lipschitz continuous, but they are not uniformly Lipschitz continuous, so the results of [\[28\]](#) cannot be applied. Nevertheless, we can apply [Theorem 4.2](#). Indeed, after some calculus, we get that

$$\begin{aligned} \partial f_n(0, 0) &= \{0\} \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right], \\ \partial^\infty f_n(0, 0) &= \{(0, 0)\}. \end{aligned}$$

We compute the function

$$f(x, y) = \log(|y| + 1) + \delta_{(-\infty, 0]}(x) = \begin{cases} +\infty & \text{if } x > 0, \\ \log(|y| + 1) & \text{if } x \leq 0, \end{cases}.$$

Then, $\partial f(0, 0) = [0, +\infty) \times [-1, 1]$ and $\partial^\infty f(0, 0) = [0, +\infty) \times \{(0, 0)\}$. In order to apply [Theorem 4.2](#) we notice that \mathbb{N} is a subset of the compact space $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ with the metric $d(a, b) = |\frac{1}{a} - \frac{1}{b}|$. Straightforwardly the assumptions [Items \(a\) and \(b\)](#) of [Theorem 4.2](#) are satisfied, furthermore, $\mathbb{N}(0, 0) = \emptyset$.

Now, we calculate $\partial f_\infty(0, 0)$ and $\partial^\infty f_\infty(0, 0)$. First we notice that

$$\hat{\partial} f_n(x, y) \subseteq [0, +\infty) \times \left[-\frac{n}{n-1}, \frac{n}{n-1}\right].$$

Then $\partial f_\infty(0, 0) = [0, +\infty) \times [-1, 1]$ and $\partial^\infty f_\infty(0, 0) = [0, +\infty) \times \{0\}$. In particular, [assumption Item \(c\)](#) of [Theorem 4.2](#) holds. Then, [Theorem 4.2](#) gives us

$$\partial f(0, 0) = \text{co}(\partial f_\infty(0, 0)) + \text{co}\left(\bigcup_{n \in \mathbb{N}_\infty} \partial^\infty f_n(0, 0)\right) = [0, +\infty) \times [-1, 1],$$

$$\partial^\infty f(0, 0) = \text{co}\left(\bigcup_{n \in \mathbb{N}_\infty} \partial^\infty f_n(0, 0)\right) = [0, +\infty) \times \{0\},$$

which are exact estimations of the limiting and singular subdifferential of the function f at $(0, 0)$.

4.2. Infinite-dimensional spaces. In this section we study the limiting subdifferential of the supremum function in an arbitrary Asplund space X .

The first result of this Subsection generalizes the *Fuzzy Intersection Rule for Fréchet Normals to Countable Intersections of Cones* established in [\[30, Theorem 5.2\]](#).

THEOREM 4.5. Let $\{A_t\}_{t \in T}$ be an arbitrary family of closed subsets of X and $\Lambda := \bigcap_{t \in T} A_t$. Then given $\bar{x} \in X$, $x^* \in \hat{N}(\Lambda, \bar{x})$, $\varepsilon > 0$ and $V \in \mathcal{N}_0(w^*)$ there are $F \in \mathcal{P}_{\mathfrak{f}}(T)$, $w_t \in \mathbb{B}(\bar{x}, \varepsilon)$ and $w_t^* \in \hat{N}(A_t, w_t)$ such that

$$(15) \quad x^* \in \sum_{t \in F} w_t^* + V.$$

Consequently, if $\{A_t\}_{t \in T}$ is a family of closed cones $\hat{N}(A_t, w_t) \subseteq N(A_t, 0)$ for all $t \in T$ and

$$(16) \quad \hat{N}(A, \bar{x}) \subseteq \text{cl}^{w^*} \left\{ \sum_{t \in F} w_t^* \mid w_t^* \in N(A_t, 0) \text{ and } t \in F \in \mathcal{P}_{\mathfrak{f}}(T) \right\}.$$

Proof. The first part corresponds to a straightforward application of [Theorem 3.8](#). Now if one considers a closed cone $K \subseteq X$ and $u \in K$ one has that

$$\hat{N}(K, u) \subseteq \hat{N}(K, n^{-1}u), \quad \forall n \in \mathbb{N}.$$

Therefore $\hat{N}(A_t, u) \subseteq N(A_t, 0)$ for every $t \in T$ and $u \in A_t$, consequently [\(15\)](#) implies [\(16\)](#). \square

REMARK 4.6. *It is important to notice that the results of [\[8\]](#) cannot be applied to derive the above formulae, since imposing uniform Lipschitz continuity of an indicator function of the set Λ at a point \bar{x} is equivalent to assume that the point \bar{x} is an interior point of Λ , which give us a trivial conclusion.*

The next definition is the notion of *sequential normal epi-compactness* (SNEC) of functions defined for the limiting subdifferential (see, e.g., [\[25, Definition 1.116 and Corollary 2.39\]](#)).

DEFINITION 4.7. *A real extended valued function f finite at x is said to be SNEC at x if for any sequences $(\lambda_k, x_k, x_k^*) \in [0, +\infty) \times X \times X^*$ satisfying $\lambda_k \rightarrow 0$, $x_k \xrightarrow{f} x$, $x_k^* \in \hat{\partial}f(x_k)$ and $\lambda_k x_k^* \xrightarrow{*} 0$ one has $\|\lambda_k x_k^*\| \rightarrow 0$. A family of functions $\{f_t\}_{t \in T}$ is said to be SNEC on a neighborhood of a point \bar{x} if there exists a neighborhood U of \bar{x} such that for all $x \in U$ all but one of these are SNEC at x .*

We say that the family of functions $\{f_t : t \in T\}$ satisfy the *limiting condition* on a neighborhood of a point \bar{x} if there exists a neighborhood U of \bar{x} such that for all $x \in U$ and $F \in \mathcal{P}_{\mathfrak{f}}(T)$

$$(17) \quad w_t^* \in \partial^\infty f_t(x), \quad t \in F \text{ and } \sum_{t \in F} w_t^* = 0 \text{ implies } w_t^* = 0, \text{ for all } t \in F.$$

It is worth mentioning that the SNEC property is immediately satisfied if the space X is finite-dimensional. Moreover, the family of functions $\{f_t\}_{t \in T}$ is SNEC and satisfies the *limiting condition* on a neighborhood of a point \bar{x} , provided that the functions are locally Lipschitz (not necessarily uniform) on a neighborhood U of \bar{x} .

The next theorem corresponds to the main result of this paper; in this result we give an upper-estimation of the subdifferential of the supremum function only using the above definitions, without the assumption of uniformly locally Lipschitz continuity.

THEOREM 4.8. *Consider a family of lsc functions $\{f_t : t \in T\}$. If the family $\{f_t : t \in T\}$ is SNEC and satisfy the limiting condition [\(17\)](#) on a neighborhood of \bar{x} . Then*

$$(18) \quad \partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left(\mathcal{S}(\bar{x}, \varepsilon) \right), \text{ and } \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left([0, \varepsilon] \cdot \mathcal{S}(\bar{x}, \varepsilon) \right).$$

Where

$$(19) \quad \mathcal{S}(\bar{x}, \varepsilon) := \left\{ \sum_{t \in F} \lambda_t \circ \partial f_t(x') : \begin{array}{l} F \in \mathcal{P}_{\mathfrak{f}}(T), x' \in \mathbb{B}(\bar{x}, \varepsilon), \\ |f_F(x') - f(\bar{x})| \leq \varepsilon, \lambda \in \Delta(F) \\ \text{and } f_t(x') = f_F(x') \text{ for all } t' \in \text{supp } \lambda \end{array} \right\},$$

and

$$\lambda \circ \partial f_t(x) := \begin{cases} \lambda \partial f_t(x), & \text{if } \lambda > 0, \\ \partial^\infty f_t(x), & \text{if } \lambda = 0. \end{cases}$$

Proof. Consider $\varepsilon > 0$ and $V \in \mathcal{N}_0(w^*)$. Pick $x^* \in \partial f(\bar{x})$ ($y^* \in \partial^\infty f(\bar{x})$, resp.). Hence, there exist sequences $x_j \xrightarrow{f} \bar{x}$ and $x_j^* \xrightarrow{w^*} x^*$ ($\nu_j \rightarrow 0^+$ and $\nu_j x_j^* \xrightarrow{w^*} y^*$, resp.) with $x_j^* \in \hat{\partial} f(x_j)$. Now, take $j_0 \in \mathbb{N}$ such that $x^* \in x_{j_0}^* + V$ ($x^* \in \nu_{j_0} x_{j_0}^* + V$ and $\nu_{j_0} \leq \varepsilon$, resp.) and $x_{j_0} \in \mathbb{B}(\bar{x}, f, \varepsilon)$. Hence, by [Theorem 3.8](#) there exist some $F \in \mathcal{T}_\varepsilon(x_{j_0})$ and $x' \in \mathbb{B}(x_{j_0}, f_F, \varepsilon)$ such that $x_{j_0}^* = w^* + v^*$ with

$$w^* \in \bigcap_{\gamma > 0} \text{cl}^{w^*} \left\{ \sum_{t \in T} \lambda_t \partial f_t(x_t) : x_t \in \mathbb{B}(x', f_t, \gamma), (\lambda_t) \in \Delta(F, x', \gamma) \right\},$$

and $v^* \in V$. One gets $x' \in \mathbb{B}(\bar{x}, 2\varepsilon)$ and $|f_F(x') - f(\bar{x})| \leq 3\varepsilon$. Now, we show that

$$(20) \quad w^* \in \mathcal{S}(\bar{x}, 3\varepsilon)$$

For this purpose let us introduce the following notation; by the symbol $S(X \times X^*)$ we understand the family of set $U \times Y$ where U and Y are (norm-) separable closed linear subspaces of X and X^* , a set $\mathcal{A} \subseteq S(X \times X^*)$ is called a *rich family* if (i) for every $U \times Y \in \mathcal{A}$, there exists $V \times Z \in \mathcal{A}$ such that $U \subseteq V$ and $Y \subseteq Z$, and (ii) $\bigcup_{n \in \mathbb{N}} U_n \times \bigcup_{n \in \mathbb{N}} Y_n \in \mathcal{A}$, whenever the sequence $(U_n \times Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ satisfies $U_n \subseteq U_{n+1}$ and $Y_n \subseteq Y_{n+1}$ (see, e.g., [\[9, 10\]](#) and the references therein). We claim that under our assumptions there exists a rich family \mathcal{A} such that for all $V \times Y$ and any sequence $y_n^* \in Y$ with $y_n^* \xrightarrow{w^*} v^*$ and v^* is zero on V , then v^* is zero in the whole X . Indeed, by [\[9, Theorem 13\]](#) there exists a rich family $\mathcal{A} \subseteq S(X \times X^*)$ such that for every $\mu := V \times Y \in \mathcal{A}$ there exists a projection $P_\mu : X^* \rightarrow X^*$ satisfying that $P_\mu(X^*) = Y$, $P_\mu^{-1}(0) = V^\perp$ and $P_\mu^*(X^{**}) = \overline{V}^{w(X^{**}, X^*)}$. Hence, consider $v_k^* \in Y$ such that $v_k^* \xrightarrow{w^*} v^*$ and $v^* = 0$ on V , so $v^* = 0$ on $\overline{V}^{w(X^{**}, X^*)}$. Moreover, because $v_k^* \in Y$ and P_μ is a projection onto Y one has $P_\mu(v_k^*) = v_k^*$, then $\langle v^*, x - P_\mu(x) \rangle = \lim \langle v_k^*, x - P_\mu(x) \rangle = \lim \langle P_\mu(v_k^*), x - P_\mu(x) \rangle = \lim \langle v_k^*, P_\mu(x) - P_\mu(x) \rangle = 0$ for every $x \in X$, which implies (using that $\langle v^*, P_\mu(x) \rangle = 0$) $\langle v^*, x \rangle = 0$.

Now, we choose a decreasing sequence of positive numbers $\gamma_n \searrow 0^+$, consider $V_1 \times Y_1 \in \mathcal{A}$ containing (x', w^*) , let $\{e(1, i)\}_{i \in \mathbb{N}}$ be a dense set in $\mathbb{B} \cap V_1$ and define

$$W(1, p) := \{y^* \in X^* : |\langle y^*, e(1, i) \rangle| \leq \gamma_p, \text{ for all } i = 1, \dots, p\}.$$

Whence for all $p \geq 1$ and $t \in F$ we can pick points $x_t(1, p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(1, p) \in \hat{\partial} f_t(x_t(1, p))$, $\lambda(1, p) \in \Delta(F, x', \gamma_p)$ and $v(1, p)^* \in W(1, p)$ such that $w^* = \sum \lambda_t(1, p) x_t^*(1, p) + v^*(1, p)$.

Now assume that we have selected $V_n \times Y_n \in \mathcal{A}$ containing all $V_k \times Y_k$ for $k \leq n$, families of points $\{e(n, i)\}_{i \in \mathbb{N}}$ dense in $\mathbb{B} \cap V_n$, which contains all previous $\{e(k, i)\}_{i \in \mathbb{N}}$ for $k \leq n$, points $x_t(i, p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(i, p) \in \hat{\partial} f_t(x_t(i, p))$, $\lambda(i, p) \in \Delta(F, x', \gamma_p)$ and $v(i, p)^* \in W(i, p)$ such that

$$(21) \quad w^* = \sum \lambda_t(i, p) x_t^*(i, p) + v^*(i, p), \text{ for } i \leq n \text{ and } p \geq 1.$$

Then, take $V_{n+1} \times Y_{n+1} \in \mathcal{A}$ such that $V_n \times Y_n \subseteq V_{n+1} \times Y_{n+1}$, $x_t(i, p) \in V_{n+1}$, $x_t^*(i, p) \in Y_{n+1}$ for all $t \in F$, $i \leq n$, $p \in \mathbb{N}$, consider $\{e(n+1, i)\}_{i \in \mathbb{N}}$ a dense set in $B \cap V_{n+1}$, and define

$$W(n+1, p) := \{y^* \in X^* : |\langle y^*, e(k, i) \rangle| \leq \gamma_p, \text{ for all } k = 1, \dots, n+1 \text{ and } i = 1, \dots, p\}.$$

Then for all $p \geq 1$ and $t \in F$ we can pick points $x_t(n+1, p) \in \mathbb{B}(x', f_t, \gamma_p)$, subgradients $x_t^*(n+1, p) \in \hat{\partial} f_t(x_t(n+1, p))$, $\lambda(n+1, p) \in \Delta(F, x', \gamma_p)$ and $v(n+1, p)^* \in W(n+1, p)$ such that $w^* = \sum \lambda_t(n+1, p) x_t^*(n+1, p) + v^*(n+1, p)$.

Now we define $\bigcup_{n \in \mathbb{N}} V_n \times \bigcup_{n \in \mathbb{N}} Y_n =: V \times Y \in \mathcal{A}$, $x_t(n) := x_t(n, n)$, $x_t^*(n) := x_t^*(n, n)$, $\lambda_t(n) := \lambda_t(n, n)$, $v^*(n) := v^*(n, n)$. Then, by our construction $x_t(n) \xrightarrow{f} x'$. Since $\lambda(n) \in \Delta(F, x', \gamma_n)$ we can assume that $\lambda_t(n) \xrightarrow{n \rightarrow \infty} \lambda_t \in [0, 1]$ for every $t \in F$, and $\sum_{t \in F} \lambda_t = 1$; moreover $f_t(x') = f_F(x')$ for every $t \in \text{supp } \lambda$.

Then, on the one hand if (there exist some subsequence such that) $\lambda_t(n) x_t^*(n)$ is bounded for all $t \in F$, in this case we can assume that

- If $t \in \text{supp } \lambda$, $\lambda_t(n) x_t^*(n)$ converge to some $\lambda_t x_t^*$ with $x_t^* \in \partial f_t(x')$.
- If $t \notin \text{supp } \lambda$, $\lambda_t(n) x_t^*(n)$ converge to some $x_t^* \in \partial^\infty f_t(x')$.
- $v^*(k) \xrightarrow{w^*} v^*$.

Furthermore, v^* is zero on V . Indeed, the set $\{e(i, j)\}_{i, j}$ is dense in V , then for every $n \geq \max\{i, j\}$ we have that $|\langle v^*(n), e(i, j) \rangle| \leq \gamma_n$ (recall $v^*(n) \in W(n, n)$), so taking the limits $\langle v^*, e(i, j) \rangle = 0$ for every i, j , therefore v^* is zero on V . Thus, by the property of \mathcal{A} necessarily v^* is zero on the whole X , hence using (21) we have that (20) holds.

On the other hand, if there exists some $t \in F$ such that $\|\lambda_t(n) \cdot x_t^*(n)\|_* \rightarrow +\infty$, we define $\eta_n := (\max_{t \in F} \{\|\lambda_t(n) x_t^*(n)\|_*, \|v^*(n)\|_*\})^{-1}$. We have $\eta_n w^* \rightarrow 0$ and (by passing to a subsequence) $\eta_n \lambda_t(n) x_t^*(n) \xrightarrow{w^*} w_t^* \in \partial^\infty f(x')$; and by a similar argument as in the first case $\eta_n v^*(n) \rightarrow 0$, so $\sum_{t \in F} w_t^* = 0$. Moreover, by the limiting condition (17) we have $w_t^* = 0$. Finally, since all the functions but one of f_t 's are SNEC at x' we have $\eta_n \lambda_t(n) x_t^*(n)$ converge in norm topology to zero, which is a contradiction.

Therefore $x^* \in \mathcal{S}(\bar{x}, 3\varepsilon) + V + V$ ($x^* \in [0, \varepsilon] \mathcal{S}(\bar{x}, 3\varepsilon) + V + V$, resp.), and by the arbitrariness of V and $\varepsilon > 0$ we conclude (18). \square

The next result gives us a simplification of the main formulae in [Theorem 4.8](#) under the additional assumption that the data is Lipschitz continuous. The case when the data is uniformly Lipschitz continuous was proved in [\[28, Theorem 3.2\]](#).

THEOREM 4.9. *Let $\{f_t : t \in T\}$ be a family of locally Lipschitz functions on a neighborhood of a point $\bar{x} \in \text{dom } f$. Then*

$$(22) \quad \partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left(\mathcal{S}(\bar{x}, \varepsilon) \right), \text{ and } \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left([0, \varepsilon] \cdot \mathcal{S}(\bar{x}, \varepsilon) \right),$$

where $\mathcal{S}(\bar{x}, \varepsilon)$ was defined in (19). In addition, if the family is uniformly locally Lipschitz at \bar{x} , then

$$(23) \quad \partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \sum_{t \in F} \lambda_t \partial f_t(x') : \begin{array}{l} F \in \mathcal{P}_{\mathbf{t}}(T_\varepsilon(\bar{x})), x' \in \mathbb{B}(\bar{x}, \varepsilon), \\ \lambda \in \Delta(F) \text{ and} \\ f_t(x') = f_F(x') \text{ for all } t \in F \end{array} \right\}.$$

Proof. Consider $V \in \mathcal{N}_0(w^*)$, $\varepsilon > 0$, a finite-dimensional subspace $L \ni \bar{x}$ such that $L^\perp \subseteq V$ and $x^* \in \partial f(x)$ (respectively, $y^* \in \partial^\infty f(\bar{x})$), let $P : X \rightarrow L$ be a continuous linear projection and define $W = (P^*)^{-1}(V)$. Hence, $x_{|L}^* \in \partial f_{|L}(x)$ (respectively, $y_{|L}^* \in \partial^\infty f_{|L}(x)$). Hence, we apply [Theorem 4.8](#) and we conclude the existence of some $F \in \mathcal{T}_\varepsilon(\bar{x})$, $x' \in \mathbb{B}(\bar{x}, \varepsilon)$, $\lambda \in \Delta(F)$ such that $x_{|L}^* \in \sum_{t \in F} \lambda_t \partial(f_{|L})_t(x') + W$ and

$(f|_L)_{t'}(x') = (f|_L)_{t''}(x')$ for all $t', t'' \in \text{supp } \lambda$, then

$$P^*(x|_L^*) \in \sum_{t \in F} \lambda_t \partial(f_t + \delta_L)(x') + V = \sum_{t \in F} \lambda_t \partial f_t(x') + L^\perp + V,$$

where the last equality follows from the sum rule for Lipschitz functions (see [17, 18, 25]). Therefore $x^* = P(x_L^*) + x^* - P(x_L^*) \in \sum_{t \in F} \lambda_t \partial f_t(x') + V$, which implies $x^* \in \mathcal{S}(\bar{x}, \varepsilon) + V$. Similarly, for $y|_L^* \in \partial^\infty f|_L(x)$ one concludes that $y^* \in [0, \varepsilon] \cdot \mathcal{S}(\bar{x}, \varepsilon) + V$, and from the arbitrariness of $\varepsilon > 0$ and $V \in \mathcal{N}_0(w^*)$ we conclude the proof of (22).

Finally to prove (23) we notice that if the functions are uniformly locally Lipschitz at \bar{x} with constant K , then assuming that $\varepsilon > 0$ is small enough, we have that for any $t \in T$, $x \in \mathbb{B}(\bar{x}, \varepsilon)$ and $|f_t(x) - f(\bar{x})| \leq \varepsilon$ we also have $f_t(\bar{x}) \geq f(\bar{x}) - (K + 1)\varepsilon$, which means $t \in T_{(K+1)\varepsilon}(\bar{x})$. \square

The next example shows an application of the above results with a family which is not uniformly locally Lipschitz. This example is important because, on the one hand it provides an exact upper-estimation of the supremum function of a family of functions which are not uniformly locally Lipschitz, and, on the other hand it gives us a nonconvex upper-estimation.

EXAMPLE 4.10. Consider $T = (0, 1)$ and the family of functions $f_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f_t(x, y) = tx^2 - \frac{|y| + 1}{t}.$$

Here, it is important to notice that all the functions are Lipschitz continuous, but not uniformly Lipschitz continuous, so the results of [28] cannot be applied. Nevertheless, we can apply Theorem 4.9. Indeed, first the supremum function is given by $f(x, y) = x^2 - |y| - 1$. The limiting subdifferential of f at $(\bar{x}, \bar{y}) = (0, 0)$ is $\partial f(0, 0) = \{0\} \times \{-1, 1\}$ and the value of f at this point is $f(0, 0) = -1$. Now, we compute the limiting subdifferential of f at (\bar{x}, \bar{y}) using Theorem 4.9. Pick z^* in the right-hand side of (22), then there exist $\varepsilon_n \rightarrow 0^+$, $F_n \in \mathcal{P}_f(T)$, $(x_n, y_n) \in \varepsilon_n \mathbb{B}$, and $\lambda_n \in \Delta(F_n)$ such that $|f_{t_n}(x_n, y_n) - f(0, 0)| \leq \varepsilon_n$, $f_t(x_n, y_n) = f_{F_n}(x_n, y_n)$ for all $t \in F_n$ and $z_n^* \in \sum_{s \in F_n} \lambda_s \partial f_s(x_n, y_n) + \varepsilon_n \mathbb{B}^*$. Now the equation

$$tx_n^2 - \frac{|y_n| + 1}{t} = sx_n^2 - \frac{|y_n| + 1}{s}$$

implies $t = s$, and consequently $F_n = \{t_n\}$.

Now, using the inequality $|f_{t_n}(x_n, y_n) - f(0, 0)| = |f_{t_n}(x_n, y_n) + 1| \leq \varepsilon_n$ one gets $t_n \rightarrow 1$. Therefore, $z_n^* \in \{(2t_n x_n^2, \frac{1}{t_n}), (2t_n x_n^2, -\frac{1}{t_n})\} + \varepsilon_n \mathbb{B}^*$ with $t_n \rightarrow 1$, $x_n \rightarrow 0$ and $\varepsilon \rightarrow 0$, consequently $z^* \in \{0\} \times \{-1, 1\}$.

In order to derive a more precise estimation of the subdifferential of the supremum function in [28, Definition 3.4], the authors introduced the definition of *equicontinuous subdifferentiability*. This notion involves some *uniform continuity* of the subdifferentials of the data functions f_t 's for points close to the active index set.

DEFINITION 4.11. Let $f_t : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a family of lsc functions indexed by $t \in T$. The family is called *equicontinuously subdifferentiable* at $\bar{x} \in X$ if for any *weak**-neighborhood V of the origin in X^* there is some $\varepsilon > 0$ such that

$$(24) \quad \partial f_t(x) \subseteq \partial f_t(\bar{x}) + V, \text{ for all } t \in T_\varepsilon(\bar{x}) \text{ and all } x \in \mathbb{B}(\bar{x}, \varepsilon).$$

Although this definition is precisely for the framework of [28], our formulae involves the singular subdifferential of the nominal data for points close to the point of interest, due to the possible lack of Lipschitz continuity of our data. For that reason we introduce the following definition, which is satisfied trivially when the nominal data is Lipschitz continuous in a neighborhood of the point of interest.

DEFINITION 4.12. *Let $f_t : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a family of lsc functions indexed by $t \in T$. The family is called *singular equicontinuously subdifferentiable* at $\bar{x} \in X$ if for any weak*-neighborhood V of the origin in X^* there is some $\varepsilon > 0$ such that*

$$(25) \quad \partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V, \text{ for all } t \in T \text{ and all } x \in \mathbb{B}(\bar{x}, \varepsilon).$$

Finally, we say that the family of functions $\{f_t : t \in T\}$ is *total equicontinuously subdifferentiable* at $\bar{x} \in X$ if $\{f_t : t \in T\}$ is equicontinuously subdifferentiable and singular equicontinuously subdifferentiable at $\bar{x} \in X$.

Using the notion of *total equicontinuously subdifferentiable* we have the following tighter formulae, which represents an extension of [28, Proposition 3.5].

THEOREM 4.13. *In the setting of [Theorem 4.8](#) assume that the family of functions $\{f_t\}_{t \in T}$ is total equicontinuously subdifferentiable at \bar{x} and*

$$(26) \quad \limsup_{x \rightarrow \bar{x}} \sup_{t \in T} |f_t(x) - f_t(\bar{x})| = 0.$$

Then

$$(27) \quad \partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left\{ \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{l} \lambda \in \Delta(T) \text{ and} \\ \text{supp } \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} \text{ and}$$

$$(28) \quad \partial^\infty f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left([0, \varepsilon] \cdot \left\{ \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{l} \lambda \in \Delta(T) \text{ and} \\ \text{supp } \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} \right).$$

Proof. Consider $x^* \in \partial f(\bar{x})$, $\varepsilon > 0$ and V a weak*-neighborhood of the origin. First, by (24) and (25) we can take $\gamma_1 > 0$ such that for all $x \in \mathbb{B}(\bar{x}, \gamma_1)$

$$(29) \quad \partial f_t(x) \subseteq \partial f_t(\bar{x}) + V, \text{ for all } t \in T_{\gamma_1}(\bar{x}) \text{ and}$$

$$(30) \quad \partial^\infty f_t(x) \subseteq \partial^\infty f_t(\bar{x}) + V \text{ for all } t \in T.$$

Second, by (26) we can take $\gamma_2 > 0$ such that

$$(31) \quad |f_t(x) - f_t(\bar{x})| \leq \gamma_1/2, \forall t \in T, \forall x \in \mathbb{B}(\bar{x}, \gamma_2).$$

Now, by [Theorem 4.8](#) we have that for $\gamma = \min\{\gamma_1/2, \gamma_2, \varepsilon/2\}$

$$x^* \in \mathcal{S}(\bar{x}, \gamma) + V.$$

Whence, there exists $F \in \mathcal{P}_f(T)$, $\lambda \in \Delta(F)$ and $x' \in \mathbb{B}(\bar{x}, \gamma)$ such that $|f_F(x') - f(\bar{x})| \leq \gamma$ and $f_F(x') = f_t(x')$ for all $t \in \text{supp } \lambda$ and

$$(32) \quad x^* \in \sum_{t \in F} \lambda_t \circ \partial f_t(x') + V.$$

Hence, by (31) we have that for all $t \in \text{supp } \lambda$

$$f(\bar{x}) \leq f_F(x') + \gamma = f_t(x') + \gamma \leq f_t(\bar{x}) + \gamma_1/2 + \gamma \leq f_t(\bar{x}) + \gamma_1,$$

which means that $t \in T_{\gamma_1}(\bar{x})$ and consequently $\text{supp } \lambda \subseteq T_{\gamma_1}(\bar{x})$. Now, by (29), (30), and (32) we have

$$\begin{aligned} x^* &\in \sum_{\lambda_t > 0} \lambda_t \cdot \partial f_t(x') + \sum_{\lambda_t = 0} \partial^\infty f_t(x') + V \\ &\subseteq \sum_{\lambda_t > 0} \lambda_t \circ \partial f_t(\bar{x}) + \sum_{\lambda_t = 0} \partial^\infty f_t(\bar{x}) + V + V + V \\ &\subseteq \left\{ \sum_{t \in T} \lambda_t \circ \partial f_t(\bar{x}) : \begin{array}{l} \lambda \in \Delta(T) \text{ and} \\ \text{supp } \lambda \subseteq T_\varepsilon(\bar{x}) \end{array} \right\} + V + V + V. \end{aligned}$$

Finally, from the arbitrariness of ε and V we conclude (27). The proof of (28) is similar, so we omit the proof. \square

5. The convex subdifferential. This section is devoted to giving formulae for the convex subdifferential. Due to the closure of the graph of the convex subdifferential under bounded nets with respect to the $\|\cdot\| \times w^*$ -topology in $X \times X^*$, we can obtain a similar result to Theorem 4.8 by changing the SNEC assumption for a similar one using nets instead of sequences. For this purpose, it is better to express the limiting condition of Theorem 4.8 in terms of the normal cone of the domain of each function f_t , more precisely, we recall that for any lsc convex function h , the normal cone to the domain of h at a point x is given by

$$N_{\text{dom } h}(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \forall y \in \text{dom } h\}.$$

Using this notation we establish the following result.

THEOREM 5.1. *Let $\{f_t : t \in T\}$ be a family of proper convex lsc functions satisfying the following assumptions: There exists a neighborhood U of \bar{x} such that*

- a) *For all $x \in U$, all but one of the functions $\{f_t : t \in T\}$ and every net $(\lambda_\nu, x_\nu, x_\nu^*) \in [0, +\infty) \times X \times X^*$ satisfying $\lambda_\nu \rightarrow 0$, $x_\nu \xrightarrow{f} x$, $x_\nu^* \in \partial f(x_\nu)$ and $\lambda_\nu x_\nu^* \xrightarrow{*} 0$ one has $\|\lambda_\nu x_\nu^*\|_* \rightarrow 0$.*
- b) *For all $x \in U$ and all $F \in \mathcal{P}_f(T)$*

$$w_t^* \in N_{\text{dom } f_t}(x), t \in F \text{ and } \sum_{t \in F} w_t^* = 0 \text{ implies } w_t^* = 0, \text{ for all } t \in F.$$

Then

$$(33) \quad \partial f(\bar{x}) \subseteq \bigcap_{\varepsilon > 0} \text{cl}^{w^*} A(\bar{x}, \varepsilon).$$

Where

$$A(\bar{x}, \varepsilon) := \bigcup \left\{ \text{co} \left(\bigcup_{t \in F_1} \partial f_t(x') \right) + \sum_{t \in F_2} N_{\text{dom } f_t}(x') \right\}$$

and the union is over all $F_1, F_2 \in \mathcal{P}_f(T)$ and $x' \in \mathbb{B}(\bar{x}, \varepsilon)$ such that $|f_t(x') - f(\bar{x})| \leq \varepsilon$ and $f_t(x') = f_{F_1 \cup F_2}(x')$ for all $t' \in F_1$. Moreover, the equality holds, whenever the function f is continuous at some point, or the space X is finite-dimensional.

Proof. Since the proof of (33) relies on similar arguments as Theorem 4.8 (but without the use of techniques of separable reduction) we prefer to omit the proof.

Now, any point in the right-hand side of (33) is the limit of a net w^* , which has the form of $w_\nu^* = \sum \lambda_\nu(t)v_\nu(t)^* + \sum w_\nu^*(t)$ with $v_\nu \in \partial f_t(x_\nu)$, $w_\nu^*(t) \in N_{\text{dom } f_t}(x_\nu)$, $\sum \lambda_\nu(t) = 1$ and $\lambda_\nu(t) \geq 0$, then one gets for every $y \in X$

$$\langle w_\nu^*, y - \bar{x} \rangle \leq f(y) - f(x) + |f_t(x') - f(\bar{x})| + \langle w_\nu^*, x_\nu - \bar{x} \rangle.$$

Therefore, we can conclude the equality in (33) whenever the $\lim \langle w_\nu^*, x_\nu - \bar{x} \rangle = 0$, and this holds in particular when the function f is continuous at some point, or the space X is finite-dimensional, because in these cases the net $\{w_\nu^*\}$ is bounded. \square

The following results have the intention of establishing formulae without any qualification. This is possible by reducing the analysis to subspaces with nice properties for the family of functions. For that reason we denoted by \mathcal{F}_x the set of all finite-dimensional affine subspaces containing x . This class of sets allows us to give formulae in any (Hausdorff) locally convex topological vector space (lcs for short). It is useful to recall some simple facts about lcs available in pioneer books such as [5, 39]: The topology on every lcs X is generated by a family of seminorms $\{\rho_i : i \in \mathcal{I}\}$, which will be always assumed to be *up-directed*, i.e., for every two points $i_1, i_2 \in \mathcal{I}$ there exists $i_3 \in \mathcal{I}$ such that $\rho_{i_3}(x) \geq \max\{\rho_{i_1}(x), \rho_{i_2}(x)\}$ for all $x \in X$. For a point \bar{x} in X , $r \geq 0$ and a seminorm ρ we define $\mathbb{B}_\rho(\bar{x}, r) := \{x \in X : \rho(x - \bar{x}) \leq r\}$. In the (topological) dual of X , denoted by X^* , some examples of topologies are the *w*-topology* denoted by $w(X^*, X)$ (w^* , for short), which is the topology generated by the pointwise convergence, and the *strong topology* denoted by $\beta(X^*, X)$ (β , for short), which is the topology generated by the uniform convergence on bounded sets. For a set $A \subseteq X^*$, the symbol $\beta\text{-seq-}A$ denotes the set of points which are the limit, with respect to the β -topology, of some sequence lying in A . Finally, for a function $g : X \rightarrow \overline{\mathbb{R}}$, $\overline{\text{co}}g$ denotes the convex lsc envelope of g . For more details about the theory of convex analysis in lcs we refer to [22, 33, 44].

Now, let us establish the first general formula without any qualification condition.

THEOREM 5.2. *Let X be an lcs, let \mathcal{I} be a family of seminorms which generate the topology on X . Consider a family of proper convex lsc functions $\{f_t : t \in T\}$. Then, for all $\bar{x} \in X$*

$$(34) \quad \partial f(\bar{x}) = \bigcap_{\substack{\varepsilon > 0, \rho \in \mathcal{I} \\ L \in \mathcal{F}_x}} \beta\text{-seq-cl } A_{\varepsilon, L, \rho}(\bar{x}),$$

where

$$A_{\varepsilon, L, \rho}(\bar{x}) := \bigcup \left\{ \text{co} \left(\bigcup_{t \in F_1} \partial f_{t, L}(x') \right) + \sum_{t \in F_2} N_{\text{dom } f_t \cap L}(x') \right\}.$$

Where $f_{t, L} := f_t + \delta_{\text{aff}(\text{dom } f \cap L)}$ and the union is over all $x' \in \mathbb{B}_\rho(\bar{x}, \varepsilon) \cap L$ and $F_1, F_2 \in \mathcal{P}_f(T)$ such that $f_t(x') = f_{F_1 \cup F_2}(x')$ for all $t \in F_1$ and $|f_t(x') - f(\bar{x})| \leq \varepsilon$.

Proof. W.l.o.g. we may assume that $\bar{x} = 0$. Consider $\varepsilon > 0$, $L \in \mathcal{F}_x$, and ρ a seminorm on X , also we can assume that ρ is a norm on L , because $A_{\varepsilon, L, \rho_1}(0) \subseteq A_{\varepsilon, L, \rho}(0)$, for any $\rho_1 \geq \rho$. Consider $W := \text{aff}(\text{dom } f \cap L)$, let us show that

$$(35) \quad \partial(f + \delta_W)(0) \subseteq \beta\text{-seq-cl } A_{\varepsilon, L, \rho}(0).$$

Indeed, take $x^* \in \partial(f + \delta_W)(0)$ and let $P : X \rightarrow (W, \rho)$ be a continuous linear projection. Hence, $x^*|_W$ (the restriction of x^* to W) belongs to $\partial f|_W(0)$. The finite-dimensionality of W gives us the continuity of $f|_W$ at some point (see [38]), so the

family $(f_t)|_W$ satisfies the hypotheses of [Theorem 5.1](#). Whence, there exists a sequence $w_n^* \rightarrow x^*|_W$ where

$$w_n^* \in \text{co} \left(\bigcup_{t \in F_{1,n}} \partial((f_t)|_W)(x'_n) \right) + \sum_{t \in F_{2,n}} N_{\text{dom}(f_t)|_W}(x'_n)$$

with $F_{1,n}, F_{2,n} \in \mathcal{P}_{\mathfrak{f}}(T)$, $x'_n \in \mathbb{B}_\rho(0, \varepsilon) \cap W$ such that $|f_t(x'_n) - f(\bar{x})| \leq \varepsilon$ and $f_t(x'_n) = \max_{F_{1,n} \cup F_{2,n}} f_t(x')$ for all $t \in F_{1,n}$.

Now we define $x_n^* := P^*(w_n^*) + x^* - P^*(x^*|_W)$, it follows that $x_n^* \in A_{\varepsilon, L, \rho}(0)$. Moreover, considering $V := P^{-1}(\mathbb{B}_W)$, where \mathbb{B}_W is the unit ball in W , we get

$$\begin{aligned} \sigma_V(x^* - y_n^*) &= \sup_{v \in V} \langle x^* - y_n^*, v \rangle = \sup_{v \in V} \langle P^*(w_n^*) - P^*(x^*|_W), v \rangle \\ &= \sup_{h \in \mathbb{B}_W} \langle z_n^* - x^*|_W, h \rangle \rightarrow 0. \end{aligned}$$

Which concludes [\(35\)](#), then using that

$$\partial f(0) = \bigcap_{L \in \mathcal{F}_0} \partial(f + \delta_{\text{aff}(\text{dom } f \cap L)})(0) \subseteq \bigcap_{\substack{\varepsilon > 0, \rho \in \mathcal{I} \\ L \in \mathcal{F}_0}} \beta\text{-seq-cl } A_{\varepsilon, L, \rho}(\bar{0}),$$

we get the first inclusion in [\(34\)](#).

Now, pick $x^* \in \bigcap_{\substack{\varepsilon > 0, \rho \in \mathcal{I} \\ L \in \mathcal{F}_0}} \beta\text{-seq-cl } A_{\varepsilon, L, \rho}(0)$ and $y \in \text{dom } f$. Then, take a sequence $\varepsilon_n \rightarrow 0$ and pick $L \in \mathcal{F}_0$ which contains y and consider $\rho \in \mathcal{I}$ such that ρ is a norm on L and $\rho(x_n) \rightarrow 0$ implies $|\langle x^*, x \rangle| \rightarrow 0$. Hence, there exist sequences $F_{1,n}, F_{2,n} \in \mathcal{P}_{\mathfrak{f}}(T)$, $x_n \in \mathbb{B}_\rho(0, \varepsilon_n) \cap L$ and $w_n^* \in X^*$ such that $w_n^* \xrightarrow{\beta} x^*$,

$$w_n^* \in \text{co} \left(\bigcup_{t \in F_{1,n}} \partial f_{t,L}(x_n) \right) + \sum_{t \in F_{2,n}} N_{\text{dom } f_t \cap L}(x_n)$$

and $|f_t(x_n) - f(0)| \leq \varepsilon_n$, $f_t(x_n) = \max_{F_{1,n} \cup F_{2,n}} f_t(x_n)$ for all $t \in F_{1,n}$, which implies

$$(36) \quad \langle w_n^*, y - x_n \rangle \leq f(y) - f(0) + \varepsilon_n.$$

We claim that $\langle w_n^*, y - x_n \rangle \rightarrow \langle x^*, y \rangle$. Indeed, because ρ is a norm in L , $x_n \in L$ and $\rho(x_n) \rightarrow 0$ necessarily $x_n \rightarrow 0$ with respect to the topology on X . Hence, the set $B := \{y - x_n : n \in \mathbb{N}\}$ is bounded, so

$$\begin{aligned} |\langle w_n^*, y - x_n \rangle - \langle x^*, y \rangle| &= |\langle w_n^* - x^*, y - x_n \rangle - \langle x^*, x_n \rangle| \\ &\leq \sigma_B(w_n^* - x^*) + |\langle x^*, x_n \rangle| \rightarrow 0. \end{aligned}$$

Finally, taking $n \rightarrow \infty$ in [\(36\)](#) it yields $\langle x^*, y - x \rangle \leq f(y) - f(0)$, which concludes the proof due to the arbitrariness of $y \in \text{dom } f$. \square

The final goal of this paper is to give an alternative proof of [\[8, Corollary 6\]](#), which, as far as we know, appears to be the most general extension of [\[14, Theorem 4\]](#). Before presenting this proof we need the following lemma. This result is interesting by itself, since it allows us to understand the subdifferential of any function in terms of the subdifferential of another function.

LEMMA 5.3. *Let X be an lcs, let $h, g : X \rightarrow \overline{\mathbb{R}}$ be two convex lsc proper functions and let $D \subseteq \text{dom } h$ be a convex subset such that*

$$h(x) = g(x) \text{ for all } x \in D.$$

Then for every $\bar{x} \in X$

$$(37) \quad \partial(h + \delta_D)(\bar{x}) = \bigcap_{L \in \mathcal{F}_{\bar{x}}} \left\{ \text{co} \{S_L(\bar{x})\} + N_{D \cap L}(\bar{x}) \right\},$$

where $S_L(\bar{x}) := \limsup \partial(g + \delta_{\text{aff}(D \cap L)})(x')$, the lim sup is understood to be the set of all $x^* \in X^*$, which are the limit (in the β -topology) of some sequence $x_n^* \in \partial(g + \delta_{\text{aff}(D \cap L)})(x_n)$ with $x_n \in \text{ri}_L(D)$, $x_n \xrightarrow{g} \bar{x}$ and $|\langle x_n^*, x_n - \bar{x} \rangle| \rightarrow 0$. Here, $\text{ri}_L(D)$ denotes the interior of $D \cap L$ with respect to $\text{aff}(D \cap L)$.

Proof. W.l.o.g. we may assume that $\bar{x} = 0$. First we notice that

$$(38) \quad \partial(h + \delta_D)(0) = \bigcap_{L \in \mathcal{L}_0} \partial(h + \delta_{D \cap L})(0) = \bigcap_{L \in \mathcal{L}_0} \partial(h + \delta_{\text{cl}(D \cap L)})(0).$$

Indeed, the first inequality is straightforward and the second follows from the fact that $\partial(h + \delta_{D \cap L})(0) = \partial(h + \delta_{\text{cl}(D \cap L)})(0)$ thanks to the *accessibility lemma* (see, e.g., [1]). Now, fix $L \in \mathcal{F}_0$, define $W = \text{aff}(L \cap D)$ and consider a continuous linear projection $P : X \rightarrow W$. We claim that

$$(39) \quad \partial(h + \delta_{\text{cl}(D \cap L)})(0) \subseteq \text{co} \{S_L(0)\} + N_{\text{dom } f \cap D \cap L}(0).$$

Indeed, take $x^* \in \partial(h + \delta_{\text{cl}(L \cap D)})(0)$, using the same finite-dimensional representation as in the proof of [Theorem 5.2](#), one gets the existence of a point $y^* \in \partial(h + \delta_{D \cap L})|_W(0)$ and $z^* \in W^\perp$ such that $x^* = P^*(y^*) + z^*$. Then, by the finite-dimensionality of W $\text{ri}_{\text{aff}(D \cap L)}$ is not empty and consequently $(h + \delta_{\text{cl}(L \cap D)})|_W$ has a point of continuity (relative to its domain). Then, we apply [\[38, Theorem 25.6\]](#) and we get the existence of sequences $u_{n,i} \in \text{ri}(\text{dom}(h + \delta_{\text{cl}(L \cap D)})|_W)$, $y_n^*, u_{n,i}^* \in W^*$, $\alpha_{n,i} \geq 0$ with $\sum_{i=1}^N \alpha_i = 1$ and a point $\theta^* \in N_{\text{dom}(h|_W)}(0)$ such that $y^* = \lim y_n^* + \theta^*$, $y_n^* = \sum_{i=1}^N \alpha_{n,i} u_{n,i}^*$, $u_{n,i}^* \in \partial(h + \delta_{\text{cl}(L \cap D)})|_W(u_{n,i})$ and $u_{n,i} \rightarrow 0$, where the number $N = \dim W + 1$ is fixed by virtue of Carathéodory's Theorem.

Now, $\partial(h + \delta_{\text{cl}(L \cap D)})|_W(u_{n,i}) = \partial h|_W(u_{n,i})$, because $u_{n,i} \in \text{ri}_L(D)$. Furthermore, $h(x') = g(x')$ for every $x' \in \text{ri}_L(D)$, which implies that $u_{n,i}^* \in \partial g|_W(u_{n,i})$.

Moreover, the vectors $\alpha_{n,i} u_{n,i}^*$ must be bounded (to prove this fact, one can argue by contradiction following the proof of [Theorem 4.8](#), and then one shows that $N_{\text{dom}(h + \delta_{\text{cl}(D \cap L)})|_W}(0)$ contains a line, which is not possible due to the continuity of $(h + \delta_{\text{cl}(D \cap L)})|_W$). Hence, we may assume that $\alpha_{n,i} u_{n,i}^*$ converges and $\alpha_{n,i} \xrightarrow{n \rightarrow \infty} \alpha_i$. More precisely, on the one hand for each index i such that $\alpha_i = 0$, one has that $\alpha_{n,i} u_{n,i}^* \rightarrow v_i^*$ and $v_i^* \in N_{\text{dom } f|_W}(0)$. Indeed, for every $y \in \text{dom } h|_W$

$$\begin{aligned} \langle v_i^*, y - 0 \rangle &= \lim \langle \alpha_{n,i} u_{n,i}^*, y - u_{n,i} \rangle + \lim \langle \alpha_{n,i} u_{n,i}^*, u_{n,i} - 0 \rangle \\ &\leq \lim \alpha_{n,i} (h(y) - h(u_{n,i})) + \lim \langle \alpha_{n,i} u_{n,i}^*, u_{n,i} - 0 \rangle = 0. \end{aligned}$$

On the other hand, we have that for every index i such that $\alpha_i \neq 0$, $u_{n,i}^* \rightarrow v_i^*$ and $|\langle u_{n,i}^*, u_{n,i} \rangle| \rightarrow 0$, then using that $u_{n,i}^* \in \partial g|_W(u_{n,i})$ we get $g(u_{n,i}) \rightarrow g(0)$. Therefore,

$$y^* = \sum_{\{i | \alpha_i \neq 0\}} \alpha_i v_i^* + \sum_{\{i | \alpha_i = 0\}} v_i^* + \theta^*,$$

with $v_i^* \in \limsup \partial f|_W(u_{n,i})$ and $q^* := \sum_{\{i|\alpha_i=0\}} v_i^* + \theta^* \in N_{\text{dom } f|_W}(0)$.

Now define $w_i^* := P^*(v_i^*)$, $\lambda^* := z^* + P^*(q^*)$, $w^* := \sum_{\{i|\alpha_i \neq 0\}} \alpha_i w_i^*$, $w_{n,i} = P^*(u_{n,i}^*)$,

it follows that $w_{n,i}^* \xrightarrow{\beta} w_i^*$, $|\langle w_{n,i}^*, u_{n,i} \rangle| \rightarrow 0$ and $w_{n,i}^* \in \partial(g + \delta_W)(u_{n,i})$, $u_{n,i} \in \text{ri}_L(\text{dom } h)$, $g(u_{n,i}) \rightarrow g(0)$, $\lambda^* \in N_{\text{dom } h \cap L}(0)$ and $x^* = w^* + \lambda^*$, which concludes the proof of (39). Then, using (38) and (39) we conclude the first inclusion in (37).

To prove the opposite inclusion, consider x^* in the right-hand side of (37) and $y \in D$, and consider L as the subspace generated by y . Then, there are $\alpha_i \geq 0$ (with $\sum_i \alpha_i = 1$), $x_{n,i}^* \in \partial(g + \delta_{\text{aff}(D \cap L)})(x_{n,i})$ and $x_{n,i} \in \text{ri}_L(D)$ such that $x_{n,i} \xrightarrow{g} 0$, $x_{n,i}^* \xrightarrow{\beta} y_i^*$, $|\langle x_{n,i}^*, x_{n,i} \rangle| \rightarrow 0$ and $x^* = \sum_i \alpha_i y_i^* + \lambda^*$. Moreover, because $x_n \in \text{ri}_L(D)$ and $h = g$ in D , we get $\partial(g + \delta_{\text{aff}(D \cap L)})(x_n) = \partial(h + \delta_{\text{aff}(D \cap L)})(x_n)$. Then,

$$\begin{aligned} \langle x^*, y \rangle &= \left\langle \sum_i \alpha_i y_i^* + \lambda^*, y \right\rangle \leq \sum_i \alpha_i \lim_n \langle x_{n,i}^*, y - x_{n,i} \rangle + \lim_n \langle x_{n,i}^*, x_{n,i} \rangle \\ &\leq \sum_i \alpha_i \lim_n (h(y) - h(x_{n,i})) = h(y) - h(0). \end{aligned}$$

From the arbitrariness of y we conclude that $x^* \in \partial(h + \delta_D)(0)$, which concludes the proof of (37). \square

THEOREM 5.4. *Let X be an lcs and let $\{f_t : t \in T\}$ be an arbitrary family of functions and let $D \subset \text{dom } \overline{\text{co}} f$ be a convex set such that*

$$\overline{\text{co}}(f + \delta_D)(x) = \sup_{t \in T} \overline{\text{co}} f_t(x) \text{ for all } x \in D.$$

Then for all $\bar{x} \in X$

$$(40) \quad \partial(f + \delta_D)(\bar{x}) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}_{\bar{x}}}} \text{cl}^{w^*} \left(\text{co} \left(\bigcup_{t \in T_\varepsilon(\bar{x})} \partial_\varepsilon f_t(\bar{x}) \right) + N_{D \cap L}(\bar{x}) \right).$$

Proof. W.l.o.g we can assume that $\bar{x} = 0$. Because the inclusion \supseteq is direct, we focus on the opposite one. To prove this inclusion, we can assume that $\partial(f + \delta_D)(0) \neq \emptyset$, in particular $(f + \delta_D)(x) = \overline{\text{co}}(f + \delta_D)(x)$. First, we denote by $h = \overline{\text{co}}(f + \delta_D)$, $g_t := \overline{\text{co}} f_t$ and $g = \sup_{t \in T} g_t$, then we apply Lemma 5.3 and we get

$$(41) \quad \partial(f + \delta_D)(0) \subseteq \partial h(0) = \bigcap_{L \in \mathcal{F}_0} \left\{ \text{co} \{S_L(0)\} + N_{D \cap L}(0) \right\}.$$

We claim that for every $L \in \mathcal{F}_0$, $\varepsilon > 0$ and $U \in \mathcal{N}_0(w^*)$

$$(42) \quad S_L(0) \subseteq \text{co} \left(\bigcup_{t \in T_\varepsilon(0)} \partial_\varepsilon f_t(0) \right) + N_{D \cap L}(0) + U + U,$$

where $S_L(0)$ was defined in Lemma 5.3. Indeed, consider $x^* \in S_L(0)$, then by definition there exist sequences $y_n \in \text{ri}_{\text{aff}(D \cap L)}(D)$ and $y_n^* \in \partial(g + \delta_{\text{aff}(D \cap L)})(y_n)$ such that $y_n^* \rightarrow x^*$, $|\langle y_n^*, y_n \rangle| \rightarrow 0$ and $|g(y_n) - g(0)| \rightarrow 0$.

Now, the restriction of each y_n^* to $W := \text{aff}(D \cap L)$ belongs to $\partial g|_W(y_n)$ and $y_n \in \text{ri}_W(\text{dom } g|_W)$. Since the function $g|_W$ is locally bounded at y_n we can find a constant M_n and a closed convex neighborhood V_n of zero (relative to W) such that

$$g_t(x) \leq g_t(y_n) + M_n - g_t(y_n), \quad \forall x \in y_n + V_n.$$

Consequently, by [44, Corollary 2.2.12]

$$|g_t(x) - g(x')| \leq 3M_{t,n}\rho_{V_n}(x - y), \forall x, x' \in y_n + \frac{1}{2}V_n,$$

where $M_{t,n} := M_n - g_t(y)$ and ρ_{V_n} is the *Minkowski's functional* associated to V_n , that is, $\rho_{V_n}(u) := \inf\{s > 0 : u \in sV_n\}$. In particular, each function $(g_t)|_W$ is Lipschitz continuous on $\frac{1}{2}V_n$, it allows us to apply [Theorem 4.9](#) and by a diagonal argument we yield that there exists a sequence of sets $F_n \in \mathcal{P}_f(T)$, and there are sequences of vectors $x_n \in W$, $x_t^*(n) \in \partial(g_t)|_W(x_n)$ together with scalars $(\lambda_t(n)) \in \Delta(F_n)$ such that $x_n \rightarrow 0$, $|g_{F_n}(x_n) - g(0)| \rightarrow 0$ and $g_t(x_n) = g_{F_n}(x_n)$ for all $t \in F_n$ and $x_n^* = \sum_{t \in F_n} \lambda_t(n)x_t^*(n) \rightarrow x_{|W}^*$. From the fact that the dimension of W is finite, we can assume that $\#F_n \leq \dim(W) + 1$. Hence, necessarily the points $x_t^*(n)$ are uniformly bounded in W , otherwise $N_{\text{dom } f_{|W}}(0)$ contains a line, which is not possible due to $\text{ri}_{\text{aff}(L \cap \text{dom } g)}(\text{dom } g_{|W}) \neq \emptyset$ (it can be seen using similar arguments as those given in the proof of [Theorem 5.2](#)). Then, we can assume that there exists $F \in \mathcal{P}_f(T)$, $x \in W$, $x_t^* \in \partial(g_t)|_W(x)$ and $(\lambda_t) \in \Delta(F)$ such that $\max_{t \in F} |\langle x_t^*, x \rangle| \leq \varepsilon/5$, $|g_t(x) - g(0)| \leq \varepsilon/5$, $g_t(x) = f_F(x)$ for all $t \in F$ and

$$x_{|W}^* \in \sum_{t \in F} \lambda_t x_t^* + (P^*)^{-1}(U),$$

where P is a continuous projection from X to W . Then,

$$(43) \quad x^* \in \sum_{t \in F} \lambda_t w_t^* + x^* - P^*(y_{|W}^*) + U,$$

here $w_t^* := P^*(x_t^*)$ and $w_t^* \in \partial(g_t + \delta_W)(x)$. Furthermore, for all $t \in F$

$$(44) \quad \begin{aligned} f_t(0) + 2\varepsilon/5 &\geq g_t(0) + 2\varepsilon/5 \geq g_t(0) + |\langle x_t^*, x \rangle| + \varepsilon/5 \\ &\geq g_t(x) + \varepsilon/5 \geq g(0) = f(0), \end{aligned}$$

Now by Hirriat-Hurruty-Phelps' formula [15, Theorem 2.1]

$$\partial(g_t + \delta_W)(x) \subseteq \partial_{\varepsilon/5} g_t(x) + W^\perp + U,$$

which implies the existence of some point $\tilde{w}_t^* \in \partial_{\varepsilon/5} g_t(x)$ such that

$$(45) \quad w_t^* \in \tilde{w}_t^* + W^\perp + U.$$

Now, let us show that $\tilde{w}_t^* \in \partial_\varepsilon f_t(0)$. Indeed, consider $z \in X$, then

$$\begin{aligned} \langle \tilde{w}_t^*, z \rangle &= \langle \tilde{w}_{i,t}^*, z - x \rangle + |\langle w_t^*, x \rangle| \leq g_t(z) - g_t(x) + \varepsilon/5 + \varepsilon/5 \\ &\leq g_t(z) - g_t(0) + g_t(0) - g_t(x) + 2\varepsilon/5 \leq g_t(z) - g_t(0) + g(0) - g_t(x) + 2\varepsilon/5 \\ &\leq g_t(z) - g_t(0) + 3\varepsilon/5 \leq f_t(x) - f_t(0) + f_t(0) - g_t(0) + 3\varepsilon/5 \\ &\leq f_t(z) - f_t(0) + \varepsilon \text{ (by (44)).} \end{aligned}$$

Now, according to (43)–(45) we get (42) and from the arbitrariness of $\varepsilon > 0$ and U we conclude that

$$(46) \quad S_L(0) + N_{D \cap L}(0) \subset \bigcap_{\varepsilon > 0} \text{cl}^{w^*} \left(\text{co} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) \right) + N_{D \cap L}(x) \right).$$

Finally, using (41) and (46) we conclude the desired inclusion in (40). \square

REMARK 5.5. *It is worth mentioning that [Theorem 5.4](#) represents a slight extension of [8, Corollary 6], because in this result the authors have assumed that the data functions f_t 's are convex and proper.*

6. Conclusions. In this paper, we have provided general formulae for the supremum function of an arbitrary family of lsc functions.

In [Section 3](#), we provided general fuzzy calculus rules in terms of the Fréchet subdifferential. Our approach follows from establishing these fuzzy calculus rules for an increasing family of functions (see [Proposition 3.7](#)), where the key tool is the introduction of the notation of *robust infimum*. Later, in [Theorem 3.8](#), we used the power set ordered by inclusion to get general fuzzy calculus rules of an arbitrary family of functions, without any qualification condition, as far as we know this approach is novel.

In [Section 4](#) we established the main results of the paper, where we replaced the Lipschitz continuous assumption of the data by some limiting condition in terms of the singular subdifferentials (see [Item \(c\)](#) and [\(17\)](#)). It has not escape our notice that these kind of conditions are becoming more popular in providing subdifferential calculus rules (see, e.g., [\[2, 3, 16–18, 25, 26, 36\]](#)). This section was divided into [Subsection 4.1](#) and [Subsection 4.2](#), which focused attention on finite-dimensional and infinite-dimensional settings respectively. In both subsections we gave formulae for the subdifferential of the supremum function under different conditions. Here, It is worth comparing [Theorem 4.2](#) and [Theorem 4.8](#). The main difference between these two results is that the first one is a convex upper-estimate, and the second one corresponds to a non-convex upper-estimate (as we showed in [Example 4.10](#)). This difference can be explained, because [Theorem 4.2](#) uses a limiting condition only at the point of interest (see, [Item \(c\)](#)), but [Theorem 4.8](#) uses the information of the subdifferential at a neighborhood of the point of interest (see [\(17\)](#)).

Finally, in [Section 5](#) we shown that our approach can be used to get new formulae for the convex subdifferential, with and without qualification conditions, of the supremum function (see [Theorem 5.1](#) and [Theorem 5.2](#)), and also, it allows us to recover [\[8, Corollary 6\]](#) using [Theorem 4.8](#) (see [Theorem 5.4](#)), which in particular shows a unifying approach to the study of the subdifferential of the supremum function.

REFERENCES

- [1] J. M. Borwein and R. Goebel. Notions of relative interior in Banach spaces. *J. Math. Sci. (N. Y.)*, 115(4):2542–2553, 2003. Optimization and related topics, 1.
- [2] J. M. Borwein, B. S. Mordukhovich, and Y. Shao. On the equivalence of some basic principles in variational analysis. *J. Math. Anal. Appl.*, 229(1):228–257, 1999.
- [3] J. M. Borwein and Q. J. Zhu. *Techniques of variational analysis*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 20. Springer-Verlag, New York, 2005.
- [4] J. M. Borwein and D. Zhuang. On Fan’s minimax theorem. *Math. Programming*, 34(2):232–234, 1986.
- [5] N. Bourbaki. *Topological vector spaces. Chapters 1–5*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1987. Translated from the French by H. G. Eggleston and S. Madan.
- [6] F. H. Clarke. *Optimization and nonsmooth analysis*, volume 5 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1990.
- [7] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 1998.
- [8] R. Correa, A. Hantoute, and M. A. López. Towards supremum-sum subdifferential calculus free of qualification conditions. *SIAM J. Optim.*, 26(4):2219–2234, 2016.
- [9] M. Cúth and M. Fabian. Rich families and projectional skeletons in Asplund WCG spaces. *J. Math. Anal. Appl.*, 448(2):1618–1632, 2017.
- [10] M. Fabian and A. D. Ioffe. Separable reductions and rich families in the theory of Fréchet subdifferentials. *J. Convex Anal.*, 23(3):631–648, 2016.
- [11] K. Fan. Minimax theorems. *Proc. Nat. Acad. Sci. U. S. A.*, 39:42–47, 1953.

- [12] A. Hantoute. Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull intersection property. *Top*, 14(2):355–374, 2006.
- [13] A. Hantoute and M. A. López. A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions. *J. Convex Anal.*, 15(4):831–858, 2008.
- [14] A. Hantoute, M. A. López, and C. Zălinescu. Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions. *SIAM J. Optim.*, 19(2):863–882, 2008.
- [15] J.-B. Hiriart-Urruty and R. R. Phelps. Subdifferential calculus using ε -subdifferentials. *J. Funct. Anal.*, 118(1):154–166, 1993.
- [16] A. D. Ioffe. Approximate subdifferentials and applications. I. The finite-dimensional theory. *Trans. Amer. Math. Soc.*, 281(1):389–416, 1984.
- [17] A. D. Ioffe. Approximate subdifferentials and applications. II. *Mathematika*, 33(1):111–128, 1986.
- [18] A. D. Ioffe. Approximate subdifferentials and applications. III. The metric theory. *Mathematika*, 36(1):1–38, 1989.
- [19] A. D. Ioffe. On the theory of subdifferentials. *Adv. Nonlinear Anal.*, 1(1):47–120, 2012.
- [20] M. Ivanov. Sequential representation formulae for G -subdifferential and Clarke subdifferential in smooth Banach spaces. *J. Convex Anal.*, 11(1):179–196, 2004.
- [21] F. Jules and M. Lassonde. Dense subdifferentiability and trustworthiness for arbitrary subdifferentials. *Serdica Math. J.*, 36(4):387–402, 2010.
- [22] P.-J. Laurent. *Approximation et optimisation*. Hermann, Paris, 1972. Collection Enseignement des Sciences, No. 13.
- [23] C. Li and K. F. Ng. Subdifferential calculus rules for supremum functions in convex analysis. *SIAM J. Optim.*, 21(3):782–797, 2011.
- [24] M. A. López and G. Still. Semi-infinite programming. *European J. Oper. Res.*, 180(2):491–518, 2007.
- [25] B. S. Mordukhovich. *Variational analysis and generalized differentiation. I*, volume 330 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Basic theory.
- [26] B. S. Mordukhovich. *Variational analysis and generalized differentiation. II*, volume 331 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Applications.
- [27] B. S. Mordukhovich. *Variational Analysis and Applications*, volume 8. Springer, Cham, 2018.
- [28] B. S. Mordukhovich and T. T. A. Nghia. Subdifferentials of nonconvex supremum functions and their applications to semi-infinite and infinite programs with Lipschitzian data. *SIAM J. Optim.*, 23(1):406–431, 2013.
- [29] B. S. Mordukhovich and T. T. A. Nghia. Nonsmooth cone-constrained optimization with applications to semi-infinite programming. *Math. Oper. Res.*, 39(2):301–324, 2014.
- [30] B. S. Mordukhovich and H. M. Phan. Tangential extremal principles for finite and infinite systems of sets, I: basic theory. *Math. Program.*, 136(1, Ser. B):3–30, 2012.
- [31] B. S. Mordukhovich and Y. H. Shao. Nonsmooth sequential analysis in Asplund spaces. *Trans. Amer. Math. Soc.*, 348(4):1235–1280, 1996.
- [32] B. S. Mordukhovich and B. Wang. Generalized differentiation of parameter-dependent sets and mappings. *Optimization*, 57(1):17–40, 2008.
- [33] J. J. Moreau. *Fonctionnelles convexes*. Number 2. Lecture notes Séminaire "Equations aux dérivées partielles", 1966.
- [34] H. V. Ngai and M. Théra. A fuzzy necessary optimality condition for non-Lipschitz optimization in Asplund spaces. *SIAM J. Optim.*, 12(3):656–668, 2002.
- [35] T. T. A. Nghia. A nondegenerate fuzzy optimality condition for constrained optimization problems without qualification conditions. *Nonlinear Anal.*, 75(18):6379–6390, 2012.
- [36] J.-P. Penot. *Calculus without derivatives*, volume 266 of *Graduate Texts in Mathematics*. Springer, New York, 2013.
- [37] P. Pérez-Aros. Formulae for the conjugate and the subdifferential of the supremum function. *Journal of Optimization Theory and Applications*, Jul 2018.
- [38] R. T. Rockafellar. *Convex analysis*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.
- [39] H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1999.
- [40] S. Simons. Maximinimax, minimax, and antiminimax theorems and a result of R. C. James. *Pacific J. Math.*, 40:709–718, 1972.
- [41] A. Stefănescu. A general min-max theorem. *Optimization*, 16(4):497–504, 1985.

- [42] L. Thibault. Sequential convex subdifferential calculus and sequential Lagrange multipliers. *SIAM J. Control Optim.*, 35(4):1434–1444, 1997.
- [43] L. Thibault. Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential. In *Constructive, experimental, and nonlinear analysis (Limoges, 1999)*, volume 27 of *CMS Conf. Proc.*, pages 279–289. Amer. Math. Soc., Providence, RI, 2000.
- [44] C. Zălinescu. *Convex analysis in general vector spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.