

CERTIFYING UNSTABILITY OF SWITCHED SYSTEMS USING* SUM OF SQUARES PROGRAMMING

BENOÎT LEGAT[†], PABLO A. PARRILO[‡], AND RAPHAËL M. JUNGERS[§]

Abstract. The joint spectral radius (JSR) of a set of matrices characterizes the maximal asymptotic growth rate of an infinite product of matrices of the set. This quantity appears in a number of applications including the stability of switched and hybrid systems. A popular method used for the stability analysis of these systems searches for a Lyapunov function with convex optimization tools. We investigate dual formulations for this approach and leverage these dual programs for developing new analysis tools for the JSR.

We show that the dual of this convex problem searches for the *occupations measures* of trajectories with high asymptotic growth rate. We both show how to generate a sequence of guaranteed high asymptotic growth rate and how to detect cases where we can provide lower bounds to the JSR. We deduce from it a new guarantee for the upper bound provided by the sum of squares Lyapunov program. We end this paper with a method to reduce the computation of the JSR of low rank matrices to the computation of the constrained JSR of matrices of small dimension.

All results of this paper are presented for the general case of constrained switched systems, that is, systems for which the switching signal is constrained by an automaton.

Key words. Joint spectral radius, Sum of squares programming, Switched Systems, Path-complete Lyapunov functions

AMS subject classifications. 93D05, 93D20, 93D30

1. Introduction. In recent years, the study of the stability of hybrid systems has been the subject of extensive research using methods based on classical ideas from Lyapunov theory and modern mathematical optimization techniques. Even for switched linear systems, arguably the simplest class of hybrid systems, determining stability is undecidable and approximating the maximal asymptotic growth rate that a trajectory can have is NP-hard [6]. Despite these negative results, the vast range of applications has motivated a wealth of algorithms to approximate this maximal asymptotic growth rate.

A switched linear system is characterized by a finite set of matrices $\mathcal{A} \triangleq \{A_1, A_2, \dots, A_m\} \subset \mathbb{R}^{n \times n}$ and the iteration

$$(1) \quad x_k = A_{\sigma_k} x_{k-1}, \quad \sigma_k \in [m].$$

The maximal asymptotic growth rate of this iteration is given by the *joint spectral radius* (JSR). The JSR $\rho(\mathcal{A})$ of a finite set of matrices \mathcal{A} is defined as

$$\rho(\mathcal{A}) = \lim_{k \rightarrow \infty} \max_{\sigma \in [m]^k} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\|^{1/k}.$$

This definition is independent of the norm used.

The JSR was introduced by Rota and Strang [33] and has many other applications such as wavelets, the capacity of some particular codes, zero-order stability of ordinary differential equations, congestion control in computer networks, curve design

*Submitted to the editors September 28, 2017. A preliminary version of this work appeared in Proceedings of Hybrid Systems: Computation and Control, 2016 [24].

[†]ICTEAM, Université catholique de Louvain, 4 Av. G. Lemaître, 1348 Louvain-la-Neuve, Belgium (benoit.legat@uclouvain.be).

[‡]Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge MA 02139, USA (parrilo@mit.edu).

[§]ICTEAM, Université catholique de Louvain, 4 Av. G. Lemaître, 1348 Louvain-la-Neuve, Belgium (raphael.jungers@uclouvain.be).

and networked and delayed control systems; see [17] for a survey on the JSR and its applications. Many algorithms exist for estimating the JSR but not much is known on how to generate an infinite sequence of matrices with an asymptotic growth rate close to the JSR. However generating such sequence can be of particular interest, depending on the application, such as exhibiting unstable trajectories for switched linear systems. The currently known algorithms generate a sequence of matrices with high spectral radius using brute force (or branch-and-bound variants) and repeat this sequence infinitely [12, 13, 18].

Approximating the JSR usually consists in certifying upper bounds $\bar{\gamma}$ to the JSR by exhibiting a Lyapunov function or invariant set for the matrices $A_i/\bar{\gamma}$. The search for such Lyapunov functions can naturally be written as a convex optimization program; see Program 2.2. Certifying lower bounds $\underline{\gamma}$ is currently either achieved using the guarantees we have on the accuracy of the upper bound to the JSR or by exhibiting trajectories of asymptotic growth rate $\underline{\gamma}$. In this paper, we introduce a new way to certify lower bounds by exhibiting nonnegative measures satisfying some invariance condition parametrized by the matrices $A_i/\underline{\gamma}$; see (9). This invariance condition is linear on the measure hence the search of measures on the convex cone of nonnegative measures is a *convex* program; see Program 2.3. It turns out that this program is the dual of Program 2.2.

We revisit the sum-of-squares program proposed by Parrilo and Jadbabaie [28] and show that its dual formulation is the moment relaxation of the search of the measures satisfying the invariance condition.

Thanks to this duality, solving this pair of programs with a given candidate value γ for the JSR either returns Lyapunov functions certifying that $\rho(\mathcal{A}) \leq \gamma$ or returns moments that are solution of the moment relaxation. These moments are not necessarily the moments of measures satisfying the invariance conditions. However, we give a rounding procedure to extract a (infinite) switching sequence from these moments and provide a guarantee on the asymptotic growth rate of this sequence. As a by-product of the rounding procedures, the spectral radius of a finite part of this infinite sequence can be used to give lower bounds on the JSR. In addition, we give a way to sometimes detect when the moments belongs to measures that satisfy the invariance conditions. This happens when the measures are the convex combination of the occupation measures of several periodic trajectories. Since the trajectories are periodic, the measures are atomic and we can recover them from moments of sufficiently high degree. We show on numerical examples that these techniques work well in practice.

In some applications the values that σ_k can take in (1) may depend on $\sigma_{k-1}, \sigma_{k-2}, \dots$. These constraints are often conveniently represented using a *finite automaton* and the JSR under such constraints is called *constrained joint spectral radius* (CJSR) [9]; an example of constrained switched system is given by Example 1.1 and its automaton is illustrated by Figure 1.

The following will serve as a running example.

Example 1.1 (Running example¹). We borrow the example of [30, Section 4].

¹The source code and instructions are available at the author's web site to reproduce the numerical results obtained for the running example. The switched system considered in Example 3.5, Example 4.2 and Example 4.9 is simpler than the running example and the results given in these examples can be obtained by hand.

The set of matrices \mathcal{A} is composed of the following four matrices

$$\begin{aligned} A_1 &= A + B \begin{pmatrix} k_1 & k_2 \\ 0 & 0 \end{pmatrix}, & A_2 &= A + B \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix}, \\ A_3 &= A + B \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix}, & A_4 &= A. \end{aligned}$$

where $k_1 = -0.49$, $k_2 = 0.27$,

$$A = \begin{pmatrix} 0.94 & 0.56 \\ 0.14 & 0.46 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The automaton is represented by Figure 1.

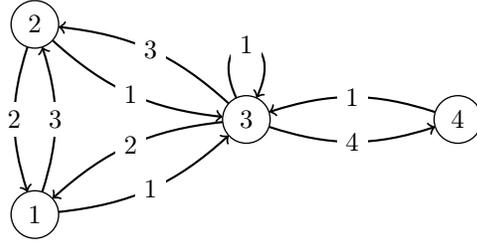


Figure 1: Automaton for the running example. The numbers on the edges are their respective labels.

The automaton representing the constraints can be represented by a strongly connected labelled directed graph $G(V, E)$, possibly with parallel edges. The labels are elements of the set $[m]$ and E is a subset of $V \times V \times [m]$. We say that $(u, v, \sigma) \in E$ if there is an edge between node u and node v with label σ .

$$(2) \quad x_k = A_{\sigma_k} x_{k-1}, \quad (\sigma_1, \dots, \sigma_k) \in G_k.$$

The *arbitrary switching* case (1) can be seen as the particular case when the automaton has only one node and m self-loops with labels $1, \dots, m$. On the other side, any constrained switched system can be replaced by an arbitrary switching system with the same CJSR; see [Lemma B.12](#). Our techniques are well suited for analysing the more general constrained systems as well.

We also provide a new estimate of the accuracy of the SOS-based approximation algorithm for the CJSR which is better than the previously existing one for sufficiently large SOS degree. The existing estimate only depends on the dimension of the matrices while our new one relates the accuracy of the SOS-based approximation algorithm with the combinatorial structure of the automaton representing the constraints.

In [1], Ahmadi and Parrilo show how to reduce the computation of the JSR of low rank matrices to a combinatorial problem, the CJSR of 1×1 matrices (i.e. scalars). As a final contribution, we generalize this approach and give a reduction of the computation of the JSR (or CJSR) of rank r matrices to the computation of the CJSR of $r \times r$ matrices.

The paper is organized as follows. In Section 2, we give the program searching for Lyapunov functions, the program searching for measures satisfying the invariance condition, prove the duality between the two programs and show that they respectively provides upper and lower bounds to the CJSR.

In Section 3, we give the SOS program searching for Lyapunov functions and we give our new estimate for its accuracy. The new bounds explicitly depend on the allowable transitions, through the graph $G(V, E)$.

In Section 4, we give the moment relaxation of the program searching for moment measures satisfying the invariance condition. We both show how to generate the sequence of high asymptotic growth rate and detect cases where we can provide lower bounds to the CJSR as mentioned in this introduction.

In Section 5, we give the low rank reduction mentioned above.

Notations. We define the automaton $G^\top(V, E^\top)$ where $E^\top = \{(v, u, \sigma) : (u, v, \sigma) \in E\}$.

We denote as E_k the subset of E^k that represents valid paths of length k . The k -tuple $(\sigma_1, \sigma_2, \dots, \sigma_k)$ is said to be G -admissible if $\sigma_1, \dots, \sigma_k$ are the respective labels of a path of length k . The *arbitrary switching* case, that is, when every tuple is G -admissible, can be seen as the particular case when the automaton has only one node and m self-loops with labels $1, \dots, m$. We denote the set $\{1, \dots, m\}$ as $[m]$ and the set of all k -tuples of $[m]^k$ that are G -admissible as G_k . The sequence $\sigma_1, \sigma_2, \dots$ is G -admissible (resp. G^\top -admissible) if $(\sigma_1, \dots, \sigma_k)$ (resp. $(\sigma_k, \dots, \sigma_1)$) is G -admissible for any $k \geq 1$. We denote $A_{\sigma_k} \cdots A_{\sigma_1}$ as A_s where $s = (\sigma_1, \dots, \sigma_k)$ or s is a path with these respective labels.

To shorten the notation we denote the i th node of a path s as $s(i)$ and the i th edge as $s[i]$. Also, for a given k -tuple s , we denote $(s(i), \dots, s(k))$ by $s(i :)$. We define

$$\begin{aligned} E_k(u, v) &= \{s \in E_k^- \mid s(1) = u, s(k+1) = v\} \\ E_k^-(v) &= \{s \in E_k^- \mid s(k+1) = v\} \\ E_k^+(v) &= \{s \in E_k^+ \mid s(1) = v\} \\ E_k^-[e] &= \{s \in E_k^- \mid s[k] = e\} \\ E_k^+[e] &= \{s \in E_k^+ \mid s[1] = e\}. \end{aligned}$$

We denote the indegree (resp. outdegree) of a node $v \in V$ as $d^-(v)$ (resp. $d^+(v)$) and the maximum indegree (resp. outdegree) of G as $\Delta^-(G) = \max_{v \in V} d^-(v)$ (resp. $\Delta^+(G) = \max_{v \in V} d^+(v)$). We also denote the number of paths of length k ending (resp. starting) at a node $v \in V$ as $d_k^-(v) \triangleq |E_k^-(v)|$ (resp. $d_k^+(v) \triangleq |E_k^+(v)|$) and define $\Delta_k^-(G) = \max_{v \in V} d_k^-(v)$ and $\Delta_k^+(G) = \max_{v \in V} d_k^+(v)$. Note that $\Delta_1^-(G) = \Delta^-(G)$, $\Delta_1^+(G) = \Delta^+(G)$ and for any k , $\Delta_k^+(G^\top) = \Delta_k^-(G)$.

2. Instability certificate using measures. The definition of the JSR is generalized as follows for constrained systems.

DEFINITION 2.1 ([9]). *The constrained joint spectral radius (CJSR) of a finite set of matrices \mathcal{A} constrained by an automaton G , denoted as $\rho(G, \mathcal{A})$, is*

$$(3) \quad \limsup_{k \rightarrow \infty} \rho_k(G, \mathcal{A}) = \rho(G, \mathcal{A}) = \lim_{k \rightarrow \infty} \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$$

where

$$(4) \quad \rho_k(G, \mathcal{A}) = \max \{ \rho(c) : c \in G_k, c \text{ is a cycle} \}, \quad \rho(c) = [\rho(A_c)]^{1/k},$$

and

$$(5) \quad \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|) = \max \{ \|A_s\|^{1/k} : s \in G_k \}.$$

We can readily see that

$$\rho_k(G, \mathcal{A}) \leq \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$$

for any k and norm $\|\cdot\|$. Equality (3) is called the *Joint Spectral Radius Theorem* and was proved in 1992 by Berger and Wang [4] in the unconstrained case. Elsner [10] provided a somewhat simpler self contained proof in 1995. Both proofs use rather involved results on the joint spectral radius.

A popular method for proving stability of a dynamical system is to find a Lyapunov function. In this section, we introduce measures playing a role dual to Lyapunov function for switched system. These measures provide a certificate for instability. Finding Lyapunov functions and finding these measures are in fact two dual programs, they are respectively provided by Program 2.2 and Program 2.3. We will be succinct in our definition of measure-theoretic concepts but the interested reader can find an good introduction to writing programs using measures and functions as decision variables in [22].

Consider the dual pair $(\mathcal{B}, \mathcal{M})$ where \mathcal{B} is the space of bounded measurable functions on \mathbb{S}^{n-1} and \mathcal{M} is the space of *finite*² *signed*³ Borel measures on \mathbb{S}^{n-1} . Given a function $f(x) \in \mathcal{B}$, we can define the homogeneous⁴ function $h(f) \triangleq x \mapsto \|x\|_2 f(x/\|x\|_2)$ on \mathbb{R}^n . We define $\mathcal{F} = \{h(f) \mid f \in \mathcal{B}\}$ with the scalar product $\langle h(f), \mu \rangle = \langle f, \mu \rangle$ for $f \in \mathcal{B}, \mu \in \mathcal{M}$.

Given an application A and a measure $\mu \in \mathcal{M}$, the *pushforward measure* $A\#\mu$ is often defined to be the measure given by $(A\#\mu)(B) = \mu(A^{-1}(B))$ for $B \in \mathbb{S}^{n-1}$. However, since \mathbb{S}^{n-1} may not be invariant under application of the matrices of \mathcal{A} , we will use an alternative definition. Given an application A and a measure μ , the pushforward measure $A\#\mu$ is defined to be the measure such that $\langle f, A\#\mu \rangle = \langle f \circ A, \mu \rangle$ for any $f \in \mathcal{F}$. Moreover, given $B \subseteq \mathbb{S}^{n-1}$, we define $\mu(B) = \langle x \mapsto \|x\|_2 \mathbf{1}_B(x), \mu \rangle$ so that $(A\#\mu)(B)$ is well defined. Using these definition, one can verify that for any application A , measure $\mu \in \mathcal{M}$ and set $B \subseteq \mathbb{S}^{n-1}$,

$$(6) \quad A\#\mu(B) \leq \mu(B) \max_{x \in B} \|Ax\|_2.$$

Let \mathcal{F}_+ (resp. \mathcal{B}_+) be the set of nonnegative functions of \mathcal{F} (resp. \mathcal{B}), \mathcal{M}_+ be the set of (nonnegative) measures of \mathcal{M} and \mathcal{F}_{++} be the set of positive functions of \mathcal{F} . Given two functions $f, g \in \mathcal{F}$, $f \geq 0$ denotes $f \in \mathcal{F}_+$ and $f \geq g$ denotes $f - g \in \mathcal{F}_+$. Similarly, given two measures $\mu, \nu \in \mathcal{M}$, $\mu \geq 0$ denotes $\mu \in \mathcal{M}_+$ and $\mu \geq \nu$ denotes $\mu - \nu \in \mathcal{M}_+$.

Program 2.2 (Primal).

$$(7) \quad \inf_{f_v \in \mathcal{F}, \bar{\gamma} \in \mathbb{R}} \bar{\gamma} \\ f_v(A_\sigma x) \leq \bar{\gamma} f_u(x), \quad \forall (u, v, \sigma) \in E, \\ f_v(x) \in \mathcal{F}_{++}, \quad \forall v \in V,$$

$$(8) \quad \sum_{v \in V} \int_{\mathbb{S}^{n-1}} f_v(x) dx = 1.$$

²The measure μ is *finite* if $\mu(\mathbb{S}^{n-1})$ is finite.

³A *signed* measure is a difference between two measures, i.e. $\mu - \nu$ where μ and ν are measures is a signed measure.

⁴A function f is homogeneous if $f(\alpha x) = \alpha f(x)$ for any scalar value α .

Program 2.3 (Dual of [Program 2.2](#)).

$$(9) \quad \sup_{\mu_{uv\sigma} \in \mathcal{M}, \underline{\gamma} \in \mathbb{R}} \underline{\gamma} \\ \sum_{(u,v,\sigma) \in E} A_\sigma \# \mu_{uv\sigma} \geq \underline{\gamma} \sum_{(v,w,\sigma) \in E} \mu_{vw\sigma}, \quad \forall v \in V,$$

$$(10) \quad \mu_{uv\sigma} \in \mathcal{M}_+, \quad \forall (u,v,\sigma) \in E, \\ \sum_{(u,v,\sigma) \in E} \mu_{uv\sigma}(\mathbb{S}^{n-1}) = 1.$$

The constraint (7) is the Lyapunov constraint. The constraint (9) is similar to the *measure invariance constraint* $A\#\mu = \mu$ of a linear dynamical system $x_{k+1} = Ax_k$ and to the *mass balance constraint* of a *circulation problem* [2]. Without constraint (8) (resp. (10)), the feasible set of [Program 2.2](#) (resp. [Program 2.3](#)) is a cone. These constraints have no effect on the optimal objective value but they make the feasible set bounded.

The main result of this section is summarized in the following theorem.

THEOREM 2.4. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G . Let $\bar{\gamma}^*$ (resp. $\underline{\gamma}^*$) be the optimal value of [Program 2.2](#) (resp. [Program 2.3](#)). The following identity holds:*

$$\underline{\gamma}^* = \rho(G, \mathcal{A}) = \bar{\gamma}^*.$$

As a consequence of [Theorem 2.4](#), we have a new criterion for lower bounds on the CJSR using measures.

COROLLARY 2.5. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. If there exist non-trivial⁵ measures $\mu_{uv\sigma}$ for each $(u, v, \sigma) \in E$ such that*

$$\sum_{(u,v,\sigma) \in E} A_\sigma \# \mu_{uv\sigma} \geq \underline{\gamma} \sum_{(v,w,\sigma) \in E} \mu_{vw\sigma}, \quad \forall v \in V$$

then $\underline{\gamma} \leq \rho(G, \mathcal{A})$.

Since $\underline{\gamma}^* = \bar{\gamma}^*$ by [Lemma A.2](#), one could prove [Theorem 2.4](#) by using that $\rho(G, \mathcal{A}) = \bar{\gamma}^*$ which is classical; see [Lemma 2.6](#) and [Theorem A.1](#). However, to illustrate the relation between atomic solutions of [Program 2.3](#) and periodic trajectories, we instead prove that $\bar{\gamma}^* \leq \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$ and $\rho_k(G, \mathcal{A}) \leq \underline{\gamma}^*$ for all k . These relations somehow suggest that [Program 2.2](#) is related to the definition of the CJSR with norms while [Program 2.3](#) is related to the definition of the CJSR with the spectral radius.

LEMMA 2.6. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. For any natural number k and norm $\|\cdot\|$, we have*

$$\bar{\gamma}^* \leq \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$$

where $\hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$ is defined in (5).

Proof. Let $f_v(x) = \max_{s \in E_{k-1}^+(v)} \|A_s x\|$. For any edge $(u, v, \sigma) \in E$,

$$f_v(A_\sigma x) = \max_{s \in E_{k-1}^+(v)} \|A_s A_\sigma x\| \leq \max_{s \in G_k} \|A_s x\| \leq [\hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)]^k \|x\|.$$

so the Lyapunov functions f_v are solution for $\bar{\gamma} = \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)$. \square

⁵At least one $\mu_{uv\sigma}$ must be nonzero.

LEMMA 2.7. Consider a finite set of matrices \mathcal{A} constrained by an automaton G and a cycle $c = (\sigma_1, \dots, \sigma_k)$ of length k with intermediary nodes $v_0, \dots, v_{k-1}, v_k = v_0 \in V$ such that $(v_{i-1}, v_i, \sigma_i) \in E$ for $i = 1, \dots, k$. Let x_0 be such that $A_c x_0 = \lambda x_0$ with $|\lambda| = \rho(A_c)$ and $\|x_0\|_2 = 1$, consider the following iteration

$$x_i = A_{\sigma_i} x_{i-1} \quad \hat{x}_i = x_i / \|x_i\|_2 \quad \alpha_i = \|x_i\|_2 / \rho(c)^i$$

where $\rho(c)$ is defined in (4). The following solution

$$\left(\mu_{uv\sigma} = \sum_{i=1, v_i=v}^k \alpha_i \delta_{\hat{x}_i} \right)_{(u,v,\sigma) \in E}$$

is feasible for Program 2.3 with any $\underline{\gamma} \geq \rho(c)$ and it satisfies the constraints (9) as equality for $\underline{\gamma} = \rho(c)$.

Proof. By construction, $\alpha_k = 1$ so $\alpha_k \delta_{\hat{x}_k} = \delta_{x_0}$ and for each $i = 0, \dots, k-1$, we have

$$A_{\sigma_i} \# (\alpha_i \delta_{\hat{x}_i}) = \alpha_i \frac{\|x_{i+1}\|_2}{\|x_i\|_2} \delta_{\hat{x}_i} = \rho(c) \alpha_{i+1} \delta_{\hat{x}_{i+1}} \leq \underline{\gamma} \alpha_{i+1} \delta_{\hat{x}_{i+1}}$$

which equality if $\rho(c) = \underline{\gamma}$. \square

LEMMA 2.8. Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. For any natural number k , we have

$$\rho_k(G, \mathcal{A}) \leq \underline{\gamma}^*$$

where $\rho_k(G, \mathcal{A})$ is defined in (4).

Proof. Let $c^* \in \arg \max \{ \rho(c) : c \in G_k, c \text{ is a cycle} \}$, by Lemma 2.7 we can build a feasible solution of Program 2.3 with $\underline{\gamma} = \rho(c)$. \square

In some sense, Lemma 2.7 is encoding a trajectory in the measures $\mu_{uv\sigma}$. We say that the resulting measures are the *occupation measures* of the trajectory x_0, x_1, \dots, x_k as defined in Lemma 2.7.

Example 2.9. Consider the unconstrained system [1, Example 2.1] with $m = 2$:

$$\mathcal{A} = \{A_1 = e_1 e_2^\top, A_2 = e_2 e_1^\top\}$$

where e_i denotes the i th canonical basis vector.

A solution to Program 2.2 is given by

$$(f(x), \bar{\gamma}) = (\|x\|_2, 1).$$

This means that $f(x)$ is a Lyapunov function for the system so as it is well known this certifies that $\rho(\mathcal{A}) \leq 1$.

A dual solution μ_1 (resp. μ_2)⁶ for the first (resp. second) matrix has the measure $\mu_1 = \delta_{(0,1)}/2$ (resp. $\mu_2 = \delta_{(1,0)}/2$). This is the solution obtained by applying Lemma 2.7 to the cycle (1, 2). This is shown in Figure 2.

Proof of Theorem 2.4. By Lemma 2.6, Lemma 2.8 and (3), we know that $\bar{\gamma}^* \leq \rho(G, \mathcal{A}) \leq \underline{\gamma}^*$. By weak duality between Program 2.2 and Program 2.3, proved in Lemma A.2, $\underline{\gamma}^* \leq \bar{\gamma}^*$. Therefore equality holds. \square

⁶In the arbitrary switching case, we write μ_σ instead of $\mu_{uv\sigma}$ for short

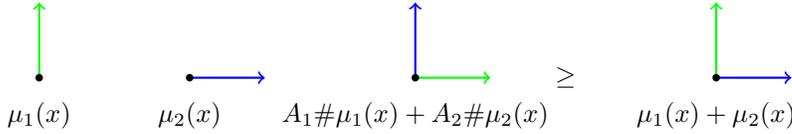


Figure 2: A representation of the optimal dual solution of [Example 2.9](#) with the constraint [\(9\)](#).

Remark 2.10. Occupation measures for continuous switched systems are studied in [\[8\]](#). These measures are supported on the cartesian product of the state space and a finite interval of time $t \in [0, T]$ while in this paper, the measures are only supported on the subset \mathbb{S}^{n-1} of the state space. Indeed, since the system [\(1\)](#) is homogeneous and time-invariant, we can encode trajectories in a measure on \mathbb{S}^{n-1} ([Lemma 2.7](#)) and still be able to recover it ([Corollary 2.5](#)).

The measures studied in [\[15\]](#) are supported on the paths in G . They are related to the measures studied in this paper since given a cycle c , we can compute the measures of the trajectory using this switching cycle and the eigenvector of A_c with [Lemma 2.7](#).

One may wonder whether [Lemma 2.7](#) also works in the reverse direction to give a *constructive* proof for [Corollary 2.5](#) when the measures $\mu_{uv\sigma}$ are atomic. Namely, can we extract a periodic trajectory with $\rho(c) \geq \underline{\gamma}$ from any atomic feasible solution of [Program 2.3](#) with $\underline{\gamma}$. As such solution may be the convex hull of solutions obtained by the construction of [Lemma 2.7](#), we may recover several periodic trajectory, from which there might be only one that satisfies $\rho(c) \geq \underline{\gamma}$. The following Lemma provides a constructive way to recover a periodic trajectory \bar{c} satisfying $\rho(c) \geq \underline{\gamma}$ in the scalar case⁷, i.e. $n = 1$

LEMMA 2.11. *Consider a finite set of matrices $\mathcal{A} \subseteq \mathbb{R}^{1 \times 1}$ constrained by an automaton G . If there exists a feasible solution μ of [Program 2.3](#) with $\underline{\gamma}$, then there exists a cycle c of length k with $\rho(c) \geq \underline{\gamma}$.*

Proof. Let (μ, γ) be the solution. By [\(10\)](#) and [\(9\)](#), we can find a cycle c for which each edge e has a nonzero measure μ_e .

If $\rho(c) \geq \underline{\gamma}$, we are done. Otherwise, if $\rho(c) < \underline{\gamma}$, using [Lemma 2.7](#), we can build a feasible solution ν such that [\(9\)](#) is satisfied with equality for $\underline{\gamma} = \rho(c)$. This means that $\mu - \lambda\nu$ is feasible with $\underline{\gamma}$ for any $\lambda \geq 0$ such that $\mu - \lambda\nu \geq 0$. Let λ^* be the maximum value of λ such that $\max_{\lambda} \mu - \lambda\nu \geq 0$. Since $n = 1$, \mathbb{S}^{n-1} is zero dimensional so for at least one edge e of the cycle c , $\mu_e - \lambda^*\nu_e$ is zero. Moreover, since μ_e is nonzero for all edge e of the cycle, $\lambda > 0$. Therefore, the number of edge with nonzero measure has decreased and at least one of the constraints [\(9\)](#) is now satisfied with strict inequality.

This process can only be repeated finitely many times until we have a zero measure since the number of edges with nonzero measure decrease each time. Moreover we will have $\rho(c) \geq \underline{\gamma}$ at least once since the constraints [\(9\)](#) cannot be satisfied with strict inequality for the zero measure. \square

Given a feasible solution of [Program 2.3](#) and a common partition of the support of the measures, we show in [Proposition 2.12](#) how to transform the solution into a solution of a scalar switched system. Using this transformation, we can always recover

⁷Note that in this case, any measure is atomic since \mathbb{S}^{n-1} is zero-dimensional

a cycle c for which $\rho(c) = \gamma$ from a solution of [Program 2.3](#) with $\underline{\gamma} = \gamma$ for which the measures are atomic.

PROPOSITION 2.12. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. Suppose that there exists a feasible solution μ of [Program 2.3](#) with $\underline{\gamma} = \gamma$ and a finite family \mathcal{S} of disjoint subsets of \mathbb{S}^{n-1} such that the support of each measure is included in the union of the sets of the family \mathcal{S} . Then there exists sets $B_1, \dots, B_k \in \mathcal{S}$ and a cycle $\sigma_1, \dots, \sigma_k$ of G such that*

$$\prod_{i=1}^k \max_{x \in B_i} \|A_{\sigma_i} x\|_2 \geq \gamma^k$$

and $A_{\sigma_i} B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \dots, k$ where $B_{k+1} = B_1$.

Proof. Given a set $B \in \mathcal{S}$ and an edge $e \in E$, let μ_e^B denote the measure defined as $\mu_e^B(C) = \mu_e(C \cap B)$. We consider a new constrained switched system with matrices $\mathcal{A}' \subseteq \mathbb{R}^{1 \times 1}$ and automaton $G'(V', E')$ where $V' = \{(v, B) \mid v \in V, B \in \mathcal{S}\}$, $e'((u, v, \sigma), B, C) = ((u, B), (v, C), (\sigma, B))$, $E' = \{e'(e, B, C) \mid e \in E, B, C \in \mathcal{S}, A_e B \cap C \neq \emptyset\}$, and $A'_{(\sigma, B)} = \max_{x \in B} \|A_\sigma x\|_2$. From any solution μ of the original system feasible for $\underline{\gamma}$, the following solution of the system with matrices \mathcal{A}' and automaton G'

$$\mu'_{e'(e, B, C)} = \frac{(A_e \# \mu_e^B)(C)}{(A_e \# \mu_e)(B)} \mu_e(B)$$

is also feasible with $\underline{\gamma}$. Indeed, by construction, for any $v \in V, B, C \in \mathcal{S}$, we have

$$(11) \quad \sum_{e \in E_1^-(v), B \in \mathcal{S}} A_e \# \mu_e^B(C) = \sum_{e \in E_1^-(v), B \in \mathcal{S}} \mu'_{e'(e, B, C)} \frac{(A_e \# \mu_e)(B)}{\mu_e(B)} \stackrel{(6)}{\leq} \sum_{e \in E_1^-(v, C)} A'_e \# \mu'_e$$

$$(12) \quad \sum_{e \in E_1^+(v)} \mu_e(C) = \sum_{e \in E_1^+(v), D \in \mathcal{S}} \frac{(A_e \# \mu_e^C)(D)}{(A_e \# \mu_e)(C)} \mu_e(C) = \sum_{e \in E_1^+(v, C)} \mu'_e.$$

By (9) on μ , the left-hand side of (12) is smaller than the left-hand side of (11). Therefore, the right-hand side of (12) is smaller than the right-hand side of (11) hence μ' satisfies (9) on the new switched system.

Therefore, by [Lemma 2.11](#), there is a cycle $(\sigma_1, B_1), \dots, (\sigma_k, B_k)$ of G' such that the modes σ_i and sets B_i are as required. \square

Example 2.13. Consider the dual solution obtained in [Example 2.9](#).

The supports of μ_1 and μ_2 are respectively $B_1 = \{(0, 1)\}$ and $B_2 = \{(1, 0)\}$. The automaton $G'(V', E')$ obtained by the transformation of [Proposition 2.12](#) is defined by $V' = \{(1, B_1), (1, B_2)\}$ and $E' = \{((1, B_1), (1, B_2), (1, B_1)), ((1, B_2), (1, B_1), (2, B_2))\}$. The new 1×1 matrices are $A'_{(1, B_1)} = 1$ and $A'_{(2, B_2)} = 1$.

The computation of the CJSR of this scalar system is a *maximum cycle mean* problem as outlined in [1]. The cycle of maximum geometric mean is $((1, B_1), (2, B_2))$ which geometric mean $\sqrt{1 \cdot 1} = 1$. We recover the cycle (1, 2) found in [Example 2.9](#).

Remark 2.14. For some systems, the *finiteness property* does not hold, that is, there is no finite cycle c for which $\rho(c)$ equals the CJSR. For these system, the optimal solutions of [Program 4.1](#) cannot be atomic. Can [Proposition 2.12](#) be used to provide a constructive proof of [Corollary 2.5](#) in this case ?

For an arbitrarily small ϵ , using [Proposition 2.12](#) with \mathcal{S} equal to an ϵ -covering of \mathbb{S}^{n-1} , we obtain a cycle c but instead of an eigenvector, we have an “eigenset” B_1 . Since the length of the cycle can be large, $A_c B_1$ can have a large diameter too. It is not clear how to obtain an eigenvector from this.

3. Automaton-dependent bounds. In this section, we introduce a method to approximate the CJSR using SOS programming and provide a new guarantee relating its accuracy with the spectral radius of the adjacency matrix and the p -radius; the definition of the p -radius can be found in the [Appendix B](#).

3.1. Sum of squares programming. Deciding whether a multivariate polynomial of degree $2d \geq 4$ is nonnegative is known to be NP-hard. However a sufficient condition for a polynomial to be nonnegative is easy to check. We say that a polynomial is a *sum of squares* (SOS) if there exist polynomials q_1, \dots, q_M such that

$$p(x) = \sum_{k=1}^M q_k^2(x).$$

If a polynomial is SOS, then it is obviously nonnegative.

It is well known that if $p(x)$ is an homogeneous polynomial of degree $2d$ then each $q_k(x)$ must be an homogeneous polynomial of degree d ; this can be shown easily using the Newton polytope of $p(x)$ and [\[32, Theorem 1\]](#). Let $x^{[d]}$ represent a basis of the homogeneous polynomials of degree d . We can check whether a polynomial is SOS using semidefinite programming thanks to the following theorem.

THEOREM 3.1 ([\[7, 25, 27, 29, 34\]](#)). *A homogeneous multivariate polynomial $p(x)$ of degree $2d$ is a sum of squares if and only if*

$$p(x) = (x^{[d]})^\top Q x^{[d]}$$

where Q is a symmetric positive semidefinite matrix.

From the exact arithmetic viewpoint, the basis $x^{[d]}$ chosen in [Theorem 3.1](#) does not affect whether $p(x)$ is SOS or not. A specific choice of basis may however improve the numerical behaviour of the corresponding semidefinite program.

We denote the set of homogeneous polynomials of degree $2d$ as $\mathbb{R}_{2d}[x]$, the cone of homogeneous SOS polynomials of degree $2d$ as Σ_{2d} and the dual of Σ_{2d} as Σ_{2d}^* .

A common interpretation of the dual space \mathbb{R}_{2d}^* of linear functionals on homogeneous polynomials of degree $2d$ is the space of moments of monomials of degree $2d$. If $p(x) = a^\top x^{[d]}$ and m is the vector of moments of $x^{[d]}$ of a measure μ then

$$\langle m, a \rangle = \int p(x) d\mu = \langle \mu, p \rangle.$$

As a sum of squares polynomial is nonnegative, this integral is nonnegative for any measure. Therefore, given a moment vector m , a necessary condition for a measure to exist with these moments is that $\langle m, a \rangle \geq 0$ for any vector of coefficients a of a sum of squares polynomial. That is, Σ_{2d}^* is a superset of the set of moments of measures. The members of Σ_{2d}^* are often called *pseudo-measures* and denoted $\tilde{\mu}$; see [\[3\]](#).

3.2. CJSR Approximation via SOS. The $2d$ th root of homogeneous polynomials of degree $2d$ can be used as Lyapunov function.

PROPOSITION 3.2. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. If there exist $|V|$ strictly positive homogeneous polynomials $p_v(x)$ of degree $2d$ such that*

$$p_v(A_\sigma x) \leq \bar{\gamma}^{2d} p_u(x)$$

holds for all edge $(u, v, \sigma) \in E$. Then $\rho(G, \mathcal{A}) \leq \bar{\gamma}$.

Proof. Define $f_v(x) = [p_v(x)]^{\frac{1}{2d}}$ and use [Theorem A.1](#). \square

We relax the positivity condition of [Proposition 3.2](#) by the more tractable sum of squares (SOS) condition and define $\rho_{\text{SOS-}2d}(G, \mathcal{A})$ as the solution of the following SOS restriction of [Program 2.2](#).

Program 3.3 (Primal).

$$(13) \quad \begin{aligned} & \inf_{p_v(x) \in \mathbb{R}_{2d}[x], \bar{\gamma} \in \mathbb{R}} \bar{\gamma} \\ & \bar{\gamma}^{2d} p_u(x) - p_v(A_\sigma x) \text{ is SOS}, \quad \forall (u, v, \sigma) \in E, \\ & p_v(x) \text{ is SOS}, \quad \forall v \in V, \end{aligned}$$

$$(14) \quad p_v(x) \text{ is strictly positive}, \quad \forall v \in V,$$

$$\sum_{v \in V} \int_{\mathbb{S}^{n-1}} p_v(x) dx = 1.$$

Remark 3.4. In practice we can replace (13) and (14) by “ $p_v(x) - \epsilon \|x\|_2^{2d}$ is SOS” for any $\epsilon > 0$. This constrains $p_v(x)$ to be in the interior of the SOS cone, which is sufficient for $p_v(x)$ to be strictly positive. The bounds given in [Section 3.3](#) are valid if $p_v(x)$ is in the interior of the SOS cone.

By [Proposition 3.2](#), a feasible solution of [Program 3.3](#) gives an upper bound for $\rho(G, \mathcal{A})$, and thus, for any positive degree $2d$,

$$(15) \quad \rho(G, \mathcal{A}) \leq \rho_{\text{SOS-}2d}(G, \mathcal{A}).$$

Example 3.5. Consider the unconstrained system [[1](#), [Example 2.1](#)] with $m = 3$:

$$\mathcal{A} = \{A_1 = e_1 e_2^\top, A_2 = e_2 e_3^\top, A_3 = e_3 e_1^\top\}$$

where e_i denotes the i th canonical basis vector.

For any d , a solution to [Program 3.3](#) is given by

$$(p(x), \gamma) = (x_1^{2d} + x_2^{2d} + x_3^{2d}, 1).$$

Example 3.6. Let us reconsider our running example; see [Example 1.1](#). The optimal solution of [Program 3.3](#) is represented by [Figure 3](#) for $2d = 2, 4, 6, 8, 10$ and 12.

3.3. Approximation guarantees. In this section, we provide a new bound that relates the accuracy of [Program 3.3](#) to the p -radius and the spectral radius of the adjacency matrix of the automaton. The p -radius is a generalization of the joint spectral radius; we recover the JSR with $p = \infty$. The definition of the p -radius and its property can be found in [Appendix B](#); see [[19](#)] for an introduction.

An important property of the p -radius is that it is increasing in p .

LEMMA 3.7. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G . For any integers $p \leq q$,*

$$(16) \quad \rho_p(G, \mathcal{A}) \leq \rho_q(G, \mathcal{A}) \leq \rho(G, \mathcal{A}) \leq \rho(A(G))^{\frac{1}{q}} \rho_q(G, \mathcal{A}) \leq \rho(A(G))^{\frac{1}{p}} \rho_p(G, \mathcal{A}).$$

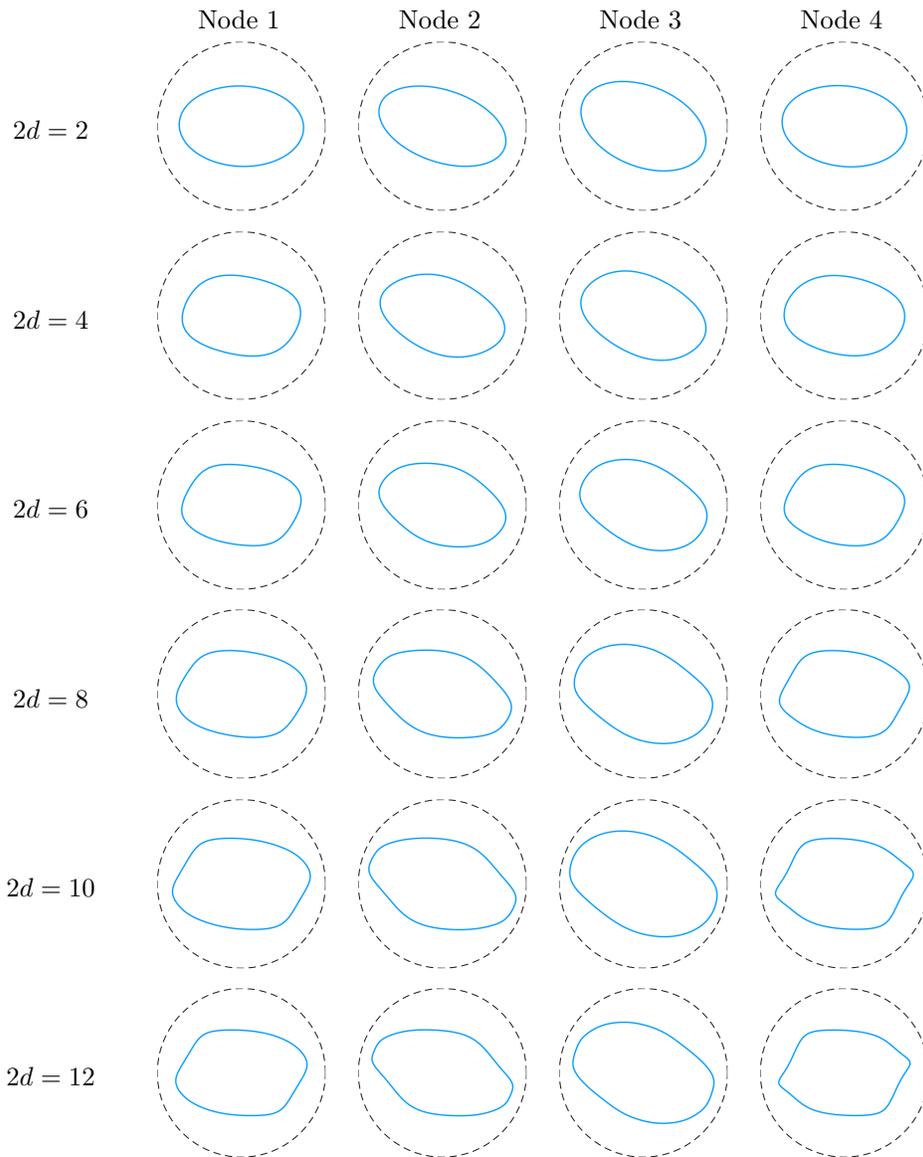


Figure 3: Representation of the solutions to Program 3.3 with different values of d for the running example. The blue curve represents the boundary of the 1-sublevel set of the optimal solution p_v at each node $v \in V$. The dashed curve is the boundary of the unit circle. Observe that some sets are not convex.

The proof can be found in Appendix B. This Lemma is already known in the unconstrained case [36].

Remark 3.8. Lemma 3.7 shows that the p -radius provides an upper and lower bound to the CJSR. It is known that the $2d$ -radius can be computed either by computing a spectral radius [5] or by solving a linear program [28] (see [26] for computation

algorithms when p is not an even integer). We need to do either of these two operations on either the sum of the d th Kronecker power of the veronese 2-lift of the matrices or the sum of the veronese $2d$ -lift of the matrices. However, the size of the matrices obtained by taking the d th Kronecker power grows rapidly and the veronese lifting needs to compute permanents which is very computationally demanding. Usually, for the same d , [Program 3.3](#) gives approximations of the CJSR with a much higher accuracy at the cost of solving an SDP on smaller matrices. While solving an SDP is more demanding than computing a spectral radius or solving a linear program, since the matrices are smaller than with the Kronecker powering and we do not need to compute permanents, this operation this method is actually more scalable.

In this section, we show the following bound stating that the solution found by [Program 3.3](#) is at least as good as the bound obtained by computing the $2d$ -radius.

THEOREM 3.9. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G and a positive integer d . The approximation given by [Program 3.3](#) using homogeneous polynomials of degree $2d$ satisfies:*

$$(17) \quad \rho_{SOS-2d}(G, \mathcal{A}) \leq \rho(A(G))^{\frac{1}{2d}} \rho_{2d}(G, \mathcal{A}) \leq \rho(A(G))^{\frac{1}{2d}} \rho(G, \mathcal{A})$$

where $A(G)$ is the adjacency matrix of G .

Note that the second inequality in (17) is simply (16). This theorem is proved at the end of this section.

As a corollary of [Theorem 3.9](#), in the trivial cases such that $\rho(A(G)) = 1$, the approximation is exact. This corresponds to the case where every node of G has indegree and outdegree 1. In that case, the graph forms a cycle of some length k and the CJSR is simply the k th root of the spectral radius of the product of the matrices along this cycle.

In the general case, the following approximation guarantee is known (note that the bound does not take into account the particular structure of the automaton):

THEOREM 3.10 ([\[30, Theorem 3.6\]](#)). *Consider a finite set of matrices $\mathcal{A} \subset \mathbb{R}^{n \times n}$ constrained by an automaton G and a positive integer d . The approximation $\rho_{SOS-2d}(G, \mathcal{A})$ given by [Program 3.3](#) using homogeneous polynomials of degree $2d$ satisfies:*

$$\rho_{SOS-2d}(G, \mathcal{A}) \leq \binom{n+d-1}{d}^{\frac{1}{2d}} \rho(G, \mathcal{A}).$$

The results of [Theorem 3.9](#), [Theorem 3.10](#) and (15) are summarized by the following corollary.

COROLLARY 3.11. *Consider a finite set of matrices $\mathcal{A} \subset \mathbb{R}^{n \times n}$ constrained by an automaton G and a positive integer d , the approximation given by [Program 3.3](#) using homogeneous polynomials of degree $2d$ satisfies:*

$$\min \left\{ \binom{n+d-1}{d}, \rho(A(G)) \right\}^{-\frac{1}{2d}} \rho_{SOS-2d}(G, \mathcal{A}) \leq \rho(G, \mathcal{A}) \leq \rho_{SOS-2d}(G, \mathcal{A}).$$

where $A(G)$ is the adjacency matrix of the automaton G .

We see that we can have arbitrary accuracy by increasing d .

For the arbitrary switching case, $\rho(A(G))$ is equal to the number of matrices m . [Theorem 3.9](#) was already known in this particular case [[28, Theorem 4.3](#)].

Our proof technique relies on the analysis of an iteration in the vector space of polynomials of degree $2d$. When this iteration converges, it converges to a feasible solution of Program 3.3. By analysing this iteration as affine iterations in this vector space, we derive a sufficient condition for its convergence and thus an upper bound for $\rho_{\text{SOS-}2d}(G, \mathcal{A})$.

Consider the iteration

$$(18) \quad \begin{aligned} p_{v,0}(x) &= 0, \\ p_{v,k+1}(x) &= q_v(x) + \frac{1}{\tau} \sum_{(u,v,\sigma) \in E} p_{u,k}(A_\sigma x), \quad v \in V \end{aligned}$$

for fixed homogeneous polynomials $q_v(x)$ of degree $2d$ in n variables (not necessarily different) and a constant $\tau > 0$.

When this iteration converges, it converges to a feasible solution of Program 3.3.

LEMMA 3.12. *Consider a constant $\tau > 0$. If there exist homogeneous polynomials $q_v(x)$ in the interior of the SOS cone such that iteration (18) converges then*

$$\rho_{\text{SOS-}2d}(G, \mathcal{A}) \leq \tau^{\frac{1}{2d}}.$$

Proof. Suppose the iteration converges to the polynomials $p_{v,\infty}(x)$. It is easy to show by induction that $p_{v,k}(x)$ is SOS for all k . It is trivial for $k = 0$ and if it is true for k then it is also true for $k + 1$ by (18). Since the SOS cone is closed, $p_{v,\infty}$ is SOS. Now by (18), for each $v \in V$,

$$p_{v,\infty}(x) = q_v(x) + \frac{1}{\tau} \sum_{(u,v,\sigma) \in E} p_{u,\infty}(A_\sigma x)$$

so $p_{v,\infty}(x)$ is also in the interior of the SOS cone. For each edge (u, v, σ) , by manipulating the above equation, we have

$$\tau p_{v,\infty}(x) - p_{u,\infty}(A_\sigma x) = \tau q_v(x) + \sum_{(u',v',\sigma') \in E} p_{u',\infty}(A_{\sigma'} x)$$

so $\tau p_{v,\infty}(x) - p_{u,\infty}(A_\sigma x)$ is SOS. Therefore $(\{p_{v,\infty}(x) : v \in V\}, \tau^{\frac{1}{2d}})$ is a feasible solution of Program 3.3. \square

In view of Lemma 3.12, it is thus natural to analyse under which condition iteration 18 converges. Recall that iteration 18 is an affine map on the vector space of homogeneous polynomials of degree $2d$.

Proof of Theorem 3.9. Iteration 18 is an affine map on the vector space of homogeneous polynomials of degree $2d$. It is well known that if the convergence is guaranteed when we only retain the linear part of the affine map then it is also guaranteed for the affine iteration.

Therefore we can analyse instead the following iteration

$$\begin{aligned} p_{v,0}(x) &= q_v(x), \\ p_{v,k+1}(x) &= \frac{1}{\tau} \sum_{(u,v,\sigma) \in E} p_{u,k}(A_\sigma x), \quad v \in V \end{aligned}$$

We can see that

$$(19) \quad p_{v,k+1}(x) = \frac{1}{\tau^k} \sum_{s \in E_k^-(v)} q_{s(1)}(A_s x) \leq \frac{1}{\tau^k} \sum_{s \in E_k^-(v)} q_{s(1)}(A_s x).$$

Consider a norm $\|\cdot\|$ of \mathbb{R}^n and its corresponding induced matrix norm of $\mathbb{R}^{n \times n}$. For each $v \in V$, we know by continuity of $q_v(x)$ that there exist $\beta_v > 0$ such that

$$q_v(x) \leq \beta_v \|x\|^{2d}$$

for all $x \in \mathbb{R}^n$. Let $\beta = \max_{v \in V} \beta_v$.

Therefore,

$$\begin{aligned} p_{v,k+1}(x) &\leq \frac{1}{\tau^k} \sum_{s \in E_k^-(v)} \beta_v \|A_s\|^{2d} \|x\|^{2d} \\ &\leq \frac{\beta}{\tau^k} \|x\|^{2d} \sum_{s \in E_k^-(v)} \|A_s\|^{2d}. \end{aligned}$$

By Lemma B.6 and Remark B.2, if $\tau > \rho(A(G))\rho_{2d}(G, \mathcal{A})$, $\lim_{k \rightarrow \infty} p_{v,k}(x) = 0$. We obtain the result by Lemma 3.12. \square

4. Finding high-growth sequences. In Section 3.2, we introduced the SOS restriction of Program 2.2 with Program 3.3. In this section, we introduce Program 4.1, the moment relaxation of Program 2.3. It turns out that Program 3.3 and Program 4.1 are dual to each other. Indeed, the proof of Lemma A.2 can be translated verbatim in order to prove that Program 4.1 is the dual of Program 3.3.

4.1. Dual SOS program.

Program 4.1 (Dual of Program 3.3).

$$(20) \quad \sum_{(u,v,\sigma) \in E} A_\sigma \# \tilde{\mu}_{uv\sigma} - \underline{\gamma}^{2d} \sup_{\tilde{\mu}_{uv\sigma} \in \mathbb{R}_{2d}^*, \underline{\gamma} \in \mathbb{R}} \underline{\gamma} \sum_{(v,w,\sigma) \in E} \tilde{\mu}_{vw\sigma} \in \Sigma_{2d}^*, \forall v \in V,$$

$$(21) \quad \tilde{\mu}_{uv\sigma} \in \Sigma_{2d}^*, \quad \forall (u,v,\sigma) \in E,$$

$$(22) \quad \sum_{(u,v,\sigma) \in E} \tilde{\mu}_{uv\sigma}(\mathbb{S}^{n-1}) = 1.$$

It is important to note that a solution of Program 4.1 is not necessarily a solution of Program 2.3. First $\tilde{\mu}_{uv\sigma}$ may not be a measure even if it belongs to Σ_{2d}^* as discussed in Section 3.1. Second, the left-hand side of (20) may also not be a measure. For this second concern, it helps to be more explicit. Suppose for instance that we are in the quadratic case, i.e. $d = 1$. In that case, if $\tilde{\mu} \in \Sigma_2^*$, there always exists a measure μ that has the moments of the pseudo-measure $\tilde{\mu}$. We can take for instance a Gaussian distribution with these second order moments. Hence we can find Gaussian distributions $\mu_{uv\sigma}$ that have the second order moments $\tilde{\mu}_{uv\sigma}$ and Gaussian distributions ν_v that have the second order moments given by the left-hand side of (20). However, we may have

$$\sum_{(u,v,\sigma) \in E} A_\sigma \# \mu_{uv\sigma} - \underline{\gamma}^{2d} \sum_{(v,w,\sigma) \in E} \mu_{vw\sigma} \neq \nu_v$$

as we only know that the left-hand side and right-hand side of the above equation have the same second order moments; see [Example 4.3](#).

However, in some cases, we can recover a feasible solution of [Program 2.3](#) from a feasible solution of [Program 4.1](#). In these cases, by [Corollary 2.5](#), this provides a lower bound to the CJSR. Moreover, there exist efficient techniques allowing to detect situations where the solution is moments of an atomic measure; see [\[16, 23\]](#). Then, using the transformation of [Proposition 2.12](#), we can transform these atomic measures into a feasible solution of a constrained scalar switched systems. For such system, we could use the algorithm described in [Lemma 2.11](#) but as pointed out in [\[1\]](#), computing the CJSR of a scalar system can easily be done by solving a maximum cycle mean problem for which efficient algorithm exists [\[20\]](#).

If we recover a feasible solution of [Program 2.3](#) from a feasible solution of [Program 4.1](#) with $\underline{\gamma} = \rho_{\text{SOS-2d}}(G, \mathcal{A})$, we can directly conclude that $\rho_{\text{SOS-2d}}(G, \mathcal{A}) = \rho(G, \mathcal{A})$. This is somewhat similar to the minimization of a multivariate polynomial using SOS where we can detect that we have reached the optimum when the measure is atomic and recover the minimizers of the polynomial from the atoms of the measure.

However, we may also check for atomic feasible solutions of [Program 2.3](#) with $\underline{\gamma} < \rho_{\text{SOS-2d}}(G, \mathcal{A})$ to provide lower bounds. Moreover, in practice, $\rho_{\text{SOS-2d}}(G, \mathcal{A})$ is computed by binary search on $\underline{\gamma}$ so we often have several such solutions.

Example 4.2. Consider [Example 3.5](#). For $i = 1, 2, 3$, let $\tilde{\mu}_i$ be the solution of [Program 4.1](#) corresponding to the matrix A_i . For any d , we can see that the dual solution for $\gamma = 1$ is such that the only monomial x^α such that $\langle \tilde{\mu}_1, x^\alpha \rangle$ (resp. $\langle \tilde{\mu}_2, x^\alpha \rangle$, $\langle \tilde{\mu}_3, x^\alpha \rangle$) is non-zero is x_1^{2d} (resp. x_2^{2d}, x_3^{2d}) and $\langle \tilde{\mu}_1, x_1^{2d} \rangle = \langle \tilde{\mu}_2, x_2^{2d} \rangle = \langle \tilde{\mu}_3, x_3^{2d} \rangle = 1/3$. Note that it means that $\tilde{\mu}_1 = \delta_{(1,0,0)}/3$, $\tilde{\mu}_2 = \delta_{(0,1,0)}/3$ and $\tilde{\mu}_3 = \delta_{(0,0,1)}/3$ where δ_x is the Dirac measure centered on x . Since these measures are solution to [Program 2.3](#) with $\underline{\gamma} = 1$, by [Corollary 2.5](#), this means that $\rho(\mathcal{A}) \geq 1$.

Example 4.3. We continue the running example; see [Example 1.1](#) and [Example 3.6](#).

For all d , $\tilde{\mu}_{212} = \tilde{\mu}_{323} = \tilde{\mu}_{344} = \tilde{\mu}_{431} = 0$ hence the node 4 is “unused” by the dual. For $2d = 2, 4, 6, 8$, $\tilde{\mu}_{123} = \tilde{\mu}_{231} = 0$ so the node 2 is “unused” for low degree.

At first, one could think that the dual variables can be used to reduce the systems, e.g. remove nodes or edges. However, as we will see, it would be a mistake to remove the node 2.

For $2d = 10$, $\tilde{\mu}_{123}$ and $\tilde{\mu}_{231}$ are not zero and are of the “same order or magnitude” than $\tilde{\mu}_{131}$ and $\tilde{\mu}_{312}$. Then for $2d = 12$, $\tilde{\mu}_{123}$ and $\tilde{\mu}_{231}$ have “larger magnitude” than $\tilde{\mu}_{131}$ and $\tilde{\mu}_{312}$. This observation will be useful for [Example 4.10](#).

We can see that while the shape of the primal variables changes a lot between $2d = 2$ and $2d = 4$ as mentioned in [Example 3.6](#), the “important” change for the dual variables happens around $2d = 10$.

It is also interesting to notice that the matrices corresponding to the dual variables have low rank. For example, for $2d = 2$, $\tilde{\mu}_{131}$ (resp. $\tilde{\mu}_{312}, \tilde{\mu}_{331}$) is the Dirac measure $0.324 \cdot \delta_{(0.917, 0.399)}$ (resp. $0.229 \cdot \delta_{(0.875, 0.485)}$, $0.447 \cdot \delta_{(0.757, -0.653)}$). However, this is not a feasible solution of [Program 2.3](#). Indeed, while [\(9\)](#) is satisfied for node 1 since $A_3 \# \delta_{(0.875, 0.485)}$ gives $\delta_{(0.917, 0.399)}$, $A_1 \# \delta_{(0.917, 0.399)}$ gives $\delta_{(1, -0.0273)}$ and $A_1 \# \delta_{(0.757, -0.653)}$ gives $\delta_{(0.423, -0.906)}$ so [\(9\)](#) is not satisfied for node 3.

4.2. Generating high growth sequence. In this section we give an algorithm that generates an infinite sequence of matrices such that the asymptotic growth rate of the product of the matrices is arbitrarily close to the CJSR. Note that by [Defini-](#)

tion 2.1, this asymptotic growth rate must be smaller than the CJSR.

Given an edge $e \in E$, let $\tilde{\mathbb{E}}_e[p(x)] = \langle \tilde{\mu}_e, p(x) \rangle$. Given a polynomial $p_0(x) \in \text{int}(\Sigma_{2\mathbf{d}})$ and an initial edge (v_1, v_0, σ_1) , the algorithm builds a G^\top -admissible sequence $(v_1, v_0, \sigma_1), (v_2, v_1, \sigma_2), \dots$ such that

$$(23) \quad \theta_k \triangleq \tilde{\mathbb{E}}_{v_k v_{k-1} \sigma_k} [p_0(A_{\sigma_1} \cdots A_{\sigma_k} x)]$$

remains “large” for increasing k . As we will see, using Lemma 4.5, this implies that $A_{\sigma_1} \cdots A_{\sigma_k}$ has a “large” norm.

LEMMA 4.4 ([24, Lemma 6]). *For any polynomial $p(x) \in \text{int}(\Sigma_{2\mathbf{d}})$, there exists a constant β such that for any matrix A ,*

$$\beta \|A\|_2^{2d} p(x) - p(Ax) \text{ is SOS}$$

where $\|A\|_2 = \rho(A^\top A)^{1/2}$ is the Euclidean norm.

LEMMA 4.5. *Let us consider a solution $(\tilde{\mu}_e : e \in E)$ of Program 4.1. For any polynomial $p(x) \in \text{int}(\Sigma_{2\mathbf{d}})$, there exists a positive constant τ such that for any matrix $A \in \mathbb{R}^{n \times n}$ and edge $e \in E$,*

$$\tilde{\mathbb{E}}_e[p(Ax)] \leq \tau \|A\|_2^{2d}$$

Proof. If all pseudo-expectations are zero, the result is trivially true. Therefore we can suppose that at least one is nonzero. By Lemma 4.4, there exists a constant $\beta > 0$ such that

$$\beta \|A\|_2^{2d} p(x) - p(Ax) \text{ is SOS.}$$

Hence

$$\tilde{\mathbb{E}}_e[p(Ax)] \leq \beta \|A\|_2^{2d} \tilde{\mathbb{E}}_e[p(x)].$$

We obtain the result with the constant $\tau = \beta \max_{e \in E} \tilde{\mathbb{E}}_e[p(x)]$. Since at least one pseudo-expectation is nonzero and $p(x)$ is in the interior of the SOS cone, $\tau > 0$. \square

Algorithm 1 Generates a sequence of large asymptotic growth using paths of length l .

Given a feasible solution $(\tilde{\mu}_e : e \in E)$ of Program 4.1
 Pick an arbitrary polynomial $p_0(x) \in \text{int}(\Sigma_{2\mathbf{d}})$
 Pick an edge $(v_0, v_{-1}, \sigma_0) \in E$ such that $\tilde{\mathbb{E}}_{v_0 v_{-1} \sigma_0} [p_0(x)] > 0$
for $k = 0, l, 2l, \dots$ **do**
 Pick $s \in \arg \max_{s \in E_l^-(v_k)} \tilde{\mathbb{E}}_{s[1]} [p_k(A_s x)]$
 Set $(v_{k+l}, \sigma_{k+l}, \dots, \sigma_{k+1}, v_k) \leftarrow s$
 Set $p_{k+1} \leftarrow p_k(A_s x)$
end for

Lemma 4.7 provides a guarantee on the growth rate of θ_k , defined in (23), using the dual constraint (20).

LEMMA 4.6. *Given a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$, if $\tilde{\mu}$ is a feasible solution of Program 4.1 then, for any edge $(u, v, \sigma) \in E$, the following holds:*

$$(24) \quad \sum_{s \in E_k^-(u)} A_s \# \tilde{\mu}_{s[1]} \succeq \gamma^{2dk} \tilde{\mu}_{uv\sigma}.$$

Proof. We prove (24) by induction, the case of $k = 0$ being trivial. We can rewrite the left-hand side of (24) as

$$(25) \quad \sum_{s \in E_k^-(u)} A_s \# \tilde{\mu}_{s[1]} = \sum_{s \in E_{k-1}^-} A_s \# \sum_{(u, s(1), \sigma) \in E} A_\sigma \# \tilde{\mu}_{us(1)\sigma}.$$

By (20),

$$\sum_{(u, s(1), \sigma) \in E} A_\sigma \# \tilde{\mu}_{us(1)\sigma} \succeq \gamma^{2d} \sum_{(s(1), w, \sigma) \in E} \tilde{\mu}_{s(1)w\sigma}.$$

Since the dual variables $\tilde{\mu}_{s(1)w\sigma}$ of the right-hand side are in the dual of the SOS cone, and one of them is $\tilde{\mu}_{s[1]}$, we have

$$\sum_{(u, s(1), \sigma) \in E} A_\sigma \# \tilde{\mu}_{us(1)\sigma} \succeq \gamma^{2d} \tilde{\mu}_{s[1]}.$$

Applying $A_s \#$ on both sides and using (25) gives

$$\sum_{s \in E_k^-(u)} A_s \# \tilde{\mu}_{s[1]} \succeq \gamma^{2d} \sum_{s \in E_{k-1}^-(u)} A_s \# \tilde{\mu}_{s[1]}.$$

□

LEMMA 4.7. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. For any positive integers d and l , using Program 4.1 with any $\gamma < \rho_{\text{SOS-2d}}(G, \mathcal{A})$, Algorithm 1 with paths of length l produces a G^\top -admissible sequence $(v_1, v_0, \sigma_0), (v_2, v_1, \sigma_1), \dots$ for which the sequence of θ_k defined in (23) satisfies the following inequality for all $k \geq 1$:*

$$\theta_k \geq \frac{\gamma^{2dl}}{d^-(v_{k-l+1})} \theta_{k-l}$$

Proof. By Lemma 4.6,

$$\sum_{s \in E_k^-(v_{k-l+1})} \tilde{\mathbb{E}}_{s[1]}[p_{k-l}(A_s)] \geq \frac{\gamma^{2dl}}{\Delta_l^-} \theta_{k-l}.$$

Since the value of s chosen by Algorithm 1 maximises $\tilde{\mathbb{E}}_{s[1]}[p_{k-l}(A_s)]$, the left-hand side of the above inequality is smaller or equal to $d^-(v_{k-l+1})\theta_k$. □

Theorem 4.8 translates the guarantee on θ_k to a guarantee on $A_{\sigma_1} \cdots A_{\sigma_k}$ using Lemma 4.5.

THEOREM 4.8. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. For any positive integer d and l , using Program 4.1 with any $\gamma < \rho_{\text{SOS-2d}}(G, \mathcal{A})$, Algorithm 1 with paths of length l produces a G^\top -admissible sequence $(v_1, v_0, \sigma_0), (v_2, v_1, \sigma_1), \dots$ that satisfies the following inequality for all $k \geq 1$:*

$$\lim_{k \rightarrow \infty} \|A_{s_k}\|_2^{\frac{1}{k}} \geq \frac{\gamma}{\|(A(G)^\top)^l\|_\infty^{\frac{1}{2dl}}}$$

where $s_k = (\sigma_k, \dots, \sigma_1)$.

Proof. By Lemma 4.7, for any k multiple of l ,

$$\tilde{\mathbb{E}}_{s_k[1]}[p_0(A_{s_k}x)] \geq \frac{\gamma^{2dk}}{(\Delta_l^-)^{\frac{k}{l}}} \tilde{\mathbb{E}}_{v_0 v_{-1} \sigma_0}[p_0(x)]$$

By Lemma 4.5, there exists a constant $\tau > 0$ such that

$$\tilde{\mathbb{E}}_{s_k[1]}[p_0(A_{s_k}x)] \leq \tau \|A_{s_k}\|^{2d}.$$

Combining these two inequalities, we obtain

$$\tau \|A_{s_k}\|^{2d} \geq \frac{\gamma^{2dk}}{(\Delta_l^-)^{\frac{k}{l}}} \tilde{\mathbb{E}}_{v_0 v_{-1} \sigma_0}[p_0(x)].$$

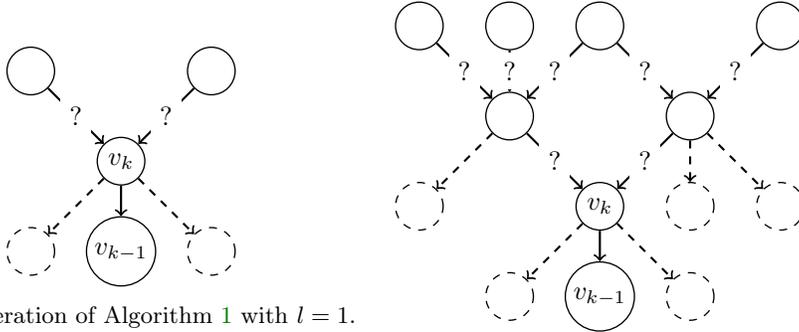
Since $\tilde{\mathbb{E}}_{v_0 v_{-1} \sigma_0}$ is nonzero, $\tilde{\mathbb{E}}_{v_0 v_{-1} \sigma_0}[p_0(x)] > 0$. Therefore taking the $(2dk)$ th root and the limit $k \rightarrow \infty$ we obtain the result. \square

Taking the limit $l \rightarrow \infty$ and using (30), we see that Theorem 3.9 is a corollary of Theorem 4.8.

Example 4.9. Suppose that we apply Algorithm 1 with $l = 1$ to Example 4.2 and let us denote by c_α the coefficient of the monomial x^α in the polynomial $p_0(x)$ chosen arbitrarily by the algorithm. The start of the sequence produced depends on the order between the coefficients $c_{(2d,0,0)}, c_{(0,2d,0)}, c_{(0,0,2d)}$. If $c_{(2d,0,0)}$ is the largest then the G -admissible left-infinite sequence found is

$$\dots, 1, 2, 3, 1, 2, 3, 1, 2, 3.$$

The product $A_{\sigma_1} A_{\sigma_2} A_{\sigma_3} \dots = A_3 A_2 A_1 A_3 A_2 A_1 \dots$ is periodic and has an asymptotic growth rate $\rho(A_{\sigma_1} A_{\sigma_2} A_{\sigma_3})^{1/3} = 1$. Hence $1 \leq \rho(G, \mathcal{A})$. We saw in Example 4.2 that $\rho_{\text{SOS-}2d}(G, \mathcal{A}) = 1$ for any d . Therefore $\rho(G, \mathcal{A}) = 1$.



(a) One iteration of Algorithm 1 with $l = 1$.

(b) One iteration of Algorithm 1 with $l = 2$.

Figure 4: Comparison between Algorithm 1 with $l = 1$ and $l = 2$. The solid edge denotes the the edge (v_k, v_{k-1}, σ_k) and the edges with question marks denote the incoming path considered by the iteration of the algorithm.

4.3. Producing lower bounds. By definition of the CJSR, the asymptotic growth rate of the norm of the product of any G -admissible (or G^\top -admissible) sequence of matrices gives a lower bound for the CJSR. In particular the sequence produced by Algorithm 1 provides a lower bound for the CJSR.

If there are two integers \bar{k}, k such that the sequence after \bar{k} is periodic of period k , the asymptotic growth rate of the norm is equal to the k th root of the spectral radius of the product of the matrices of one period. This is due to the Gelfand's formula $\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$. From the same identity, we see that the spectral radius of the product of the matrices of one G -admissible cycle gives a lower bound for the CJSR.

To find lower bounds for the CJSR, one could generate all the cycles of length smaller than some maximum length and compute the spectral radius for all of them. This brute force approach is not scalable because the number of paths considered grows exponentially with the maximum length.⁸

Gripenberg [12] proposes a branch-and-bound algorithm that prunes the search using an a priori fixed absolute error. Two other branch-and-bound variant exists: the balanced complex polytope algorithm [13] and the invariant conitope algorithm [18].

These algorithms can also be used to produce a G -admissible sequence of matrices of high asymptotic growth rate by reproducing the cycles of high spectral radius infinitely. The advantage of Algorithm 1 is that it provides a guarantee of accuracy given in Theorem 4.8. Algorithm 1 provides at the same time a high growth infinite trajectory and lower bounds of guaranteed accuracy.

We can compute lower bounds using the upper bound provided by Program 3.3 and Corollary 3.11 but in practice the trajectories are periodic after some time \bar{k} so we are able to compute much better lower bounds than the pessimistic bound provided by Corollary 3.11. This is shown by the following example.

Example 4.10. We tried the atom extraction procedure and Algorithm 1 for $l = 1$ and $l = 3$ on our running example; see Example 1.1, Example 3.6 and Example 4.3. The code used to obtain the results of this example can be found on the author's website. The result is shown in Figure 5. We showed in [24] that the CJSR of the system is equal to 0.97482. We can see that this lower bound is found for $d = 4$ for $l = 1$ and for $d = 1$ for $l = 3$. The atom extraction finds the lower bound 0.939255.

4.4. Improving the automaton-dependent bounds. Summarizing our results above, after solving Program 3.3, we obtain an upper bound on the CJSR and a lower bound thanks to Corollary 3.11. Running Algorithm 1 provides lower bounds that we can compute if the sequence produced is periodic and this lower bound will always be at least as high than the lower bound produced by Theorem 3.9.

In this subsection we show that there is another way to improve the lower bound provided by Theorem 3.9 using the dual solution. The improved lower bound will never be higher than the bound provided by Algorithm 1 but it only requires checking which dual values are zero so it almost does not require any computation. This lower bound is provided by the Theorem 4.12 which is a generalization of [30, Theorem 3.10].

It is based on the fact that if in an optimal dual solution the dual variable of an edge is 0 then removing the dual variable and the primal constraint of this edge does not affect $\rho_{\text{SOS-}2d}(G, \mathcal{A})$. We could prove the following theorem using this fact but we give an alternative proof using Algorithm 1.

⁸The exponential growth of the brute force approach is the reason why one should choose a small l for Algorithm 1.

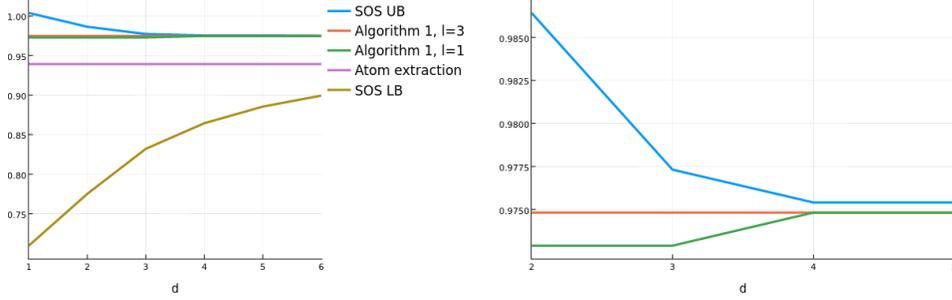


Figure 5: Result of Example 4.10. The SOS UB is the upper bound found by Program 3.3 and the SOS LB is obtained from this upper bound the guarantee given in Corollary 3.11. The value d of horizontal axis corresponds to using polynomials of degree $2d$. The right figure is a zoom of the left figure.

DEFINITION 4.11. Consider a finite set of matrices \mathcal{A} constrained by an automaton G and a positive integer d . The set of edges E_{2d} is the set of edges $e \in E$ such that there exists $\delta > 0$ such that for all $0 < \epsilon < \delta$, there is a solution of Program 4.1 with $\gamma \geq \rho_{SOS-2d}(G, \mathcal{A}) - \epsilon$ such that $\tilde{\mu}_e = 0$. We define the graph $G_{2d}(V, E_{2d})$ using this set of edges.

THEOREM 4.12. Consider a finite set of matrices \mathcal{A} constrained by an automaton G and a positive integer d . The approximation given by Program 3.3 using homogeneous polynomials of degree $2d$ satisfies:

$$\rho_{SOS-2d}(G, \mathcal{A}) \leq \rho(G_{2d})^{\frac{1}{2d}} \rho(G, \mathcal{A})$$

where $A(G_{2d})$ is the adjacency matrix of G_{2d} .

Proof. For any edge $e \in E$ such that $\tilde{\mu}_e = 0$, removing this edge does not violate any dual constraint (20).

Using Theorem 4.8 once we have removed all these edges and taking the limit $l \rightarrow \infty$ we obtain the result. \square

Remark 4.13. Note that it is not true that $G_{2d_2} \subseteq G_{2d_1}$ when $d_2 \geq d_1$. Indeed, as we have seen with Example 4.10 the dual variables of edges that were needed to have an s.m.p. are zero for $2d < 10$ so preventing Algorithm 1 to choose these edges even for $l > 1$ prevents it to find the s.m.p. for $2d < 10$.

5. Low rank reduction. Suppose we want to compute the CJSR of a finite set of matrices $\mathcal{A} \triangleq \{A_1, \dots, A_m\} \subset \mathbb{R}^{n \times n}$ of rank at most r constrained by an automaton $G(V, E)$. For $i = 1, \dots, m$, since the matrix A_i has rank at most r , there exists $X_i, Y_i \in \mathbb{R}^{n \times r}$ such that $A_i = X_i Y_i^T$. This can be used to build a new system with matrices of $\mathbb{R}^{r \times r}$ with the same CJSR. This new system can therefore be used to reduce the computation the CJSR of system of low rank matrices to a system system matrices of small size. Note that in the case $r = 1$, it is known that the CJSR is computable in polynomial time [1].

THEOREM 5.1 (Low Rank Reduction). Consider a finite set of matrices $\mathcal{A} \triangleq \{A_1, \dots, A_m\} \subset \mathbb{R}^{n \times n}$ of rank at most r constrained by an automaton $G(V, E)$.

For a fixed decomposition $A_\sigma = X_\sigma Y_\sigma^T$ for $\sigma = 1, \dots, m$ where $X_\sigma, Y_\sigma \in \mathbb{R}^{n \times r}$, denote the set of matrices $\mathcal{A}' \triangleq \{A'_{\sigma_1 \sigma_2} \mid \sigma_1, \sigma_2 = 1, \dots, m\} \subset \mathbb{R}^{r \times r}$ where $A'_{\sigma_1 \sigma_2} =$

$Y_{\sigma_1}^T X_{\sigma_2}$. Define the graph $G'(V', E')$ with $V' \triangleq E$ and

$$E' \triangleq \{((u, v, \sigma_1), (v, w, \sigma_2), \sigma_2 \sigma_1) \mid (u, v, \sigma_1), (v, w, \sigma_2) \in E\}.$$

Then the two CJSR are the same:

$$\rho(G, \mathcal{A}) = \rho(G', \mathcal{A}').$$

Proof. As the CJSR does not depend on the norm used, we choose a norm $\|\cdot\|$ that is *submultiplicative*, that is $\|AB\| \leq \|A\|\|B\|$ for all matrices A, B . For example, any norm induced by a vector norm is submultiplicative.

Let $\beta = \max_{i=1}^m \max\{\|X_i\|, \|Y_i^T\|\}$. If $\beta = 0$, then $\rho(G, \mathcal{A}) = 0 = \rho(G', \mathcal{A}')$. Therefore we may assume that $\beta > 0$. Consider a positive integer k . We first show that $[\hat{\rho}_k(G, \mathcal{A}, \|\cdot\|)]^k \leq \beta^2 [\hat{\rho}_{k-1}(G', \mathcal{A}')]^{k-1}$. For any G -admissible $(\sigma_1, \dots, \sigma_k)$, we have

$$(26) \quad A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1} = X_{\sigma_k} A'_{\sigma_k \sigma_{k-1}} \cdots A'_{\sigma_3 \sigma_2} A'_{\sigma_2 \sigma_1} Y_{\sigma_1}^T.$$

using the submultiplicativity of the norm chosen, we have

$$\begin{aligned} \|A_{\sigma_k} \cdots A_{\sigma_2} A_{\sigma_1}\| &\leq \|X_{\sigma_k}\| \|A'_{\sigma_k \sigma_{k-1}} \cdots A'_{\sigma_3 \sigma_2} A'_{\sigma_2 \sigma_1}\| \|Y_{\sigma_1}^T\| \\ &\leq \beta^2 \|A'_{\sigma_k \sigma_{k-1}} \cdots A'_{\sigma_3 \sigma_2} A'_{\sigma_2 \sigma_1}\| \\ &\leq \beta^2 [\hat{\rho}_{k-1}(G', \mathcal{A}')]^{k-1}. \end{aligned}$$

The same way, we now show that $[\hat{\rho}_{k-1}(G', \mathcal{A}')]^{k-1} \leq \beta^2 [\hat{\rho}_{k-2}(G, \mathcal{A}, \|\cdot\|)]^{k-2}$. For any G' -admissible $(\sigma_2 \sigma_1, \dots, \sigma_k \sigma_{k-1})$,

$$\begin{aligned} \|A'_{\sigma_k \sigma_{k-1}} \cdots A'_{\sigma_3 \sigma_2} A'_{\sigma_2 \sigma_1}\| &\leq \|Y_{\sigma_1}^T\| \|A_{\sigma_{k-1}} \cdots A_{\sigma_2}\| \|X_{\sigma_1}\| \\ &\leq \beta^2 [\hat{\rho}_{k-2}(G, \mathcal{A}, \|\cdot\|)]^{k-2}. \end{aligned}$$

In summary, we have

$$\hat{\rho}_k(G, \mathcal{A}, \|\cdot\|) \leq \beta^{\frac{2}{k}} [\hat{\rho}_{k-1}(G', \mathcal{A}')]^{\frac{k-1}{k}} \leq \beta^{\frac{4}{k}} [\hat{\rho}_{k-2}(G, \mathcal{A}, \|\cdot\|)]^{\frac{k-2}{k}}.$$

Taking the limit $k \rightarrow \infty$ we get $\rho(G, \mathcal{A}) \leq \rho(G', \mathcal{A}') \leq \rho(G, \mathcal{A})$. \square

Example 5.2. Consider an unconstrained switched system with 2 rank r matrices A_1, A_2 . This system is equivalent to the constrained switched system with automaton represented in Figure 6a. Its low rank reduction is represented in Figure 6b.

Remark 5.3. The matrices X_σ, Y_σ of the factorization $A_\sigma = X_\sigma Y_\sigma^T$ are not unique. For any invertible matrix $S \in \mathbb{R}^{r \times r}$, $A_\sigma = (X_\sigma S)(S^{-1} Y_\sigma^T)$ also gives a factorization. However, if $\rho(G', \mathcal{A}')$ is approximated using the sum of squares algorithm of Section 3.2, any two factorizations will give the same approximation. The effect of using $X_\sigma S$ and $Y_\sigma S^{-T}$ instead of X_σ and Y_σ will simply be a linear change of variable of the polynomial p_σ ; see Section 3.2.

We now discuss the reduction quantitatively. Suppose first that we want to approximate the unconstrained JSR of m matrices of dimension n and rank at most r . Using Theorem 5.1, we see that the unconstrained JSR is equal to the CJSR of m^2 matrices of rank r constrained by an automaton with m nodes and m^2 edges,

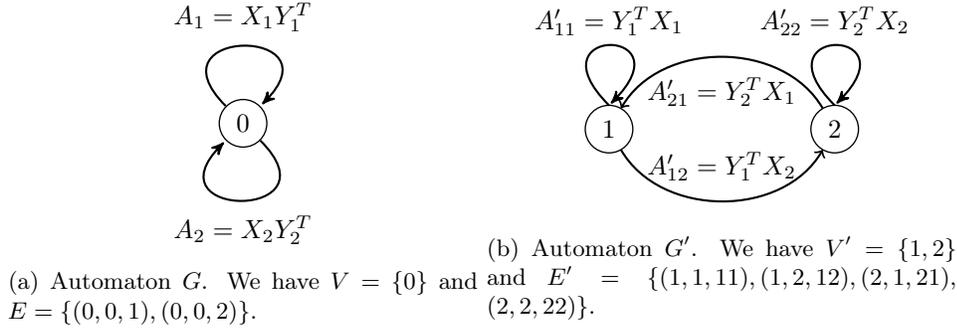


Figure 6: Simple example of the low rank reduction.

the underlying directed graph of the automaton is the complete graph with a loop at every node. This is summarized in Table 1a.

Suppose now that we want to approximate the CJSR of m matrices of dimension n and rank at most r constrained by an automaton of $|V|$ nodes and $|E|$ edges. Using Theorem 5.1, we see that this CJSR is equal to the CJSR of matrices of dimension r . The number of matrices in this CJSR might be lower than m^2 because if (σ_1, σ_2) is not G -admissible, then we do not need $A'_{\sigma_2 \sigma_1}$. The size of each important quantity before and after the reduction is given by Table 1b. We have just explained why $|\mathcal{A}'| \leq m^2$. We can see that by definition of E' , $|E'| \leq |E|^2$ with equality if and only if $|V| = 1$, that is the unconstrained case. Intuitively, the “more” the original CJSR is constrained, the “sparser” G' will be.

Quantity	\mathcal{A}	(G', \mathcal{A}')	Quantity	(G, \mathcal{A})	(G', \mathcal{A}')
dimension	n	r	dimension	n	r
rank	r	r	rank	r	r
matrices	m	m	matrices	m	$\leq m^2$
nodes		m	nodes	$ V $	$ E $
edges		m^2	edges	$ E $	$\leq E ^2$

(a) Unconstrained case.
(b) Constrained case.

 Table 1: Quantification of the low rank reduction of Theorem 5.1. The quantity of the CJSR (G', \mathcal{A}') obtained by the reduction are expressed as a function of the quantity of the original CJSR (G, \mathcal{A}) .

6. Conclusions. We have analysed the dual of the SOS Lyapunov program for switched systems and shown how to leverage it to study the system stability. We also generalized the whole approach to the constrained switched systems, a class of systems that has attracted increasing attention recently.

It turns out from our analysis that these two concepts are intrinsically related: Our Theorem 4.8, which leverages the dual of the classical JSR algorithm, actually naturally applies to the constrained case; Even more, Proposition 2.12 transforms an unconstrained system into a scalar constrained one for the purpose of computing a

lower bound. Finally, we show in [Theorem 5.1](#) that unconstrained systems with low rank matrices naturally lead to the definition of an auxiliary *constrained* system.

We have introduced two techniques to generate lower bounds from the solution of the SOS dual program. In practice, these techniques provide periodic trajectories of high asymptotic growth rate. Since the SOS program can be solved efficiently, does this give an efficient algorithm to generate lower bounds on the CJSR with guaranteed accuracy? This is not clear, because our algorithm provides firm guarantees only when the computed measures are atomic, which is not always the case.

More generally, the techniques developed in this work, based on generating “bad” trajectories for a dynamical system via dual solutions, naturally extend to many other problems in systems theory. We are currently exploring such possibilities.

REFERENCES

- [1] A. A. AHMADI AND P. A. PARRILO, *Joint spectral radius of rank one matrices and the maximum cycle mean problem.*, in CDC, 2012, pp. 731–733.
- [2] R. K. AHUJA, T. L. MAGNANTI, AND J. B. ORLIN, *Network Flows: Theory, Algorithms, and Applications*, Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1993.
- [3] B. BARAK, F. G. BRANDAO, A. W. HARROW, J. KELNER, D. STEURER, AND Y. ZHOU, *Hypercontractivity, sum-of-squares proofs, and their applications*, in Proceedings of the forty-fourth annual ACM Symposium on Theory of Computing, ACM, 2012, pp. 307–326.
- [4] M. A. BERGER AND Y. WANG, *Bounded semigroups of matrices*, Linear Algebra and its Applications, 166 (1992), pp. 21–27.
- [5] V. D. BLONDEL AND Y. NESTEROV, *Computationally efficient approximations of the joint spectral radius*, SIAM Journal on Matrix Analysis and Applications, 27 (2005), pp. 256–272.
- [6] V. D. BLONDEL AND J. N. TSITSIKLIS, *The boundedness of all products of a pair of matrices is undecidable*, Systems & Control Letters, 41 (2000), pp. 135–140.
- [7] M.-D. CHOI, T. Y. LAM, AND B. REZNICK, *Sums of squares of real polynomials*, in Proceedings of Symposia in Pure mathematics, vol. 58, American Mathematical Society, 1995, pp. 103–126.
- [8] M. CLAEYS, J. DAAFOUZ, AND D. HENRION, *Modal occupation measures and lmi relaxations for nonlinear switched systems control*, Automatica, 64 (2016), pp. 143–154.
- [9] X. DAI, *A Gel'fand-type spectral radius formula and stability of linear constrained switching systems*, Linear Algebra and its Applications, 436 (2012), pp. 1099–1113.
- [10] L. ELSNER, *The generalized spectral-radius theorem: an analytic-geometric proof*, Linear Algebra and its Applications, 220 (1995), pp. 151–159.
- [11] M. FEKETE, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Mathematische Zeitschrift, 17 (1923), pp. 228–249.
- [12] G. GRIPENBERG, *Computing the joint spectral radius*, Linear Algebra and its Applications, 234 (1996), pp. 43–60.
- [13] N. GUGLIELMI AND M. ZENNARO, *An algorithm for finding extremal polytope norms of matrix families*, Linear Algebra and its Applications, 428 (2008), pp. 2265–2282.
- [14] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge university press, 1952.
- [15] K. G. HARE, I. D. MORRIS, N. SIDOROV, AND J. THEYS, *An explicit counterexample to the Lagarias–Wang finiteness conjecture*, Advances in Mathematics, 226 (2011), pp. 4667 – 4701, <https://doi.org/http://dx.doi.org/10.1016/j.aim.2010.12.012>, <http://www.sciencedirect.com/science/article/pii/S0001870810004457>.
- [16] D. HENRION AND J.-B. LASSERRE, *Detecting global optimality and extracting solutions in gloptipoly*, in Positive polynomials in control, Springer, 2005, pp. 293–310.
- [17] R. JUNGERS, *The joint spectral radius: theory and applications*, vol. 385, Springer Science & Business Media, 2009.
- [18] R. M. JUNGERS, A. Cicone, AND N. GUGLIELMI, *Lifted polytope methods for computing the joint spectral radius*, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 391–410.
- [19] R. M. JUNGERS AND V. Y. PROTASOV, *Fast methods for computing the p-radius of matrices*, SIAM Journal on Scientific Computing, 33 (2011), pp. 1246–1266.
- [20] R. M. KARP, *A characterization of the minimum cycle mean in a digraph*, Discrete Mathe-

- matics, 23 (1978), pp. 309–311, [https://doi.org/http://dx.doi.org/10.1016/0012-365X\(78\)90011-0](https://doi.org/http://dx.doi.org/10.1016/0012-365X(78)90011-0), <http://www.sciencedirect.com/science/article/pii/0012365X78900110>.
- [21] V. KOZYAKIN, *The Berger–Wang formula for the Markovian joint spectral radius*, Linear Algebra and its Applications, 448 (2014), pp. 315–328.
- [22] J. B. LASSERRE, *Moments, positive polynomials and their applications*, World Scientific, 2009.
- [23] M. LAURENT, *Sums of squares, moment matrices and optimization over polynomials*, in Emerging applications of algebraic geometry, Springer, 2009, pp. 157–270.
- [24] B. LEGAT, R. M. JUNGERS, AND P. A. PARRILO, *Generating unstable trajectories for Switched Systems via Dual Sum-Of-Squares techniques*, in Proceedings of the 19th International Conference on Hybrid Systems: Computation and Control, HSCC '16, ACM, 2016, pp. 51–60, <https://doi.org/10.1145/2883817.2883821>, <http://doi.acm.org/10.1145/2883817.2883821>.
- [25] Y. NESTEROV, *Squared functional systems and optimization problems*, in High performance optimization, Springer, 2000, pp. 405–440.
- [26] M. OGURA, V. M. PRECIADO, AND R. M. JUNGERS, *Efficient method for computing lower bounds on the p -radius of switched linear systems*, Systems & Control Letters, 94 (2016), pp. 159–164.
- [27] P. A. PARRILO, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*, PhD thesis, Citeseer, 2000.
- [28] P. A. PARRILO AND A. JADBABAIE, *Approximation of the joint spectral radius using sum of squares*, Linear Algebra and its Applications, 428 (2008), pp. 2385–2402.
- [29] P. A. PARRILO AND S. LALL, *Semidefinite programming relaxations and algebraic optimization in control*, European Journal of Control, 9 (2003), pp. 307–321.
- [30] M. PHILIPPE, R. ESSICK, G. E. DULLERUD, AND R. M. JUNGERS, *Stability of discrete-time switching systems with constrained switching sequences*, Automatica, 72 (2016), pp. 242–250.
- [31] V. Y. PROTASOV, *The generalized joint spectral radius. A geometric approach*, Izvestiya: Mathematics, 61 (1997), p. 995, <http://stacks.iop.org/1064-5632/61/i=5/a=A05>.
- [32] B. REZNICK, *Extremal PSD forms with few terms*, Duke Math. J., 45 (1978), pp. 363–374, <https://doi.org/10.1215/S0012-7094-78-04519-2>, <http://dx.doi.org/10.1215/S0012-7094-78-04519-2>.
- [33] G.-C. ROTA AND W. STRANG, *A note on the joint spectral radius*, Proceedings of the Netherlands Academy, (1960). 22:379–381.
- [34] N. SHOR, *Class of global minimum bounds of polynomial functions*, Cybernetics and Systems Analysis, 23 (1987), pp. 731–734.
- [35] Y. WANG, N. ROOHI, G. E. DULLERUD, AND M. VISWANATHAN, *Stability of linear autonomous systems under regular switching sequences*, in Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on, IEEE, 2014, pp. 5445–5450.
- [36] D.-X. ZHOU, *The p -norm joint spectral radius and its applications in wavelet analysis*, AMS IP Studies in Advanced Mathematics, 25 (2002), pp. 305–326.

Appendix A. Stability certificates and duality.

THEOREM A.1. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. We have*

$$\lim_{k \rightarrow \infty} \hat{\rho}_k(G, \mathcal{A}, \|\cdot\|) \leq \bar{\gamma}^*.$$

Proof. Consider a norm $\|\cdot\|$ of \mathbb{R}^n and its corresponding induced matrix norm of $\mathbb{R}^{n \times n}$. For each $v \in V$, we know by compactness of the unit ball in \mathbb{R}^n , continuity and strict positivity of $f_v(x)$ that there exist $0 < \alpha_v \leq \beta_v$ such that

$$\alpha_v \|x\| \leq f_v(x) \leq \beta_v \|x\|$$

for all $x \in \mathbb{R}^n$. Let $\alpha = \min_{v \in V} \alpha_v$ and $\beta = \max_{v \in V} \beta_v$.

For a G -admissible k -uple $(\sigma_1, \sigma_2, \dots, \sigma_k)$,

$$\|A_{\sigma_k} \cdots A_{\sigma_1}\| = \sup_{x \neq 0} \frac{\|A_{\sigma_k} \cdots A_{\sigma_1} x\|}{\|x\|}.$$

Consider a path such that the i th edge has label σ_i for $i = 1, \dots, k$ and denote the intermediary nodes of that path as v_0, v_1, \dots, v_k . For any $x \in \mathbb{R}^n$, we have

$$\|A_{\sigma_k} \cdots A_{\sigma_1} x\| \leq \alpha_{v_k} f_{v_k}(A_{\sigma_k} \cdots A_{\sigma_1} x) \leq \alpha_{v_k} \bar{\gamma} f_{v_{k-1}}(A_{\sigma_{k-1}} \cdots A_{\sigma_1} x) \leq \alpha_{v_k} \bar{\gamma}^k f_{v_0}(x)$$

and

$$\|x\| \geq \beta_{v_0} p_{v_0}(x)$$

hence

$$\|A_{\sigma_k} \cdots A_{\sigma_1}\| \leq \frac{\beta_{v_0} \bar{\gamma}^k}{\alpha_{v_k}} \leq \frac{\beta}{\alpha} \bar{\gamma}^k.$$

Taking the k th root, the limit $k \rightarrow \infty$ and using [Definition 2.1](#) we obtain the result. \square

LEMMA A.2 (No duality gap). *For a fixed γ ,*

Weak duality *If [Program 2.2](#) (resp. [Program 2.3](#)) is feasible for $\bar{\gamma} = \gamma$ (resp. $\underline{\gamma} = \gamma$) then [Program 2.3](#) (resp. [Program 2.2](#)) is infeasible for all $\underline{\gamma} < \gamma$ (resp. $\bar{\gamma} > \gamma$).*

Strong duality *If [Program 2.2](#) (resp. dual) is infeasible for $\bar{\gamma} = \gamma$ (resp. $\underline{\gamma} = \gamma$) then [Program 2.3](#) (resp. [Program 2.2](#)) is feasible for $\underline{\gamma} = \gamma$ (resp. $\bar{\gamma} = \gamma$).*

In other words, there exists a value γ^ such that for every $\gamma > \gamma^*$, there exists a feasible solution to [Program 2.2](#) [Program 2.2](#) and for every $\gamma < \gamma^*$, there exists a feasible solution to [Program 2.3](#) [Program 2.3](#). Moreover, either [Program 2.2](#) program, [Program 2.3](#) program or both have a feasible solution with $\gamma = \gamma^*$.*

Proof. Consider the hyperplane

$$C \triangleq \left\{ (f_v : v \in V) \in \mathcal{F}^{|V|} \mid \sum_{v \in V} \int_{\mathbb{S}^{n-1}} f_v(x) dx = 1 \right\}$$

and the map

$$\mathcal{D}_\gamma : \mathcal{F}^{|V|} \rightarrow \mathcal{F}^{|E|} : (f_v : v \in V) \mapsto (\gamma f_u(x) - f_v(A_\sigma x) : (u, v, \sigma) \in E).$$

Given a fixed γ , [Program 2.2](#) has no solution for $\bar{\gamma} = \gamma$ if and only if $\mathcal{D}_\gamma(\mathcal{F}_{++}^{|V|} \cap C) \cap \mathcal{F}_+^{|E|} = \emptyset$. Since $\mathcal{F}_{++}^{|V|} \cap C$ is compact, so is $\mathcal{D}_\gamma(\mathcal{F}_{++}^{|V|} \cap C)$. We know that a compact set and a closed set have no intersection if and only if there exist a strict separating hyperplane separating the two sets. That is, a measure $\mu \in \mathcal{M}$ such that $\langle \mu, f \rangle \geq 0$ for all $f \in \mathcal{F}_+^{|E|}$ and $\langle \mu, f \rangle < 0$ for all $f \in \mathcal{D}_\gamma(\mathcal{F}_{++}^{|V|} \cap C)$. The first condition is simply $\mu \in \mathcal{M}_+$. For the second condition, we remark that

$$\mathcal{D}_\gamma(\mathcal{F}_{++}^{|V|} \cap C) = \mathcal{D}_\gamma(\text{int}(\mathcal{F}_+^{|V|}) \cap C) = \text{ri } \mathcal{D}_\gamma(\mathcal{F}_+^{|V|} \cap C).$$

We have $\langle \mu, f \rangle < 0$ for all $f \in \text{ri } \mathcal{D}_\gamma(\mathcal{F}_+^{|V|} \cap C)$ if and only if $\langle \mu, f \rangle \leq 0$ for all $f \in \mathcal{D}_\gamma(\mathcal{F}_+^{|V|} \cap C)$ and

$$(27) \quad \exists f \in \mathcal{D}_\gamma(\mathcal{F}_+^{|V|} \cap C) : \langle \mu, f \rangle \neq 0.$$

Therefore, if [Program 2.2](#) has no solution for $\bar{\gamma} = \gamma$ then there exists a *nonzero* measure $\mu \in (\mathcal{M}_+)^{|E|}$ such that for all $f \in C$ and $(u, v, \sigma) \in E$,

$$(28) \quad \sum_{v \in V} \sum_{(v, u, \sigma) \in E} \bar{\gamma} \mathbb{E}_{vu\sigma}[f_v(x)] \leq \sum_{v \in V} \sum_{(u, v, \sigma) \in E} \mathbb{E}_{uv\sigma}[f_v(A_\sigma x)]$$

and (27) holds.

Note that if the inequality (28) is respected for some $f \in C$, it is also respected for λf for all $\lambda > 0$. So we can impose that the inequality should be respected for all $f \in \mathcal{F}_+^{|V|} \setminus \{0\}$.

The constraint (28) must be true for all $f \in \mathcal{F}_+^{|V|} \setminus \{0\}$ so in particular in the case where there is a node $v \in V$ such that $f_u(x) = 0$ for all $u \neq v$. Therefore we must have

$$\gamma \sum_{(v,u,\sigma) \in E} \mathbb{E}_{vu\sigma}[f_v(x)] \leq \sum_{(u,v,\sigma) \in E} \mathbb{E}_{uv\sigma}[f_v(A_\sigma x)], \quad \forall f_v \in \mathcal{F}_+ \quad \square$$

for all $v \in V$. This is (9) so the strong duality is proven.

To show the weak duality, we show that if there exists a dual solution μ for $\underline{\gamma} = \gamma$ then (9) and (27) are satisfied for all $\underline{\gamma} < \gamma$. We know that (9) is satisfied for γ so the constraint (9) is also satisfied for any $\underline{\gamma} < \gamma$. Using (28) and (10) with $f_v(x) = \|x\|$ for all $v \in V$, we have $\langle \mu, f \rangle < 0$ for all $\underline{\gamma} < \gamma$.

Appendix B. The p -radius. We extend the definition of the p -radius to the constrained case.

DEFINITION B.1 (Constrained p -radius). *The constrained p -radius of a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$, denoted as $\rho_p(G, \mathcal{A})$, is*

$$\rho_p(G, \mathcal{A}) = \lim_{k \rightarrow \infty} \left[|E_k|^{-1} \sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \right]^{\frac{1}{pk}}$$

where

$$\hat{\rho}_{p;k;v}(G, \mathcal{A}) = \sum_{s \in E_k^+(v)} \|A_s\|^p.$$

Thus, the CJSR can be defined as the constrained p -radius for $p = \infty$.

Remark B.2. Since G is assumed to be strongly connected, we could give the following equivalent definition

$$(29) \quad \rho_p(G, \mathcal{A}) = \lim_{k \rightarrow \infty} \left[\max_{v \in V} [d_k^+(v)]^{-1} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \right]^{\frac{1}{pk}}$$

or the same definition with “ $d_k^-(v)$ ” instead of “ $d_k^+(v)$ ” and “ $s \in E_k^-(v)$ ” instead of “ $s \in E_k^+(v)$ ” in the definition of $\hat{\rho}_{p;k;v}(G, \mathcal{A})$.

By the equivalence of norms, the definition of the p -radius does not depend on the norm used.

We can show that the p -radius is well defined using the following classical result, known as *Fekete’s Lemma* [11].

LEMMA B.3. *Let $\{a_n\} : n \geq 1$ be a sequence of real numbers such that*

$$a_{m+n} \leq a_m + a_n.$$

Then the limit

$$\lim_{n \rightarrow \infty} \frac{a_n}{n}$$

exists and is equal to $\inf \left\{ \frac{a_n}{n} \right\}$.

LEMMA B.4. Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$ and the sequence $(a_k)_k = \max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A})$ with a submultiplicative norm. The sequence $\sqrt[k]{a_k}$ converges when $k \rightarrow \infty$. Moreover,

$$\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = \inf \{ \sqrt[k]{a_k} \}.$$

Proof. By submultiplicativity, for any $v \in V$, k and any $k_1, k_2 \geq 0$ such that $k_1 + k_2 = k$,

$$\begin{aligned} \hat{\rho}_{p;k;v}(G, \mathcal{A}) &= \sum_{u \in V} \sum_{s_1 \in E_{k_1}(v, u), s_2 \in E_{k_2}^+(u)} \|A_{s_2} A_{s_1}\|^p \\ &\leq \sum_{u \in V} \sum_{s_1 \in E_{k_1}(v, u), s_2 \in E_{k_2}^+(u)} \|A_{s_2}\|^p \|A_{s_1}\|^p \\ &= \sum_{u \in V} \hat{\rho}_{p;k_2;u}(G, \mathcal{A}) \sum_{s_1 \in E_{k_1}^-(u), s_1(1)=v} \|A_{s_1}\|^p \\ &\leq a_{k_2} \sum_{u \in V} \sum_{s_1 \in E_{k_1}^-(u), s_1(1)=v} \|A_{s_1}\|^p \\ &\leq \hat{\rho}_{p;k_1;v}(G, \mathcal{A}) a_{k_2} \end{aligned}$$

hence, in particular, $a_k \leq a_{k_1} a_{k_2}$ and $\log a_k \leq \log a_{k_1} + \log a_{k_2}$. We can conclude by [Lemma B.3](#). \square

COROLLARY B.5. The following holds

$$\lim_{k \rightarrow \infty} \left[\max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \right]^{\frac{1}{k}} = \lim_{k \rightarrow \infty} \left[\sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \right]^{\frac{1}{k}}$$

and, in particular, the limit on the right-hand side converges.

Proof. For a finite set of nonnegative numbers, their maximum is always between their average and their sum:

$$\frac{1}{|V|} \sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \leq \max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \leq \sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A})$$

or equivalently

$$\max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \leq \sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}) \leq |V| \max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A}).$$

By [Lemma B.4](#), $\max_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A})$ converges for $k \rightarrow \infty$ hence $\sum_{v \in V} \hat{\rho}_{p;k;v}(G, \mathcal{A})$ converges too. Taking the k th root and the limit $k \rightarrow \infty$ gives the identity. \square

LEMMA B.6. Consider a finite set of matrices \mathcal{A} constrained by an automaton G . The following relation holds

$$\rho_p(G, \mathcal{A}) = \rho(A(G)) \lim_{k \rightarrow \infty} \left[\sum_{s \in E_k} \|A_s\|^p \right]^{\frac{1}{pk}}.$$

Proof. By [Corollary B.5](#),

$$\lim_{k \rightarrow \infty} \left[\sum_{s \in E_k} \|A_s\|^p \right]^{\frac{1}{pk}}$$

converges. It remains to show that $\lim_{k \rightarrow \infty} |E_k|^{-\frac{1}{pk}}$ converges to $\rho(A(G))$.

Consider the matrix norm $\|\cdot\|_\infty$ on $\mathbb{R}^{n \times n}$ induced by the infinity norm on \mathbb{R}^n . It is well known that

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

where a_{ij} is (i, j) entry of A . It is also well known that the (u, v) entry of $A(G)^k$ gives the number of paths of length k starting at node u and ending at node v in G . Hence $\|A(G)^k\|_\infty = \Delta_k^+(G)$. By Gelfand's formula,

$$(30) \quad \rho(A(G)) = \lim_{k \rightarrow \infty} \|A(G)^k\|_\infty^{1/k}.$$

Since $|E_k|/|V| \leq \Delta_k^+(G) \leq |E_k|$, and $|V|^{1/k} \rightarrow 1$ as $k \rightarrow \infty$, we are done. \square

Proof of Lemma 3.7. The Lemma is a consequence of the inequality between ordinary means and the inequality between the p -norms ($p \geq 1$)

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}. \quad \square$$

LEMMA B.7 ([14]). *For any nonnegative integers a_1, \dots, a_n and positive real numbers $p \leq q$,*

$$\left(\frac{1}{n} \sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{n} \sum_{i=1}^n a_i^q \right)^{\frac{1}{q}} \leq \max\{a_i \mid i = 1, \dots, n\}.$$

LEMMA B.8. *For any real numbers $1 \leq p \leq q$,*

$$\|x\|_\infty \leq \|x\|_q \leq \|x\|_p.$$

Let $x^{[d]}$ denote the *scaled monomial basis*. The elements of this basis are

$$\frac{d!}{\alpha_1! \alpha_2! \cdots \alpha_n!} x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

for each n -tuples of nonnegative integers α such that $\alpha_1 + \cdots + \alpha_n = d$. For this basis, $\|x^{[d]}\|_2 = \|x\|_2^d$ where $\|\cdot\|_2$ is the Euclidean norm.

For any matrix $A \in \mathbb{R}^{n \times n}$, the map $x \mapsto x^{[d]}$ induces an associated map $A^{[d]} \in \mathbb{R}^{N_d \times N_d}$ which is the unique matrix that satisfies $(Ax)^{[d]} = A^{[d]}x^{[d]}$. We also denote $\mathcal{A}^{[d]} \triangleq \{A_1^{[d]}, \dots, A_m^{[d]}\}$.

Since $\|Ax\|^{[d]} = \|A\|^{[d]}\|x\|^{[d]}$, we have the following Lemma that is known in the unconstrained case or for the constrained case with $p = \infty$.

LEMMA B.9. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G , then*

$$\rho_p(G, \mathcal{A}) = \rho_1(G, \mathcal{A}^{[p]})^{\frac{1}{p}}$$

and

$$\rho(G, \mathcal{A}) = \rho(G, \mathcal{A}^{[p]})^{\frac{1}{p}}.$$

We say that a cone \mathcal{K} is *proper* if it is closed, solid, convex and pointed. We say that a matrix A *leaves a set S invariant* if $AS \subseteq S$ and we say that a set of matrices \mathcal{A} *leaves a proper cone invariant* if there exists a proper cone \mathcal{K} such that each matrix of \mathcal{A} leaves \mathcal{K} invariant.

LEMMA B.10 ([5, 31]). *If a set of m matrices leaves a proper cone \mathcal{K} invariant, then*

$$\begin{aligned} \rho_1(\mathcal{A}) &= \frac{1}{m} \lim_{k \rightarrow \infty} \left\| \sum_{s \in [m]^k} A_s \right\|^{\frac{1}{k}} \\ &= \frac{1}{m} \rho \left(\sum_{A \in \mathcal{A}} A \right). \end{aligned}$$

We deduce the following corollary of Lemma B.9 and Lemma B.10.

COROLLARY B.11. *If $\mathcal{A}^{[p]}$ leaves a proper cone \mathcal{K} invariant, then*

$$\begin{aligned} \rho_p(\mathcal{A}) &= \frac{1}{m^{\frac{1}{p}}} \lim_{k \rightarrow \infty} \left\| \sum_{s \in [m]^k} A_s^{[p]} \right\|^{\frac{1}{pk}} \\ &= \frac{1}{m^{\frac{1}{p}}} \rho \left(\sum_{A \in \mathcal{A}} A^{[p]} \right)^{\frac{1}{p}}. \end{aligned}$$

We generalize it to the constrained case using the lifting procedure introduced independantly by Kozyakin [21] and Wang [35].

LEMMA B.12. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. The following identity holds for any $p \in [1, +\infty]$*

$$\sum_{s \in E_k} \|A_s\|^p = \sum_{s \in E^k} [\|A'_s\|']^p$$

where

$$\mathcal{A}' = \{ A'_{uv\sigma} = (e_v e_u^\top) \otimes A_\sigma \mid (u, v, \sigma) \in E \}.$$

Proof. Consider a vector norm $\|\cdot\|$ of \mathbb{R}^n and the vector norm $\|\cdot\|'$ of $\mathbb{R}^{n|V|}$ such that

$$\|e_1 \otimes x_1 + \cdots + e_{|V|} \otimes x_{|V|}\| = \|x_1\| + \cdots + \|x_{|V|}\|.$$

Consider the induced matrix norms $\|\cdot\|$ and $\|\cdot\|'$. It is easy to see that for any nodes $u, v \in V$ and any matrix $B \in \mathbb{R}^{n \times n}$, $\|(e_v e_u^\top) \otimes B\|' = \|B\|$. In particular, given a path $s \in E_k$,

$$\begin{aligned} \|A'_s\|' &= \left\| \prod_{i=1}^k (e_{s(i+1)} e_{s(i)}^\top) \otimes A_{s[i]} \right\|' \\ &= \|(e_{s(k+1)} e_{s(1)}^\top) \otimes A_s\|' = \|A_s\| \end{aligned}$$

and given $s \notin E_k$, $\|A'_s\| = 0$. □

It is easy to see that if \mathcal{A} leaves the proper cone \mathcal{K} invariant then the set of matrices \mathcal{A}' of [Lemma B.12](#) leaves the proper cone $\mathcal{K}^{|V|}$ invariant.

LEMMA B.13. *Consider a finite set of matrices \mathcal{A} constrained by an automaton $G(V, E)$. If \mathcal{A} leaves a proper cone invariant, then*

$$\begin{aligned} \rho_1(G, \mathcal{A}) &= \frac{1}{[\rho(A(G))]^{\frac{1}{p}}} \lim_{k \rightarrow \infty} \left\| \sum_{s \in E_k} (e_{s(k+1)} e_{s(1)}^\top) \otimes A_s \right\|^{\frac{1}{k}} \\ &= \frac{1}{[\rho(A(G))]^{\frac{1}{p}}} \rho \left(\sum_{(u,v,\sigma) \in E} (e_v e_u^\top) \otimes A_\sigma \right). \end{aligned}$$

Proof. Combine [Lemma B.6](#), [Lemma B.12](#) and [Lemma B.10](#). \square

THEOREM B.14. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G . If $\mathcal{A}^{[p]}$ leaves a proper cone invariant, the following identities hold*

$$\begin{aligned} \rho_p(G, \mathcal{A}) &= \frac{1}{[\rho(A(G))]^{\frac{1}{p}}} \lim_{k \rightarrow \infty} \left\| \sum_{s \in E_k} (e_{s(k+1)} e_{s(1)}^\top) \otimes A_s^{[p]} \right\|^{\frac{1}{pk}} \\ &= \frac{1}{[\rho(A(G))]^{\frac{1}{p}}} \rho \left(\sum_{(u,v,\sigma) \in E} (e_v e_u^\top) \otimes A_\sigma^{[p]} \right)^{\frac{1}{p}}. \end{aligned}$$

[Theorem B.14](#) shows that when there is an invariant proper cone, $\rho_p(G, \mathcal{A})$ is as easy to obtain as computing a spectral radius.

It turns out that if p is even then there exists an invariant proper cone.

LEMMA B.15. *Consider a finite set of matrices \mathcal{A} constrained by an automaton G . For any positive integer d , $\mathcal{A}^{[2d]}$ leaves an invariant proper cone. Moreover this cone is the cone of SOS polynomials in the scaled monomial basis.*

Proof. Consider an homogeneous SOS polynomial $p(x)$ of degree $2d$ and its coordinates p in the scaled monomial basis. That is, $p(x) = \langle p, x^{[2d]} \rangle$. For any matrix A , we have

$$\langle A^{[2d]} p, x^{[2d]} \rangle = \langle p, (A^{[2d]})^\top x^{[2d]} \rangle = \langle p, (A^\top x)^{[2d]} \rangle = p(A^\top x).$$

Therefore if p is the coordinate vector of an SOS polynomial then $A^{[2d]} p$ is also the coordinate vector of an SOS polynomial. \square