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INTERNAL CONTROLLABILITY OF NONLOCALIZED SOLUTION FOR THE KADOMTSEV–PETVIASHVILI II EQUATION*

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Abstract. The internal control problem for the Kadomtsev–Petviashvili II equation, better known as KP-II, is the object of study in this paper. The controllability in $L^2(\mathbb{T}^2)$ from a vertical strip is proved using the Hilbert uniqueness method through the techniques of semiclassical and microlocal analysis. Additionally, a negative result for the controllability in $L^2(\mathbb{T}^2)$ from a horizontal strip is also shown.

Key words. control theory, semiclassical analysis, microlocal analysis, dispersive equations

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1. Introduction.

The Kadomtsev–Petviashvili equations, better known as KP, is

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) \pm \partial_y^2 u = 0, \tag{1.1}$$

and they was introduced by Kadomtsev and Petviashvili (see [17]) in 1970 from the study of transverse stability of the solitary wave solution of the Korteweg–de Vries (KdV) equation. The KP equations are completely integrable and they can be solved by inverse scattering transform. Moreover, (1.1) has been studied separately depending on the sign that is used; with a negative sign it is known as the KP-I equation, and otherwise it is the KP-II equation. The propagation of the trajectories behaves very differently from one equation to the other and they do not allow study of them at the same time. In this paper, we concentrate on the KP-II equation.

Concerning the Cauchy problem, the KP-II equation has been well studied. In a pioneering work, Bourgain [3] proved the global well-posedness of the KP-II equation in $L^2(\mathbb{T}^2)$ by using the Fourier restriction norm he introduced in [2]. For the nonperiodic setting, Takaoka and Tzvetkov in [15] proved local well-posedness in anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{3}$ and $s_2 \geq 0$. Hadac, Kerr, and Koch in [8] proved global well-posedness and scattering for small data in critical functional space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$. Molinet, Saut, and Tzvetkov in [16] showed the local and global well-posedness for partially periodic data.

We will address the problem of exact controllability for the KP-II equation. Before getting into this problem, we observe that (1.1) can be written as

$$\partial_t u + \partial_x^3 u + u\partial_x u \pm \partial_x^{-1}\partial_y^2 u = 0,$$

where the Fourier multiplier ∂_x^{-1} is defined by

$$\widehat{\partial_x^{-1} v}(k,\eta) = \frac{1}{ik}\widehat{v}(k,\eta)$$

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for all distributions with horizontal mean value

$$v \in \mathcal{D}_0'(\mathbb{T}^2) := \left\{ v \in \mathcal{D}'(\mathbb{T}^2) : \widehat{v}(0,l) = 0 \ \forall l \in \mathbb{Z} \right\}.$$

For any $s \in \mathbb{R}$, we denote by $H_0^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \cap \mathcal{D}_0'(\mathbb{T}^2)$ a closed subspace of $H^s(\mathbb{T}^2)$. In particular, $L_0^2(\mathbb{T}^2) := H_0^0(\mathbb{T}^2)$. Additionally, for an open set $\omega \subset \mathbb{T}^2$, we denote

$$C_\omega^2(\mathbb{T}^2) := \left\{ g \in C^2(\mathbb{T}^2) : \mathrm{supp}(g) \subset \omega \right\}.$$

When ω is of the form $(a,b)_x \times \mathbb{T}_y$ or $\mathbb{T}_x \times (a,b)_y$, the functions in $C_\omega^2(\mathbb{T}^2)$ can be identified as functions of a single variable, supported in (a,b). In these cases, we will use the notation C_ω^2 for short.

The internal control problem that we are interested in this paper is, given $T > 0$ and $u_0, u_1 \in L_0^2$, does there exist a control input $f \in L^2((0,T); L^2(\mathbb{T}^2))$, supported on some open subset $\omega \subset \mathbb{T}^2$, such that the solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = f, \quad (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{1.2}$$

satisfies $u(T,\cdot) = u_1$?

Additionally, we face the difficulty that the control input f should be localized in ω while keeping the horizontal mean value. However, if the control region ω is either a horizontal strip or a vertical strip, we can define the control operator as follows.

For a vertical control region of the form $\omega = (a,b) \times \mathbb{T}$, we fix a nonnegative real-valued function $g \in C_\omega^2(\mathbb{T})$ such that $\int_{\mathbb{T}} g = 1$. In this case, we define the control input $\mathcal{G}\mathfrak{h}$, where $\mathcal{G}$ is the linear operator:

$$\mathcal{G}\mathfrak{h}(x,y) := g(x) \left(\mathfrak{h}(x,y) - \int_{\mathbb{T}} g(x')\mathfrak{h}(x',y)dx' \right). \tag{1.3}$$

If the control region is a horizontal strip of the form $\omega = \mathbb{T} \times (a,b)$, we define the control input as $\mathcal{K}\mathfrak{h}$, where $\mathcal{K}$ is the operator:

$$\mathcal{K}\mathfrak{h}(x,y) := g(y) \left(\mathfrak{h}(x,y) - \int_{\mathbb{T}} g(y')\mathfrak{h}(x,y')dy' \right). \tag{1.4}$$

Our first result concerns the internal controllability of the linearized KP-II equation on the vertical region:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{G}\mathfrak{h}, \quad (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2). \end{cases} \tag{1.5}$$

THEOREM 1.1. *Given $T > 0$, and $u_0, u_1 \in L_0^2(\mathbb{T}^2)$, there exists $\mathfrak{h} \in L^2((0,T); L^2(\mathbb{T}^2))$ such that the solution u of (1.5) satisfies $u(T) = u_1$.*

For the nonlinear control system,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}\mathfrak{h}, \quad (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{1.6}$$

by adapting a pertubative argument, relying on the Cauchy theory for the KP-II equation, we obtain the following result of the exact controllability in a local sense.

THEOREM 1.2. *Given $T > 0$, there exists $R > 0$ such that for any $u_0, u_1 \in L^2_0(\mathbb{T}^2)$ satisfying $\|u_0\|_{L^2(\mathbb{T}^2)} \leq R$ and $\|u_1\|_{L^2(\mathbb{T}^2)} \leq R$, there exists a control $\mathfrak{h} \in L^2((0,T); L^2(\mathbb{T}^2))$ such that the solution u of (1.6) with $\mathcal{G}$ satisfies $u(T) = u_1$.*

Remark 1.3. In [3], Bourgain proved that the KP-II equation is globally well-posed in $H^s_0(\mathbb{T}^2)$ for all $s \geq 0$. Our results in Theorems 1.1 and 1.2 also hold for any data in $H^s_0(\mathbb{T}^2)$. The main reason to consider $L^2(\mathbb{T}^2)$ here is that the quantity

$$\int_{\mathbb{T}^2} |u(t,x,y)|^2 dxdy$$

is conserved along the KP-II flow (1.1) and hence $L^2(\mathbb{T}^2)$ is a natural functional space to study the problem of controllability.

On the contrary, for the controllability from the horizontal region

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{K}\mathfrak{h}, & (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L^2_0(\mathbb{T}^2), \end{cases} \tag{1.7}$$

we have a negative answer which shows that the exact controllability for the linearized KP-II equation cannot hold at any time $T > 0$ when the control region is a horizontal strip.

THEOREM 1.4. *Given $T > 0$ and $u_0 \in L^2_0(\mathbb{T}^2)$, there exists $u_1 \in L^2(\mathbb{T}^2)$ and there does not exist $\mathfrak{h} \in L^2((0,T); L^2_0(\mathbb{T}))$ such that the solution u of (1.7) satisfies $u(T) = u_1$.*

The proofs of Theorems 1.1 and 1.4 rely on the propagation of singularities for the KP-II flow. It turns out that the propagation on the horizontal direction is much stronger than on the vertical direction. The heuristic is that the singularities will travel into some vertical control region in a very short time; however, for a horizontal control region the singularities move too slowly to enter. This can be interpreted physically, since the KP equations describe the regime where the wavelengths in the transverse direction (in y) are much larger than in the direction of propagation (in x).

The paper is organized as follows. In section 2, some results of well-posedness are mentioned; they will be important in the proof of the controllability of the full control system. In section 3, the linear controllability is established by proving the observability inequality. In section 4, the local controllability of the nonlinear equation is proved by fixed point arguments. In section 5, we construct a counterexample to complete the proof of Theorem 1.4.

2. Notation and preliminaries.

Throughout this article, we use the identification $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) = [-\pi, \pi]/\mathbb{Z}_2$. We will adapt the standard convention for constancy in PDE. The constant C will denote a positive constant that can change from line to line and the dependency will be specified if there is any risk of confusion.

We need the following classical inequality of Ingham.

PROPOSITION 2.1 (Proposition 4.3 in [10]). *Let $(\omega_k)_{k\in\mathbb{Z}}$ be a family of real numbers, satisfying the uniform gap condition*

$$\gamma := \inf_{k_1 \neq k_2} |\omega_{k_1} - \omega_{k_2}| > 0.$$

If $I \subset \mathbb{R}$ is a bounded interval of length $|I| > \frac{2\pi}{\gamma}$, then there exists $C_\gamma > 0$, depending only on γ and the length $|I|$, such that for all $(a_k)_{k\in\mathbb{Z}} \subset l^2(\mathbb{Z})$, we have

$$\frac{1}{C_\gamma}\sum_{k\in\mathbb{Z}}|a_k|^2 \le \int_I \left|\sum_{k\in\mathbb{Z}} a_k e^{i\omega_k t}\right|^2 dt \le C_\gamma \sum_{k\in\mathbb{Z}}|a_k|^2.$$

Next we briefly review the Cauchy theory for KP-II following [16]. The initial value problem

$$\begin{cases}\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, & (t,x,y)\in\mathbb{R}\times\mathbb{T}^2,\\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2),\end{cases} \tag{2.1}$$

is proved in [3] by Bourgain to be globally well-posed when $u_0 \in H_0^s(\mathbb{T}^2)$ for $s \ge 0$.

In [3], Bourgain introduced a Fourier restriction norm

$$\|u\|^2_{X^{s,b,b_1}} = \int_{\mathbb{R}} \sum_{(k,l)\in\mathbb{Z}^2} \left\langle \frac{\langle\sigma(\tau,k,l)\rangle}{\langle k\rangle^3}\right\rangle^{2b_1} \langle\sigma(\tau,k,l)\rangle^{2b}\langle(k,l)\rangle^{2s}|\widehat{u}(\tau,k,l)|^2 d\tau,$$

where $\sigma(\tau,k,l) = \tau - k^3 + \frac{l^2}{k}$ and $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. For $T > 0$, the norm in the localized time interval $[0,T]$ is defined by

$$\|u\|_{X_T^{s,b,b_1}} := \inf\{\|w\|_{X^{s,b,b_1}} : w(t) = u(t) \,\text{on}\, (0,T)\}.$$

Denoting by $S(t) = e^{-it(\partial_x^3+\partial_x^{-1}\partial_y^2)}$ the linear semigroup, we have the following estimate.

Proposition 2.2 (Proposition 2.6 in [3]). *For $s \ge 0$, $-\frac12 < b' \le 0 < \frac12 < b \le b'+1$, $b_1 \in \mathbb{R}$, and $T \le 1$, we have*

$$\left\|\int_0^t S(t-t')F(t')dt'\right\|_{X_T^{s,b,b_1}} \le CT^{1-(b-b')}\|F\|_{X_T^{s,b',b_1}}$$

for any $F \in X_T^{s,b',b_1}$.

The proposition above is false for the end points $b' = -\frac12$ and $b = \frac12$. However, for the periodic problem, it seems that we cannot avoid use of these end points. The way to resolve this issue is to define an auxiliary norm

$$\|u\|_{Z^{s,b}} := \|\langle\sigma\rangle^{b-\frac12}\langle(k,l)\rangle^s\widehat{u}\|_{l^2_{(k,l)}L^1_\tau}.$$

We denote by $Z_T^{b,s}$ the restricted spaces, defined in the same manner. The analogue of Proposition 2.2 is as follows.

Proposition 2.3 (Propositions 2.5 and 2.6 in [3]). *Under the same conditions as in Proposition 2.2, we have*

$$\left\|S(t)u_0 + \int_0^t S(t-t')F(t')dt'\right\|_{X_T^{s,\frac12,b_1}\cap Z_T^{s,\frac12}} \le C\|u_0\|_{H^s} + C\|F\|_{X_T^{s,-\frac12,b_1}\cap Z_T^{s,-\frac12}}.$$

The proof can be found, for example, in [18]. In order to show that (2.1) is locally well-posed in the Fourier restriction spaces, we write it in the integral form:

$$u(t) = S(t)u_0 + \int_0^t S(t-t')(u\partial_x u)(t')dt'. \tag{2.2}$$

To use the fixed point argument, the following bilinear estimate is crucial.

PROPOSITION 2.4 (section 3.3 in [16]). *There exist $\frac{1}{4} < b_1 < \frac{3}{8}$, $C > 0$, and $\nu > 0$ such that for all $0 < T \leq 1, s \geq 0$, the bilinear estimate*

$$\|\partial_x(uv)\|_{X_T^{s,-\frac{1}{2},b_1} \cap Z_T^{s,-\frac{1}{2}}} \leq CT^{\nu}\|u\|_{X_T^{s,\frac{1}{2},b_1}}\|v\|_{X_T^{s,\frac{1}{2},b_1}}$$

holds for functions $u, v \in X_T^{s,\frac{1}{2},b_1}$ satisfying

$$\int_{\mathbb{T}} u(t,x,y)dx = \int_{\mathbb{T}} v(t,x,y)dx = 0.$$

This bilinear estimate is established by Bourgain in [3]. We use the adapted version of [16], in which the authors dealt with partially periodic data.

3. Linear controllability on vertical strip. In this section, the study of the internal controllability of linear system (1.5) is addressed by defining a linear operator in Proposition 3.9, which characterizes the control input of the linear system and drives the solution from an initial state u_0 to a final state u_1. Notice that by reversibility, the exact controllability is equivalent to null controllability: given any initial state $u_0 \in L_0^2$, find a function $\mathfrak{h} \in L^2((0,T) \times \mathbb{T}^2)$ so that the equation satisfies $u(0,\cdot) = u_0$ and $u(T,\cdot) = 0$. Hence, we will study the null controllability.

The classical strategy to study the null controllability is to show the observability inequality for the adjoint system associated to the equation; in the KP-II case, it matches with the homogeneous linearized KP-II equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2). \end{cases} \tag{3.1}$$

From the classical Hilbert uniqueness method (HUM), one can deduce that the null controllability is equivalent to the observability for its adjoint system.

PROPOSITION 3.1 (see [14]). *Given $T > 0$, the system* (1.5) *is null controllable at T if and only if given $u_0 \in L^2(\mathbb{T}^2)$, there exists a unique solution u to* (3.1) *such that*

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C_T \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t,x,y)|^2 dxdydt, \tag{3.2}$$

where the constant $C_T > 0$ does not depend on u_0.

The region where the control will be placed is a vertical strip given by

$$\omega := (a,b) \times \mathbb{T}$$

and the operator $\mathcal{G}$ is given by (1.3). The region ω will allow us to get a reduction of the KP-II equation (3.1) in one dimension. Indeed, by expanding the solution $u(t,x,y)$ to (3.1) in Fourier series in the y variable

$$u(t,x,y) = \sum_{l \in \mathbb{Z}} a_l(t,x)e^{ily},$$

we find that for each $l \in \mathbb{Z}$, a_l satisfies the equation

$$\partial_t a_l + \partial_x^3 a_l - l^2 \partial_x^{-1} a_l = 0.$$

Therefore, by changing the notation, it is reduced to the study of the following λ-dependent equations:

$$\begin{cases} \partial_t u + \partial_x^3 u - \lambda^2 \partial_x^{-1} u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}). \end{cases} \tag{3.3}$$

3.1. Observability inequality. Due to Proposition 3.1, the proof of Theorem 1.1 is reduced to the proof of (3.2). From the one-dimensional reduction and Plancherel's theorem, we can further reduce the observability (3.2) to the following uniform observability for the family of equations (3.3).

PROPOSITION 3.2. *Given $T > 0$, there exists $C_T > 0$ such that for all $\lambda > 0$,*

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C_T \int_0^T \int_{\mathbb{T}} |\mathcal{G}u(t,x)|^2 dx dt \tag{3.4}$$

holds for all solution u of (3.3)*.*

The rest of this section is devoted to the proof of Proposition 3.2. The strategy is as follows. First, we reduce the inequality (3.4) to a weaker one (3.5), which is the observability for high frequencies and it does not consider the normalization part which simplifies the operator $\mathcal{G}$. Next, inspired by the work of Lebeau in [12], we rescale the time to change it to the semiclassical scale. This reduces the weak observability for system (3.3) to an inequality of the same form but for another semiclassical system (3.8). The third step is to reduce the inequality in the previous step to a frequency-localized one. Finally, we use the propagation argument to prove the frequency-localized semiclassical observability, namely (3.10).

3.1.1. Reduction to weak observability. The weak observability takes the form, uniformly in $\lambda \geq 0$,

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C_T \int_0^T \int_{\mathbb{T}} |g(x)u(t,x)|^2 dx dt + C\|u_0\|_{H^{-1}(\mathbb{T})}^2. \tag{3.5}$$

First, we prove a lemma concerning the commutator of a high-frequency cut-off and the operator $\mathcal{G}$.

LEMMA 3.3. *Take $\chi \in C^\infty(\mathbb{R})$ with* $\mathrm{supp}(\chi) \subset \{|\xi| > 1\}$ *and $\chi|_{|\xi|\geq 2} = 1$. Then there exist $h_0 > 0, C > 0$ such that for all $0 < h < h_0$, we have*

$$\int_0^T \|[\chi(hD_x), \mathcal{G}]u(t,\cdot)\|_{L^2(\mathbb{T})}^2 dt \leq Ch^2 \|u(0)\|_{L^2(\mathbb{T})}^2$$

with the notation $D_x := \frac{1}{i}\frac{\partial}{\partial x}$ and the commutator $[A,B] := AB - BA$.

Proof. We write

$$\int_0^T \|[\chi(hD_x), \mathcal{G}]u(t,\cdot)\|_{L^2(\mathbb{T})}^2 dt \leq C(\mathrm{I} + \mathrm{II}),$$

where

$$\mathrm{I} = \int_0^T \int_{\mathbb{T}} |[g(x), \chi(hD_x)]u(t,x)|^2 dx dt,$$

$$\mathrm{II} = \int_0^T \int_{\mathbb{T}} \left| g(x) \int_{\mathbb{T}} g(x')\chi(hD_x)u(t,x')dx' - \chi(hD_x)\left(g(x) \int_{\mathbb{T}} g(x')u(t,x')dx' \right) \right|^2 dx dt.$$

From symbolic calculus,[1] we have

$$\|[g(x),\chi(hD_x)]\|_{L^2\to L^2}\leq Ch,$$

and by conservation of the L^2 norm, we have

$$\mathrm{I}\leq Ch^2\int_0^T\|u(t)\|_{L^2(\mathbb{T})}^2dt=Ch^2T\|u(0)\|_{L^2(\mathbb{T})}^2.$$

For II, we first calculate (to simplify the notation, we omit the variable t here)

$$\begin{aligned}&\left(g(x)\int_{\mathbb{T}}g(x')(\chi(hD_x)u)(x')dx'\right)^{\widehat{}}(l)-\chi(hD_x)\left(g(x)\int_{\mathbb{T}}g(x')u(x')dx'\right)^{\widehat{}}(l)\\&\quad=\widehat{g}(l)\sum_{l_1\neq 0}\left(\chi(hl_1)-\chi(hl)\right)\widehat{g}(l_1)\widehat{u}(l_1).\end{aligned}$$

Since $|\chi(hl_1)-\chi(hl)|\leq\|\chi'\|_{L^\infty}h|l_1-l|$, we have

$$\begin{aligned}\mathrm{II}&\leq Ch^2\sum_l|\widehat{g}(l)|^2\left|\sum_{l_1\neq 0}|l_1-l|\widehat{g}(l_1)\widehat{u}(l_1)\right|^2\\&\leq Ch^2\sum_l|\widehat{g}(l)|^2\left(\sum_{l_1\neq 0}|l_1-l|^2|\widehat{g}(l_1)|^2\right)\left(\sum_{l_1\neq 0}|\widehat{u}(l_1)|^2\right)\\&\leq Ch^2\|u\|_{L^2(\mathbb{T})}^2\sum_{l,l_1\neq 0}|l_1-l|^2|\widehat{g}(l_1)|^2|\widehat{g}(l)|^2\\&=Ch^2\|u\|_{L^2(\mathbb{T})}^2,\end{aligned}$$

where we used the fact that $g\in C^2(\mathbb{T})$. ∎

PROPOSITION 3.4. (3.5) *implies the following full observability inequality:*

$$\|u_0\|_{L^2(\mathbb{T})}^2\leq C_T\int_0^T\int_{\mathbb{T}}|\mathcal{G}u(t,x)|^2dxdt. \tag{3.6}$$

Proof. The proof is essentially a unique continuation argument. However, it is more delicate since we need a uniform estimate with respect to λ. The proof will be divided into two steps.

The first step is to show that for any fixed $\lambda>0$, (3.6) holds with constant $C(\lambda)>0$ which may depend on λ. We argue by contradiction, assuming that (3.6) is not true; then we can select a sequence u_n of solutions to (3.3) so that

$$\|u_n(0)\|_{L^2(\mathbb{T})}=1\quad\text{and}\quad\lim_{n\to\infty}\int_0^T\int_{\mathbb{T}}|\mathcal{G}u_n(t,x)|^2dxdt=0.$$

Up to a subsequence, we may assume that $u_n(0)\rightharpoonup u_0$, weakly in $L^2(\mathbb{T})$. One can easily verify that $u_0\in L_0^2(\mathbb{T})$. Moreover, from the semigroup property, $u_n(t)\rightharpoonup u(t)$ weakly in $C([0,T];L^2(\mathbb{T}))$ and $u(t)$ is the distributional solution to (3.3) with initial data u_0. Since $\mathcal{G}:L_0^2(\mathbb{T})\to L_0^2(\mathbb{T})$ is a bounded operator, we have that $\mathcal{G}u(t,\cdot)=0$ in

[1] Though g is not assumed to be smooth, the following estimate is still valid.

$L^2_0(\mathbb{T})$ for a.e. $t \in [0,T]$. This means that $u(t,x)|_\omega = C(t)$ in $\mathcal{D}'(\omega)$ for a.e. $t \in [0,T]$. Moreover, from the strong continuity of the semigroup on $L^2_0(\mathbb{T})$,

$$C(t) = \int_{\mathbb{T}} g(x)u(t,x)dx \quad \forall t \in [0,T],$$

and $C(t)$ is a continuous function in t. Therefore we have that

$$g(x)\left(u(t,x) - C(t)\right) = 0, \text{ in } C([0,T];L^2(\mathbb{T})).$$

Thus $u(t,x)|_{x\in\omega} = C(t)$ in $\mathcal{D}'(\omega)$ for all $t \in [0,T]$. Now, if we rewrite (3.3) as $\partial_x(\partial_t u + \partial_x^3 u) + \lambda u = 0$ and evaluate u for $x \in \omega$, we have that $u|_\omega = 0$ in $\mathcal{D}'(\omega)$. Next we claim that $u \equiv 0$. Indeed, following [1], we consider the following family of sets (depending on T'):

$$\mathcal{N}_{T'} := \left\{u_0 \in L^2_0(\mathbb{T}) : S(t)u_0|_\omega = 0 \quad \forall t \in [0,T']\right\}.$$

For any $T' > T/2$, applying inequality (3.5) (with $T/2$), we have that for any $u_0 \in \mathcal{N}_{T'}$,

$$\|u_0\|_{L^2(\mathbb{T})} \leq C\|u_0\|_{H^{-1}(\mathbb{T})}.$$

This implies that the subspace $\mathcal{N}_{T'}$ in $L^2_0(\mathbb{T})$ is finite dimensional. Moreover, $\mathcal{N}_{T_1} \subset \mathcal{N}_{T_2}$ if $T_1 > T_2$. Now for $\delta > 0$ small, $S(\delta) : \mathcal{N}_{T'} \subset \mathcal{N}_{T'-\delta}$ is a linear mapping. Since for $T' > T/2$, $\dim\mathcal{N}_{T'-\delta} < \infty$, there exists $\delta_0 > 0$ such that for all $\delta \leq \delta_0$, $\mathcal{N}_{T'-\delta} = \mathcal{N}_{T'}$. Therefore, $(S(\delta) - \mathrm{Id})\delta^{-1} : \mathcal{N}_{T'} \to \mathcal{N}_{T'}$ is a linear mapping. Passing $\delta \to 0$, we have that $(\partial_t S(t))|_{t=0} : \mathcal{N}_{T'} \to \mathcal{N}_{T'}$. Denoting by σ and v_0 any of its eigenvalue and the corresponding eigenfunction of $\partial_t S(t)|_{t=0}$ on $\mathcal{N}_{T'}$, since $(\partial_t S(t)v_0)|_{t=0} = (-\partial_x^3 + \lambda^2\partial_x^{-1})v_0$, we have

$$\left(-\partial_x^3 + \lambda^2\partial_x^{-1}\right)v_0 = \sigma v_0.$$

This implies that v_0 has only a finite number of nonvanishing Fourier modes. Thus v_0 has an analytic extension near the real axis. Therefore, $v_0|_\omega = 0$ yields $v_0 \equiv 0$. Hence $\mathcal{N}_{T'} = \{0\}$.

Since the weak limit of $u_n(0)$ is 0, we have

$$\int_{\mathbb{T}} g(x)u_n(t,x)dx \to 0 \text{ and } \|gu_n\|_{L^2([0,T]\times\mathbb{T})} \to 0.$$

Moreover, up to a subsequence, $\|u_n(0)\|_{H^{-1}(\mathbb{T})} \to 0$, due to the Rellich theorem. This is a contradiction to the assumption that $\|u_n(0)\|_{L^2(\mathbb{T})} = 1$.

The second step is to prove that (3.6) is uniformly on λ. Again, we assume that (3.6) is not true. Then there exists a sequence of positive numbers $\lambda_n > 0$ and a sequence of solutions u_n to (3.3) with parameters λ_n such that

$$\|u_n(0)\|_{L^2(\mathbb{T})} = 1 \quad \text{and} \quad \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}} |\mathcal{G}u_n(t,x)|^2dxdt = 0.$$

Up to a subsequence, we may assume that $\lambda_n \to \lambda_\infty \in [0,\infty]$. Suppose $\lambda_\infty < \infty$; a similar argument as in the first step will lead to a contradiction.

The last possibility is $\lambda_\infty = \infty$. We write

$$u_n(0) = \sum_{l\neq 0} a_{n,l}e^{ilx}$$

and the corresponding solution of (3.3) is

$$u_n(t,x) = \sum_{l\neq 0} a_{n,l} e^{it\left(l^3 - \frac{\lambda_n^2}{l}\right)} e^{ilx}.$$

For any $\epsilon_0 > 0$, we set

$$u_n^{(\epsilon_0)} := \sum_{|l|\geq \frac{1}{\epsilon_0}} a_{n,l} e^{it\left(l^3 - \frac{\lambda_n^2}{l}\right)} e^{ilx}, \quad v_n^{(\epsilon_0)} = u_n - u_n^{(\epsilon_0)}.$$

From Lemma 3.3, we have

$$\int_0^T \|\mathcal{G} u_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2 \|u_n(0)\|_{L^2(\mathbb{T})}^2 + C\int_0^T \|(\mathcal{G}u_n)^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt.$$

Thus, there exists $C > 0$ such that for any $\epsilon_0 > 0$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \int_0^T \|\mathcal{G} u_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2, \\ &\limsup_{n\to\infty} \int_0^T \|\mathcal{G} v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2. \end{aligned} \tag{3.7}$$

For any $\epsilon > 0$ small, we can find $\epsilon_0 > 0$ small enough such that

$$\sum_{|l|\geq \frac{1}{\epsilon_0}} |\widehat{g}(l)|^2 \leq \epsilon^2,$$

and then

$$\left\| g(x) \int_{\mathbb{T}} g(x') u_n^{(\epsilon_0)}(t,x')dx' \right\|_{L^2(\mathbb{T})}^2 \leq \epsilon^2 \|g\|_{L^2(\mathbb{T})}^2 \|u_n^{(\epsilon_0)}(0)\|_{L^2(\mathbb{T})}^2.$$

Thus, from (3.5),

$$\begin{aligned} \|u_n^{(\epsilon_0)}(0)\|_{L^2(\mathbb{T})}^2 &\leq C\epsilon^2 + C\epsilon_0^2 + \|u_n^{(\epsilon_0)}(0)\|_{H^{-1}(\mathbb{T})}^2 \\ &\leq C(\epsilon^2 + \epsilon_0^2) \end{aligned}$$

for n large enough.

On the other hand, direct calculation yields

$$\begin{aligned} \int_0^T \|\mathcal{G} v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt &= \int_0^T \sum_l \left| \sum_{1\leq |l_1|\leq 1/\epsilon_0} (\widehat{g}(l-l_1) - \widehat{g}(l)\widehat{g}(l_1)) a_{n,l_1} e^{it\left(l_1^3 - \frac{\lambda_n^2}{l_1}\right)} \right|^2 dt \\ &\geq C \sum_l \sum_{1\leq |l_1|\leq 1/\epsilon_0} |\widehat{g}(l-l_1) - \widehat{g}(l)\widehat{g}(l_1)|^2 |a_{n,l_1}|^2 \\ &= C \sum_{1\leq |l_1|\leq 1/\epsilon_0} c_{l_1} |a_{n,l_1}|^2 \end{aligned}$$

with $c_{l_1} = \sum_l |\widehat{g}(l-l_1) - \widehat{g}(l)\widehat{g}(l_1)|^2$, by the Ingham inequality (Proposition 2.1), due to the assumption that $\lambda_n \to \infty$. Notice that the constant C can be chosen independent of n and ϵ_0, provided that if n is large enough, then

$$\sum_{1\leq |l_1|\leq 1/\epsilon_0} \left| (l_1+1)^3 - l_1^3 - \frac{\lambda_n^2}{l_1} + \frac{\lambda_n^2}{l_1+1} \right| \geq \gamma > 0 \quad \text{and} \quad T > \frac{2\pi}{\gamma}.$$

Noting that $c_{l_1} \geq |\widehat{g}(0) - \widehat{g}(l_1)^2|^2$ and $\widehat{g}(0) = 1$, there exists a constant $c_0 > 0$, independent of ϵ_0, ϵ, and n, so that $c_{l_1} \geq c_0$ for all $1 \leq |l_1| \leq 1/\epsilon_0$. Thus, for n sufficiently large,

$$\|v_n^{(\epsilon_0)(0)}\|_{L^2(\mathbb{T})}^2 \leq \frac{C}{c_0}\int_0^T \|\mathcal{G}v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2.$$

Therefore,

$$1 = \limsup_{n\to\infty} \|u_n(0)\|_{L^2(\mathbb{T})}^2 = \|u_n^{\epsilon_0}(0)\|_{L^2(\mathbb{T})}^2 + \|v_n^{\epsilon_0}(0)\|_{L^2(\mathbb{T})}^2 \leq C(\epsilon_0^2 + \epsilon^2) < 1,$$

which cannot happen. ∎

3.2. Reduction to semiclassical observability. Now, we consider the semiclassical equation of the following form:

$$\begin{cases} h\partial_t u + (h\partial_x)^3 u - (h\partial_x)^{-1} u = 0, & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}). \end{cases} \tag{3.8}$$

PROPOSITION 3.5. *Assume that there exist $T_0 > 0, h_0 > 0$ such that the semiclassical observability*

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C_{T_0}\int_0^{T_0}\int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + C\|u_0\|_{H^{-1}(\mathbb{T})}^2 \tag{3.9}$$

holds for any h-dependent solutions u of (3.8) *with initial data $u_0 \in L_0^2(\mathbb{T})$, uniformly for $0 < h < h_0$. Then for any $T > 0$, the observability inequality* (3.5) *holds for the λ-dependent solutions of* (3.3)*, uniformly in $\lambda \geq 0$.*

Proof. It would be sufficient to prove (3.5) when $\lambda > 1$ is large enough since for bounded $\lambda \geq 0$, (3.3) can be viewed as a pertubation of the linear KdV equation and the constant C on the right-hand side of (3.5) can be chosen to be continuously depended on λ. For $\lambda \geq \frac{1}{h_0^2}$, we write $\lambda^2 = \frac{1}{h^4}$ and (3.3) becomes

$$h^3\partial_t u + (h\partial_x)^3 u - (h\partial_x)^{-1} u = 0.$$

Setting $w(t,x) = u(h^2 t, x)$, it satisfies the equation

$$h\partial_t w + (h\partial_x)^3 w - (h\partial_x)^{-1} w = 0.$$

Now from (3.9), we have

$$\|w(0)\|_{L^2(\mathbb{T})}^2 \leq C\int_0^{T_0}\int_{\mathbb{T}} |g(x)w(t,x)|^2 dxdt + C\|w(0)\|_{H^{-1}(\mathbb{T})}^2.$$

Changing back to $u(t,x)$, it holds that

$$\|u(0)\|_{L^2(\mathbb{T})}^2 \leq \frac{C}{h^2}\int_0^{h^2 T_0}\int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(0)\|_{H^{-1}(\mathbb{T})}^2.$$

Due to the invariance of the time-translation and the conservation of the H^s norm of the linear equation, we have for any $M \in \mathbb{N}$,

$$\begin{aligned} \|u(Mh^2T_0)\|_{L^2(\mathbb{T})}^2 &= \|u(0)\|_{L^2(\mathbb{T})}^2 \\ &\leq \frac{C}{h^2}\int_{Mh^2T_0}^{(M+1)h^2T_0}\int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(Mh^2T_0)\|_{H^{-1}(\mathbb{T})}^2 \\ &= \frac{C}{h^2}\int_{Mh^2T_0}^{(M+1)h^2T_0}\int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(0)\|_{H^{-1}(\mathbb{T})}^2. \end{aligned}$$

Summing for M from 0 to $\epsilon_0 h^{-2}$, with $\epsilon_0 T_0 \leq T$, we have

$$\|u(0)\|^2_{L^2(\mathbb{T})} \leq \frac{C}{\epsilon_0}\int_0^T \int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + \frac{C}{\epsilon_0}\|u(0)\|^2_{H^{-1}(\mathbb{T})}.$$

This completes the proof of Proposition 3.5. ∎

3.2.1. Reduction to frequency-localized semiclassical observability. We use a homogeneous Littlewood–Paley decomposition. Take $\psi \in C_c^\infty(\mathbb{R})$ with support $\text{supp}(\psi) \subset \{1/2 \leq |\xi| \leq 2\}$ and $\psi_k \in C_c^\infty(\mathbb{R})$ such that

$$\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1 \quad \forall \xi \neq 0,$$

where $\psi_k(\xi) = \psi(2^k\xi)$. We will reduce the proof of the inequality (3.9) to the following.

Proposition 3.6. *There exist $\epsilon_0 > 0, h_0 > 0$, small and $T_0 > 0$, $C_0 = C_0(\epsilon_0) > 0$ such that for all $k \in \mathbb{Z}$, with $2^k h \leq \epsilon_0$,*

$$\|\psi_k(hD_x)u(0)\|^2_{L^2(\mathbb{T})} \leq C_0 \int_0^{T_0}\int_{\mathbb{T}} |g(x)\psi_k(hD_x)u(t,x)|^2 dxdt \tag{3.10}$$

holds for all solutions $u(t,x)$ of (3.8), uniformly in $h \in (0,h_0)$.

This proposition will be proved in the next subsection. In fact, from the proof, we can deduce that if Proposition 3.6 holds true for some $\epsilon_0 > 0, h_0 > 0$, it is also true for any other parameters ϵ_1, h_1 such that $\epsilon_1 < \epsilon_0$ and $h_1 < h_0$ with possible change in the dependency of constant C_0.

Lemma 3.7. *Proposition 3.6 implies the inequality (3.9).*

Indeed, applying Lemma 3.3, we have

$$\begin{aligned}\|g\psi_k(hD_x)u\|^2_{L^2(\mathbb{T})} &\leq 2\|\psi_k(hD_x)(gu)\|^2_{L^2(\mathbb{T})} + 2\|[\psi(2^k hD_x), g]u\|^2_{L^2(\mathbb{T})}\\ &\leq 2\|\psi_k(hD_x)(gu)\|^2_{L^2(\mathbb{T})} + C(2^k h)^2\|u(t)\|^2_{L^2(\mathbb{T})},\end{aligned}$$

thus

$$\begin{aligned}\sum_{k\leq \log_2(\epsilon_0/h)} \|\psi_k(hD_x)u(0)\|^2_{L^2(\mathbb{T})} &\leq C\sum_{k\leq \log_2(\epsilon_0/h)}\int_0^{T_0} \|\psi_k(hD_x)(gu(t))\|^2_{L^2(\mathbb{T})}\\ &\quad + CT_0 \sum_{k\leq \log_2(\epsilon_0/h)} (2^k h)^2 \|u(0)\|^2_{L^2(\mathbb{T})}\\ &\leq C\int_0^{T_0} \|gu(t)\|^2_{L^2(\mathbb{T})}dt + CT_0\epsilon_0^2\|u(0)\|^2_{L^2(\mathbb{T})}.\end{aligned}$$

Therefore,

$$\|u(0)\|^2_{L^2(\mathbb{T})} \leq C\int_0^{T_0}\int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + CT_0\epsilon_0^2\|u(0)\|^2_{L^2(\mathbb{T})} + C\|u(0)\|^2_{H^{-1}(\mathbb{T})}.$$

To complete the proof, we choose $\epsilon_0^2 < \frac{CT_0}{2}$ and (3.9) follows.

In summary, we have shown that in order to prove the uniform observability inequality (3.6) for all solutions of (3.3), it suffices to prove the observability (3.10) for all solutions of (3.8), uniformly in $0 < h \ll 1$ and $k \in \mathbb{Z}$ such that $2^k h < \epsilon_0$.

3.2.2. Propagation estimate with parameter dependence symbol. This section is devoted to the proof of Proposition 3.6. We recall some basic notation and results about $\tilde{h}$-pseudodifferential calculus. For $m \in \mathbb{R}$, let S^m be the set of $\tilde{h}$-dependent functions $a(x,\xi,\tilde{h})$ with parameter $\tilde{h} \in (0,1)$ such that for any indices α, β,

$$\sup_{(x,\xi,\tilde{h})\in\mathbb{R}^{2d}\times(0,1)} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi,\tilde{h})| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.$$

For $a \in S^m$, we denote by $\mathrm{Op}_{\tilde{h}}(a)$ the $\tilde{h}$-pseudodifferential operator acting on Schwartz functions via

$$\mathrm{Op}_{\tilde{h}}(a)f(x) := \frac{1}{(2\pi\tilde{h})^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-y)\cdot\xi}{\tilde{h}}} a(x,\xi,\tilde{h})f(y)dyd\xi.$$

We refer to [19] for symbolic calculus and other basic properties about the $\tilde{h}$-pseudodifferential operator. For functions on a compact Riemannian manifold, we can also define the $\tilde{h}$-pseudodifferential operator by using local coordinate and partition of unity.

Now let us consider the following ϵ-dependence symbols:

$$p_\epsilon(x,\xi) = \left(\frac{\epsilon^4}{\xi} - \xi^3\right)\chi(\xi), \quad q_\epsilon(x,\xi) = \left(\frac{1}{\xi} - \epsilon^4\xi^3\right)\chi(\xi),$$

where $\chi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\chi) \subset \{\alpha < |\xi| < \beta\}$ for some $0 < \alpha < \frac{1}{2}$, $\beta > 2$, and $\chi \equiv 1$ in a neighborhood of $\{1/2 \leq |\xi| \leq 2\}$. Denote $P_\epsilon = \mathrm{Op}_{\widetilde{h}}(p_\epsilon)$ and $Q_\epsilon = \mathrm{Op}_{\widetilde{h}}(q_\epsilon)$. Denote $U_\epsilon(t)$ and $V_\epsilon(t)$ solutions of the operator equations

$$\begin{cases} \frac{\widetilde{h}}{i}\partial_t U_\epsilon(t) + U_\epsilon(t)P_\epsilon = 0, \\ U_\epsilon(0) = I, \end{cases} \tag{3.11}$$

$$\begin{cases} \frac{\widetilde{h}}{i}\partial_t V_\epsilon(t) + V_\epsilon(t)Q_\epsilon = 0, \\ V_\epsilon(0) = I. \end{cases} \tag{3.12}$$

The flows associated to the vector fields $H_{p_\epsilon}, H_{q_\epsilon}$ are explicitly given by

$$\phi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right)\chi(\xi_0)t + \left(\frac{\epsilon^4}{\xi_0} - \xi_0^3\right)\chi'(\xi_0)t, \xi_0\right),$$

$$\varphi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{1}{\xi_0^2} + 3\epsilon^4\xi_0^2\right)\chi(\xi_0)t + \left(\frac{1}{\xi_0} - \epsilon^4\xi_0^3\right)\chi'(\xi_0)t, \xi_0\right).$$

From Egorov's theorem (see [19]), for any symbol $a(x,\xi) \in C_c^\infty(T^*\mathbb{T})$, we have

$$U_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)U_\epsilon(t) = \mathrm{Op}_{\widetilde{h}}(a\circ\phi_{\epsilon,t}) + O_{L^2\to L^2}(\widetilde{h}),$$
$$V_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)V_\epsilon(t) = \mathrm{Op}_{\widetilde{h}}(a\circ\varphi_{\epsilon,t}) + O_{L^2\to L^2}(\widetilde{h}).$$

Moreover, the remainders $O_{L^2\to L^2}(\widetilde{h})$ can be written more precisely as some operator $A_{\widetilde{h}}$ on $L^2(\mathbb{T})$ with the operator norm bounded by

$$\|A_{\widetilde{h}}\|_{L^2\to L^2} \leq C_N\widetilde{h}\sum_{|\alpha|\leq N}\widetilde{h}^{|\alpha|}\left(\sup_{(x,\xi)}|\partial_{x,\xi}^\alpha(a\circ\varphi_{\epsilon,t})| + \sup_{(x,\xi)}|\partial_{x,\xi}^\alpha(a\circ\phi_{\epsilon,t}|\right)$$

for some $N \in \mathbb{N}$ and for some $h_0 > 0$, $0 < h < h_0$. Therefore, we remark that the bounds $O_{L^2\to L^2}(\widetilde{h})$ are independent of $\epsilon \leq 1$, due to the explicit formulas of $\varphi_{\epsilon,t}$ and $\phi_{\epsilon,t}$.

Now we prove the following localized observability estimates.

PROPOSITION 3.8. *There exist* $C_0 > 0, T_0 > 0, \widetilde{h}_0 > 0$ *such that for all* $u_0 \in L^2_0(\mathbb{T})$, *all* $\widetilde{h} \le \widetilde{h}_0$,

$$\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})} \le C_0 \int_0^{T_0} \|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}dt, \tag{3.13}$$

$$\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})} \le C_0 \int_0^{T_0} \|gV_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}dt. \tag{3.14}$$

Proof. Here we only prove the first inequality, since the second one follows in the same manner. Consider the symbol $a(x,\xi) = g(x)^2\widetilde{\psi}(\xi)$ (strictly speaking, g is not smooth and we need to approximate it by smoothing functions) and its quantization $\mathrm{Op}_{\widetilde{h}}(a) = (g(x))^2\widetilde{\psi}(\widetilde{h}D_x)$, where $\widetilde{\psi}$ is a slight enlargement of ψ such that $\widetilde{\psi}\psi = \psi$ and $\operatorname{supp}\widetilde{\psi} \subset \{\alpha < |\xi| < \beta\}$. From Egorov's theorem, we have

$$U_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)U_\epsilon(t) = \mathrm{Op}_{\widetilde{h}}(a\circ\phi_{\epsilon,t}) + O_{L^2\to L^2}(\widetilde{h}), \text{ uniformly in } \epsilon \le 1.$$

Note that on the support of a, $\chi'(\xi) = 0$, so we have

$$\phi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right)t, \xi_0\right).$$

Notice that $|\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2| \ge c_0 > 0$, uniformly in ϵ, on the ξ-support of $\widetilde{\psi}$. Since $g \ge 0$ is a nonzero continuous function, for some $T_0 = T_0(c_0) > 0$, and $c_1 > 0$, we have[2]

$$\int_0^{T_0} a\circ\phi_{\epsilon,t}dt \ge c_1 > 0.$$

Now we calculate

$$\begin{aligned}
&\int_0^{T_0} \|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}dt \\
&\quad = \int_0^{T_0} \left(gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0, gU_\epsilon(t)\widetilde{\psi}(\widetilde{h}D_x)\psi(\widetilde{h}D_x)u_0\right)_{L^2(\mathbb{T})} dt \\
&\quad = \int_0^{T_0} \left(U_\epsilon(-t)\widetilde{\psi}(\widetilde{h}D_x)g^2U_\epsilon(t)u_0, \psi(\widetilde{h}D_x)u_0\right)_{L^2(\mathbb{T})} dt \\
&\quad = \left(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0, \psi(\widetilde{h}D_x)u_0\right)_{L^2(\mathbb{T})}
\end{aligned}$$

with $b_{T_0}(x,\xi) = \int_0^{T_0} a\circ\phi_{\epsilon,t}dt$ modulo $\widetilde{h}S^0$. Thus, from the sharp Gårding inequality (see [19]), we have

$$\left(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0, \psi(\widetilde{h}D_x)u_0\right)_{L^2(\mathbb{T})} \ge \frac{c_1}{2}\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})} - C\widetilde{h}\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}.$$

To conclude the proof, we choose $\widetilde{h}_0 < \min\{\frac{c_1}{4C}, 1\}$. ∎

[2]This is just the geometric control condition, which is trivially satisfied in the one-dimensional situation.

Proof of Proposition 3.6. For fixed $h \ll 1$, we analyze the three regimes for $k \in \mathbb{Z}$.

Case 1*: $|k| \leq N_0$ for some large natural number N_0.* This corresponds to the case $|\xi| \sim 1$. Let $u_k = \psi_k(hD_x)u$; the equation satisfied by u_k is (3.8). We can use either (3.13) or (3.14) with parameter $\epsilon = 1$ to obtain that (note that $\widetilde{h} = 2^k\widetilde{h} \sim h$ in this regime)

$$\|\psi_k(hD_x)u_0\|^2_{L^2(\mathbb{T})} \leq C_0 \int_0^{T_0} \|g\psi_k(hD_x)u(t)\|^2_{L^2(\mathbb{T})}dt.$$

Case 2*: $k \leq -N_0$ for some large constant N_0.* This case corresponds to $|\xi| \sim 2^{-k} \gg 1$. Defining a new semiclassical parameter $\widetilde{h}_k = 2^k h \ll 1$ and to rescale the time variable we set $w_k(t,x) := \psi(\widetilde{h}_k D_x)u(2^{2k}t, x)$ and $u_k = \psi(\widetilde{h}_k D_x)u$. The equation satisfied by w_k is

$$\widetilde{h_k}\partial_t w_k + (\widetilde{h_k}\partial_x)^3 w_k + 2^{4k}(\widetilde{h}\partial_x)^{-1}w_k = 0.$$

Applying (3.13) to w_k with $\epsilon = 2^k \ll 1$ and $\widetilde{h} = \widetilde{h}_k$ we obtain

$$\|w_k(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|gw_k(t)\|^2_{L^2(\mathbb{T})}dt.$$

From conservation of the L^2 norm, we apply the inequality above $2^{-2k} - 1$ times and obtain that

$$\begin{aligned}
\frac{1}{2^{2k}}\|u_k(0)\|^2_{L^2(\mathbb{T})} &\leq \frac{C}{2^{2k}} \sum_{M=0}^{2^{-2k}-1} \int_{M2^{2k}T_0}^{(M+1)2^{2k}T_0} \|gu_k(t)\|^2_{L^2(\mathbb{T})}dt \\
&= \frac{C}{2^{2k}} \int_0^{T_0} \|gu_k(t)\|^2_{L^2(\mathbb{T})}dt.
\end{aligned}$$

This is exactly

$$\|\psi_k(hD_x)u(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|g\psi_k(hD_x)u(t)\|^2_{L^2(\mathbb{T})}dt.$$

Case 3*: $k \geq N_0$.* This case corresponds to $|\xi| \sim 2^{-k} \ll 1$. Define the new small semiclassical parameter $\tilde{h}_k = 2^k h$. The $\widetilde{h}$-pseudodifferential calculus applies, by the restriction $2^k h \leq \epsilon_0 \ll 1$.

Denote by $u_k = \psi(\widetilde{h}_k D_x)u$ and define $v_k(t,x) = u_k(2^{-2k}t, x)$. v_k solves the equation

$$\widetilde{h}_k\partial_t v_k + 2^{-4k}(\widetilde{h}_k\partial_x)^3 v_k + (\widetilde{h}_k\partial)^{-1}v_k = 0.$$

Applying (3.14) with $\widetilde{h} = \widetilde{h}_k, \epsilon = 2^{-k}$, we obtain that

$$\|v_k(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|gv_k(t)\|^2_{L^2(\mathbb{T})}dt.$$

Again by conservation of the L^2 norm as in the argument of Case 2, we finally have

$$\|u_k(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|gu_k(t)\|^2_{L^2(\mathbb{T})}dt.$$

This completes the proof of Proposition 3.6. Hence the proof of Proposition 3.2 and the observability inequality (3.2) for the linearized KP-II equation are also complete. ∎

As a consequence of Proposition 3.1, the internal controllability for the linear KP II is obtained. We conclude this section by summarizing it in the following proposition.

PROPOSITION 3.9. *Given $T > 0$, there exists a bounded linear operator*

$$\Upsilon : (L_0^2(\mathbb{T}^2))^2 \to L^2(0, T; L^2(\mathbb{T}^2))$$

such that for any $u_0, u_1 \in L_0^2(\mathbb{T}^2)$, the control defined by $\mathfrak{h} := \Upsilon(u_0, u_1)$ drives the solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{G}\mathfrak{h}, & (t, x) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0, \end{cases} \tag{3.15}$$

to $u(T) = u_1$. Moreover, we have

$$\|\Upsilon(u_0, u_1)\|_{L^2(0,T;L^2(\mathbb{T}^2))} \leq C\|(u_0, u_1)\|_{(L^2(\mathbb{T}^2))^2}.$$

4. Local controllability of nonlinear equation. For the full KP-II control system

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}\mathfrak{h}, & (t, x) \in \mathbb{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0,\ u|_{t=T} = u_1, \end{cases} \tag{4.1}$$

in order to prove the existence of $u \in L^2(0, T; L_0^2(\mathbb{T}^2))$ solving $u|_{t=0} = u_0$, $u|_{t=T} = u_1$, we will reduce it to a fixed point problem by a standard argument.

Proof of Theorem 1.2. The solution of (4.1) with control input h is given by

$$u(t) = S(t)u_0 + v(t, u) + \int_0^t S(t - t')\mathcal{G}\mathfrak{h}(t')dt'$$

with

$$v(t, u) = \int_0^t S(t - t')u\partial_x u dt'.$$

It must satisfy

$$u_1 = S(T)u_0 + v(T, u) + \int_0^T S(T - t')\mathcal{G}\mathfrak{h}(t')dt.$$

Choosing the control input of the form $\mathfrak{h} = \Upsilon(u_0, w)$, this implies that

$$S(T)u_0 + \int_0^T S(T - t')\mathcal{G}\mathfrak{h}(t')dt' = w.$$

This indicates that $w = u_1 - v(T, u)$. In summary, defining the nonlinear map Γ by

$$\Gamma(u) = S(t)u_0 + v(t, u) + \int_0^t S(t - t')\mathcal{G}\mathfrak{h}_u(t')dt'$$

with

$$\mathfrak{h}_u = \Upsilon(u_0, u_1 - v(T, u)),$$

we need to find a fixed point of Γ.

We need show that $\Gamma : X_T^{0,\frac{1}{2},b_1} \cap Z_T^{0,\frac{1}{2}} \to X_T^{0,\frac{1}{2},b_1} \cap Z_T^{0,\frac{1}{2}}$ is a contraction in a bounded ball. From Propositions 2.4 and 2.3, we have

$$
\begin{aligned}
\|\Gamma(u)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} &\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)}+\|\mathcal{G}\mathfrak{h}_u\|_{X_T^{0,-\frac{1}{2},b_1}}+\|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right)\\
&\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)}+\|u_1\|_{L^2(\mathbb{T}^2)}+\|\upsilon(T,u)(T)\|_{L^2(\mathbb{T}^2)}+\|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right)\\
&\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)}+\|u_1\|_{L^2(\mathbb{T}^2)}+\|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right),
\end{aligned}
$$

where $C>0$ does not depend on u_0. For $R>0$, let $B_R=B_R(0)$ be the ball centered at zero with radius R, that is,

$$
B_R:=\left\{u\in X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}:\|u\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}<R\right\}.
$$

Then

$$
\|\Gamma(u)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)}+\|u_1\|_{L^2(\mathbb{T}^2)}+R^2\right). \tag{4.2}
$$

Additionally, for $u,v\in B_R$ we have

$$
\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} &\leq C\left\|\int_0^t S(t-\tau)(\mathcal{G}\mathfrak{h}_u-\mathcal{G}\mathfrak{h}_v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&\quad+\left\|\int_0^t S(t-t')(u\partial_x u-v\partial_x v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&\leq C\left\|\Upsilon(u_0,u_1-\upsilon(T,u))-\Upsilon(u_0,u_1-\upsilon(T,v))\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&\quad+C\left\|\int_0^t S(t-t')(u\partial_x u-v\partial_x v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&\leq C\|\upsilon(T,u)-\upsilon(T,v)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&\quad+\|u-v\|_{X_T^{0,\frac{1}{2},b_1}}\|u+v\|_{X_T^{0,\frac{1}{2},b_1}}\\
&\leq C\|u-v\|_{X_T^{0,\frac{1}{2},b_1}}\|u+v\|_{X_T^{0,\frac{1}{2},b_1}}\\
&\leq \frac{1}{2}\|u-v\|_{X_T^{0,\frac{1}{2},b_1}} \qquad (4.3)
\end{aligned}
$$

by using properties of the bounded linear operator Υ. Choose $\delta>0$ and $R>0$ such that $2C\delta+CR^2\leq R$ and $CR<\frac{1}{2}$ with $\|u_0\|_{L^2(\mathbb{T}^2)}<\delta$ and $\|u_1\|_{L^2(\mathbb{T}^2)}<\delta$. We can conclude from (4.2) that the image of B_R through Γ stays in the ball B_R and from (4.3) that Γ is a contraction. The proof of Theorem 1.2 is complete. ∎

5. Noncontrollability in horizontal strip. In this section, we prove Theorem 1.4 by disproving the observability for the linearized KP-II equation (1.5) on the horizontal control region. By translation, we may assume that the horizontal control region is $\omega=(-\pi,-\alpha)\cup(\alpha,\pi]$ for some $0<\alpha<\pi$. Recall that $\mathcal{K}$ is defined by (1.4). By the HUM method, the proof of Theorem 1.4 reduces to prove the following.

PROPOSITION 5.1. *For any $T>0$, there does not exist a finite constant $C_T>0$ such that the observability inequality*

$$
\|u(0)\|^2_{L^2(\mathbb{T}^2)}\leq C_T\int_0^T\int_{\mathbb{T}^2}|\mathcal{K}u(t,x,y)|^2dxdydt \tag{5.1}
$$

holds for every solution $u \in L^2((0,T); L_0^2(\mathbb{T}^2))$ *of the linearized KP-II equation*

$$\partial_t u + \partial_x^3 u + \partial_x^{-1} \partial_y^2 u = 0.$$

The building block for proving Proposition 5.1 is the following lemma for the one-dimensional semiclassical Schrödinger equation.

LEMMA 5.2. *Assume that* $\omega = (-\pi, -\alpha) \cup (\alpha, \pi]$ *for* $0 < \alpha < \pi$. *Then for any* $T > 0$, *there exists a sequence of solutions* u_n *to*

$$\begin{cases} ih_n \partial_t u_n + h_n^2 \partial_x^2 u_n = 0, \\ u_n|_{t=0} = u_{n,0} \in L^2(\mathbb{T}), \end{cases} \tag{5.2}$$

such that

$$\liminf_{n\to\infty} \|u_{n,0}\|_{L^2(\mathbb{T})} > 0$$

and

$$\lim_{n\to\infty} \int_0^T \int_\omega |u_n(t,x)|^2 dx dt = 0.$$

Proof. Take $G(x) = e^{-\frac{x^2}{2}}$ and define $G^{\epsilon_n}(x) = \frac{1}{\sqrt{\epsilon_n}} G(\frac{x}{\epsilon_n})$. Denote the Fourier coefficient of G^{ϵ_n} by

$$g^{\epsilon_n}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G^{\epsilon_n}(x) e^{-ikx} dx = \frac{\sqrt{\epsilon_n}}{2\pi} \int_{-\frac{\pi}{\epsilon_n}}^{\frac{\pi}{\epsilon_n}} G(z) e^{-i\epsilon_n k z} dz.$$

The coefficient function $g^{\epsilon_n}(z)$ satisfies the following estimates:

$$\|g^{\epsilon_n}\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{1/2}),\ \|(g^{\epsilon_n})'\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{3/2}),\ \|(g^{\epsilon_n})''\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{5/2}). \tag{5.3}$$

Take an even cut-off function $\psi \in C_c^\infty(\mathbb{R})$ with supp $\psi \subset [-B, B]$ with $0 < b < B$ and $0 \le \psi \le 1$, $\psi(z) \equiv 1$, for all $|z| \le b$. We define

$$u_{n,0}(x) = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k) \psi(h_n k) e^{ikx},$$

and then the corresponding solution to (5.2) is given explicitly by

$$u_n(t,x) = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k) \psi(h_n k) e^{i(kx - k^2 h_n t)}.$$

We need to estimate the mass of initial data. First,

$$\|G^{\epsilon_n}\|_{L^2(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} |g^{\epsilon_n}(k)|^2 \sim 1$$

holds from the Plancherel theorem and the definition of $g^{\epsilon_n}(k)$. We next estimate the mass away from the frequency scale h_n^{-1}, that is,

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |(1 - \psi(h_n k)) g^{\epsilon_n}(k)|^2 &\le \sum_{|k| > h_n^{-1} b} |g^{\epsilon_n}(k)|^2 \\ &\le \sum_{|k| > h_n^{-1} b} \frac{\epsilon_n}{4\pi^2} \left| \int_{\mathbb{R}} G(z) e^{-ik\epsilon_n} z dz \right|^2 \end{aligned}$$

$$= \sum_{|k|>h_n^{-1}b} \frac{\epsilon_n}{4\pi^2} \left| \int_{\mathbb{R}} G(z) \frac{1}{-ik\epsilon_n} \frac{d}{dz} e^{-ik\epsilon_n} z dz \right|^2$$
$$\leq \sum_{|k|>h_n^{-1}b} \frac{1}{4k^2\pi^2\epsilon_n} \|G'\|^2_{L^1(\mathbb{R})}.$$

By setting $\epsilon_n = \sqrt{h_n} \ll 1$, we have $\|(1-\psi(h_n D_x))G^{\epsilon_n}\|_{L^2(\mathbb{T})} \ll 1$ and then $\|u_{n,0}\|_{L^2(\mathbb{T})} \sim 1$. It remains to estimate the term on the right-hand side of observability inequality (5.1).

Observe that $u_{n,0}$ is localized by $|k| \leq \frac{B}{h_n}$ in frequency and by $|x| \leq \epsilon_n$ in space obeying the uncertainty principle ($\epsilon_n h_n^{-1} \gtrsim 1$). Since the wave packet of the frequency scale smaller than Bh_n^{-1} moves at velocity bigger than $2Bh_n^{-1}$, it will remain small for $|t| < T$ in ω. More precisely, we need a decay estimate for $|u_n(t,x)|$ when $x \in \omega$ and $|t| < T$. Now we choose $B > 0$ such that $|x - 2Bt| \geq c_0 > 0 \mod 2\pi$ for all $x \in \omega$ and $|t| \leq T$. Write

$$u_n(t,x) = \sum_{k\in\mathbb{Z}} K_{t,x}^{(n)}(k)$$

with

$$K_{t,x}^{(n)}(z) = g^{\epsilon_n}(z)\psi(h_n z) e^{i(zx - h_n z^2 t)}.$$

From the Poisson summation formula, we have

$$u_n(t,x) = \sum_{m\in\mathbb{Z}} \widehat{K_{t,x}^{(n)}}(2\pi m).$$

For fixed $m \in \mathbb{Z}$,

$$\widehat{K_{t,x}^{(n)}}(2\pi m) = \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_n z) e^{i\varphi_{t,x}(z)} dz$$
$$= \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_n z) \mathcal{L}^2(e^{i\varphi_{t,x}(z)}) dz$$

with $\mathcal{L} = \frac{1}{i\varphi'_{t,x}(z)} \frac{d}{dz}$ and $\varphi_{t,x}(z) = (x - 2\pi m)z - h_n z^2 t$. By integration by parts, we have

$$\widehat{K_{t,x}^{(n)}}(2\pi m) = \int_{\mathbb{R}} \frac{d}{dz} \left(\frac{1}{i\varphi'_{t,x}(z)} \frac{d}{dz} \left(\frac{g^{\epsilon_n}(z)\psi(h_n z)}{i\varphi'_{t,x}(z)} \right) \right) e^{i\varphi_{t,x}(z)} dz.$$

After tedious calculation, we obtain that

$$\frac{d}{dz} \left(\frac{1}{i\varphi'_{t,x}(z)} \frac{d}{dz} \left(\frac{g^{\epsilon_n}(z)\psi(h_n z)}{i\varphi'_{t,x}(z)} \right) \right)$$
$$= \frac{(g^{\epsilon_n})''\psi(h_n z) + 2h_n (g^{\epsilon_n})'\psi'(h_n z) + h_n^2 \psi''(h_n z) g^{\epsilon_n}}{(\varphi'_{t,x})^2}$$
$$- \frac{3((g^{\epsilon_n})'\psi(h_n z) + h_n \psi'(h_n z) g^{\epsilon_n})\varphi''_{t,x}}{(\varphi'_{t,x})^3} - \frac{3g^{\epsilon_n}\psi(h_n z)(\varphi''_{t,x})^2}{(\varphi'_{t,x})^4}.$$

From (5.3), we have

$$|\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq \sup_{|h_n z|\leq B} \frac{C\epsilon_n^{1/2}\|\psi\|_{W^{2,1}(\mathbb{R})}}{|(x - 2h_n z t) - 2\pi m|^2}.$$

For any $x \in 2\pi p + (-\pi, -\alpha) \cup (\alpha, \pi]$, $|x - 2h_n zt| \geq c_0 > 0 \bmod 2\pi$ with $p \in \mathbb{Z}$, it holds that

$$\sum_{m\in\mathbb{Z}} |\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq C \sum_{m\in\mathbb{Z}} \frac{C\epsilon_n^{1/2}}{|c_0 - 2\pi(m-p)|^2} \leq C\epsilon_n^{1/2}.$$

Therefore,

$$\int_0^T \int_\omega |u_n(t,x)|^2 dxdt \leq C\epsilon_n^{1/2} T|\omega| \to 0, \text{ as } n \to \infty.$$

This completes the proof of Lemma 5.2. ∎

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. For any $T > 0$, we will construct a sequence of solutions u_n to the linearized KP-II equation such that

$$\|u_n(0)\|_{L^2(\mathbb{T}^2)} \sim 1 \quad \text{and} \quad \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^2} |\mathcal{K}u_n(t,x,y)|^2 dxdydt = 0.$$

Denote by $v_n(t,y)$ the sequence of solutions to the semiclassical Schrödinger equation which satisfies the conditions in Lemma 5.2. Define

$$u_n(t,x,y) = v_n(t,y) e^{\frac{it}{h_n^3}} e^{\frac{ix}{h_n}} = \sum_{k\in\mathbb{Z}} \widehat{v_n}(k) e^{i(ky - h_n k^2 t)} e^{i(\frac{x}{h_n} + \frac{t}{h_n^3})}.$$

Then u_n solves the linearized KP-II equation. Moreover,

$$\|u_n(0)\|_{L^2(\mathbb{T}^2)} = \|v_n(0)\|_{L^2(\mathbb{T})} \sim 1,$$

and

$$\int_0^T \int_\omega |u_n(t,x,y)|^2 dxdydt = \int_0^T \int_{(-\pi,\alpha)\cup(\alpha,\pi]} |v_n(t,y)|^2 dtdy \to 0, \text{as } n \to \infty.$$

Now we claim that

$$\lim_{n\to\infty} \int_{\mathbb{T}} g(y')v_n(t,y')dy' \to 0 \text{ in } L^\infty([0,T]; L^2(\mathbb{T})).$$

Indeed,

$$\begin{aligned}
\left| \int_{\mathbb{T}} g(y')v_n(t,y')dy' \right| &= \left| \sum_{k\in\mathbb{Z}} \overline{\widehat{g}(k)} g^{\epsilon_n}(k)\psi(h_n k) e^{-ik^2 t} \right| \\
&= \left| \left(\sum_{|k|\leq M} + \sum_{|k|>M} \right) \overline{\widehat{g}(k)} g^{\epsilon_n}(k)\psi(h_n k) e^{-ik^2 t} \right| \\
&\leq \epsilon_n^{1/2} \|g\|_{L^2(\mathbb{T})} M^{1/2} + \|G^{\epsilon_n}\|_{L^2(\mathbb{T})} \left(\sum_{|k|>M} |\widehat{g}(k)|^2 \right)^{1/2}
\end{aligned}$$

and the right-hand side tends to 0 as $n \to \infty$ since we can choose M to be arbitrarily large before taking the limit in n. The validity of the claim implies that $g(y) \int_{\mathbb{T}} g(y')u_n(t,x,y')dy' \to 0$ in $L^2([0,T] \times \mathbb{T}^2)$. This completes the proof of Proposition 5.1, as well as Theorem 1.4. ∎

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