# On the size-Ramsey number of tight paths 

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#### Abstract

For any $r \geq 2$ and $k \geq 3$, the $r$-color size-Ramsey number $\hat{R}(\mathcal{G}, r)$ of a $k$-uniform hypergraph $\mathcal{G}$ is the smallest integer $m$ such that there exists a $k$-uniform hypergraph $\mathcal{H}$ on $m$ edges such that any coloring of the edges of $\mathcal{H}$ with $r$ colors yields a monochromatic copy of $\mathcal{G}$. Let $\mathcal{P}_{n, k-1}^{(k)}$ denote the $k$-uniform tight path on $n$ vertices. Dudek, Fleur, Mubayi and Rődl showed that the size-Ramsey number of tight paths $\hat{R}\left(\mathcal{P}_{n, k-1}^{(k)}, 2\right)=O\left(n^{k-1-\alpha}(\log n)^{1+\alpha}\right)$ where $\alpha=\frac{k-2}{\binom{k-1}{2}+1}$. In this paper, we improve their bound by showing that $\hat{R}\left(\mathcal{P}_{n, k-1}^{(k)}, r\right)=O\left(r^{k}(n \log n)^{k / 2}\right)$ for all $k \geq 3$ and $r \geq 2$.


## 1 Introduction

Given two simple graphs $G$ and $H$ and a positive integer $r$, say that $H \rightarrow(G)_{r}$ if every $r$-edge-coloring of $H$ results in a monochromatic copy of $G$ in $H$. In this notation, the Ramsey number $R(G)$ of G is the minimum $n$ such that $K_{n} \rightarrow(G)_{2}$. The size-Ramsey number $\hat{R}(G, r)$ of $G$ is defined as the minimum number of edges in a graph $H$ such that $H \rightarrow(G)_{r}$, i.e.

$$
\hat{R}(G, r)=\min \left\{|E(H)|: H \rightarrow(G)_{r}\right\}
$$

When $r=2$, we ignore $r$ and simply use $\hat{R}(G)$.
Size-Ramsey number was first studied by Erdős, Faudree, Rousseau and Schelp [8] in 1978. By the definition of $R(G)$, we have

$$
\hat{R}(G) \leq\binom{ R(G)}{2}
$$

Chvátal (see, e.g. []) showed that this bound is tight for complete graphs, i.e. $\hat{R}\left(K_{n}\right)=\binom{R\left(K_{n}\right)}{2}$. Answering a question of Erdős [9], Beck [3] showed by a probabilistic construction that

$$
\hat{R}\left(P_{n}\right)=O(n)
$$

[^0]Alon and Chung [1] gave an explicit construction of a graph $G$ with $O(n)$ edges such that $G \rightarrow P_{n}$. Recently, Dudek and Prałat [6] provided a simple alternative proof for this result (See also [10]). The best upper bound $\hat{R}\left(P_{n}\right) \leq 74 n$ is due to Dudek and Prałat [7] by considering a random 27 -regular graph of a proper order.

Dudek, Fleur, Mubayi, and Rődl 11 first initiated the study of size-Ramsey number in hypergraphs. A $k$-uniform hypergraph $\mathcal{G}$ on a vertex set $V(\mathcal{G})$ is a family of $k$-element subsets (called edges) of $V(\mathcal{G})$. We use $E(\mathcal{G})$ to denote the edge set. Given $k$-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$, we say that $\mathcal{H} \rightarrow(\mathcal{G})_{r}$ if every $r$-edge-coloring of $\mathcal{H}$ results in a monochromatic copy of $\mathcal{G}$ in $\mathcal{H}$. Define the size-Ramsey number $\hat{R}(\mathcal{G}, r)$ of a $k$-uniform hypergraph $\mathcal{G}$ as

$$
\hat{R}(\mathcal{G}, r)=\min \left\{|E(\mathcal{H})|: \mathcal{H} \rightarrow(\mathcal{G})_{r}\right\}
$$

When $r=2$, we simply use $\hat{R}(\mathcal{G})$ for the ease of reference.
Given integers $1 \leq l<k$ and $n \equiv l(\bmod k-l)$, an $l$-path $\mathcal{P}_{n, l}^{(k)}$ is a $k$-uniform hypergraph with vertex set $[n]$ and edge set $\left\{e_{1}, \cdots, e_{m}\right\}$, where $e_{i}=\{(i-1)(k-l)+1,(i-1)(k-l)+2, \cdots,(i-1)(k-l)+k\}$ and $m=\frac{n-l}{k-l}$, i.e. the edges are intervals of length $k$ in $[n]$ and consecutive edges intersect in exactly $l$ vertices. A $\mathcal{P}_{n, 1}^{(k)}$ is commonly referred as a loose path and a $\mathcal{P}_{n, k-1}^{(k)}$ is called a tight path.

Dudek, Fleur, Mubayi and Rődl [11] showed that when $l \leq \frac{k}{2}$, the sizeRamsey number of a path $\mathcal{P}_{n, l}^{(k)}$ can be easily reduced to the graph case. In particular, they showed that if $1 \leq l \leq \frac{k}{2}$, then

$$
\hat{R}\left(\mathcal{P}_{n, l}^{(k)}\right) \leq \hat{R}\left(P_{n}\right)=O(n)
$$

For tight paths, they showed in the same paper that for fixed $k \geq 3$,

$$
\hat{R}\left(\mathcal{P}_{n, k-1}^{(k)}\right)=O\left(n^{k-1-\alpha}(\log n)^{1+\alpha}\right)
$$

where $\alpha=(k-2) /\left(\binom{k-2}{2}+1\right)$. Observe that $\hat{R}\left(\mathcal{P}_{n, l}^{(k)}\right) \leq \hat{R}\left(\mathcal{P}_{n, k-1}^{(k)}\right)$. Thus any upper bound on the size-Ramsey number of tight paths is also an upper bound for other $l$-path $\mathcal{P}_{n, l}^{(k)}$.

Motivated by their approach, we use a different probabilistic construction and improve the upper bound to $O\left((n \log n)^{k / 2}\right)$. In particular, we show the following result on the multi-color size-Ramsey number of tight paths in hypergraphs:

Theorem 1. For any fixed $k \geq 3$, any $r \geq 2$, and sufficiently large $n$, we have

$$
\hat{R}\left(\mathcal{P}_{n, k-1}^{(k)}, r\right)=O\left(r^{k}(n \log n)^{\frac{k}{2}}\right) .
$$

## 2 Proof of Theorem 1

The approach of our proof is inspired by Dudek, Fleur, Mubayi and Rődl's approach in their proof of Theorem 2.8 in [11]. In their proof, they constructed their hypergraph by setting edges to be the $k$-cliques of an Erdős-Rényi random graph. Then they use a greedy algorithm to show that the number of edges of each color is smaller than $\frac{1}{r}$ fraction of the total number of edges, which gives a contradiction. Motivated by their approach, we use the same greedy algorithm but a different probabilistic construction of the hypergraph. Instead of using $k$-cliques of an Erdős-Rényi random graph as edges, we use $k$-cycles of a random $C_{k}$-colorable graph (which will be defined later) as edges.

Throughout the paper, we will use the following version of Chernoff inequalities for the binomial random variables $X \sim \operatorname{Bin}(n, p)$ (for details, see, e.g. (4) ):

$$
\begin{gather*}
\operatorname{Pr}(X \leq E(X)-\lambda) \leq \exp \left(-\frac{\lambda^{2}}{2 E(X)}\right)  \tag{1}\\
\operatorname{Pr}(X \geq E(X)+\lambda) \leq \exp \left(-\frac{\lambda^{2}}{2(E(X)+\lambda / 3)}\right) \tag{2}
\end{gather*}
$$

We follow a similar notation as 11]. A graph $G$ is $C_{k}$-colorable if there is a graph homomorphism $\pi$ mapping $G$ to the cycle $C_{k}$. That is, $V(G)$ can be partitioned into $k$-parts $V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ so that $E(G) \subseteq \bigcup_{i=1}^{k} E\left(V_{i}, V_{i+1}\right)$ with $V_{k+1}=V_{1}$ and $E\left(V_{i}, V_{i+1}\right)$ denoting the set of edges between a vertex in $V_{i}$ and a vertex in $V_{i+1}$. For such a graph $G$, we say a $k$-cycle $C$ in $G$ is proper if it intersects each $V_{i}$ by exactly one vertex. For $1 \leq l \leq k-1$, we say a path $P_{l}$ of $l$ vertices in $G$ is proper if it intersects each $V_{i}$ by at most one vertex. Let $\mathcal{T}_{k-1}(G)$ denote the set of all proper $(k-1)$-paths in $G$. Let $\mathcal{B} \subseteq \mathcal{T}_{k-1}$ be a family of pairwise vertex-disjoint proper $(k-1)$-paths. Let $t_{\mathcal{B}}$ be the total number of proper $k$-cycles in $G$ that extend some $B \in \mathcal{B}$. For $A \subseteq V$, define $y_{A, \mathcal{B}}$ as the number of proper $k$-cycles in $G$ that extend a proper $(k-1)$-path $B \in \mathcal{B}$ with a vertex $v \in A \cup \bigcup_{B \in \mathcal{B}} V(B)$. Given $C \subseteq V(G)$, we use $z_{C}$ to denote the number of proper $k$-cycles in $G$ that intersect $C$. We use $t_{k}$ to denote the total number of proper $k$-cycles in $G$.

We say an event in a probability space holds a.a.s. (aka, asymptotically almost surely) if the probability that it holds tends to 1 as $n$ goes to infinity. Finally, we use $\log n$ to denote natural logarithms.

Proposition 1. For every $r \geq 2, k \geq 3$, and sufficiently large $n$, there exists a $C_{k}$-colorable graph $G=(V, E)$ satisfying the following:
(i) For every $\mathcal{B}$ consisting of $n$ pairwise vertex-disjoint proper $(k-1)$-paths, and every $A \subseteq V \backslash \bigcup_{B \in \mathcal{B}} V(B)$ with $|A| \leq n$, we have

$$
y_{A, \mathcal{B}}<\frac{1}{2 k r} t_{\mathcal{B}} .
$$

(ii) For every $C \subseteq V$ with $|C| \leq(k-1) n$, we have

$$
z_{C}<\frac{t_{k}}{2 r}
$$

(iii) The total number of proper $k$-cycles satisfies

$$
t_{k}=O\left(r^{k}(n \log n)^{k / 2}\right)
$$

Proof. Set $c=16 k^{2} r$ and $p=\frac{\sqrt{\log n}}{\sqrt{n}}$. Consider the following random $C_{k^{-}}$ colorable graph $G$. Let $V(G)=V_{1} \cup V_{2} \cup \cdots \cup V_{k}$ be the disjoint union of $k$ sets. Each $V_{i}$ (for $1 \leq i \leq k$ ) has the same size $c n$. For any pair of vertices $\{u, v\}$ in two consecutive parts, i.e., there is an $i \in[k]$, such that $u \in V_{i}$ and $v \in V_{i+1}$ (with the convention $V_{k+1}=V_{1}$ ), add $u v$ as an edge of $G$ with probability $p$ independently. There is no edge inside each $V_{i}$ or between two non-consecutive parts.

We will show that this random $C_{k}$-colorable graph $G$ satisfies a.a.s. (i)-(iii).
First we show that $G$ a.a.s. satisfies $(i)$. For a fixed family $\mathcal{B}$ of $n$ pairwise vertex-disjoint proper $(k-1)$-paths, we would like to give a lower bound of $t_{\mathcal{B}}$. For each proper $(k-1)$-path $B \in \mathcal{B}$, there are $c n$ vertices that can extend $B$ into a proper $k$-cycle, each with probability $p^{2}$ independently. Thus, we have $t_{\mathcal{B}} \sim \operatorname{Bin}\left(c n^{2}, p^{2}\right)$ with

$$
E\left[t_{\mathcal{B}}\right]=c n^{2} p^{2}=c n \log n=16 k^{2} r n \log n
$$

Applying Chernoff inequality, we have

$$
\begin{aligned}
\operatorname{Pr}\left(t_{\mathcal{B}} \leq \frac{\left.E\left[t_{\mathcal{B}}\right]\right)}{2}\right) & \leq \exp \left(-\frac{1}{8} E\left[t_{\mathcal{B}}\right]\right) \\
& =\exp \left(-2 k^{2} r n \log n\right)
\end{aligned}
$$

Now for fixed $A \subseteq V \backslash \bigcup_{B \in \mathcal{B}} V(B)$, we estimate the upper bound of $y_{A, \mathcal{B}}$. Without loss of generality, we can assume that $|A|=n$. We have $y_{A, \mathcal{B}} \leq Y \sim$ $\operatorname{Bin}\left(2 n^{2}, p^{2}\right)$, thus

$$
E[Y]=2 n^{2} p^{2}=2 n \log n
$$

Thus if we apply the Chernoff bound (2) with $\lambda=(2 k-1) E[Y]$, then

$$
\begin{aligned}
\operatorname{Pr}\left(Y \geq \frac{1}{4 k r} E\left[t_{\mathcal{B}}\right]\right) & =\operatorname{Pr}(Y \geq 2 k E[Y]) \\
& =\operatorname{Pr}(Y \geq E[Y]+\lambda) \\
& \leq \exp \left(-\frac{\lambda^{2}}{2(\mathrm{E}[Y]+\lambda / 3)}\right) \\
& \leq \exp \left(-\frac{3(2 k-1)^{2}}{2 k+2} n \log n\right)
\end{aligned}
$$

The number of possible choices of $\mathcal{B}$ is upper bounded by $\left(\binom{c n}{n} \cdot n!\right)^{k}$. The number of possible choices of $A$ and $\mathcal{B}$ is upper bounded by $\left(\binom{c n}{n,\lceil n / k\rceil} \cdot n!\right)^{k} \leq$ $\left(\binom{c n}{n, n} \cdot n!\right)^{k}$. Stirling approximation of binomial coefficient gives us that

$$
\begin{aligned}
\log \left(\binom{c n}{n} \cdot n!\right)^{k} & =(1+o(1))(k n \log n), \\
\log \left(\binom{c n}{n, n} \cdot n!\right)^{k} & =(1+o(1))(k n \log n) .
\end{aligned}
$$

Therefore by the union bound, we have

$$
\begin{aligned}
\operatorname{Pr}\left(\bigcup_{\mathcal{B}}\left\{t_{\mathcal{B}} \leq \frac{E\left[t_{\mathcal{B}}\right]}{2}\right\}\right) & \leq\left(\binom{c n}{n} \cdot n!\right)^{k} \operatorname{Pr}\left(t_{\mathcal{B}} \leq \frac{E\left[t_{\mathcal{B}}\right]}{2}\right) \\
& \leq \exp \left((1+o(1)) k n \log n-2 k^{2} r n \log n\right) \\
& =o(1)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\operatorname{Pr}\left(\bigcup_{A, \mathcal{B}}\left\{y_{A, \mathcal{B}} \geq \frac{1}{4 k r} E\left[t_{\mathcal{B}}\right]\right\}\right) & \leq\left(\binom{c n}{n, n} \cdot n!\right)^{k} \operatorname{Pr}\left(Y \geq \frac{1}{4 k r} E\left[t_{\mathcal{B}}\right]\right) \\
& \leq \exp \left((1+o(1)) k n \log n-\frac{3(2 k-1)^{2}}{2 k+2} n \log n\right) \\
& =o(1) .
\end{aligned}
$$

In the last step, we observe $\frac{3(2 k-1)^{2}}{2 k+2}>k$ for all $k \geq 3$.
Therefore, combining previous inequalities, it follows that for all $A, \mathcal{B}$ satisfying the condition in $(i)$, we have, a.a.s.,

$$
y_{A, \mathcal{B}}<\frac{1}{4 k r} E\left[t_{\mathcal{B}}\right] \leq \frac{1}{2 k r} t_{\mathcal{B}} .
$$

This finishes the proof of $(i)$.
Now we will prove that $G$ satisfies (ii) and (iii) a.a.s.
We will use the Kim-Vu inequality [12] stated as below:
Let $H$ be a (weighted) hypergraph with $V(H)=[n]$. Edge edge $e$ has some weight $w(e)$. Suppose $\left\{t_{i}: i \in[n]\right\}$ is a set of Bernoulli independent random variables with probability p of being 1. Consider the polynomial

$$
Y_{H}=\sum_{e \in E(H)} w(e) \prod_{s \in e} t_{s}
$$

Furthermore, for a subset $A$ of $V(H)$, define

$$
Y_{H_{A}}=\sum_{e, A \subset e} w(e) \prod_{i \in e \backslash A} t_{i} .
$$

$$
\begin{align*}
& \text { If we define } E_{i}(H)=\max _{A \subset V(H),|A|=i} E\left(Y_{H_{A}}\right), E(H)=\max _{i \geq 0} E_{i}(H) \text { and } \\
& E^{\prime}(H)=\max _{i \geq 1} E_{i}(H) \text {, then } \\
& \operatorname{Pr}\left(\left|Y_{H}-E_{0}(H)\right|>a_{k}\left(E(H) E^{\prime}(H)\right)^{1 / 2} \lambda^{k}\right)=O(\exp (-\lambda+(k-1) \log n)) \tag{3}
\end{align*}
$$

for any positive number $\lambda>1$ and $a_{k}=8^{k}(k!)^{1 / 2}$.
In our context, for a fixed $v \in V(G)$, let $H$ be the $k$-uniform hypergraph constructed by the proper $k$-cycles of $G$ containing $v$. The edge set of $H$ is the collection of all $k$-tuples $\left\{v v_{1}, v_{1} v_{2}, \cdots, v_{k-2} v_{k-1}, v_{k-1} v\right\}$ such that $v v_{1} v_{2} \cdots v_{k-1} v$ is a proper $k$-cycle in $G$ and all edges have weight 1 .

Fix $v \in V(G)$. we let $X_{v}$ denote the number of proper $k$-cycles in $G$ that contain $v$. Then it's not hard to see that

$$
\begin{gathered}
E_{0}\left(X_{v}\right)=E\left(X_{v}\right)=(c n)^{k-1} p^{k}=c^{k-1} n^{\frac{k-2}{2}}(\log n)^{\frac{k}{2}} . \\
E^{\prime}\left(X_{v}\right)=(c n)^{k-2} p^{k-1}=c^{k-2} n^{\frac{k-3}{2}}(\log n)^{\frac{k-1}{2}} .
\end{gathered}
$$

Applying Kim- Vu inequality with $\lambda=2(k-1) \log n$, we get that for each $v \in V(G)$,

$$
\operatorname{Pr}\left(\left|X_{v}-E_{0}\left(X_{v}\right)\right|>a_{k}\left(E\left(X_{v}\right) E^{\prime}\left(X_{v}\right)\right)^{1 / 2} \lambda^{k}\right)=O(\exp (-(k-1) \log n))
$$

Observe that $a_{k}\left(E\left(X_{v}\right) E^{\prime}\left(X_{v}\right)\right)^{1 / 2} \lambda^{k}=o\left(E_{0}\left(X_{v}\right)\right)$. Applying union bound for all $v \in V(G)$, we obtain that a.a.s that

$$
X_{v}=(1 \pm o(1))(c n)^{k-1} p^{k}=(1 \pm o(1)) c^{k-1} n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}}
$$

Recall that $t_{k}$ denotes the total number of proper $k$-cycles in $G$ and $z_{C}$ denotes the number of proper $k$-cycles in $G$ that intersect $C$. Suppose $|C| \leq(k-1) n$. Then

$$
z_{C} \leq(1+o(1))(k-1) n c^{k-1} n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}}=(1+o(1))(k-1) c^{k-1}(n \log n)^{\frac{k}{2}} .
$$

Note that $t_{k}=\frac{1}{k} \sum_{v \in V(G)} X_{v}$. Thus

$$
\begin{aligned}
t_{k} & \geq \frac{1}{k}(1-o(1)) k c n \cdot c^{k-1} n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}} \\
& \geq(1-o(1)) c^{k}(n \log n)^{\frac{k}{2}} .
\end{aligned}
$$

Since $c=16 k^{2} r$, we have that for n sufficiently large,

$$
z_{C}<\frac{t_{k}}{2 r}
$$

Moreover, similar to the above calculation, we have that a.a.s.,

$$
t_{k} \leq(1+o(1)) c^{k}(n \log n)^{\frac{k}{2}}=O\left(r^{k}(n \log n)^{\frac{k}{2}}\right)
$$

Now we will prove the main result. We use the same greedy algorithm approach by Dudek, Fleur, Mubayi and Rődl in 11 .

Proof of Theorem 1: We show that there exists a $k$-uniform hypergraph $\mathcal{H}$ with $|E(\mathcal{H})|=O\left(r^{k} n^{\frac{k}{2}}(\log n)^{\frac{k}{2}}\right)$ such that any $r$-coloring of the edges of $\mathcal{H}$ yields a monochromatic copy of $\mathcal{P}_{n, k-1}^{(k)}$.

Let $G$ be the graph constructed from Proposition 1 for $n$ sufficiently large. Let $\mathcal{H}$ be a $k$-uniform hypergraph such that $V(\mathcal{H})=V(G)$ and $E(\mathcal{H})$ be the collection of all proper $k$-cycles in $G$.

Take an arbitrary $r$-coloring of the edges $\mathcal{H}_{0}=\mathcal{H}$ and assume that there is no monochromatic $\mathcal{P}_{n, k-1}^{(k)}$. Without loss of generality, suppose the color class with the most number of edges is blue. We will consider the following greedy algorithm:
(1) Let $\mathcal{B}=\emptyset$ be a trash set of proper $(k-1)$-paths in $G$. Let $A$ be a blue tight path in $\mathcal{H}$ that we will iteratively modify. Throughout the process, let $U=V(\mathcal{H}) \backslash\left(V(A) \cup \bigcup_{B \in \mathcal{B}} V(B)\right)$ be the set of unused vertices. If at any point $|\mathcal{B}|=n$, terminate.
(2) If possible, choose a blue edge $v_{1} v_{2} \cdots v_{k-1} v_{k}$ from $U$ and put these vertices into $A$ and set the pointer to $v_{k}$. Otherwise, if not possible, terminate.
(3) Suppose the pointer is at $v_{i}$ and $v_{i-k+2}, \cdots, v_{i-1}, v_{i}$ are the last $k-1$ vertices of the constructed blue path $A$. There are two cases:

Case 1: If there exists a vertex $u \in U$ such that $v_{i-k+2}, \cdots, v_{i-1}, v_{i}, u$ form a blue edge in $\mathcal{H}$, then we extend $P$, i.e. add $v_{i+1}=u$ into $A$. Set the pointer to $v_{i+1}$ and restart Step (3).
Case 2: Otherwise, remove the last $k-1$ vertices from $A$ and set $\mathcal{B}=\mathcal{B} \cup$ $\left\{\left\{v_{i-k+2}, \cdots, v_{i-1}, v_{i}\right\}\right\}$. Set the pointer to $v_{i-k+1}$. Now if $|A|<k$, then set $A=\emptyset$ and go to Step (2). Otherwise, restart Step (3).

Note that this procedure will terminate under two circumstances: either $|\mathcal{B}|=n$ or there is no blue edge in $U$.

Let us first consider the case when $|\mathcal{B}|=n$, i.e. there are $n$ pairwise vertexdisjoint proper $(k-1)$-paths in $\mathcal{B}$. Moreover, $|A| \leq n$ since there is no blue path of $n$ vertices. Applying Proposition 1 with sets $A$ and $\mathcal{B}$, we obtain that

$$
y_{A, \mathcal{B}}<\frac{1}{2 k r} t_{\mathcal{B}} .
$$

Observe that every edge of $\mathcal{H}$ that extends some $B \in \mathcal{B}$ with a vertex from $V\left(\mathcal{H}_{0}\right) \backslash\left(V(A) \cup \bigcup_{B \in \mathcal{B}_{m}} B\right)$ must be non-blue. Therefore, the number of blue edges of $\mathcal{H}$ that contain some $B \in \mathcal{B}$ as subgraph is at most $y_{A, \mathcal{B}}$.

Consider $A, \mathcal{B}$ as $A_{0}, \mathcal{B}_{0}$ respectively. Now remove all the blue edges from $\mathcal{H}_{0}$ that contain some $B \in \mathcal{B}_{0}$ as subgraph and denote the resulting hypergraph
as $\mathcal{H}_{1}$. Perform the greedy procedure again on $\mathcal{H}_{1}$. This will generate a new $A_{1}$ and $\mathcal{B}_{1}$. Applying Proposition 1 again, we have $y_{A_{1}, \mathcal{B}_{1}} \leq \frac{1}{2 k r} t_{\mathcal{B}_{1}}$. Keep repeating the procedure until it is no longer possible. Observe that $\mathcal{B}_{i} \cap \mathcal{B}_{j}=\emptyset$ for $i \neq j$.

When the above procedure can not be repeated anymore, we are in the case that $\left|\mathcal{B}_{m}\right|<n$ for some positive integer $m$ and there are no more blue edges in $V(\mathcal{H}) \backslash \bigcup_{B \in \mathcal{B}_{m}} B$. In this case, $A_{m}=\emptyset$ and all the blue edges remaining in $\mathcal{H}_{m}$ have to intersect the set $C=\bigcup_{B \in \mathcal{B}_{m}} B$. By Proposition it follows that

$$
z_{C}<\frac{1}{2 r} t_{k} .
$$

Let $e_{b}(\mathcal{H})$ denote the total number of blue edges in $\mathcal{H}$. We have

$$
\begin{aligned}
e_{b}(\mathcal{H}) & \leq \sum_{i=0}^{m-1} y_{A_{i}, \mathcal{B}_{i}}+z_{C} \\
& <\sum_{i=0}^{m-1} \frac{1}{2 k r} t_{\mathcal{B}_{i}}+\frac{1}{2 r} t_{k}
\end{aligned}
$$

Note that every proper $k$-cycle can extend exactly $k$ proper $(k-1)$-paths. We have $\sum_{i=0}^{m-1} t_{\mathcal{B}_{i}} \leq k t_{k}$. Thus,

$$
\begin{aligned}
e_{b}(\mathcal{H}) & <\frac{1}{2 k r} \sum_{i=0}^{m-1} t_{\mathcal{B}_{i}}+\frac{1}{2 r} t_{k} \\
& \leq \frac{1}{2 r} t_{k}+\frac{1}{2 r} t_{k} \\
& =\frac{1}{r}|E(\mathcal{H})| .
\end{aligned}
$$

The conclusion is that the number of blue edges in $\mathcal{H}$ is strictly smaller than $\frac{1}{r}$ of the total number of edges in $\mathcal{H}$, which contradicts that blue is the color class with the most number of edges of $\mathcal{H}$.

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