# Length of the longest common subsequence between overlapping words 

Boris Bukh* Raymond Hogenson ${ }^{\dagger}$

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#### Abstract

Given two random finite sequences from $[k]^{n}$ such that a prefix of the first sequence is a suffix of the second, we examine the length of their longest common subsequence. If $\ell$ is the length of the overlap, we prove that the expected length of an LCS is approximately $\max \left(\ell, \mathbf{E}\left[L_{n}\right]\right)$, where $L_{n}$ is the length of an LCS between two independent random sequences. We also obtain tail bounds on this quantity.


## 1 Introduction

A word is a finite sequence of symbols over some alphabet. We write $|W|$ for the length of a word $W$. We write $W[i]$ for the $i$ th symbol of $W$, indexing starting with 0 . A subsequence of a word $W$ is a word obtained by deleting symbols from $W$. A common subsequence between two words $V$ and $W$ is a subsequence of both $V$ and $W$. A natural notion of similarity between two words is the length of the longest common subsequence (LCS) for the two. We write $\operatorname{LCS}(V, W)$ for the length of an LCS between words $V$ and $W$. A subword of a word $W$ is a subsequence consisting of contiguous symbols from $W$. We denote the subword of $W$ consisting of symbols $a$ through $b-1$ by $W[a, b)$. For a set $A=\left\{0 \leq i_{1}<i_{2}<\cdots\right\}$, we write $W[A]$ for the subsequence given by $W\left[i_{1}\right] W\left[i_{2}\right] \cdots$.

To take a concrete example, let $V=$ abedbba and $W=$ aabdca. Then $\operatorname{LCS}(V, W)=4$, as evidenced by the common subsequence abda. In this paper we are interested in the LCS between words chosen randomly.

As the nature of the symbols will not be important to us, will use $[k]:=$ $\{1,2, \ldots, k\}$ for the alphabet. We'll write $W \sim[k]^{n}$ to indicate that $W$ is a word chosen uniformly at random from $[k]^{n}$.

[^0]Let $L_{n}:=\operatorname{LCS}(V, W)$ where $V, W \sim[k]^{n}$. Then we define

$$
\gamma_{k}:=\lim _{n \rightarrow \infty} \frac{\mathbf{E}\left[L_{n}\right]}{n}
$$

See [3] for a proof that this limit exists, as well as upper and lower bounds on $\gamma_{k}$. Refer to [8] for the best known bounds on $\gamma_{2}$ and a deterministic method to determine accurate bounds for $\gamma_{k}$ for $k>2$. In [6] it is shown that $\gamma_{k} \sqrt{k} \rightarrow 2$ as $k \rightarrow \infty$.

In this paper, we examine a related problem: the LCS between two random words which overlap. Namely, let $0 \leq \alpha \leq 1$, pick $Z \sim[k]^{n+\alpha n}$, and choose $V=Z[0, n)$ and $W=Z[\alpha n, n+\alpha n)$. Thus a suffix of $V$ is the same as a prefix of $W$. We say that $W$ is shifted from $V$ by $\alpha$. We will examine $\operatorname{SHIFT}(n, k, \alpha n):=\operatorname{LCS}(Z[0, n), Z[\alpha n, n+\alpha n))$ where $Z \sim[k]^{n+\alpha n}$.

This is motivated in part by an application to DNA sequencing. In this process, we have two sections of DNA which can be regarded as words over the alphabet of nucleotides. The pieces of DNA may overlap, and we wish to determine whether the similarity between them is more than coincidence, i.e., if they are indeed from the same section of the genome.

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## 2 Results

For $W$ shifted from $V$ by $\alpha n$, the length of the overlap $(\alpha n)$ is a lower bound on the length of the LCS since the overlapping section is a common subsequence between the two. When $\alpha n$ is much less than $\mathbf{E}\left[L_{n}\right]$, we might think that the overlap does not matter and $\operatorname{LCS}(W, V)$ behaves like $L_{n}$. This is indeed so, as the following two theorems show.

Theorem 1. There exists a constant $c_{k}$ such that for any $t \geq 6 \sqrt{n}$,

$$
\operatorname{Pr}\left[\operatorname{SHIFT}(n, k, \alpha n) \geq \max \left(n-\alpha n+1, \gamma_{k} n+t\right)\right] \leq \exp \left(-c_{k} t^{2} / n\right)
$$

when $n$ is sufficiently large.
Theorem 2. There exists a constant $c_{k}$ such that for any $t \geq 5 n^{3 / 4} \sqrt{\log n}$,

$$
\operatorname{Pr}\left[\operatorname{SHIFT}(n, k, \alpha n) \leq \gamma_{k} n-t\right] \leq \exp \left(-c_{k} t^{2} / n^{3 / 4}\right)
$$

when $n$ is sufficiently large.
All logarithms in this paper are to base $e=2.71 \ldots$
We expect that $\operatorname{SHIFT}(n, k, \alpha n)=\max \left(n-\alpha n, \mathbf{E}\left[L_{n}\right]+O(\sqrt{n})\right)$ with high probability. This is supported by [7] which shows that the standard deviation of $L_{n}$ is $O(\sqrt{n})$.

## 3 Tools

Here we collect several auxiliary results.
Lemma 1 (1], Theorem 1.1).

$$
\gamma_{k} n \geq \mathbf{E}\left[L_{n}\right] \geq \gamma_{k} n-4 \sqrt{n \log n}
$$

Let $\Omega=\prod_{i=1}^{n} \Omega_{i}$ where each $\Omega_{i}$ is a probability space and $\Omega$ has the product measure. Let $h: \Omega \rightarrow \mathbb{R}$. Let $X$ be a random variable given by $X=h(\cdot)$.

We call $h: \Omega \rightarrow \mathbb{R}$ Lipschitz if $|h(x)-h(y)| \leq 1$ whenever $x, y$ differ in at most one coordinate.

Lemma 2 (Azuma's inequality, [2], Theorem 7.4.2). If $h$ is Lipschitz, then

$$
\operatorname{Pr}[|X-\mathbf{E}[X]|>\lambda] \leq e^{-t^{2} / 4}
$$

Lemma 3 (Hoeffding's inequality, [5]). Suppose $X_{i}$ are independent random variables with $a_{i} \leq X_{i} \leq b_{i}$. Then for all $t>0$,

$$
\operatorname{Pr}\left[\sum_{i} X_{i} \leq \mathbf{E}[X]-t\right] \leq \exp \left(\frac{-2 t^{2}}{\sum_{i}\left(b_{i}-a_{i}\right)^{2}}\right)
$$

Lemma 4. Let $X_{1}, \ldots, X_{m}$ be independent random variables, each of which is exponential with mean $k$. Let $X=X_{1}+\cdots+X_{m}$. Then

$$
\operatorname{Pr}[X \leq m(k-\lambda)] \leq \exp \left(-m \lambda^{2} / 2 k^{2}\right)
$$

Proof. We may assume that $\lambda<k$, for otherwise the result is trivial. With hindsight set $t=\lambda / k(k-\lambda)$. Then

$$
\begin{aligned}
\mathbf{E}\left[\exp \left(-t X_{i}\right)\right] & =\sum_{j \geq 1} \exp (-t j)(1-1 / k)^{j-1}(1 / k) \\
& =\frac{1}{1+\left(e^{t}-1\right) k} \leq \frac{1}{1+k t}
\end{aligned}
$$

From this it follows that $\mathbf{E}[\exp (t m k-t X)] \leq\left(e^{t k} /(1+k t)\right)^{m}$. Therefore by Markov's inequality we have

$$
\begin{aligned}
\operatorname{Pr}[m k-X \geq \lambda m] & \leq \frac{\left(e^{t k} /(1+k t)\right)^{m}}{\exp (t \lambda m)}=\left(\frac{e^{t(k-\lambda)}}{1+k t}\right)^{m} \\
& =\left(e^{\lambda / k}(1-\lambda / k)\right)^{m} \leq\left(1-(\lambda / k)^{2} / 2\right)^{m} \\
& \leq \exp \left(-m \lambda^{2} / 2 k^{2}\right)
\end{aligned}
$$

where in the penultimate line we used the inequality $e^{x}(1-x) \leq 1-x^{2} / 2$, which can be established by considering the Taylor expansion of $e^{x}(1-x)$.

In this paper, we will weaken this bound to

$$
\begin{equation*}
\operatorname{Pr}[X \leq m(k-\lambda)] \leq \exp (-m \lambda / 4 k+m / 32) \tag{1}
\end{equation*}
$$

using $x^{2} \geq x / 2-1 / 16$.


Figure 1: A visual representation of a common subsequence between shifted words.


Figure 2: An example where the common subsequence is the overlapping section.

## 4 Proof of Theorem 1

There is a geometric way to interpret a common subsequence. Consider a line segment from $(0,0)$ to $(n-1,0)$, and a second from $(\alpha n, 1)$ to $(n+\alpha n-1,1)$. Now we place the symbols from $V$ on the first line segment and those from $W$ on the second. For each pair of symbols in $V$ and $W$, connect them with an edge if the symbols are equal. The LCS, then, is the largest set of noncrossing edges. Furthermore, symbols aligned vertically will be certainly equal by the nature of the shift. See Figs. 1 and 2 for examples.
Lemma 5. Suppose we have indices $0 \leq i_{1}<\cdots<i_{t}<n$. Let $Z \sim[k]^{n+\alpha n}$ and take $V=Z[0, n)$ and $W=Z[\alpha n, n+\alpha n)$. Define $A=V\left[\left\{i_{1}, \ldots, i_{t}\right\}\right]$, $B_{\text {start }}=\max \left(i_{t}-t+1-\alpha n, 0\right)$, and $B=W\left[B_{\text {start }}, n\right)$. Then $A[\ell]$ is to the left of $B[\ell]$ for every $\ell$, and

$$
\operatorname{Pr}[\operatorname{LCS}(A, B)=t] \leq \exp \left(\left(|B|-\frac{7}{8} k|A|\right) / 4 k\right)
$$

Proof. See Fig. 3 for a depiction of this lemma in our geometric model. We will use the notation $W_{1} \unlhd W_{2}$ to mean that $W_{1}$ is a subsequence of $W_{2}$.

We prove first that $A[\ell]$ is always strictly to the left of $B[\ell]$ for every $\ell$. Let $\ell_{V}=i_{\ell}$, and $\ell_{W}=\ell+B_{\text {start }} \geq \ell+i_{t}-t+1-\alpha n$. These are the positions of $A[\ell]$ and $B[\ell]$ in $V$ and $W$ respectively. The horizontal position of $\ell_{W}$ is then at least $\ell+i_{t}-t+1$ (from the left of $V$ ), and the position of $\ell_{V}$ is at most $i_{t}-t+\ell$.


Figure 3: The dots represent the symbols of $V$ that are in $A$ and the bold line represents the symbols in $B$.

To prove the bound on $\operatorname{Pr}[\operatorname{LCS}(A, B)=t]$, we introduce an equivalent way of generating random word $Z$. Let $R, S \sim[k]^{\infty}$. Imagine $n+\alpha n$ placeholders corresponding to symbols of $Z$. In the beginning, the placeholders are empty. We will use symbols from $S$ and from $R$ in order to fill the placeholders using the following process. Start with $\ell_{A}=\ell_{B}=0$. At each step, if the placeholder for $A\left[\ell_{A}\right]$ is empty, use the next symbol from $S$ to fill it. Then we examine successive symbols from $R$ until the last examined symbol is equal to $A\left[\ell_{A}\right]$; we use the examined symbols to fill placeholders in $B$ starting from $B\left[\ell_{B}\right]$ (if we run out of empty placeholders, we simply discard symbols from $R$ ). Finally, we increment $\ell_{A}$ and increase $\ell_{B}$ appropriately so that $B\left[\ell_{B}\right]$ is the first unfilled placeholder in $B$.

Note that at each step $\ell_{B}$ increases by at least 1 . Since $\ell_{A}$ increases by exactly 1 , it follows that $B\left[\ell_{B}\right]$ is to the right of $A\left[\ell_{A}\right]$ at all times in this process.

Finally, after all placeholders in $A$ are filled, we fill the rest of $Z$ with symbols from $S$.

During this process, each next filled symbol is independent of all the ones before. Therefore, the word $Z$ we obtain is a uniformly random word.

Let $X_{i}$ be the number of symbols from $R$ consumed after we match the $(i-1)$ th symbol of $A$ but before we match the $i$ th symbol of $A$. Let $X=\sum_{i} X_{i}$. Then $\operatorname{Pr}[A \unlhd B]=\operatorname{Pr}[X \leq|B|]$.

We apply (11) (which is a weakening of Lemma (4) with $m=|A|, \lambda=k-$ $|B| /|A|$ :

$$
\operatorname{Pr}[A \unlhd B]=\operatorname{Pr}[X \leq|B|] \leq \exp \left(\left(|B|-\frac{7}{8} k|A|\right) / 4 k\right)
$$

Pick $Z \sim[k]^{n+\alpha n}$, and $V=Z[0, n), W=Z[\alpha n, n+\alpha n$ ) (so $W$ is shifted from $V$ by $\alpha n$ ).

We define the span of an edge $e$ between the $i$ th symbol from $V$ and the $j$ th symbol from $W$ to be $\operatorname{span}(e):=j+\alpha n-i$. This is the difference in $x$ coordinates in the geometric model. If the span is positive, the slope of the edge will be positive, and conversely a negative span indicates a negative-sloping edge. In particular, if the span is 0 , the symbols will be equal due to the nature of the shift. We say that symbols are overlapping if the span between them is 0 .

With hindsight, set $\varepsilon=0.01$. We will break our analysis into several cases, according to the shape of the LCS. For each shape, we'll bound the probability that there is a long LCS between $V$ and $W$ of that shape. Each case can be described by the edge with the least span. Let $e$ be an edge connecting a symbol of $V$ with a symbol of $W$, and define the following random variable
$\operatorname{BigSHIFT}_{e}:=$ "Maximum length of a common subsequence between $V$ and $W$ that uses the edge $e$, and the edge $e$ is an edge of largest span in this subsequence".


Figure 4: Illustration of the case when $0<\operatorname{span}(e) \leq \varepsilon n$. The words $V$ and $W$ are divided by the edge which is assumed to exist in this case.

Note that the probability of the event in Theorem 1 can be bounded by

$$
\begin{align*}
& \operatorname{Pr}\left[\operatorname{SHIFT}(n, k, \alpha n) \geq \max \left(n-\alpha n+1, \gamma_{k} n+t\right)\right] \\
& \quad \leq \sum_{e} \operatorname{Pr}\left[\operatorname{BigSHIFT}_{e} \geq \max \left(n-\alpha n+1, \gamma_{k} n+t\right)\right] \tag{2}
\end{align*}
$$

where the sum is over all edges $e$ connecting a symbol of $V$ with a symbol of $W$.
Let $e$ be an arbitrary edge, connecting $V[i]$ and $W[j]$. We shall estimate the $\operatorname{Pr}\left[\operatorname{BigSHIFT}{ }_{e} \geq \max \left(n-\alpha n+1, \gamma_{k} n+t\right)\right]$.

Case $\operatorname{span}(e) \leq 0$. There are $n-i$ symbols to the right of $e$ in $V$ and $j-1$ symbols to the left of $e$ in $W$. Therefore the length of the LCS is at most $n-i+j \leq n-\alpha n$ so the event $\operatorname{BigSHIFT}_{e}$ cannot occur.

Case $0<\operatorname{span}(e) \leq \varepsilon n$. Write $A=V[i+1, n), B=W[0, j)$, so that $A$ is the subword of $V$ after the symbol in position $i$ and $B$ is the subword of $W$ before position $j$. Let $s=j+\alpha n-i$. We have $s \leq \varepsilon n$.

If $W_{2}$ is a subsequence of $W_{1}$, then there are indices $i_{1}, \ldots, i_{\left|W_{2}\right|}$ such that $W_{2}[\eta]=W_{1}\left[i_{\eta}\right]$ for each $1 \leq \eta \leq\left|W_{2}\right|$. We use the notation $W_{1} \backslash W_{2}$ to mean $W_{1}$ with symbols $i_{1}, \ldots, i_{\left|W_{2}\right|}$ deleted.

Because $|A|+|B|=n-\alpha n+s$, for the length of the LCS to be greater than or equal to $n-\alpha n$, we must have

$$
A \backslash L_{1} \unlhd W[j+1, n) \text { and } B \backslash L_{2} \unlhd V[0, i)
$$

for some $L_{1} \unlhd A, L_{2} \unlhd B$ satisfying $\left|L_{1}\right|+\left|L_{2}\right|=s$.
We can then bound $\operatorname{Pr}[\operatorname{SHIFT}(n, \alpha, k)>n-\alpha n]$ from above by

$$
\begin{equation*}
\sum_{\substack{L_{1} \subseteq A \\ L_{2} \subseteq B \\\left|L_{1}\right|+\left|L_{2}\right|=s}} \operatorname{Pr}\left[A \backslash L_{1} \unlhd W[j+1, n) \text { and } B \backslash L_{2} \unlhd V[0, i)\right] . \tag{3}
\end{equation*}
$$

Let $A^{\prime}=A \backslash L_{1}$ and $W^{\prime}=W[j+1, n)$. Note that in our geometrical interpretation of the LCS, we have $A^{\prime}$ positioned above $W^{\prime}$, and slightly to the left. See Fig. 4 .

Consider the starting position of $W^{\prime}$. It is $j+1$ into $W . j$ is symbols right of $i$. So $W^{\prime}$ is $i+s+1$ from the beginning of $V$. In the terminology of Lemma 5.
$i_{t}=i+|A|$ and $t=|A|-s+1$, so $i_{t}-t+2=i+s+1$, and we can thus apply Lemma 5 to $A^{\prime}$ and $W^{\prime}$. So we have

$$
\begin{equation*}
\operatorname{Pr}\left[A \backslash L_{1} \unlhd W[j+1, n)\right] \leq \exp \left(\left(n-|B|-\frac{7}{8} k\left(|A|-\left|L_{1}\right|\right)\right) / 4 k\right) \tag{4}
\end{equation*}
$$

By the same reasoning

$$
\begin{equation*}
\operatorname{Pr}\left[B \backslash L_{2} \unlhd V[0, i)\right] \leq \exp \left(\left(n-|A|-\frac{7}{8} k\left(|B|-\left|L_{2}\right|\right)\right) / 4 k\right) \tag{5}
\end{equation*}
$$

Therefore we combine (3), (4), and (5) to get

$$
\begin{align*}
& \operatorname{Pr}[\operatorname{BigSHIFT} \\
& \leq \sum_{\substack{L_{1} \subseteq A \\
L_{2} \subseteq B \\
\left|L_{1}\right|+\left|L_{2}\right|=s}} \exp \left(\left(2 n-|A|-|B|-\frac{7}{8} k\left(|A|+|B|-\left|L_{1}\right|-\left|L_{2}\right|\right)\right) / 4 k\right)  \tag{6}\\
& \leq \sum_{\substack{L_{1} \subseteq A \\
L_{2} \subseteq B \\
\left|L_{1}\right|+\left|L_{2}\right|=s}} \exp \left(\left(1+\alpha-\varepsilon-\frac{7}{8} k+\frac{7}{8} \alpha k\right) n / 4 k\right) \tag{7}
\end{align*}
$$

Where we simplified (77) using $|A|+|B|=n-\alpha n+s$ and $\left|L_{1}\right|+\left|L_{2}\right|=s$.
Now we simplify the last sum as

$$
\text { (77) } \begin{align*}
& =\binom{|A|+|B|}{s} \exp \left(\left(1+\alpha-\varepsilon-\frac{7}{8} k+\frac{7}{8} \alpha k\right) n / 4 k\right) \\
& \leq \varepsilon n \exp \left(\varepsilon n \log (e(1-\alpha+\varepsilon) / \varepsilon)+\left(1+\alpha-\varepsilon-\frac{7}{8} k+\frac{7}{8} \alpha k\right) n / 4 k\right)  \tag{8}\\
& \leq \exp \left(\left(1+\alpha-\frac{7}{8} k+\frac{7}{8} \alpha k+\varepsilon-\varepsilon \log \varepsilon\right) n / 4 k\right)
\end{align*}
$$

$\operatorname{using}\binom{n}{k} \leq\left(\frac{e n}{k}\right)^{k}$.
Sub-case $n-\alpha n+\varepsilon n \leq \gamma_{k} n$. An edge with span less than $\varepsilon n$ limits the length of the LCS to $n-\alpha n+\varepsilon n \leq \gamma_{k} n$, so the event $\operatorname{BigSHIFT}_{e} \geq \gamma_{k} n+t$ does not occur, i.e., the probability is 0 in this case.

Sub-case $n-\alpha n+\varepsilon n>\gamma_{k} n$. We can bound (8) above by

$$
\begin{aligned}
& \exp \left(\left(1+\left(1+\varepsilon-\gamma_{k}\right)-\frac{7}{8} k+\frac{7}{8}\left(1+\varepsilon-\gamma_{k}\right) k+\varepsilon-\varepsilon \log \varepsilon\right) n / 4 k\right) \\
& \quad \leq \exp \left(\left(2-\left(\frac{7}{8} k+1-\varepsilon\right) \gamma_{k}+2 \varepsilon-\varepsilon \log \varepsilon\right) n / 4 k\right)
\end{aligned}
$$

since $\alpha \leq 1+\varepsilon-\gamma_{k}$ in this case.
Since there are no more than $n^{2}$ choices for edge $e$, the contribution of this case to the right-hand side of (2) is at most

$$
\begin{aligned}
& n^{2} \exp \left(\left(2-\left(\frac{7}{8} k+1-\varepsilon\right) \gamma_{k}+2 \varepsilon-\varepsilon \log \varepsilon\right) n / 4 k\right) \\
& \quad \leq \exp \left(\left(2-\left(\frac{7}{8} k+1-\varepsilon\right) \gamma_{k}+3 \varepsilon-\varepsilon \log \varepsilon\right) n / 4 k\right)
\end{aligned}
$$



Figure 5: Edges from a common subsequence are shown by solid lines and the block boundaries determined by these are shown by dashed lines. Alternating blocks are also shaded for easy viewing.

We claim that $2-\left(\frac{7}{8} k+1-\varepsilon\right) \gamma_{k}+3 \varepsilon-\varepsilon \log \varepsilon<0$. Recall that earlier we set $\varepsilon=0.01$. So we must show that $2.08-\left(\frac{7}{8} k+0.99\right) \gamma_{k}<0$. For $k \geq 3$ the bound $\gamma_{k} \geq 1 / \sqrt{k}$ from [4] suffices. For $k=2$, the lower bound on $\gamma_{k}$ from [8] suffices to show the inequality. Therefore our upper bound in this case is

$$
\exp \left(-c_{1} n\right)
$$

for some positive constant $c_{1}$ depending only on $k$.
Case $\operatorname{span}(e)>\varepsilon n$. To bound the probability of $\operatorname{BigSHIFT}_{e}$ in this case, we will estimate the probability that there is a common subsequence of a certain approximate shape.

Formally, we say that a pair $\left(A_{i}, B_{i}\right)$ is a block if $A_{i}$ is a subword of $V_{i}$ and $B_{i}$ is a subword of $W_{i}$. We say that $\left(A_{i}, B_{i}\right)_{i}$ is a block partition if $\left(A_{i}\right)_{i}$ is a partition of $V$ and $\left(B_{i}\right)_{i}$ is a partition of $W$. In this case, we call the edges between symbols of $A_{i}$ and $B_{i}$ for some $i$ dominated by the partition. We say that a common subsequence $C$ of $V$ and $W$ is dominated by the partition if all the edges in geometric model are dominated by the partition.

A block partition $\left(A_{i}, B_{i}\right)_{i}$ is said to be nonoverlapping if, for each $i$, the symbols in $A_{i}$ and $B_{i}$ are disjoint.

Given a common subsequence $C$ in which every edge satisfies $\operatorname{span}(E)>\varepsilon n$, we construct a partition that dominates it as follows. We first partition $V$ into subwords of length exactly $\varepsilon n$. Let $\left(V_{i}\right)$ be this partition. Consider the first edge in $S$ from $V_{i}$ to $W$. Let $W\left[S_{i}\right]$ be its endpoint in $W$. Then $W$ is partitioned into subwords $W_{i}:=\left[S_{i}, S_{i+1}\right)$. See Fig. 5 for an example.

Because $\operatorname{span}(e)>\varepsilon n$ for every edge $e$ and each $V_{i}$ has length $\varepsilon n$, we see that $V_{i}$ and $W_{i}$ do not overlap. Hence, the resulting block partition is nonoverlapping.

So, it suffices to bound that probability that, given a nonoverlapping partition $\left(V_{i}, W_{i}\right)_{i}$ into $1 / \varepsilon$ blocks, there is a long common subsequence dominated by this partition. The key observation is that because $V_{i}$ and $W_{i}$ do not overlap the symbols in $V_{i}$ and $W_{i}$ are independent.

Let $V^{\prime}$ and $W^{\prime}$ be two random words of length $n$ each, which are disjoint from one another and from $Z$. Partition $V^{\prime}$ into $\left(V_{i}^{\prime}\right)_{i}$ and $W^{\prime}$ into $\left(W_{i}^{\prime}\right)_{i}$ in
such a way that $\left|V_{i}^{\prime}\right|=\left|V_{i}\right|$ and $\left|W_{i}^{\prime}\right|=\left|W_{i}\right|$. Consider

$$
\begin{aligned}
X & =\sum_{i} \operatorname{LCS}\left(V_{i}, W_{i}\right) \\
X^{\prime} & =\sum_{i} \operatorname{LCS}\left(V_{i}^{\prime}, W_{i}^{\prime}\right)
\end{aligned}
$$

Since $\mathbf{E}\left[\operatorname{LCS}\left(V_{i}, W_{i}\right)\right]=\mathbf{E}\left[\operatorname{LCS}\left(V_{i}^{\prime}, W_{i}^{\prime}\right)\right]$, by the linearity of expectation it follows that $\mathbf{E}[X]=\mathbf{E}\left[X^{\prime}\right]$. It is also clear that $X \leq \operatorname{LCS}\left(V^{\prime}, W^{\prime}\right)$, implying that $\mathbf{E}\left[X^{\prime}\right] \leq \mathbf{E}\left[L_{n}\right]$. Therefore, $\mathbf{E}[X] \leq \mathbf{E}\left[L_{n}\right]$.

From Azuma's inequality (Lemma 2) and the bound on $\mathbf{E}\left[L_{n}\right]$ in Lemma 1 we obtain that

$$
\operatorname{Pr}\left[X \geq \gamma_{k} n+t\right] \leq \operatorname{Pr}\left[X \geq \mathbf{E}\left[L_{n}\right]+t\right] \leq \exp \left(-t^{2} / 4 n\right)
$$

Random variable $X$ is the longest length of a common subsequence that is dominated by a given $\left(V_{i}, W_{i}\right)_{i}$. Since that there no more than $\binom{n}{1 / \varepsilon}^{2}$ block partitions into $1 / \varepsilon$ blocks, from the union bound it follows that the contribution of the case $\operatorname{span}(e)>\varepsilon n$ to (2) is at most

$$
\binom{n}{1 / \varepsilon}^{2} \exp \left(\frac{-t^{2}}{4 n}\right) \leq \exp \left(\frac{-t^{2}}{5 n}\right)
$$

since $\varepsilon=0.01$ and $n$ is large enough.
Summing all the contributions from all the cases, we see that the right side of (2) is bounded by

$$
\exp \left(-c_{1} n\right)+\exp \left(-c_{2} t^{2} / n\right) \leq \exp \left(-c_{k} t^{2} / n\right)
$$

for some constant $c_{k}$ as long as $n$ is large enough.

## 5 Proof of Theorem 2

We have two words $V$ and $W$ shifted by $\alpha n$. Divide each word into blocks of size $\alpha n^{1 / 2}$. Write $V_{i}$ for the $i$ th block, $V_{i}=V\left[i \alpha n^{1 / 2},(i+1) \alpha n^{1 / 2}\right)$. Similarly write $W_{i}$ for the $i$ th block from $W$.

Note that $\operatorname{LCS}(V, W) \geq \sum_{i} \operatorname{LCS}\left(V_{i}, W_{i}\right)$, therefore we can bound

$$
\begin{equation*}
\operatorname{Pr}\left[\operatorname{LCS}(V, W) \leq \gamma_{k} n-\ell\right] \leq \operatorname{Pr}\left[\sum_{i} \operatorname{LCS}\left(V_{i}, W_{i}\right) \leq \gamma_{k} n-\ell\right] \tag{9}
\end{equation*}
$$

Applying Lemma 1 we have that

$$
\begin{aligned}
\gamma_{k} n & =\frac{\sqrt{n}}{\alpha} \gamma_{k} \alpha \sqrt{n} \\
& \leq \frac{\sqrt{n}}{\alpha}\left(\mathbf{E}\left[L_{\alpha \sqrt{n}}\right]+4 \sqrt{\alpha \sqrt{n} \log (\alpha \sqrt{n})}\right) \\
& \leq \mathbf{E}\left[\frac{\sqrt{n}}{\alpha} L_{\alpha \sqrt{n}}\right]+4 n^{3 / 4} \sqrt{\log n}
\end{aligned}
$$

Thus we can upper-bound (19) by

$$
\begin{equation*}
\operatorname{Pr}\left[\sum_{i} \operatorname{LCS}\left(V_{i}, W_{i}\right) \leq \mathbf{E}\left[\sum_{i} \operatorname{LCS}\left(V_{i}, W_{i}\right)\right]+4 n^{3 / 4} \sqrt{\log n}-t\right] \tag{10}
\end{equation*}
$$

To bound (10) we apply Lemma (3)

$$
\exp \left(\frac{-2\left(t-4 n^{3 / 4} \sqrt{\log n}\right)^{2}}{n^{1 / 2}\left(\alpha n^{1 / 2}\right)^{2} / \alpha}\right) \leq \exp \left(\frac{-c_{k} t^{2}}{n^{3 / 4}}\right)
$$

for an appropriately small $c_{k}$ if $n$ is large.

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