Short-term at-the-money asymptotics under stochastic volatility models

Omar El Euch
École Polytechnique

Masaaki Fukasawa
Graduate School of Engineering Science, Osaka University
1-3 Machikaneyama, Toyonaka, Osaka, JAPAN
fukasawa@sigmath.es.osaka-u.ac.jp

Jim Gatheral
Baruch College, The City University of New York

Mathieu Rosenbaum
École Polytechnique

22 March 2019

Abstract

A small-time Edgeworth expansion of the density of an asset price is given under a general stochastic volatility model, from which asymptotic expansions of put option prices and at-the-money implied volatilities follow. A limit theorem for at-the-money implied volatility skew and curvature is also given as a corollary. The rough Bergomi model is treated as an example.

1 Introduction

Stochastic volatility models are extensions of the Black-Scholes model that explain a number of empirical evidences. The Heston and SABR models among them are popular in financial practice owing to (semi-)analytic (approximation) formulas for the vanilla option prices or the option-implied volatilities. For a practical guide on stochastic volatility modeling, we refer to [12]. Recently, attracting much attention is a class of stochastic volatility models where the volatility is driven by a fractional Brownian motion with the Hurst parameter smaller than 1/2. This is due to their consistency to a power law for the term structure of the implied volatility skew that has been empirically recognized; see [1, 3, 6, 8, 10, 11, 13, 14]. The small Hurst parameter implies in particular that the volatility path is rougher than a Brownian motion and so, this class is often referred as the rough volatility models. Since the models do not admit of explicit expressions for option prices or implied volatilities, the above mentioned consistency has been discussed through asymptotic analyses.

The aim of this paper is to provide a general framework under which the short-term asymptotics of the at-the-money implied volatility is studied. Here, by short-term asymptotics, we mean the asymptotics with time-to-maturity $\theta \to 0$. By at-the-money, we mean a regime of log-moneyness $k = O(\sqrt{\theta})$. The framework is for a general continuous stochastic volatility model. The rough Bergomi model introduced by [3] is treated as an example. The asymptotic expansion of the at-the-money implied volatility is given up to the second-order.

The first-order expansion was already given in [10] by a different method. For the SABR model, Osajima [19] gave the second-order expansion based on the Watanabe-Yoshida theory; see e.g., [15, 22]. For a Markov stochastic volatility model with jumps, Medvedev and Scaillet [16, 17] derived the expansion by a formal computation. For the Markov diffusion case, Pagliarani and Pascucci [20] proved the validity of the Taylor expansion. An expansion of the at-the-money implied volatility skew is derived under a Lévy jump model with Markov stochastic volatility by Figueroa-López and Ólafsson [5]. Beside these results for the at-the-money regime, considering near-the-money, that is, a moderate deviation regime, Friz et al. [7] derived the asymptotic skew and curvature of the implied volatility by assuming the asymptotic behavior of the density function of the underlying asset price. Recently, Bayer et al. [4] has extended the moderate deviation analysis to a rough volatility model.

In this paper, we introduce a novel approach based on the conditional Gaussianity of a continuous stochastic volatility model to prove the validity of a second order density expansion, from which follow expansions of the option prices and the implied volatility as well as the asymptotic skew and curvature formula. In contrast to [15, 19, 22], we do not rely on the Malliavin calculus, which enables us to treat effectively the rough volatility models. In contrast to the elementary method of [10], our approach can be extended to higher-order expansions without any additional theoretical difficulty. We choose the square root of the forward variance, that is, the fair strike of a variance swap, as the leading term of our asymptotic expansion, while a recent work [2] studies the difference between the implied volatility and the fair strike of a volatility swap in terms of the Malliavin derivatives.

The paper is organized as follows. In Section 2, we describe assumptions and general results. In Section 3, we give the proofs of the general results. In Section 4, we treat regular stochastic volatility models. In Section 5, we show that the rough Bergomi model fits into the framework as well and compute the coefficients of the expansion for this particular model.

2 Framework

2.1 Assumptions

Let $(\Omega, \mathcal{F}, Q)$ be a probability space equipped with a filtration $\{\mathcal{F}_t; t \geq 0\}$ satisfying the usual assumptions. A log price process Z is assumed to follow

$$\mathrm{d}Z_t = r\mathrm{d}t - \frac{1}{2}v_t\mathrm{d}t + \sqrt{v_t}\mathrm{d}B_t,$$

where $r \in \mathbb{R}$ stands for an interest rate and v is a positive continuous process adapted to a smaller filtration $\{\mathcal{G}_t; t \geq 0\}$, of which the square root is called the volatility of Z. The Brownian motion B is decomposed as

$$\mathrm{d}B_t = \rho_t\mathrm{d}W_t + \sqrt{1-\rho_t^2}\mathrm{d}W'_t,$$

where W' is an $\{\mathcal{F}_t\}$-Brownian motion independent of $\mathcal{G}_t$ for all $t \geq 0$, W is a $\{\mathcal{G}_t\}$-Brownian motion and ρ is a progressively measurable processes with respect to $\{\mathcal{G}_t\}$ and taking values in $[-1, 1]$. A typical situation for stochastic volatility models, including the Heston, SABR and rough Bergomi models, is that (W, W') is a two dimensional $\{\mathcal{F}_t\}$-Brownian motion and $\{\mathcal{G}_t\}$ is the filtration generated by W, that is,

$$\mathcal{G}_t = \mathcal{N} \vee \sigma(W_s; s \leq t),$$

where $\mathcal{N}$ is the null sets of $\mathcal{F}$. Denote by $\|\cdot\|_p$ the L^p norm under Q. Our key assumption is the following: for any $p > 0$,

$$\sup_{\theta\in(0,1)} \left\| \frac{1}{\theta}\int_0^\theta v_t\mathrm{d}t \right\|_p < \infty, \quad \sup_{\theta\in(0,1)} \left\| \left\{ \frac{1}{\theta}\int_0^\theta v_t(1-\rho_t^2)\mathrm{d}t \right\}^{-1} \right\|_p < \infty. \tag{1}$$

This is satisfied by standard stochastic volatility models (with correlation parameter $|\rho| < 1$) but not by local volatility models that correspond to $\rho \equiv 1$.

An arbitrage-free price $p(K, \theta)$ of a put option at time 0 with strike $K > 0$ and maturity $\theta > 0$ is given by

$$p(K, \theta) = e^{-r\theta}E[(K - \exp(Z_\theta))_+] = e^{-r\theta}\int_0^K Q(\log x \geq Z_\theta)\mathrm{d}x.$$

The forward variance curve $v_0(t)$ at time 0 is defined by $v_0(t) = E[v_t]$. Changing variable as

$$x = F\exp\left(\zeta\sigma_0(\theta)\right), \quad F = \exp(r\theta + Z_0),$$

where

$$\sigma_0(\theta) = \sqrt{\int_0^\theta v_0(t)\mathrm{d}t},$$

we have

$$\frac{p(Fe^{z\sigma_0(\theta)},\theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z} Q\left(\zeta \geq X_\theta\right) e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta,$$

where

$$X_\theta = -\frac{1}{2\sigma_0(\theta)}\langle M\rangle_\theta + \frac{1}{\sigma_0(\theta)}M_\theta,\ \ M_\theta = \int_0^\theta \sqrt{v_t}\mathrm{d}B_t,\ \ \langle M\rangle_\theta = \int_0^\theta v_t\mathrm{d}t.$$

Based on this expression, the asymptotic behavior of put option prices is studied through the asymptotic distribution of X_θ. From the martingale central limit theorem[1], it is not difficult to see that X_θ converges in law to the standard normal distribution as $\theta \to 0$. To determine higher-order asymptotic distribution, we assume the following structure: there exists a family of random vectors

$$\left\{(M_\theta^{(0)}, M_\theta^{(1)}, M_\theta^{(2)}, M_\theta^{(3)}); \theta \in (0,1)\right\}$$

such that

1. the law of $M_\theta^{(0)}$ is standard normal for all $\theta > 0$,
2. $$\sup_{\theta\in(0,1)} \|M_\theta^{(i)}\|_p < \infty,\ \ i = 1,2,3 \tag{2}$$
   for all $p > 0$ and
3. for some $H \in (0, 1/2]$ and $\epsilon \in (0, H)$,
$$\begin{aligned} &\lim_{\theta\to 0}\theta^{-2H-2\epsilon}\left\|\frac{M_\theta}{\sigma_0(\theta)} - M_\theta^{(0)} - \theta^H M_\theta^{(1)} - \theta^{2H}M_\theta^{(2)}\right\|_{1+\epsilon} = 0,\\ &\lim_{\theta\to 0}\theta^{-H-2\epsilon}\left\|\frac{\langle M\rangle_\theta}{\sigma_0(\theta)^2} - 1 - \theta^H M_\theta^{(3)}\right\|_{1+\epsilon} = 0. \end{aligned} \tag{3}$$

Further, we assume the existence of the derivatives

$$\begin{aligned} a_\theta^{(i)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{E[M_\theta^{(i)}|M_\theta^{(0)} = x]\phi(x)\right\},\ \ i = 1,2,3,\\ b_\theta(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{E[M_\theta^{(1)}|M_\theta^{(0)} = x]\phi(x)\right\}\\ c_\theta(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{E[|M_\theta^{(1)}|^2|M_\theta^{(0)} = x]\phi(x)\right\} \end{aligned} \tag{4}$$

in the Schwartz space (i.e., the space of the rapidly decreasing smooth functions), where ϕ is the standard normal density.

[1] The martingale central limit theorem for one-dimensional continuous local martingales is proved as follows. Let M^n be a continuous local martingale with $\langle M^n\rangle_1 \to 1$ in probability. By the Dambis-Dubins-Schwarz theorem, $M^n = W^n_{\langle M^n\rangle}$ for a Brownian motion W^n. Since $(W^n, \langle M^n\rangle_1) \to (W, 1)$ in law, by the continuous mapping theorem, we conclude $M_1^n \to W_1$ in law.

As will be discussed in Section 4, regular stochastic volatility models satisfy these assumptions with $H = 1/2$, where (3) is a consequence of the Itô-Taylor expansion. In Section 5, we see that the rough Bergomi model, where the volatility is driven by a fractional Brownian motion, satisfies these assumptions with H being the Hurst parameter of the fractional Brownian motion.

2.2 General results

The fundamental result in this paper is the following.

Theorem 2.1 *The law of X_θ admits a density p_θ, and for any $\alpha \in \mathbb{N} \cup \{0\}$,*

$$\sup_{x\in\mathbb{R}} (1 + x^2)^\alpha |p_\theta(x) - q_\theta(x)| = o(\theta^{2H}) \tag{5}$$

as $\theta \to 0$, where

$$\begin{aligned} q_\theta(x) =& \phi\left(x + \frac{\sigma_0(\theta)}{2}\right) - \theta^H \left(a_\theta^{(1)}\left(x + \frac{\sigma_0(\theta)}{2}\right) - \frac{\sigma_0(\theta)}{2} a_\theta^{(3)}\left(x + \frac{\sigma_0(\theta)}{2}\right)\right) \\ &- \theta^{2H}\left(a_\theta^{(2)}(x) - \frac{1}{2}c_\theta(x)\right). \end{aligned} \tag{6}$$

The proof is given in Section 3.2. In order to derive a neat asymptotic expansion of the put option prices, we introduce an additional assumption which is satisfied by the models in Sections 4 and 5.

Theorem 2.2 *Suppose we have (5) with q_θ of the form*

$$\begin{aligned} q_\theta(x) =& \phi\left(x + \frac{\sigma_0(\theta)}{2}\right)\left\{1 + \kappa_3(\theta)\left(H_3\left(x + \frac{\sigma_0(\theta)}{2}\right) - \sigma_0(\theta) H_2\left(x + \frac{\sigma_0(\theta)}{2}\right)\right)\theta^H\right\} \\ &+ \phi(x)\left(\kappa_4(\theta) H_4(x) + \frac{\kappa_3(\theta)^2}{2} H_6(x)\right)\theta^{2H} \end{aligned} \tag{7}$$

with bounded functions $\kappa_3(\theta)$ and $\kappa_4(\theta)$ of θ, where H_k is the kth Hermite polynomial:

$$H_1(x) = x,\ \ H_2(x) = x^2 - 1,\ \ H_3(x) = x^3 - 3x,\ \ H_4(x) = x^4 - 6x^2 + 3, \dots$$

Then, for any $z_0 \in \mathbb{R}$,

$$\begin{aligned} \frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{Fe^{-r\theta}\sigma_0(\theta)} = & \frac{1}{\sigma_0(\theta)}\left(\Phi\left(z + \frac{\sigma_0(\theta)}{2}\right)e^{\sigma_0(\theta)z} - \Phi\left(z - \frac{\sigma_0(\theta)}{2}\right)\right) \\ &+ \kappa_3(\theta)\phi\left(z + \frac{\sigma_0(\theta)}{2}\right) H_1\left(z + \frac{\sigma_0(\theta)}{2}\right) e^{\sigma_0(\theta)z}\theta^H \\ &+ \phi(z)\left(\kappa_4(\theta) H_2(z) + \frac{\kappa_3(\theta)^2}{2} H_4(z)\right)\theta^{2H} + o(\theta^{2H}) \end{aligned}$$

uniformly in $z \le z_0$.

The proof is given in Section 3.3. Under the same assumption, an asymptotic expansion of the Black-Scholes implied volatility follows. Denote by $p_{\mathrm{BS}}(K, \theta, \sigma)$ the put option price with strike price K and maturity θ under the Black-Scholes model with volatility parameter $\sigma > 0$. Given a put option price $p(K, \theta)$, $K = Fe^k$, the Black-Scholes implied volatility $\sigma_{\mathrm{BS}}(k, \theta)$ is defined through

$$p_{\mathrm{BS}}(K, \theta, \sigma_{\mathrm{BS}}(k, \theta)) = p(K, \theta).$$

The at-the-money implied volatility skew and curvature are defined respectively as the first and the second derivatives in k of the Black-Scholes implied volatility at $k = 0$. The skew behavior is especially important in order to argue the consistency of a model to the empirically observed power law.

Theorem 2.3 *Suppose we have (5) with q_θ of the form (7). Then, for any $z \in \mathbb{R}$,*

$$\begin{aligned}
&\sigma_{\mathrm{BS}}(\sqrt{\theta} z, \theta) \\
&= \kappa_2 \left\{ 1 + \kappa_3 \left(\frac{z}{\kappa_2} + \frac{\kappa_2 \sqrt{\theta}}{2} \right) \theta^H + \left(\frac{3\kappa_3^2}{2} - \kappa_4 + (\kappa_4 - 3\kappa_3^2) \frac{z^2}{\kappa_2^2} \right) \theta^{2H} \right\} + o(\theta^{2H}),
\end{aligned}$$

where $\kappa_2 = \kappa_2(\theta) = \sigma_0(\theta)/\sqrt{\theta}$, $\kappa_3 = \kappa_3(\theta)$ and $\kappa_4 = \kappa_4(\theta)$.

Theorem 2.4 *Suppose we have (5) with q_θ of the form (7). Then,*

$$\begin{aligned}
\partial_k \sigma_{\mathrm{BS}}(0, \theta) &= \kappa_3(\theta) \theta^{H-1/2} + o(\theta^{2H-1/2}), \\
\partial_k^2 \sigma_{\mathrm{BS}}(0, \theta) &= 2 \frac{\kappa_4(\theta) - 3\kappa_3(\theta)^2}{\kappa_2(\theta)} \theta^{2H-1} + o(\theta^{2H-1}).
\end{aligned}$$

The proofs are given in Sections 3.4 and 3.5 respectively.

Remark: The above asymptotic estimates are not uniform in H; the assumed stochastic expansion (3) is not uniform in H and so, there seems no hope to have uniformity. We would have uniformity in $H \in [H_0, 1/2]$ for some $H_0 > 0$ if we could strengthen the condition (3) to uniform convergence on $[H_0, 1/2]$. It seems impossible to argue the uniformity in $H \in (0, 1/2]$ because Lemma 3.2 below requires some ϵ-δ argument depending on H.

3 Proofs

3.1 Characteristic function expansion

Here we give an asymptotic expansion of the characteristic function of X_θ. Let

$$Y_\theta = M_\theta^{(0)} + \theta^H M_\theta^{(1)} + \theta^{2H} M_\theta^{(2)} - \frac{\sigma_0(\theta)}{2} \left(1 + \theta^H M_\theta^{(3)} \right).$$

Lemma 3.1 *Let $H \in (0, 1/2]$ and $\epsilon \in (0, H)$ be constants under which (3) holds. Then, for any $\alpha \in \mathbb{N} \cup \{0\}$,*

$$\sup_{|u| \leq \theta^{-\epsilon}} |E[X_\theta^\alpha e^{iuX_\theta}] - E[Y_\theta^\alpha e^{iuY_\theta}]| = o(\theta^{2H+\epsilon}).$$

Proof: Since $|e^{ix} - 1| \leq |x|$, we have

$$\begin{aligned} |E[X_\theta^\alpha e^{iuX_\theta}] - E[Y_\theta^\alpha e^{iuY_\theta}]| &\leq |E[(X_\theta^\alpha - Y_\theta^\alpha) e^{iuX_\theta}]| + |E[Y_\theta^\alpha e^{iuY_\theta}(e^{iu(X_\theta - Y_\theta)} - 1)]| \\ &\leq E[|X_\theta^\alpha - Y_\theta^\alpha|] + uE[|Y_\theta|^\alpha |X_\theta - Y_\theta|] \end{aligned}$$

By (1) and (2) respectively, X_θ and Y_θ have moments of any order. Therefore by the Hölder inequality,

$$E[|Y_\theta|^\alpha |X_\theta - Y_\theta|] \leq C_1(\alpha, \epsilon) \|X_\theta - Y_\theta\|_{1+\epsilon}$$

for a constant $C_1(\alpha, \epsilon) > 0$. Since $X_\theta^\alpha - Y_\theta^\alpha = (X_\theta - Y_\theta) \sum_{\beta=0}^{\alpha-1} (-1)^\beta X_\theta^{\alpha-1-\beta} Y_\theta^\beta$, the Hölder inequality gives also

$$E[|X_\theta^\alpha - Y_\theta^\alpha|] \leq C_2(\alpha, \epsilon) \|X_\theta - Y_\theta\|_{1+\epsilon}$$

for a constant $C_2(\alpha, \epsilon) > 0$. Since $\sigma_0(\theta) = O(\theta^{1/2})$, we have $\|X_\theta - Y_\theta\|_{1+\epsilon} = o(\theta^{2H+2\epsilon})$ by (3), from which the result follows. ////

Lemma 3.2 *Let H and ϵ be as in Lemme 3.1. Then, for any $\delta \in [0, (H - \epsilon)/3)$,*

$$\sup_{|u| \leq \theta^{-\delta}} \left| E[Y_\theta^\alpha e^{iuY_\theta}] - E\left[e^{iuM_\theta^{(0)}} \left((M_\theta^{(0)})^\alpha + A(\alpha, u, M_\theta^{(0)}) + B(\alpha, u, M_\theta^{(0)}) \right) \right] \right| = o(\theta^{2H+\epsilon}),$$

where

$$\begin{aligned} A_\theta(\alpha, u, x) &= \left(iux^\alpha + \alpha x^{\alpha-1} \right) (E[Y_\theta | M_\theta^{(0)} = x] - x), \\ B_\theta(\alpha, u, x) &= \left(-\frac{u^2}{2} x^\alpha + iu\alpha x^{\alpha-1} + \frac{\alpha(\alpha-1)}{2} x^{\alpha-2} \right) \\ &\quad \times \left(\theta^{2H} E[|M_\theta^{(1)}|^2 | M_\theta^{(0)} = x] - \sigma_0(\theta) \theta^H E[M_\theta^{(1)} | M_\theta^{(0)} = x] + \frac{\sigma_0(\theta)^2}{4} \right). \end{aligned}$$

Proof: This follows from the fact that

$$\left| e^{ix} - 1 - ix + \frac{x^2}{2} \right| \leq \frac{|x|^3}{6}$$

for all $x \in \mathbb{R}$. Indeed, this implies that

$$\sup_{|u| \leq \theta^{-\delta}} \left| E[Y_\theta^\alpha e^{iuY_\theta}] - E\left[Y_\theta^\alpha e^{iuM_\theta^{(0)}} \left(1 + iu(Y_\theta - M_\theta^{(0)}) - \frac{u^2}{2} (Y_\theta - M_\theta^{(0)})^2 \right) \right] \right| = o(\theta^{2H+\epsilon}).$$

Expand $Y_\theta^\alpha = (M_\theta^{(0)})^\alpha + \alpha (M_\theta^{(0)})^{\alpha-1} (Y_\theta - M_\theta^{(0)}) + \ldots$ and take the conditional expectation given $M_\theta^{(0)}$ to obtain the result. ////

Lemma 3.3 *Define $\bar{q}_\theta(x)$ by*

$$\begin{aligned}\bar{q}_\theta(x) =& \phi(x) - \theta^H a_\theta^{(1)}(x) - \theta^{2H} a_\theta^{(2)}(x) - \frac{\sigma_0(\theta)}{2}(x\phi(x) - \theta^H a_\theta^{(3)}(x)) \\ &+ \frac{\theta^{2H}}{2} c_\theta(x) - \frac{\theta^H \sigma_0(\theta)}{2} b_\theta(x) + \frac{\sigma_0(\theta)^2}{8}(x^2-1)\phi(x),\end{aligned} \tag{8}$$

where $a_\theta^{(i)}$, b_θ and c_θ are defined by (4). Then,

$$\int_{\mathbb{R}} e^{iux} x^\alpha \bar{q}_\theta(x)\mathrm{d}x = E\left[e^{iuM_\theta^{(0)}}\left((M_\theta^{(0)})^\alpha + A(\alpha, u, M_\theta^{(0)}) + B(\alpha, u, M_\theta^{(0)})\right)\right].$$

Proof: Since the density of $M_\theta^{(0)}$ is ϕ by the assumption, this simply follows from integration by parts. ////

3.2 Density expansion

Here we derive an asymptotic expansion of the density of X_θ.

Lemma 3.4 *There exists a density of X_θ and for any $\alpha, j \in \mathbb{N} \cup \{0\}$,*

$$\sup_{\theta \in (0,1)} \int |u|^j |E[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u < \infty$$

Proof: Note that the distribution of X_θ is Gaussian conditionally on $\mathcal{G}_\theta$, with conditional mean

$$U_\theta := -\frac{1}{2\sigma_0(\theta)}\langle M\rangle_\theta + \frac{1}{\sigma_0(\theta)}\int_0^\theta \sqrt{v_t}\rho_t \mathrm{d}W_t$$

and conditional variance

$$V_\theta := \frac{1}{\sigma_0(\theta)^2}\int_0^\theta v_t(1-\rho_t^2)\mathrm{d}t.$$

Therefore, for any bounded continuous function f, we have

$$E[f(X_\theta)] = E[E[f(X_\theta)|\mathcal{G}_\theta]] = E[\int f(x)\phi(x, U_\theta, V_\theta)\mathrm{d}x],$$

where $\phi(\cdot, u, v)$ is the density of the normal distribution with mean u and variance v. This means that X_θ admits a density

$$p_\theta(x) = E[\phi(x, U_\theta, V_\theta)].$$

Furthermore, the density function is in the Schwartz space $\mathcal{S}$ and each Schwartz semi-norm is uniformly bounded in θ by (1). Therefore,

$$\begin{aligned}\sup_{\theta\in(0,1)} \int |u|^j |E[X_\theta^\alpha e^{iuX_\theta}]| \mathrm{d}u &= \sup_{\theta\in(0,1)} \int \left| \int u^j x^\alpha e^{iux} p_\theta(x)\mathrm{d}x \right| \mathrm{d}u \\ &= \sup_{\theta\in(0,1)} \int \left| \int e^{iux} \partial_x^j (x^\alpha p_\theta(x))\mathrm{d}x \right| \mathrm{d}u < \infty\end{aligned}$$

since the Fourier transform is a continuous linear mapping from $\mathcal{S}$ to $\mathcal{S}$. ////

Proof of Theorem 2.1: As seen in the proof of Lemma 3.4, the density p_θ exists in the Schwartz space. Note that for a function f in the Schwartz space, by Taylor's theorem,

$$\begin{aligned}\left|f(x+a) - f(x) - f'(x)a - f''(x)\frac{a^2}{2}\right| &\leq \frac{a^3}{2}\sup_{|b|\leq|a|}|f'''(x+b)| \\ &\leq \frac{a^3}{2}\sup_{|b|\leq|a|}\frac{1}{(1+(x+b)^2)^\alpha}\sup_{y\in\mathbb{R}}(1+y^2)^\alpha|f'''(y)|\end{aligned}$$

and so,

$$\sup_{x\in\mathbb{R}}(1+x^2)^\alpha\left|f(x+a) - f(x) - f'(x)a - f''(x)\frac{a^2}{2}\right| = O(a^3).$$

This gives

$$\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|q_\theta(x) - \bar{q}_\theta(x)| = O(\theta^{1+H}) = o(\theta^{2H}),$$

where $\bar{q}_\theta$ is given by (8). By the Fourier identity,

$$(1+x^2)^\alpha(p_\theta(x) - \bar{q}_\theta(x)) = \frac{1}{2\pi}\int\int e^{iuy}(1+y^2)^\alpha(p_\theta(y) - \bar{q}_\theta(y))\mathrm{d}y e^{-iux}\mathrm{d}u$$

Combining the lemmas in the previous section, taking $\delta \in (0, \min\{\epsilon, (H-\epsilon)/3\})$, we have

$$\int_{|u|\leq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha(p_\theta(y) - \bar{q}_\theta(y))\mathrm{d}y\right|\mathrm{d}u = o(\theta^{2H}).$$

On the other hand,

$$\begin{aligned}\int_{|u|\geq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha p_\theta(y)\mathrm{d}y\right|\mathrm{d}u &\leq \theta^{j\delta}\int_{|u|\geq\theta^{-\delta}}|u|^j|E[(1+X_\theta^2)^\alpha e^{iuX_\theta}]|\mathrm{d}u \\ &= O(\theta^{j\delta})\end{aligned}$$

for any $j \in \mathbb{N}$ by Lemma 3.4. The remainder

$$\int_{|u|\geq\theta^{-\delta}}\left|\int e^{iuy}(1+y^2)^\alpha \bar{q}_\theta(y)\mathrm{d}y\right|\mathrm{d}u$$

is handled in the same manner. ////

3.3 Put option price expansion

Here we consider put option prices. Denote by p_θ the density of X_θ as before and consider a normalized put option price

$$\frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta}p_\theta(x)\mathrm{d}x e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta.$$

Lemma 3.5 *Let $q_\theta(x)$, $\theta > 0$ be a family of functions on $\mathbb{R}$ (not necessarily the one given by (8)). If*

$$\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|p_\theta(x) - q_\theta(x)| = o(\theta^\beta)$$

for some $\alpha > 5/4$ and $\beta > 0$, then for any $z_0 \in \mathbb{R}$,

$$\frac{p(Fe^{\sigma_0(\theta)z}, \theta)}{F\sigma_0(\theta)} = e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta} q_\theta(x)\mathrm{d}x e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta + o(\theta^\beta)$$

uniformly in $z \le z_0$.

Proof: By the Cauchy-Schwarz inequality,

$$\begin{aligned}
& e^{-r\theta}\int_{-\infty}^{z}\int_{-\infty}^{\zeta}|p_\theta(x) - q_\theta(x)|\mathrm{d}z e^{\sigma_0(\theta)\zeta}\mathrm{d}\zeta \\
\le\ & e^{-r\theta}\int_{-\infty}^{z}\sqrt{\int_{-\infty}^{\zeta}\frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}}\sqrt{\int_{-\infty}^{\zeta}(1+x^2)^{2\alpha-1}|p_\theta(x) - q_\theta(x)|^2\mathrm{d}z e^{\sigma_0(\theta)\zeta}}\mathrm{d}\zeta \\
\le\ & \sqrt{\pi}e^{-r\theta+\sigma_0(\theta)z}\sup_{x\in\mathbb{R}}(1+x^2)^\alpha|p_\theta(x) - q_\theta(x)|\int_{-\infty}^{z}\sqrt{\int_{-\infty}^{\zeta}\frac{\mathrm{d}x}{(1+x^2)^{2\alpha-1}}}\mathrm{d}\zeta,
\end{aligned}$$

which is $o(\theta^\beta)$ if $\alpha > 5/4$. ////

Proof of Theorem 2.2: This is a direct consequence of the previous lemma. For example,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{e^{-\sigma_0(\theta)z}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{1}{\sigma_0(\theta)}\left(\Phi\left(z+\frac{\sigma_0(\theta)}{2}\right)e^{\sigma_0(\theta)z} - \Phi\left(z-\frac{\sigma_0(\theta)}{2}\right)\right)\right\}\right\} = \phi\left(z+\frac{\sigma_0(\theta)}{2}\right).$$

The derivative of $H_k(z)\phi(z)$ is $-H_{k+1}(z)\phi(z)$. Recall also $\sigma_0(\theta) = O(\sqrt{\theta})$. ////

3.4 Implied volatility expansion

Here we prove an expansion formula for the Black-Scholes implied volatility.

Proof of Theorem 2.3: Step 1). Fix $z \in \mathbb{R}$. Note that

$$P_\theta(\sigma) := \frac{p_{\mathrm{BS}}(Fe^{\sqrt{\theta}z}, \theta, \sigma)}{Fe^{-r\theta}\sqrt{\theta}} = \frac{1}{\sqrt{\theta}}\left(\Phi\left(\frac{z}{\sigma}+\frac{\sigma\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z} - \Phi\left(\frac{z}{\sigma}-\frac{\sigma\sqrt{\theta}}{2}\right)\right) \tag{9}$$

and that

$$P_\theta : [0,\infty] \to \left[\frac{(e^{\sqrt{\theta}z}-1)_+}{\sqrt{\theta}}, \frac{e^{\sqrt{\theta}z}}{\sqrt{\theta}}\right]$$

is a strictly increasing function. From (9) and Proposition 2.2, we have

$$
\begin{aligned}
\frac{p(Fe^{\sqrt{\theta}z},\theta)}{Fe^{-r\theta}\sqrt{\theta}} =& P_\theta(\kappa_2) + \kappa_2\kappa_3\phi\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)H_1\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z}\theta^H \\
&+\kappa_2\phi\left(\frac{z}{\kappa_2}\right)\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}+o(\theta^{2H}) \\
=& P_\theta(\kappa_2)+O(\theta^H).
\end{aligned}
$$

Therefore

$$
\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta) = P_\theta^{-1}(P_\theta(\kappa_2)+O(\theta^H)).
$$

By (1), κ_2 is bounded in θ, say, by $L>0$. The function P_θ converges as $\theta\to 0$ to

$$
P_0(\sigma) := z\Phi\left(\frac{z}{\sigma}\right)+\sigma\phi\left(\frac{z}{\sigma}\right)
$$

pointwise, and by Dini's theorem, this convergence is uniform on $[0,L]$. Since the limit function P_0 is strictly increasing, the inverse functions P_θ^{-1} converges to P_0^{-1}. Again by Dini's theorem, this convergence is uniform and in particular, P_θ^{-1} are equicontinuous. Thus we conclude $\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2\to 0$ as $\theta\to 0$. Then, write $\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)=\kappa_2+\beta(\theta)$ and substitute this to the equation $P_\theta(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)) = P_\theta(\kappa_2)+O(\theta^H)$. The Taylor expansion gives $\beta(\theta)=O(\theta^H)$.

Step 2). From (9) we have

$$
P_\theta(\sigma) = \sigma F_1\left(\frac{z}{\sigma}\right)+\frac{\sigma^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\sigma}\right)+\frac{\sigma^3\theta}{6}F_3\left(\frac{z}{\sigma}\right)+o(\theta),
$$

where

$$
F_1(x)=x\Phi(x)+\phi(x),\ \ F_2(x)=x^2\Phi(x)+x\phi(x),\ \ F_3(x)=x^3\Phi(x)+\left(x^2-\frac{1}{4}\right)\phi(x).
$$

Using that

$$
\partial_\sigma\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}=\phi\left(\frac{z}{\sigma}\right),
$$

we have

$$
\begin{aligned}
&\kappa_2F_1\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_2^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\kappa_2}\right)+\kappa_2\phi\left(\frac{z}{\kappa_2}\right)\kappa_3H_1\left(\frac{z}{\kappa_2}\right)e^{\sqrt{\theta}z}\theta^H \\
&=\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)F_1\left(\frac{z}{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)}\right)+\frac{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)}\right)+O(\theta^{2H}) \\
&=\kappa_2F_1\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_2^2\sqrt{\theta}}{2}F_2\left(\frac{z}{\kappa_2}\right)+\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+O(\theta^{2H}),
\end{aligned}
$$

from which we conclude $\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)=\kappa_2+\kappa_3 z e^{\sqrt{\theta}z}\theta^H+O(\theta^{2H})$.

Step 3). Using that

$$\partial_\sigma^2\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}=\frac{z^2}{\sigma^3}\phi\left(\frac{z}{\sigma}\right),\quad \partial_\sigma\left\{\sigma^2 F_2\left(\frac{z}{\sigma}\right)\right\}=z\phi\left(\frac{z}{\sigma}\right),$$

we obtain

$$\begin{aligned}
&\kappa_2\phi\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)\left(\kappa_3 H_1\left(\frac{z}{\kappa_2}+\frac{\kappa_2\sqrt{\theta}}{2}\right)e^{\sqrt{\theta}z}\theta^H+\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}\right)\\
&=\frac{p(Fe^{\sqrt{\theta}z},\theta)}{Fe^{-r\theta}\sqrt{\theta}}-P_\theta(\kappa_2)+o(\theta^{2H})\\
&=P_\theta(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,z))-P_\theta(\kappa_2)+o(\theta^{2H})\\
&=\partial_\sigma\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+\frac{1}{2}\partial_\sigma^2\left\{\sigma F_1\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)^2\\
&\quad+\frac{\sqrt{\theta}}{2}\partial_\sigma\left\{\sigma^2 F_2\left(\frac{z}{\sigma}\right)\right\}\Big|_{\sigma=\kappa_2}(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+o(\theta^{2H})\\
&=\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)+\frac{\sqrt{\theta}}{2}z\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)\\
&\quad+\frac{z^2}{2\kappa_2^3}\phi\left(\frac{z}{\kappa_2}\right)(\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2)^2+o(\theta^{2H})
\end{aligned}$$

from Theorem 2.2 and Step 2. The left hand side is further expanded as

$$\begin{aligned}
\kappa_2\phi\left(\frac{z}{\kappa_2}\right)\Bigg\{&\kappa_3 H_1\left(\frac{z}{\kappa_2}\right)e^{\sqrt{\theta}z}\theta^H-\kappa_3 H_2\left(\frac{z}{\kappa_2}\right)\frac{\kappa_2}{2}\theta^{H+1/2}\\
&+\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}\Bigg\}+o(\theta^{2H}).
\end{aligned}$$

Denote $\gamma(\theta)=\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta)-\kappa_2-\kappa_3 z e^{\sqrt{\theta}z}\theta^H$ and substitute this to obtain

$$\begin{aligned}
\gamma(\theta)=&-\kappa_3 H_2\left(\frac{z}{\kappa_2}\right)\frac{\kappa_2^2}{2}\theta^{H+1/2}+\kappa_2\left(\kappa_4 H_2\left(\frac{z}{\kappa_2}\right)+\frac{\kappa_3^2}{2}H_4\left(\frac{z}{\kappa_2}\right)\right)\theta^{2H}\\
&-\frac{\kappa_3}{2}z^2\theta^{H+1/2}-\frac{\kappa_3^2}{2\kappa_2^3}z^4\theta^{2H}+o(\theta^{2H})\\
=&\left(\frac{\kappa_2^2}{2}-z^2\right)\kappa_3\theta^{H+1/2}+\kappa_2\left((\kappa_4-3\kappa_3^2)\frac{z^2}{\kappa_2^2}+\frac{3}{2}\kappa_3^2-\kappa_4\right)\theta^{2H}+o(\theta^{2H}),
\end{aligned}$$

from which we conclude the result. ////

3.5 Asymptotics for at-the-money skew and curvature

Here we prove Theorem 2.4.

Proof of Theorem 2.4: It is known (see e.g., Fukasawa [9]) that

$$\begin{aligned}
\partial_k \sigma_{\mathrm{BS}}(k,\theta) &= \frac{Q(k \geq \sigma_0(\theta)X_\theta) - \Phi(f_2(k,\theta))}{\sqrt{\theta}\phi(f_2(k,\theta))}, \\
\partial_k^2 \sigma_{\mathrm{BS}}(k,\theta) &= \frac{p_\theta(k/\sigma_0(\theta))}{\sigma_0(\theta)\sqrt{\theta}\phi(f_2(k,\theta))} - \sigma_{\mathrm{BS}}(k,\theta)\partial_k f_1(k,\theta)\partial_k f_2(k,\theta),
\end{aligned} \tag{10}$$

where

$$f_1(k,\theta) = \frac{k}{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)} - \frac{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)}{2}, \quad f_2(k,\theta) = \frac{k}{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)} + \frac{\sqrt{\theta}\sigma_{\mathrm{BS}}(k,\theta)}{2}.$$

Since the condition of Theorem 2.2 is met, we have

$$Q(0 \geq X_\theta) = \Phi\left(\frac{\sigma_0(\theta)}{2}\right) + \kappa_3(\theta)\phi\left(\frac{\sigma_0(\theta)}{2}\right)\theta^H + o(\theta^{2H}).$$

On the other hand, by Theorem 2.3,

$$f_2(0,\theta) = \frac{\sqrt{\theta}}{2}\kappa_2(\theta) + O(\theta^{2H+1/2})$$

and so,

$$\begin{aligned}
\Phi(f_2(0,\theta)) &= \Phi\left(\frac{\sigma_0(\theta)}{2}\right) + O(\theta^{2H+1/2}), \\
\phi(f_2(0,\theta)) &= \phi(0) - \phi(0)\frac{\theta}{8}\kappa_2(\theta)^2 + O(\theta^{2H+1}).
\end{aligned}$$

Then, it follows from (10) that

$$\partial_k \sigma_{\mathrm{BS}}(0,\theta) = \kappa_3(\theta)\theta^{H-1/2} + o(\theta^{2H-1/2}). \tag{11}$$

Further, under the condition, we have

$$p_\theta(0) = \phi\left(\frac{\sigma_0(\theta)}{2}\right)\left\{1 - \frac{\kappa_3(\theta)}{2}\sigma_0(\theta)\theta^H + \left(3\kappa_4(\theta) - 15\frac{\kappa_3(\theta)^2}{2}\right)\theta^{2H}\right\} + o(\theta^{2H}).$$

On the other hand, by Theorem 2.3 and (11),

$$\begin{aligned}
&\sigma_{\mathrm{BS}}(0,\theta)\partial_k f_1(0,\theta)\partial_k f_2(0,\theta) \\
&\quad = \frac{1}{\sigma_{\mathrm{BS}}(0,\theta)\theta} + O(\theta^{2H}) \\
&\quad = \frac{1}{\kappa_2(\theta)\theta}\left(1 - \frac{1}{2}\kappa_2(\theta)\kappa_3(\theta)\theta^{H+1/2} - \left(\frac{3}{2}\kappa_3(\theta)^2 - \kappa_4(\theta)\right)\theta^{2H}\right) + o(\theta^{2H-1}).
\end{aligned}$$

Then, it follows from (10) that

$$\partial_k^2 \sigma_{\mathrm{BS}}(0,\theta) = \frac{2\kappa_4(\theta) - 6\kappa_3(\theta)^2}{\kappa_2(\theta)}\theta^{2H-1} + o(\theta^{2H-1}),$$

which completes the proof. ////

4 Regular stochastic volatility models

Here we briefly discuss that regular stochastic volatility models satisfy all the assumptions in Section 2.1 with $H = 1/2$. Consider the volatility process $v_t = v(X_t)$, where X is a Markov process satisfying a stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + c(X_t)\mathrm{d}W_t$$

and v is a smooth positive function defined on the state space of X. Let $\rho \in (-1, 1)$ be a constant and $\{\mathcal{G}_t\}$ be the augmented filtration generated by W. We assume (1), which is satisfied in the usual cases including the log-normal SABR and Heston models. Denote by L the generator of X. Put $f = \sqrt{v}$, $g = f'c$ and $h = v'c$. Then, by Itô's formula, we have

$$M_\theta = f(X_0)B_\theta + \int_0^\theta \int_0^t g(X_s)\mathrm{d}W_s\mathrm{d}B_t + \int_0^\theta \int_0^t Lf(X_s)\mathrm{d}s\mathrm{d}B_t,$$
$$\langle M\rangle_\theta = v(X_0)\theta + \int_0^\theta \int_0^t h(X_s)\mathrm{d}W_s\mathrm{d}t + \int_0^\theta \int_0^t Lv(X_s)\mathrm{d}s\mathrm{d}t.$$

Let $\bar{B}^\theta_t = \theta^{-1/2}B_{\theta t}$, $\bar{W}^\theta_t = \theta^{-1/2}W_{\theta t}$ and $X^\theta_t = X_{\theta t}$. Then

$$\frac{M_\theta}{\sqrt{\theta}} = f(X_0)\bar{B}^\theta_1 + \sqrt{\theta}\int_0^1 \int_0^u g(X^\theta_v)\mathrm{d}\bar{W}^\theta_v\mathrm{d}\bar{B}^\theta_u + \theta\int_0^1 \int_0^u Lf(X^\theta_v)\mathrm{d}v\mathrm{d}\bar{B}^\theta_u,$$
$$\frac{\langle M\rangle_\theta}{\theta} = v(X_0) + \sqrt{\theta}\int_0^1 \int_0^u h(X^\theta_v)\mathrm{d}\bar{W}^\theta_v\mathrm{d}u + \theta\int_0^1 \int_0^u Lv(X^\theta_v)\mathrm{d}v\mathrm{d}u.$$

It would follow that

$$\frac{\sigma_0(\theta)^2}{\theta} = \frac{E[\langle M\rangle_\theta]}{\theta} = v(X_0) + \frac{1}{2}Lv(X_0)\theta + O(\theta^{3/2}),$$

and so

$$\frac{\sigma_0(\theta)}{\sqrt{\theta}} = f(X_0) + \frac{1}{4}\frac{Lv(X_0)}{f(X_0)}\theta + O(\theta^{3/2})$$

under a mild regularity condition. Then, we have (3) with $H = 1/2$, $M^{(0)}_\theta = \bar{B}^\theta_1$ and

$$M^{(1)}_\theta = \frac{g(X_0)}{f(X_0)}\int_0^1 \bar{W}^\theta_u\mathrm{d}\bar{B}^\theta_u,$$
$$M^{(2)}_\theta = -\frac{Lv(X_0)}{4v(X_0)}\bar{B}^\theta_1 + \frac{g'(X_0)c(X_0)}{f(X_0)}\int_0^1 \int_0^u \bar{W}^\theta_v\mathrm{d}\bar{W}^\theta_v\mathrm{d}\bar{B}^\theta_u + \frac{Lf(X_0)}{f(X_0)}\int_0^1 u\mathrm{d}\bar{B}^\theta_u,$$
$$M^{(3)}_\theta = 2\frac{g(X_0)}{f(X_0)}\int_0^1 \bar{W}^\theta_u\mathrm{d}u$$

again under a mild regularity condition. By Nualart et al. [18] or Appendix A below,

$$E[M^{(1)}_\theta | M^{(0)}_\theta = x] = \frac{g(X_0)}{f(X_0)}\frac{\rho}{2}H_2(x), \quad E[M^{(3)}_\theta | M^{(0)}_\theta = x] = \frac{g(X_0)}{f(X_0)}\rho H_1(x)$$

and so,

$$
\begin{aligned}
&a_\theta^{(1)}\left(x+\frac{\sigma_0(\theta)}{2}\right)-\frac{\sigma_0(\theta)}{2}a_\theta^{(3)}\left(x+\frac{\sigma_0(\theta)}{2}\right)\\
&\quad=-\kappa_3\left(H_3\left(x+\frac{\sigma_0(\theta)}{2}\right)-\sigma_0(\theta)H_2\left(x+\frac{\sigma_0(\theta)}{2}\right)\right)\phi\left(x+\frac{\sigma_0(\theta)}{2}\right)
\end{aligned}
$$

with

$$
\kappa_3=\frac{\rho}{2}\frac{g(X_0)}{f(X_0)}.
$$

Further,

$$
\begin{aligned}
E[M_\theta^{(2)}|M_\theta^{(0)}=x]&=-\frac{Lv}{4v}(X_0)x+\frac{g'c}{f}(X_0)\frac{\rho^2}{6}H_3(x)+\frac{Lf}{2f}(X_0)x\\
&=-\frac{g^2}{4f^2}(X_0)x+\frac{g'c}{f}(X_0)\frac{\rho^2}{6}H_3(x)
\end{aligned}
$$

and

$$
E[(M_\theta^{(1)})^2|M_\theta^{(0)}=x]=\frac{g^2}{f^2}(X_0)\left(\rho^2\left(\frac{1}{2}+H_2(x)+\frac{1}{4}H_4(x)\right)+(1-\rho^2)\left(\frac{1}{2}+\frac{1}{3}H_2(x)\right)\right).
$$

Therefore,

$$
a_\theta^{(2)}(x)-\frac{1}{2}c_\theta(x)=-\kappa_4H_4(x)\phi(x)-\frac{\kappa_3^2}{2}H_6(x)\phi(x)
$$

with

$$
\kappa_4=\frac{g'c}{f}(X_0)\frac{\rho^2}{6}+\frac{g^2}{f^2}(X_0)\frac{1+2\rho^2}{6}.
$$

Thus we have observed that (6) has the form of (7) and so, all the theorems in Section 2.2 are applied. In particular, Theorem 2.3 proves the Medvedev-Scaillet formula (Proposition 1 of [16] that was obtained by a formal computation when f is the identity function).

5 The rough Bergomi model

Here we show that the rough Bergomi model proposed by [3] fits into the framework and compute the expansion terms. Let $\rho_t=\rho\in(-1,1)$ be a constant and

$$
v_t=v_0(t)\exp\left\{\eta\sqrt{2H}\int_0^t(t-s)^{H-1/2}\mathrm{d}W_s-\frac{\eta^2}{2}t^{2H}\right\}.
$$

The deterministic function $v_0(t)=E[v_t]$ is assumed to be continuous.

Theorem 5.1 *We have (5) for q_θ given by (7) with*

$$\kappa_3(\theta) = \rho\eta\sqrt{\frac{H}{2}}\frac{1}{\theta^H\sigma_0(\theta)^3}\int_0^\theta\int_0^t (t-s)^{H-1/2}\sqrt{v_0(s)}\mathrm{d}s v_0(t)\mathrm{d}t,$$
$$\kappa_4(\theta) = \frac{(1+2\rho^2)\eta^2 H}{(2H+1)^2(2H+2)} + \frac{\rho^2\eta^2 H\beta(H+3/2, H+3/2)}{2(H+1/2)^2},$$

where β is the beta function.

Proof: Since v_t is log-normally distributed, (1) holds by Jensen's inequality. The conditions (2) and (3) follow from Lemma 5.1 below. The functions $a_\theta^{(i)}$ and c_θ are computed in Lemmas 5.2, 5.3, 5.4 and 5.5 below. The function b_θ is obtained as the the derivative of $a_\theta^{(1)}$. They are apparently rapidly decreasing smooth functions. Then, by Theorem 2.1, it suffices to show that q_θ defined by (8) has the form (7) up to $o(\theta^{2H})$ with $\kappa_3(\theta)$ and $\kappa_4(\theta)$ specified above. ////

Theorems 2.2, 2.3 and 2.4 are therefore valid here as well. When $H < 1/2$ and the forward variance curve is flat (i.e., v_0 is constant), Theorem 2.3 gives a similar formula to the Bergomi-Guyon expansion formally derived in [3][2]. In fact, the expansion of $O(\eta)$ given in [3] coincides with our expansion of $O(\theta^H)$ when v_0 is constant. When v_0 is not constant, or when looking at the second-order terms, the formulas are not the same; this is not surprising because the asymptotics are $\eta \to 0$ in [3] while $\theta \to 0$ here. Further, when v_0 is constant, the same formula of $O(\theta^H)$ can be obtained by expanding the rate function of the large deviation result of [6] as in [4]. To be more precise, note that by Theorem 5.1,

$$\kappa_3(\theta) = \rho\frac{\eta\sqrt{2H}}{2(H+1/2)(H+3/2)}$$

when v_0 is constant and Theorem 2.3 implies

$$\frac{\sigma_{\mathrm{BS}}(\sqrt{\theta}z,\theta) - \sigma_{\mathrm{BS}}(\sqrt{\theta}\zeta,\theta)}{\sqrt{\theta}z - \sqrt{\theta}\zeta} = \kappa_3(\theta)\theta^{H-1/2} + O(\theta^{2H-1/2})$$

for $z \neq \zeta$. A weaker assertion, where $O(\theta^{2H-1/2})$ is replaced with $o(\theta^{H-1/2})$, was already shown in [10] by a different method. What is shown in [4] via an expansion of the rate function is that this formula up to $o(\theta^{H-1/2})$ remains valid even if $\sqrt{\theta}z$ and $\sqrt{\theta}\zeta$ are replaced with $\theta^\beta z$ and $\theta^\beta\zeta$ respectively for $\beta \in (1/2-H, 1/2)$.

How small θ has to be for reasonable accuracy of our asymptotic formulas should be examined via numerical experiments. Our extensive experiments suggest $\eta\theta^H < 1$ as a rough criterion[3]. Here we present only a few examples of the volatility surfaces. In Figures 1 and 2, the points are by the Monte Carlo

[2] Note however that there is a typo in the second order term in [3].

[3] Note that $\eta\theta^H$ is the standard deviation of log-spot-variance.

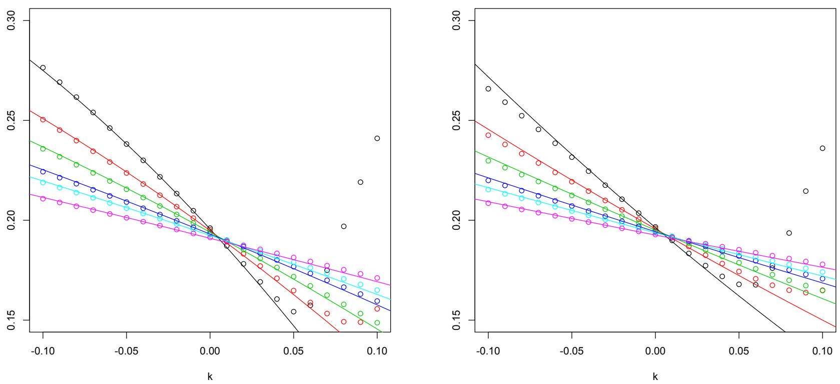

Figure 1: The Black-Scholes implied volatility under the rough Bergomi model with $v_0 \equiv .04$ and $(H, \rho, \eta) = (.07, -.9, .9)$ (left) or $(H, \rho, \eta) = (.07, -.7, .9)$ (right).

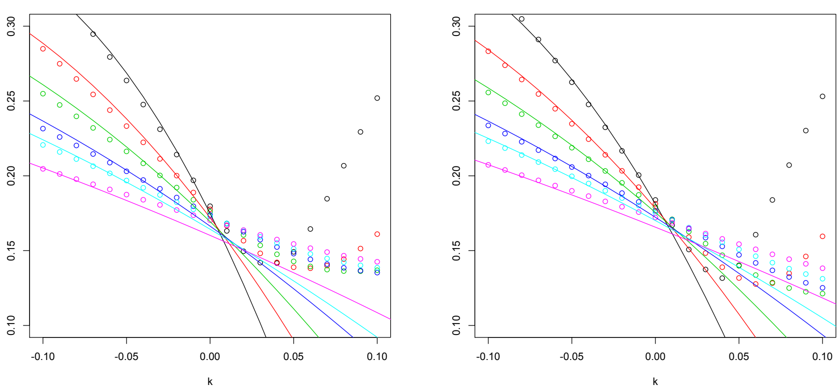

Figure 2: The Black-Scholes implied volatility under the rough Bergomi model with $v_0 \equiv .04$ and $(H, \rho, \eta) = (.05, -.9, 2.3)$ (left) or $(H, \rho, \eta) = (.07, -.9, 1.9)$ (right).

and the curves are by the asymptotic formula given in Theorems 2.3 and 5.1. The different colors are for different time-to-maturities; black for $\theta = .02$, red for $\theta = .05$, green for $\theta = .1$, blue for $\theta = .2$, cyan for $\theta = .3$ and magenta for $\theta = .6$. Note that the sets of parameters in Figure 2 are those calibrated from option data by Bayer et al [3].

In order to prove Lemmas below, we need some preparation. Let H_k, $k = 0, 1, \dots$ be the Hermite polynomials as before:

$$H_k(x) = (-1)^k e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2/2}$$

and $H_k(x, a) = a^{k/2} H_k(x/\sqrt{a})$ for $a > 0$. As is well-known, we have

$$\exp\left\{ux - \frac{au^2}{2}\right\} = \sum_{k=0}^{\infty} H_k(x, a) \frac{u^k}{k!}$$

and for any continuous local martingale M and $n \in \mathbb{N}$,

$$\mathrm{d}L_t^{(n)} = nL_t^{(n-1)} \mathrm{d}M_t, \tag{12}$$

where $L^{(k)} = H_k(M, \langle M \rangle)$ for $k \in \mathbb{N}$. See, e.g., Revuz and Yor [21].

Define $\hat{W}$, $\hat{W}'$, $\hat{B}$ by

$$\hat{W}_t = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)} \mathrm{d}W_s, \quad \hat{W}_t' = \frac{1}{\sigma_0(\theta)} \int_0^{\tau^{-1}(t)} \sqrt{v_0(s)} \mathrm{d}W_s'$$

and $\hat{B} = \rho \hat{W} + \sqrt{1 - \rho^2} \hat{W}'$, where

$$\tau(s) = \frac{1}{\sigma_0(\theta)^2} \int_0^s v_0(t) \mathrm{d}t.$$

Then, $(\hat{W}, \hat{W}')$ is a 2-dimensional Brownian motion under E and for any square-integrable function f,

$$\int_0^a f(s) \mathrm{d}W_s = \sigma_0(\theta) \int_0^{\tau(a)} \frac{f(\tau^{-1}(t))}{\sqrt{v_0(\tau^{-1}(t))}} \mathrm{d}\hat{W}_t.$$

Therefore,

$$M_\theta = \sigma_0(\theta) \int_0^1 \exp\left\{\theta^H F_t^t - \frac{\eta^2}{4} |\tau^{-1}(t)|^{2H}\right\} \mathrm{d}\hat{B}_t$$

where

$$F_u^t = \eta \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^u \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} \mathrm{d}\hat{W}_s, \quad u \in [0, t].$$

Let

$$G_t^{(k)} = H_k(F_t^t, \langle F^t \rangle_t).$$

Then, we have

$$\begin{aligned}
M_\theta &= \sigma_0(\theta)\int_0^1 \exp\left\{-\frac{\eta^2}{8}|\tau^{-1}(t)|^{2H}\right\}\exp\left\{\theta^H F_t^t - \frac{\theta^{2H}}{2}\langle F^t\rangle_t\right\}\mathrm{d}\hat{B}_t \\
&= \sigma_0(\theta)\int_0^1 \exp\left\{-\frac{\eta^2}{8}|\tau^{-1}(t)|^{2H}\right\}\sum_{k=0}^\infty G_t^{(k)}\frac{\theta^{Hk}}{k!}\mathrm{d}\hat{B}_t.
\end{aligned}$$

Lemma 5.1 *We have (3) with*

$$\begin{aligned}
M_\theta^{(0)} &= \hat{B}_1, \\
M_\theta^{(1)} &= \int_0^1 h_\theta(t)G_t^{(1)}\mathrm{d}\hat{B}_t, \\
M_\theta^{(2)} &= \int_0^1 \left\{\frac{h_\theta(t)-1}{\theta^{2H}} + h_\theta(t)\frac{G_t^{(2)}}{2}\right\}\mathrm{d}\hat{B}_t, \\
M_\theta^{(3)} &= 2\int_0^1 F_t^t\mathrm{d}t,
\end{aligned}$$

where

$$h_\theta(t) = \exp\left\{-\frac{\eta^2}{8}|\tau^{-1}(t)|^{2H}\right\}.$$

Proof: For $M_\theta^{(i)}$, $i = 0, 1, 2$, it suffices to show

$$\left\|\int_0^1 h_\theta(t)\sum_{k=J}^\infty G_t^{(k)}\frac{\theta^{Hk}}{k!}\mathrm{d}\hat{B}_t\right\|_2 = O(\theta^{HJ})$$

for any $J \geq 3$. The proof for $M_\theta^{(3)}$ is similar and so omitted. It suffices to show

$$E\left[\int_0^1 \left|\sum_{k=J}^\infty G_t^{(k)}\frac{\theta^{Hk}}{k!}\right|^2\mathrm{d}t\right] = O(\theta^{2HJ}).$$

By the Cauchy-Schwarz inequality, the left hand side is dominated by

$$\sum_{k=J}^\infty \theta^{Hk}\sum_{k=J}^\infty \frac{\theta^{Hk}}{(k!)^2}\int_0^1 E[|G_t^{(k)}|^2]\mathrm{d}t$$

Let

$$G_{t,s}^{(k)} = H_k(F_s^t, \langle F^t\rangle_s), \quad s \in [0,t].$$

Then, by (12),

$$
\begin{aligned}
E[|G_t^{(k)}|^2] &= E[|G_{t,t}^{(k)}|^2] \\
&= k^2 \int_0^t E[|G_{t,s}^{(k-1)}|^2] \mathrm{d}\langle F^t \rangle_s \\
&= k^2 (k-1)^2 \int_0^t \int_0^{s_1} E[|G_{t,s_2}^{(k-2)}|^2] \mathrm{d}\langle F^t \rangle_{s_2} \mathrm{d}\langle F^t \rangle_{s_1} \\
&\le (k!)^2 \langle F^t \rangle_t^k = (k!)^2 \left(\frac{\eta^2}{4} \frac{|\tau^{-1}(t)|^{2H}}{\theta^{2H}} \right)^k .
\end{aligned}
$$

Note that $\tau^{-1}(t) \le \tau^{-1}(1) = \theta$. Therefore, for sufficiently small θ,

$$
\sum_{k=J}^{\infty} \theta^{Hk} \sum_{k=J}^{\infty} \frac{\theta^{Hk}}{(k!)^2} \int_0^1 E[|G_t^{(k)}|^2] \mathrm{d}t \le \left(\frac{\eta^2}{4} \right)^J \frac{\theta^{2HJ}}{(1-\theta^H)(1-\theta^H \eta^2/4)},
$$

which completes the proof. ////

Now we compute $a_\theta^{(i)}$, b_θ and c_θ based on Lemma 5.1. The following lemmas follow from the results in Section A by straightforward computations.

Lemma 5.2

$$
\begin{aligned}
a_\theta^{(1)}(x) &= -H_3(x)\phi(x)\rho\eta \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^1 h_\theta(t) \int_0^t \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} \mathrm{d}s\mathrm{d}t \\
&= -H_3(x)\phi(x)\rho\eta \sqrt{\frac{H}{2}} \\
&\quad \times \frac{1}{\theta^H \sigma_0(\theta)^3} \int_0^\theta \exp\left\{ -\frac{\eta^2}{8} t^{2H} \right\} \int_0^t (t-s)^{H-1/2} \sqrt{v_0(s)} \mathrm{d}s v_0(t) \mathrm{d}t \\
&\sim -H_3(x)\phi(x) \frac{\rho\eta\sqrt{2H}}{2(H+1/2)(H+3/2)}.
\end{aligned}
$$

Lemma 5.3

$$
\begin{aligned}
a_\theta^{(2)}(x) &= -H_2(x)\phi(x) \int_0^1 \frac{h_\theta(t) - 1}{\theta^{2H}} \mathrm{d}t \\
&- H_4(x)\phi(x)\rho^2 \frac{\eta^2 H}{4} \frac{\sigma_0(\theta)^2}{\theta^{2H}} \int_0^1 h_\theta(t) \left(\int_0^t \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} \mathrm{d}s \right)^2 \mathrm{d}t \\
&\sim H_2(x)\phi(x) \int_0^1 \frac{\eta^2}{8\theta^{2H}} |\tau^{-1}(t)|^{2H} \mathrm{d}t - H_4(x)\phi(x)\rho^2 \frac{\eta^2 H}{(2H+1)^2(2H+2)}.
\end{aligned}
$$

Lemma 5.4

$$
\begin{aligned}
a_\theta^{(3)}(x) &= -2H_2(x)\phi(x)\rho\eta \sqrt{\frac{H}{2}} \frac{\sigma_0(\theta)}{\theta^H} \int_0^1 \int_0^t \frac{(\tau^{-1}(t) - \tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}} \mathrm{d}s\mathrm{d}t \\
&\sim -2H_2(x)\phi(x) \frac{\rho\eta\sqrt{2H}}{2(H+1/2)(H+3/2)}.
\end{aligned}
$$

Lemma 5.5

$$
\begin{aligned}
c_\theta(x) &= H_6(x)\phi(x)\rho^2\frac{\eta^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\left(\int_0^1 h_\theta(t)\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}}\mathrm{d}s\mathrm{d}t\right)^2\\
&\quad + H_4(x)\phi(x)\rho^2\frac{\eta^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\left(\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}}\mathrm{d}s\right)^2\mathrm{d}t\\
&\quad + H_4(x)\phi(x)\rho^2\eta^2 H\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}}\mathrm{d}s\\
&\qquad\qquad \times\int_t^1 h_\theta(u)\frac{(\tau^{-1}(u)-\tau^{-1}(t))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(t))}}\mathrm{d}u\mathrm{d}t\\
&\quad + H_4(x)\phi(x)\frac{\eta^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\left(\int_s^1 \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{H-1/2}}{\sqrt{v_0(\tau^{-1}(s))}}\mathrm{d}t\right)^2\mathrm{d}s\\
&\quad + H_2(x)\phi(x)\frac{\eta^2 H}{2}\frac{\sigma_0(\theta)^2}{\theta^{2H}}\int_0^1 h_\theta(t)^2\int_0^t \frac{(\tau^{-1}(t)-\tau^{-1}(s))^{2H-1}}{v_0(\tau^{-1}(s))}\mathrm{d}s\mathrm{d}t\\
&\sim H_6(x)\phi(x)\rho^2\frac{\eta^2 H}{2(H+1/2)^2(H+3/2)^2} + H_4(x)\phi(x)\frac{2(1+\rho^2)\eta^2 H}{(2H+1)^2(2H+2)}\\
&\quad + H_4(x)\phi(x)\frac{\rho^2\eta^2 H\beta(H+3/2,H+3/2)}{(H+1/2)^2}\\
&\quad + H_2(x)\phi(x)\int_0^1 \frac{\eta^2}{4\theta^{2H}}|\tau^{-1}(t)|^{2H}\mathrm{d}t.
\end{aligned}
$$

A Conditional expectations of Wiener-Itô integrals

Here we collect results on the conditional expectations of Wiener-Itô integrals that follow from Proposition 3 of Nualart et al [18]. Let $x \in \mathbb{R}$ and B be a standard Brownian motion ($B_0 = 0$). Let f be a continuous function on

$$\left\{(s,t)\in(0,1)^2; s<t\right\}$$

with

$$\int_0^1\int_0^t |f(s,t)|^2\mathrm{d}s\mathrm{d}t < \infty.$$

Lemma A.1

$$E\left[\int_0^1\int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}t\,\Big|\,B_1=x\right]=H_1(x)\int_0^1\int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,$$

$$E\left[\int_0^1\int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}B_t\,\Big|\,B_1=x\right]=H_2(x)\int_0^1\int_0^t f(s,t)\mathrm{d}s\mathrm{d}t,$$

$$\begin{aligned}E\left[\int_0^1\left(\int_0^t f(s,t)\mathrm{d}B_s\right)^2\mathrm{d}B_t\,\Big|\,B_1=x\right]&=H_3(x)\int_0^1\left(\int_0^t f(s,t)\mathrm{d}s\right)^2\mathrm{d}t\\&\quad+H_1(x)\int_0^1\int_0^t f(s,t)^2\mathrm{d}s\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}E\left[\int_0^1\left(\int_s^1 f(s,t)\mathrm{d}B_t\right)^2\mathrm{d}s\,\Big|\,B_1=x\right]&=H_2(x)\int_0^1\left(\int_s^1 f(s,t)\mathrm{d}t\right)^2\mathrm{d}s\\&\quad+\int_0^1\int_s^1 f(s,t)^2\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Lemma A.2

$$\begin{aligned}&E\left[\left(\int_0^1\int_0^t f(s,t)\mathrm{d}B_s\mathrm{d}B_t\right)^2\,\Big|\,B_1=x\right]-\int_0^1\int_0^t f(s,t)^2\mathrm{d}s\mathrm{d}t\\&=H_4(x)\left(\int_0^1\int_0^t f(s,t)\mathrm{d}s\mathrm{d}t\right)^2+H_2(x)\int_0^1\left(\int_0^t f(s,t)\mathrm{d}s+\int_t^1 f(t,u)\mathrm{d}u\right)^2\mathrm{d}t.\end{aligned}$$

References

[1] Alòs, E., León, J.A. and Vives, J., On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility, *Finance Stoch.* 11, (2007), 571-589.

[2] Alòs, E. and Shiraya, K.,Estimating the Hurst parameter from short term volatility swaps, to appear in *Finance Stoch.*.

[3] Bayer, C., Friz, P.K. and Gatheral, J., Pricing under rough volatility, *Quant. Finance* 16:6, (2016), 887-904.

[4] Bayer, C. Friz, P.K., Gulisashvili, A., Horvath, B. and Stemper B., Short-time near-the-money skew in rough fractional volatility models, to appear in *Quant. Finance*.

[5] Figueroa-López, J.E. and Ólafsson, S., Short-term asymptotics for the implied volatility skew under a stochastic volatility model with Lévy jumps, *Finance Stoch.* 20 (2016), 973-1020.

[6] Forde, M. and Zhang, H., Asymptotics for rough stochastic volatility models, *SIAM J. Finan. Math.*, (2017), 8(1), 114-145.

[7] Friz, P., Gerhold, S and Pinter, A., Option pricing in the moderate deviations regime, *Math. Finance*, (2018), 28(3), 962-988.

[8] Fukasawa, M., Asymptotic analysis for stochastic volatility: Martingale expansion, *Finance Stoch.* 15, (2011), 635-654.

[9] Fukasawa, M. The normalizing transformation of the implied volatility smile, *Math. Finance* 22, (2012) 753-762.

[10] Fukasawa, M., Short-time at-the-money skew and rough fractional volatility, *Quant. Finance.* 17:2, (2017), 189-198.

[11] Garnier, J. and Solna, K., Correction to Black-Scholes formula due to fractional stochastic volatility, *SIAM J. Finan. Math.*, (2017), 8(1) 560-588.

[12] Gatheral, J., *The Volatility Surface: A Practioner's Guide*. (2006) (John Wiley & Sons Inc: Hoboken, NJ).

[13] Gatheral, J., Jaisson, T. and Rosenbaum, M., Volatility is rough, *Quant. Finance.* 18:6, (2018), 933-949.

[14] Guennoun, H., Jacquier, A. and Roome, P., Asymptotic behaviour of the fractional Heston model, *SIAM J. Finan. Math.*, (2018), 9(3), 1017-1045.

[15] Kunitomo, N. and Takahashi, A., On validity of the asymptotic expansion approach in contingent claim analysis, *Ann. Appl. Probab.*, (2003), 13(3), 914-952.

[16] Medvedev, A. and Scaillet, O., A Simple Calibration Procedure of Stochastic Volatility Models with Jumps by Short Term Asymptotics, Research Paper No. 93, September 2003, FAME - International Center for Financial Asset Management and Engineering.

[17] Medvedev, A. and Scaillet, O., Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility, *The Review of Financial Studies* , (2006), 20(2), 427-459.

[18] Nualart, D., Üstünel, A. S. and Zakai, M., On the moments of a multiple Wiener-Itô integral and the space induced by the polynomials of the integral. *Stochastics* 25 (1988), no. 4, 233-240.

[19] Osajima, Y., The asymptotic expansion formula of implied volatility for dynamic SABR model and FX hybrid model, 2006, UTMS 2006-29, The University of Tokyo.

[20] Pagliarani, S. and Pascucci, A., The exact Taylor formula of the implied volatility, *Finance Stoch.* 21 (2017), 661-718.

[21] Revuz, D. and Yor, M., *Continuous Martingales and Brownian Motion*, (1999), Springer-Verlag Berlin Heidelberg.

[22] Yoshida, N., Asymptotic expansion for statistics related to small diffusions, *J. Jpn. Stat. Soc.*, (1992) 22, 139-159.

[23] Yoshida, N., Malliavin calculus and martingale expansion, *Bull. Sci. math.* 125, 6-7 (2001) 431-456.