# About the quadratic Szegő hierarchy 

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March $31^{\text {st }}$, 2018


#### Abstract

The purpose of this paper is to go further into the study of the quadratic Szegó equation, which is the following Hamiltonian PDE : $$
i \partial_{t} u=2 J \Pi\left(|u|^{2}\right)+\bar{J} u^{2}, \quad u(0, \cdot)=u_{0},
$$ where $\Pi$ is the Szegő projector onto nonnegative modes, and $J=J(u)$ is the complex number given by $J=\int_{\mathbb{T}}|u|^{2} u$. We exhibit an infinite set of new conservation laws $\left\{\ell_{k}\right\}$ which are in involution. These laws give us a better understanding of the "turbulent" behavior of certain rational solutions of the equation : we show that if the orbit of a rational solution is unbounded in some $H^{s}, s>\frac{1}{2}$, then one of the $\ell_{k}$ 's must be zero. As a consequence, we characterise growing solutions which can be written as the sum of two solitons.


MSC 2010: 37K10, 35B40.
Keywords : Szegő equation, complete integrability, growth of Sobolev norms.

## 1 Introduction

The equation and its Hamiltonian structure. In this paper, we consider the following quadratic Szegó equation on the torus $\mathbb{T}=(\mathbb{R} / 2 \pi \mathbb{Z})$ :

$$
\begin{equation*}
i \partial_{t} u=2 J \prod\left(|u|^{2}\right)+\bar{J} u^{2} \tag{1}
\end{equation*}
$$

where $u:(t, x) \in \mathbb{R} \times \mathbb{T} \mapsto u(t) \in \mathbb{C}, J$ is a complex number depending on $u$ and given by $J(u):=\int_{\mathbb{T}}|u|^{2} u$, and $\Pi$ is the Szegő projector onto functions with only nonnegative modes :

$$
\Pi\left(\sum_{k \in \mathbb{Z}} a_{k} e^{i k x}\right)=\sum_{k=0}^{+\infty} a_{k} e^{i k x} .
$$

In particular, $\Pi$ acts on $L^{2}(\mathbb{T})$ equipped with its usual inner product $(f \mid g):=\int_{\mathbb{T}} f \bar{g}$. We call $L_{+}^{2}(\mathbb{T})$ the closed subspace of $L^{2}$ which is made of square-integrable functions on $\mathbb{T}$ whose Fourier series is supported on nonnegative frequencies. Then $\Pi$ induces the orthogonal projection from $L^{2}$ onto $L_{+}^{2}$. In the sequel, if $G$ is a subspace of $L^{2}$, we denote by $G_{+}$the subspace $G \cap L_{+}^{2}$ of $L_{+}^{2}$.

Equation (1) appears to be a Hamiltonian PDE : consider $L_{+}^{2}$ as the phase space, endowed with the standard symplectic structure given by $\omega\left(h_{1}, h_{2}\right):=\operatorname{Im}\left(h_{1} \mid h_{2}\right)$. The Hamiltonian associated to (1) is then the following functional :

$$
\mathcal{H}(u):=\frac{1}{2}|J(u)|^{2}=\left.\left.\left|\int_{\mathbb{T}}\right| u\right|^{2} u\right|^{2}
$$

Indeed, if $u$ is regular enough (say $u \in L_{+}^{4}$ ), and $h \in L_{+}^{2}$, we see that $\langle d \mathcal{H}(u), h\rangle=\operatorname{Re}\left(2 J|u|^{2}+\right.$ $\left.\bar{J} u^{2} \mid h\right)=\omega\left(h \mid X_{\mathcal{H}}(u)\right)$, where $X_{\mathcal{H}}(u):=-2 i J \Pi\left(|u|^{2}\right)-i \bar{J} u^{2} \in L_{+}^{2}$ is called the symplectic gradient of $\mathcal{H}$. Equation (1) can be restated as

$$
\begin{equation*}
\dot{u}=X_{\mathcal{H}}(u), \tag{2}
\end{equation*}
$$

where the dot stands for a time-derivative : in other words, (1) is the flow of the vector field $X_{\mathcal{H}}$.
If now $\mathcal{F}$ is some densely-defined differentiable functional on $L_{+}^{2}$, and if $u$ is a smooth solution of (2), then

$$
\frac{d}{d t} \mathcal{F}(u)=\langle d \mathcal{F}(u), \dot{u}\rangle=\omega\left(\dot{u}, X_{\mathcal{F}}(u)\right)=\omega\left(X_{\mathcal{H}}(u), X_{\mathcal{F}}(u)\right)
$$

Defining the Poisson bracket $\{\mathcal{H}, \mathcal{F}\}$ to be the functional given by $\{\mathcal{H}, \mathcal{F}\}=\omega\left(X_{\mathcal{H}}, X_{\mathcal{F}}\right)$, the evolution of $\mathcal{F}$ along flow lines of (2) is thus given by the equation $\dot{\mathcal{F}}=\{\mathcal{H}, \mathcal{F}\}$. In particular, $\dot{\mathcal{H}}=$ $\{\mathcal{H}, \mathcal{H}\}=0$, which means that the Hamiltonian $\mathcal{H}$ is conserved (at least for smooth solutions). Hence the factor $J$ in (1) only evolves through its argument, explaining the terminology of "quadratic equation". Two other conservation laws arise from the invariances of $\mathcal{H}$ : the mass $Q$ and the momentum $M$ defined by

$$
\begin{aligned}
Q(u) & :=\int_{\mathbb{T}}|u|^{2} \\
M(u) & :=\int_{\mathbb{T}} \bar{u} D u, \quad D:=-i \partial_{x}
\end{aligned}
$$

We have $\{\mathcal{H}, Q\}=\{\mathcal{H}, M\}=0$. Moreover, as $u$ only has nonnegative modes, these conservation laws control the $H^{1 / 2}$ regularity of $u$, namely $(Q+M)(u) \simeq\|u\|_{H^{1 / 2}}^{2}$.

Observe that replacing the variable $e^{i x} \in \mathbb{T}$ by $z \in \mathbb{D}$ in the Fourier series induces an isometry between $L_{+}^{2}$ and the Hardy space $\mathbb{H}^{2}(\mathbb{D})$, which is the set of holomorphic functions on the unit open $\operatorname{disc} \mathbb{D}:=\{z \in \mathbb{C}| | z \mid<1\}$ whose trace on the boundary $\partial \mathbb{D}$ lies in $L^{2}$. Therefore, we will often consider solutions of equation (1) as functions of $t \in \mathbb{R}$ and $z \in \mathbb{D}$.

Invariant manifolds. Equation (1) was first introduced in [15], following the seminal work of Gérard and Grellier on the cubic Szegő equation $[4,5,6,7]$ :

$$
\begin{equation*}
i \partial_{t} u=\Pi\left(|u|^{2} u\right) \tag{3}
\end{equation*}
$$

As (3), equation (1) can be considered as a toy model of a nonlinear non-dispersive equation, whose study is expected to give hints on physically more relevant equations, such as the conformal flow on $\mathbb{S}^{3}$ [2] or the cubic Lowest-Landau-Level (LLL) equation [3].

In $[15,16]$, using the formalism of equation (3), the author proved that there is a Lax pair structure associated to the quadratic equation (1), that we are now going to recall, as well as some of its consequences.

If $u \in H_{+}^{1 / 2}(\mathbb{T})$, we define the Hankel operator of symbol $u$ by $H_{u}: L_{+}^{2} \rightarrow L_{+}^{2}, h \mapsto \Pi(u \bar{h})$. It is a bounded $\mathbb{C}$-antilinear operator over $L_{+}^{2}$, and $H_{u}^{2}$ is trace class (hence compact), selfadjoint and positive. Consequently, we can write its spectrum as a decreasing sequence of nonnegative eigenvalues

$$
\rho_{1}^{2}(u)>\rho_{2}^{2}(u)>\cdots>\rho_{n}^{2}(u)>\cdots \longrightarrow 0
$$

each of them having some multiplicity greater than 1 . Analogous to the family of Hankel operators is the one of Toeplitz operators : given a symbol $b \in L^{\infty}(\mathbb{T})$, we define $T_{b}: L_{+}^{2} \rightarrow L_{+}^{2}, h \mapsto \Pi(b h)$, which is $\mathbb{C}$-linear and bounded on $L_{+}^{2}$. Its adjoint is $\left(T_{b}\right)^{*}=T_{\bar{b}}$. A special Toeplitz operator is called the (right) shift $S:=T_{e^{i x}}$. Thus we can define another operator which turns out to be of great importance in the study of (1) : for $u \in H_{+}^{1 / 2}$, the shifted Hankel operator is defined by $K_{u}:=H_{u} S$. An easy computation shows that $K_{u}$ also satisfies $K_{u}=S^{*} H_{u}=H_{S^{*} u}$. As a consequence,

$$
\begin{equation*}
K_{u}^{2}(h)=H_{u}^{2}(h)-(h \mid u) u, \quad \forall h \in L_{+}^{2} \tag{4}
\end{equation*}
$$

$K_{u}^{2}$ is compact as well, selfadjoint and positive, hence we can denote its eigenvalues by the decreasing sequence $\sigma_{1}^{2}(u)>\cdots>\sigma_{n}^{2}(u)>\cdots \rightarrow 0$. In fact, (4) leads to a more accurate interlacement property :

$$
\begin{equation*}
\rho_{1}^{2}(u) \geq \sigma_{1}^{2}(u) \geq \rho_{2}^{2}(u) \geq \sigma_{2}^{2}(u) \geq \cdots \geq \rho_{n}^{2}(u) \geq \sigma_{n}^{2}(u) \geq \cdots \longrightarrow 0 \tag{5}
\end{equation*}
$$

where there cannot be two consecutive equality signs.
The idea of a Lax pair is to look at the evolution of a solution $t \mapsto u(t)$ of (1) by associating to each $u(t)$ an operator $L_{u(t)}$ acting on some Hilbert space, and by computing the evolution of this operator rather than that of the function $u(t)$ itself. First of all, thanks to the conservation laws $\mathcal{H}, Q$ and $M$, it can be shown that the flow of $(1)$ is well defined on every $H_{+}^{s}(\mathbb{T})$ for $s>\frac{1}{2}$ [15]. We refer to solutions belonging to theses spaces as smooth solutions. The statement of the Lax pair theorem is then the following :

Theorem 1 ([15, 16]). Let $t \mapsto u(t)$ be a smooth solution of the quadratic Szegô equation (1). Then the evolution of $H_{u(t)}$ and $K_{u(t)}$ is given by

$$
\begin{aligned}
\frac{d}{d t} K_{u} & =B_{u} K_{u}-K_{u} B_{u} \\
\frac{d}{d t} H_{u} & =B_{u} H_{u}-H_{u} B_{u}+i \bar{J}(u \mid \cdot) u
\end{aligned}
$$

where $B_{u}:=-i\left(T_{\bar{J} u}+T_{J \bar{u}}\right)$ is a bounded anti-selfadjoint operator over $L_{+}^{2}$.
Only the first identity concerning $K_{u}$ is a rigorous Lax pair, but the second one turns out [16] to give helpful informations about $H_{u}$ as well. In particular, we have the following corollary :

Corollary 1.1. If $t \mapsto u(t)$ is a smooth solution of (1), then $\operatorname{rk}\left(K_{u}\right)$ and $\operatorname{rk}\left(H_{u}\right)$ are conserved. For any $j \geq 1, \sigma_{j}^{2}(u(t))$ is also conserved.

This corollary is of particular interest when $H_{u}$ has finite rank. In that case, since $K_{u}=H_{u} S$, we have $\operatorname{rk}\left(K_{u}\right) \leq \operatorname{rk}\left(H_{u}\right)$. Because $\operatorname{rk}\left(H_{u}\right)=\operatorname{rk}\left(H_{u}^{2}\right)$ and the same for $K_{u}$ and $K_{u}^{2}$, we must have by (4) that $\operatorname{rk}\left(K_{u}\right) \in\left\{\operatorname{rk}\left(H_{u}\right), \operatorname{rk}\left(H_{u}\right)-1\right\}$. Therefore, for $d \in \mathbb{N}$, we designate by $\mathcal{V}(d)$ the set of symbols $u \in H_{+}^{1 / 2}$ such that $\operatorname{rk}\left(H_{u}\right)+\operatorname{rk}\left(K_{u}\right)=d$.

It turns out that $\mathcal{V}(d)$ can be explicitely characterized (see [4]): it is the set of rational functions of the variable $z$ of the form

$$
u(z)=\frac{A(z)}{B(z)}
$$

where $A$ and $B$ are complex polynomials, such that $A \wedge B=1, B(0)=1$ and $B$ has no root in the closed disc $\overline{\mathbb{D}}$, and such that

- (case $d=2 N)$ the degree of $B$ is exactly $N$ and the degree of $A$ is at most $N-1$,
- (case $d=2 N+1$ ) the degree of $A$ is exactly $N$ and the degree of $B$ is at most $N$.

Since functions of $\mathcal{V}(d)$ obviously belong to $C_{+}^{\infty}$, they give rise to smooth solutions of (1), and by the previous corollary, $\mathcal{V}(d)$ is left invariant by the flow of the quadratic Szegő equation.

Geometrically speaking, $\mathcal{V}(d)$ is a complex manifold of dimension $d$. Moreover, restricting the scalar product $(\cdot \mid \cdot)$ to the tangent space $T_{u} \mathcal{V}(d)$ for each $u \in \mathcal{V}(d)$ defines a Hermitian metric on $\mathcal{V}(d)$ whose imaginary part induces a symplectic structure on the $2 d$-dimensional real manifold $\mathcal{V}(d)$. In other words, $\mathcal{V}(d)$ is a Kähler manifold.

Additional conservation laws. It is a natural question to ask whether the finite-dimensional ODE induced by (1) on $\mathcal{V}(d)$ is integrable or not, in the sense of the classical Hamiltonian mechanics. The celebrated Arnold-Jost-Liouville-Mineur theorem [1, 10, 12, 13] states that this problem first consists in finding $d$ conservation laws (for a $2 d$-dimensional manifold) that are generically independent and in involution (i.e. such that $\{F, G\}=0$ for any choice of $F, G$ among these laws).

In the case of the cubic Szegó equation (3), the $\mathcal{V}(d)$ 's are also invariant by the flow, and such conservation laws were first found in [4]. Relying on the fact that $H_{u}$ and $K_{u}$ satisfy an exact Lax pair, it can be proved [5] that both the $\rho_{j}^{2}$ 's and the $\sigma_{k}^{2}$ 's are generically independent conservation laws for solutions of (3), and they satisfy in addition

$$
\left\{\rho_{j}^{2}, \rho_{k}^{2}\right\}=0, \quad\left\{\sigma_{j}^{2}, \sigma_{k}^{2}\right\}=0, \quad\left\{\rho_{j}^{2}, \sigma_{k}^{2}\right\}=0
$$

for any choice of indices $j, k \geq 1$.
In our case, Corollary 1.1 states that the $\sigma_{k}^{2}$, s are conservation laws for (1). But the $\rho_{j}^{2}$,s are no more conserved, that is why the Lax pair theorem only provides $\lfloor d / 2\rfloor$ conservation laws on $\mathcal{V}(d)$. The purpose of this paper is then to investigate and find the missing ones, to get the full quadratic Szegő hierarchy.

We can now state the main theorem of this paper. Let $u \in H_{+}^{1 / 2}$, and recall that

$$
\sigma_{1}^{2}(u)>\sigma_{2}^{2}(u)>\cdots>\sigma_{k}^{2}(u)>\cdots
$$

is the decreasing list of the distinct eigenvalues of $K_{u}^{2}$. For $k \geq 1$, we set $F_{u}\left(\sigma_{j}(u)\right):=\operatorname{ker}\left(K_{u}^{2}-\right.$ $\left.\sigma_{k}^{2}(u) I\right)$, and we introduce

$$
\begin{aligned}
u_{k}^{K} & :=\mathbb{1}_{\left\{\sigma_{k}^{2}(u)\right\}}\left(K_{u}^{2}\right)(u), \\
w_{k}^{K} & :=\mathbb{1}_{\left\{\sigma_{k}^{2}(u)\right\}}\left(K_{u}^{2}\right)\left(\Pi\left(|u|^{2}\right)\right),
\end{aligned}
$$

in the sense of the functional calculus. In other terms, $u_{k}$ (resp. $w_{k}$ ) is the orthogonal projection of $u$ (resp. $\left.\Pi\left(|u|^{2}\right)\right)$ onto the finite-dimensional subspace $F_{u}\left(\sigma_{k}\right)$ of $L_{+}^{2}$. Finally, we set

$$
\ell_{k}(u):=\left(2 Q+\sigma_{k}^{2}\right)\left\|u_{k}^{K}\right\|_{L^{2}}^{2}-\left\|w_{k}^{K}\right\|_{L^{2}}^{2}
$$

By convention, we call $\ell_{\infty}$ the quantity that we obtain by replacing $\sigma_{k}^{2}$ by 0 in the above functional (thus considering the projection of $u$ and $\Pi\left(|u|^{2}\right)$ onto the kernel of $\left.K_{u}^{2}\right)$.

The main result reads as follows :
Theorem 2. We have the following identities on $H_{+}^{1 / 2}$ :

$$
\left\{\ell_{j}, \ell_{k}\right\}=0, \quad\left\{\ell_{j}, \sigma_{k}^{2}\right\}=0, \quad\left\{\sigma_{j}^{2}, \sigma_{k}^{2}\right\}=0
$$

for any $j, k \geq 1$.
Furthermore, the $\ell_{k}$ 's are conservation laws for the quadratic Szegó equation (1).
Let us comment on this result :

- The question of finding additional conservation laws was first raised in [4] for the cubic Szegó equation on $\mathbb{T}$, at a time when the Lax pair for $K_{u}$ and the conservation laws $\sigma_{k}^{2}$ had not been discovered. These laws were found to be the $J_{2 n}(u):=\left(H_{u}^{2 n}(1) \mid 1\right), n \geq 1$. A similar inquiry turned out to be necessary in the study of related equations for which only one Lax pair is available, such as the cubic Szegó equation on $\mathbb{R}$ [14], or the cubic Szegő equation with a linear perturbative term on $\mathbb{T}[17,18]$.
- We will prove in the beginning of Section 4 that the knowledge of the $\ell_{j}$ 's and the $\sigma_{k}^{2}$ 's enables to reconstruct the a priori conservation laws $M, Q$ and $\mathcal{H}$. We have for instance

$$
\begin{aligned}
Q^{2} & =\sum_{k \geq 1} \ell_{k} \\
|J|^{2} & =\sum_{k \geq 1}\left(Q+\sigma_{k}^{2}\right) \ell_{k}
\end{aligned}
$$

However, the question of the generic independence of the $\ell_{k}$ 's is left unanswered.

- The proof of Theorem 2 relies on generating series. For rational data (i.e. having a finite sequence of $\sigma_{k}^{2}$ of cardinality $N$ ) and an appropriate $x \in \mathbb{R}$, we will show that

$$
\sum_{k=1}^{N} \frac{\ell_{k}}{1-x \sigma_{k}^{2}}=\frac{x^{2} \mathscr{J}^{(4)}(x)^{2}-x\left|\mathscr{J}^{(3)}(x)\right|^{2}-Q^{2}}{\mathscr{J}^{(0)}(x)}
$$

where for $m \geq 0$,

$$
\mathscr{J}^{(m)}(x):=\left(\left(I-x H_{u}^{2}\right)^{-1}\left(H_{u}^{m}(1)\right) \mid 1\right)=\sum_{j=0}^{+\infty} x^{j} J_{m+2 j}
$$

and $J_{p}:=\left(H_{u}^{p}(1) \mid 1\right)$ as above. Using the commutation relations between $\rho_{j}^{2}$ and $\sigma_{k}^{2}$ as well as the action-angle coordinates coming from the cubic Szegó equation [7], we will find that

$$
\left\{\sum_{k=1}^{N} \frac{\ell_{k}}{1-x \sigma_{k}^{2}}, \sum_{k=1}^{N} \frac{\ell_{k}}{1-y \sigma_{k}^{2}}\right\}=0
$$

for all $x \neq y$.

Connection with the growth of Sobolev norms for rational solutions. An important question in the study of Hamiltonian PDEs is the question of the existence of "turbulent" trajectories : provided that $M$ and $Q$ are conserved, does there exist initial data $u_{0} \in C_{+}^{\infty}$ giving rise to solutions of (1) such that

$$
\limsup _{t \rightarrow+\infty}\|u(t)\|_{H^{s}}=+\infty
$$

for some $s>\frac{1}{2}$ ?
A positive answer to this question is given in [15], where however it is shown that such a growth cannot happen faster than exponentially in time. An explicit computation tells us that this rate of growth is indeed achieved for solutions on $\mathcal{V}(3)$ satisfying the following condition :

$$
\begin{equation*}
|J|^{2}=Q^{3} \tag{6}
\end{equation*}
$$

More precisely, solutions of the form

$$
u(z)=b+\frac{c z}{1-p z}
$$

with $b, c, p \in \mathbb{C}, c \neq 0, b-c p \neq 0$ and $|p|<1$, which also satisfy (6), are such that for any $s>1 / 2$, there exists a constant $C_{s}>0$ such that $\|u(t)\|_{H^{s}} \sim C_{s} e^{C_{s}|t|}$.

As in [18], it appears that the possible growth of Sobolev norms can be detected in terms of the new conservation laws $\ell_{k}$.

Proposition 1.2. Let $v^{n}$ be some sequence in $\mathcal{V}(d)$ for some $d \in \mathbb{N}$. Assume that it is bounded in $H_{+}^{1 / 2}$ and that $\operatorname{Sp} K_{v^{n}}^{2}$ does not depend on $n$. Then the following statements are equivalent :
(i) There exists $s_{0}>\frac{1}{2}$ such that $v^{n}$ is unbounded in $H_{+}^{s_{0}}$.
(ii) For every $s>\frac{1}{2}, v^{n}$ is unbounded in $H_{+}^{s}$.
(iii) There exists a subsequence $\left\{n_{k}\right\}$ and $v_{\text {bad }} \in \mathcal{V}\left(d^{\prime}\right)$ (where $d^{\prime} \leq d-1$ if $d$ is even, and $d^{\prime} \leq d-2$ if $d$ is odd), such that

$$
v^{n_{k}} \rightharpoonup v_{\mathrm{bad}} \quad \text { in } H_{+}^{\frac{1}{2}}
$$

This proposition implies a necessary condition on initial data for some growth of Sobolev norm to occur for solutions of the quadratic Szegó equation (1).
Corollary 1.3. Assume that $u_{0} \in \mathcal{V}(d)$ for some $d \in \mathbb{N}$, and assume that there exists $s_{0}>\frac{1}{2}$ such that the corresponding solution $u(t)$ of (1) is unbounded in $H_{+}^{s_{0}}$. Then $u(t)$ is unbounded in every $H^{s}, s>\frac{1}{2}$. Furthermore, for some $k \geq 1$ such that $\sigma_{k}^{2}$ is the $k$-th non-zero eigenvalue of $K_{u_{0}}^{2}$, we must have

$$
\ell_{k}\left(u_{0}\right)=0 .
$$

Remark 1. The proof of Proposition 1.2 relies on a connection between growth of Sobolev norms and loss of compactness, quantified by equipping $H_{+}^{1 / 2}$ with the weak topology and studying the cluster points of the strongly bounded sequence $u^{n}$. This idea can be illustrated by the following basic example. Pick some $\ell^{2}$ sequence of positive numbers $\left(a_{k}\right)$, and consider the periodic functions defined by

$$
f_{n}(x):=\sum_{k=0}^{+\infty} a_{k} e^{i(k n) x}, \quad n \in \mathbb{N}, x \in \mathbb{T}
$$

Then the sequence $f_{n}$ is uniformly bounded in $L^{2}$, but the $L^{2}$-energy of $f_{n}$ obviously moves toward high frequencies (or equivalently, the $H^{s}$ norm of $f_{n}, s>0$, is morally going to grow like $n^{s}$ ). This phenomenon can be described saying that the only weak cluster point of the sequence $f_{n}$ in $L^{2}$ is $a_{0}$, and $\left|a_{0}\right|^{2}<\left\|f_{n}\right\|_{L^{2}}^{2}$. This energy loss through high-frequency energy transfer is precisely what is captured by Proposition 1.2.

This result enables to find the right counterpart of condition (6) for solutions in $\mathcal{V}(4)$ :
Theorem 3. A solution $t \mapsto u(t)$ of (1) in $\mathcal{V}(4)$ is unbounded in some $H^{s}, s>\frac{1}{2}$, if and only if $K_{u(t)}^{2}$ has two distinct eigenvalues of multiplicity $1, \sigma_{1}^{2}>\sigma_{2}^{2}$, and if $\ell_{1}(u)=0$ or equivalently

$$
\begin{equation*}
|J|^{2}=Q^{2}\left(Q+\sigma_{2}^{2}\right) \tag{7}
\end{equation*}
$$

In that case, for all $s>1 / 2$, there exists constants $C_{s}, C_{s}^{\prime}>0$ such that we have

$$
\frac{1}{C_{s}} e^{C_{s}^{\prime}|t|} \leq\|u(t)\|_{H^{s}} \leq C_{s} e^{C_{s}^{\prime}|t|}, \quad \text { as } t \rightarrow \pm \infty
$$

Example. A concrete example of a function of $\mathcal{V}(4)$ satisfying (7) is given by

$$
v(z):=\frac{z}{(1-p z)^{2}}, \quad \forall z \in \mathbb{D},
$$

whenever $|p|^{2}=3 \sqrt{2}-4 \simeq 0,2426 \ldots$
Remark 2. The interest of Theorem 3 is to display the case of an interaction between two solitons. Traveling waves for equation (1) are classified in [16], and this turbulent solution $u$ appears to be the exact sum of two solitons. Indeed, a turbulent solution such as the one described above will be after some time $T$ of the form

$$
u(t, z)=\frac{\alpha}{1-p z}+\frac{\beta}{1-q z},
$$

with $\alpha, \beta, p, q \in \mathbb{C}$, with $|p|,|q|<1$ and $p \neq q$. One of the two poles approaches the unit circle $\partial \mathbb{D}$ exponentially fast, while the other remains inside a disc of radius $r<1$.

Open questions. The picture we draw here remains far from being complete. First of all, now that we have $d$ conservation laws in involution on $\mathcal{V}(d)$, we would like to apply the ArnoldLiouville theorem. For that purpose, we should give a description of the level sets of the $\ell_{j}$ 's and the $\sigma_{k}^{2}$ 's on $\mathcal{V}(d)$, and find which ones are compact in $H_{+}^{1 / 2}$. Obviously, some are not, since we found solutions in $\mathcal{V}(3)$ and $\mathcal{V}(4)$ that leave every compact of $H^{1 / 2}$.

Then, to solve explicitely the quadratic Szegó equation (1) on $\mathcal{V}(d)$, we should find angle coordinates in $\mathbb{T}^{d}$ (for compact level sets) or in $\mathbb{T}^{d^{\prime}} \times \mathbb{R}^{d-d^{\prime}}$ (in the general case), for some $d^{\prime}<d$. Angle coordinates for the cubic Szegő equation are found in [5] for the torus, and in [14] for the real line. For the case of action-angle coordinates for other integrable PDEs, one can refer to $[8,11]$. Noteworthy is that the angle associated to $\sigma_{k}^{2}$ in the case of the cubic Szegó coordinates does not evolve linearly in time through the flow of the quadratic equation (1) (see Lemma 3.7 below, where we compute its evolution).

The exact situation on $\mathcal{V}(4)$ is not completely understood either. Whereas on $\mathcal{V}(3)$, only $\ell_{1}$ can cancel out, corresponding to (6), and $\ell_{\infty}(u)>0$ for all $u \in \mathcal{V}(3)$, it is not certain whether $\ell_{2}$ can be zero on $\mathcal{V}(4)$. In any case, Theorem 3 is enough to say that solutions of (1) on $\mathcal{V}(4)$ such that $\ell_{2}=0$, if any, are bounded in every $H^{s}$ topology.

A broadly open question naturally concerns the case of the $\mathcal{V}(d)$ 's for $d \geq 5$. By the substitution principle that is stated in [16, Proposition 3.5], replacing $z$ by $z^{N}, N \geq 2$, in turbulent solutions of $\mathcal{V}(3)$ and $\mathcal{V}(4)$ will allow us to give examples of exponentially growing solutions on each of the $\mathcal{V}(d)$ 's, $d \geq 5$. However, can we completely classify such growing solutions? Is it possible to find other types or rates of growth, such as a polynomial one, or an intermittent one (i.e. a solution satisfying both $\lim \sup \|u(t)\|_{H^{s}}=\infty$ and $\left.\lim \inf \|u(t)\|_{H^{s}}<\infty\right)$ ?

Going from rational solutions to general data in $H_{+}^{1 / 2}$ is our long-term objective. We would like to understand, as in [7] for the cubic Szegő equation or in [9] for the resonant NLS, which is the generic behaviour of solutions of (1) on that space. To this end, it seems unlikely that we can get around the construction of action-angle variables.

Plan of the paper. After some preliminaries in Section 2 about the spectral theory of $H_{u}$ and $K_{u}$, we will see in Section 3 how to prove simply that the $\ell_{j}$ 's are conserved along the evolution of (1), and we prove that the cancellation of at least one $\ell_{j}$ is a necessary condition for growth of Sobolev norms to occur. In Section 4, we analyse the case of $\mathcal{V}(4)$. Section 5 is finally devoted to the proof of the commutation of the $\ell_{j}$ 's.

Acknowledgements. The author would like to express his gratitude to Pr. Patrick Gérard, who provided him much insightful advice during this work.

## 2 Preliminaries: spectral theory of $H_{u}$ and $K_{u}$

For the sake of completeness, we recall in this section some of the results of [7], where the spectral theory of compact Hankel operators is studied in great detail.

We begin with a definition :

Definition (Finite Blaschke products). A function $\Psi \in L_{+}^{2}$ is called a Blaschke product of degree $m \geq 0$ if there exists $\psi \in \mathbb{T}$ as well as $m$ complex numbers $a_{j} \in \mathbb{D}, j \in \llbracket 1, m \rrbracket$, such that

$$
\Psi(z)=e^{i \psi} \prod_{j=1}^{m} \frac{z-\overline{a_{j}}}{1-a_{j} z}, \quad \forall z \in \mathbb{D} .
$$

$\psi$ is called the angle of $\Psi$, and $D(z)=\prod_{j=1}^{m}\left(1-a_{j} z\right)$ is called the normalized denominator of $\Psi$ (i.e. with $D(0)=1$ ).

Observe that a Blaschke product of degree $m$ belongs to $\mathcal{V}(2 m+1)$, but more importantly, if $\Psi$ is a Blaschke product, then $\left|\Psi\left(e^{i x}\right)\right|^{2}=1$ for all $x \in \mathbb{T}$. In particular, $\Psi \in L_{+}^{\infty}$.

Singular values. Now, fix $u \in H_{+}^{1 / 2}$. For $s \geq 0$, we introduce two subspaces of $L_{+}^{2}$ defined by

$$
\begin{aligned}
E_{u}(s) & :=\operatorname{ker}\left(H_{u}^{2}-s^{2} I\right), \\
F_{u}(s) & :=\operatorname{ker}\left(K_{u}^{2}-s^{2} I\right)
\end{aligned}
$$

We denote by $\Xi_{u}^{H}$ (resp. $\Xi_{u}^{K}$ ) the set of $s>0$ such that $E_{u}(s)$ (resp. $\left.F_{u}(s)\right)$ is not $\{0\}$. It is the set of the square-roots of the non-zero eigenvalues of $H_{u}^{2}$ (resp. $K_{u}^{2}$ ). We call them the singular values associated to $u$. The link between $\Xi_{u}^{H}$ and $\Xi_{u}^{K}$ can be described more precisely :
Proposition 2.1 ([7, Lemma 3.1.1]). Let $s \in \Xi_{u}^{H} \cup \Xi_{u}^{K}$. Then one of the following holds:
(i) $\operatorname{dim} E_{u}(s)=\operatorname{dim} F_{u}(s)+1, u \not \perp E_{u}(s)$, and $F_{u}(s)=E_{u}(s) \cap u^{\perp}$;
(ii) $\operatorname{dim} F_{u}(s)=\operatorname{dim} E_{u}(s)+1, u \not \perp F_{u}(s)$, and $E_{u}(s)=F_{u}(s) \cap u^{\perp}$.

In the first case, we say that $s$ is $H$-dominant, and we write $s \in \Sigma_{u}^{H}$.
In the second case, we say that $s$ is $K$-dominant, and we write $s \in \Sigma_{u}^{K}$.
It also appears that, writing $\Xi_{u}^{H} \cup \Xi_{u}^{K}$ as a decreasing sequence (with no repetition), $H$ dominant singular values are given by the odd terms, and $K$-dominant by the even ones.

Projections. Let $\left\{\rho_{j}\right\}_{j \geq 1}$ (resp. $\left\{\sigma_{k}\right\}_{k \geq 1}$ ) be the decreasing list of the elements of $\Xi_{u}^{H}$ (resp. $\left.\Xi_{u}^{K}\right)$. We define

$$
\begin{array}{c|l}
u_{j}^{H} & \text { the projection of } u \text { onto } E_{u}\left(\rho_{j}\right) \\
u_{k}^{K} & \text { the projection of } u \text { onto } F_{u}\left(\sigma_{k}\right) \\
w_{k}^{K} & \text { the projection of } \Pi\left(|u|^{2}\right) \text { onto } F_{u}\left(\sigma_{k}\right)
\end{array}
$$

The notation $u_{j}^{H}$ should be read as "the projection of $u$ onto the $j$-th eigenspace of $H_{u}^{2 " \text { ". }}$
By Proposition 2.1, $u_{j}^{H} \neq 0$ if and only if $\rho_{j} \in \Sigma_{u}^{H}$, and $u_{k}^{K} \neq 0$ if and only if $\sigma_{k} \in \Sigma_{u}^{K}$. In particular,

$$
\begin{equation*}
u=\sum_{k \geq 1} u_{k}^{K}+u_{\infty}^{K}=\sum_{\substack{k \geq 1 \\ \sigma_{k} \in \sum_{u}^{K}}} u_{k}^{K}+u_{\infty}^{K}, \tag{8}
\end{equation*}
$$

where $u_{\infty}^{K}$ stands for the projection of $u$ onto $\operatorname{ker} K_{u}^{2}$. The same formula holds for $u_{j}^{H}$, but the extra term is no more needed, since $u \perp \operatorname{ker} H_{u}^{2}$.

These decompositions of $u$ appears to be very useful, for we can describe how $H_{u}$ and $K_{u}$ act on $E_{u}(s)$ and $F_{u}(s), s>0$. This is what is summed up in the next proposition :

Proposition 2.2 ([7, Proposition 3.5.1]). - If $s \in \Sigma_{u}^{H}$, write $s=\rho_{j}$ for some $j \geq 1$. Let $m=\operatorname{dim} E_{u}\left(\rho_{j}\right)=\operatorname{dim} F_{u}\left(\rho_{j}\right)+1$. Then there exists $\Psi_{j}^{H}$, a Blaschke product of degree $m-1$, such that

$$
\rho_{j} u_{j}^{H}=\Psi_{j}^{H} H_{u}\left(u_{j}^{H}\right) .
$$

In addition, if $D$ is the normalized denominator of $\Psi_{j}^{H}$, then

$$
\begin{aligned}
& E_{u}\left(\rho_{j}\right)=\left\{\left.\frac{f}{D} H_{u}\left(u_{j}^{H}\right) \right\rvert\, f \in \mathbb{C}_{m-1}[z]\right\}, \\
& F_{u}\left(\rho_{j}\right)=\left\{\left.\frac{g}{D} H_{u}\left(u_{j}^{H}\right) \right\rvert\, g \in \mathbb{C}_{m-2}[z]\right\},
\end{aligned}
$$

and $H_{u}$ (resp. $K_{u}$ ) acts on $E_{u}\left(\rho_{j}\right)$ (resp. $\left.F_{u}\left(\rho_{j}\right)\right)$ by reversing the order of the coefficients of the polynomial $f$ (resp. g), conjugating them, and multiplying the result by $\rho_{j} e^{i \psi_{j}}$, where $\psi_{j}$ is the angle of $\Psi_{j}^{H}$.

- If $s \in \Sigma_{u}^{K}$, write $s=\sigma_{k}$ for some $k \geq 1$. Let $m^{\prime}=\operatorname{dim} F_{u}\left(\sigma_{k}\right)=\operatorname{dim} E_{u}\left(\sigma_{k}\right)+1$. Then there exists $\Psi_{k}^{K}$, a Blaschke product of degree $m^{\prime}-1$, such that

$$
K_{u}\left(u_{k}^{K}\right)=\sigma_{k} \Psi_{k}^{K} u_{k}^{K}
$$

In addition, if $D$ is the normalized denominator of $\Psi_{k}^{K}$, then

$$
\begin{aligned}
& F_{u}\left(\sigma_{k}\right)=\left\{\left.\frac{f}{D} u_{k}^{K} \right\rvert\, f \in \mathbb{C}_{m^{\prime}-1}[z]\right\}, \\
& E_{u}\left(\sigma_{k}\right)=\left\{\left.\frac{z g}{D} u_{k}^{K} \right\rvert\, g \in \mathbb{C}_{m^{\prime}-2}[z]\right\},
\end{aligned}
$$

and $K_{u}\left(\right.$ resp. $\left.H_{u}\right)$ acts on $F_{u}\left(\sigma_{k}\right)$ (resp. $\left.E_{u}\left(\sigma_{k}\right)\right)$ by reversing the order of the coefficients of the polynomial $f$ (resp. $g$ ), conjugating them, and multiplying the result by $\sigma_{k} e^{i \psi_{k}}$, where $\psi_{k}$ is the angle of $\Psi_{k}^{K}$.

We also recall a formula which enables to compute $\left\|u_{k}^{K}\right\|_{L^{2}}^{2}$ and $\left\|u_{j}^{H}\right\|_{L^{2}}^{2}$ in terms of the singular values. For $s \in \Sigma_{u}^{H}$, we call $\sigma(s)$ the biggest element of $\Sigma_{u}^{K}$ which is smaller than $s$, if it exists, or 0 otherwise. With this notation, we have the following formulae :

Proposition 2.3 ([7, Proposition 3.2.1]). Let $s=\rho_{j} \in \Sigma_{u}^{H}$ and let $\sigma_{k}=\sigma(s)$. We have

$$
\begin{gathered}
\left\|u_{j}^{H}\right\|_{L^{2}}^{2}=\left(s^{2}-\sigma(s)^{2}\right) \prod_{s^{\prime} \neq s} \frac{s^{2}-\sigma\left(s^{\prime}\right)^{2}}{s^{2}-s^{\prime 2}}, \\
\left\|u_{k}^{K}\right\|_{L^{2}}^{2}=\left(s^{2}-\sigma(s)^{2}\right) \prod_{s^{\prime} \neq s} \frac{\sigma(s)^{2}-s^{\prime 2}}{\sigma(s)^{2}-\sigma\left(s^{\prime}\right)^{2}},
\end{gathered}
$$

where the products are taken over $s^{\prime} \in \Sigma_{u}^{H}$.

The inverse spectral formula. In this paragraph, we state a weaker version of the main result in [7]. In the previous propositions, we have associated to each $u \in H_{+}^{1 / 2}$ a set of $H$ dominant of $K$-dominant singular values, each of them being linked to some finite Blaschke product. Conversely, let $q \in \mathbb{N} \backslash\{0\}$ and $s_{1}>s_{2}>\cdots>s_{2 q-1}>s_{2 q} \geq 0$ some real numbers. Let also $\Psi_{n}, n \in \llbracket 1,2 q \rrbracket$, be finite Blaschke products. We define a matrix $\mathscr{C}(z)$, where $z \in \mathbb{D}$ is a parameter, by its coefficients

$$
c_{j, k}=\frac{s_{2 j-1}-z s_{2 k} \Psi_{2 j-1}(z) \Psi_{2 k}(z)}{s_{2 j-1}^{2}-s_{2 k}^{2}}, \quad 1 \leq j, k \leq q
$$

Theorem $4([7$, Theorem 1.0.3]). For all $z \in \mathbb{D}, \mathscr{C}(z)$ is invertible, and if we set

$$
u(z):=\sum_{1 \leq j, k \leq q}\left[\mathscr{C}(z)^{-1}\right]_{j, k} \Psi_{2 k-1}(z)
$$

then $u \in \mathcal{V}(2 q)$ (or $u \in \mathcal{V}(2 q-1)$ if $\left.s_{2 q}=0\right)$.
Furthermore, it is the unique function in $H_{+}^{1 / 2}$ such that the $H$-dominant and $K$-dominant singular values associated to $u$ are given respectively by the $s_{2 j-1}$ 's, $j \in \llbracket 1, q \rrbracket$, and by the $s_{2 k}$ 's, $k \in \llbracket 1, q \rrbracket$, and such that the Blaschke products associated to these singular values are given respectively by $\Psi_{2 j-1}, j \in \llbracket 1, q \rrbracket$, and by $\Psi_{2 k}, k \in \llbracket 1, q \rrbracket$.

## 3 The additional conservation laws $\ell_{j}$

In the sequel, we show how to prove simply that $\ell_{k}(u)$ is conserved along solutions of the quadratic Szegő equation (1). We then intend to prove Proposition 1.2 and its corollary : we give a necessary condition for growth of Sobolev norms to occur in the rational case. Let us mention that this condition will be an adaptation of the results of [18] in our context.

### 3.1 Evolution of $u_{k}^{K}$ and $w_{k}^{K}$

Recall that if $\sigma_{k}^{2}$ is the $k$-th eigenvalue of $K_{u}^{2}$ (by convention, we set $\sigma_{\infty}=0$ ), we have called $F_{u}\left(\sigma_{k}\right):=\operatorname{ker}\left(K_{u}^{2}-\sigma_{k}^{2} I\right)$, and we have defined $u_{k}^{K}\left(\right.$ resp. $\left.w_{k}^{K}\right)$ to be the orthogonal projection of $u\left(\right.$ resp. $\left.H_{u}(u)\right)$ onto $F_{u}\left(\sigma_{k}\right)$.

We first calculate the evolution of $u_{k}^{K}$ and $w_{k}^{K}$.
Lemma 3.1. Suppose that $t \mapsto u(t)$ is a smooth solution of (1). Then we have

$$
\begin{gather*}
\dot{u}_{k}^{K}=B_{u} u_{k}^{K}-i J w_{k}^{K}  \tag{9}\\
\dot{w}_{k}^{K}=B_{u} w_{k}^{K}+i \bar{J}\left(2 Q+\sigma_{k}^{2}\right) u_{k}^{K} \tag{10}
\end{gather*}
$$

Proof. The proof of these identities relies on the Lax pair. First observe that in view of the expression of $B_{u}$ and of equation (1), we have

$$
\begin{equation*}
\dot{u}=B_{u}(u)-i J H_{u}(u) \tag{11}
\end{equation*}
$$

For the evolution of $u_{k}^{K}$, set $f:=\mathbb{1}_{\left\{\sigma_{k}^{2}\right\}}$ and write

$$
\begin{aligned}
\frac{d}{d t} u_{k}^{K}=\frac{d}{d t} f\left(K_{u}^{2}\right) u & =\left[B_{u}, f\left(K_{u}^{2}\right)\right] u+f\left(K_{u}^{2}\right)\left(B_{u} u-i J H_{u} u\right) \\
& =B_{u} f\left(K_{u}^{2}\right) u-i J f\left(K_{u}^{2}\right) H_{u} u \\
& =B_{u} u_{k}^{K}-i J w_{k}^{K},
\end{aligned}
$$

which corresponds to (9). As for $w_{k}^{K}$,

$$
\begin{aligned}
\frac{d}{d t} f\left(K_{u}^{2}\right) H_{u} u & =\left[B_{u}, f\left(K_{u}^{2}\right)\right] H_{u} u+f\left(K_{u}^{2}\right)\left(\left[B_{u}, H_{u}\right] u+i \bar{J} Q u\right)+f\left(K_{u}^{2}\right) H_{u}\left(B_{u} u-i J H_{u} u\right) \\
& =B_{u} f\left(K_{u}^{2}\right) H_{u} u+i \bar{J} Q f\left(K_{u}^{2}\right) u+i \bar{J} f\left(K_{u}^{2}\right) H_{u}^{2} u \\
& =B_{u} w_{k}^{K}+i \bar{J} Q u_{k}^{K}+i \bar{J} f\left(K_{u}^{2}\right)\left(K_{u}^{2} u+Q u\right) \\
& =B_{u} w_{k}^{K}+i \bar{J}\left(2 Q+\sigma_{k}^{2}\right) u_{k}^{K},
\end{aligned}
$$

where we used the relation (4) between $H_{u}^{2}$ and $K_{u}^{2}$. Now (10) is proved.
Proposition 3.2. With the hypothesis of the preceding lemma, setting

$$
\ell_{k}(t):=\left(2 Q+\sigma_{k}^{2}\right)\left\|u_{k}^{K}(t)\right\|_{L^{2}}^{2}-\left\|w_{k}^{K}(t)\right\|_{L^{2}}^{2},
$$

we have $\frac{d}{d t} \ell_{k}=0$.
Proof. As $\sigma_{k}^{2}$ and $Q$ are constant, it suffices to compute the time derivative of $\left\|u_{k}^{K}(t)\right\|_{L^{2}}^{2}$ and $\left\|w_{k}^{K}(t)\right\|_{L^{2}}^{2}$. On the one hand, by (9),

$$
\begin{equation*}
\frac{d}{d t}\left\|u_{k}^{K}\right\|_{L^{2}}^{2}=2 \operatorname{Re}\left(\dot{u}_{k}^{K} \mid u_{k}^{K}\right)=2 \operatorname{Im}\left(J\left(w_{k}^{K} \mid u_{k}^{K}\right)\right) \tag{12}
\end{equation*}
$$

since $B_{u}$ is anti-selfadjoint. On the other hand, by (10),

$$
\frac{d}{d t}\left\|w_{k}^{K}\right\|_{L^{2}}^{2}=2 \operatorname{Re}\left(\dot{w}_{k}^{K} \mid w_{k}^{k}\right)=-2\left(2 Q+\sigma_{k}^{2}\right) \operatorname{Im}\left(\bar{J}\left(u_{k}^{K} \mid w_{k}^{K}\right)\right)
$$

Thus $\frac{d}{d t}\left\|w_{k}^{K}\right\|_{L^{2}}^{2}=\left(2 Q+\sigma_{k}^{2}\right) \frac{d}{d t}\left\|u_{k}^{K}\right\|_{L^{2}}^{2}$, which yields the conservation of $\ell_{k}$.
Hereafter, we give another expression of $\ell_{k}$ by means of the spectral theory of $H_{u}$ and $K_{u}$ (see Section 2). Fix some $u \in H_{+}^{1 / 2}$.
Lemma 3.3. Let $\sigma_{k}^{2}$ be a non-zero eigenvalue of $K_{u}^{2}$.
(i) Suppose $\sigma_{k} \in \Sigma_{u}^{K}$. Then $u_{k}^{K} \neq 0$ and $w_{k}^{K}$ is colinear to $u_{k}^{K}$. Consequently,

$$
\ell_{k}(u)=\left\|u_{k}^{K}\right\|_{L^{2}}^{2}\left(\left(2 Q+\sigma_{k}^{2}\right)-\left|\xi_{k}\right|^{2}\right),
$$

where

$$
\begin{equation*}
\xi_{k}:=\left(\Pi\left(|u|^{2}\right) \left\lvert\, \frac{u_{k}^{K}}{\left\|u_{k}^{K}\right\|_{L^{2}}^{2}}\right.\right) . \tag{13}
\end{equation*}
$$

(ii) Suppose $\sigma_{k} \in \Sigma_{u}^{H}$. Then $u_{k}^{K}=0$ and $w_{k}^{K} \neq 0$, hence

$$
\ell_{k}(u)<0
$$

Proof. Let us first examine the case when $\sigma_{k} \in \Sigma_{u}^{K}$. Then $u_{k}^{K} \neq 0$ by Proposition 2.1. Now, if $h \in F_{u}\left(\sigma_{k}\right)$ and $h \perp u_{k}$, it means that $(h \mid u)=0$ and $h \in E_{u}\left(\sigma_{k}\right)$, i.e. $H_{u}^{2} h=\sigma_{k}^{2} h$. We have thereby

$$
\left(w_{k}^{K} \mid h\right)=\left(\Pi\left(|u|^{2}\right) \mid h\right)=\left(H_{u} u \mid h\right)=\left(H_{u} h \mid u\right)=0
$$

since $H_{u} h \in E_{u}\left(\sigma_{k}\right)$, so $H_{u} h \perp u$. This proves that $w_{k}^{K}$ and $u_{k}^{K}$ are colinear, and the formula with $\xi_{k}$ immediately follows, since

$$
w_{k}^{K}=\left(w_{k}^{K} \left\lvert\, \frac{u_{k}^{K}}{\left\|u_{k}^{K}\right\|_{L^{2}}}\right.\right) \frac{u_{k}^{K}}{\left\|u_{k}^{K}\right\|_{L^{2}}}
$$

In the case when $\sigma_{k} \in \Sigma_{u}^{H}$, by Proposition 2.1 again, we have $u_{k}^{K}=0$. Let us turn to $w_{k}^{K}$. First, setting $f=\mathbb{1}_{\left\{\sigma_{k}^{2}\right\}}$, we observe that $f\left(H_{u}^{2}\right)\left(\Pi\left(|u|^{2}\right)\right)=H_{u} f\left(H_{u}^{2}\right)(u)=H_{u} u_{k}^{H} \neq 0$, because $u_{k}^{H} \neq 0$ and $H_{u}$ is one-to-one on $E_{u}\left(\sigma_{k}\right)$. Now observe that $H_{u} u_{k}^{H} \notin \mathbb{C} u_{k}^{H}$, otherwise we would have $\operatorname{dim} E_{u}\left(\sigma_{k}\right)=1$ by Proposition 2.2, and thus $\operatorname{dim} F_{u}\left(\sigma_{k}\right)=0$, which contradicts the assumption that $\sigma_{k}^{2}$ is an eigenvalue of $K_{u}^{2}$. Therefore,

$$
w_{k}^{K}=f\left(K_{u}^{2}\right) f\left(H_{u}^{2}\right)\left(\Pi\left(|u|^{2}\right)\right)=f\left(K_{u}^{2}\right) H_{u} u_{k}^{H} \neq 0
$$

since $F_{u}\left(\sigma_{k}\right)=E_{u}\left(\sigma_{k}\right) \cap\left(u_{k}^{H}\right)^{\perp}$. The second part of the lemma is proved.
Remark 3. Let us make a series of remarks on the case $k=\infty$. For $\sigma_{\infty}=0$, it is also true that $w_{\infty}^{K}$ and $u_{\infty}^{K}$ are colinear. Indeed, if $h \in \operatorname{ker} K_{u}^{2}$ and $h \perp u$, then

$$
0=\left(K_{u}^{2}(h) \mid 1\right)=\left(h \mid K_{u}^{2}(1)\right)=\left(h \mid H_{u}^{2}(1)+(1 \mid u) u\right)=\left(h \mid H_{u}(u)\right)
$$

so $h \perp H_{u}(u)$. In particular, if $u \perp \operatorname{ker} K_{u}^{2}$, then $H_{u}(u) \perp \operatorname{ker} K_{u}^{2}$.
When $u_{\infty}^{K} \neq 0$, then

$$
\xi_{\infty}^{K}=\left(H_{u}^{2}(1) \left\lvert\, \frac{u_{\infty}^{K}}{\left\|u_{\infty}^{K}\right\|_{L^{2}}^{2}}\right.\right)=\left(1 \left\lvert\, \frac{K_{u}^{2}\left(u_{\infty}^{K}\right)+\left(u_{\infty}^{K} \mid u\right) u}{\left\|u_{\infty}^{K}\right\|_{L^{2}}^{2}}\right.\right)=(1 \mid u)
$$

thus (even if $u \perp \operatorname{ker} K_{u}^{2}$ ),

$$
\begin{equation*}
\ell_{\infty}=\left\|u_{\infty}^{K}\right\|_{L^{2}}^{2}\left(2 Q-|(u \mid 1)|^{2}\right) \tag{14}
\end{equation*}
$$

It is worth noticing that identity (14) yields another proof of the fact that the submanifold $\left\{\mathrm{rk} H_{u}^{2}=D\right\}$ of $L_{+}^{2}$ is stable by the flow of (1) (see [16, Corollary 2.2]). Indeed, it suffices to show that when $\operatorname{rk} K_{u}^{2}=D^{\prime}<+\infty$, rk $H_{u}^{2}$ cannot pass from $D^{\prime}$ to $D^{\prime}+1$ or conversely. The condition $\operatorname{rk} H_{u}^{2}=\operatorname{rk} K_{u}^{2}=D^{\prime}$ for some $u \in H_{+}^{1 / 2}$ means that $\operatorname{Im} H_{u}^{2}=\operatorname{Im} K_{u}^{2}$ (since the inclusion $\supseteq$ is always true). As $u=H_{u}(1) \in \operatorname{Im} H_{u}=\operatorname{Im} H_{u}^{2}$, we then have $u \in \operatorname{Im} K_{u}^{2}$. Hence $u_{\infty}^{K}=0$. But since $\left\|u_{\infty}^{K}\right\|_{L^{2}}^{2}\left(2 Q-|(u \mid 1)|^{2}\right)$ is conserved by the flow, and as $2 Q-|(u \mid 1)|^{2} \geq Q$ by Cauchy-Schwarz, we see that if $u_{\infty}^{K}=0$ at time 0 , then it must remain true for all times.

Now, if $u_{\infty}^{K}=0$ for some $u \in H_{+}^{1 / 2}$, it means that $u \in\left(\operatorname{ker} K_{u}^{2}\right)^{\perp}=\operatorname{Im} K_{u}^{2}$, so writing $K_{u}^{2}=H_{u}^{2}-(\cdot \mid u) u$ shows that $\operatorname{Im} H_{u}^{2}=\operatorname{Im} K_{u}^{2}$. Conversely, if $u_{\infty}^{K} \neq 0$ at time 0 , it will never cancel.

### 3.2 About the dominance of eigenvalues of $K_{u}^{2}$

During the proof of Corollary 1.3, we will need to know how often $u_{k}^{K}$ may be zero. Indeed, the eigenvalues of $K_{u}^{2}$ are conserved, but as the eigenvalues of $H_{u}^{2}$ have a non trivial evolution in time, it could perfectly happen that a $K$-dominant singular value associated to $u$ transforms into a $H$-dominant one : such a phenomenon is called crossing in [18], and we follow this terminology. The purpose of this section is then to prove the following proposition.

Proposition 3.4. Suppose that $t \mapsto u(t)$ is a solution of the quadratic Szegó equation (1) in $\mathcal{V}(d)$, and suppose that $u$ is not constant in time. Then there exists a discrete set $\Lambda \subset \mathbb{R}$ such that when $t \notin \Lambda$, all the eigenvalues of $K_{u}^{2}$ are $K$-dominant.

To put it in a different way, if $t \notin \Lambda$, then

$$
\Xi_{u(t)}^{K}=\Sigma_{u(t)}^{K},
$$

and every $H$-dominant singular value associated to $u(t)$ is therefore of multiplicity 1 . It means that crossing cannot happen outside a discrete set of times.

To prove this proposition, we start from a lemma which applies to all smooth solutions (not only the rational ones) :

Lemma 3.5. Let $s>\frac{1}{2}$ and $u_{0} \in H_{+}^{s}(\mathbb{T})$. Then the solution $t \mapsto u(t)$ of (1) such that $u(0)=u_{0}$ is real analytic in the variable $t \in \mathbb{R}$, taking values in the Hilbert space $H^{s}$.

Proof. It is enough to prove the lemma on compact sets of $\mathbb{R}$, so we fix $T>0$, and if $f:[-T, T] \rightarrow$ $H^{s}$ is a continuous function, we denote by

$$
\|f\|_{T}:=\max _{t \in[-T, T]}\|f(t)\|_{H^{s}}
$$

Recall that for $s>\frac{1}{2}, H_{+}^{s}(\mathbb{T})$ is an algebra, and that $\Pi: H^{s} \rightarrow H_{+}^{s}$ is bounded and has norm 1. Therefore, the proof we are going to give only resorts to an ODE framework.

First of all, from the Cauchy-Lipschitz theorem, $t \mapsto u(t)$ is $C^{\infty}$ on $\mathbb{R}$. It then suffices to prove that there exists constants $c_{0}, C>0$ such that

$$
\left\|\frac{\partial^{n} u}{\partial t^{n}}\right\|_{T} \leq c_{0} C^{n} n!, \quad \forall n \geq 0
$$

Write equation (1) in the following way :

$$
\begin{equation*}
\partial_{t} u=-i\left(\int_{\mathbb{T}}|u|^{2} u\right) \Pi\left(|u|^{2}\right)-i \bar{J}\left(\int_{\mathbb{T}}|u|^{2} \bar{u}\right) u^{2}-i J\left(\int_{\mathbb{T}}|u|^{2} u\right) \Pi\left(|u|^{2}\right) \tag{15}
\end{equation*}
$$

so that $\partial_{t} u$ appears to be a sum of three terms, each of them being a "product" of five copies of $u$. Now, it is clear that for $n \geq 0, \partial_{t}^{n} u$ will be a sum of $c_{n}$ terms, each of which contains a "product" of $d_{n}$ copies of $u$.

Let us find the induction relation between $c_{n+1}$ and $c_{n}$, and between $d_{n+1}$ and $d_{n}$. If we differentiate $\partial_{t}^{n} u$, the time-derivative is going to hit, one after another, each of the $d_{n}$ factors $u$ of the $c_{n}$ terms of the sum, and for each of them, it will create three terms by (15). Thus

$$
c_{n+1}=3 d_{n} c_{n}
$$

As for $d_{n+1}$, the time-derivative will remove one the $u$ factors and replace it by 5 others, so

$$
d_{n+1}=d_{n}+4
$$

Consequently, $d_{n}=4 n+d_{0}=4 n+1$, and $c_{n+1}=3(4 n+1) c_{n} \leq 12(n+1) c_{n}$, for all $n \geq 0$. By an easy induction, using $c_{0}=1$, we thus have

$$
c_{n} \leq 12^{n}(n!)
$$

Finally, we bound each of the $d_{n}$ factors of the $c_{n}$ terms by $\|u\|_{T}$, so we get

$$
\left\|\frac{\partial^{n} u}{\partial t^{n}}\right\|_{T} \leq 12^{n}\|u\|_{T}^{4 n+1}(n!)
$$

which gives the result.
Corollary 3.6. Let $\sigma_{k} \in \Xi_{u}^{K}$. Then $t \mapsto[u(t)]_{k}^{K}$ is real analytic.
Proof. It suffices to choose $\varepsilon>0$ small enough so that

$$
\left[\sqrt{\sigma_{k}^{2}-\varepsilon}, \sqrt{\sigma_{k}^{2}+\varepsilon}\right] \cap \Xi_{u(t)}^{K}=\left\{\sigma_{k}\right\}
$$

for all $t \in \mathbb{R}$ (which is possible, since $\Xi_{u(t)}^{K}$ does not depend on $t$ by the Lax pair). Then, denoting by $\mathcal{C}\left(\sigma_{k}^{2}, \varepsilon\right)$ the circle of center $\sigma_{k}^{2}$ and of radius $\varepsilon$ in $\mathbb{C}$, we have

$$
[u(t)]_{k}^{K}=\frac{1}{2 i \pi} \int_{\mathcal{C}\left(\sigma_{k}^{2}, \varepsilon\right)}\left(z I-K_{u(t)}^{2}\right)^{-1}(u(t)) d z
$$

by the residue formula. This proves the corollary.
Now we turn to the proof of the main proposition of this section :
Proof of Proposition 3.4. Assume first that there exists $\sigma_{k} \in \Xi_{k}^{K}$ such that, for some accumulating sequence of times $\left\{t_{n}\right\}, \sigma_{k}$ is an $H$-dominant singular value associated to $u\left(t_{n}\right)$. Then by Proposition 2.1, we then have $\left[u\left(t_{n}\right)\right]_{k}^{K}=0$ for all $n \in \mathbb{N}$. Therefore, by the real analyticity of this function, it imposes that

$$
u_{k}^{K} \equiv 0 \quad \text { on } \mathbb{R}
$$

which means that $\sigma_{k}$ remains $H$-dominant for all times. Besides, by Lemma 3.3, we know that, in that case, $\ell_{k}=-\left\|w_{k}\right\|_{L^{2}}^{2}$ is conserved and negative. This proves that $w_{k}^{K} \neq 0$ for all $t \in \mathbb{R}$.

Now, recall from (9) that the evolution of $u_{k}^{K}$ is given by

$$
\dot{u}_{k}^{K}=B_{u} u_{k}^{K}-i J w_{k}^{K}
$$

In our case, this means that $-i J w_{k}^{K}$ is identically zero. As $w_{k}^{K} \neq 0$, we must have

$$
J \equiv 0
$$

and this is equivalent to $u$ being a steady solution (i.e. $\partial_{t} u=0$ ).
We therefore have proved that if $t \mapsto u(t)$ is not constant-in-time, the following set $\{t \in \mathbb{R} \mid$ $\left.[u(t)]_{k}^{K}=0\right\}$ is discrete in $\mathbb{R}$. But since our solution belongs to $\mathcal{V}(d)$, the set $\Xi_{k}^{K}$ is finite, so the times for which one at least of the $u_{k}^{K}$ 's, $k \geq 1$, cancels out, lie in a finite union of discrete sets, so they form a discrete subset of $\mathbb{R}$. This proves the proposition.

### 3.3 About the motion of the Blaschke products $\Psi_{k}^{K}$

Now we turn to another evolution law. Recall from Proposition 2.2 that if $\sigma_{k}$ is a $K$-dominant singular value associated to $u$, then there exists $\Psi_{k}^{K}$, a Blaschke product of degree $m\left(\sigma_{k}\right)-1$, where $m\left(\sigma_{k}\right)$ is the dimension of $F_{u}\left(\sigma_{k}\right)$, such that

$$
\begin{equation*}
K_{u}\left(u_{k}^{K}\right)=\sigma_{k} \Psi_{k}^{K} u_{k}^{K} \tag{16}
\end{equation*}
$$

The evolution equation for $\Psi_{k}^{K}$ plays an important role and can be computed :
Lemma 3.7. Choose $t_{0} \in \mathbb{R} \backslash \Lambda$ (where $\Lambda$ is given by Proposition 3.4), and let $I$ be a maximal interval such that $t_{0} \in I \subseteq(\mathbb{R} \backslash \Lambda)$. Let $\sigma_{k} \in \Xi_{u}^{K}$. Then for all $t \in I, \Psi_{k}^{K}(t)$ is well defined by (16), and there exists a smooth function $\psi_{k, I}: I \rightarrow \mathbb{T}$ with $\psi_{k, I}\left(t_{0}\right)=0$, such that

$$
\Psi_{k}^{K}(t)=e^{i \psi_{k, I}(t)} \Psi_{k}^{K}\left(t_{0}\right), \quad \forall t \in I
$$

Proof. The fact that $\Psi_{k}^{K}$ is well-defined comes from the fact that for all $t \in I, \sigma_{k} \in \Sigma_{u(t)}^{K}$. Differentiate (16), using (9) and the Lax pair :

$$
B_{u} K_{u}\left(u_{k}^{K}\right)+i \bar{J} K_{u}\left(w_{k}^{K}\right)=\sigma_{k} \dot{\Psi}_{k}^{K} u_{k}^{K}+\sigma_{k} \Psi_{k}^{K} B_{u} u_{k}^{K}-i \sigma_{k} J \Psi_{k}^{K} w_{k}^{K}
$$

By (16) again, and the fact that $w_{k}^{K}=\left\|u_{k}^{K}\right\|^{-2}\left(\Pi\left(|u|^{2}\right) \mid u_{k}^{K}\right) u_{k}^{K}$, we get

$$
\begin{equation*}
\dot{\Psi}_{k}^{K} u_{k}^{K}=\left(B_{u} \Psi_{k}^{K}-\Psi_{k}^{K} B_{u}\right) u_{k}^{K}+2 i \operatorname{Re}\left(J \frac{\left(\Pi\left(|u|^{2}\right) \mid u_{k}^{K}\right)}{\left\|u_{k}^{K}\right\|^{2}}\right) \Psi_{k}^{K} u_{k}^{K} \tag{17}
\end{equation*}
$$

Our goal is to show that $\left(B_{u} \Psi_{k}^{K}-\Psi_{k}^{K} B_{u}\right) u_{k}^{K}=0$. This is obvious when $m\left(\sigma_{k}\right)=1$ (because in that case $\Psi_{k}^{K}$ is only a complex number), so we assume that $m\left(\sigma_{k}\right) \geq 2$. Since $u \in L_{+}^{2}$, it is clear that $T_{\bar{J} u}\left(\Psi_{k}^{K} u_{k}^{K}\right)=\bar{J} u \Psi_{k}^{K} u_{k}^{K}=\Psi_{k}^{K} T_{\bar{J} u}\left(u_{k}^{K}\right)$. So it is enough to show that $T_{\bar{u}}\left(\Psi_{k}^{K} u_{k}^{K}\right)-$
$\Psi_{k}^{K} T_{\bar{u}}\left(u_{k}^{K}\right)=0$, and then multiply this identity by $J$. This cancellation follows from a direct computation:

$$
\begin{aligned}
T_{\bar{u}}\left(\Psi_{k}^{K} u_{k}^{K}\right)-\Psi_{k}^{K} T_{\bar{u}}\left(u_{k}^{K}\right) & =\Pi\left(\Psi_{k}^{K}(I-\Pi)\left(\bar{u} u_{k}^{K}\right)\right) \\
& \left.=\Pi\left(\Psi_{k}^{K} \bar{z} \overline{\Pi\left(\bar{z} u u_{k}^{K}\right.}\right)\right) \\
& =\Pi\left(\bar{z} \Psi_{k}^{K} \overline{K_{u}\left(u_{k}^{K}\right)}\right) \\
& =\sigma_{k} \Pi\left(\bar{z}\left|\Psi_{k}^{K}\right|^{2} \overline{u_{k}^{K}}\right) \\
& =\sigma_{k} \Pi\left(\bar{z} \overline{u_{k}^{K}}\right) \\
& =0
\end{aligned}
$$

where we used the elementary fact that for $h \in L_{+}^{2}$, we have $(I-\Pi)(h)=\bar{z} \overline{\Pi(\bar{z} \bar{h})}$.
Going back to (17), since $u_{k}^{K} \neq 0$, we find

$$
\dot{\Psi}_{k}^{K}=2 i \operatorname{Re}\left(J \frac{\left(\Pi\left(|u|^{2}\right) \mid u_{k}^{K}\right)}{\left\|u_{k}^{K}\right\|^{2}}\right) \Psi_{k}^{K}=2 i \operatorname{Re}\left(J \xi_{k}\right) \Psi_{k}^{K}
$$

with the notation of (13). This gives the yielded result, with $\psi_{k, I}(t)=2 \operatorname{Re}\left(\int_{t_{0}}^{t} J(s) \xi_{k}(s) d s\right)$.
From Lemma 3.7, we deduce an important corollary : the zeros of the Blaschke product associated to some $\sigma_{k} \in \Sigma_{u}^{K}$ remain unchanged from one connected component of $\mathbb{R} \backslash \Lambda$ to another. As a consequence, the Blaschke products associated to $K$-dominant values can be defined for all times.

Corollary 3.8. Fix $t_{0} \in \mathbb{R} \backslash \Lambda^{1}$. For each $\sigma_{k} \in \Xi_{u}^{K}$, the Blaschke product $\Psi_{k}^{K}$ is well-defined by (16) for every $t \in \mathbb{R} \backslash \Lambda$, and there exists a continuous function $\psi_{k}: \mathbb{R} \rightarrow \mathbb{T}$ with $\psi_{k}\left(t_{0}\right)=0$, such that

$$
\Psi_{k}^{K}(t)=e^{i \psi_{k}(t)} \Psi_{k}^{K}\left(t_{0}\right), \quad \forall t \in \mathbb{R} \backslash \Lambda
$$

Proof. Pick $\sigma_{k} \in \Xi_{u}^{K}$, and assume that there exists a time $\tilde{t} \in \mathbb{R}$ such that $\sigma_{k}$ is an $H$-dominant singular value associated to $u(\tilde{t})$. Pick also $\varepsilon>0$ such that $[\tilde{t}-\varepsilon, \tilde{t}+\varepsilon] \cap \Lambda=\{\tilde{t}\}$.

Now, we know from Lemma 3.3 that $w_{k}^{K}(\tilde{t}) \neq 0$. On the other hand, it can be shown as in Corollary 3.6 that $w_{k}^{K}$ is a real analytic function. Up to changing $\varepsilon$, we assume that $w_{k}^{K}(t) \neq 0$ if $|t-\tilde{t}| \leq \varepsilon$, and for such $t$, we can define

$$
\Psi^{\sharp}(t):=\frac{K_{u(t)}\left(w_{k}^{K}(t)\right)}{\sigma_{k} w_{k}^{K}(t)} .
$$

$\Psi^{\sharp}$ is a continuous function of $t$ on the interval $[\tilde{t}-\varepsilon, \tilde{t}+\varepsilon]$ which takes values into rational functions of $z \in \mathbb{D}$.

Besides, recall that $w_{k}^{K}$ is colinear to $u_{k}^{K}$ when $\sigma_{k} \in \Sigma_{u(t)}^{K}$ (see Lemma 3.3). Thus, if $t \in$ $[\tilde{t}-\varepsilon, \tilde{t}+\varepsilon] \backslash\{\tilde{t}\}$, then $\Psi^{\sharp}(t)$ coincides with $\Psi_{k}^{K}(t)$, i.e. $e^{i \psi_{k, I_{1}}(t)} \Psi_{1}$ on the left of $\tilde{t}$, and $e^{i \psi_{k, I_{2}}(t)} \Psi_{2}$

1. In the sequel, we will assume without loss of generality that $t_{0}=0$.
on the right (where $\psi_{k, I_{j}}: \mathbb{R} \rightarrow \mathbb{T}$ are smooth, and $\Psi_{j}$ are some constant-in-time finite Blaschke products of identical degree, by Lemma 3.7). Therefore, $\Psi^{\sharp}$ enables to extend continuously each of the two functions, which imposes that $\Psi_{1}=\Psi_{2}$ and that the $\psi_{k, I_{j}}$ coincide with a function which is continuous in $\tilde{t}$.

### 3.4 Weakly convergent sequences in $H_{+}^{1 / 2}$

Before coming back to equation (1), let us prove three useful preliminary results about weakly convergent sequences in $H_{+}^{1 / 2}(\mathbb{T})$.

Lemma 3.9. Let $v_{n} \in H_{+}^{1 / 2}$ such that $\left\{v_{n}\right\}$ converges weakly to some $v$ in $H_{+}^{1 / 2}$. Then, for any $h \in L_{+}^{2}, H_{v_{n}}(h) \rightarrow H_{v}(h)$ strongly in $L_{+}^{2}$.

Proof. Replacing $v_{n}$ by $v_{n}-v$, we can assume that $v=0$. By Rellich's theorem, since $v_{n} \rightharpoonup 0$ in $H^{1 / 2}(\mathbb{T})$, we have $v_{n} \rightarrow 0$ in every $L^{p}(\mathbb{T}), p<\infty$. Thus, given $h \in L_{+}^{4}$,

$$
\left\|H_{v_{n}}(h)\right\|_{L^{2}} \leq\left\|v_{n} \bar{h}\right\|_{L^{2}} \leq\left\|v_{n}\right\|_{L^{4}}\|h\|_{L^{4}} \longrightarrow 0
$$

when $n \rightarrow+\infty$. Now set $\varepsilon>0$. If $h \in L_{+}^{2}$, there exists $\tilde{h} \in L_{+}^{4}$ such that $\|h-\tilde{h}\|_{L^{2}} \leq \varepsilon$. Furthermore, by the principle of uniform boundedness, there exists $C>0$ such that $\left\|v_{n}\right\|_{H^{1 / 2}} \leq C$ for all $n \geq 0$, hence $\left\|H_{v_{n}}\right\| \leq C$. Then, for $n$ large enough,

$$
\left\|H_{v_{n}}(h)\right\|_{L^{2}} \leq\left\|H_{v_{n}}(h-\tilde{h})\right\|_{L^{2}}+\left\|H_{v_{n}}(\tilde{h})\right\|_{L^{2}} \leq(C+1) \varepsilon
$$

which proves the lemma.
Lemma 3.10. Let $d \in \mathbb{N}$. Suppose $v_{n} \in \mathcal{V}(d)$ and $v_{n} \rightharpoonup v$ in $H_{+}^{1 / 2}$. Then $v \in \mathcal{V}\left(d^{\prime}\right)$ for some $d^{\prime} \leq d$.

Proof. This is in fact a completely general result on sequences of bounded operators $\mathcal{T}_{n}$ on some Hilbert space $H$, such that $\sup _{n}\left\|\mathcal{T}_{n}\right\|<+\infty$, and $\operatorname{rk} \mathcal{T}_{n}=k$. Assume that for all $h \in H$, $\mathcal{T}_{n}(h) \rightarrow \mathcal{T}(h)$ strongly. Then $\operatorname{rk} \mathcal{T} \leq k$.

Indeed, for any choice of $k+1$ vectors $h_{1}, \ldots, h_{k+1} \in H$, the Gram matrix

$$
\left(\begin{array}{cccc}
\left(\mathcal{T}_{n}\left(h_{1}\right) \mid \mathcal{T}_{n}\left(h_{1}\right)\right) & \left(\mathcal{T}_{n}\left(h_{1}\right) \mid \mathcal{T}_{n}\left(h_{2}\right)\right) & \cdots & \left(\mathcal{T}_{n}\left(h_{1}\right) \mid \mathcal{T}_{n}\left(h_{k+1}\right)\right) \\
\left(\mathcal{T}_{n}\left(h_{2}\right) \mid \mathcal{T}_{n}\left(h_{1}\right)\right) & \left(\mathcal{T}_{n}\left(h_{2}\right) \mid \mathcal{T}_{n}\left(h_{2}\right)\right) & \cdots & \left(\mathcal{T}_{n}\left(h_{2}\right) \mid \mathcal{T}_{n}\left(h_{k+1}\right)\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\mathcal{T}_{n}\left(h_{k+1}\right) \mid \mathcal{T}_{n}\left(h_{1}\right)\right) & \left(\mathcal{T}_{n}\left(h_{k+1}\right) \mid \mathcal{T}_{n}\left(h_{2}\right)\right) & \cdots & \left(\mathcal{T}_{n}\left(h_{k+1}\right) \mid \mathcal{T}_{n}\left(h_{k+1}\right)\right)
\end{array}\right)
$$

has determinant 0 , since $\mathcal{T}_{n}\left(h_{1}\right), \ldots, \mathcal{T}_{n}\left(h_{k+1}\right)$ are not linearly independent. Passing to the limit $n \rightarrow+\infty$ in this determinant shows that $\mathcal{T}\left(h_{1}\right), \ldots, \mathcal{T}\left(h_{k+1}\right)$ are not independant either, whatever the choice of $h_{j}$. So rk $\mathcal{T} \leq k$. Applying this general result both to $H_{v_{n}}$ and $K_{v_{n}}$ gives the result.

We will also need a refinement of Lemma 3.10 in the case of sequences of functions such that the corresponding shifted Hankel operator has constant spectrum.

Lemma 3.11. Let $v_{n} \in H_{+}^{1 / 2}$ such that $v_{n} \rightharpoonup v$ in $H_{+}^{1 / 2}$. Suppose that $\operatorname{Sp} K_{v_{n}}^{2}$ does not depend of $n \geq 1$. Then if $\sigma^{2} \in \operatorname{Sp} K_{v}^{2}$ with multiplicity $m$, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $\sigma^{2} \in \operatorname{Sp} K_{v_{n}}^{2}$ with multiplicity at least $m$.

Proof. For $\sigma \in \Xi_{v}^{K}$, denote by $\pi^{\sigma}$ the projection onto $\operatorname{ker}\left(K_{v}^{2}-\sigma^{2} I\right)$. By the residue theorem, given $\sigma \in \Xi_{v}^{K}$ and $0<\varepsilon<\sigma^{2}$ such that $\sigma^{2} \pm \varepsilon \notin \operatorname{Sp} K_{v}^{2}$, we have

$$
\begin{equation*}
\frac{1}{2 i \pi} \int_{\mathcal{C}\left(\sigma^{2}, \varepsilon\right)}\left(z I-K_{v}^{2}\right)^{-1} d z=\sum_{\substack{\tilde{\sigma}^{2} \in \Xi_{v}^{K} \\\left|\tilde{\sigma}^{2}-\sigma^{2}\right|<\varepsilon}} \pi^{\tilde{\sigma}} \tag{18}
\end{equation*}
$$

where $\mathcal{C}\left(\sigma^{2}, \varepsilon\right)$ is the circle of center $\sigma^{2}$ and of radius $\varepsilon$.
If $\sigma^{2}$ is a non-zero eigenvalue of $K_{v}^{2}$, let $\left\{e_{j} \mid j=1, \ldots, m\right\}$ be an orthonormal basis of the corresponding eigenspace, which must be of finite dimension for $K_{v}^{2}$ is compact. Let $\varepsilon>0$ be sufficiently small so that $\left\{z \in \mathbb{C}\left|\left|z-\sigma^{2}\right| \leq \varepsilon\right\}\right.$ does not contain any other eigenvalue of $K_{v}^{2}$, and contains at most one eigenvalue $\tilde{\sigma}^{2}$ of $K_{v_{n}}^{2}$ for all $n \geq 1$. For each $1 \leq j \leq m$,

$$
e_{j}^{n}:=\frac{1}{2 i \pi} \int_{\mathcal{C}\left(\sigma^{2}, \varepsilon\right)}\left(z I-K_{v_{n}}^{2}\right)^{-1}\left(e_{j}\right) d z
$$

is well defined, and by Lemma 3.9 and formula (18), we have

$$
e_{j}^{n} \longrightarrow \frac{1}{2 i \pi} \int_{\mathcal{C}\left(\sigma^{2}, \varepsilon\right)}\left(z I-K_{v}^{2}\right)^{-1}\left(e_{j}\right)=e_{j}
$$

as $n \rightarrow \infty$. Thus, if $n$ is large enough, the $e_{j}^{n}, j=1, \ldots, m$ form a family of (non-zero) independant vectors that all belong to $\operatorname{ker}\left(K_{v_{n}}^{2}-\tilde{\sigma}^{2} I\right)$. As this is true for any $\varepsilon>0$ small enough, we must have $\tilde{\sigma}=\sigma$. So $\sigma \in \Xi_{v_{n}}^{K}$, and the dimension of $\operatorname{ker}\left(K_{v_{n}}^{2}-\sigma^{2} I\right)$ is at least $m$ when $n$ is large enough.

### 3.5 An equivalent condition for the growth of Sobolev norms in $\mathcal{V}(d)$

Let us now fix an integer $d \geq 2$, and let $u$ be a solution of (1) in $\mathcal{V}(d)$. Write

$$
u(t, z)=\frac{A(t, z)}{B(t, z)}, \quad \forall z \in \mathbb{D}, \forall t \in \mathbb{R}
$$

where $A(t, \cdot)$ and $B(t, \cdot)$ are polynomials whose degree depends on $N:=\left\lfloor\frac{d}{2}\right\rfloor$ in the following way : $\operatorname{deg} A \leq N-1$ and $\operatorname{deg} B=N$ when $d$ is even, and $\operatorname{deg} A=N$ and $\operatorname{deg} B \leq N$ when $d$ is odd. Moreover, $A$ and $B$ are relatively prime, with $B(t, 0)=1$ and $B$ having no roots inside the closed unit disc of $\mathbb{C}$. With these notations, we have $\operatorname{rk} K_{u}=N$, and rk $H_{u}=d-N$. Write $B(t, z)=\prod_{j=1}^{N}\left(1-p_{j}(t) z\right)$, with $\left|p_{j}(t)\right|<1$ for all $1 \leq j \leq N, t \in \mathbb{R}$.

Observe that, as a smooth solution of (1), by the conservation of $M$ and $Q$, the function $t \mapsto u(t)$ remains bounded in $H^{1 / 2}$, so by the Banach-Alaoglu theorem, the following set

$$
\mathcal{A}^{\infty}(u)=\left\{\left.v \in H_{+}^{\frac{1}{2}} \right\rvert\, \exists t_{n} \rightarrow \pm \infty \text { s.t. } u\left(t_{n}\right) \stackrel{H^{\frac{1}{2}}}{\rightharpoonup} v\right\}
$$

is non-empty ${ }^{2}$.
We are ready to state our proposition in terms of solutions of (1) - but it can be formulated and proved as well in the general framework of Proposition 1.2:

Proposition 3.12. The following statements are equivalent:
(i) $u$ is bounded in $H_{+}^{s_{0}}$ for some $s_{0}>\frac{1}{2}$.
(ii) $u$ is bounded in every $H_{+}^{s}, s>\frac{1}{2}$.
(iii) $\mathcal{A}^{\infty}(u) \subseteq \mathcal{V}(d)$ when $d$ is even, and $\mathcal{A}^{\infty}(u) \subseteq \mathcal{V}(d) \cup \mathcal{V}(d-1)$ when $d$ is odd.

Proof. Start with an observation. Writing $A(t, z)=\sum_{j=0}^{N} a_{j}(t) z^{j}$, we have by Cauchy-Schwarz

$$
\sum_{j=0}^{N}\left|a_{j}(t)\right| \leq \sqrt{N} \cdot\|A(t, \cdot)\|_{L^{2}} \leq \sqrt{N}\|B(t, \cdot)\|_{L^{\infty}}\|u(t)\|_{L^{2}} \leq 2^{N} \sqrt{N}\left\|u_{0}\right\|_{L^{2}}
$$

which proves that all the coefficients of $A$ remain bounded uniformly in time. So if $\left\{t_{n}\right\}$ is a sequence of times with $t_{n} \rightarrow \pm \infty$, we can assume up to an extraction that, for each $z \in \mathbb{D}$,

$$
u\left(t_{n}, z\right) \longrightarrow \frac{\sum_{j=0}^{N} a_{j}^{\infty} z^{j}}{\prod_{j=1}^{N}\left(1-p_{j}^{\infty} z\right)},
$$

where $a_{j}^{\infty}, p_{j}^{\infty} \in \mathbb{C}$ with $\left|p_{j}^{\infty}\right| \leq 1$. Besides, if $u\left(t_{n}, \cdot\right) \rightharpoonup v \in \mathcal{A}^{\infty}(u)$, then $\forall k \in \mathbb{N}, \hat{u}\left(t_{n}, k\right) \rightarrow \hat{v}(k)$, which implies that, for each $z \in \mathbb{D}$, we also have

$$
u\left(t_{n}, z\right)=\sum_{k=0}^{\infty} \hat{u}\left(t_{n}, k\right) z^{k} \longrightarrow \sum_{k=0}^{\infty} \hat{v}(k) z^{k}=v(z) .
$$

Now, if (iii) is satisfied, then there must be some $\rho<1$ such that $\left|p_{j}(t)\right| \leq \rho$ for all $t \in \mathbb{R}$ and $1 \leq j \leq N$, otherwise, choosing an appropriate sequence $\left\{t_{n}\right\}$, one of the $p_{j}^{\infty}$ at least would be of modulus 1 (say $p_{1}^{\infty}=e^{i \theta}$ ). Hence considering $v$ a cluster point of $\left\{u\left(t_{n}\right)\right\}$ for the weak $H^{1 / 2}$ topology, we would have

$$
v(z)=\frac{\sum_{j=0}^{N} a_{j}^{\infty} z^{j}}{\left(1-e^{i \theta} z\right) \prod_{j=2}^{N}\left(1-p_{j}^{\infty} z\right)},
$$

by the previous remark. But $v \in L_{+}^{2}$, so $1-e^{i \theta} z$ would have to divide the numerator. After simplification, we would get $v \in \mathcal{V}(d-\ell)$ with $\ell \geq 2$, and $v \in \mathcal{A}^{\infty}(u)$, which contradicts (iii). But once we have such a $\rho<1$, it is possible to control the $H_{+}^{s}$ norm of $u$. Indeed, $\|A(t, \cdot)\|_{H^{s}} \leq\left(1+N^{2}\right)^{s / 2}\|A(t, \cdot)\|_{L^{2}} \leq C\left(N, s, u_{0}\right)$ for all time $t \in \mathbb{R}$. In addition,

$$
\frac{1}{B(t, z)}=\prod_{j=1}^{N}\left(\sum_{k \geq 0} p_{j}^{k} z^{k}\right)=\sum_{k \geq 0} z^{k}\left(\sum_{\substack{\left(k_{1}, \ldots, k_{N}\right) \in \mathbb{N}^{N} \\ k_{1}+\cdots+k_{N}=k}} p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{N}^{k_{N}}\right),
$$

2. Here, the letter $\mathcal{A}$ stands for the French word adhérence, which means "closure".
so the coefficient of $z^{k}$ is controlled by $k^{N} \rho^{k}$. This proves that

$$
\left\|\frac{1}{B(t, \cdot)}\right\|_{H^{s}}
$$

is uniformly bounded for any $s>1 / 2$. Hence (ii) is proved.
Let us now prove that (i) implies (iii). If $u$ is bounded in some $H^{s_{0}}, s_{0}>\frac{1}{2}$, then its orbit belongs to a compact set of $H^{1 / 2}$, for the injection $H^{s} \hookrightarrow H^{1 / 2}$ is compact. Therefore, for each $v \in \mathcal{A}^{\infty}(u)$, there exists a sequence of times $\left\{t_{n}\right\}$ such that $u\left(t_{n}\right) \rightarrow v$ strongly in $H_{+}^{1 / 2}$. But by the min-max formula, we know the $k$-th eigenvalue of $K_{u}^{2}$ depends continuously on $u$ with respect to the $H^{1 / 2}$ topology, and as it is a conservation law of (1), we get in particular that $N=\operatorname{rk} K_{u\left(t_{n}\right)}^{2}=\operatorname{rk} K_{v}^{2}$. Furthermore, by Lemma 3.10, we get rk $H_{v}^{2} \leq \liminf \lim _{n \rightarrow+\infty} \operatorname{rk} H_{u\left(t_{n}\right)}^{2}$. Since $\operatorname{rk} H_{v}^{2} \geq \operatorname{rk} K_{v}^{2}=N$, we have $\operatorname{rk} H_{v}^{2}=N$ if $d$ is even, and $\operatorname{rk} H_{v}^{2} \in\{N, N+1\}$ if $d$ is odd. This finishes the proof.

### 3.6 Proof of Corollary 1.3

Now we translate Proposition 3.12 into a blow-up criterion for solutions of (1) in $\mathcal{V}(d)$ :
Proposition 3.13. Let $t \mapsto u(t)$ be a solution of (1) in $\mathcal{V}(d)$. The following alternative holds :

- either the trajectory $\{u(t) \mid t \in \mathbb{R}\}$ is bounded in every $H^{s}, s>1 / 2$.
- or there exists $\sigma_{k} \in \Xi_{u}^{K}$ and a sequence $t_{n}$ going to $\pm \infty$ such that $u_{k}^{K}\left(t_{n}\right) \neq 0$ for all $n \geq 1$, and

$$
\left\{\begin{array}{l}
u_{k}^{K}\left(t_{n}\right) \rightarrow 0, \\
w_{k}^{K}\left(t_{n}\right) \rightarrow 0
\end{array} \quad \text { in } L_{+}^{2}\right.
$$

Proof. Suppose that $t \mapsto u(t)$ is not bounded in some $H^{s_{0}}, s_{0}>1 / 2$. By continuity of the solution in $H^{s_{0}}$ and by Proposition 3.4, we can find a sequence $t_{n}$ such that for all $n \geq 1, t_{n} \in \mathbb{R} \backslash \Lambda$ and $\left\|u\left(t_{n}\right)\right\|_{H^{s_{0}}} \rightarrow+\infty$. By Proposition 1.2, it means that up to passing to a subsequence, we can assume that there exists $v \in H_{+}^{1 / 2}$ such that $\operatorname{rk} K_{v}<\operatorname{rk} K_{u(t)}=N$ and $u\left(t_{n}\right) \rightharpoonup v$ in $H^{1 / 2}$.

We set $u^{n}:=u\left(t_{n}\right)$. By Rellich's theorem, we have $u^{n} \rightarrow v$ strongly in $L_{+}^{2}$. Let $\sigma_{k} \in \Xi_{u^{n}}^{K}$, and denote by $\pi^{n}$ (resp. $\pi^{\infty}$ ) the orthogonal projection onto $F_{u^{n}}\left(\sigma_{k}\right)$ (resp. $F_{v}\left(\sigma_{k}\right)$ ). With this notation, $\left(u^{n}\right)_{k}^{K}=\pi^{n}\left(u^{n}\right)$ and $v_{k}^{K}=\pi^{\infty}(v)$. Since $K_{u^{n}}(h) \rightarrow K_{v}(h)$ for any fixed $h \in L_{+}^{2}$, adapting formula (18), we also have $\pi^{n}(h) \rightarrow \pi^{\infty}(h)$. As $\left\|\pi^{n}\right\| \leq 1$, we thus get
$\left\|\left(u^{n}\right)_{k}^{K}-v_{k}^{K}\right\|_{L^{2}} \leq\left\|\pi^{n}\left(u^{n}\right)-\pi^{n}(v)\right\|_{L^{2}}+\left\|\left(\pi^{n}-\pi^{\infty}\right)(v)\right\|_{L^{2}} \leq\left\|u^{n}-v\right\|_{L^{2}}+\left\|\left(\pi^{n}-\pi^{\infty}\right)(v)\right\|_{L^{2}}$, so $\left(u^{n}\right)_{k}^{K} \rightarrow v_{k}^{K}$ strongly in $L_{+}^{2}$.

Now, since all the eingevalues of $K_{u^{n}}^{2}$ are $K$-dominant by the hypothesis on $t_{n}$, we can write

$$
\begin{gathered}
K_{u^{n}}^{2}\left(\left(u^{n}\right)_{k}^{K}\right)=\sigma_{k}^{2}\left(u^{n}\right)_{k}^{K}, \\
K_{u^{n}}\left(\left(u^{n}\right)_{k}^{K}\right)=\sigma_{k} \Psi^{n} \cdot\left(u^{n}\right)_{k}^{K},
\end{gathered}
$$

where $\Psi^{n}:=\Psi_{k}^{K}\left(t_{n}\right)$, and $\left(u^{n}\right)_{k}^{K} \neq 0$ for all $n \geq 1$. We would like to pass to the limit in these identities. Since $\left\|K_{u^{n}}\right\| \leq C$, we see that $\left\|K_{u^{n}}\left(\left(u^{n}\right)_{k}^{K}\right)-K_{v}\left(v_{k}^{K}\right)\right\|_{L^{2}} \leq C\left\|\left(u^{n}\right)_{k}^{K}-v_{k}^{K}\right\|_{L^{2}}+$ $\left\|K_{u^{n}}\left(v_{k}^{K}\right)-K_{v}\left(v_{k}^{K}\right)\right\|_{L^{2}}$, so $K_{u^{n}}\left(\left(u^{n}\right)_{k}^{K}\right) \rightarrow K_{v}\left(v_{k}^{K}\right)$ strongly in $L_{+}^{2}$. The same holds replacing $K_{u^{n}}$ by $K_{u^{n}}^{2}$ and $K_{v}$ by $K_{v}^{2}$. Eventually, by Lemma 3.8, we have $\Psi_{k}^{K}\left(t_{n}\right)=e^{i \psi_{k}\left(t_{n}\right)} \Psi_{k}^{K}(0)$. So up to passing to a subsequence, $\Psi^{n} \rightarrow e^{i \psi^{\infty}} \Psi_{k}^{K}(0)$ for some $\psi^{\infty} \in \mathbb{T}$. Hence, taking $n$ to $\infty$, we get

$$
\begin{gather*}
K_{v}^{2}\left(v_{k}^{K}\right)=\sigma_{k}^{2} v_{k}^{K}  \tag{19}\\
K_{v}\left(v_{k}^{K}\right)=\sigma_{k} e^{i \psi^{\infty}} \Psi_{k}^{K}(0) v_{k}^{K} \tag{20}
\end{gather*}
$$

If now $v_{k}^{K} \neq 0$ for every $\sigma_{k} \in \Xi_{u^{n}}^{K}=\Sigma_{u^{n}}^{K}$, then the previous equality shows that $\sigma_{k}$ also belongs to $\Sigma_{v}^{K}$, and more precisely, as the dimension of $F_{v}\left(\sigma_{k}\right)$ is given by the degree of the associated Blaschke product plus 1, we get from (20) that $\operatorname{dim} F_{u^{n}}\left(\sigma_{k}\right)=\operatorname{dim} F_{v}\left(\sigma_{k}\right)$. This proves that

$$
\operatorname{rk} K_{v} \geq \sum_{\sigma_{k} \in \Sigma_{u^{n}}^{K}} \operatorname{dim}\left(F_{v}\left(\sigma_{k}\right)\right)=\sum_{\sigma_{k} \in \Sigma_{u^{n}}^{K}} \operatorname{dim}\left(F_{u^{n}}\left(\sigma_{k}\right)\right)=\operatorname{rk} K_{u^{n}}=N,
$$

since $t_{n} \notin \Lambda$. This is a contradiction. Consequently, for some $\sigma_{k} \in \Xi_{u^{n}}^{K}$, we must have $\left[u\left(t_{n}\right)\right]_{k}^{K} \rightarrow 0$ in $L_{+}^{2}$.

Besides, for such $\sigma_{k}$ 's, we call $\left(w^{n}\right)_{k}^{K}$ the projection of $\Pi\left(\left|u^{n}\right|^{2}\right)$ onto $F_{u^{n}}\left(\sigma_{k}\right)$. We know from Lemma 3.3 that $\left(w^{n}\right)_{k}^{K}$ is colinear to $\left(u^{n}\right)_{k}^{K}$. Denote by $y_{k}^{K}$ the projection of $\Pi\left(|v|^{2}\right)$ onto $F_{v}\left(\sigma_{k}\right)$. Since $\Pi\left(\left|u^{n}\right|^{2}\right) \rightarrow \Pi\left(|v|^{2}\right)$ strongly in $L^{2}$, we get that $\left(w^{n}\right)_{k}^{K} \rightarrow y_{k}^{K}$ strongly in $L^{2}$, by the same argument as above. Then, passing to the limit in the expression of $K_{u^{n}}\left(\left(w^{n}\right)_{k}^{K}\right)$ and $K_{u^{n}}^{2}\left(\left(w^{n}\right)_{k}^{K}\right)$, we get as before

$$
\begin{gathered}
K_{v}^{2}\left(y_{k}^{K}\right)=\sigma_{k}^{2} y_{k}^{K}, \\
K_{v}\left(y_{k}^{K}\right)=\sigma_{k} e^{i \psi^{\infty}} \Psi_{k}^{K}(0) y_{k}^{K} .
\end{gathered}
$$

Let us show that these equalities impose on $y_{k}^{K}$ to be 0 for at least one $k$. Assume that $y_{k}^{K} \neq 0$. Together with $v_{k}^{K}$, it means that $\sigma_{k} \in \Xi_{v}^{K} \backslash \Sigma_{v}^{K}$, i.e. $\sigma_{k}=\rho_{j}$ is $H$-dominant. Denote by $m_{k}$ the dimension of $F_{u^{n}}\left(\sigma_{k}\right)$ and by $n_{j}$ the dimension of $E_{v}\left(\rho_{j}\right)$. By Proposition 2.2, since $y_{k}^{K} \in F_{v}\left(\rho_{j}\right)$, there exists a non-zero polynomial $g \in \mathbb{C}_{n_{j}-2}[z]$ as well as an polynomial $D(z)$ such that

$$
y_{k}^{K}=\frac{g(z)}{D(z)} H_{v}\left(v_{j}^{H}\right),
$$

and there exists $\varphi \in \mathbb{T}$ such that

$$
K_{v}\left(y_{k}^{K}\right)=\rho_{j} e^{-i \varphi} \frac{\tilde{g}(z)}{D(z)} H_{v}\left(v_{j}^{H}\right),
$$

where $\tilde{g}$ is the polynomial of degree at most $n_{j}-2$ obtained by reversing the order of the coefficients of $g$. Thus, combining all the informations we have,

$$
\frac{K_{v}\left(y_{k}^{K}\right)}{y_{k}^{K}}=\rho_{j} e^{-i \varphi} \frac{\tilde{g}}{g}=\sigma e^{i \psi^{\infty}} \Psi_{k}^{K}(0) .
$$

Since $\Psi_{k}^{K}$ is an irreducible rational function whose numerator and denominator are both of degree $m_{k}-1$, it means that $n_{j}-2 \geq \operatorname{deg} \tilde{g}=\operatorname{deg} g \geq m_{k}-1$, hence

$$
n_{j}-1=\operatorname{dim} F_{v}\left(\sigma_{k}\right) \geq \operatorname{dim} F_{u^{n}}\left(\sigma_{k}\right)=m_{k}
$$

But this cannot happen for all $\sigma_{k}$ 's for which $v_{k}^{K}=0$, otherwise we would still have rk $K_{v} \geq N$. Therefore, there exists $\sigma_{k} \in \Xi_{u}^{K}$ such that both $v_{k}^{K}$ and $y_{k}^{K}$ are zero.

Conversely, if $t \mapsto u(t)$ is bounded in some $H^{s_{0}}, s_{0}>1 / 2$, we have seen during the proof of Proposition 3.12 that for any $v \in \mathcal{A}^{\infty}(u)$, we have $\Xi_{u}^{K}=\Xi_{v}^{K}$. Pick some $\sigma_{k} \in \Xi_{u}^{K}$. Then either $\sigma_{k}$ is $K$-dominant for $v$, and then $v_{k}^{K} \neq 0$, or $\sigma_{k}$ is $H$-dominant for $v$, but then $\left[\Pi\left(|v|^{2}\right)\right]_{k}^{K} \neq 0$ by Lemma 3.3. So in both cases, denoting by $t_{n}$ a sequence of times such that $u^{n}:=u\left(t_{n}\right) \rightharpoonup v$ in $H^{1 / 2}$, and defining $w^{n}:=\Pi\left(\left|u^{n}\right|^{2}\right)$ as above, we cannot have $\left(u^{n}\right)_{k}^{K} \rightarrow 0$ and $\left(w^{n}\right)_{k}^{K} \rightarrow 0$ at the same time.

Remark 4. As a by-product of the proof of Proposition 3.13, it appears that whenever $u$ is a solution of (1) in $\mathcal{V}(d), v \in \mathcal{A}^{\infty}(u)$, and $\sigma \in \Xi_{u(t)}^{K}$ with multiplicity $m(\sigma)$,

- either $\sigma \in \Xi_{v}^{K}$ with multiplicity $m(\sigma)$,
- or $\sigma \notin \Xi_{v}^{K}$.

Indeed, if $\sigma \in \Xi_{v}^{K}$, then it is either $H$-dominant or $K$-dominant, so one at least of the vectors $\mathbb{1}_{\left\{\sigma^{2}\right\}}\left(K_{v}^{2}\right) \Pi\left(|v|^{2}\right)$ and $\mathbb{1}_{\left\{\sigma^{2}\right\}}\left(K_{v}^{2}\right)(v)$ is non-zero. Then, a Blaschke product argument as in proof above shows that $\sigma$ has multiplicity at least $m(\sigma)$. Of course, it cannot be strictly bigger than $m(\sigma)$ (by Lemma 3.11).

Corollary 1.3 is now a mere consequence of Proposition 3.13. We restate it for the convenience of the reader :

Corollary 3.14 (Necessary condition for norm explosion). Let $t \mapsto u(t)$ be a solution of (1) in $\mathcal{V}(d)$, and suppose that it is not bounded in some $H^{s}$ topology, $s>\frac{1}{2}$. Then there exists $\sigma_{k} \in \Xi_{u}^{K}$ such that

$$
\ell_{k}(t):=\left(2 Q+\sigma_{k}^{2}\right)\left\|u_{k}^{K}(t)\right\|_{L^{2}}^{2}-\left\|w_{k}^{K}(t)\right\|_{L^{2}}^{2}=0, \quad \forall t \in \mathbb{R}
$$

Proof. The quantity $\ell_{k}$ is conserved by Proposition 3.2 , and if $t \mapsto u(t)$ is unbounded in some $H^{s}, s>\frac{1}{2}$, it tends to zero along a sequence of times by Proposition 3.13. Thus it is identically zero.

Remark 5. Thanks to Corollary 3.14 together with Proposition 3.13, if one wants to prove that a rational solution has growing Sobolev norms, it suffices to study the evolution of $[u(t)]_{k}^{K}$ if $\ell_{k}=0$. If it tends to zero along a sequence of times, so does automatically $[w(t)]_{k}^{K}$ along the same sequence, and the conditions of Proposition 3.13 are then fulfilled. The convergence to zero of both $u_{k}^{K}$ and $w_{k}^{K}$ is what makes this situation very different from of crossing (where only $u_{k}^{K}$ goes to zero).

## 4 The particular case of $\mathcal{V}(4): 2$-soliton turbulence

### 4.1 A priori analysis

We begin this section by proving identities that make a link between all the objects we have defined so far :
Lemma 4.1. Let $u \in H_{+}^{1 / 2}$. Write $\Xi_{u}^{K}=\left\{\sigma_{1}>\sigma_{2}>\ldots>\sigma_{k}>\ldots\right\}$. Then

$$
\begin{aligned}
M(u) & =\sum_{1 \leq k<\infty} \sigma_{k}^{2} \cdot \operatorname{dim} F_{u}\left(\sigma_{k}\right), \\
Q(u)^{2} & =\sum_{1 \leq k \leq \infty} \ell_{k}, \\
|J(u)|^{2} & =\sum_{1 \leq k \leq \infty}\left(Q+\sigma_{k}^{2}\right) \ell_{k} .
\end{aligned}
$$

In addition, if $\sigma_{k} \in \Sigma_{u}^{K}$, we have set $\xi_{k}:=\left\|u_{k}^{K}\right\|_{L^{2}}^{-2}\left(\Pi\left(|u|^{2}\right) \mid u_{k}^{K}\right)$. Then

$$
\overline{J(u)}=\sum_{\sigma_{k} \in \Sigma_{u}^{K}} \xi_{k}\left\|u_{k}^{K}\right\|_{L^{2}}^{2}
$$

Remark 6. The above formulae must take "infinity" terms into account, with the convention already mentionned that $\sigma_{\infty}=0$.

Proof of Lemma 4.1. The first identity is proved in [5], but we recall it here. In the canonical basis of $L_{+}^{2}$, the matrix of $H_{u}$ reads

$$
H_{u}=\left(\begin{array}{cccc}
\widehat{u}(0) & \widehat{u}(1) & \widehat{u}(2) & \cdots \\
\widehat{u}(1) & \widehat{u}(2) & \widehat{u}(3) & \cdots \\
\widehat{u}(2) & \widehat{u}(3) & \widehat{u}(4) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

where $\widehat{u}$ is the Fourier transform of $u$. So taking the $\mathbb{C}$-antilinearity of $H_{u}$ into account, the trace norm of $H_{u}^{2}$ is given by

$$
\operatorname{Tr} H_{u}^{2}=\sum_{j \geq 0} \sum_{m \geq 0}|\widehat{u}(j+m)|^{2}=\sum_{n \geq 0}(1+n)|\widehat{u}(n)|^{2}=Q(u)+M(u)
$$

Therefore, by (4) (i.e. $\left.H_{u}^{2}=K_{u}^{2}+(\cdot \mid u) u\right)$, we find $\operatorname{Tr} K_{u}^{2}=\operatorname{Tr} H_{u}^{2}-\operatorname{Tr}((\cdot \mid u) u)=(Q+M)-Q=M$, and the first formula follows, computing $\operatorname{Tr} K_{u}^{2}$ in an orthonormal basis of eigenvectors.

Secondly, note that $\sigma_{k}^{2}\left\|u_{k}^{K}\right\|_{L^{2}}=\left(K_{u}^{2}\left(u_{k}^{K}\right) \mid u_{k}^{K}\right)=\left(K_{u}^{2}(u) \mid u_{k}^{K}\right)$. Decomposing $u$ and $\Pi\left(|u|^{2}\right)$ along all the eigenspaces of $K_{u}^{2}$, it yields that

$$
\begin{aligned}
\sum_{1 \leq k \leq \infty} \ell_{k} & =\sum_{1 \leq k \leq \infty}\left(2 Q+\sigma_{k}^{2}\right)\left\|u_{k}^{K}\right\|_{L^{2}}^{2}-\left\|w_{k}^{K}\right\|_{L^{2}}^{2} \\
& =2 Q^{2}+\left(K_{u}^{2}(u) \mid u\right)-\left\|H_{u}(u)\right\|_{L^{2}}^{2} \\
& =2 Q^{2}+\left(H_{u}^{2}(u)-Q u \mid u\right)-\left(H_{u}^{2}(u) \mid u\right) \\
& =Q^{2},
\end{aligned}
$$

where we used the orthogonality of the eigenspaces of $K_{u}^{2}$ to sum the squared norms of $u_{k}^{K}$ and $w_{k}^{K}$.

Then, using extensively (4) again,

$$
\begin{aligned}
\sum_{1 \leq k \leq \infty}\left(Q+\sigma_{k}^{2}\right) \ell_{k} & =Q^{3}+\sum_{1 \leq k \leq \infty}\left(2 Q+\sigma_{k}^{2}\right) \sigma_{k}^{2}\left\|u_{k}^{K}\right\|_{L^{2}}^{2}-\sigma_{k}^{2}\left\|w_{k}^{K}\right\|_{L^{2}}^{2} \\
& =Q^{3}+2 Q\left(K_{u}^{2}(u) \mid u\right)+\left(K_{u}^{4}(u) \mid u\right)-\left(K_{u}^{2}\left(H_{u}(u)\right) \mid H_{u}(u)\right) \\
& =Q^{3}+Q\left(K_{u}^{2}(u) \mid u\right)+\left(H_{u}^{2}\left(K_{u}^{2}(u)\right) \mid u\right)-\left[\left(H_{u}^{3}(u) \mid H_{u}(u)\right)-|J|^{2}\right] \\
& =Q^{3}+Q\left(K_{u}^{2}(u) \mid u\right)-Q\left(H_{u}^{2}(u) \mid u\right)+|J|^{2} \\
& =|J|^{2} .
\end{aligned}
$$

It remains to prove the alternative expression of $J(u)$. Since $w_{k}^{K}$ is colinear to $u_{k}^{K}$ with $w_{k}^{K}=\xi_{k} u_{k}^{K}$ (when this last projection is not zero), and because of the decomposition (8),

$$
\overline{J(u)}=\left(\Pi\left(|u|^{2}\right) \mid u\right)=\sum_{\sigma_{k} \in \Sigma_{u}^{K}}\left(\Pi\left(|u|^{2}\right) \mid u_{k}^{K}\right)=\sum_{\sigma_{k} \in \Sigma_{u}^{K}}\left(w_{k}^{K} \mid u_{k}^{K}\right)=\sum_{\sigma_{k} \in \Sigma_{u}^{K}} \xi_{k}\left\|u_{k}^{K}\right\|_{L^{2}}^{2},
$$

as announced in Lemma 4.1.
Let us now make a few considerations on $\mathcal{V}(4)$. On $\mathcal{V}(4)$, we have $\operatorname{rk} K_{u}^{2}=2$ and $u \perp \operatorname{ker} K_{u}^{2}$ $\left(\right.$ since $\operatorname{Ran} H_{u}^{2}=\operatorname{Ran} K_{u}^{2}$ ).

Corollary 4.2. Let $u \in \mathcal{V}(4) \backslash\{0\}$. There exists $\sigma_{k} \in \Xi_{u}^{K}$ such that $\ell_{k}(u)=0$ if and only if $\Xi_{u}^{K}$ has two distinct elements $\sigma_{1}>\sigma_{2}$, and

$$
|J|^{2}=Q^{2}\left(Q+\sigma_{k}^{2}\right),
$$

for one $k \in\{1,2\}$.
Proof. Suppose that $K_{u}^{2}$ has a unique eigenvalue $\sigma_{1}^{2}$, and $\ell_{1}=Q^{2} \neq 0$ by Lemma 4.1. So for one of the $\ell_{k}$ to cancel out, $K_{u}^{2}$ must have two distinct eigenvalues $\sigma_{1}^{2}>\sigma_{2}^{2}$. In that case, we have

$$
\left\{\begin{array}{l}
\ell_{1}+\ell_{2}=Q^{2}, \\
\ell_{1}\left(Q+\sigma_{1}^{2}\right)+\ell_{2}\left(Q+\sigma_{2}^{2}\right)=|J|^{2}
\end{array}\right.
$$

This system can be solved, and we find

$$
\left\{\begin{array}{l}
\ell_{1}=\frac{|J|^{2}-Q^{2}\left(Q+\sigma_{2}^{2}\right)}{\sigma_{1}^{2}-\sigma_{2}^{2}}, \\
\ell_{2}=\frac{Q^{2}\left(Q+\sigma_{1}^{2}\right)-|J|^{2}}{\sigma_{1}^{2}-\sigma_{2}^{2}}
\end{array}\right.
$$

which proves the corollary.

Remark 7. Suppose that for some solution $t \mapsto u(t)$ in $\mathcal{V}(4), u\left(t_{n}\right)$ is not bounded in $H^{s}$ for some $s>\frac{1}{2}$ and some sequence of times $t_{n}$. Then by Proposition 3.12, there exists $v \in \mathcal{V}(d)$, $d \leq 2$, such that $u\left(t_{n}\right) \rightharpoonup v$ in $H^{1 / 2}$ up to extraction. In fact, since $J\left(u\left(t_{n}\right)\right)=J(v)$ and $Q\left(u\left(t_{n}\right)\right)=Q(v)$ by Rellich's theorem, we cannot have $v \in \mathcal{V}(d)$ for $d \leq 1$, otherwise we would have $\left|J\left(u\left(t_{n}\right)\right)\right|^{2}=Q\left(u\left(t_{n}\right)\right)^{3}$, which is not the case by the preceding corollary. Therefore,

$$
v(z)=\frac{\alpha_{\infty}}{1-p_{\infty} z}
$$

with $\alpha_{\infty}, p_{\infty} \in \mathbb{C}, 0<\left|p_{\infty}\right|<1$. It means that one of the two poles of $u\left(t_{n}\right)$ goes to $\mathbb{T}$, whereas the other stays away from $\mathbb{T}$ and from infinity.

### 4.2 Growing Sobolev norms in $\mathcal{V}(4)$

The purpose of this paragraph is to prove the first part of Theorem 3: solutions in $\mathcal{V}(4)$ have growing Sobolev norms if and only $\ell_{1}=0$.

Throughout we fix $u_{0} \in \mathcal{V}(4)$ such that $\Xi_{u_{0}}^{K}=\left\{\sigma_{1}>\sigma_{2}\right\}$, and

$$
\left(\ell_{1}\left(u_{0}\right), \ell_{2}\left(u_{0}\right)\right) \in\left\{\left(0, Q\left(u_{0}\right)^{2}\right),\left(Q\left(u_{0}\right)^{2}, 0\right)\right\}
$$

We denote by $u(t)$ the solution of (1) such that $u(0)=u_{0}$. Begin with an obvious consequence of the previous results :

Lemma 4.3. For all $t \in \mathbb{R}, \sigma_{1}$ and $\sigma_{2}$ are $K$-dominant.
Proof. By Proposition 3.3, if there was a phenomenon of crossing at some time $t$, we would have $\ell_{1}<0$ or $\ell_{2}<0$, which is excluded by our hypothesis.

Thus we call $\rho_{1}^{2}(t), \rho_{2}^{2}(t)$ the simple eigenvalues of $H_{u(t)}^{2}$, satisfying $\forall t \in \mathbb{R}$,

$$
\rho_{1}(t)>\sigma_{1}>\rho_{2}(t)>\sigma_{2}>0
$$

We can also define $\xi_{1}$ and $\xi_{2}$ as in (13) for all times, and we denote by $u_{1}:=[u(t)]_{1}^{K}, u_{2}:=[u(t)]_{2}^{K}$ with an implicit time-dependence. In particular, we have by Lemma 4.1 and Proposition 3.3 :

$$
\begin{gather*}
\bar{J}=\xi_{1}\left\|u_{1}\right\|_{L^{2}}^{2}+\xi_{2}\left\|u_{2}\right\|_{L^{2}}^{2}  \tag{21}\\
\ell_{1}=\left\|u_{1}\right\|_{L^{2}}^{2}\left(2 Q+\sigma_{1}^{2}-\left|\xi_{1}\right|^{2}\right)  \tag{22}\\
\ell_{2}=\left\|u_{2}\right\|_{L^{2}}^{2}\left(2 Q+\sigma_{2}^{2}-\left|\xi_{2}\right|^{2}\right) \tag{23}
\end{gather*}
$$

The main lemma of this paragraph is the following :
Lemma 4.4. - Suppose that $\ell_{1}\left(u_{0}\right)=0$. Then $\left\|u_{1}\right\|_{L^{2}}^{2}$ goes exponentially fast to zero in both time directions.

- Suppose that $\ell_{2}\left(u_{0}\right)=0$. Then there exists a constant $C>0$ such that

$$
\left\|u_{2}\right\|_{L^{2}}^{2} \geq C>0
$$

uniformly in time.

Proof. Let us denote by $x:=\left\|u_{1}\right\|_{L^{2}}^{2}$. We have $\left\|u_{2}\right\|_{L^{2}}^{2}=Q-x$. Recall from (12) that $\dot{x}=$ $2 x \operatorname{Im}\left(J \xi_{1}\right)$. Using (21), we then have

$$
\dot{x}=2 x(Q-x) \operatorname{Im}\left(\xi_{1} \overline{\xi_{2}}\right) .
$$

Moreover, (21) shows that $|J|^{2}=\left|\xi_{1}\right|^{2} x^{2}+\left|\xi_{2}\right|^{2}(Q-x)^{2}+2 x(Q-x) \operatorname{Re}\left(\xi_{1} \overline{\xi_{2}}\right)$. Therefore, we get

$$
\begin{equation*}
(\dot{x})^{2}=4 x^{2}(Q-x)^{2}\left|\xi_{1}\right|^{2}\left|\xi_{2}\right|^{2}-\left(|J|^{2}-\left|\xi_{1}\right|^{2} x^{2}-\left|\xi_{2}\right|^{2}(Q-x)^{2}\right)^{2} . \tag{24}
\end{equation*}
$$

Suppose now that $\ell_{1}=0$. Corollary 4.2 says that then $|J|^{2}=Q^{2}\left(Q+\sigma_{2}^{2}\right)$ and $\ell_{2}=Q^{2}$. Then by (22) and (23), we have

$$
\begin{gathered}
\left|\xi_{1}\right|^{2}=2 Q+\sigma_{1}^{2} \\
\left|\xi_{2}\right|^{2}(Q-x)=\left(2 Q+\sigma_{2}^{2}\right)(Q-x)-Q^{2}
\end{gathered}
$$

Coming back to (24), this gives

$$
\begin{aligned}
(\dot{x})^{2}=4 x^{2}(Q-x)(2 Q & \left.+\sigma_{1}^{2}\right)\left(\left(2 Q+\sigma_{2}^{2}\right)(Q-x)-Q^{2}\right) \\
& -\left(Q^{2}\left(Q+\sigma_{2}^{2}\right)-\left(2 Q+\sigma_{1}^{2}\right) x^{2}-(Q-x)\left(\left(2 Q+\sigma_{2}^{2}\right)(Q-x)-Q^{2}\right)\right)^{2}
\end{aligned}
$$

Since $\left(2 Q+\sigma_{2}^{2}\right)(Q-x)-Q^{2}=\left(Q+\sigma_{2}^{2}\right) Q-x\left(2 Q+\sigma_{2}^{2}\right)$, we find a simplification in the large squared parenthesis, and we get

$$
\left(\frac{\dot{x}}{x}\right)^{2}=4\left(2 Q+\sigma_{1}^{2}\right)(Q-x)\left(\left(Q+\sigma_{2}^{2}\right) Q-x\left(2 Q+\sigma_{2}^{2}\right)\right)-\left(Q\left(3 Q+2 \sigma_{2}^{2}\right)-x\left(4 Q+\sigma_{1}^{2}+\sigma_{2}^{2}\right)\right)^{2}
$$

If we now develop the different terms crudely, we end up at

$$
\begin{equation*}
\left(\frac{\dot{x}}{x}\right)^{2}=Q^{2} P\left(\frac{\sigma_{1}^{2}-\sigma_{2}^{2}}{Q} x\right), \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
P(X)=\left[4\left(Q+\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)-Q^{2}\right]-2 X\left(3 Q+2 \sigma_{2}^{2}\right)-X^{2} . \tag{26}
\end{equation*}
$$

We thus find an equation which is of the same type of the one on $\mathcal{V}(3)$ while analysing the case $|J|^{2}=Q^{3}$ (see [15]). The analysis here goes the same. We see that $P(X) \rightarrow-\infty$ as $X \rightarrow \pm \infty$, and equation (25) implies that $P$ also takes at least one nonnegative value on $(0,+\infty)$ (because $x(t)>0$ for all $t \in \mathbb{R})$. So $P$ is real-rooted, and its roots $\lambda_{1}, \lambda_{2}$ cannot be both non-positive. Furthermore, they satisfy

$$
\lambda_{1}+\lambda_{2}=-2\left(3 Q+2 \sigma_{2}^{2}\right)<0 .
$$

This equation implies that $\lambda_{1}$ and $\lambda_{2}$ cannot be both non-negative either. Hence they must have different signs : one of them is strictly negative, and the other must be strictly positive. In particular,

$$
P(0)=4\left(Q+\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)-Q^{2}>0
$$

Setting $y:=\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right) Q^{-1} x$, we can then write equation (25) in the form :

$$
(\dot{y})^{2}=A^{2} y^{2}(y+a)(b-y)
$$

for some constants $A, a, b>0$.
This equation can be solved explicitely : there exists $t_{0} \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, we have

$$
y(t)=\frac{2 a b}{(a-b)+(a+b) \cosh \left(\tau\left(t-t_{0}\right)\right)}, \quad \tau:=A \sqrt{a b} .
$$

We see on this formula that $y$ (hence $x=\left\|u_{1}\right\|_{L^{2}}^{2}$ ) decreases exponentially fast in both time directions. The rate is given by

$$
\tau=Q \sqrt{\left|\lambda_{1} \lambda_{2}\right|}=Q \sqrt{4\left(Q+\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)-Q^{2}}
$$

It remains to treat the case when $\ell_{2}=0$. Taking the computation back from the beginning, we see that the equation on $x:=\left\|u_{2}\right\|_{L^{2}}^{2}$ is given by (25), changing $x$ into $-x$ and exchanging the indices 1 and 2. Thus $\dot{x}$ satisfies the same equation as (25), but the polynomial is now $\tilde{P}$ and can be deduced from (26) :

$$
\tilde{P}(X)=\left[-4\left(Q+\sigma_{1}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)-Q^{2}\right]-2 X\left(3 Q+2 \sigma_{1}^{2}\right)-X^{2}
$$

Yet it can been seen directly now that $\tilde{P}(0)<0$, so it imposes that $x=\left\|u_{2}\right\|_{L^{2}}^{2}$ remains bounded away from 0 .

At this point, Lemma 4.4 shows that when $\ell_{1}\left(u_{0}\right)=0$ in $\mathcal{V}(4)$, the the corresponding solution satisfies

$$
\begin{equation*}
\forall s>\frac{1}{2}, \quad \lim _{t \rightarrow \pm \infty}\|u(t)\|_{H^{s}}=+\infty \tag{27}
\end{equation*}
$$

Indeed, the existence of a sequence $t_{n}$ such that $\left\|u\left(t_{n}\right)\right\|_{H \underline{s}} \leq C<+\infty$ for some $\underline{s}>\frac{1}{2}$ would imply that $u_{1}$ would not go to zero along this sequence $t_{n}$, which would contradict the result of Lemma 4.4.

### 4.3 Determination of the rate of growth

To prove Theorem 3, it remains to show that the growth of Sobolev norms is exponential in time in the case when $\ell_{1}=0$. This can be seen through the inverse formula of Theorem 4, which will enable us to prove the following result :

Lemma 4.5. Let $u_{0} \in \mathcal{V}(4)$ such that $\ell_{1}\left(u_{0}\right)=0$. Let $t \mapsto u(t)$ be the corresponding solution of (1). Then there exists a time $t_{0}>0$ such that for all $t \in \mathbb{R}$ with $|t| \geq t_{0}, u(t)$ has two distinct poles, one of which comes close to the unit circle $\partial \mathbb{D} \subset \mathbb{C}$ exponentially fast in time.
Proof. As above, we denote the singular values associated to $u(t)$ by $\rho_{1}>\sigma_{1}>\rho_{2}>\sigma_{2}$, where $\rho_{1}$ and $\rho_{2}$ depend on time. Under the hypothesis of the lemma, we have seen that $u_{1}^{K}:=[u(t)]_{1}^{K}$ goes to zero exponentially fast. Together with the formula coming from Proposition 2.3 :

$$
\left\|u_{1}^{K}\right\|_{L^{2}}^{2}=\frac{\left(\rho_{1}^{2}-\sigma_{1}^{2}\right)\left(\sigma_{1}^{2}-\rho_{2}^{2}\right)}{\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)}
$$

it means that at least one of the $\rho_{j}$ 's shrinks exponentially fast to $\sigma_{1}$. Notice that we have seen during the proof of Lemma 4.1 that $Q=\operatorname{Tr} H_{u}^{2}-\operatorname{Tr} K_{u}^{2}$, so in our case,

$$
\begin{equation*}
Q=\rho_{1}^{2}-\sigma_{1}^{2}+\rho_{2}^{2}-\sigma_{2}^{2} . \tag{28}
\end{equation*}
$$

In particular, the $\rho_{j}$ 's both converge exponentially fast to some limit.
Define the angles $\varphi_{1}, \varphi_{2}, \psi_{1}, \psi_{2} \in \mathbb{T}$ (depending on time) so that

$$
\begin{array}{ll}
H_{u}\left(u_{1}^{H}\right)=\rho_{1} e^{i \varphi_{1}} u_{1}^{H}, & K_{u}\left(u_{1}^{K}\right)=\sigma_{1} e^{i \psi_{1}} u_{1}^{K}, \\
H_{u}\left(u_{2}^{H}\right)=\rho_{2} e^{i \varphi_{2}} u_{2}^{H}, & K_{u}\left(u_{2}^{K}\right)=\sigma_{2} e^{i \psi_{2}} u_{2}^{K} .
\end{array}
$$

Adapting Theorem 4 to this context of simple singular values (involving Blaschke products of degree 0 only) shows that $u(t, z)$ is simply given by the sum of the coefficients of the inverse of the following matrix :

$$
\mathscr{C}(z):=\left(\begin{array}{cc}
\frac{\rho_{1} e^{i \varphi_{1}}-\sigma_{1} e^{i \psi_{1}} z}{\rho_{1}^{2}-\sigma_{1}^{2}} & \frac{\rho_{1} e^{i \varphi_{1}}-\sigma_{2} e^{i \psi_{2}} z}{\rho_{1}^{2}-\sigma_{2}^{2}} \\
\frac{\rho_{2} e^{i \varphi_{2}}-\sigma_{1} e^{i \psi_{1}} z}{\rho_{2}^{2}-\sigma_{1}^{2}} & \frac{\rho_{2} e^{i \varphi_{2}}-\sigma_{2} e^{i \psi_{2}} z}{\rho_{2}^{2}-\sigma_{2}^{2}}
\end{array}\right) .
$$

Since all the coefficients of $\mathscr{C}(z)$ are polynomials in $z$, computing $\mathscr{C}^{-1}(z)$ thanks to the cofactor matrix, we see that the poles of $u(t, \cdot)$ will be given by the inverse of the roots of $\operatorname{det} \mathscr{C}(z)$. As we are only interested by the modulus of these roots, we can change $z$ into $z e^{-i \theta}$ and $\operatorname{det} \mathscr{C}(z)$ into $e^{-i \theta^{\prime}} \operatorname{det} \mathscr{C}(z)$, for $\theta, \theta^{\prime} \in \mathbb{T}$. So we only have to look for the zeros of

$$
z \mapsto \frac{\rho_{1} e^{i \varphi}-\sigma_{1} e^{i \psi} z}{\rho_{1}^{2}-\sigma_{1}^{2}} \frac{\rho_{2}-\sigma_{2} z}{\rho_{2}^{2}-\sigma_{2}^{2}}-\frac{\rho_{1} e^{i \varphi}-\sigma_{2} z}{\rho_{1}^{2}-\sigma_{2}^{2}} \frac{\rho_{2}-\sigma_{1} e^{i \psi} z}{\rho_{2}^{2}-\sigma_{1}^{2}},
$$

where we have set $\varphi:=\varphi_{1}-\varphi_{2}$ and $\psi:=\psi_{1}-\psi_{2}$. Multiplying this polynomial by $\left(\rho_{1}^{2}-\sigma_{1}^{2}\right)\left(\rho_{1}^{2}-\right.$ $\left.\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\rho_{2}^{2}\right)\left(\rho_{2}^{2}-\sigma_{2}^{2}\right)$ means that we only have to seek the roots of

$$
\begin{aligned}
P_{t}(z):=\left(\rho_{1}^{2}-\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\rho_{2}^{2}\right)\left(\rho_{1} e^{i \varphi}-\sigma_{1} e^{i \psi} z\right) & \left(\rho_{2}-\sigma_{2} z\right) \\
& +\left(\rho_{1}^{2}-\sigma_{1}^{2}\right)\left(\rho_{2}^{2}-\sigma_{2}^{2}\right)\left(\rho_{1} e^{i \varphi}-\sigma_{2} z\right)\left(\rho_{2}-\sigma_{1} e^{i \psi} z\right) .
\end{aligned}
$$

We now have to distinguish three cases :
First case : $\rho_{1}^{2} \rightarrow \sigma_{1}^{2}$, but $\rho_{2}^{2} \rightarrow \tau^{2}$, where $\sigma_{2}<\tau<\sigma_{1}$. In that case, it appears that if we define

$$
P_{t}^{\lim , 1}(z):=\sigma_{1} e^{i \psi}\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)\left(\sigma_{1}^{2}-\tau^{2}\right)\left(e^{i(\varphi-\psi)}-z\right)\left(\tau-\sigma_{2} z\right),
$$

then the coefficients of $P_{t}$ and $P_{t}^{\text {lim, } 1}$ become exponentially close to each other. But so do their roots, because $P_{t}^{\text {lim, } 1}$ has distinct roots (one of them is of modulus 1 and the other one is of modulus $\tau / \sigma_{2}>1$ ), and the discriminant formulae are differentiable in the coefficients in this case. So one of the roots of $P_{t}$ converges exponentially fast to the unit circle $\partial \mathbb{D}$ (and so does the corresponding pole of $u(z)$ ).

Second case : $\rho_{2}^{2} \rightarrow \sigma_{1}^{2}$, but $\rho_{1}^{2} \rightarrow\left(\tau^{\prime}\right)^{2}$, where $\tau^{\prime}>\sigma_{1}$. This case goes as the preceding one, by considering the second term in $P_{t}$ as the leading order.

Third case : $\rho_{1}^{2} \rightarrow \sigma_{1}^{2}$ and $\rho_{2}^{2} \rightarrow \sigma_{1}^{2}$. By the formula (28) for $Q$, it implies that $Q=\sigma_{1}^{2}-\sigma_{2}^{2}$, and then we obtain $\rho_{1}^{2}-\sigma_{1}^{2}=\sigma_{1}^{2}-\rho_{2}^{2}$. So the coefficients of $\left(\rho_{1}^{2}-\sigma_{1}^{2}\right)^{-1} P_{t}(z)$ come exponentially
close to those of

$$
P_{t}^{\lim , 3}(z):=\sigma_{1} e^{i \psi}\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)\left[\left(e^{i(\varphi-\psi)}-z\right)\left(\sigma_{1}-\sigma_{2} z\right)+\left(\sigma_{1} e^{i \varphi}-\sigma_{2} z\right)\left(e^{-i \psi}-z\right)\right]
$$

In that case, we need something more to get to the conclusion, and this is precisely what preserves the asymptotic behaviour of $u(t)$ of just disclosing a simple crossing phenomenon, namely the fact that we also have $w_{1}^{K}:=\left[\Pi\left(|u(t)|^{2}\right)\right]_{1}^{K} \rightarrow 0$.

We first compute $w_{1}^{K}$ in terms of the variables $\rho_{j}, \varphi_{j}$ and $\sigma_{k}$. Writing $u=u_{1}^{H}+u_{2}^{H}$, we have

$$
w_{1}^{K}=\left(H_{u}(u) \left\lvert\, \frac{u_{1}^{K}}{\left\|u_{1}^{K}\right\|_{L^{2}}^{2}}\right.\right) u_{1}^{K}=\left(\rho_{1} e^{i \varphi_{1}} u_{1}^{H}+\rho_{2} e^{i \varphi_{2}} u_{2}^{H} \left\lvert\, \frac{u_{1}^{K}}{\left\|u_{1}^{K}\right\|_{L^{2}}^{2}}\right.\right) u_{1}^{K}
$$

But $\left(u_{j}^{H} \mid u_{1}^{K}\right)$ is easy to compute. Indeed,

$$
\begin{aligned}
\rho_{j}^{2}\left(u_{j}^{H} \mid u_{1}^{K}\right)=\left(H_{u}^{2}\left(u_{j}^{H}\right) \mid u_{1}^{K}\right)= & \left(u_{j}^{H} \mid H_{u}^{2}\left(u_{1}^{K}\right)\right) \\
& =\left(u_{j}^{H} \mid K_{u}^{2}\left(u_{1}^{K}\right)+\left(u_{1}^{K} \mid u\right) u\right)=\sigma_{1}^{2}\left(u_{j}^{H} \mid u_{1}^{K}\right)+\left\|u_{j}^{H}\right\|_{L^{2}}^{2}\left\|u_{1}^{K}\right\|_{L^{2}}^{2}
\end{aligned}
$$

so

$$
\left(u_{j}^{H} \mid u_{1}^{K}\right)=\frac{\left\|u_{j}^{H}\right\|_{L^{2}}^{2}\left\|u_{1}^{K}\right\|_{L^{2}}^{2}}{\rho_{j}^{2}-\sigma_{1}^{2}}
$$

Going back to the expression of $w_{1}^{K}$, with the help of Proposition 2.3 again, we find

$$
\begin{aligned}
w_{1}^{K} & =\left(\rho_{1} e^{i \varphi_{1}} \frac{\left\|u_{1}^{H}\right\|_{L^{2}}^{2}}{\rho_{1}^{2}-\sigma_{1}^{2}}-\rho_{2} e^{i \varphi_{2}} \frac{\left\|u_{2}^{H}\right\|_{L^{2}}^{2}}{\sigma_{1}^{2}-\rho_{2}^{2}}\right) u_{1}^{K} \\
& =\left(\rho_{1} e^{i \varphi_{1}} \frac{\rho_{1}^{2}-\sigma_{2}^{2}}{\rho_{1}^{2}-\rho_{2}^{2}}-\rho_{2} e^{i \varphi_{2}} \frac{\rho_{2}^{2}-\sigma_{2}^{2}}{\rho_{1}^{2}-\rho_{2}^{2}}\right) u_{1}^{K}
\end{aligned}
$$

Now, since $\ell_{1}(u)=\left(2 Q+\sigma_{1}^{2}\right)\left\|u_{1}^{K}\right\|_{L^{2}}^{2}-\left\|w_{1}^{K}\right\|_{L^{2}}^{2}=0$, it implies that for all times,

$$
\left(2 Q+\sigma_{1}^{2}\right)=\frac{\left\|w_{1}^{K}\right\|_{L^{2}}^{2}}{\left\|u_{1}^{K}\right\|_{L^{2}}^{2}}
$$

hence

$$
\begin{aligned}
\left(2 Q+\sigma_{1}^{2}\right)\left(\rho_{1}^{2}-\rho_{2}^{2}\right)^{2} & =\left|\rho_{1} e^{i \varphi_{1}}\left(\rho_{1}^{2}-\sigma_{2}^{2}\right)-\rho_{2} e^{i \varphi_{2}}\left(\rho_{2}^{2}-\sigma_{2}^{2}\right)\right|^{2} \\
& =\left|\rho_{1} e^{i \varphi_{1}}\left(\rho_{1}^{2}-\rho_{2}^{2}\right)+\left(\rho_{1} e^{i \varphi_{1}}-\rho_{2} e^{i \varphi_{2}}\right)\left(\rho_{2}^{2}-\sigma_{2}^{2}\right)\right|^{2}
\end{aligned}
$$

Therefore, $\left|\rho_{1} e^{i \varphi_{1}}-\rho_{2} e^{i \varphi_{2}}\right|^{2}$ has to go to zero exponentially fast. Developping this expression, we see that in particular, $\cos \left(\varphi_{1}-\varphi_{2}\right)=\cos \varphi \rightarrow 0$ exponentially fast. So $\varphi \rightarrow 0$ in $\mathbb{T}$ at an exponential rate. So we can replace the polynomial $P_{t}^{\text {lim,3 }}$ above by

$$
\tilde{P}_{t}^{\lim , 3}(z):=2 \sigma_{1} e^{i \psi}\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)\left(\sigma_{1}-\sigma_{2} z\right)\left(e^{-i \psi}-z\right)
$$

This finishes to show that one of the roots of $P_{t}$ in $\mathbb{C}$ has to approach $\partial \mathbb{D}$ exponentially fast.

Thanks to Lemma 4.5, we can come to the conclusion of the proof of the growth result on $\mathcal{V}(4)$.

Proof of Theorem 3. Let $t_{0} \in \mathbb{R}$ as in Lemma 4.5, and write $u(t)$ as

$$
u(t, z)=\frac{\alpha}{1-p z}+\frac{\beta}{1-q z},
$$

where $1-|p| \sim e^{-\kappa|t|}$ as $|t| \rightarrow+\infty$, and $|q| \leq q_{\max }<1$. We also know that $|\alpha|$ and $|\beta|$ are bounded functions (for instance, by the proof of Proposition 3.12).

Observe that for some constant $C>0$, we have

$$
\frac{1}{C} \leq\left\|\frac{\alpha}{1-p z}\right\|_{H^{1 / 2}}^{2} \leq C
$$

The right bound is immediate with the one on $q$ and on $u$. The left bound comes from the fact that $u$ cannot come arbitrarily close to $\mathcal{V}(2)$ in $H^{1 / 2}$ : if there was a sequence of times $t_{n}$ such that

$$
\left\|u\left(t_{n}\right)-\frac{\beta\left(t_{n}\right)}{1-q\left(t_{n}\right) z}\right\|_{H^{1 / 2}} \longrightarrow 0
$$

then by compacity (since $\beta$ is bounded and $q$ is bounded away from $\partial \mathbb{D}$ ), $\beta\left(t_{n}\right) /\left(1-q\left(t_{n}\right) z\right)$ would converge along some subsequence to some $v_{\infty} \in \mathcal{V}(2)$ strongly in $H^{1 / 2}$. Then $\left\|u\left(t_{n}\right)-v_{\infty}\right\|_{H^{1 / 2}}$ would go to zero, but this cannot happen, since $K_{u\left(t_{n}\right)}^{2}$ has to distinct constant eigenvalues, whereas $K_{v_{\infty}}^{2}$ has only one. This fact can also be proved by invoking the stability of $\mathcal{V}(2)$ in $H_{+}^{1 / 2}$, which is established in [16, Section 4].

As a consequence,

$$
\frac{1}{C}\left(1-|p|^{2}\right)^{2} \leq|\alpha|^{2} \leq C\left(1-|p|^{2}\right)^{2}
$$

Besides, the study of the power series $\sum x^{j} j^{2 s}$ as $x \rightarrow 1^{-}$shows that

$$
\left\|\frac{1}{1-p z}\right\|_{H^{s}}^{2}=\sum_{j=0}^{\infty}|p|^{2 j}\left(1+j^{2}\right)^{s} \simeq C_{s}\left(1-|p|^{2}\right)^{-(1+2 s)}
$$

as $|p| \rightarrow 1$, for some constant $C_{s}>0$. Therefore, if $s>\frac{1}{2}$,

$$
\frac{1}{C^{\prime}}\left(\frac{1}{1-|p|^{2}}\right)^{2 s-1} \leq\left\|\frac{\alpha}{1-p z}\right\|_{H^{s}}^{2} \simeq\|u\|_{H^{s}}^{2} \leq C^{\prime}\left(\frac{1}{1-|p|^{2}}\right)^{2 s-1}
$$

which concludes the proof.

### 4.4 Example of an initial data in $\mathcal{V}(4)$ with $\ell_{1}=0$

We conclude this chapter by giving an example showing that the condition $\ell_{1}=0$ can indeed occur on $\mathcal{V}(4)$. This will finish the proof of the existence of unbounded orbits in $H^{s}$ inside $\mathcal{V}(4)$.

Proposition 4.6. Let $p \in \mathbb{C}$ with $0<|p|<1$, and fix

$$
u(z):=\frac{z}{(1-p z)^{2}}, \quad \forall z \in \mathbb{D} .
$$

Then $u \in \mathcal{V}(4), K_{u}^{2}$ has two distinct eigenvalues, and $\ell_{2}(u) \neq 0$. In addition, $\ell_{1}(u)=0$ if and only if $|p|^{2}=3 \sqrt{2}-4$.

Proof. We first compute $Q$ and $J$. Observe that

$$
Q(u)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \frac{e^{2 i x}}{\left(1-p e^{i x}\right)^{2}\left(e^{i x}-\bar{p}\right)^{2}} d x=\frac{1}{2 i \pi} \int_{\mathcal{C}} \frac{z}{(1-p z)^{2}(z-\bar{p})^{2}} d z
$$

where $\mathcal{C}$ denotes the unit circle in $\mathbb{C}$. To calculate this contour integral, we use the residue formula, so we compute

$$
\operatorname{Res}_{z=\bar{p}}\left[\frac{z}{(1-p z)^{2}(z-\bar{p})^{2}}\right]=\left.\frac{d}{d z}\right|_{z=\bar{p}}\left[\frac{z}{(1-p z)^{2}}\right] .
$$

Let $r:=|p|^{2}$. Then we find $Q(u)=\frac{1+r}{(1-r)^{3}}$. Similarly,

$$
J(u)=\frac{1}{2 i \pi} \int_{\mathcal{C}} \frac{z^{2}}{(1-p z)^{4}(z-\bar{p})^{2}} d z=\left.\frac{d}{d z}\right|_{z=\bar{p}}\left[\frac{z^{2}}{(1-p z)^{4}}\right]=2 Q(u) \frac{\bar{p}}{\left(1-|p|^{2}\right)^{2}} .
$$

So the following expressions are established :

$$
|J(u)|^{2}=\frac{4 r(1+r)^{2}}{(1-r)^{10}}, \quad \frac{|J(u)|^{2}}{Q(u)^{2}}=\frac{4 r}{(1-r)^{4}} .
$$

It remains to find the expression of the eigenvalues of $K_{u}^{2}$. Since $K_{u}=H_{S^{*} u}$, we define

$$
\tilde{u}(z):=\frac{1}{(1-p z)^{2}},
$$

and we study $H_{\tilde{u}}^{2}$. We know from [4, Appendix 4] that the image of $H_{\tilde{u}}$ is generated by $e_{1}(z):=$ $\frac{1}{1-p z}$ and by $e_{2}(z):=\frac{1}{(1-p z)^{2}}$. By the means of a partial fraction decomposition, we find

$$
\begin{aligned}
& H_{\tilde{u}}\left(e_{1}\right)=\frac{|p|^{2}}{\left(1-|p|^{2}\right)^{2}} e_{1}+\frac{1}{1-|p|^{2}} e_{2} \\
& H_{\tilde{u}}\left(e_{2}\right)=\frac{2|p|^{2}}{\left(1-|p|^{2}\right)^{3}} e_{1}+\frac{1}{\left(1-|p|^{2}\right)^{2}} e_{2}
\end{aligned}
$$

We can compute the matrix of $H_{\tilde{u}}^{2}$ in the basis $\left((1-r)^{-1} e_{1}, e_{2}\right)$. It reads

$$
\frac{1}{(1-r)^{4}}\left[\begin{array}{cc}
r(2+r) & 1+r \\
2 r(1+r) & 1+2 r
\end{array}\right] .
$$

We have $\operatorname{Tr} H_{\tilde{u}}^{2}=(1-r)^{-4}\left(1+4 r+r^{2}\right)$, and $\operatorname{det} H_{\tilde{u}}^{2}=(1-r)^{-8} r\left[(2+r)(1+2 r)-2(1+r)^{2}\right]=$ $(1-r)^{-8} r^{2}$. Thus the characteristic polynomial of $H_{\tilde{u}}^{2}$ equals

$$
\chi(X)=\frac{1}{(1-r)^{8}} P\left((1-r)^{4} X\right),
$$

where $P(X)=r^{2}-\left(1+4 r+r^{2}\right) X+X^{2}$. We deduce from $P$ the eigenvalues of $K_{u}^{2}$ :

$$
\frac{1+4 r+r^{2} \pm(1+r) \sqrt{1+6 r+r^{2}}}{2(1-r)^{4}}
$$

where the $+\operatorname{sign}$ corresponds to $\sigma_{1}^{2}$ and the $-\operatorname{sign}$ to $\sigma_{2}^{2}$. Note that $\sigma_{1}>\sigma_{2}$ indeed.
Compute

$$
Q+\sigma_{j}^{2}=\frac{3+4 r-r^{2} \pm(1+r) \sqrt{1+6 r+r^{2}}}{2(1-r)^{4}}
$$

so $|J|^{2}=Q^{2}\left(Q+\sigma_{j}^{2}\right)$ if and only if

$$
\begin{equation*}
8 r=3+4 r-r^{2} \pm(1+r) \sqrt{1+6 r+r^{2}} . \tag{29}
\end{equation*}
$$

This implies that $\left(3-4 r-r^{2}\right)^{2}=(1+r)^{2}\left(1+6 r+r^{2}\right)$, and developping this expression, the terms in $r^{4}$ and $r^{3}$ cancel out. We end up with an equation of degree 2 on $r$ :

$$
r^{2}+8 r-2=0
$$

This equation ony has one positive solution, $r=3 \sqrt{2}-4$. Going back to (29), we see that only the - sign is consistent. Consequently, if $|p|^{2}=3 \sqrt{2}-4$, then $|J|^{2}=Q^{2}\left(Q+\sigma_{2}^{2}\right)$ and therefore $\ell_{1}(u)=0$, whereas $\ell_{2}(u)=0$ never occurs for functions of the type $\frac{z}{(1-p z)^{2}}$.

## 5 Computation of the Poisson brackets

In the last part of this paper, we intend to finish the proof of Theorem 2 by proving the Poisson-commutation of the conservation laws of the quadratic Szegó equation. Throughout this section, the notation $\|\cdot\|$ will always refer to the $L^{2}$ norm.

### 5.1 The generating series

Recall some notations : for $u \in H_{+}^{1 / 2}$, and $n \geq 1$, we set $J_{n}(u)=\left(H_{u}^{n}(1) \mid 1\right)$. In particular, $J_{2}=Q$ and $J_{3}=J$ - in the sequel, we prefer these harmonized notations. We also define, for $x \in \mathbb{R}$ such that $\frac{1}{x} \notin \operatorname{Sp}\left(H_{u}^{2}\right)$, and $m \geq 0$,

$$
\mathscr{J}^{(m)}(x):=\left(\left(I-x H_{u}^{2}\right)^{-1} H_{u}^{m}(1) \mid 1\right)=\sum_{j=0}^{+\infty} x^{j} J_{m+2 j} .
$$

The first result we establish is the following alternative form for the generating series :

Proposition 5.1. Let $u \in H_{+}^{1 / 2}$, and denote by $\sigma_{k}, \ell_{k}, k \geq 1$, the conservation laws associated to $u$ as defined above. Then

$$
\begin{equation*}
\sum_{k \geq 1} \frac{\ell_{k}}{1-x \sigma_{k}^{2}}=\mathscr{R}(x):=\frac{J_{2}^{2}+x\left|\mathscr{J}^{(3)}(x)\right|^{2}-x^{2} \mathscr{J}^{(4)}(x)^{2}}{\mathscr{J}^{(0)}(x)} \tag{30}
\end{equation*}
$$

Remark 8. We should observe that to some extent, Lemma 5.1 is a generalization of the formulae of Lemma 4.1, that we can recover here by developping $\mathscr{R}(x)$ as a power series.

We are going to express the right hand side of (30) in terms of the resolvant of $K_{u}$. As above, we set, for appropriate $x \in \mathbb{R}$,

$$
\begin{aligned}
\mathscr{K}^{(0)}(x) & :=\left(\left(I-x K_{u}^{2}\right)^{-1}(1) \mid 1\right) \\
\mathscr{K}^{(1)}(x) & :=\left(\left(I-x K_{u}^{2}\right)^{-1}(u) \mid 1\right) \\
\mathscr{K}^{(2)}(x) & :=\left(\left(I-x K_{u}^{2}\right)^{-1}(u) \mid u\right)
\end{aligned}
$$

Lemma 5.2. For all $x \in \mathbb{R}$ such that it is defined, we have

$$
2+2 x J_{2}-x^{2} \mathscr{R}(x)=\mathscr{K}^{(0)}(x)+2 x \operatorname{Re}\left(\overline{J_{1}} \mathscr{K}^{(1)}(x)\right)+\left(1-x \mathscr{K}^{(2)}(x)\right)\left(1+x\left(2 J_{2}-\left|J_{1}\right|^{2}\right)\right)
$$

Proof. The proof relies on identities discovered in [7, 18] that we recall here. Since $K_{u}^{2}=H_{u}^{2}-$ $(\cdot \mid u) u$ (see (4)), we have, for $h \in L_{+}^{2}$,

$$
\begin{aligned}
\left(I-x K_{u}^{2}\right)^{-1}(h)-\left(I-x H_{u}^{2}\right)^{-1}(h) & =\left(I-x K_{u}^{2}\right)^{-1}\left[\left(I-x H_{u}^{2}\right)-\left(I-x K_{u}^{2}\right)\right]\left(I-x H_{u}^{2}\right)^{-1}(h) \\
& =-x\left(h \mid\left(I-x H_{u}^{2}\right)^{-1}(u)\right)\left(I-x K_{u}^{2}\right)^{-1}(u)
\end{aligned}
$$

Taking $h=u$ yields

$$
\begin{equation*}
\left(I-x H_{u}^{2}\right)^{-1}(u)=\mathscr{J}^{(0)}(x) \cdot\left(I-x K_{u}^{2}\right)^{-1}(u) \tag{31}
\end{equation*}
$$

and taking $h=1$ gives, once we have made the scalar product with 1 ,

$$
\left(\left(I-x K_{u}^{2}\right)^{-1}(1) \mid 1\right)-\left(\left(I-x H_{u}^{2}\right)^{-1}(1) \mid 1\right)=-x \overline{\mathscr{J}^{(1)}(x)} \mathscr{K}^{(1)}(x)
$$

Since $\mathscr{J}^{(1)}(x)=\mathscr{J}^{(0)}(x) \mathscr{K}^{(1)}(x)$ by (31), this can also be written as

$$
\begin{equation*}
\mathscr{K}^{(0)}(x)=\mathscr{J}^{(0)}(x)\left(1-x\left|\mathscr{K}^{(1)}(x)\right|^{2}\right)=\mathscr{J}^{(0)}(x)-x \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)} \tag{32}
\end{equation*}
$$

Observe also that $\mathscr{J}^{(0)}(x)=1+x \mathscr{J}^{(2)}(x)=1+x \mathscr{J}^{(0)}(x) \mathscr{K}^{(2)}(x)$, hence

$$
\begin{equation*}
\frac{1}{\mathscr{J}^{(0)}(x)}=1-x \mathscr{K}^{(2)}(x) \tag{33}
\end{equation*}
$$

Finally, we have $J_{2}+x \mathscr{J}^{(4)}(x)=\mathscr{J}^{(2)}(x)$ and $J_{1}+x \mathscr{J}^{(3)}(x)=\mathscr{J}^{(1)}(x)$.

Now we are ready to transform the expression of $\mathscr{R}(x)$, using (32), and (33) together with (31) :

$$
\begin{aligned}
& x^{2} \mathscr{R}(x)=\frac{-x^{4} \mathscr{J}^{(4)}(x)^{2}+x^{3}\left|\mathscr{J}^{(3)}(x)\right|^{2}+x^{2} J_{2}^{2}}{\mathscr{J}^{(0)}(x)} \\
&=-\frac{x^{2}\left[\mathscr{J}^{(2)}(x)-J_{2}\right]^{2}}{\mathscr{J}^{(0)}(x)}+\frac{x^{3}}{\mathscr{J}^{(0)}(x)}\left|\frac{\mathscr{J}^{(1)}(x)-J_{1}}{x}\right|^{2}+\frac{x^{2} J_{2}^{2}}{\mathscr{J}^{(0)}(x)} \\
&=-\frac{x^{2} \mathscr{J}^{(2)}(x)^{2}}{\mathscr{J}^{(0)}(x)}+2 x^{2} J_{2} \mathscr{K}^{(2)}(x)-\left(\mathscr{K}^{(0)}(x)-\mathscr{J}^{(0)}(x)\right)-\frac{2 x \operatorname{Re}\left(\overline{J_{1}} \mathscr{J}^{(1)}(x)\right)}{\mathscr{J}^{(0)}(x)}+\frac{x\left|J_{1}\right|^{2}}{\mathscr{J}^{(0)}(x)} \\
& \quad=2-\frac{1}{\mathscr{J}^{(0)}(x)}+2 x^{2} J_{2} \mathscr{K}^{(2)}(x)-\mathscr{K}^{(0)}(x)-2 x \operatorname{Re}\left(\overline{J_{1}} \mathscr{K}^{(1)}(x)\right)+\frac{x\left|J_{1}\right|^{2}}{\mathscr{J}^{(0)}(x)} \\
& \quad=2+2 x J_{2}-\left(1-x \mathscr{K}^{(2)}(x)\right)\left(1+x\left(2 J_{2}-\left|J_{1}\right|^{2}\right)\right)-\mathscr{K}^{(0)}(x)-2 x \operatorname{Re}\left(\overline{J_{1}} \mathscr{K}^{(1)}(x)\right),
\end{aligned}
$$

and the lemma is proved.
Now we can turn to the
Proof of Proposition 5.1. Let us first restrict to a convenient framework: we assume that $u$ is a rational function, with $\operatorname{rk} H_{u}=\operatorname{rk} K_{u}$ (i.e. $u \perp \operatorname{ker} K_{u}$ ), and denoting by $\left\{\rho_{j}\right\}$ (resp. $\left\{\sigma_{k}\right\}$ ) the elements of $\Xi_{u}^{H}$ (resp. $\Xi_{u}^{K}$ ), we also assume that all these singular values are of multiplicity one. In particular, this imposes all the eigenvalues of $K_{u}^{2}$ to be $K$-dominant. Then the general result will follow by density of such functions in $H_{+}^{1 / 2}$.

We are going to study the poles of $\mathscr{R}(x)$. Recall that if $h \in L_{+}^{2}$ is given, $h_{k}^{K}$ refers to the orthogonal projection of $h$ onto $\operatorname{ker}\left(K_{u}^{2}-\sigma_{k}^{2} I\right)$. In particular, we have

$$
\begin{aligned}
u & =\sum_{k} u_{k}^{K}, \\
1 & =\sum_{k} 1_{k}^{K},
\end{aligned}
$$

all the sums being finite. As a consequence, we can write

$$
\begin{aligned}
& \mathscr{K}^{(0)}(x)=\sum_{k} \frac{\left\|1_{k}^{K}\right\|^{2}}{1-x \sigma_{k}^{2}}, \\
& \mathscr{K}^{(1)}(x)=\sum_{k} \frac{\left(u_{k}^{K} \mid 1\right)}{1-x \sigma_{k}^{2}}, \\
& \mathscr{K}^{(2)}(x)=\sum_{k} \frac{\left\|u_{k}^{K}\right\|^{2}}{1-x \sigma_{k}^{2}} .
\end{aligned}
$$

By Lemma 5.2 and the previous expressions, we then see that $\mathscr{R}(x)$ is a rational function of $x$, that it has simple poles at each $\frac{1}{\sigma_{k}^{2}}$, and that its limit as $x \rightarrow+\infty$ equals 0 .

Besides, multiplying the equality in Lemma 5.2 by $\left(1-x \sigma_{k}^{2}\right)$ and evaluating at $x=1 / \sigma_{k}^{2}$ gives the following formula for the poles of $\mathscr{R}(x)$ :

$$
\alpha_{k}:=\left\|u_{k}^{K}\right\|^{2}\left(2 J_{2}-\left|J_{1}\right|^{2}+\sigma_{k}^{2}\right)-2 \sigma_{k}^{2} \operatorname{Re}\left(\overline{J_{1}}\left(u_{k}^{K} \mid 1\right)\right)-\sigma_{k}^{4}\left\|1_{k}^{K}\right\|^{2}
$$

It remains to show that $\alpha_{k}=\ell_{k}$ for all $k \geq 1$. Using the fact that $1_{k}^{K}$ is colinear to $u_{k}^{K}$ because of our assumption on the dimension of the eigenspaces of $K_{u}^{2}$, we have

$$
\begin{aligned}
\alpha_{k} & =\left\|u_{k}^{K}\right\|^{2}\left(2 Q-|(u \mid 1)|^{2}+\sigma_{k}^{2}\right)-2 \sigma_{k}^{2} \operatorname{Re}\left((1 \mid u)\left(u_{k}^{K} \mid 1\right)\right)-\sigma_{k}^{4}\left\|1_{k}^{K}\right\|^{2} \\
& =\left\|u_{k}^{K}\right\|^{2}\left(2 Q+\sigma_{k}^{2}-|(u \mid 1)|^{2}-2 \sigma_{k}^{2} \operatorname{Re}\left((1 \mid u) \frac{\left(u_{k}^{K} \mid 1\right)}{\left\|u_{k}^{K}\right\|^{2}}\right)-\sigma_{k}^{4} \frac{\left|\left(u_{k}^{K} \mid 1\right)\right|^{2}}{\left\|u_{k}^{K}\right\|^{4}}\right) \\
& =\left\|u_{k}^{K}\right\|^{2}\left(2 Q+\sigma_{k}^{2}-\left|(u \mid 1)+\sigma_{k}^{2} \frac{\left(u_{k}^{K} \mid 1\right)}{\left\|u_{k}^{K}\right\|^{2}}\right|^{2}\right) .
\end{aligned}
$$

Because of formula $K_{u}^{2}+(\cdot \mid u) u=H_{u}^{2}$, we have

$$
(u \mid 1)+\sigma_{k}^{2} \frac{\left(u_{k}^{K} \mid 1\right)}{\left\|u_{k}^{K}\right\|^{2}}=(u \mid 1)+\frac{\left(H_{u}^{2}\left(u_{k}^{K}\right)-\left(u_{k}^{K} \mid u\right) u \mid 1\right)}{\left\|u_{k}^{K}\right\|^{2}}=\frac{\left(u_{k}^{K} \mid H_{u}^{2}(1)\right)}{\left\|u_{k}^{K}\right\|^{2}}=\overline{\xi_{k}}
$$

hence $\alpha_{k}=\left\|u_{k}^{K}\right\|^{2}\left(\left(2 Q+\sigma_{k}^{2}\right)-\left|\xi_{k}\right|^{2}\right)=\ell_{k}$ by Lemma 3.3. The proof of Proposition 5.1 is now complete.

In the sequel, we prefer manipulating another generating function, coming from $\mathscr{R}(x)$, which involves functionals $\mathscr{J}^{(m)}$ that are of lower order. We thus define

$$
\mathscr{F}(x):=2 J_{2}-x \mathscr{R}(x) .
$$

Since $J_{2}^{2}-x^{2} \mathscr{J}^{(4)}(x)^{2}=\left(J_{2}-x \mathscr{J}^{(4)}(x)\right)\left(J_{2}+x \mathscr{J}^{(4)}(x)\right)=\left(2 J_{2}-\mathscr{J}^{(2)}(x)\right) \mathscr{J}^{(2)}(x)$, we get

$$
\begin{aligned}
\mathscr{F}(x) & =\frac{2 J_{2} \mathscr{J}^{(0)}(x)-x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}-x\left(J_{2}^{2}-x^{2} \mathscr{J}^{(4)}(x)^{2}\right)}{\mathscr{J}^{(0)}(x)} \\
& =\frac{2 J_{2}\left(\mathscr{J}^{(0)}(x)-x \mathscr{J}^{(2)}(x)\right)-x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}+x \mathscr{J}^{(2)}(x)^{2}}{\mathscr{J}^{(0)}(x)} \\
& =\frac{2 J_{2}+x \mathscr{J}^{(2)}(x)^{2}-x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}
\end{aligned}
$$

Since $\mathscr{R}(x)$ is invariant by rotation of $u$ by $e^{i \theta}$, we have $\left\{J_{2}, \mathscr{R}(x)\right\}=0$, hence

$$
\left\{\mathscr{F}(x), \sigma_{k}^{2}\right\}=-x\left\{\mathscr{R}(x), \sigma_{k}^{2}\right\}, \quad\{\mathscr{F}(x), \mathscr{F}(y)\}=x y\{\mathscr{R}(x), \mathscr{R}(y)\}
$$

so from now on, we only study $\mathscr{F}(x)$.

### 5.2 A Lax pair for $\mathscr{F}(x)$

As in [4, 5], it is of high importance to study the evolution of the Hamiltonian system generated by $\mathscr{F}(x)$ (where $x \in \mathbb{R}$ is fixed). In particular, we are going to prove that the evolution given by $\dot{u}=X_{\mathscr{F}(x)}(u)$ also admits a Lax pair for $K_{u}$. As a consequence, the $k$-th eigenvalue of $K_{u}^{2}$ will be conserved by this flow, so we will obtain the identity

$$
\left\{\mathscr{F}(x), \sigma_{k}^{2}\right\}=0, \quad \forall k \geq 1
$$

In view of (30), and of the fact that $\left\{\sigma_{j}^{2}, \sigma_{k}^{2}\right\}=0$ for any $j, k \geq 1$, we will get that

$$
\left\{\ell_{j}, \sigma_{k}^{2}\right\}=0, \quad \forall j, k \geq 1
$$

We first introduce th following notations :

$$
\begin{aligned}
w^{0}(x) & :=\left(I-x H_{u}^{2}\right)^{-1}(1), \\
w^{1}(x) & :=\left(I-x H_{u}^{2}\right)^{-1}(u) .
\end{aligned}
$$

Note that $w^{1}(x)=H_{u}\left(w^{0}(x)\right)$, and that we also recover $w^{0}$ from $w^{1}$ thanks to the formula $1+x H_{u}\left(w^{1}(x)\right)=w^{0}(x)$.

Theorem 5. The Hamiltonian vector field associated to the functional $\mathscr{F}(x)$ (where $x \in \mathbb{R}$ is fixed) is given by

$$
\begin{align*}
X_{\mathscr{F}(x)}(u)=\frac{-i}{\mathscr{J}^{(0)}}(4 u+ & x\left(4 \mathscr{J}^{(2)}-2 \mathscr{F}\right) w^{0} H_{u}\left(w^{0}\right) \\
& \left.-2 x^{2} \overline{\mathcal{J}^{(3)}}\left(H_{u}\left(w^{0}\right)\right)^{2}-2 x^{3} \mathscr{J}^{(3)}\left(H_{u}\left(w^{1}\right)\right)^{2}-4 x^{2} \mathscr{J}^{(3)} H_{u}\left(w^{1}\right)\right) \tag{34}
\end{align*}
$$

In addition, for any solution to the evolution equation $\dot{u}=X_{\mathscr{F}(x)}(u)$, we have

$$
\frac{d}{d t} K_{u}=\left[B_{u}^{x}, K_{u}\right],
$$

where $B_{u}^{x}$ is a skew-symmetric operator given by

$$
\begin{gather*}
B_{u}^{x}=-i A_{u}^{x},  \tag{35}\\
A_{u}^{x}:=\frac{1}{\mathscr{J}^{(0)}}\left(2 I+\left(2 \mathscr{J}^{(2)}-\mathscr{F}\right) \cdot\left(x T_{w^{0}} T_{\overline{w^{0}}}+x^{2} T_{w^{1}} T_{\overline{w^{1}}}\right)-2 x^{2}\left(\mathscr{J}^{(3)} T_{w^{0}} T_{\overline{w^{1}}}+\overline{\mathscr{J}^{(3)}} T_{w^{1}} T_{\overline{w^{0}}}\right)\right) \tag{36}
\end{gather*}
$$

Notice that in (34), (36), as well as in the rest of this paragraph, we omit the $x$ dependence of functionals and functions, in order to shorten our formulae.

We will make use of an elementary lemma which we recall here :
Lemma 5.3. Let $h \in L_{+}^{2}$. Then $(I-\Pi)(\bar{z} h)=\bar{z} \overline{\Pi(\bar{h})}$.

Proof of Theorem 5. Recall that

$$
\mathscr{F}=\frac{2 Q+x\left(\mathscr{J}^{(2)}\right)^{2}-x^{2}\left|\mathscr{J}^{(3)}\right|^{2}}{1+x \mathscr{J}^{(2)}}
$$

First of all, we compute, for $h \in L_{+}^{2}$,

$$
\begin{aligned}
d_{u} \mathscr{J}^{(2)} \cdot h & =\left(x\left(I-x H_{u}^{2}\right)^{-1}\left(H_{u} H_{h}+H_{h} H_{u}\right)\left(I-x H_{u}^{2}\right)^{-1}(u) \mid u\right)+2 \operatorname{Re}\left(\left(I-x H_{u}^{2}\right)^{-1}(u) \mid h\right) \\
& =2 x \operatorname{Re}\left(H_{u}\left(w^{1}\right) \mid H_{h}\left(w^{1}\right)\right)+2 \operatorname{Re}\left(w^{1} \mid h\right) \\
& =2 \operatorname{Re}\left(w^{0} w^{1} \mid h\right)=2 \operatorname{Re}\left(w^{0} H_{u}\left(w^{0}\right) \mid h\right)
\end{aligned}
$$

Similarly, as $\mathscr{J}^{(3)}=\left(\left(I-x H_{u}^{2}\right)^{-1}(u) \mid H_{u}(u)\right)$,

$$
\begin{aligned}
d_{u} \mathscr{J}^{(3)} \cdot h & =\left(x\left(H_{u} H_{h}+H_{h} H_{u}\right) w^{1} \mid H_{u}\left(w^{1}\right)\right)+2\left(w^{1} \mid H_{u}(h)\right)+\left(w^{1} \mid H_{h}(u)\right) \\
& =x\left(H_{u}^{2}\left(w^{1}\right) \mid H_{h}\left(w^{1}\right)\right)+x\left(h \mid\left(H_{u}\left(w^{1}\right)\right)^{2}\right)+2\left(h \mid H_{u}\left(w^{1}\right)\right)+\left(u w^{1} \mid h\right) \\
& =\left(\left(H_{u}\left(w^{0}\right)\right)^{2} \mid h\right)+x\left(h \mid\left(H_{u}\left(w^{1}\right)\right)^{2}\right)+2\left(h \mid H_{u}\left(w^{1}\right)\right)
\end{aligned}
$$

where we used that $x H_{u}^{2}\left(w^{1}\right)=w^{1}-u=H_{u}\left(w^{0}\right)-u$.
We are now ready to compute

$$
\begin{aligned}
d_{u} \mathscr{F} \cdot h=- & \frac{\mathscr{F}}{\mathscr{J}^{(0)}} \cdot 2 x \operatorname{Re}\left(h \mid w^{0} H_{u}\left(w^{0}\right)\right)+\frac{1}{\mathscr{J}^{(0)}}\left(4 \operatorname{Re}(h \mid u)+2 x \mathscr{J}^{(2)} \cdot 2 \operatorname{Re}\left(h \mid w^{0} H_{u}\left(w^{0}\right)\right)\right) \\
& -\frac{2 x^{2}}{\mathscr{J}^{(0)}} \operatorname{Re}\left(\mathscr{J}^{(3)}\left(h \mid\left(H_{u}\left(w^{0}\right)\right)^{2}\right)+x \overline{\mathscr{J}^{(3)}}\left(h \mid\left(H_{u}\left(w^{1}\right)\right)^{2}\right)+2 \overline{\mathscr{J}^{(3)}}\left(h \mid H_{u}\left(w^{1}\right)\right)\right) .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d_{u} \mathscr{F} \cdot h=\frac{1}{\mathscr{J}^{(0)}} \operatorname{Re}(h \mid 4 u+ & x\left(4 \mathscr{J}^{(2)}-2 \mathscr{F}\right) w^{0} H_{u}\left(w^{0}\right) \\
& \left.-2 x^{2} \overline{\mathscr{J}^{(3)}}\left(H_{u}\left(w^{0}\right)\right)^{2}-2 x^{3} \mathscr{J}^{(3)}\left(H_{u}\left(w^{1}\right)\right)^{2}-4 x^{2} \mathscr{J}^{(3)} H_{u}\left(w^{1}\right)\right),
\end{aligned}
$$

which is equivalent to formula (34).
Now, assume that $\dot{u}=X_{\mathscr{F}}(u)$, and compute $i \frac{d}{d t} K_{u}(h)$, for $h \in L_{+}^{2}$. Step by step, we have, as in [4], and using Lemma 5.3:

$$
\begin{aligned}
\Pi\left(\bar{z} w^{0} H_{u}\left(w^{0}\right) \bar{h}\right) & =\Pi\left(\bar{z}\left(1+x H_{u}^{2}\left(w^{0}\right)\right) H_{u}\left(w^{0}\right) \bar{h}\right) \\
& =\Pi\left(\bar{z} u \overline{w^{0} h}\right)+x \Pi\left(H_{u}^{2}\left(w^{0}\right)[\Pi+(I-\Pi)]\left(\bar{z} H_{u}\left(w^{0}\right) \bar{h}\right)\right) \\
& =T_{\overline{w^{0}}} K_{u}(h)+x H_{u}^{2}\left(w^{0}\right) \Pi\left(\bar{z} u \overline{w^{0} h}\right)+x \Pi\left(H_{u}^{2}\left(w^{0}\right) \bar{z} \overline{\Pi\left(\overline{H_{u}\left(w^{0}\right)} h\right)}\right) \\
& =\left(1+x H_{u}^{2}\left(w^{0}\right)\right) T \overline{w^{0}} K_{u}(h)+x \Pi\left(\bar{z} u \overline{w^{1}} \overline{\Pi\left(\overline{w^{1}} h\right)}\right) \\
& =T_{w^{0}} T \overline{w^{0}} K_{u}(h)+x K_{u} T_{w^{1}} T \overline{w^{1}}(h),
\end{aligned}
$$

which we can symmetrize as

$$
K_{w^{0} H_{u}\left(w^{0}\right)}=\frac{1}{2}\left(T_{w^{0}} T_{\overline{w^{0}}} K_{u}+K_{u} T_{w^{0}} T_{\overline{w^{0}}}\right)+\frac{x}{2}\left(T_{w^{1}} T_{\overline{w^{1}}} K_{u}+K_{u} T_{w^{1}} T_{\overline{w^{1}}}\right)
$$

Then,

$$
\begin{aligned}
\Pi\left(\bar{z}\left(H_{u}\left(w^{0}\right)\right)^{2} \bar{h}\right) & =H_{u}\left(w^{0}\right) \Pi\left(\bar{z} H_{u}\left(w^{0}\right) \bar{h}\right)+\Pi\left(u \overline{w^{0}}(I-\Pi)\left(\bar{z} H_{u}\left(w^{0}\right) \bar{h}\right)\right) \\
& =w^{1} \Pi\left(\overline{w^{0}} \Pi(\bar{z} u \bar{h})\right)+\Pi\left(\bar{z} u \overline{w^{0} \Pi\left(\overline{w^{1}} h\right)}\right),
\end{aligned}
$$

so

$$
K_{\left(H_{u}\left(w^{0}\right)\right)^{2}}=T_{w^{1}} T_{\overline{w^{0}}} K_{u}+K_{u} T_{w^{0}} T_{\overline{w^{1}}} .
$$

Replacing $w^{0}$ by $w^{1}$ in the previous expression, we get

$$
\begin{aligned}
x K_{\left(H_{u}\left(w^{1}\right)\right)^{2}} & =x T_{H_{u}\left(w^{1}\right)} T_{\overline{w^{1}}} K_{u}+x K_{u} T_{w^{1}} T_{\overline{H_{u}\left(w^{1}\right)}} \\
& =T_{w^{0}} T \overline{w^{1}} K_{u}+K_{u} T_{w^{1}} T \overline{w^{0}}-T \overline{w^{1}} K_{u}-K_{u} T_{w^{1}} .
\end{aligned}
$$

The minus terms will exactly be compensated by

$$
2 \Pi\left(\bar{z} H_{u}\left(w^{1}\right) \bar{h}\right)=2 \Pi\left(\bar{z} u \overline{w^{1} h}\right)=\Pi\left(\overline{w^{1}} \Pi(\bar{z} u \bar{h})\right)+\Pi\left(\bar{z} u \overline{\Pi\left(w^{1} h\right)}\right)
$$

or equivalently, $2 K_{H_{u}\left(w^{1}\right)}=T \overline{w^{1}} K_{u}+K_{u} T_{w^{1}}$. This completes the proof of (35)-(36).

### 5.3 Commutation between the additional conservation laws

In this paragraph, we conclude the proof of Theorem 2 by proving that $\{\mathscr{F}(x), \mathscr{F}(y)\}=0$ when $x \neq y \in \mathbb{R}$ are fixed. Because of (30) and by the preceding commutation identities, it will be enough to show that $\left\{\ell_{j}, \ell_{k}\right\}=0$ when $j, k \geq 1$.

Theorem 6. For any $x \neq y \in \mathbb{R}$, we have $\{\mathscr{F}(x), \mathscr{F}(y)\}=0$.
To prove such a result, we will restrict again on the dense subset of $H_{+}^{1 / 2}$ which consists of symbols $u \in H_{+}^{1 / 2}$ such that both $K_{u}$ and $H_{u}$ have finite rank $N$ for some $N \in \mathbb{N}$, and so that the singular values of $H_{u}$ and $K_{u}$ satisfy

$$
\rho_{1}^{2}>\sigma_{1}^{2}>\rho_{2}^{2}>\sigma_{2}^{2}>\cdots>\rho_{N}^{2}>\sigma_{N}^{2}>0
$$

Recall that, under this assumption of genericity, we can write

$$
u=\sum_{j=1}^{N} u_{j}^{H}=\sum_{k=1}^{N} u_{k}^{K},
$$

where $u_{j}^{H}$ (resp. $u_{k}^{K}$ ) is the projection of $u$ onto the one-dimensional eigenspace of $H_{u}^{2}$ (resp. $K_{u}^{2}$ ) associated to $\rho_{j}^{2}$ (resp. $\sigma_{k}^{2}$ ). In that case, Proposition 2.2 also simplifies, and Blaschke products are just real numbers modulo $2 \pi$ : there exists angles $\left(\varphi_{1}, \ldots, \varphi_{N}, \psi_{1}, \ldots, \psi_{N}\right) \in \mathbb{T}^{2 N}$, such that $H_{u}\left(u_{j}^{H}\right)=\rho_{j} e^{i \varphi_{j}} u_{j}^{H}$ and $K_{u}\left(u_{k}^{K}\right)=\sigma_{k} e^{i \psi_{k}} u_{k}^{K}$, for $j \in \llbracket 1, N \rrbracket$ and $k \in \llbracket 1, N \rrbracket$. Moreover, on this open subset of generic sates of $\mathcal{V}(2 N)$, the symplectic form reads $\omega=\sum_{j=1}^{N} d\left(\rho_{j}^{2} / 2\right) \wedge d \varphi_{j}+$ $\sum_{k=1}^{N} d\left(\sigma_{k}^{2} / 2\right) \wedge d \psi_{k}($ see $[5,7])$.

We begin by proving a lemma inspired by the work of Haiyan Xu [18].

Lemma 5.4. For any $x \neq y \in \mathbb{R}$, we have

$$
\begin{aligned}
& \left\{\left|\mathscr{J}^{(1)}(x)\right|^{2},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}= \\
& \quad \frac{4 \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\left.\mathscr{J}^{(1)}(y)\right)}\right.}{x-y}\left[x \mathscr{J}^{(0)}(x)^{2}-y \mathscr{J}^{(0)}(y)^{2}+x^{2}\left|\mathscr{J}^{(1)}(x)\right|^{2}-y^{2}\left|\mathscr{J}^{(1)}(y)\right|^{2}\right] .
\end{aligned}
$$

Proof. Recall that we defined $\mathscr{K}^{(0)}(x)=\left(\left(I-x K_{u}^{2}\right)^{-1}(1) \mid 1\right)$, and that it obeys

$$
\mathscr{K}^{(0)}(x)=\mathscr{J}^{(0)}(x)-x \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}
$$

by (32).
From the theory of the cubic Szegó equation, it is known that $\left\{\mathscr{J}^{(0)}(x), \mathscr{J}^{(0)}(y)\right\}=0$ (see [4]) and $\left\{\mathscr{K}^{(0)}(x), \mathscr{K}^{(0)}(y)\right\}=0$ (see [18]). In view of (32), this last identity gives

$$
\begin{aligned}
0 & =\left\{\mathscr{J}^{(0)}(x)-x \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}, \mathscr{J}^{(0)}(y)-y \frac{\left|\mathscr{J}^{(1)}(y)\right|^{2}}{\mathscr{J}^{(0)}(y)}\right\} \\
& =-y\left\{\mathscr{J}^{(0)}(x), \frac{\left|\mathscr{J}^{(1)}(y)\right|^{2}}{\mathscr{J}^{(0)}(y)}\right\}+x\left\{\mathscr{J}^{(0)}(y), \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}\right\}+x y\left\{\frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}, \frac{\left|\mathscr{J}^{(1)}(y)\right|^{2}}{\mathscr{J}^{(0)}(y)}\right\} .
\end{aligned}
$$

First of all, we have to compute $\left\{\mathscr{J}^{(0)}(x),\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}$. This is done in [18], but for the seek of completeness, we recall the argument. We have

$$
\mathscr{J}^{(1)}(x)=\left(\sum_{j=1}^{N} \frac{u_{j}^{H}}{1-x \rho_{j}^{2}} \left\lvert\, \sum_{j=1}^{N} \frac{\left(1 \mid u_{j}^{H}\right) u_{j}^{H}}{\left\|u_{j}^{H}\right\|^{2}}\right.\right)=\sum_{j=1}^{N} \frac{\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\rho_{j}\left(1-x \rho_{j}^{2}\right)},
$$

because $\left(u_{j}^{H} \mid 1\right)=\rho_{j}^{-1} e^{-i \varphi_{j}}\left(H_{u}\left(u_{j}^{H}\right) \mid 1\right)$, and $\left(H_{u}\left(u_{j}^{H}\right) \mid 1\right)=\left(H_{u}(1) \mid u_{j}^{H}\right)=\left\|u_{j}^{H}\right\|^{2}$. Besides, we know $([7])$ that $\mathscr{J}^{(0)}(x)=\prod_{j=1}^{N} \frac{1-x \sigma_{j}^{2}}{1-x \rho_{j}^{2}}$, we can compute directly from the expression of $\omega$ :

$$
\begin{aligned}
\left\{\mathscr{J}^{(0)}(x), \mathscr{J}^{(1)}(y)\right\} & =\sum_{j=1}^{N} \frac{2 x \mathscr{J}^{(0)}(x)}{1-x \rho_{j}^{2}} \cdot\left(-i \frac{\left\|u_{j}^{H}\right\|^{2} e^{-\varphi_{j}}}{\rho_{j}\left(1-y \rho_{j}^{2}\right)}\right) \\
& =-2 i x \mathscr{J}^{(0)}(x) \sum_{j=1}^{N} \frac{\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\rho_{j}(x-y)}\left(\frac{x}{1-x \rho_{j}^{2}}-\frac{y}{1-y \rho_{j}^{2}}\right) \\
& =-\frac{2 i x}{x-y} \mathscr{J}^{(0)}(x)\left(x \mathscr{J}^{(1)}(x)-y \mathscr{J}^{(1)}(y)\right) .
\end{aligned}
$$

This yields

$$
\begin{equation*}
\left.\left\{\mathscr{J}^{(0)}(x),\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}=2 \operatorname{Re} \overline{\mathscr{J}^{(1)}(y)}\left\{\mathscr{J}^{(0)}(x), \mathscr{J}^{(1)}(y)\right\}\right)=\frac{4 x^{2} \mathscr{J}^{(0)}(x)}{x-y} \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathscr{J}^{(1)}(y)}\right) . \tag{37}
\end{equation*}
$$

Secondly, we write

$$
\begin{aligned}
\left\{\frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(x)}, \frac{\left|\mathscr{J}^{(1)}(y)\right|^{2}}{\mathscr{J}^{(0)}(y)}\right\} & =\left\{\frac{1}{\mathscr{J}^{(0)}(x)},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\} \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(y)} \\
& -\left\{\frac{1}{\mathscr{J}^{(0)}(y)},\left|\mathscr{J}^{(1)}(x)\right|^{2}\right\} \frac{\left|\mathscr{J}^{(1)}(y)\right|^{2}}{\mathscr{J}^{(0)}(x)}+\frac{\left\{\left|\mathscr{J}^{(1)}(x)\right|^{2},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}}{\mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)},
\end{aligned}
$$

and we have, using (37),

$$
\left\{\frac{1}{\mathscr{J}^{(0)}(x)},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\} \frac{\left|\mathscr{J}^{(1)}(x)\right|^{2}}{\mathscr{J}^{(0)}(y)}=-\frac{4 x^{2}\left|\mathscr{J}^{(1)}(x)\right|^{2}}{(x-y) \mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)} \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\left.\mathscr{J}^{(1)}(y)\right)} .\right.
$$

Now we can go back to $0=\left\{\mathscr{K}^{(0)}(x), \mathscr{K}^{(0)}(y)\right\}$, and get

$$
\begin{aligned}
& 0=\left(-\frac{4 x^{2} y \mathscr{J}^{(0)}(x)}{(x-y) \mathscr{J}^{(0)}(y)}+\frac{4 x y^{2} \mathscr{J}^{(0)}(y)}{(x-y) \mathscr{J}^{(0)}(x)}\right) \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathscr{J}^{(1)}(y)}\right) \\
+ & \left(\frac{-4 x^{3} y\left|\mathscr{J}^{(1)}(x)\right|^{2}+4 x y^{3}\left|\mathscr{J}^{(1)}(y)\right|^{2}}{(x-y) \mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)}\right) \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathscr{J}^{(1)}(y)}\right)+\frac{x y\left\{\left|\mathscr{J}^{(1)}(x)\right|^{2},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}}{\mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)},
\end{aligned}
$$

and this ends the proof of Lemma 5.4.
Lemma 5.5. For $x \neq y \in \mathbb{R}$, we have

$$
\begin{align*}
& \left\{\mathscr{J}^{(3)}(x), \mathscr{J}^{(3)}(y)\right\}=-\frac{2 i}{x-y}\left[x \mathscr{J}^{(3)}(x)-y \mathscr{J}^{(3)}(y)\right]^{2}  \tag{38}\\
& \left\{\mathscr{J}^{(3)}(x), \overline{\mathscr{J}^{(3)}(y)}\right\}=\frac{2 i}{x-y}\left[\frac{\mathscr{J}^{(0)}(x)^{2}}{x}-\frac{\mathscr{J}^{(0)}(y)^{2}}{y}-\frac{1}{x}+\frac{1}{y}\right] . \tag{39}
\end{align*}
$$

Proof. We expand $\mathscr{J}^{(3)}(x)$ thanks to the decomposition $u=\sum_{j} u_{j}^{H}$ :

$$
\begin{equation*}
\mathscr{J}^{(3)}(x)=\sum_{j=1}^{N} \frac{\rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{1-x \rho_{j}^{2}} . \tag{40}
\end{equation*}
$$

and the expression of $\left\|u_{j}^{H}\right\|^{2}$ is given in Proposition 2.3 :

$$
\left\|u_{j}^{H}\right\|^{2}=\frac{\prod_{l=1}^{N}\left(\rho_{j}^{2}-\sigma_{l}^{2}\right)}{\prod_{l \neq j}\left(\rho_{j}^{2}-\rho_{l}^{2}\right)} .
$$

Thus we compute

$$
\frac{\partial \mathscr{J}^{(3)}(x)}{\partial\left(\rho_{j}^{2} / 2\right)}=\frac{\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\rho_{j}\left(1-x \rho_{j}^{2}\right)}+\frac{2 x \rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\left(1-x \rho_{j}^{2}\right)^{2}}+\frac{\rho_{j} e^{-i \varphi_{j}}}{1-x \rho_{j}^{2}} \frac{\partial\left\|u_{j}^{H}\right\|^{2}}{\partial\left(\rho_{j}^{2} / 2\right)}+2 \sum_{l \neq j} \frac{\rho_{l}\left\|u_{l}^{H}\right\|^{2} e^{-i \varphi_{l}}}{\left(\rho_{l}^{2}-\rho_{j}^{2}\right)\left(1-x \rho_{l}^{2}\right)}
$$

We also have $\frac{\partial \mathcal{L}^{(3)}(y)}{\partial \varphi_{j}}=-i \frac{\rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{1-y \rho_{j}^{2}}$, hence, symmetrizing in $x$ and $y$, we get

$$
\begin{aligned}
& \frac{\partial \mathscr{J}^{(3)}(x)}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \mathscr{J}^{(3)}(y)}{\partial \varphi_{j}}-\frac{\partial \mathscr{J}^{(3)}(y)}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \mathscr{J}^{(3)}(x)}{\partial \varphi_{j}}= \\
&-2 i \rho_{j}^{2}\left\|u_{j}^{H}\right\|^{4} e^{-2 i \varphi_{j}}\left[\frac{x}{\left(1-x \rho_{j}^{2}\right)^{2}\left(1-y \rho_{j}^{2}\right)}-\frac{y}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)^{2}}\right] \\
&-2 i \sum_{l \neq j} \frac{\rho_{j} \rho_{l}\left\|u_{j}^{H}\right\|^{2}\left\|u_{l}^{H}\right\|^{2} e^{-i\left(\varphi_{j}+\varphi_{l}\right)}}{\rho_{l}^{2}-\rho_{j}^{2}}\left[\frac{1}{\left(1-x \rho_{l}^{2}\right)\left(1-y \rho_{j}^{2}\right)}-\frac{1}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{l}^{2}\right)}\right] .
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{x}{\left(1-x \rho_{j}^{2}\right)^{2}\left(1-y \rho_{j}^{2}\right)}-\frac{y}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)^{2}} & =\frac{x-y}{\left(1-x \rho_{j}^{2}\right)^{2}\left(1-y \rho_{j}^{2}\right)^{2}}, \\
\frac{1}{\left(1-x \rho_{l}^{2}\right)\left(1-y \rho_{j}^{2}\right)}-\frac{1}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{l}^{2}\right)} & =\frac{\left(\rho_{l}^{2}-\rho_{j}^{2}\right)(x-y)}{\left(1-x \rho_{j}^{2}\right)\left(1-x \rho_{l}^{2}\right)\left(1-y \rho_{j}^{2}\right)\left(1-y \rho_{l}^{2}\right)},
\end{aligned}
$$

which yields

$$
\begin{aligned}
& \frac{\partial \mathscr{J}^{(3)}(x)}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \mathscr{J}^{(3)}(y)}{\partial \varphi_{j}}-\frac{\partial \mathscr{J}^{(3)}(y)}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \mathscr{J}^{(3)}(x)}{\partial \varphi_{j}} \\
&=-2 i(x-y) \sum_{l=1}^{N} \frac{\rho_{j} \rho_{l}\left\|u_{j}^{H}\right\|^{2}\left\|u_{l}^{H}\right\|^{2} e^{-i\left(\varphi_{j}+\varphi_{l}\right)}}{\left(1-x \rho_{j}^{2}\right)\left(1-x \rho_{l}^{2}\right)\left(1-y \rho_{j}^{2}\right)\left(1-y \rho_{l}^{2}\right)} .
\end{aligned}
$$

Summing over $j$ then gives

$$
\begin{aligned}
\left\{\mathscr{J}^{(3)}(x), \mathscr{J}^{(3)}(y)\right\} & =-2 i(x-y)\left(\sum_{j=1}^{N} \frac{\rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)}\right)^{2} \\
& =-2 i(x-y)\left(\left(I-x H_{u}^{2}\right)^{-1}\left(I-y H_{u}^{2}\right)^{-1}(u) \mid H_{u}(u)\right)^{2} \\
& =-\frac{2 i}{x-y}\left(\left(I-x H_{u}^{2}\right)^{-1}\left[\left(I-y H_{u}^{2}\right)-\left(I-x H_{u}^{2}\right)\right]\left(I-y H_{u}^{2}\right)^{-1}(u) \mid 1\right)^{2} \\
& =-\frac{2 i}{x-y}\left(\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y)\right)^{2}=-\frac{2 i}{x-y}\left(x \mathscr{J}^{(3)}(x)-y \mathscr{J}^{(3)}(y)\right)^{2} .
\end{aligned}
$$

This is the first part of Lemma 5.5.
We turn to the second part. We will first compute $\left\{\mathscr{J}^{(1)}(x), \mathscr{J}^{(1)}(y)\right\}$, then we will deduce $\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}$ from Lemma 5.4 and finally get the expression of $\left\{\mathscr{J}^{(3)}(x), \overline{\mathscr{J}^{(3)}(y)}\right\}$. The
same computation as above also provides a formula :

$$
\begin{aligned}
\left\{\mathscr{J}^{(1)}(x), \mathscr{J}^{(1)}(y)\right\} & =-2 i(x-y)\left(\left(I-x H_{u}^{2}\right)^{-1}\left(I-y H_{u}^{2}\right)^{-1}(u) \mid 1\right)^{2} \\
& =-2 i(x-y)\left(\left(I-x H_{u}^{2}\right)^{-1}\left[\left(I-x H_{u}^{2}\right)+x H_{u}^{2}\right]\left(I-y H_{u}^{2}\right)^{-1}(u) \mid 1\right)^{2} \\
& =-2 i(x-y)\left(\mathscr{J}^{(1)}(y)+\frac{x}{x-y}\left(\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y)\right)\right)^{2} \\
& =-\frac{2 i}{x-y}\left(x \mathscr{J}^{(1)}(x)-y \mathscr{J}^{(1)}(y)\right)^{2} .
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left\{\left|\mathscr{J}^{(1)}(x)\right|^{2},\left|\mathscr{J}^{(1)}(y)\right|^{2}\right\}=2 \operatorname{Re}\left(\overline{\mathscr{J}^{(1)}(x) \mathscr{J}^{(1)}(y)}\{ \right. & \left.\mathscr{J}^{(1)}(x), \mathscr{J}^{(1)}(y)\right\} \\
& \left.+\overline{\mathscr{J}^{(1)}(x)} \mathscr{J}^{(1)}(y)\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}\right),
\end{aligned}
$$

and
$2 \operatorname{Re}\left(\overline{\mathscr{J}^{(1)}(x) \mathscr{J}^{(1)}(y)}\left\{\mathscr{J}^{(1)}(x), \mathscr{J}^{(1)}(y)\right\}\right)=\frac{4 \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathscr{J}^{(1)}(y)}\right)}{x-y}\left(x^{2}\left|\mathscr{J}^{(1)}(x)\right|^{2}-y^{2}\left|\mathscr{J}^{(1)}(y)\right|^{2}\right)$, so by Lemma 5.4,
$2 \operatorname{Re}\left(\overline{\mathscr{J}^{(1)}(x)} \mathscr{J}^{(1)}(y)\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}\right)=\frac{4 \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathscr{J}^{(1)}(y)}\right)}{x-y}\left[x \mathscr{J}^{(0)}(x)^{2}-y \mathscr{J}^{(0)}(y)^{2}\right]$.
Denote by $f(x, y):=\left\{\mathscr{J}^{(1)}(x), \overline{\mathcal{J}^{(1)}(y)}\right\}$. As above, we compute, for $j \in \llbracket 1, N \rrbracket$,

$$
\begin{aligned}
\frac{\partial \mathscr{J}^{(1)}(x)}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \overline{\mathscr{J}^{(1)}(y)}}{\partial \varphi_{j}}- & \frac{\partial \overline{\mathscr{J}^{(1)}(y)}}{\partial\left(\rho_{j}^{2} / 2\right)} \frac{\partial \mathscr{J}^{(1)}(x)}{\partial \varphi_{j}}= \\
\frac{-2 i\left\|u_{j}^{H}\right\|^{4}}{\rho_{j}^{4}\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)}+ & \frac{2 i x\left\|u_{j}^{H}\right\|^{4}}{\rho_{j}^{2}\left(1-x \rho_{j}^{2}\right)^{2}\left(1-y \rho_{j}^{2}\right)}+\frac{2 i y\left\|u_{j}^{H}\right\|^{4}}{\rho_{j}^{2}\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)^{2}}+\frac{4 i\left\|u_{j}^{H}\right\|^{2} \frac{\partial\left\|u_{j}^{H}\right\|^{2}}{\partial \rho_{j}^{2}}}{\rho_{j}^{2}\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{j}^{2}\right)} \\
& +2 i \sum_{l \neq j} \frac{\left\|u_{j}^{H}\right\|^{2}\left\|u_{l}^{H}\right\|^{2}}{\rho_{j} \rho_{l}\left(\rho_{l}^{2}-\rho_{j}^{2}\right)}\left(\frac{e^{i\left(\varphi_{j}-\varphi_{l}\right)}}{\left(1-x \rho_{l}^{2}\right)\left(1-y \rho_{j}^{2}\right)}+\frac{e^{-i\left(\varphi_{j}-\varphi_{l}\right)}}{\left(1-x \rho_{j}^{2}\right)\left(1-y \rho_{l}^{2}\right)}\right) .
\end{aligned}
$$

The crucial fact is the following : when we sum over $j$, the term of the last line (involving $\sum_{l \neq j}$ ) vanishes. All the remaining terms are purely imaginary, and we proved that $f(x, y) \in i \mathbb{R}$. We write $f(x, y)=i g(x, y)$. Therefore,

$$
2 \operatorname{Re}\left(\overline{\mathscr{J}^{(1)}(x)} \mathscr{J}^{(1)}(y)\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}\right)=2 \operatorname{Im}\left(\mathscr{J}^{(1)}(x) \overline{\mathcal{J}^{(1)}(y)}\right) \cdot g(x, y) .
$$

and by (41),

$$
\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}=\frac{2 i}{x-y}\left[x \mathscr{J}^{(0)}(x)^{2}-y \mathscr{J}^{(0)}(y)^{2}\right] .
$$

To conclude, observe that $\mathscr{J}^{(1)}(x)=J_{1}+x \mathscr{J}^{(3)}(x)$, with $J_{1}=\mathscr{J}^{(1)}(0)$. Hence

$$
\begin{aligned}
x y\left\{\mathscr{J}^{(3)}(x), \overline{\mathscr{J}^{(3)}(y)}\right\} & =\left\{\mathscr{J}^{(1)}(x), \overline{\mathscr{J}^{(1)}(y)}\right\}-\left\{\mathscr{J}^{(1)}(x), \overline{J_{1}}\right\}-\left\{J_{1}, \overline{\mathscr{J}^{(1)}(y)}\right\}+\left\{J_{1}, \overline{J_{1}}\right\} \\
& =\frac{2 i}{x-y}\left[x \mathscr{J}^{(0)}(x)^{2}-y \mathscr{J}^{(0)}(y)^{2}\right]-2 i \mathscr{J}^{(0)}(x)^{2}-2 i \mathscr{J}^{(0)}(y)^{2}+2 i \\
& =\frac{2 i}{x-y}\left[y \mathscr{J}^{(0)}(x)^{2}-x \mathscr{J}^{(0)}(y)^{2}+x-y\right] .
\end{aligned}
$$

Dividing by $x y$ gives the claim and completes the proof.
We are now ready to prove Theorem 6.
Proof of Theorem 6. Begin by noticing that, since $J_{2}, \mathscr{J}^{(0)}, \mathscr{J}^{(2)}$ only depend on the actions $\rho_{j}^{2} / 2$ and $\sigma_{k}^{2} / 2$, all the brackets which don't involve $\mathscr{J}^{(3)}$ are zero.

We thus only need to compute

- $\left\{J_{2},\left|\mathscr{J}^{(3)}(x)\right|^{2}\right\} \equiv 0$, since the functional $\left|\mathscr{J}^{(3)}(x)\right|^{2}$ is invariant under phase rotation of functions.
- Because of the product formula for $\mathscr{J}^{(0)}(x)$ and (40), we have

$$
\begin{aligned}
\left\{\frac{1}{\mathscr{J}^{(0)}(x)},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} & =\sum_{j=1}^{N} \frac{-4 x}{\mathscr{J}^{(0)}(x)\left(1-x \rho_{j}^{2}\right)} \operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)} \frac{\rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{1-y \rho_{j}^{2}}\right) \\
& =\sum_{j=1}^{N} \frac{-4 x}{\mathscr{J}^{(0)}(x)} \operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)} \frac{\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{\rho_{j}(x-y)}\left[\frac{1}{1-x \rho_{j}^{2}}-\frac{1}{1-y \rho_{j}^{2}}\right]\right) \\
& =\frac{-4 x}{(x-y) \mathscr{J}^{(0)}(x)} \operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)}\left[\mathscr{J}^{(1)}(x)-\mathscr{J}^{(1)}(y)\right]\right) \\
& =\frac{-4 x^{2}}{(x-y) \mathscr{J}^{(0)}(x)} \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right) .
\end{aligned}
$$

- A similar trick gives

$$
\begin{aligned}
\left\{\mathscr{J}^{(2)}(x)^{2},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} & =2 \mathscr{J}^{(2)}(x)\left\{\frac{\mathscr{J}^{(0)}(x)}{x},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} \\
& =\frac{2 \mathscr{J}^{(2)}(x)}{x} \sum_{j=1}^{N} \frac{4 x \mathscr{J}^{(0)}(x)}{1-x \rho_{j}^{2}} \operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(y)} \frac{\rho_{j}\left\|u_{j}^{H}\right\|^{2} e^{-i \varphi_{j}}}{1-y \rho_{j}^{2}}\right) \\
& =\frac{8 x \mathscr{J}^{(2)}(x) \mathscr{J}^{(0)}(x)}{x-y} \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right) .
\end{aligned}
$$

- Finally, Lemma 5.5 enables to calculate

$$
\begin{aligned}
2 \operatorname{Re}\left(\overline{\mathscr{J}^{(3)}(x)}\right. & \left.\overline{\mathscr{J}^{(3)}(y)}\left\{\mathscr{J}^{(3)}(x), \mathscr{J}^{(3)}(y)\right\}\right) \\
& =\frac{4}{x-y} \operatorname{Im}\left(\overline{\mathscr{J}^{(3)}(x) \mathscr{J}^{(3)}(y)}\left(x \mathscr{J}^{(3)}(x)-y \mathscr{J}^{(3)}(y)\right)^{2}\right) \\
& =\frac{4}{x-y}\left[x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}-y^{2}\left|\mathscr{J}^{(3)}(y)\right|^{2}\right] \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
2 \operatorname{Re}\left(\overline{\mathscr{J}^{(3)}(x)}\right. & \left.\mathscr{J}^{(3)}(y)\left\{\mathscr{J}^{(3)}(x), \overline{\mathcal{J}^{(3)}(y)}\right\}\right) \\
& =\frac{4}{x-y}\left[\frac{\mathscr{J}^{(0)}(x)^{2}}{x}-\frac{\mathscr{J}^{(0)}(y)^{2}}{y}+\frac{1}{x}-\frac{1}{y}\right] \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right),
\end{aligned}
$$

so that

$$
\begin{aligned}
& \quad\left\{\left|\mathscr{J}^{(3)}(x)\right|^{2},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\}= \\
& \frac{4}{x-y}\left[x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}-y^{2}\left|\mathscr{J}^{(3)}(y)\right|^{2}+\frac{\mathscr{\mathscr { J }}^{(0)}(x)^{2}}{x}-\frac{\mathscr{J}^{(0)}(y)^{2}}{y}+\frac{1}{x}-\frac{1}{y}\right] \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right) .
\end{aligned}
$$

At last, we can compute the main Poisson bracket, expanding it as a double product :

$$
\begin{aligned}
\{\mathscr{F}(x), \mathscr{F}(y)\}= & -\frac{y^{2}\left(2 J_{2}+x \mathscr{J}^{(2)}(x)^{2}-x^{2}\left|\mathscr{J}^{(3)}(x)\right|^{2}\right)}{\mathscr{J}^{(0)}(y)}\left\{\frac{1}{\mathscr{J}^{(0)}(x)},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} \\
& -\frac{x y^{2}}{\mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)}\left\{\mathscr{J}^{(2)}(x)^{2},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} \\
& -\frac{x^{2}\left(2 J_{2}+y \mathscr{J}^{(2)}(y)^{2}-y^{2}\left|\mathscr{J}^{(3)}(y)\right|^{2}\right)}{\mathscr{J}^{(0)}(x)}\left\{\left|\mathscr{J}^{(3)}(x)\right|^{2}, \frac{1}{\mathscr{J}^{(0)}(y)}\right\} \\
& -\frac{x^{2} y}{\mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)}\left\{\left|\mathscr{J}^{(3)}(x)\right|^{2}, \mathscr{J}^{(2)}(y)^{2}\right\} \\
& +\frac{x^{2} y^{2}}{\mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)}\left\{\left|\mathscr{J}^{(3)}(x)\right|^{2},\left|\mathscr{J}^{(3)}(y)\right|^{2}\right\} .
\end{aligned}
$$

Summing up, and taking obvious cancellations into account, we have

$$
\begin{aligned}
(x-y) \mathscr{J}^{(0)}(x) \mathscr{J}^{(0)}(y)\{\mathscr{F}(x), \mathscr{F}(y)\} & = \\
4 x y \operatorname{Im}\left(\mathscr{J}^{(3)}(x) \overline{\mathscr{J}^{(3)}(y)}\right) & {\left[x^{2} y \mathscr{J}^{(2)}(x)^{2}-2 x y \mathscr{J}^{(2)}(x) \mathscr{J}^{(0)}(x)-x y^{2} \mathscr{J}^{(2)}(y)^{2}\right.} \\
& \left.+2 x y \mathscr{J}^{(2)}(y)^{2} \mathscr{J}^{(0)}(y)+y \mathscr{J}^{(0)}(x)^{2}-x \mathscr{J}^{(0)}(y)^{2}+x-y\right] .
\end{aligned}
$$

Now remember that $x \mathscr{J}^{(2)}(x)=\mathscr{J}^{(0)}(x)-1$. So

$$
\begin{aligned}
& x^{2} y \mathscr{J}^{(2)}(x)^{2}-2 x y \mathscr{J}^{(2)}(x) \mathscr{J}^{(0)}(x)+y \mathscr{J}^{(0)}(x)^{2}-y \\
& \quad=y\left[\left(\mathscr{J}^{(0)}(x)-1\right)^{2}-2 \mathscr{J}^{(0)}(x)\left(\mathscr{J}^{(0)}(x)-1\right)+\mathscr{J}^{(0)}(x)^{2}-1\right] \\
& \quad=0,
\end{aligned}
$$

and the same holds interverting $x$ and $y$. This concludes the proof of Theorem 6.

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