

# Linear Strain Tensors on Hyperbolic Surfaces and Asymptotic Theories for Thin Shells

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**Abstract** We perform a detailed analysis of the solvability of linear strain equations on hyperbolic surfaces under a technical assumption (noncharacteristic). For regular enough hyperbolic surfaces, it is proved that smooth infinitesimal isometries are dense in the  $W^{2,2}$  infinitesimal isometries and that smooth enough infinitesimal isometries can be matched with higher order infinitesimal isometries. Then those results are applied to elasticity of thin shells for the  $\Gamma$ -limits. The recovery sequences ( $\Gamma$ -lim sup inequality) are obtained for dimensionally-reduced shell theories, when the elastic energy density scales like  $h^\beta$ ,  $\beta \in (2, 4)$ , that is, intermediate regime between pure bending ( $\beta = 2$ ) and the von-Karman regime ( $\beta = 4$ ), where  $h$  is thickness of a shell.

**Keywords** hyperbolic surface, shell, nonlinear elasticity, Riemannian geometry

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## 1 Introduction and main results

Let  $M \subset \mathbb{R}^3$  be a surface with a normal  $\vec{n}$  and let the middle surface of a shell be an open set  $\Omega \subset M$ . Let  $T^k(M)$  denote all the  $k$ -order tensor fields on  $M$  for an integer  $k \geq 0$ . Let  $T_{\text{sym}}^2(M)$  be all the 2-order symmetrical tensor fields on  $M$ . For  $y \in H^1(\Omega, \mathbb{R}^3)$ , we decompose it into  $y = W + w\vec{n}$ , where  $w = \langle y, \vec{n} \rangle$  and  $W \in T(\Omega)$ . For  $U \in T_{\text{sym}}^2(\Omega)$  given,

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linear strain tensor of a displacement  $y \in W^{1,2}(\Omega, \mathbb{R}^3)$  of the middle surface  $\Omega$  takes the form

$$\text{sym } DW + w\Pi = U \quad \text{for } x \in \Omega, \quad (1.1)$$

where  $D$  is the connection of the induced metric in  $M$ ,  $2\text{sym } DW = DW + D^T W$ , and  $\Pi$  is the second fundamental form of  $M$ . Equation (1.1) plays a fundamental role in the theory of thin shells, see [5, 6, 7, 8]. When  $U = 0$ , a solution  $y$  to (1.1) is referred to as an *infinitesimal isometry*. Under a technical assumption (noncharacteristic) on hyperbolic surfaces, we establish existence, uniqueness, and regularity of solutions for (1.1). For regular enough hyperbolic surface that satisfies a noncharacteristic assumption, it is proved that smooth infinitesimal isometries are dense in the  $W^{2,2}(\Omega, \mathbb{R}^3)$  infinitesimal isometries (Theorem 1.2) and that smooth enough infinitesimal isometries can be matched with higher order infinitesimal isometries (Theorem 1.3). This matching property is an important tool in obtaining recovery sequences ( $\Gamma$ -lim sup inequality) for dimensionally-reduced shell theories in elasticity, when the elastic energy density scales like  $h^\beta$ ,  $\beta \in (2, 4)$ , that is, intermediate regime between pure bending ( $\beta = 2$ ) and the von-Karman regime ( $\beta = 4$ ). Such results have been obtained for elliptic surfaces [8] and developable surfaces [5]. A survey on this topic is presented in [6]. Here we shall establish the similar results for hyperbolic surfaces in Theorems 1.6-1.7.

We state our main results for the hyperbolic surfaces as follows.

A region  $\Omega \subset M$  is said to be hyperbolic if its Gaussian curvature  $\kappa$  is strictly negative. We assume throughout this paper that

$$\kappa(x) < 0 \quad \text{for } x \in \overline{\Omega}.$$

We introduce the notion of a *noncharacteristic region* below, subject to the second fundamental form  $\Pi$  of the surface  $M$ .

**Definition 1.1** *A region  $\Omega \subset M$  is said to be noncharacteristic if*

$$\Omega = \{ \alpha(t, s) \mid (t, s) \in (0, a) \times (0, b) \},$$

where  $\alpha : [0, a] \times [0, b] \rightarrow M$  is an imbedding map which is a family of regular curves with two parameters  $t, s$  such that

$$\Pi(\alpha_t(t, s), \alpha_t(t, s)) \neq 0, \quad \text{for all } (t, s) \in [0, a] \times [0, b],$$

$$\Pi(\alpha_s(0, s), \alpha_s(0, s)) \neq 0, \quad \Pi(\alpha_s(a, s), \alpha_s(a, s)) \neq 0, \quad \text{for all } s \in [0, b],$$

$$\Pi(\alpha_t(0, s), \alpha_s(0, s)) = \Pi(\alpha_t(a, s), \alpha_s(a, s)) = 0, \quad \text{for all } s \in [0, b].$$

Consider a surface given by the graph of a function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$M = \{ (x, h(x)) \mid x = (x_1, x_2) \in \mathbb{R}^2 \}.$$

Under the coordinate system  $\psi(p) = x$  for  $p = (x, h(x)) \in M$ ,

$$\partial x_1 = (1, 0, h_{x_1}(x)), \quad \partial x_2 = (0, 1, h_{x_2}(x)), \quad \vec{n} = \frac{1}{\sqrt{1 + |\nabla h|^2}} (-\nabla h, 1),$$

$$\Pi = -\frac{1}{\sqrt{1 + |\nabla h|^2}} \nabla^2 h, \quad \kappa = \frac{h_{x_1 x_1} h_{x_2 x_2} - h_{x_1 x_2}^2}{(1 + |\nabla h|^2)^2}.$$

(i) Let  $h(x) = h_1(x_1) + h_2(x_2)$  where  $h_i : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^2$  functions with  $h_1''(x_1)h_2''(x_2) < 0$ . Let  $\sigma_i \in \mathbb{R}$  for  $1 \leq i \leq 4$  with  $\sigma_1 < \sigma_2$  and  $\sigma_3 < \sigma_4$ . Then

$$\Omega = \{ (x, h(x)) \mid \sigma_1 < x_1 < \sigma_2, \sigma_3 < x_2 < \sigma_4 \}$$

is noncharacteristic.

(ii) Let  $h(x) = x_1^3 - 3x_1x_2^2$ . Then

$$\kappa(p) < 0 \quad \text{for } p = (x, h(x)), \quad x \in \mathbb{R}^2, \quad |x| > 0.$$

For  $\varepsilon > 0$  and  $\sigma_1 < \sigma_2$  given

$$\Omega = \{ (x, h(x)) \mid \varepsilon < x_1 < \frac{1}{\varepsilon}, \frac{\sigma_1}{x_1} < x_2 < \frac{\sigma_2}{x_1} \}$$

is noncharacteristic. However, there exists a region on  $M$  for the  $h$ , that is not a noncharacteristic. For example, a region

$$\Omega = \{ (x, h(x)) \mid a < |x| < b \}$$

is not noncharacteristic, where  $0 < a < b$  and  $|x| = x_1^2 + x_2^2$ , since its boundaries  $|x| = a$  and  $|x| = b$  are not noncharacteristic curves.

Moreover, if  $\Omega$  is given by a single principal coordinate, i.e.,

$$\Omega = \{ \alpha(t, s) \mid (t, s) \in (a, b) \times (c, d) \},$$

where

$$\nabla_{\partial t} \vec{n} = \lambda_1 \partial t, \quad \nabla_{\partial s} \vec{n} = \lambda_2 \partial s, \quad \lambda_1 > 0, \quad \lambda_2 < 0,$$

and  $\vec{n}$  is the normal of  $M$ , then  $\Omega$  is a noncharacteristic region clearly. Since a principal coordinate exists locally [14], a noncharacteristic region also exists locally.

The notion of the noncharacteristic region is a technical assumption and a different region is given in [13]. In general, for  $U \in T_{\text{sym}}^2(\Omega)$  given, there are many solutions to (1.1). The aim of this assumption is to help us choose a regular solution for each  $U$ . In

fact, equation (1.1) can be translated into a scalar second order partial differential equation (see Theorem 2.1 later), which is elliptic for the elliptic surface [8], parabolic for the developable surface with no flat part [5], and hyperbolic for the hyperbolic surface, respectively. Here we assume  $\Omega$  to be a noncharacteristic region in order to set up appropriate boundary conditions such that the scalar equation is to be well-posedness (see Theorem 4.1 and (4.1)-(4.5)). The main observation is that if the values of a solution  $v$  to the hyperbolic equation (2.25) and its derivatives along a noncharacteristic curve are preset, then the solution  $v$  is uniquely determined in some neighborhood of this curve. We first solve (2.25) locally and then paste up the local solutions to yield a global one (see Lemma 4.4), where the noncharacteristic assumption is such that this produce is successful. We believe the corresponding results hold true for a more general region but some more complicated discusses may be involved.

We say that a noncharacteristic region  $\Omega \subset M$  is of class  $C^{m,1}$  for some integer  $m \geq 0$  if the surface  $M$  is of class  $C^{m,1}$  and all the curves  $\alpha(0, \cdot)$ ,  $\alpha(a, \cdot)$ , and  $\alpha(\cdot, s)$  for each  $s \in [0, b]$  are of class  $C^{m,1}$ . The points  $\alpha(0, 0)$ ,  $\alpha(a, 0)$ ,  $\alpha(0, b)$ , and  $\alpha(a, b)$  are angular points of  $\Omega$  even if  $\Omega$  is smooth.

**Theorem 1.1** *Let  $\Omega$  be a noncharacteristic region of class  $C^{2,1}$ . For  $U \in C^{1,1}(\Omega, T_{\text{sym}}^2)$ , there exists a solution  $y = W + w\vec{n} \in C^{0,1}(\Omega, \mathbb{R}^3)$  to equation (1.1) satisfying the bounds*

$$\|W\|_{C^{1,1}(\Omega, T)} + \|w\|_{C^{0,1}(\Omega)} \leq C\|U\|_{C^{1,1}(\Omega, T_{\text{sym}}^2)}. \quad (1.2)$$

*If, in addition,  $\Omega \in C^{m+2,1}$ ,  $U \in C^{m+1,1}(\Omega, T_{\text{sym}}^2)$  for some  $m \geq 1$ , then*

$$\|W\|_{C^{m+1,1}(\Omega, T)} + \|w\|_{C^{m,1}(\Omega)} \leq C\|U\|_{C^{m+1,1}(\Omega, T_{\text{sym}}^2)}. \quad (1.3)$$

**Remark 1.1** *For the solvability of (1.1), the noncharacteristic assumption of  $\Omega$  can be relaxed. Let  $\Omega$  be not a noncharacteristic region but there be a noncharacteristic one  $\hat{\Omega}$  such that  $\Omega \subset \hat{\Omega}$ . Then Theorem 1.1 still holds true. In fact, we can extend  $U$  such that  $\hat{U} \in C^{m+1,1}(\hat{\Omega}, T_{\text{sym}}^2)$  with the estimate*

$$\|\hat{U}\|_{C^{m+1,1}(\hat{\Omega}, T_{\text{sym}}^2)} \leq C\|U\|_{C^{m+1,1}(\Omega, T_{\text{sym}}^2)}.$$

*Then we solve (1.1) on  $\hat{\Omega}$  to obtain a solution  $y$  for which (1.3) still holds.*

For  $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ , we denote the left hand side of equation (1.1) by  $\text{sym} \nabla y$ . Let

$$\mathcal{V}(\Omega, \mathbb{R}^3) = \{V \in W^{2,2}(\Omega, \mathbb{R}^3) \mid \text{sym} \nabla V = 0\}.$$

**Theorem 1.2** *Let  $\Omega$  be a noncharacteristic region of class  $C^{m+2,1}$  for some integer  $m \geq 0$ . Then, for every  $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$  there exists a sequence  $\{V_k\} \subset \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{m,1}(\Omega, \mathbb{R}^3)$  such that*

$$\lim_{k \rightarrow \infty} \|V - V_k\|_{W^{2,2}(\Omega, \mathbb{R}^3)} = 0. \quad (1.4)$$

A one parameter family  $\{u_\varepsilon\}_{\varepsilon>0} \subset C^{0,1}(\overline{\Omega}, \mathbb{R}^2)$  is said to be a (generalized)  $m$ th order infinitesimal isometry if the change of metric induced by  $u_\varepsilon$  is of order  $\varepsilon^{m+1}$ , that

$$\|\nabla^T u_\varepsilon \nabla u_\varepsilon - g\|_{L^\infty(\Omega, T^2)} = \mathcal{O}(\varepsilon^{m+1}) \quad \text{as } \varepsilon \rightarrow 0,$$

where  $g$  is the induced metric of  $M$  from  $\mathbb{R}^3$ , see [5]. A given  $m$ th order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order  $m_1 > m$ , a property to which we refer to by *matching property of infinitesimal isometries*, [5, 8]. This property plays an important role in the construction of a recover sequence in the  $\Gamma$ -limit for thin shells.

**Theorem 1.3** *Let  $\Omega$  be a noncharacteristic region of class  $C^{2m+1,1}$ . Given  $V \in \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{2m-1,1}(\Omega, \mathbb{R}^3)$ , there exists a family  $\{w_\varepsilon\}_{\varepsilon>0} \subset C^{1,1}(\Omega, \mathbb{R}^3)$ , equibounded in  $C^{1,1}(\Omega, \mathbb{R}^3)$ , such that for all small  $\varepsilon > 0$  the family:*

$$u_\varepsilon = \text{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$$

*is a  $m$ th order infinitesimal isometry of class  $C^{1,1}$ .*

Let  $A : \Omega \rightarrow \mathbb{R}^{3 \times 3}$  be a matrix field. We define  $A \in T^2(\Omega)$  by

$$A(\alpha, \beta) = \langle A(x)\alpha, \beta \rangle \quad \text{for } \alpha, \beta \in T_x \Omega, \quad x \in \Omega.$$

For  $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$  given, there exists a unique  $A \in W^{1,2}(\Omega, T^2)$  such that

$$\nabla_\alpha V = A(x)\alpha \quad \text{for } \alpha \in T_x M, \quad A(x) = -A^T(x), \quad x \in \Omega. \quad (1.5)$$

The finite strain space is the following closed subspace of  $L^2(\Omega, T_{\text{sym}}^2)$

$$\mathcal{B}(\Omega, T_{\text{sym}}^2) = \left\{ \lim_{h \rightarrow 0} \text{sym} \nabla w_h \mid w_h \in W^{1,2}(\Omega, \mathbb{R}^3) \right\}$$

where limits are taken in  $L^2(\Omega, T_{\text{sym}}^2)$ , see [3, 7, 9].  $\mathcal{B}(\Omega, T_{\text{sym}}^2)$  and  $\mathcal{V}(\Omega, \mathbb{R}^3)$  are two basic spaces for the  $\Gamma$ -limit functional. A region  $\Omega \subset M$  is said to be *approximately robust* if

$$(A^2)_{\text{tan}} \in \mathcal{B}(\Omega, T_{\text{sym}}^2) \quad \text{for } V \in \mathcal{V}(\Omega, \mathbb{R}^3),$$

where

$$(A^2)_{\text{tan}}(\alpha, \beta) = \langle A^2 \alpha, \beta \rangle \quad \text{for } \alpha, \beta \in T_x \Omega, \quad x \in \Omega.$$

If  $\Omega$  is approximately robust, then the  $\Gamma$ -limit functional can be simplified to the bending energy. An approximately robust surface exhibits a better capacity to resist stretching so that the limit functional consists only of a bending term, see [7].

**Theorem 1.4** *Let  $\Omega \subset M$  be a noncharacteristic region of class  $C^{2,1}$ . Then  $\Omega$  is approximately robust.*

**Application to elasticity of thin shells** Let  $\vec{n}$  be the normal field of surface  $M$ . Consider a family  $\{\Omega_h\}_{h>0}$  of thin shells of thickness  $h$  around  $\Omega$ ,

$$\Omega_h = \{x + t\vec{n}(x) \mid x \in \Omega, |t| < h/2\}, \quad 0 < h < h_0,$$

where  $h_0$  is small enough so that the projection map  $\pi : \Omega_h \rightarrow \Omega$ ,  $\pi(x + t\vec{n}) = x$  is well defined. For a  $W^{1,2}$  deformation  $u_h : \Omega_h \rightarrow \mathbb{R}^3$ , we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

$$E_h(u_h) = \frac{1}{h} \int_{\Omega_h} W(\nabla u_h) dz.$$

The stored-energy density function  $W : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is  $C^2$  in an open neighborhood of  $SO(3)$ , and it is assumed to satisfy the conditions of normalization, frame indifference and quadratic growth: For all  $F \in \mathbb{R}^3 \times \mathbb{R}^3$ ,  $R \in SO(3)$ ,

$$W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq C \operatorname{dist}^2(F, SO(3)),$$

with a uniform constant  $C > 0$ . The potential  $W$  induces the quadratic forms ([1])

$$Q_3(F) = D^2W(Id)(F, F), \quad Q_2(x, F_{\tan}) = \min\{Q_3(\hat{F}) \mid \hat{F} = F_{\tan}\}.$$

We shall consider a sequence  $e_h > 0$  such that:

$$0 < \lim_{h \rightarrow 0} e_h/h^\beta < \infty \quad \text{for some } 2 < \beta \leq 4. \quad (1.6)$$

Let

$$\beta_m = 2 + 2/m.$$

Recall the following result.

**Theorem 1.5** [7] *Let  $\Omega$  be a surface embedded in  $\mathbb{R}^3$ , which is compact, connected, oriented, of class  $C^{1,1}$ , and whose boundary  $\partial\Omega$  is the union of finitely many Lipschitz curves. Let  $u_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$  be a sequence of deformations whose scaled energies  $E_h(u_h)/e_h$  are uniformly bounded. Then there exist a sequence  $Q_h \in SO(3)$  and  $c_h \in \mathbb{R}^3$  such that for the normalized rescaled deformations*

$$y_h(z) = Q_h u_h(x + \frac{h}{h_0} t \vec{n}(x)) - c_h, \quad z = x + t \vec{n}(x) \in \Omega_{h_0},$$

the following holds.

- (i)  $y_h$  converge to  $\pi$  in  $W^{1,2}(\Omega_{h_0}, \mathbb{R}^3)$ .
- (ii) The scaled average displacements

$$V_h(x) = \frac{h}{h_0 \sqrt{e_h}} \int_{-h_0/2}^{h_0/2} [y_h(x + t \vec{n}) - x] dt$$

converge to some  $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$ .

(iii)  $\liminf_{h \rightarrow 0} E_h(u_h)/e_h \geq I(V)$ , where

$$I(V) = \frac{1}{24} \int_{\Omega} \mathcal{Q}_2 \left( x, (\nabla(A\vec{n}) - A\nabla\vec{n})_{\text{tan}} \right) dg, \quad (1.7)$$

where  $A$  is given in (1.5).

The above result proves the lower bound for the  $\Gamma$ -convergence. We now state the upper bound in the  $\Gamma$ -convergence result for a smooth noncharacteristic region.

Since  $\Omega$  is approximately robust (Theorem 1.4), Theorem 1.6 below follows from [7, Theorem 2.3] immediately.

**Theorem 1.6** *Let  $\Omega \subset M$  be a noncharacteristic region of class  $C^{2,1}$ . Assume that (1.6) holds for  $\beta = 4$ . Then for every  $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$  there exists a sequence of deformations  $\{u_h\} \subset W^{1,2}(\Omega, \mathbb{R}^3)$  such that (i) and (ii) of Theorem 1.5 hold. Moreover,*

$$\lim_{h \rightarrow 0} \frac{1}{e_h} E_h(u_h) = I(V), \quad (1.8)$$

where  $I(V)$  is given in (1.7).

**Theorem 1.7** *Let assumption (1.6) hold with  $2 < \beta < 4$ . Let  $\Omega \subset M$  be a noncharacteristic region of class  $C^{2m+1,1}$ , where  $m \geq 2$  is given such that*

$$e_h = o(h^{\beta m}).$$

*Then the results in Theorem 1.6 hold.*

The rest of the paper is organized as follows.

Section 2 reduces the linear strain equations (1.1) into one scalar second order equation (2.25) (Theorem 2.1).

Section 3 makes preparations to solve problem (2.25). The main observation is that under an asymptotic coordinate system, this equation locally takes a normal form (Proposition 4.1). Thus we study solvability regions for the normal equation, in where existence, uniqueness and estimates for solutions are presented.

Section 4 is devoted to solvability of the scalar equation (2.25). Using solvability of a normal equation in Section 3, we first solve the scalar equation (2.25) locally and then patch the local solutions together (Theorems 4.1-4.4), where the noncharacteristic assumption of the region  $\Omega$  is needed to guarantee this process to be successful.

Section 5 returns to the main theorems in Section 1, and provides proofs for them, using the main results in Section 4.

## 2 Linear Strain Equations

We reformulate some expressions from [4, Section 9.2] to reduce (1.1) to a coordinate free, scalar equation which can be solved by selecting special charts.

Let  $k \geq 1$  be an integer. Let  $T \in T^k(M)$  be a  $k$ th order tensor field and let  $X \in T(M)$  be a vector field. We define a  $k - 1$ th order tensor field by

$$i_X T(X_1, \dots, X_{k-1}) = T(X, X_1, \dots, X_{k-1}) \quad \text{for } X_1, \dots, X_{k-1} \in T(M),$$

which is called an *inner product* of  $T$  with  $X$ . For any  $T \in T^2(M)$  and  $\alpha \in T_x M$ ,

$$\text{tr}_g i_\alpha DT$$

is a linear functional on  $T_x M$ , where  $\text{tr}_g i_\alpha DT$  is the trace of the 2-order tensor field  $i_\alpha DT$  in the induced metric  $g$ . Thus there is a vector, denoted by  $\Lambda(T)$ , such that

$$\langle \Lambda(T), \alpha \rangle = \text{tr}_g i_\alpha DT \quad \text{for } \alpha \in T_x M, x \in M. \quad (2.1)$$

Clearly, the above formula defines a vector field  $\Lambda(T) \in T(M)$ .

We also need another linear operator  $Q$  as follows. Let  $M$  be oriented and  $\mathcal{E}$  be the volume element of  $M$  with the positive orientation. Let  $x \in M$  be given and let  $e_1, e_2$  be an orthonormal basis of  $T_x M$  with the positive orientation, that is,

$$\mathcal{E}(e_1, e_2) = 1 \quad \text{at } x.$$

We define  $Q : T_x M \rightarrow T_x M$  by

$$Q\alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all } \alpha \in T_x M. \quad (2.2)$$

$Q$  is well defined in the following sense: Let  $\hat{e}_1, \hat{e}_2$  be a different orthonormal basis of  $T_x M$  with the positive orientation,

$$\mathcal{E}(\hat{e}_1, \hat{e}_2) = 1.$$

Let

$$\hat{e}_i = \sum_{j=1}^2 \alpha_{ij} e_j \quad \text{for } i = 1, 2.$$

Then

$$1 = \mathcal{E}(\hat{e}_1, \hat{e}_2) = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}.$$

Using the above formula, a simple computation yields

$$\langle \alpha, \hat{e}_2 \rangle \hat{e}_1 - \langle \alpha, \hat{e}_1 \rangle \hat{e}_2 = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2.$$

Clearly,  $Q : T_x M \rightarrow T_x M$  is an isometry and

$$Q^T = -Q, \quad Q^2 = -\text{Id}.$$

**Remark 2.1**  $Q : T_x M \rightarrow T_x M$  is the rotation by  $\pi/2$  along the clockwise direction.

The operator, defined above, defines an operator, still denoted by  $Q : T(M) \rightarrow T(M)$ , by

$$(QX)(x) = QX(x), \quad x \in M, \quad X \in T(M).$$

For each  $k \geq 2$ , the operator  $Q$  further induces an operator, denoted by  $Q^* : T^k(M) \rightarrow T^k(M)$  by

$$(Q^*T)(X_1, \dots, X_k) = T(QX_1, \dots, QX_k), \quad X_1, \dots, X_k \in T(M), \quad T \in T^k(M). \quad (2.3)$$

Notice that orientability of  $M$  is necessary to operators  $Q$  or  $Q^*$ .

Let  $x \in \Omega$  be given and let  $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ . Set

$$p(y)(x) = \frac{1}{2}[\nabla y(e_2, e_1) - \nabla y(e_1, e_2)] \quad \text{for } x \in \Omega, \quad (2.4)$$

where  $\nabla y(\alpha, \beta) = \langle \nabla_\beta y, \alpha \rangle$  for  $\alpha, \beta \in T_x M$ ,  $\nabla$  is the differential in the Euclidean space  $\mathbb{R}^3$ , and  $e_1, e_2$  is an orthonormal basis of  $T_x M$  with the positive orientation. It is easy to check that the value of the right hand side of (2.4) is independent of choice of a positively orientated orthonormal basis. Thus

$$p : W^{1,2}(\Omega, \mathbb{R}^3) \rightarrow L^2(\Omega)$$

is a linear operator.

For  $U \in T_{\text{sym}}^2(M)$  given, consider problem

$$\text{sym } \nabla y(\alpha, \beta) = U(\alpha, \beta) \quad \text{for } \alpha, \beta \in T_x M, \quad x \in M, \quad (2.5)$$

where  $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ .

Let  $x \in \Omega$  be given. To simplify computation we use many times the following special frame field: Let  $E_1, E_2$  be a positively orientated frame field normal at  $x$  with following properties

$$\begin{aligned} \langle E_i, E_j \rangle &= \delta_{ij} \quad \text{in some neighbourhood of } x, \\ D_{E_i} E_j &= 0, \quad \nabla_{E_i} \vec{n} = \lambda_i E_i \quad \text{at } x \text{ for } 1 \leq i, j \leq 2, \end{aligned} \quad (2.6)$$

where  $\nabla$  is the connection of the Euclidean space  $\mathbb{R}^3$ ,  $D$  is the connection of  $M$  in the induced metric,  $\vec{n}$  is the normal field of  $M$ , and  $\lambda_1 \lambda_2 = \kappa$  is the Gaussian curvature. It follows from (2.6) that

$$\Pi(E_i, E_j) = \lambda_i \delta_{ij}, \quad \nabla_{E_i} E_j = -\lambda_i \delta_{ij} \vec{n} \quad \text{at } x \text{ for } 1 \leq i, j \leq 2, \quad (2.7)$$

where  $\Pi(\alpha, \beta) = \langle \nabla_\alpha \vec{n}, \beta \rangle$  is the second fundamental form of  $M$ . We need to deal with the relation between the connections  $\nabla$  and  $D$ , carefully.

Let  $y \in W^{1,2}(\Omega, \mathbb{R}^3)$  be a solution to problem (2.5). Then (2.5) reads

$$\begin{cases} \nabla y(E_1, E_1) = U(E_1, E_1), \\ \nabla y(E_2, E_1) + \nabla y(E_1, E_2) = 2U(E_1, E_2), \\ \nabla y(E_2, E_2) = U(E_2, E_2), \end{cases} \quad \text{in some neighbourhood of } x. \quad (2.8)$$

Let

$$v = p(y)$$

and define

$$u = \nabla y(\vec{n}, E_1)E_1 + \nabla y(\vec{n}, E_2)E_2. \quad (2.9)$$

We can check easily that  $u$  is a globally defined vector field on  $\Omega$ . Moreover,  $v$  satisfies

$$v + U(E_2, E_1) = \nabla y(E_2, E_1), \quad v - U(E_1, E_2) = -\nabla y(E_1, E_2) \quad (2.10)$$

in some neighbourhood of  $x$ . Therefore,  $\{v, u\}$  determines  $\nabla_\alpha y$  for  $\alpha \in T_x M$ , that is,

$$\begin{cases} \nabla_{E_1} y = U(E_1 E_1)E_1 + [v + U(E_1, E_2)]E_2 + \langle u, E_1 \rangle \vec{n}, \\ \nabla_{E_2} y = [-v + U(E_1, E_2)]E_1 + U(E_2, E_2)E_2 + \langle u, E_2 \rangle \vec{n}. \end{cases} \quad (2.11)$$

The relation (2.11) can be rewritten as in a form of coordinate free

$$\nabla_\alpha y = i_\alpha U - vQ\alpha + \langle u, \alpha \rangle \vec{n} \quad \text{for } \alpha \in T_x M, x \in \Omega.$$

The function  $v$  and the vector field  $u$  are the new dependent variables and we proceed to find the differential equations they satisfy.

Differentiating the first equation in (2.10) with respect to  $E_2$  and using the relations (2.6) and (2.7), we have

$$\begin{aligned} E_2(v) + DU(E_2, E_1, E_2) &= \nabla^2 y(E_2, E_1, E_2) + \nabla y(\nabla_{E_2} E_2, E_1) \\ &= E_1[\nabla y(E_2, E_2)] - \lambda_2 \nabla y(\vec{n}, E_1) \\ &= DU(E_2, E_2, E_1) - \lambda_2 \langle u, E_1 \rangle \quad \text{at } x, \end{aligned} \quad (2.12)$$

where the following formula has been used

$$\nabla^2 y(E_2, E_1, E_2) = \nabla^2 y(E_2, E_2, E_1) \quad \text{at } x.$$

Similarly, we obtain

$$E_1(v) - DU(E_1, E_2, E_1) = -DU(E_1, E_1, E_2) + \lambda_1 \langle u, E_2 \rangle \quad \text{at } x. \quad (2.13)$$

Combining (2.12), (2.13) and (2.6) yields

$$\begin{aligned}
Dv &= [DU(E_1, E_2, E_1) - DU(E_1, E_1, E_2)]E_1 + \lambda_1 \langle u, E_2 \rangle E_1 \\
&\quad + [DU(E_2, E_2, E_1) - DU(E_2, E_1, E_2)]E_2 - \lambda_2 \langle u, E_1 \rangle E_2 \\
&= Q\{[DU(E_2, E_1, E_1) + DU(E_2, E_2, E_2)]E_2 - [DU(E_2, E_2, E_2) + DU(E_1, E_1, E_2)]E_2 \\
&\quad + [DU(E_1, E_2, E_2) + DU(E_1, E_1, E_1)]E_1 - [DU(E_1, E_1, E_1) + DU(E_2, E_2, E_1)]E_1\} \\
&\quad + \nabla \vec{n} Qu \\
&= Q[\Lambda(U) - D(\text{tr}_g U)] + \nabla \vec{n} Qu \quad \text{for } x \in \Omega,
\end{aligned} \tag{2.14}$$

where the operator  $Q : T_x M \rightarrow T_x M$  is defined in (2.2),  $\Lambda(U) \in \mathcal{X}(\Omega)$  is given in (2.1), and  $\nabla \vec{n} : T_x M \rightarrow T_x M$  is the shape operator, defined by

$$\nabla \vec{n} \alpha = \nabla_\alpha \vec{n} \quad \text{for } \alpha \in T_x M, x \in M.$$

Now we proceed to derive the differential equations for which the function  $v$  satisfies. Since

$$\kappa = \Pi(E_1, E_1)\Pi(E_2, E_2) - \Pi^2(E_1, E_2) \quad \text{in a neighbourhood of } x,$$

from (2.6) and (2.7) we compute

$$\begin{aligned}
D\kappa &= [D\Pi(E_1, E_1, E_1)\lambda_2 + \lambda_1 D\Pi(E_2, E_2, E_1)]E_1 \\
&\quad + [D\Pi(E_1, E_1, E_2)\lambda_2 + \lambda_1 D\Pi(E_2, E_2, E_2)]E_2 \quad \text{at } x.
\end{aligned} \tag{2.15}$$

Using (2.14), (2.6) and (2.7), we have

$$\begin{aligned}
D(\nabla \vec{n} Qu)(E_1, E_1) &= E_1 \langle \nabla \vec{n} Qu, E_1 \rangle = E_1 \langle u, Q^T \nabla_{E_1} \vec{n} \rangle \\
&= Du(Q^T \nabla_{E_1} \vec{n}, E_1) + \langle u, D_{E_1}(Q^T \nabla_{E_1} \vec{n}) \rangle \\
&= \lambda_1 Du(E_2, E_1) + D\Pi(E_1, E_1, E_1) \langle u, E_2 \rangle - D\Pi(E_1, E_1, E_2) \langle u, E_1 \rangle \quad \text{at } x,
\end{aligned} \tag{2.16}$$

where the symmetry of  $D\Pi$  is used. A similar computation yields

$$\begin{aligned}
D(\nabla \vec{n} Qu)(E_2, E_2) &= -\lambda_2 Du(E_1, E_2) \\
&\quad + D\Pi(E_1, E_2, E_2) \langle u, E_2 \rangle - D\Pi(E_2, E_2, E_2) \langle u, E_1 \rangle \quad \text{at } x.
\end{aligned} \tag{2.17}$$

Multiplying (2.16) by  $\lambda_2$  and (2.17) by  $\lambda_1$ , respectively, summing them, and using (2.15), we obtain

$$\langle D(\nabla \vec{n} Qu), Q^* \Pi \rangle = \kappa [Du(E_2, E_1) - Du(E_1, E_2)] + \langle Qu, D\kappa \rangle. \tag{2.18}$$

Note that the function  $Du(E_2, E_1) - Du(E_1, E_2)$  is globally defined on  $\Omega$  which is independent of choice of a positively orientated orthonormal basis when the vector field  $u$  is given. From (2.14) and (2.18), we obtain

$$\langle D^2 v, Q^* \Pi \rangle = \langle D\{Q[\Lambda(U) - D(\text{tr}_g U)]\}, Q^* \Pi \rangle + \kappa [Du(E_2, E_1) - Du(E_1, E_2)] + \langle Qu, D\kappa \rangle.$$

Next, let us consider the compatibility conditions which insure that a  $y$  to satisfy (2.11) exists when the function  $v$  and the vector field  $u$  are given to satisfy (2.14). We define  $B : T_x M \rightarrow T_x M$  for  $x \in \Omega$  by

$$B\alpha = i_\alpha U - vQ\alpha + \langle u, \alpha \rangle \vec{n} \quad \text{for } \alpha \in T_x M. \quad (2.19)$$

It is easy to check that there is a  $y : \Omega \rightarrow \mathbb{R}^3$  such that

$$\nabla_\alpha y = B\alpha \quad \text{for } \alpha \in T_x M, x \in \Omega$$

if and only if the operator  $B$  satisfies

$$\nabla_X(BY) = \nabla_Y(BX) + B[X, Y] \quad \text{for } X, Y \in \mathcal{X}(\Omega). \quad (2.20)$$

Using (2.6), (2.7), (2.13), and (2.19), we have

$$\begin{aligned} \nabla_{E_1}(BE_2) &= [DU(E_2, E_1, E_1) - E_1(v) + \lambda_1 \langle u, E_2 \rangle] E_1 + DU(E_2, E_2, E_1) E_2 \\ &\quad + [Du(E_2, E_1) - \lambda_1 U(E_2, E_1) + v\lambda_1] \vec{n} \\ &= DU(E_1, E_1, E_2) E_1 + DU(E_2, E_2, E_1) E_2 \\ &\quad + [Du(E_2, E_1) - \lambda_1 U(E_2, E_1) + v\lambda_1] \vec{n} \quad \text{at } x. \end{aligned} \quad (2.21)$$

Similarly, we obtain

$$\begin{aligned} \nabla_{E_2}(BE_1) &= DU(E_1, E_1, E_2) E_1 + DU(E_2, E_2, E_1) E_2 \\ &\quad + [Du(E_1, E_2) - \lambda_2 U(E_1, E_2) - v\lambda_2] \vec{n} \quad \text{at } x. \end{aligned} \quad (2.22)$$

It follows from (2.21) and (2.22) that the relation (2.20) holds if and only if

$$Du(E_2, E_1) - Du(E_1, E_2) + \text{tr}_g U(Q\nabla \vec{n} \cdot, \cdot) + v \text{tr}_g \Pi = 0 \quad \text{for } x \in \Omega.$$

Moreover, we assume that

$$\kappa(x) \neq 0 \quad \text{for all } x \in \overline{\Omega}. \quad (2.23)$$

From (2.14), we obtain

$$u = Q(\nabla \vec{n})^{-1} Q[\Lambda(U) - D(\text{tr}_g U)] - Q(\nabla \vec{n})^{-1} Dv \quad \text{for } x \in \Omega. \quad (2.24)$$

The above derivation yields the following.

**Theorem 2.1** ([4]) *Suppose that (2.23) holds. Let  $v$  be a solution to problem*

$$\langle D^2 v, Q^* \Pi \rangle = P(U) - v\kappa \text{tr}_g \Pi + X(v) \quad \text{for } x \in \Omega, \quad (2.25)$$

where

$$P(U) = \langle D\{Q[\Lambda(U) - D(\operatorname{tr}_g U)]\}, Q^*\Pi \rangle - \langle Q[\Lambda(U) - D(\operatorname{tr}_g U)], (\nabla\vec{n})^{-1}D\kappa \rangle - \kappa \operatorname{tr}_g U(Q\nabla\vec{n}\cdot, \cdot), \quad (2.26)$$

$$X = (\nabla\vec{n})^{-1}D\kappa. \quad (2.27)$$

Let  $u$  be given by (2.24). Then there is a  $y$  to satisfy (2.5) such that (2.11) holds. Moreover,

$$|\nabla y|^2(x) = |U|^2(x) + 2v^2(x) + |u(x)|^2 \quad \text{for } x \in \Omega.$$

If, in addition,  $y = W + w\vec{n}$ ,  $w = \langle y, \vec{n} \rangle$ , then

$$u = Dw - i_W\Pi,$$

$$Dw = i_W\Pi - Q(\nabla\vec{n})^{-1}Dv + Q(\nabla\vec{n})^{-1}Q[\Lambda(U) - D(\operatorname{tr}_g U)].$$

**Remark 2.2** A solution  $y$ , modulo a constant vector, in Theorem 2.1 is unique when a solution  $v$  to (2.25) is given.

**Remark 2.3** If  $\Omega$  is elliptic and  $\Pi > 0$ , then  $\hat{g} = \Pi$  is another metric on  $\Omega$ . From [11] we have

$$\langle D^2v, Q^*\Pi \rangle = \kappa\Delta_{\hat{g}}v + \frac{1}{2\kappa}\Pi(QD\kappa, QDv) \quad \text{for } x \in \Omega,$$

where  $\Delta_{\hat{g}}$  is the Laplacian of the metric  $\hat{g}$ . Thus, in this case equation (2.25) becomes

$$\Delta_{\hat{g}}v = \frac{1}{\kappa}P(U) - v \operatorname{tr}_g \Pi + \frac{1}{2\kappa}X(v) \quad \text{for } x \in \Omega.$$

### 3 Solvability Regions for Normal Equations

Under an asymptotic coordinate system, equation (2.25) on a hyperbolic surface takes the form of a normal equation in  $\mathbb{R}^2$  locally, such as in (3.1) below. Thus the local solvability of equation (2.25) transfers to that of equation (3.1) in the Euclidean space  $\mathbb{R}^2$ . We study the solvability of the normal equation (3.1) in the space  $\mathbb{R}^2$  in this section.

We consider the following normal equation

$$w_{x_1x_2}(x) = \eta(f, w) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2 \quad (3.1)$$

where

$$\eta(f, w) = f + f_0(x)w(x) + X(w),$$

$f_0$  is a function, and  $X = (X_1, X_2)$  is a vector field on  $\mathbb{R}^2$ .

**Regions**  $E(\gamma)$ ,  $R(z, a, b)$ ,  $P_i(\beta)$ ,  $\Xi_i(\beta, \gamma)$  and  $\Phi(\beta, \gamma, \hat{\beta})$  In appropriate asymptotic coordinate systems, the problem (2.25) can transfer to solvability of (3.1) on  $E(\gamma)$ ,  $R(z, a, b)$ ,  $P_i(\beta)$ ,  $\Xi_i(\beta, \gamma)$  or  $\Phi(\beta, \gamma, \hat{\beta})$  with appropriate boundary data. We now introduce those regions to establish the corresponding solvability.

### 3.1 Regions $E(\gamma)$ and $R(z, a, b)$

Let  $k \geq 0$  be an integer. Let  $f_0$  and  $X$  be of class  $C^{k-1,1}$ , where  $C^{-1,1} = L^\infty$ . A curve  $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : [a, b] \rightarrow \mathbb{R}^2$  is said to be *noncharacteristic* if

$$\gamma'_1(t)\gamma'_2(t) \neq 0 \quad \text{for } t \in [a, b].$$

We define a linear operator  $\mathcal{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\mathcal{F}x = (x_2, -x_1) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \quad (3.2)$$

Let  $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : [0, t_0] \rightarrow \mathbb{R}^2$  be a noncharacteristic curve with  $\gamma'_1(0)\gamma'_2(0) < 0$ . We assume that

$$\gamma'_1(t) > 0, \quad \gamma'_2(t) < 0 \quad \text{for } t \in [0, t_0]. \quad (3.3)$$

Otherwise, we consider the curve  $z(t) = \gamma(-t + t_0)$ . Set

$$E(\gamma) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid \gamma_1 \circ \gamma_2^{-1}(x_2) < x_1 < \gamma_1(t_0), \gamma_2(t_0) < x_2 < \gamma_2(0) \}. \quad (3.4)$$

Consider the boundary data

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F}\dot{\gamma} \rangle \circ \gamma(t) = q_1(t) \quad \text{for } t \in (0, t_0). \quad (3.5)$$

Next, we consider a rectangle. For  $z = (z_1, z_2) \in \mathbb{R}^2$ ,  $a > 0$ , and  $b > 0$  given, let

$$R(z, a, b) = (z_1, z_1 + a) \times (z_2, z_2 + b). \quad (3.6)$$

Consider the boundary data

$$w(x_1, z_2) = p_1(x_1), \quad w(z_1, x_2) = p_2(x_2) \quad (3.7)$$

for  $x_1 \in [z_1, z_1 + a]$  and  $x_2 \in [z_2, z_2 + b]$ , respectively.

Let  $f$  be a function with its domain  $E$ . For simplicity, we denote  $\|f\|_{C^{k,1}} = \|f\|_{C^{k,1}(\overline{E})}$ ,  $\|f\|_{W^{k,2}} = \|f\|_{W^{k,2}(E)}$ , and so on.

**Proposition 3.1** *Let  $q_0$  be of  $C^{k,1}$  and  $q_1, f$  be of  $C^{k-1,1}$ , respectively. Then problem (3.1) admits a unique solution  $w \in C^{k,1}(\overline{E(\gamma)})$  with the data (3.5). Moreover, there is a  $C > 0$ , independent of solutions  $w$ , such that*

$$\|w\|_{C^{k,1}} \leq C(\|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}). \quad (3.8)$$

**Proposition 3.2** *Let  $p_1$  and  $p_2$  be of class  $C^{k,1}$  with  $p_1(z_1) = p_2(z_2)$ . Let  $f$  be of class  $C^{k-1,1}$ . Then there is a unique solution  $w \in C^{k,1}(\overline{R(z, a, b)})$  to (3.1) with the data (3.7) satisfying*

$$\|w\|_{C^{k,1}} \leq C(\|p_1\|_{C^{k,1}} + \|p_2\|_{C^{k,1}} + \|f\|_{C^{k-1,1}}).$$

The proofs of Propositions 3.1 and 3.2 will be given after Lemma 3.2.

**Lemma 3.1** *Let  $T > 0$  be given. There is a  $\varepsilon_T > 0$  such that if  $|\gamma(0)| \leq T$  and  $\max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T$ , Theorem 3.1 holds true.*

**Proof.** The proof is broken into several steps as follows.

**Step 1.** Let  $k = 0$  and let  $w \in C^{0,1}(\overline{E(\gamma)})$  be a solution to (3.1) with the data (3.5). It follows from (3.5) that

$$w_{x_1} \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} [\gamma'_1(t)q'_0(t) + \gamma'_2(t)q_1(t)], \quad w_{x_2} \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} [\gamma'_2(t)q'_0(t) - \gamma'_1(t)q_1(t)].$$

Let  $x = (x_1, x_2) \in E(\gamma)$  be given. We integrate (3.1) with respect to the first variable  $\zeta_1$  over  $(\gamma_1 \circ \gamma_2^{-1}(\zeta_2), x_1)$  for  $\zeta_2 \in (\gamma_2 \circ \gamma_1^{-1}(x_1), x_2)$  to have

$$w_{x_2}(x_1, \zeta_2) = w_{x_2} \circ \gamma(\gamma_2^{-1}(\zeta_2)) + \int_{\gamma_1 \circ \gamma_2^{-1}(\zeta_2)}^{x_1} \eta(f, w)(\zeta_1, \zeta_2) d\zeta_1. \quad (3.9)$$

Then integrating the above identity over  $(\gamma_2 \circ \gamma_1^{-1}(x_1), x_2)$  with respect to the second variable  $\zeta_2$  yields

$$w(x_1, x_2) = \mathcal{B}(q_0, q_1) + \int_{E(x)} \eta(f, w) d\zeta,$$

where

$$\mathcal{B}(q_0, q_1) = q_0 \circ \gamma_1^{-1}(x_1) + \int_{\gamma_1^{-1}(x_1)}^{\gamma_2^{-1}(x_2)} \frac{\gamma'_2(t)}{|\gamma'(t)|^2} [\gamma'_2(t)q'_0(t) - \gamma'_1(t)q_1(t)] dt, \quad (3.10)$$

$$E(x) = \{(\zeta_1, \zeta_2) \mid \gamma_1 \circ \gamma_2(\zeta_2) < \zeta_1 < x_1, \gamma_2 \circ \gamma_1^{-1}(x_1) < \zeta_2 < x_2\}. \quad (3.11)$$

**Step 2.** We define an operator  $I : C^{0,1}(\overline{E(\gamma)}) \rightarrow C^{0,1}(\overline{E(\gamma)})$  by

$$I(w) = \mathcal{B}(q_0, q_1) + \int_{E(x)} \eta(f, w) d\zeta \quad \text{for } w \in C^{0,1}(\overline{E(\gamma)}). \quad (3.12)$$

It is easy to check that  $w \in C^{0,1}(\overline{E(\gamma)})$  solves (3.1) with the data (3.5) if and only if  $I(w) = w$ .

Next, we show that there is a  $0 < \varepsilon_T \leq 1$  such that when  $|\gamma(0)| \leq T$  and  $0 < \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T$ , the map  $I : C^{0,1}(\overline{E(\gamma)}) \rightarrow C^{0,1}(\overline{E(\gamma)})$  is contractible. Thus the existence and uniqueness of solutions in the case  $k = 0$  follows from Banach's fixed point theorem.

A simple computation shows that for  $w \in C^{0,1}(\overline{E(\gamma)})$

$$[I(w)]_{x_1} = \frac{1}{|\gamma'(t)|^2} [\gamma'_1(t)q'_0(t) + \gamma'_2(t)q_1(t)] \Big|_{t=\gamma_1^{-1}(x_1)} + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} \eta(f, w)(x_1, \zeta_2) d\zeta_2,$$

$$[I(w)]_{x_2} = \frac{1}{|\gamma'(t)|^2} [\gamma'_2(t)q'_0(t) - \gamma'_1(t)q_1(t)] \Big|_{t=\gamma_2^{-1}(x_2)} + \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)}^{x_1} \eta(f, w)(\zeta_1, x_2) d\zeta_1.$$

The above formulas yield for  $w_1, w_2 \in C^{0,1}(\overline{E(\gamma)})$ ,

$$\|I(w_1) - I(w_2)\|_{C^{0,1}(\overline{E(\gamma)})} \leq C_T \max\{\lambda, \lambda^2\} \|w_1 - w_2\|_{C^{0,1}(\overline{E(\gamma)})},$$

where

$$\lambda = \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\}, \quad C_T = \|f_0\|_{L^\infty(|x| \leq 2T)} + \|X\|_{L^\infty(|x| \leq 2T)}.$$

Thus, the map  $I : C^{0,1}(\overline{E(\gamma)}) \rightarrow C^{0,1}(\overline{E(\gamma)})$  is contractible if  $\lambda > 0$  is small.

**Step 3.** Consider the case  $k = 1$ . Let  $q_0 \in C^{1,1}[0, t_0]$ ,  $q_1 \in C^{0,1}[0, t_0]$ , and  $f \in C^{0,1}(\overline{E(\gamma)})$  be given. By Step 2, there is a  $\varepsilon_T > 0$  such that if  $|\gamma(0)| \leq T$  and  $0 < \lambda < \varepsilon_T$ , problem (3.1) has a unique solution  $w \in C^{0,1}(\overline{E(\gamma)})$  with the data (3.5). A formal computation shows that  $u = w_{x_1}$  solves problem

$$u_{x_1 x_2} = \eta(\hat{f}, u) \quad \text{for } x \in E(\gamma), \quad (3.13)$$

with the data

$$u \circ \gamma(t) = \hat{q}_0(t), \quad \langle \nabla u, \mathcal{F}\dot{\gamma} \rangle \circ \gamma(t) = \hat{q}_1(t) \quad \text{for } t \in (0, t_0), \quad (3.14)$$

where

$$\begin{aligned} \hat{f} &= f_{x_1} + f_{0x_1} w + \nabla_{\partial x_1} X(w), \quad \hat{q}_0(t) = \frac{1}{|\gamma'(t)|^2} [\gamma'_2(t) q'_0(t) + \gamma'_1(t) q_1(t)], \\ \hat{q}_1(t) &= \frac{\gamma'_2(t)}{\gamma'_1(t)} \hat{q}'_0(t) - \frac{|\gamma'(t)|^2}{\gamma'_1(t)} \eta(f, w) \circ \gamma(t). \end{aligned}$$

We apply Step 2 to problem (3.13) and (3.14) to obtain  $u = w_{x_1} \in C^{0,1}(\overline{E(\gamma)})$  when  $0 < \lambda < \varepsilon_T$ . A similar argument yields  $w_{x_2} \in C^{0,1}(\overline{E(\gamma)})$ . Thus  $w \in C^{1,1}(\overline{E(\gamma)})$ .

By repeating the above procedure, the existence and uniqueness of the solutions in the cases  $k \geq 2$  are obtained.

**Step 4.** Let map  $I : C^{k,1}(\overline{E(\gamma)}) \rightarrow C^{k,1}(\overline{E(\gamma)})$  be defined in Step 2 and let  $w \in C^{k,1}(\overline{E(\gamma)})$  be the solution to problem (3.1) with the data (3.5). Then

$$\begin{aligned} \|w\|_{C^{k,1}} &= \|I(w)\|_{C^{k,1}} \leq \|I(0)\|_{C^{k,1}} + \|I(w) - I(0)\|_{C^{k,1}} \\ &\leq C(\|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}) + C_T \max\{\lambda, \lambda^2\} \|w\|_{C^{k,1}}. \end{aligned}$$

Thus, the estimate (3.8) follows if  $\lambda > 0$  is small.  $\square$

By a similar argument as for Lemma 3.1, we have the following lemmas.

**Lemma 3.2** *Let  $T > 0$ . There is  $\varepsilon_T > 0$  such that if  $|z| \leq T$  and  $0 < \max\{a, b\} < \varepsilon_T$ , then Proposition 3.2 holds.*

**Proof of Proposition 3.1.** We shall show that the assumptions  $|\gamma(0)| \leq T$  and  $\max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\} < \varepsilon_T$  in Lemma 3.1 are unnecessary. Let  $T > 0$  be given such that

$$E(\gamma) \subset \{x \in \mathbb{R}^2 \mid |x| \leq T\}.$$

Let  $\varepsilon_T > 0$  be given such that Lemmas 3.1 and 3.2 hold. We divide the curve  $\gamma$  into  $m$  parts with the points  $\tau_0 = 0, \tau_0 < \tau_1 < \dots < \tau_m = t_0$  such that

$$|\gamma(\tau_{i+1}) - \gamma(\tau_i)| = \frac{\varepsilon_T}{2}, \quad 0 \leq i \leq m-2, \quad |\gamma(t_0) - \gamma(\tau_{m-1})| \leq \frac{\varepsilon_T}{2}.$$

For simplicity, we assume that  $m = 3$ . The other cases can be treated by a similar argument.

In the case of  $m = 3$ , we have

$$\overline{E(\gamma)} = (\cup_{i=0}^2 \overline{E}_i) \cup (\cup_{i=1}^3 \overline{R}_i) \quad (3.15)$$

where

$$E_i = \{x \in E(\gamma) \mid \gamma_1(\tau_i) \leq x_1 \leq \gamma_1(\tau_{i+1}), \gamma_2(\tau_{i+1}) \leq x_2 \leq \gamma_2(\tau_i)\} \quad i = 0, 1, 2,$$

$$R_1 = [\gamma_1(\tau_1), \gamma_1(\tau_2)] \times [\gamma_2(\tau_1), \gamma_2(0)], \quad R_2 = [\gamma_1(\tau_2), \gamma_1(t_0)] \times [\gamma_2(\tau_2), \gamma_2(\tau_1)],$$

$$R_3 = [\gamma_1(\tau_2), \gamma_1(t_0)] \times [\gamma_2(\tau_1), \gamma_2(0)].$$

From Lemma 3.1, problem (3.1) admits a unique solution  $w_i \in C^{k,1}(\overline{E}_i)$  for each  $i = 0, 1$ , and  $2$ , respectively, with the corresponding data and the corresponding estimates. We define  $w \in C^{k,1}(\cup_{i=0}^2 \overline{E}_i)$  by

$$w(x) = w_i(x) \quad \text{for } x \in \overline{E}_i \quad \text{for } i = 0, 1, 2.$$

We extend the domain of  $w$  from  $\cup_{i=0}^2 \overline{E}_i$  to  $\overline{E(\gamma)}$  by the following way. By Lemma 3.2, we define  $w \in C^{k,1}(\overline{R}_i)$  to be the solution  $u_i \in C^{k,1}(\overline{R}_i)$  to problem (3.1) with the data

$$u_i(\gamma_1(\tau_i), x_2) = w_{i-1}(\gamma_1(\tau_i), x_2) \quad \text{for } x_2 \in [\gamma_2(\tau_i), \gamma_2(\tau_{i-1})],$$

$$u_i(x_1, \gamma_2(\tau_i)) = w_i(x_1, \gamma_2(\tau_i)) \quad \text{for } x_1 \in [\gamma_1(\tau_i), \gamma_1(\tau_{i+1})],$$

for  $i = 1$ , and  $2$ , respectively. Then we extend  $w$  on  $C^{k,1}(\overline{R}_3)$  to be the solution  $u_3$  of (3.1) with the data

$$u_3(\gamma_1(\tau_2), x_2) = u_1(\gamma_1(\tau_2), x_2) \quad \text{for } x_2 \in [\gamma_2(\tau_1), \gamma_2(0)],$$

$$u_3(x_1, \gamma_2(\tau_2)) = u_2(x_1, \gamma_2(\tau_2)) \quad \text{for } x_1 \in [\gamma_1(\tau_2), \gamma_1(t_0)].$$

To complete the proof, we have to show that  $w$  is a  $C^{k,1}$  solution on all the connection segments between any two subregions above. Consider the subregion

$$\overline{\tilde{E}} = \overline{E}_0 \cup \overline{E}_1 \cup \overline{R}_1.$$

Since  $|\gamma(\tau_2) - \gamma(0)| \leq \varepsilon_T$ , Lemma 3.1 insures that problem (3.1) admits a unique solution  $\tilde{w} \in C^{k,1}(\overline{E})$  with the corresponding data. Then the uniqueness implies that  $w(x) = \tilde{w}(x)$  for  $x \in \overline{E}$ . In particular,  $w$  is  $C^{k,1}$  on the segments  $\{(\gamma_1(\tau_1), x_2) \mid x_2 \in [\gamma_2(\tau_1), \gamma_2(0)]\}$  and  $\{(x_1, \gamma_2(\tau_1)) \mid x_1 \in [\gamma_1(\tau_1), \gamma_1(\tau_2)]\}$ , respectively. By a similar argument, we show that  $w$  is also  $C^{k,1}$  on all the other segments.

The estimates in (3.8) follow from the ones in Lemmas 3.1 and 3.2.  $\square$

**Proof of Proposition 3.2.** We divided  $R(z, a, b)$  into a sum of small rectangles and apply Lemma 3.2 to paste the solutions together.  $\square$

To have density results in Theorem 1.2, we also need estimates of some (boundary) traces of the solutions. For  $\sigma \in (0, t_0)$ , let

$$\beta_\sigma(t) = \gamma(\sigma) - t\mathcal{F}\dot{\gamma}(\sigma) \quad \text{for } t \in (0, t_\sigma),$$

where  $t_\sigma > 0$  is such that  $\beta_\sigma(t_\sigma) \in \partial E(\gamma)$ .

**Proposition 3.3** *Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Let  $q_0$  be of class  $W^{2,2}$  and  $q_1, f$  be of class  $W^{1,2}$ , respectively. Then problem (3.1) admits a unique solution  $w \in W^{2,2}$  with the data (3.5). Moreover, there is a  $C > 0$ , independent of solutions  $w$ , such that*

$$\|w\|_{W^{2,2}}^2 + \|w_{x_2} \circ \beta_\sigma\|_{W^{1,2}}^2 \leq C(\|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}), \quad (3.16)$$

where  $W^{i,2} = W^{i,2}(E(\gamma))$  for  $1 \leq i \leq 2$ .

**Proof** A similar argument as for Theorem 3.1 shows that a unique solution  $w \in W^{2,2}(E(\gamma))$  with the data (3.5) exists, and the estimate

$$\|w\|_{W^{2,2}}^2 \leq C(\|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}) \quad (3.17)$$

holds.

Let  $\beta_\sigma(t) = (\beta_{\sigma_1}(t), \beta_{\sigma_2}(t))$ . Using equation (3.1), we have

$$w_{x_2 x_2} \circ \beta_\sigma(t) = w_{x_2 x_2} \circ \gamma \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t) + \int_{\gamma_1 \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)}^{\beta_{\sigma_1}(t)} [\eta(f, w)]_{x_2}(\zeta_1, \beta_{\sigma_2}(t)) d\zeta_1,$$

which yields

$$\begin{aligned} |w_{x_2 x_2} \circ \beta_\sigma(t)|^2 &\leq 2|w_{x_2 x_2} \circ \gamma \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)|^2 \\ &+ 2[\gamma_1(t_0) - \gamma_1(0)]C \int_{\gamma_1 \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)}^{\beta_{\sigma_1}(t)} (|f|^2 + |\nabla f|^2 + |w|^2 + |\nabla w|^2 + |\nabla^2 w|^2)(\zeta_1, \beta_{\sigma_2}(t)) d\zeta_1. \end{aligned}$$

Integrating the above inequality over  $(0, t_\sigma)$  with respect to  $t$ , we obtain

$$\|w_{x_2 x_2} \circ \beta_\sigma\|_{L^2}^2 \leq C(\|f\|_{W^{1,2}}^2 + \|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2).$$

A similar computation shows that  $\|\nabla w_{x_1} \circ \beta_\sigma\|_{L^2}^2$ ,  $\|\nabla w \circ \beta_\sigma\|_{L^2}^2$ , and  $\|w \circ \beta\|_{L^2}^2$  can be bounded also by the right hand side of the above inequality. Thus estimate (3.16) follows from (3.17).  $\square$

Let

$$\Gamma(\gamma, w) = \sum_{j=0}^1 \|\nabla^j w \circ \gamma\|_{L^2(0,t_0)}^2 + \int_0^{t_0} [|w_{x_1 x_1} \circ \gamma(t)|^2 t + |w_{x_2 x_2} \circ \gamma(t)|^2 (t_0 - t)] dt. \quad (3.18)$$

**Proposition 3.4** *Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there are  $0 < c_1 < c_2$  such that for all solutions  $w \in W^{2,2}$  to problem (3.1)*

$$c_1 \Gamma(\gamma, w) \leq \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)], \quad (3.19)$$

$$\|w(\cdot, \gamma_2(0))\|_{W^{1,2}(\gamma_1(0), \gamma_1(t_0))}^2 + \int_{\gamma_1(0)}^{\gamma_1(t_0)} |w_{x_1 x_1}(x_1, \gamma_2(0))|^2 (x_1 - \gamma_1(0)) dx_1 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)],$$

$$\|w(\gamma_1(t_0), \cdot)\|_{W^{1,2}(\gamma_2(t_0), \gamma_2(0))}^2 + \int_{\gamma_2(t_0)}^{\gamma_2(0)} |w_{x_2 x_2}(\gamma_1(t_0), x_2)|^2 (x_2 - \gamma_2(t_0)) dx_2 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)],$$

where  $W^{i,2} = W^{i,2}(E(\gamma))$  for  $1 \leq i \leq 2$ .

**Proposition 3.5** *Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there is  $C > 0$  such that for all solutions  $w \in W^{2,2}(R(z, a, b))$  to problem (3.1)*

$$\|w\|_{W^{2,2}}^2 \leq C (\|f\|_{W^{1,2}}^2 + \|p_1\|_{W^{2,2}(z_1, z_1+a)}^2 + \|p_2\|_{W^{2,2}(z_2, z_2+b)}^2). \quad (3.20)$$

The proofs of the above two propositions will complete from Lemmas 3.3 and 3.4 below by an argument as for Proposition 3.1. We omit the details.

**Lemma 3.3** *Let  $T > 0$  be given. There is  $\varepsilon_T > 0$  such that if  $|\gamma(0)| \leq T$  and  $\max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T$ , Proposition 3.4 holds.*

**Proof Step 1** Using (3.1) we have

$$w_{x_1 x_1}(x) = w_{x_1 x_1} \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} [\eta(f, w)]_{x_1}(x_1, \zeta_2) d\zeta_2,$$

which yields

$$|w_{x_1 x_1}(x)|^2 \leq 2|w_{x_1 x_1} \circ \gamma \circ \gamma_1^{-1}(x_1)|^2 + 2|x_2 - \gamma_2 \circ \gamma_1^{-1}(x_1)| \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{\gamma_2(0)} |[\eta(f, w)]_{x_1}(x_1, \zeta_2)|^2 d\zeta_2,$$

and

$$|w_{x_1 x_1} \circ \gamma \circ \gamma_1^{-1}(x_1)|^2 \leq 2|w_{x_1 x_1}(x)|^2 + 2|x_2 - \gamma_2 \circ \gamma_1^{-1}(x_1)| \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{\gamma_2(0)} |[\eta(f, w)]_{x_1}(x_1, \zeta_2)|^2 d\zeta_2,$$

respectively. Integrating thee above two inequalities, first with respect to  $x_2$  over  $(\gamma_2 \circ \gamma_1^{-1}(x_1), \gamma_2(0))$  and then with respect to  $x_1$  over  $(\gamma_1(0), \gamma_1(t_0))$  respectively, we obtain

$$\|w_{x_1 x_1}\|_{L^2}^2 \leq 2\sigma_{12} \int_0^{t_0} |w_{x_1 x_1} \circ \gamma(t)|^2 t dt + \varepsilon_T^2 C_T (\|f\|_{\mathbb{W}^{1,2}}^2 + \|w\|_{\mathbb{W}^{2,2}}^2)$$

and

$$\sigma_{11} \int_0^{t_0} |w_{x_1 x_1} \circ \gamma(t)|^2 t dt \leq 2\|w_{x_1 x_1}\|_{L^2}^2 + \varepsilon_T^2 C_T (\|f\|_{\mathbb{W}^{1,2}}^2 + \|w\|_{\mathbb{W}^{2,2}}^2),$$

where

$$\sigma_{11} = \inf_{t \in (0, t_0)} [\gamma_2(0) - \gamma_2(t)] \gamma_1'(t) / t, \quad \sigma_{12} = \sup_{t \in (0, t_0)} [\gamma_2(0) - \gamma_2(t)] \gamma_1'(t) / t,$$

$$C_T = \sup_{|x| \leq 2T} (1 + f_0^2 + |\nabla f_0|^2 + |X|^2 + |\nabla X|^2).$$

By similar arguments, we establish the following

$$\|w_{x_2 x_2}\|_{L^2}^2 \leq 2\sigma_{22} \int_0^{t_0} |w_{x_2 x_2} \circ \gamma(t)|^2 (t_0 - t) dt + \varepsilon_T^2 C_T (\|f\|_{\mathbb{W}^{1,2}}^2 + \|w\|_{\mathbb{W}^{2,2}}^2),$$

$$\sigma_{21} \int_0^{t_0} |w_{x_2 x_2} \circ \gamma(t)|^2 (t_0 - t) dt \leq 2\|w_{x_2 x_2}\|_{L^2}^2 + \varepsilon_T^2 C_T (\|f\|_{\mathbb{W}^{1,2}}^2 + \|w\|_{\mathbb{W}^{2,2}}^2),$$

where

$$\sigma_{21} = \inf_{t \in (0, t_0)} [\gamma_1(t_0) - \gamma_1(t)] [-\gamma_2'(t)] / (t_0 - t), \quad \sigma_{22} = \sup_{t \in (0, t_0)} [\gamma_1(t_0) - \gamma_1(t)] [-\gamma_2'(t)] / (t_0 - t).$$

**Step 2** As in Step 1, we have

$$\begin{aligned} \|w_{x_1}\|_{L^2}^2 &\leq 2\sigma_{12} \int_0^{t_0} |w_{x_1} \circ \gamma(t)|^2 t dt + \varepsilon_T^2 C_T (\|f\|_{L^2}^2 + \|w\|_{\mathbb{W}^{1,2}}^2) \\ &\leq 2\sigma_{12} t_0 \|w_{x_1} \circ \gamma\|_{L^2(0, t_0)}^2 + \varepsilon_T^2 C_T (\|f\|_{L^2}^2 + \|w\|_{\mathbb{W}^{1,2}}^2), \\ \sigma_{11} \int_0^{t_0} |w_{x_1} \circ \gamma(t)|^2 t dt &\leq 2\|w_{x_1}\|_{L^2}^2 + \varepsilon_T^2 C_T (\|f\|_{L^2}^2 + \|w\|_{\mathbb{W}^{1,2}}^2). \end{aligned} \quad (3.21)$$

In addition, since

$$w_{x_1} \circ \gamma \circ \gamma_2^{-1}(x_2) = w_{x_1}(x) - \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)}^{x_1} w_{x_1 x_1}(\zeta_1, x_2) d\zeta_1 \quad \text{for } x_2 \in (\gamma_2(t_0), \gamma_2(0)),$$

it follows that

$$\sigma_{21} \int_0^{t_0} |w_{x_1} \circ \gamma(t)|^2 (t_0 - t) dt \leq 2\|w_{x_1}\|_{L^2}^2 + \varepsilon_T^2 \|w_{x_1 x_1}\|_{L^2}^2. \quad (3.22)$$

Combing (3.21) and (3.22), we have

$$\begin{aligned} \min\{\sigma_{11}, \sigma_{21}\} \|w_{x_1} \circ \gamma\|_{L^2(0, t_0)}^2 &\leq \frac{1}{t_0} (\sigma_{21} \int_0^{t_0/2} |w_{x_1} \circ \gamma(t)|^2 (t_0 - t) dt + \sigma_{11} \int_{t_0/2}^{t_0} |w_{x_1} \circ \gamma(t)|^2 t dt) \\ &\leq \frac{1}{t_0} [4 + \varepsilon_T^2 (C_T + 1)] (\|f\|_{L^2}^2 + \|w\|_{\mathbb{W}^{2,2}}^2). \end{aligned}$$

By a similar computation, we obtain

$$\begin{aligned} \|w_{x_2}\|_{L^2}^2 &\leq 2\sigma_{12}t_0\|w_{x_2} \circ \gamma\|_{L^2(0,t_0)}^2 + C_T\varepsilon_T^2(\|f\|_{L^2}^2 + \|w\|_{W^{1,2}}^2), \\ \min\{\sigma_{11}, \sigma_{21}\}\|w_{x_2} \circ \gamma\|_{L^2(0,t_0)}^2 &\leq \frac{1}{t_0}[4 + \varepsilon_T^2(C_T + 1)](\|f\|_{L^2}^2 + \|w\|_{W^{2,2}}^2), \\ \|w\|_{L^2}^2 &\leq 2\sigma_{12}t_0\|w \circ \gamma\|_{L^2(0,a)}^2 + \varepsilon_T^2\|w_{x_2}\|_{L^2}^2, \\ \min\{\sigma_{11}, \sigma_{21}\}\|w \circ \gamma\|_{L^2(0,t_0)}^2 &\leq \frac{1}{t_0}[4 + \varepsilon_T^2(C_T + 1)]\|w\|_{W^{1,2}}^2. \end{aligned}$$

**Step 3** From Steps 1 and 2, we obtain

$$[1 - (4C_T + 1)\varepsilon_T^2]\|w\|_{W^{2,2}}^2 \leq 2[(\sigma_{12}(1 + t_0) + \sigma_{22})\Gamma(\gamma, w) + (4C_T + 1)\varepsilon_T^2\|f\|_{W^{1,2}}^2],$$

when  $\lambda$  is small, and

$$\min\{\sigma_{11}, \sigma_{21}\}\Gamma(\gamma, w) \leq 2\{2 + C_T\varepsilon_T^2 + \frac{3}{t_0}[4 + (C_T + 1)\varepsilon_T^2]\}(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2),$$

respectively. Thus (3.19) follows.

**Step 4** We have

$$w_{x_1x_1}(x_1, \gamma_2(0)) = w_{x_1x_1} \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{\gamma_2(0)} [\eta(f, w)]_{x_1}(x_1, \zeta_2) d\zeta_2,$$

which gives, by (3.19),

$$\begin{aligned} \int_{\gamma_1(0)}^{\gamma_1(t_0)} |w_{x_1x_1}(x_1, \gamma_2(0))|^2 (x_1 - \gamma_1(0)) dx_1 &\leq 2 \int_0^{t_0} |w_{x_1x_1} \circ \gamma(t)|^2 [\gamma_1(t) - \gamma_1(0)] dt \\ + C(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2) &\leq C[\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)]. \end{aligned}$$

A similar argument completes the proof of the third inequality in Proposition 3.4.  $\square$

A similar argument yields the following.

**Lemma 3.4** *Let  $T > 0$  be given. There is  $\varepsilon_T > 0$  such that if  $|z| \leq T$  and  $0 < \max\{a, b\} < \varepsilon_T$ , then Proposition 3.5 holds.*

### 3.2 Regions $P_i(\beta)$

Let  $\beta = (\beta_1, \beta_2) : [0, t_0] \rightarrow \mathbb{R}^2$  be a noncharacteristic curve with  $\beta_1'(0)\beta_2'(0) > 0$ . We assume

$$\beta_i'(t) > 0 \quad \text{for } t \in [0, t_0], \quad i = 1, 2. \quad (3.23)$$

Otherwise, we consider the curve  $z(t) = \beta(-t + t_0)$ . Set

$$P_1(\beta) = \{(x_1, x_2) \mid \beta_1 \circ \beta_2^{-1}(x_2) < x_1 < \beta_1(t_0), \beta_2(0) < x_2 < \beta_2(t_0)\}, \quad (3.24)$$

and consider the boundary data

$$w_{x_2} \circ \beta(t) = p(t), \quad t \in (0, t_0); \quad w(x_1, \beta_2(0)) = p_1(x_1), \quad x_1 \in (\beta_1(0), \beta_1(t_0)). \quad (3.25)$$

Set

$$P_2(\beta) = \{ (x_1, x_2) \mid \beta_1(0) < x_1 < \beta_1 \circ \beta_2^{-1}(x_2), \beta_2(0) < x_2 < \beta_2(t_0) \}, \quad (3.26)$$

and consider the boundary data

$$w_{x_1} \circ \beta(t) = p(t), \quad t \in (0, t_0); \quad w(\beta_1(0), x_2) = p_2(x_2), \quad x_2 \in (\beta_2(0), \beta_2(t_0)). \quad (3.27)$$

By similar arguments for the region  $E(\gamma)$ , we establish Propositions 3.6-3.8 below. The details are omitted.

**Proposition 3.6** *Let the curve  $\beta$  be of class  $C^{k-1,1}$ . Let  $p_1$  (or  $p_2$ ) be of class  $C^{k,1}$  and let  $p, f$  be of class  $C^{k-1,1}$ . Then problem (3.1) admits a unique solution  $w \in C^{k,1}(\overline{P_1(\beta)})$  (or  $C^{k,1}(\overline{P_2(\beta)})$ ) with the data (3.25) (or (3.27)) to satisfy*

$$\|w\|_{C^{k,1}} \leq C(\|p\|_{C^{k-1,1}} + \|p_1\|_{C^{k,1}} + \|f\|_{C^{k-1,1}}) \text{ (or } (\|p\|_{C^{k-1,1}} + \|p_2\|_{C^{k,1}} + \|f\|_{C^{k-1,1}})).$$

**Proposition 3.7** *Let the curve  $\beta$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Let  $p_1$  (or  $p_2$ ) be of class  $W^{2,2}$  and let  $p, f$  be of class  $W^{1,2}$ . Then problem (3.1) admits a unique solution  $w \in W^{2,2}(P_1(\beta))$  (or  $W^{2,2}(P_2(\beta))$ ) with the data (3.25) (or (3.27)) to satisfy*

$$\|w\|_{W^{2,2}} \leq C(\|p\|_{W^{1,2}} + \|p_1\|_{W^{2,2}} + \|f\|_{W^{1,2}}) \text{ (or } (\|p\|_{W^{1,2}} + \|p_2\|_{W^{2,2}} + \|f\|_{W^{1,2}})).$$

Let

$$\Gamma(P_i, w) = \int_0^{t_0} |p'(t)|^2 (t_0 - t) dt + \|p_i\|_{W^{1,2}}^2 + \int_{\beta_i(0)}^{\beta_i(t_0)} |p_i''(x_i)|^2 (x_i - z_i) dx_i, \quad i = 1, 2.$$

**Proposition 3.8** *Let the curve  $\beta$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there are  $0 < c_1 < c_2$  such that for all solutions  $w \in W^{2,2}(P_i(\beta))$  to problem (3.1) with the corresponding boundary data satisfy*

$$c_1 \Gamma(P_i, w) \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2 [\Gamma(P_i, w) + \|f\|_{W^{1,2}}^2],$$

$$\begin{aligned} c_1 \|w|_{x_i=\beta_i(t_0)}\|_{W^{2,2}}^2 &\leq \int_0^{t_0} |p'(t)|^2 dt + \int_{\beta_i(0)}^{\beta_i(t_0)} |p_i''(x_1)|^2 (x_i - \beta_i(0)) dx_i + \|p_i\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2 \\ &\leq c_2 (\|w|_{x_i=\beta_i(t_0)}\|_{W^{2,2}}^2 + \int_{\beta_i(0)}^{\beta_i(t_0)} |p_i''(x_i)|^2 (x_i - \beta_i(0)) dx_i + \|p_i\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2), \end{aligned} \quad (3.28)$$

for  $i = 1$ , and  $2$ , respectively.

**Remark 3.1** (3.28) implies that  $p \in W^{1,2}$  if and only if  $w|_{x_i=\beta_i(t_0)} \in W^{2,2}$ . However, the case of  $p \notin W^{1,2}$  may happen.

### 3.3 Regions $\Xi_i(\beta, \gamma)$

Let  $\gamma : [0, t_1] \rightarrow \mathbb{R}^2$  and  $\beta : [0, t_0] \rightarrow \mathbb{R}^2$  be two noncharacteristic curves with  $\gamma(0) = \beta(0)$  such that

$$\gamma_1(t_1) \leq \beta_1(t_0), \quad \gamma'_1(t) > 0, \quad \gamma'_2(t) < 0, \quad \beta'_1(t) > 0, \quad \beta'_2(t) > 0$$

hold. Set

$$\overline{\Xi_1(\beta, \gamma)} = \overline{P_1(\beta)} \cup \overline{R(z, a, b)} \cup \overline{E(\gamma)}, \quad (3.29)$$

where  $P_1(\beta)$ ,  $R(z, a, b)$ , and  $E(\gamma)$  are given in (3.24), (3.6), and (3.4), respectively, with  $z = (\beta_1(t_0), \gamma_2(0))$ ,  $a = \gamma(t_1) - \beta_1(t_0)$ , and  $b = \beta_2(t_0) - \gamma_2(t_0)$ . Consider the boundary data

$$w_{x_2} \circ \beta(t) = p(t) \quad \text{for } t \in [0, t_0], \quad (3.30)$$

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F}\dot{\gamma} \rangle \circ \gamma(t) = q_1(t) \quad \text{for } t \in (0, t_1), \quad (3.31)$$

where  $\mathcal{F}$  is given by (3.2).

Let  $\gamma : [0, t_1] \rightarrow \mathbb{R}^2$  and  $\beta : [0, t_0] \rightarrow \mathbb{R}^2$  be two noncharacteristic curves with  $\gamma(t_1) = \beta(0)$  such that

$$\gamma_2(0) \geq \beta_2(t_0), \quad \gamma'_1(t) > 0, \quad \gamma'_2(t) < 0, \quad \beta'_1(t) > 0, \quad \beta'_2(t) > 0$$

hold. Set

$$\overline{\Xi_2(\beta, \gamma)} = \overline{E(\gamma)} \cup \overline{R(z, a, b)} \cup \overline{P_2(\beta)}, \quad (3.32)$$

where  $E(\gamma)$ ,  $R(z, a, b)$ , and  $P_2(\beta)$  are given in (3.4), (3.6), and (3.26), respectively, with  $z = (\gamma_1(t_1), \gamma_2(t_0))$ ,  $a = \beta(t_0) - \gamma_1(t_1)$ , and  $b = \gamma_2(0) - \beta_2(t_0)$ . Consider the data

$$w_{x_1} \circ \beta(t) = p(t) \quad \text{for } t \in [0, t_0], \quad (3.33)$$

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F}\dot{\gamma} \rangle \circ \gamma(t) = q_1(t) \quad \text{for } t \in (0, t_1), \quad (3.34)$$

where  $\mathcal{F}$  is given by (3.2).

We consider solvability of (3.1) on  $\Xi_1(\beta, \gamma)$ . To have a  $C^{k,1}$  solution on  $\overline{\Xi_1(\beta, \gamma)}$ , we need some kind of *compatibility conditions at the point*  $\gamma(0) = \beta(0)$ . From Proposition 3.1, problem (3.1) admits a unique solution  $u \in C^{k,1}(\overline{E(\gamma)})$  with the data (3.31). From Proposition 3.6, there is a unique solution  $v \in C^{k,1}(\overline{P_1(\beta)})$  to problem (3.1) with the data

$$v_{x_2} \circ \beta(t) = p(t), \quad t \in (0, t_0), \quad v(x_1, \beta_2(0)) = u(x_1, \beta_2(0)), \quad x_1 \in [\beta_1(0), \beta_1(t_0)]. \quad (3.35)$$

In terms of the uniqueness, if problem (3.1) has a unique solution  $w \in C^{k,1}(\overline{\Xi_1(\beta, \gamma)})$  with the data (3.30) and (3.31) together, then

$$w(x) = \begin{cases} v(x) & \text{for } x \in \overline{P_1(\beta)}, \\ u(x) & \text{for } x \in \overline{E(\gamma)}. \end{cases} \quad (3.36)$$

Conversely, if we define  $w$  by the formula (3.36), then whether it is a  $C^{k,1}$  solution to (3.1) on  $\Xi_1(\beta, \gamma)$  depends on the  $C^{k,1}$  regularity of  $w$  at the point  $\beta(0)$ . Thus, *compatibility conditions* are something which can guarantee that  $w$  is  $C^{k,1}$  at  $\gamma(0) = \beta(0)$ , that is

$$\nabla^j u \circ \gamma(0) = \nabla^j v \circ \beta(0) \quad \text{for } 0 \leq j \leq k. \quad (3.37)$$

The solution  $u$  with the data (3.31) yields

$$\nabla u \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} (\gamma'_1(t)q'_0(t) + \gamma'_2(t)q_1(t), \gamma'_2(t)q'_0(t) - \gamma'_1(t)q_1(t)) \quad (3.38)$$

for  $t \in [0, t_1]$ . Using (3.1) and (3.38), we have

$$\begin{aligned} u_{x_2 x_1} \circ \gamma(t) &= f \circ \gamma(t) + \frac{1}{|\gamma'(t)|^2} [\gamma'_2(t)X_1 \circ \gamma(t) - \gamma'_1(t)X_2 \circ \gamma(t)]q_1(t) \\ &\quad + f_0 \circ \gamma(t)q_0(t) + \frac{1}{|\gamma'(t)|^2} [\gamma'_1(t)X_1 \circ \gamma(t) + \gamma'_2(t)X_2 \circ \gamma(t)]q'_0(t) \end{aligned} \quad (3.39)$$

for  $t \in (0, t_1)$ . Next, differentiating the second component in (3.38) with respect to variable  $t$  and using (3.39), we obtain

$$\begin{aligned} u_{x_2 x_2} \circ \gamma(t) &= -\frac{\gamma'_1}{\gamma'_2} f \circ \gamma(t) - \frac{\gamma'_1}{\gamma'_2} f_0 \circ \gamma(t)q_0 \\ &\quad - \left[ \frac{2\langle \gamma'', \gamma' \rangle}{|\gamma'|^4} + \frac{\gamma'_1}{|\gamma'|^2 \gamma'_2} (\gamma'_1 X_1 \circ \gamma + \gamma'_2 X_2 \circ \gamma) - \frac{\gamma''_2}{|\gamma'|^2 \gamma'_2} \right] q'_0 + \frac{1}{|\gamma'|^2} q''_0 \\ &\quad + \left[ \frac{2\langle \gamma'', \gamma' \rangle \gamma'_1}{|\gamma'|^4 \gamma'_2} - \frac{\gamma'_1}{|\gamma'|^2 \gamma'_2} (\gamma'_2 X_1 \circ \gamma - \gamma'_1 X_2 \circ \gamma) - \frac{\gamma''_1}{|\gamma'|^2 \gamma'_2} \right] q_1 - \frac{\gamma'_1}{|\gamma'|^2 \gamma'_2} q'_1. \end{aligned} \quad (3.40)$$

By repeating the above procedure, we have shown that, for  $1 \leq j \leq k-1$ , there are  $j$  order tensor fields  $A_{\alpha\beta}(t)$ ,  $A_\alpha^1(t)$ , and  $A_\alpha^0(t)$  such that

$$\begin{aligned} \nabla^j u_{x_2} \circ \gamma(t) &= \sum_{\alpha+\beta \leq j-1} \partial_{x_1}^\alpha \partial_{x_2}^\beta f \circ \gamma(t) A_{\alpha\beta}(t) + \sum_{\alpha \leq j} q_1^{(\alpha)}(t) A_\alpha^1(t) \\ &\quad + \sum_{\alpha \leq j+1} q_0^{(\alpha)}(t) A_\alpha^0(t) \quad \text{for } t \in [0, t_1]. \end{aligned} \quad (3.41)$$

Let  $v \in C^{k,1}(\overline{P_1(\beta)})$  be the solution to (3.1) with the data (3.35). Then

$$p'(t) = \langle \nabla v_{x_2}(\beta(t)), \dot{\beta}(t) \rangle, \quad p''(t) = \langle \nabla^2 v_{x_2}(\beta(t)), \dot{\beta}(t) \otimes \dot{\beta}(t) \rangle + \langle \nabla v_{x_2}(\beta(t)), \ddot{\beta}(t) \rangle$$

for  $t \in [0, t_0]$ . Some computations show that

$$\begin{aligned} p^{(l)}(t) &= \langle \nabla^l v_{x_2}(\beta(t)), \dot{\beta}(t) \otimes \cdots \otimes \dot{\beta}(t) \rangle \\ &\quad + \sum_{j_1 + \cdots + j_i = l, 1 \leq i \leq l-1} a_{j_1 \dots j_i} \langle \nabla^i v_{x_2}(\beta(t)), \beta^{(j_1)}(t) \otimes \cdots \otimes \beta^{(j_i)}(t) \rangle \end{aligned} \quad (3.42)$$

for  $t \in [0, t_0]$ , and  $1 \leq l \leq k$ , where  $a_{j_1 \dots j_i}$  are positive integers. Then assumption (3.37) is stated as the following.

**Definition 3.1** Let the curves  $\beta$  and  $\gamma$  be of class  $C^{k,1}$ . Let  $q_0$  be of class  $C^{k,1}$  and  $p, q_1, f$  of class  $C^{k-1,1}$ , respectively. It is said that the  $k$ th order compatibility conditions hold at  $\gamma(0) = \beta(0)$  if  $|\gamma'(0)|^2 p(0) = \gamma_2'(0)q_0'(0) - \gamma_1'(0)q_1(0)$  and

$$p^{(l)}(0) = \langle \nabla^l u_{x_2} \circ \gamma(0), \dot{\beta}(0) \otimes \cdots \otimes \dot{\beta}(0) \rangle + \sum_{j_1 + \cdots + j_l = l, 1 \leq i \leq l-1} a_{j_1 \cdots j_l} \langle \nabla^i u_{x_2} \circ \gamma(0), \beta^{(j_1)}(0) \otimes \cdots \otimes \beta^{(j_l)}(0) \rangle \quad (3.43)$$

for  $1 \leq l \leq k-1$ , where  $\nabla^i u_{x_2} \circ \gamma(0)$  and  $a_{j_1 \cdots j_l}$  are given in (3.41) and (3.42), respectively.

**Proposition 3.9** Let the curves  $\beta$  and  $\gamma$  be of class  $C^{k,1}$ . Let  $q_0$  be of class  $C^{k,1}$  and  $p, q_1, f$  of class  $C^{k-1,1}$ , respectively. If  $k \geq 1$ , we assume that the  $k$ th order compatibility conditions hold at  $\gamma(0) = \beta(0)$ . Then problem (3.1) admits a unique solution  $w \in C^{k,1}(\overline{\Xi_1(\beta, \gamma)})$  with the data (3.30) and (3.31). Moreover, the following estimates hold

$$\|w\|_{C^{k,1}} \leq C(\|p\|_{C^{k-1,1}} + \|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}).$$

**Proof** The uniqueness and the estimate follows from Propositions 3.1, 3.2, and 3.6. It is remaining to show the existence. Let  $u$  and  $v$  be given in (3.36) with the corresponding boundary data. Let  $h$  be the solution to (3.1) on  $R(z, a, b)$  with the data

$$\begin{aligned} h(x_1, \gamma_2(0)) &= u(x_1, \gamma_2(0)) \quad \text{for } x_1 \in [\beta_1(t_0), \gamma_1(t_1)], \\ h(\beta_1(t_0), x_2) &= v(\beta_1(t_0), x_2) \quad \text{for } x_2 \in [\gamma_2(0), \beta_2(t_0)], \end{aligned}$$

where  $R(z, a, b)$  is given in (3.29). We now define

$$w(x) = \begin{cases} v & \text{for } x \in \overline{P_1(\beta)}, \\ u & \text{for } x \in \overline{E(\gamma)}, \\ h & \text{for } x \in \overline{R(z, a, b)}. \end{cases}$$

Then  $w$  is a solution to (3.1) with the data (3.30) and (3.31). Next we shall show  $w \in C^{k,1}(\overline{\Xi_1(\beta, \gamma)})$ .

We proceed by induction in  $k \geq 0$ . The definition of  $w$  guarantees  $w \in C^{0,1}(\overline{\Xi_1(\beta, \gamma)})$ . Let  $w \in C^{k,1}(\overline{\Xi_1(\beta, \gamma)})$ . Next we show that the  $k+1$ th order compatibility conditions imply  $w \in C^{k+1,1}(\overline{\Xi_1(\beta, \gamma)})$ . For this purpose it is enough to show that  $w$  is  $C^{k+1}$  on the segments

$$\vartheta = \{ (x_1, \beta_2(0)), (x_2, \beta_1(t_0)) \mid x_1 \in [\beta_1(0), \gamma_1(t_0)], x_2 \in [\gamma_2(0), \beta_2(t_0)] \}.$$

By the induction assumptions, we have

$$\partial_{x_1}^i \partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_1}^i \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for } \beta_1(0) \leq x_1 \leq \beta_1(t_0), \quad (3.44)$$

for  $0 \leq i+j \leq k$ . Next we show that (3.44) are true with  $i+j = k+1$ . Since  $v(x_1, \beta_2(0)) = u(x_1, \beta_2(0))$  for all  $x_1 \in [\beta_1(0), \beta_1(t_0)]$ , it follows that

$$\partial_{x_1}^{k+1} v(x_1, \beta_2(0)) = \partial_{x_1}^{k+1} u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(t_0)].$$

Let  $i+j = k+1$  with  $j \geq 1$ . If  $i \geq 1$ , then  $j = k+1-i \leq k$  and, by the induction assumptions,

$$\partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(t_0)],$$

which yield

$$\partial_{x_1}^i \partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_1}^i \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(t_0)]. \quad (3.45)$$

Next we check the case of  $i = 0$  and  $j = k+1$ .

Using (3.1), we have

$$\begin{aligned} (\partial_{x_2}^{k+1} v(x_1, \beta_2(0)))_{x_1} &= \partial_{x_2}^k (v_{x_1 x_2})(x_1, \beta_2(0)) = \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1} + X_2 v_{x_2}](x_1, \beta_2(0)) \\ &= X_2(x_1, \beta_2(0)) \partial_{x_2}^{k+1} v(x_1, \beta_2(0)) + \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1}](x_1, \beta_2(0)) \\ &\quad + [\sum_{i=1}^k C_k^i \partial_{x_2}^i X_2 \partial_{x_2}^{k-i+1} v](x_1, \beta_2(0)). \end{aligned} \quad (3.46)$$

Let

$$\rho(x_1) = \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1}](x_1, \beta_2(0)) + [\sum_{i=1}^k C_k^i \partial_{x_2}^i X_2 \partial_{x_2}^{k-i+1} v](x_1, \beta_2(0))$$

for  $x_1 \in [\beta_1(0), \beta_1(t_0)]$ . It follows from (3.46) that  $\tau(x_1) = \partial_{x_2}^{k+1} v(x_1, \beta_2(0))$  is the solution to problem

$$\begin{cases} \tau'(x_1) = X_2(x_1, \beta_2(0))\tau(x_1) + \rho(x_1) & \text{for } x_1 \in [\beta_1(0), \beta_1(t_0)], \\ \tau(\beta_1(0)) = \partial_{x_2}^{k+1} v(\beta_1(0)). \end{cases} \quad (3.47)$$

Moreover, the induction assumptions,  $w \in C^{k,1}(\overline{\Xi_1(\beta, \gamma)})$ , yield

$$\langle \nabla^i v_{x_2}(z), \beta^{(j_1)}(0) \otimes \cdots \otimes \beta^{(j_i)}(0) \rangle = \langle \nabla^i u_{x_2}(z), \beta^{(j_1)}(0) \otimes \cdots \otimes \beta^{(j_i)}(0) \rangle$$

for  $j_1 + \cdots + j_i = l$ ,  $1 \leq i \leq l-1$ , and  $1 \leq l \leq k$ . Then the  $k+1$ th order compatibility conditions imply

$$\langle \nabla^k v_{x_2} \circ \beta(0), \dot{\beta}(0) \otimes \cdots \otimes \dot{\beta}(0) \rangle = \langle \nabla^k u_{x_2} \circ \gamma(0), \dot{\beta}(0) \otimes \cdots \otimes \dot{\beta}(0) \rangle.$$

Using (3.45) and  $\beta_2'(0) > 0$ , we obtain

$$\partial_{x_2}^{k+1} u \circ \gamma(0) = \partial_{x_2}^{k+1} v \circ \beta(0). \quad (3.48)$$

In addition, it follows from the induction assumptions and (3.45) that

$$\rho(x_1) = \partial_{x_2}^k [f + f_0 u + X_1 u_{x_1}](x_1, \beta_2(0)) + \left[ \sum_{i=1}^k C_k^i \partial_{x_2}^i X_2 \partial_{x_2}^{k-i+1} u \right](x_1, \beta_2(0)),$$

for  $x_1 \in [\beta_1(0), \beta_1(t_0)]$ . By a similar computation as in (3.46),  $\partial_{x_2}^{k+1} u(x_1, \beta_2(0))$  is also a solution to problem (3.47) with the same initial data (3.48). The uniqueness of solutions of problem (3.47) yields

$$\partial_{x_2}^{k+1} v(x_1, \beta_2(0)) = \partial_{x_2}^{k+1} u(x_1, \beta_2(0)), \quad x_1 \in [\beta_1(0), \beta_1(t_1)].$$

Thus  $w$  is  $C^{k+1}$  on the segment

$$\{(x_1, \beta_2(0)) \mid x_1 \in [\beta_1(0), \beta_1(t_0)]\}.$$

A similar argument shows that  $w$  is  $C^{k+1}$  on the rest of  $\vartheta$ . The induction is complete.  $\square$

By similar arguments, we have Propositions 3.10-3.12 below. The details are omitted.

**Proposition 3.10** *Let the curves  $\beta$  and  $\gamma$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Let  $q_0$  be of class  $W^{2,2}$  and  $p, q_1, f$  of class  $W^{1,2}$ , respectively, such that the 1th order compatibility conditions hold true at  $\gamma(0)$ . Then problem (3.1) admits a unique solution  $w \in W^{2,2}(\Xi_1(\beta, \gamma))$  with the data (3.30) and (3.31). Moreover, the following estimates hold*

$$\|w\|_{W^{2,2}} \leq C(\|p\|_{W^{1,2}} + \|q_0\|_{W^{2,2}} + \|q_1\|_{W^{1,2}} + \|f\|_{W^{1,2}}).$$

Let

$$\Gamma_i(\beta, w) = \int_0^{t_0} |[w_{x_i} \circ \beta(s)]'|^2 (t_0 - s) ds \quad \text{for } s \in (0, t_0), \quad i = 1, 2. \quad (3.49)$$

**Proposition 3.11** *Let the curves  $\beta$  and  $\gamma$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there are  $0 < c_1 < c_2$  such that for all solutions  $w \in W^{2,2}(\Xi_1(\beta, \gamma))$  to problem (3.1)*

$$c_1[\Gamma(\gamma, w) + \Gamma_2(\beta, w)] \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2[\Gamma(\gamma, w) + \Gamma_2(\beta, w) + \|f\|_{W^{1,2}}^2],$$

where  $\Gamma(\gamma, w)$  is given in (3.18).

**Proposition 3.12** *The corresponding results as in Propositions 3.9, 3.10, and 3.11 hold where  $\Xi_1(\beta, \gamma)$  and  $\Gamma_2(\beta, w)$  are replaced with  $\Xi_2(\beta, \gamma)$  and  $\Gamma_1(\beta, w)$ , respectively.*

### 3.4 Region $\Phi(\beta, \gamma, \hat{\beta})$

Let  $\beta : [0, t_0] \rightarrow \mathbb{R}^2$ ,  $\gamma : [0, t_1] \rightarrow \mathbb{R}^2$ , and  $\hat{\beta} : [0, t_2] \rightarrow \mathbb{R}^2$  be noncharacteristic curves with  $\beta(0) = \gamma(0)$  and  $\gamma(t_1) = \hat{\beta}(0)$  such that

$$\begin{aligned} \gamma_1(t_1) \geq \beta_1(t_0), \quad \gamma'_1(t) > 0, \quad \gamma'_2(t) < 0, \quad \beta'_1(t) > 0, \quad \beta'_2(t) > 0, \\ \gamma_2(t_1) \geq \hat{\beta}_2(t_2), \quad \hat{\beta}'_1(t) > 0, \quad \hat{\beta}'_2(t) > 0. \end{aligned}$$

We define

$$\overline{\Phi(\beta, \gamma, \hat{\beta})} = \overline{\Xi_1(\beta, \gamma)} \cup \overline{R(z, a, b)} \cup \overline{P_2(\hat{\beta})},$$

where  $\Xi_1(\beta, \gamma)$ ,  $R(z, a, b)$ ,  $P_2(\hat{\beta})$  are given in (3.29), (3.6), and (3.26), respectively, with  $z = (\gamma_1(t_1), \hat{\beta}_2(t_2))$ ,  $a = \hat{\beta}_1(t_2) - \gamma_1(t_1)$ , and  $b = \beta_2(t_0) - \hat{\beta}_2(t_2)$ . Consider the boundary data

$$w_{x_2} \circ \beta(t) = p_1(t), \quad t \in [0, t_0], \quad w_{x_1} \circ \hat{\beta}(t) = p_2(t), \quad t \in (0, t_2), \quad (3.50)$$

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F}\hat{\gamma} \rangle \circ \gamma(t) = q_1(t) \quad \text{for } t \in (0, t_1). \quad (3.51)$$

By similar arguments as for  $\Xi_1(\beta, \gamma)$ , we have Propositions 3.13-3.15 below. The details are omitted.

**Proposition 3.13** *Let the curves  $\beta$ ,  $\gamma$ , and  $\hat{\beta}$  be of class  $C^{k,1}$ . Let  $q_0$  be of class  $C^{k,1}$ , and  $p_1, p_2, q_1, f$  of class  $C^{k-1,1}$  such that the  $k$ th order compatibility conditions hold true at  $\gamma(0)$  and  $\gamma(t_1)$ , respectively. Then problem (3.1) admits a unique solution  $w \in C^{k,1}(\overline{\Phi(\beta, \gamma, \hat{\beta})})$  with the data (3.50) and (3.51). Moreover, the following estimates hold*

$$\|w\|_{C^{k,1}}^2 \leq C(\|p_1\|_{C^{k-1,1}}^2 + \|p_2\|_{C^{k-1,1}}^2 + \|q_0\|_{C^{k,1}}^2 + \|q_1\|_{C^{k-1,1}}^2 + \|f\|_{C^{k-1,1}}^2).$$

**Proposition 3.14** *Let the curves  $\beta$ ,  $\gamma$ , and  $\hat{\beta}$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Let  $q_0$  be of class  $W^{2,2}$ , and  $p_1, p_2, q_1, f$  of class  $W^{1,2}$ , such that the 1th order compatibility conditions hold true at  $\gamma(0)$  and  $\gamma(t_1)$ , respectively. Then problem (3.1) admits a unique solution  $w \in W^{2,2}(\overline{\Phi(\beta, \gamma, \hat{\beta})})$  with the data (3.50) and (3.51). Moreover, the following estimates hold*

$$\|w\|_{W^{2,2}}^2 \leq C(\|p_1\|_{W^{1,2}}^2 + \|p_2\|_{W^{1,2}}^2 + \|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2).$$

**Proposition 3.15** *Let the curves  $\beta$ ,  $\gamma$ , and  $\hat{\beta}$  be of class  $C^1$ . Let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there are  $0 < c_1 < c_2$  such that for all solutions  $w \in W^{2,2}(\overline{\Phi(\beta, \gamma, \hat{\beta})})$  to problem (3.1)*

$$c_1[\Gamma(\gamma, w) + \Gamma_1(\hat{\beta}, w) + \Gamma_2(\beta, w)] \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2[\Gamma(\gamma, w) + \Gamma_1(\hat{\beta}, w) + \Gamma_2(\beta, w) + \|f\|_{W^{1,2}}^2],$$

where  $\Gamma(\gamma, w)$ ,  $\Gamma_1(\hat{\beta}, w)$ , and  $\Gamma_2(\beta, w)$  are given in (3.18) and (3.49), respectively.

## 4 Solvability for Hyperbolic Surfaces

Let  $M \subset \mathbb{R}^3$  be a hyperbolic surface with the normal field  $\vec{n}$  and let  $\Omega \subset M$  be a noncharacteristic region, where

$$\Omega = \{ \alpha(t, s) \mid (t, s) \in (0, a) \times (0, b) \}.$$

We consider solvability of problem under appropriate part boundary data

$$\langle D^2w, Q^*\Pi \rangle = f + f_0w + X(w) \quad \text{for } x \in \Omega, \quad (4.1)$$

where  $f_0$  is a function on  $M$  and  $X \in T(M)$  is a vector field on  $M$ . Clearly, equation (2.25) takes the form of (4.1).

To set up boundary data, we consider some boundary operators. Let  $x \in \partial\Omega$  be given.  $\mu \in T_xM$  with  $|\mu| = 1$  is said to be the *noncharacteristic normal* outside  $\Omega$  if there is a curve  $\zeta : (0, \varepsilon) \rightarrow \Omega$  such that

$$\zeta(0) = x, \quad \zeta'(0) = -\mu, \quad \Pi(\mu, X) = 0 \quad \text{for } X \in T_x(\partial\Omega).$$

Let  $\mu$  be the the noncharacteristic normal field along  $\partial\Omega$ . Let the linear operator  $Q : T_xM \rightarrow T_xM$  be given in (2.2) for  $x \in M$ . Recall that the shape operator  $\nabla\vec{n} : T_xM \rightarrow T_xM$  is defined by  $\nabla\vec{n}X = \nabla_X\vec{n}(x)$  for  $X \in T_xM$ . We define boundary operators  $\mathcal{T}_i : T_xM \rightarrow T_xM$  by

$$\mathcal{T}_iX = \frac{1}{2} \left[ X + (-1)^i \chi(\mu, X) \rho(X) Q \nabla\vec{n}X \right] \quad \text{for } X \in T_xM, \quad i = 1, 2, \quad (4.2)$$

where

$$\chi(\mu, X) = \text{sign det}(\mu, X, \vec{n}), \quad \rho(X) = \frac{1}{\sqrt{-\kappa}} \text{sign } \Pi(X, X), \quad (4.3)$$

and  $\text{sign}$  is the sign function.

We shall consider the part boundary data

$$\langle Dw, \mathcal{T}_2\alpha_s \rangle \circ \alpha(0, s) = p_1(s), \quad \langle Dw, \mathcal{T}_2\alpha_s \rangle \circ \alpha(a, s) = p_2(s) \quad \text{for } s \in (0, b), \quad (4.4)$$

$$w \circ \alpha(t, 0) = q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw, (\mathcal{T}_2 - \mathcal{T}_1)\alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for } t \in (0, a). \quad (4.5)$$

To have a smooth solution, we need some kind of *compatibility conditions* as follows.

Let  $A$  and  $B$  be  $k$ th order and  $m$ th order tensor fields on  $M$ , respectively, with  $k \geq m$ . We define  $A(i-)B$  to be a  $(k - m)$ th order tensor field by

$$A(i-)B(X_1, \dots, X_{k-m}) = \langle i_{X_{k-m}} \cdots i_{X_1} A, B \rangle(x) \quad \text{for } x \in M, \quad (4.6)$$

where  $X_1, \dots, X_{k-m}$  are vector fields on  $M$ .

For convenience, we assume that

$$|\alpha_t(t, 0)| = 1 \quad \text{for } t \in [0, a].$$

Then  $Q\alpha_t, \alpha_t$  forms an orthonormal basis of  $T_{\alpha(t,0)}M$  with the positive orientation for all  $t \in [0, a]$  and

$$Q\nabla\vec{n}\alpha_t = \Pi(\alpha_t, \alpha_t)Q\alpha_t - \Pi(\alpha_t, Q\alpha_t)\alpha_t \quad \text{for } t \in [0, a].$$

Let  $k \geq 1$  be an integer. First, we assume that  $w$  is a  $C^{k,1}$  solution to (4.1) in a neighborhood of the curve  $\alpha(t, 0)$  with the data (4.5). Then

$$Dw(\alpha(t, 0)) = B_1(t)q'_0(t) + C_0(t)q_1(t) \quad \text{for } t \in [0, a],$$

where

$$B_1(t) = [\alpha_t + \frac{\Pi(\alpha_t, Q\alpha_t)}{\Pi(\alpha_t, \alpha_t)}Q\alpha_t], \quad C_0(t) = \frac{\sqrt{2}}{\varrho(\alpha_t)\Pi(\alpha_t, \alpha_t)}Q\alpha_t,$$

are vector fields along the curve  $\alpha(t, 0)$ , from which we obtain

$$D_{\alpha_t}Dw(\alpha(t, 0)) = D_{\alpha_t}B_1q'_0(t) + B_1(t)q''_0(t) + D_{\alpha_t}C_0q_1(t) + C_0(t)q'_1(t).$$

Using (4.1) and the above formula, we compute along the curve  $\alpha(t, 0)$  to have

$$\begin{aligned} D^2w(Q\alpha_t, Q\alpha_t)\Pi(\alpha_t, \alpha_t) &= f + f_0w + \langle Dw, X \rangle - \langle D_{\alpha_t}Dw, \alpha_t \rangle \Pi(Q\alpha_t, Q\alpha_t) \\ &\quad + 2\langle D_{\alpha_t}Dw, Q\alpha_t \rangle \Pi(Q\alpha_t, \alpha_t) \\ &= f + f_0q_0(t) + [\langle X, B_1(t) \rangle + \langle D_{\alpha_t}B_1, Z(t) \rangle]q'_0(t) + \langle B_1(t), Z(t) \rangle q''_0(t) \\ &\quad + [\langle X, C_0(t) \rangle + \langle D_{\alpha_t}C_0, Z(t) \rangle]q_1(t) + \langle C_0(t), Z(t) \rangle q'_1(t) \quad \text{for } t \in [0, a], \end{aligned} \quad (4.7)$$

where

$$Z(t) = 2\Pi(Q\alpha_t, \alpha_t)Q\alpha_t - \Pi(Q\alpha_t, Q\alpha_t)\alpha_t.$$

Since  $\Pi(\alpha_t, \alpha_t) \neq 0$  for all  $t \in [0, a]$ , we have obtained two order tensor fields,  $A^2(t), B_i^2(t)$ , and  $C_i^2(t)$ , that are given by  $f_0, X, \Pi, Q\alpha_t, \alpha_t$ , and their differentials, such that

$$D^2w(\alpha(t, 0)) = A^2(t)f + \sum_{i=0}^2 B_i^2(t)q_0^{(i)}(t) + \sum_{i=0}^1 C_i^2(t)q_1^{(i)}(t) \quad \text{for } t \in [0, a].$$

By repeating the above procedure, we obtain  $(k+i)$ th order tensors fields  $A_i^{k+i}(t)$ , and  $k$ th order tensor fields  $B_i^k(t), C_i^k(t)$ , such that

$$D^k w(\alpha(t, 0)) = \mathcal{Q}_k(q_0, q_1, f)(t) \quad \text{for } t \in [0, a],$$

where

$$\mathcal{Q}_k(q_0, q_1, f)(t) = \sum_{i=0}^{k-2} A_i^{k+i}(t)(i-)D^i f(\alpha(t, 0)) + \sum_{i=0}^k B_i^k(t)q_0^{(i)}(t) + \sum_{i=0}^{k-1} C_i^k(t)q_1^{(i)}(t) \quad (4.8)$$

for  $t \in [0, a]$  and  $k \geq 2$ , where “ $(i-)$ ” is defined in (4.6).

**Definition 4.1** Let  $q_0$  be of class  $C^{k,1}$ , and  $p_1, p_2, q_1, f$  of class  $C^{k-1,1}$  to be said to satisfy the  $k$ th order compatibility conditions at  $\alpha(0,0)$  and  $\alpha(a,0)$  if

$$p_j(t_j) = \langle B_1(t_j), \mathcal{T}_2 \alpha_s \rangle q_0'(t_j) + \langle C_0(t_j), \mathcal{T}_2 \alpha_s \rangle q_1(t_j), \quad (4.9)$$

$$\begin{aligned} p_j^{(l)}(0) &= \langle \mathcal{Q}_l(q_0, q_1, f)(t_j), \dot{\gamma}_j(0) \otimes \cdots \otimes \dot{\gamma}_j(0) \rangle \\ &+ \sum_{j_1 + \cdots + j_l = l, 1 \leq i \leq l-1} a_{j_1 \cdots j_l} \langle \mathcal{Q}_i(q_0, q_1, f)(t_j), \gamma_j^{(j_1)}(0) \otimes \cdots \otimes \gamma_j^{(j_l)}(0) \rangle \end{aligned} \quad (4.10)$$

for  $1 \leq l \leq k-1$ , where  $a_{j_1 \cdots j_l}$  are positive integers given in (3.42),  $j = 1, 2$ ,  $\gamma_1(s) = \alpha(0, s)$ ,  $\gamma_2(s) = \alpha(a, s)$ ,  $t_1 = 0$ , and  $t_2 = a$ .

Our main task in this section is to establish the following.

**Theorem 4.1** Let  $\Omega$  be a noncharacteristic region of class  $C^{m+2,1}$  and let  $f_0$  and  $X$  be of class  $C^{m-1,1}$ . Let  $q_0$  be of class  $C^{m,1}$ , and  $p_1, p_2, q_1, f$  be of  $C^{m-1,1}$ , respectively. If  $m \geq 1$ , we assume that the  $m$ th compatibility conditions holds. Then there is a unique solution  $w \in C^{m,1}(\bar{\Omega})$  to problem (4.1) with the data (4.4) and (4.5) satisfying

$$\begin{aligned} \|w\|_{C^{m,1}(\bar{\Omega})} &\leq C(\|q_1\|_{C^{m-1,1}[0,a]} + \|q_0\|_{C^{m,1}[0,a]} + \|p_1\|_{C^{m-1,1}[0,b]} \\ &+ \|p_2\|_{C^{m-1,1}[0,b]} + \|f\|_{C^{m-1,1}(\bar{\Omega})}). \end{aligned} \quad (4.11)$$

**Remark 4.1** If  $p_1, p_2 \in C_0^{m-1,1}(0, b)$ ,  $q_0 \in C_0^{m,1}(0, a)$ ,  $q_1 \in C_0^{m-1,1}(0, a)$ , and  $f \in C_0^{m-1,1}(\Omega)$  for an integer  $m \geq 0$ , then the  $m$ th order compatibility conditions are clearly true.

**Theorem 4.2** Let  $\Omega$  be a noncharacteristic region of class  $C^{2,1}$  and let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Let  $q_0$  be of class  $W^{2,2}$ , and  $p_1, p_2, q_1, f$  of class  $W^{1,2}$  to satisfy the 1th order compatibility conditions. Then there is a unique solution  $w \in W^{2,2}(\Omega)$  to problem (4.1) with the data (4.4) and (4.5). Moreover, there is  $C > 0$ , in dependent of solution  $w$ , such that

$$\begin{aligned} \|w\|_{W^{2,2}(\Omega)}^2 &\leq C(\|q_0\|_{W^{2,2}(0,a)}^2 + \|q_1\|_{W^{1,2}(0,a)}^2 + \|p_1\|_{W^{1,2}(0,b)}^2 \\ &+ \|p_2\|_{W^{1,2}(0,b)}^2 + \|f\|_{W^{1,2}(\Omega)}^2). \end{aligned} \quad (4.12)$$

We define

$$\Gamma(\Omega, w) = \int_0^b (|p_1'(s)|^2 + |p_2'(s)|^2)(b-s)ds + \Gamma(\alpha(\cdot, 0), w), \quad (4.13)$$

where  $p_1, p_2$  are given in (4.4), and

$$\begin{aligned} \Gamma(\alpha(\cdot, 0), w) &= \sum_{j=0}^1 \|\nabla^j w \circ \alpha(\cdot, 0)\|_{L^2(0,a)}^2 \\ &+ \int_0^a [|D^2 w(\mathcal{T}_1 \alpha_t, \mathcal{T}_1 \alpha_t)|^2 t + |D^2 w(\mathcal{T}_2 \alpha_t, \mathcal{T}_2 \alpha_t)|^2 (a-t)] dt. \end{aligned} \quad (4.14)$$

**Theorem 4.3** *Let  $\Omega$  be a noncharacteristic region of class  $C^{2,1}$  and let  $f_0$  and  $X$  be of class  $C^{0,1}$ . Then there are  $0 < c_1 < c_2$  such that for all solutions  $w \in W^{2,2}(\Omega)$  to problem (4.1)*

$$c_1 \Gamma(\Omega, w) \leq \|w\|_{W^{2,2}(\Omega)}^2 + \|f\|_{W^{1,2}(\Omega)}^2 \leq c_2 (\|f\|_{W^{1,2}(\Omega)}^2 + \Gamma(\Omega, w)). \quad (4.15)$$

Next, we assume that  $f = 0$  to consider problem

$$\langle D^2 w, Q^* \Pi \rangle = f_0 w + X(w) \quad \text{for } x \in \Omega. \quad (4.16)$$

Denote by  $\Upsilon(\Omega)$  all the solutions  $w \in W^{2,2}(\Omega)$  to problem (4.16). For  $w \in \Upsilon(\Omega)$ , we let

$$\Gamma(w) = \int_0^b (|p'_1(s)|^2 + |p'_2(s)|^2) ds + \|q_0\|_{W^{2,2}(0,a)}^2 + \|q_1\|_{W^{1,2}(0,a)}^2,$$

where  $p_1, p_2, q_0,$  and  $q_1$  are given in (4.4) and (4.5), respectively. We define

$$\mathcal{H}(\Omega) = \{ w \in \Upsilon(\Omega) \text{ with the 1th order compatibility conditions} \mid \Gamma(w) < \infty \}.$$

**Theorem 4.4** *Let  $\Omega$  be a noncharacteristic region of class  $C^{2,1}$  and  $X$  of class  $C^{0,1}$ . For each  $w \in \Upsilon(\Omega)$ , there exists a sequence  $w_n \in \mathcal{H}(\Omega)$  such that*

$$\lim_{n \rightarrow \infty} \|w_n - w\|_{W^{2,2}(\Omega)} = 0.$$

The remains of this section is devoted to the proofs of Theorems 4.1-4.4. The proofs of Theorems 4.1-4.2 and 4.3-4.4 are given after Lemma 4.5 and Lemma 4.7, respectively.

We shall solve (4.1) locally in asymptotic coordinate systems and then paste the local solutions together. A chart  $\psi(p) = (x_1, x_2)$  on  $M$  is said to be an *asymptotic coordinate system* if

$$\Pi(\partial x_1, \partial x_1) = \Pi(\partial x_2, \partial x_2) = 0. \quad (4.17)$$

Let  $p \in M$ . Then  $\kappa(p) < 0$  if and only if there exists an asymptotic coordinate system at  $p$  ([10]). In this system

$$\kappa(q) = -\frac{\Pi^2(\partial x_1, \partial x_2)}{\det G}, \quad \det G = |\partial x_1|^2 |\partial x_2|^2 - \langle \partial x_1, \partial x_2 \rangle^2.$$

In an asymptotic coordinate system, equation (4.1) takes a normal form. We have the following.

**Proposition 4.1** *Let  $M$  be a hyperbolic orientated surface and let  $\psi(p) = (x_1, x_2) : U(\subset M) \rightarrow \mathbb{R}^2$  be an asymptotic coordinate system on  $M$  with the positive orientation. Then*

$$\langle D^2 w, Q^* \Pi \rangle = \pm 2 \sqrt{\frac{-\kappa}{\det G}} w_{x_1 x_2}(x) + \text{the first order terms}, \quad (4.18)$$

where  $w(x) = w \circ \psi^{-1}(x)$  and the sign takes  $-$  if  $\Pi(\partial x_1, \partial x_2) > 0$  and  $+$  if  $\Pi(\partial x_1, \partial x_2) < 0$ , respectively.

**Proof** Let  $p \in U$  be fixed. Let  $\alpha_i = (\alpha_{i1}, \alpha_{i2})^T \in \mathbb{R}^2$  be such that

$$(\alpha_1, \alpha_2) \in \text{SO}(2), \quad \det(\alpha_1, \alpha_2) = 1, \quad G(p)\alpha_i = \eta_i \alpha_i \quad \text{for } i = 1, 2,$$

where  $\eta_i > 0$  are the eigenvalues of the matrix  $G(p)$ . Set

$$E_i = \alpha_{i1} \partial x_1 + \alpha_{i2} \partial x_2 \quad \text{for } i = 1, 2. \quad (4.19)$$

Since

$$\langle E_i, E_j \rangle = \alpha_i^T G(p) \alpha_j = \eta_j \delta_{ij} \quad \text{for } 1 \leq i, j \leq 2,$$

$\frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}}$  forms an orthonormal basis of  $M_p$ . Moreover,  $\frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}}$  is of the positive orientation due to

$$\det \left( \frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}}, \vec{n} \right) = \det \left[ \left( \partial x_1, \partial x_2, \vec{n} \right) \begin{pmatrix} \frac{\alpha_{11}}{\sqrt{\eta_1}} & \frac{\alpha_{21}}{\sqrt{\eta_2}} & 0 \\ \frac{\alpha_{12}}{\sqrt{\eta_1}} & \frac{\alpha_{22}}{\sqrt{\eta_2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = \det \left( \partial x_1, \partial x_2, \vec{n} \right) \frac{1}{\sqrt{\eta_1 \eta_2}} = 1.$$

It follows from (2.2) that

$$Q \frac{E_1}{\sqrt{\eta_1}} = -\frac{E_2}{\sqrt{\eta_2}}, \quad Q \frac{E_2}{\sqrt{\eta_2}} = \frac{E_1}{\sqrt{\eta_1}}.$$

Using the above relations and the formulas (4.17), we have at  $p$

$$\begin{aligned} \eta_1 \eta_2 \langle D^2 w, Q^* \Pi \rangle &= D^2 w(E_1, E_1) \Pi(E_2, E_2) - 2D^2 w(E_1, E_2) \Pi(E_1, E_2) \\ &\quad + D^2 w(E_2, E_2) \Pi(E_1, E_1) \\ &= 2[\alpha_{21} \alpha_{22} D^2 w(E_1, E_1) - (\alpha_{11} \alpha_{22} + \alpha_{12} \alpha_{21}) D^2 w(E_1, E_2) \\ &\quad + \alpha_{11} \alpha_{12} D^2 w(E_2, E_2)] \Pi(\partial x_1, \partial x_2) \\ &= 2D^2 w(\alpha_{21} E_1 - \alpha_{11} E_2, \alpha_{22} E_1 - \alpha_{12} E_2) \Pi(\partial x_1, \partial x_2) \\ &= -2(\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21})^2 D^2 w(\partial x_2, \partial x_2) \Pi(\partial x_1, \partial x_2) \\ &= -2[w_{x_1 x_2} - D_{\partial x_1} \partial x_2(w)] \Pi(\partial x_1, \partial x_2), \end{aligned} \quad (4.20)$$

where the formula

$$\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} = \det(\alpha_1, \alpha_2) = 1,$$

has been used.

$$(4.18) \text{ follows from (4.20) since } \kappa = -\frac{\Pi^2(\partial x_1, \partial x_2)}{\eta_1 \eta_2}. \quad \square$$

**Lemma 4.1** *There is a  $\sigma_0 > 0$  such that, for all  $p \in \overline{\Omega}$ , there exist asymptotic coordinate systems  $\psi : B(p, \sigma_0) \rightarrow \mathbb{R}^2$  with  $\psi(p) = (0, 0)$ , where  $B(p, \sigma_0)$  is the geodesic plate in  $M$  centered at  $p$  with radius  $\sigma_0$ .*

**Proof.** For  $p \in \overline{\Omega}$ , let  $\sigma(p)$  denote the least upper bound of the radii  $\sigma$  for which an asymptotic systems  $\psi = x : B(p, \sigma) \rightarrow \mathbb{R}^2$  with  $\psi(p) = (0, 0)$  exists. From the existence of local asymptotic coordinate systems,  $\sigma(p) > 0$  for all  $p \in \overline{\Omega}$ . Let  $p, q \in \overline{\Omega}$ , and  $q \in B(p, \sigma(p))$ . Let

$$\sigma_1(q) = \inf_{z \in M, d(z, p) = \sigma(p)} d(q, z),$$

where  $d(\cdot, \cdot)$  is the distance function on  $M \times M$  in the induced metric. Then  $\sigma_1(q) > 0$  and  $B(q, \sigma_1(q)) \subset B(p, \sigma(p))$ , since  $q \in B(p, \sigma(p))$ .

For any  $0 < \hat{\sigma} < \sigma_1(q)$ ,  $\overline{B(q, \hat{\sigma})} \subset B(p, \sigma(p))$ . Thus, there is a  $0 < \sigma < \sigma(p)$  such that  $\overline{B(q, \hat{\sigma})} \subset B(p, \sigma)$ . Let  $\psi = x : B(p, \sigma) \rightarrow \mathbb{R}^2$  be an asymptotic system with  $\psi(p) = (0, 0)$ . Set  $\hat{\psi}(z) = \psi(z) - \psi(q)$  for  $z \in B(q, \hat{\sigma})$ . Then  $\hat{\psi} : B(q, \hat{\sigma}) \rightarrow \mathbb{R}^2$  is an asymptotic coordinate system with  $\hat{\psi}(q) = (0, 0)$ , that is,

$$\sigma(q) \geq \sigma_1(q) \quad \text{for } q \in B(p, \sigma(p)).$$

Thus,  $\sigma(p)$  is lower semi-continuous in  $\overline{\Omega}$  and  $\min_{p \in \overline{\Omega}} \sigma(p) > 0$  since  $\overline{\Omega}$  is compact.  $\square$

**Lemma 4.2** *Let  $\gamma : [0, a] \rightarrow M$  be a regular curve without self intersection points. Then there is a  $\sigma_0 > 0$  such that, for all  $p \in \{\gamma(t) \mid t \in (0, a)\}$ ,  $S(p, \sigma_0)$  has at most two intersection points with  $\{\gamma(t) \mid t \in [0, a]\}$ , where  $S(p, \sigma_0)$  is the geodesic circle centered at  $p$  with radius  $\sigma_0$ . If  $p = \gamma(0)$ , or  $\gamma(a)$ , then  $S(p, \sigma_0)$  has at most one intersection point with  $\{\gamma(t) \mid t \in [0, a]\}$ .*

**Proof.** By contradiction. Let the claim in the lemma be not true. For each integer  $k \geq 1$ , there exists  $t_k < t_k^1 < t_k^2$  (or  $t_k > t_k^1 > t_k^2$ ) in  $[0, a]$  such that

$$d(\gamma(t_k), \gamma(t_k^1)) = d(\gamma(t_k), \gamma(t_k^2)) = \frac{1}{k} \quad \text{for } k \geq 1. \quad (4.21)$$

We may assume that

$$t_k \rightarrow t^0, \quad t_k^1 \rightarrow t^1, \quad t_k^2 \rightarrow t^2 \quad \text{as } k \rightarrow \infty,$$

for certain points  $t^0, t^1, t^2 \in [0, a]$ . Then  $0 \leq t^0 \leq t^1 \leq t^2 \leq a$  and

$$\gamma(t^0) = \gamma(t^1) = \gamma(t^2).$$

The assumption that the curve  $\gamma$  has no self intersection point implies that

$$t^0 = t^1 = t^2.$$

For  $k \geq 1$ , let

$$f_k(t) = \frac{1}{2} \rho_k^2(\gamma(t)), \quad \text{for } t \in [0, a],$$

where  $\rho_k(p) = d(\gamma(t_k), p)$  for  $p \in M$ . It follows from (4.21) that there is a  $\zeta_k$  with  $t_k^1 < \zeta_k < t_k^2$  such that

$$f'_k(\zeta_k) = 0.$$

On the other hand, the formula  $f'_k(t) = \rho_k(\gamma(t))\langle D\rho_k(\gamma(t)), \dot{\gamma}(t) \rangle$  implies that  $f'_k(t_k) = 0$ . Thus, we obtain  $\eta_k \in (t_k, \zeta_k)$  such that

$$f''_k(\eta_k) = 0 \quad \text{for } k \geq 1.$$

Since

$$f''_k(t) = D(\rho_k D\rho_k)(\dot{\gamma}(t), \dot{\gamma}(t)) + \rho_k(\gamma(t))\langle D\rho_k(\gamma(t)), D_{\dot{\gamma}(t)}\dot{\gamma} \rangle,$$

we have

$$|\dot{\gamma}(t^0)|^2 = f''_0(t^0) = \lim_{k \rightarrow \infty} f''_k(\eta_k) = 0,$$

which contradicts the regularity of the curve  $\gamma$ , where

$$f_0(t) = \frac{1}{2}d^2(\gamma(t^0), \gamma(t)) \quad \text{for } t \in [0, a].$$

□

We need the following.

**Proposition 4.2** (i)  $\det(Q\nabla\vec{n}X, X, \vec{n}(x)) = \Pi(X, X)(x)$  for  $X \in T_xM$ ,  $x \in M$ .  
(ii)  $\Pi(Q\nabla\vec{n}X, Q\nabla\vec{n}X) = \kappa\Pi(X, X)$  for  $X \in T_xM$ ,  $x \in M$ .

**Proof** Let  $x \in M$  be given. Let  $e_1, e_2$  be an orthonormal basis of  $T_xM$  with the positive orientation such that

$$\Pi(e_i, e_j)(x) = \lambda_i \delta_{ij} \quad \text{for } 1 \leq i, j \leq 2. \quad (4.22)$$

Then

$$\det(Q\nabla\vec{n}X, X, \vec{n}) = \det(e_1, e_2, \vec{n}) \begin{pmatrix} \lambda_2 \langle X, e_2 \rangle & \langle X, e_1 \rangle & 0 \\ -\lambda_1 \langle X, e_1 \rangle & \langle X, e_2 \rangle & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Pi(X, X). \quad (4.23)$$

In addition, using (4.22), we have

$$\begin{aligned} \Pi(Q\nabla\vec{n}X, Q\nabla\vec{n}X) &= \Pi(-\lambda_1 \langle X, e_1 \rangle e_2 + \lambda_2 \langle X, e_2 \rangle e_1, -\lambda_1 \langle X, e_1 \rangle e_2 + \lambda_2 \langle X, e_2 \rangle e_1) \\ &= \lambda_1^2 \lambda_2 \langle X, e_1 \rangle^2 + \lambda_2^2 \lambda_1 \langle X, e_2 \rangle^2 = \kappa \Pi(X, X). \end{aligned}$$

□

**Lemma 4.3** *Let  $p_0 \in M$  and let  $B(p_0, \sigma)$  be the geodesic ball centered at  $p_0$  with radius  $\sigma > 0$ . Let  $\gamma : [-a, a] \rightarrow B(p_0, \sigma)$  and  $\beta : [-b, b] \rightarrow B(p_0, \sigma)$  be two noncharacteristic curves of class  $C^1$ , respectively, with*

$$\gamma(0) = \beta(0) = p_0, \quad \Pi(\dot{\gamma}(0), \dot{\beta}(0)) = 0.$$

*Let  $\hat{\psi} : B(p_0, \sigma) \rightarrow \mathbb{R}^2$  be an asymptotic coordinate system. Then there exists an asymptotic coordinate system  $\psi : B(p_0, \sigma) \rightarrow \mathbb{R}^2$  with  $\psi(p_0) = (0, 0)$  such that*

$$\psi(\gamma(t)) = (t, -t) \quad \text{for } t \in [-a, a], \quad (4.24)$$

$$\beta'_1(s) > 0, \quad \beta'_2(s) > 0 \quad \text{for } s \in [-b, b], \quad (4.25)$$

*where  $\psi(\beta(s)) = (\beta_1(s), \beta_2(s))$ . Moreover, for  $X = X_1\partial x_1 + X_2\partial x_2$  with  $\Pi(X, X) \neq 0$ , we have*

$$\varrho(X)Q\nabla\vec{n}X = \chi(\gamma'(0), \beta'(0)) \begin{cases} X_1\partial x_1 - X_2\partial x_2, & X_1X_2 > 0, \\ -X_1\partial x_1 + X_2\partial x_2, & X_1X_2 < 0, \end{cases} \quad (4.26)$$

*where  $\varrho(X)$  is given in (4.3) and*

$$\chi(\gamma'(0), \beta'(0)) = \text{sign det}(\gamma'(0), \beta'(0), \vec{n}(p_0)).$$

**Proof.** Let  $\hat{\psi}(p_0) = (0, 0)$  and

$$\hat{\psi}(\gamma(t)) = (\gamma_1(t), \gamma_2(t)) \quad \text{for } t \in [-a, a].$$

Since  $\gamma$  is noncharacteristic,

$$\Pi(\dot{\gamma}(t), \dot{\gamma}(t)) = 2\gamma'_1(t)\gamma'_2(t)\Pi(\partial x_1, \partial x_2) \neq 0 \quad \text{for } t \in [-a, a].$$

Without loss of generality, we assume that

$$\gamma'_1(t) > 0, \quad \gamma'_2(t) < 0 \quad \text{for } t \in [-a, a]. \quad (4.27)$$

We extend the domain  $[-a, a]$  of  $\gamma(t)$  to  $\mathbb{R}$  such that

$$\lim_{t \rightarrow \pm\infty} \gamma_1(t) = \pm\infty, \quad \lim_{t \rightarrow \pm\infty} \gamma_2(t) = \mp\infty,$$

and the relations (4.27) hold for all  $t \in \mathbb{R}$ . Consider a diffeomorphism  $\varphi(x) = y : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$\varphi(x) = (\gamma_1^{-1}(x_1), -\gamma_2^{-1}(x_2)) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \quad (4.28)$$

Then  $\varphi \circ \hat{\psi} : B(p_0, \sigma) \rightarrow \mathbb{R}^2$  is an asymptotic coordinate system such that

$$\varphi \circ \hat{\psi}(\gamma(t)) = (t, -t) \quad \text{for } t \in [-a, a]. \quad (4.29)$$

Let  $\varphi \circ \hat{\psi}(\beta(s)) = (\beta_1(s), \beta_2(s))$ . Since  $\beta$  is noncharacteristic,

$$\beta'_1(s)\beta'_2(s) \neq 0 \quad \text{for } s \in [-b, b].$$

In addition, the assumption  $\Pi(\dot{\gamma}(0), \dot{\beta}(0)) = 0$  and the relation (4.29) imply that

$$0 = \Pi(\dot{\gamma}(0), \dot{\beta}(0)) = \Pi(\partial x_1 - \partial x_2, \beta'_1(0)\partial x_1 + \beta'_2(0)\partial x_2) = [\beta'_2(0) - \beta'_1(0)]\Pi(\partial x_1, \partial x_2),$$

that is,  $\beta'_1(0) = \beta'_2(0)$ . If  $\beta'_1(0) > 0$ , we let  $\psi(p) = \varphi \circ \hat{\psi}(p)$  to have (4.25). If  $\beta'_1(0) < 0$ , we define instead of (4.28)

$$\varphi(x) = (\gamma_2^{-1}(x_2), -\gamma_1^{-1}(x_1)) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Thus (4.25) follows again.

Next, we prove (4.26). Let  $Q\nabla\vec{n}X = Y_1\partial x_1 + Y_2\partial x_2$ . Since  $(Y_1X_2 + Y_2X_1)\Pi(\partial x_1, \partial x_2) = \Pi(Q\nabla\vec{n}X, X) = \langle Q\nabla\vec{n}X, \nabla\vec{n}X \rangle = 0$ , we have

$$Q\nabla\vec{n}X = \sigma(X_1\partial x_1 - X_2\partial x_2),$$

where  $\sigma$  is a function. Using Proposition 4.2 (ii), we obtain

$$\sigma^2 = -\kappa.$$

Next, from (4.24) and (4.25), we have

$$(\gamma'(0), \beta'(0), \vec{n}) = (\partial x_1, \partial x_2, \vec{n}) \begin{pmatrix} 1 & \beta'_1(0) & 0 \\ -1 & \beta'_2(0) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which yields

$$\text{sign det}(\gamma'(0), \beta'(0), \vec{n}) = \text{sign det}(\partial x_1, \partial x_2, \vec{n}).$$

Thus (4.26) follows from Proposition 4.2 (i).  $\square$

Denote

$$\Omega(0, s_0) = \{ \alpha(t, s) \mid t \in (0, a), s \in (0, s_0) \} \quad \text{for } s_0 \in [0, b]. \quad (4.30)$$

Then  $\Omega = \Omega(0, b)$ .

**Lemma 4.4** *Let the assumptions in Theorem 4.1 hold. Then there is a  $0 < \omega \leq b$  such that problem (4.1) admits a unique solution  $w \in C^{m,1}(\overline{\Omega(0,\omega)})$  with the data (4.4) where  $s \in [0, \omega]$ , and (4.5) to satisfy*

$$\begin{aligned} \|w\|_{C^{m,1}(\overline{\Omega(0,\omega)})} &\leq C(\|p_1\|_{C^{m-1,1}[0,b]} + \|p_2\|_{C^{m-1,1}[0,b]} + \|q_0\|_{C^{m,1}[0,a]} \\ &\quad + \|q_1\|_{C^{m-1,1}[0,a]} + \|f\|_{C^{m-1,1}(\overline{\Omega})}). \end{aligned} \quad (4.31)$$

**Proof.** Let  $\sigma_0 > 0$  be given small such that the claims in Lemmas 4.1 and 4.2 hold, where  $\gamma(t) = \alpha(t, 0)$  in Lemma 4.2. We divide the curve  $\alpha(t, 0)$  into  $m$  parts with the points  $\lambda_i = \alpha(t_i, 0)$  such that

$$\lambda_0 = \alpha(0, 0), \quad \lambda_m = \alpha(a, 0), \quad d(\lambda_i, \lambda_{i+1}) = \frac{\sigma_0}{3}, \quad 0 \leq i \leq m-2, \quad d(\lambda_{m-1}, \lambda_m) \leq \frac{\sigma_0}{3},$$

where  $t_0 = 0, t_1 > 0, t_2 > t_1, \dots$ , and  $t_m = a > t_{m-1}$ . For simplicity, we assume that  $m = 3$ . The other cases can be treated by a similar argument.

We shall construct a local solution in a neighborhood of  $\alpha(t, 0)$  by the following steps.

**Step 1.** Let  $\hat{\sigma}_0 > 0$  be small such that

$$\alpha(0, s) \in B(\lambda_0, \sigma_0) \quad \text{for } s \in [0, s_0].$$

From Lemma 4.3, there is asymptotic coordinate system  $\psi_0(p) = x : B(\lambda_0, \sigma_0) \rightarrow \mathbb{R}^2$  with  $\psi_0(\lambda_0) = (0, 0)$  such that

$$\psi_0(\alpha(t, 0)) = (t, -t) \quad \text{for } t \in [0, t_2], \quad (4.32)$$

$$\beta'_{01}(s) > 0, \quad \beta'_{02}(s) > 0 \quad \text{for all } s \in [0, s_0],$$

where  $\beta_0(s) = \psi_0(\alpha(0, s)) = (\beta_{01}(s), \beta_{02}(s))$ . Let  $\gamma_0(t) = (t, -t)$ . We may assume that  $s_0 > 0$  is given small such that  $\beta_{01}(s_0) \leq t_2$  since  $\beta_{01}(0) = 0$ . Set

$$\Xi_1(\beta_0, \gamma_0) = P_1(\beta_0) \cup R((\beta_{01}(s_0), 0), c_0, d_0) \cup E(\gamma_0), \quad (4.33)$$

as in (3.29) with  $c_0 = t_2 - \beta_{01}(s_0)$  and  $d_0 = \beta_{02}(s_0)$ . Then we let

$$\Omega_0 = \Omega \cap \psi_0^{-1}[\Xi_1(\beta_0, \gamma_0)].$$

Noting that for the region  $\Omega_0$

$$\chi(\mu(\alpha(t, 0)), \alpha_t(t, 0)) = \chi(-\alpha_s(0, 0), \alpha_t(0, 0)) \quad \text{for } t \in (0, t_2),$$

$$\chi(\mu(\alpha(0, s)), \alpha_s(0, s)) = \chi(-\alpha_t(0, 0), \alpha_s(0, 0)) \quad \text{for } s \in (0, s_0),$$

from (4.26), we obtain

$$\mathcal{T}_1 \alpha_s(0, s) = \beta'_{01}(s) \partial x_1, \quad \mathcal{T}_2 \alpha_s(0, s) = \beta'_{02}(s) \partial x_2 \quad \text{for } s \in (0, s_0), \quad (4.34)$$

$$\mathcal{T}_1 \alpha_t(t, 0) = \partial x_1, \quad \mathcal{T}_2 \alpha_t(t, 0) = -\partial x_2 \quad \text{for } t \in (0, t_2). \quad (4.35)$$

From Proposition 4.1, solvability of problem (4.1) on  $\Omega \cap \psi_0^{-1}(\Xi_1(\beta_0, \gamma_0))$  is equivalent to that of problem (3.1) over the region  $\Xi_1(\beta_0, \gamma_0)$ . Next, we consider the transfer of the boundary data under the chart  $\psi_0$ . The corresponding part data are

$$w_{x_2} \circ \beta_0(s) = \langle Dw, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(0, s) / \beta'_{02}(s) = p_1(s) / \beta'_{02}(s) \quad \text{for } s \in [0, s_0],$$

$$\begin{aligned}
w(t, -t) &= w \circ \psi_0^{-1}(t, -t) = w(\alpha(t, 0)) = q_0(t) \quad \text{for } t \in [0, t_2], \\
\frac{\partial}{\partial \nu} w(t, -t) &= \frac{1}{\sqrt{2}} \langle Dw, (\mathcal{T}_2 - \mathcal{T}_1)\alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for } t \in [0, a],
\end{aligned}$$

where

$$w(x) = w \circ \psi^{-1}(x).$$

It is easy to check that  $p_1/\beta'_{02}$ ,  $q_0$ ,  $q_1$ , and  $f$  are  $m$ th order compatible at  $\alpha(0, 0)$  in the sense of Definition 4.1 is equivalent to that  $p_1/\beta'_{02}$ ,  $q_0$ ,  $q_1$ , and  $\hat{f}$  do in the sense of Definition 3.1, where

$$\hat{f} = \frac{f \circ \psi_0^{-1}(x)}{2} \sqrt{\frac{\det G(x)}{-k \circ \psi_0^{-1}(x)}} \quad \text{for } x \in \psi_0(B(\lambda_0, \sigma_0)) \subset \mathbb{R}^2,$$

where  $\det G(x) = \det(\langle \partial x_i, \partial x_j \rangle)$ .

From Proposition 3.9, problem (3.1) admits a unique solution  $w \in C^{m,1}(\overline{\Xi_1(\beta_0, \gamma_0)})$  with the corresponding boundary data. Thus, we have obtained a solution, denoted by  $w_0 \in C^{m,1}(\overline{\Omega_0})$ , to problem (4.1) with the data

$$\begin{aligned}
\langle Dw_0, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(0, s) &= p_1(s) \quad \text{for } s \in [0, s_0], \\
w_0 \circ \alpha(t, 0) &= q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw_0, (\mathcal{T}_2 - \mathcal{T}_1)\alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for } t \in [0, t_2],
\end{aligned}$$

where

$$\Omega_0 = \Omega \cap \psi_0^{-1}[\Xi_1(\beta_0, \gamma_0)].$$

It follows from the estimate in Proposition 3.9 that

$$\|w_0\|_{C^{m,1}(\overline{\Omega_0})} \leq C\Gamma_{mC}(p_1, p_2, q_0, q_1, f), \quad (4.36)$$

where

$$\begin{aligned}
\Gamma_{mC}(p_1, p_2, q_0, q_1, f) &= \|p_1\|_{C^{m-1,1}(0,b)} + \|p_2\|_{C^{m-1,1}(0,b)} + \|q_0\|_{C^{m,1}(0,a)} \\
&\quad + \|q_1\|_{C^{m-1,1}(0,a)} + \|f\|_{C^{m-1,1}(\overline{\Omega})}.
\end{aligned}$$

We define a curve on  $\Omega_0$  by

$$\zeta_1(s) = \psi_0^{-1} \circ \gamma_{t_1}(s) \quad \text{for } s \in [0, s_{t_1}], \quad (4.37)$$

where

$$\gamma_{t_1}(s) = (s + t_1, s - t_1), \quad s_{t_1} = \begin{cases} t_1 & \text{if } t_1 \in (0, \frac{t_2}{2}], \\ t_2 - t_1 & \text{if } t_1 \in (\frac{t_2}{2}, t_2). \end{cases}$$

Then  $\zeta_1(s)$  is noncharacteristic and

$$\Pi(\dot{\zeta}_1(0), \alpha_t(t_1, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0. \quad (4.38)$$

**Step 2.** Let the curve  $\zeta_1$  be given in (4.37). Let  $s_1 > 0$  be small such that

$$\zeta_1(s) \in B(\lambda_1, \sigma_0) \quad \text{for } s \in [0, s_1].$$

From the noncharacteristicness of  $\zeta_1(s)$  and the relation (4.38) and Lemma 4.3 again, there exists an asymptotic coordinate system  $\psi_1(p) = x : B(\lambda_1, \sigma_0) \rightarrow \mathbb{R}^2$  with  $\psi_1(\lambda_1) = (0, 0)$  and

$$\psi_1(\alpha(t + t_1, 0)) = (t, -t) \quad \text{for } t \in [0, t_3 - t_1],$$

$$\beta'_{11}(s) > 0, \quad \beta'_{12}(s) > 0 \quad \text{for } s \in [0, s_1],$$

where  $\beta_1(s) = \psi_1(\zeta_1(s)) = (\beta_{11}(s), \beta_{12}(s))$ . We also assume that  $s_1 > 0$  has been taken small such that  $\beta_{11}(s_1) \leq t_3$ . This time, we set

$$\Xi_1(\beta_1, \gamma_1) = P_1(\beta_1) \cup R((\beta_{11}(s_1), 0), c_1, d_1) \cup E_1(\gamma_1),$$

where  $c_1 = t_3 - t_1 - \beta_{11}(s_1)$ ,  $d_1 = \beta_{12}(s_0)$ , and  $\gamma_1(t) = \psi_1(\alpha(t + t_1, 0))$ . Next, let

$$\Omega_1 = \Omega \cap \psi_1^{-1}[\Xi_1(\beta_1, \gamma_1)].$$

Since for the region  $\Omega_1$

$$\chi(\mu(\zeta_1(s)), \zeta'_1(s)) = \chi(-\alpha_t(0, t_1), \zeta'_1(0)) \quad \text{for } s \in (0, s_1),$$

$$\chi(\mu(\alpha(t_1 + t, 0)), \alpha_t(t_1 + t, 0)) = \chi(-\zeta'_1(0), \alpha_t(t_1, 0)) \quad \text{for } t \in (0, t_3 - t_1),$$

it follows from (4.26) that

$$\mathcal{T}_1 \zeta'_1(s) = \beta'_{11}(s) \partial x_1, \quad \mathcal{T}_2 \zeta'_1(s) = \beta'_{12}(s) \partial x_2 \quad \text{for } s \in (0, s_1),$$

$$\mathcal{T}_1 \alpha_t(t_1 + t, 0) = \partial x_1, \quad \mathcal{T}_2 \alpha_t(t_1 + t, 0) = -\partial x_2 \quad \text{for } t \in (0, t_3 - t_1).$$

By some similar arguments in Step 1, we obtain a unique solution  $w_1 \in C^{m,1}(\overline{\Omega_1})$  to problem (4.1) with the data

$$\langle Dw_1, \mathcal{T}_2 \dot{\zeta}_1 \rangle \circ \beta_1(s) = \langle Dw_0, \mathcal{T}_2 \dot{\zeta}_1(s) \rangle \circ \beta_1(s) \quad \text{for } s \in [0, s_1],$$

$$w_1(\alpha(t, 0)) = q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw_1, (\mathcal{T}_2 - \mathcal{T}_1) \alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for } t \in [t_1, t_3],$$

where  $w_0$  is the solution of (4.1) on  $\Omega_0$ , given in Step 1. The following estimate also holds

$$\begin{aligned} \|w_1\|_{C^{m,1}(\overline{\Omega_1})} &\leq C(\|\langle Dw_0, \mathcal{T}_2 \dot{\zeta}_1 \rangle \circ \zeta_1\|_{C^{m-1,1}[0, s_1]} + \|q_0\|_{C^{m,1}[0, a]} + \|q_1\|_{C^{m-1,1}[0, a]}) \\ &\quad + \|f\|_{C^{m-1,1}(\overline{\Omega})} \leq C\Gamma_m C(p, q_0, q_1, h). \end{aligned} \quad (4.39)$$

As in Step 1, we define a curve on  $\Omega_1$  by

$$\zeta_2(s) = \psi_1^{-1}(s + t_2 - t_1, s + t_1 - t_2) \quad \text{for } s \in [0, s_{t_2}], \quad (4.40)$$

where

$$s_{t_2} = t_2 - t_1 \quad \text{if} \quad t_2 - t_1 \leq \frac{t_3 - t_1}{2}; \quad s_{t_2} = t_3 - t_2 \quad \text{if} \quad t_2 - t_1 > \frac{t_3 - t_1}{2}.$$

Then  $\zeta_2(s)$  is noncharacteristic and

$$\Pi(\dot{\zeta}_2(0), \alpha_t(t_2, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0. \quad (4.41)$$

**Step 3.** Let the curve  $\zeta_2$  be given in (4.40). Let  $s_2 > 0$  be small such that

$$\zeta_2(s), \quad \alpha(a, s) \in B(\lambda_2, \sigma_0) \quad \text{for} \quad s \in [0, s_2].$$

Let  $\psi_2(p) = x : B(\lambda_2, \sigma_0) \rightarrow \mathbb{R}^2$  be an asymptotic coordinate system with  $\psi_2(\lambda_2) = (0, 0)$ ,

$$\psi_2(\alpha(t + t_2, 0)) = (t, -t) \quad \text{for} \quad t \in [0, a - t_2], \quad (4.42)$$

and

$$\beta'_{21}(s) > 0, \quad \beta'_{22}(s) > 0 \quad \text{for} \quad s \in [0, s_2],$$

where  $\beta_2(s) = \psi_2(\zeta_2(s)) = (\beta_{21}(s), \beta_{22}(s))$ .

Let  $\beta_3(s) = \psi_2(\alpha(a, s)) = (\beta_{31}(s), \beta_{32}(s))$ . Next, we prove that

$$\beta'_{31}(s) > 0, \quad \beta'_{32}(s) > 0 \quad \text{for} \quad s \in [0, s_2], \quad (4.43)$$

by contradiction. Since  $\alpha(a, s)$  is noncharacteristic, using (4.42) and the assumption  $\Pi(\alpha_t(a, 0), \alpha_s(a, 0)) = 0$ , we have

$$\beta'_{31}(0) = \beta'_{32}(0); \quad \text{thus} \quad \beta'_{31}(s)\beta'_{32}(s) > 0 \quad \text{for} \quad s \in [0, s_2].$$

Let

$$p(t, s) = \alpha_1(t, s) + \alpha_2(t, s), \quad \psi_2(\alpha(t + t_2, s)) = (\alpha_1(t, s), \alpha_2(t, s)).$$

Let (4.43) be not true, that is,  $\beta'_{31}(s) < 0, \beta'_{32}(s) < 0$  for  $s \in [0, s_2]$ . Thus

$$p(0, s) = \beta_{21}(s) + \beta_{22}(s) > \beta_{21}(0) + \beta_{22}(0) = 0 \quad \text{for} \quad s \in (0, s_2],$$

$$p(a - t_2, s) = \beta_{31}(s) + \beta_{32}(s) < \beta_{31}(0) + \beta_{32}(0) = 0 \quad \text{for} \quad s \in (0, s_2].$$

Let  $t(s) \in (0, a - t_2)$  be such that

$$\alpha_1(t(s), s) + \alpha_2(t(s), s) = 0 \quad \text{for} \quad s \in (0, s_2). \quad (4.44)$$

Since  $\alpha_{1t}(0, 0) = 1$  and  $\alpha(t + t_2, s)$  are noncharacteristic for all  $s \in [0, s_2]$ , we have  $\alpha_{1t}(t, s) > 0$  and

$$0 < \alpha_{1t}(0, s) < \alpha_{1t}(t(s), s) < \alpha_{1t}(a - t_2, s) = \beta_{31}(s) < \beta_{31}(0) = a - t_2.$$

Thus, equality (4.44) means that  $\alpha(\alpha_1(t(s), s) + t_2, 0) = \alpha(t(s), s)$ , which is a contradiction since  $\alpha : [0, a] \times [a, b] \rightarrow M$  is an imbedding map.

We also assume that  $s_2$  has been taken so small such that

$$\beta_{21}(s_2) < a - t_2, \quad \beta_{32}(s_2) < 0,$$

since  $\beta_{21}(0) = 0$  and  $\beta_{32}(0) = -(a - t_2) < 0$ . Let  $\gamma_2(t) = \psi_2(\alpha(t + t_2, 0)) = (t, -t)$ . We now set

$$\Phi(\beta_2, \gamma_2, \beta_3) = \Xi_1(\beta_2, \gamma_2) \cup R((a - t_2, \beta_{32}(s_2)), c_3, d_3) \cup P_2(\beta_3),$$

where  $\Xi_1(\beta_2, \gamma_2)$ ,  $R((a - t_2, \beta_{32}(s_2)), c_3, d_3)$ , and  $P_2(\beta_3)$  are given in (3.29), (3.6), and (3.26), respectively, with  $c_3 = \beta_{32}(s_2) - a + t_2$  and  $d_3 = \beta_{22}(s_2) - \beta_{32}(s_2)$ . Let

$$\Omega_2 = \Omega \cap \psi_2^{-1}[\Phi(\beta_2, \gamma_2, \beta_3)].$$

This time we use (4.26) to obtain, for the region  $\Omega_2$ ,

$$\mathcal{T}_1 \zeta_2'(s) = \beta_{21}'(s) \partial x_1, \quad \mathcal{T}_2 \zeta_2'(s) = \beta_{22}'(s) \partial x_2 \quad \text{for } s \in (0, s_2),$$

$$\mathcal{T}_1 \alpha_t(t_2 + t, 0) = \partial x_1, \quad \mathcal{T}_2 \alpha_t(t_2 + t, 0) = -\partial x_2 \quad \text{for } t \in (0, a - t_2),$$

$$\mathcal{T}_1 \alpha_s(a, s) = \beta_{32}'(s) \partial x_2, \quad \mathcal{T}_2 \alpha_s(a, s) = \beta_{31}'(s) \partial x_1 \quad \text{for } s \in (0, s_2).$$

Applying Proposition 3.13, problem (4.1) admits a unique solution  $w_2 \in C^{m,1}(\overline{\Omega_2})$  with the data

$$\langle Dw_2, \mathcal{T}_2 \zeta_2 \rangle \circ \beta_2(s) = \langle Dw_1, \mathcal{T}_2 \zeta_2 \rangle \circ \beta_2(s), \quad \langle Dw_2, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(a, s) = p_2(s) \quad \text{for } s \in [0, s_2],$$

$$w_2(\alpha(t, 0)) = q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw_2, (\mathcal{T}_2 - \mathcal{T}_1) \alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for } t \in [t_2, a].$$

Using the estimates in Proposition 3.13 and (4.39), we obtain

$$\begin{aligned} \|w_2\|_{C^{m,1}(\overline{\Omega_2})} &\leq C(\|\langle Dw_1, \mathcal{T}_2 \zeta_2 \rangle \circ \zeta_2\|_{C^{m-1,1}[0, s_2]} + \|p_2\|_{C^{m-1,1}[0, b]} + \|q_0\|_{C^{m,1}[0, a]} \\ &\quad + \|q_1\|_{C^{m-1,1}[0, a]} + \|f\|_{C^{m-1,1}(\overline{\Omega})}) \leq C\Gamma_{mC}(p_1, p_2, q_0, q_1, f). \end{aligned} \quad (4.45)$$

**Step 4.** We define

$$w = w_i \quad \text{for } p \in \Omega_i \quad \text{for } i = 0, 1, 2.$$

Let  $\omega > 0$  be small such that

$$\alpha(t, s) \in \Omega_0 \cup \Omega_1 \cup \Omega_2 \quad \text{for } (t, s) \in (0, a) \times (0, \omega).$$

Then  $w \in C^{m,1}(\overline{\Omega(0, \omega)})$  will be a solution to (4.1) with the corresponding data if we show that

$$w_0(p) = w_1(p) \quad \text{for } p \in \Omega_0 \cap \Omega_1; \quad w_1(p) = w_2(p) \quad \text{for } p \in \Omega_1 \cap \Omega_2. \quad (4.46)$$

Since

$$\begin{aligned} w_{1x_2} \circ \beta_1(s) &= w_{0x_2} \circ \beta_1(s) \quad \text{for } s \in [0, s_1], \\ w_0(t, -t) &= w_1(t, -t), \quad \frac{\partial w_0}{\partial \nu}(t, -t) = \frac{\partial w_1}{\partial \nu}(t, -t) \quad \text{for } t \in [t_1, t_2], \end{aligned}$$

from the uniqueness in Proposition 3.9, we have

$$w_0(x) = w_1(x) \quad \text{for } x \in \Xi_1(\beta_0, \gamma_0) \cap \Xi_1(\beta_1, \gamma_1),$$

which yields the first identity in (4.46). A similar argument shows that the second identity in (4.46) is true.

Finally, the estimate (4.31) follows from (4.36), (4.39), and (4.45).  $\square$

From a similar argument as for the proof of Lemma 4.4, we obtain the following.

**Lemma 4.5** *Let the assumptions in Theorem 4.2 hold. Then there is a  $0 < \omega \leq b$  such that problem (4.1) admits a unique solution  $w \in \mathbb{W}^{2,2}(\Omega(0, \omega))$  with the data (4.4) where  $s \in (0, \omega)$ , and (4.5) to satisfy*

$$\begin{aligned} \|w\|_{\mathbb{W}^{2,2}(\Omega(0, \omega))}^2 &\leq C(\|q_0\|_{\mathbb{W}^{2,2}(0, a)}^2 + \|q_1\|_{\mathbb{W}^{1,2}(0, a)}^2 + \|p_1\|_{\mathbb{W}^{2,2}(0, b)}^2 \\ &\quad + \|p_2\|_{\mathbb{W}^{2,2}(0, b)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}). \end{aligned} \quad (4.47)$$

We are now ready to prove Theorems 4.1 and 4.2.

**Proof of Theorem 4.1** Let  $\aleph$  be the set of all  $0 < \omega \leq b$  such that the claims in Lemma 4.4 hold. We shall prove

$$b \in \aleph.$$

Let  $\omega_0 = \sup_{\omega \in \aleph} \omega$ . Then  $0 < \omega_0 \leq b$ . Thus there is a unique solution  $w \in C^{m,1}(\Omega(0, \omega_0))$  to (4.1) with the data (4.4), where  $s \in [0, \omega_0)$ , and (4.5).

Next we show that  $\omega_0 = b$  by contradiction. Let  $0 < \omega_0 < b$ . By an argument as for Lemma 4.4, the solution  $w \in C^{m,1}(\Omega(0, \omega_0))$  can be extended such that  $w \in C^{m,1}(\overline{\Omega(0, \omega_0)})$ . Then by Lemma 4.4 again,  $w$  can be extended outside  $C^{m,1}(\overline{\Omega(0, \omega_0)})$ , which contradicts with the definition of  $\omega_0$ .

Let  $\lambda_0 = \alpha(0, \omega_0)$ ,  $\sigma_0$ ,  $t_0 = 0$ ,  $t_1$ ,  $t_2$ , and  $t_3 = a$  be given as in the proof of Lemma 4.4.

Let  $\psi_0(p) = x : B(\lambda_0, \sigma_0) \rightarrow \mathbb{R}^2$  be an asymptotic coordinate system with  $\psi_0(\lambda_0) = (0, 0)$  such that

$$\begin{aligned} \psi_0(\alpha(t, \omega_0)) &= (t, -t) \quad \text{for } t \in [0, t_2], \\ \zeta_1'(s) &> 0, \quad \zeta_2'(s) > 0 \quad \text{for } s \in [\omega_0 - \varepsilon_0, \omega_0], \end{aligned}$$

where  $\psi_0(\alpha(0, s)) = (\zeta_1(s), \zeta_2(s))$ .

For  $\varepsilon > 0$ , let

$$\beta_0(s) = \psi_0(\alpha(0, s + \omega_0 - \varepsilon)) = (\beta_{01}(s), \beta_{02}(s)), \quad \gamma_0(t) = \psi_0(\alpha(t, \omega_0 - \varepsilon)) = (\gamma_{01}(t), \gamma_{02}(t)),$$

for  $s \in [0, 2\varepsilon]$ , where  $\beta_{0i}(s) = \zeta_i(s + \omega_0 - \varepsilon)$  for  $i = 1, 2$ . We fixed  $\varepsilon > 0$  small such that

$$\omega_0 + \varepsilon \leq b, \quad \beta_{01}(\varepsilon) \leq \gamma_{01}(t_2), \quad \gamma'_{01}(t) > 0, \quad \gamma'_{02}(t) < 0.$$

Let  $\Xi_1(\beta_0, \gamma_0)$  be given as in (4.33). Clearly,

$$\{\alpha(t, \omega_0) \mid t \in [0, t_2]\} \subset \psi_0^{-1}(\overline{\Xi_1(\beta_0, \gamma_0)}).$$

From Proposition 3.9, we can extend a solution  $w$  such that  $w$  is  $C^{k,1}$  on the segment  $\{\alpha(t, \omega_0 + \varepsilon) \mid t \in [0, t_2]\}$ . Repeating Steps 2-4 in the proof of Lemma 4.4, the solution  $w$  can be extended such that  $w$  is  $C^{k,1}$  on the segment  $\{\alpha(t, \omega_0 + \varepsilon) \mid t \in [0, a]\}$ , which contradicts the definition of  $\omega_0$ . The proof is complete.  $\square$

**Proof of Theorem 4.2** A similar argument as in the proof of Theorem 4.1 completes the proof.  $\square$

To prove Theorems 4.3 and 4.4, we need the following lemmas.

**Lemma 4.6** *Let the assumptions in Theorem 4.3 hold. Then there are  $0 < \omega \leq b$  and  $C > 0$  such that for all solutions  $w \in W^{2,2}(\Omega)$  to problem (4.1)*

$$\|w\|_{W^{2,2}(\Omega(0,\omega))}^2 \leq C[\|f\|_{W^{1,2}(\Omega)}^2 + \Gamma(\Omega, w)], \quad (4.48)$$

where  $\Omega(0, \omega)$  and  $\Gamma(\Omega, w)$  is given in (4.30) and (4.13), respectively.

**Proof** We keep all the notion in the proof of Lemma 4.4. Let  $\omega > 0$  be given in Step 4. Then

$$w_0(x) = w \circ \psi_0^{-1}(x)$$

is a solution to problem (3.1) on the region  $\Xi_1(\beta_0, \gamma_0)$ , where  $\Xi_1(\beta_0, \gamma_0)$  is given in (4.33) and

$$\beta_0(s) = \psi_0(\alpha(0, s)) = (\beta_{01}(s), \beta_{02}(s)) \quad \text{for } s \in (0, s_0), \quad \gamma_0(t) = \psi_0(\alpha(t, 0)) = (t, -t)$$

for  $t \in [0, t_2]$ .

It follows from (4.34) and (4.35) that

$$\left| |D^2 w(\mathcal{T}_1 \alpha_t(t, 0), \mathcal{T}_1 \alpha_t(t, 0))| - |w_{0x_1 x_1} \circ \gamma_0(t)| \right| \leq C |\nabla w_0 \circ \gamma_0(t)|,$$

Similarly, we have

$$\left| |D^2 w(\mathcal{T}_2 \alpha_t(t, 0), \mathcal{T}_2 \alpha_t(t, 0))| - |w_{0x_2 x_2} \circ \gamma_0(t)| \right| \leq C |\nabla w_0 \circ \gamma_0(t)|,$$

$$\left| |D^2 w(\mathcal{T}_1 \alpha_s(0, s), \mathcal{T}_1 \alpha_s(0, s))| - |w_{0x_1 x_1} \circ \beta_0(s)| \beta_{01}'^2(s) \right| \leq C |\nabla w_0 \circ \beta_0(s)|,$$

$$\left| |D^2 w(\mathcal{T}_2 \alpha_s(0, s), \mathcal{T}_2 \alpha_s(0, s))| - |w_{0x_2 x_2} \circ \beta_0(s)| \beta_{02}'^2(s) \right| \leq C |\nabla w_0 \circ \beta_0(s)|.$$

Using the above relations, we obtain

$$\Gamma(\gamma_0, w_0) + \Gamma_2(\beta_0, w_0) \leq C\Gamma(\Omega, w), \quad (4.49)$$

where  $\Gamma(\gamma_0, w_0)$  and  $\Gamma_2(\beta_0, w_0)$  are given in (3.18) and (3.49), respectively.

Applying Proposition 3.11 to  $\Xi_1(\beta_0, \gamma_0)$  and using (4.49), we have

$$\|w\|_{\mathbb{W}^{2,2}(\Omega_0)}^2 \leq C\|w_0\|_{\mathbb{W}^{2,2}(\Xi_1(\beta_0, \gamma_0))}^2 \leq C(\|f\|_{\mathbb{W}^{1,2}}^2 + \Gamma(\Omega, w)).$$

Using (3.16) by a similar argument as for the above estimates, we obtain

$$\|w\|_{\mathbb{W}^{2,2}(\Omega_i)}^2 \leq C(\|f\|_{\mathbb{W}^{1,2}}^2 + \Gamma(\Omega, w)), \quad \text{for } i = 1, 2.$$

Thus the estimate (4.48) follows.  $\square$

**Lemma 4.7** *Let the assumptions in Theorem 4.3 hold. Then there is  $C > 0$  such that for all solutions  $w \in \mathbb{W}^{2,2}(\Omega)$  to problem (4.1)*

$$\Gamma(\Omega, w) \leq C(\|w\|_{\mathbb{W}^{2,2}(\Omega)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}^2). \quad (4.50)$$

**Proof Step 1** We claim that for each  $\varepsilon > 0$  small, there is  $C_\varepsilon > 0$  such that

$$\sum_{j=0}^2 \int_\varepsilon^{a-\varepsilon} |D^j w \circ \alpha(t, 0)|^2 dt \leq C_\varepsilon(\|w\|_{\mathbb{W}^{2,2}(\Omega)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}^2). \quad (4.51)$$

Let  $t_0 \in (0, a)$  be fixed and let  $p_0 = \alpha(t_0, 0)$ . Let  $\zeta : (0, \varepsilon) \rightarrow \Omega$  be such that

$$\zeta(0) = p_0, \quad \zeta'(0) = -\mu(p_0),$$

where  $\mu(p_0)$  is the noncharacteristic normal at the boundary point  $p_0$  outside  $\Omega$ . From Lemma 4.3, there are  $0 < \sigma_0 < \min\{t_0, a - t_0\}$  and an asymptotic coordinate system  $\psi : B(p_0, \sigma_0) \rightarrow \mathbb{R}^2$  with  $\psi(p_0) = (0, 0)$  such that

$$\psi(\alpha(t + t_0, 0)) = (t, -t) \quad \text{for } t \in (-\sigma_0, \sigma_0), \quad \zeta_1'(s) > 0, \quad \zeta_2'(s) > 0 \quad \text{for } s \in (0, \varepsilon),$$

where  $\psi(\zeta(s)) = (\zeta_1(s), \zeta_2(s))$ . Set

$$\Omega_{p_0} = \Omega \cap \psi^{-1}[E(\gamma)],$$

where

$$\gamma(t) = (t, -t), \quad E(\gamma) = \{x \mid -x_2 < x_1 < \sigma_1, \quad -\sigma_2 < x_2 < \sigma_1\}.$$

Using (4.26) for the region  $\Omega_{p_0}$ , we obtain

$$\mathcal{T}_1 \alpha_t(t + t_0, 0) = \partial x_1, \quad \mathcal{T}_2 \alpha_t(t + t_0, 0) = -\partial x_2 \quad \text{for } t \in (-\sigma_0, \sigma_0),$$

where the operators  $\mathcal{T}_i$  are given in (4.2).

Observe that  $w_0(x) = w \circ \psi^{-1}(x)$  is a solution to problem (3.1) on the region  $E(\gamma)$ . Applying Proposition 3.4, we have

$$\begin{aligned} \sum_{j=0}^2 \int_{-\sigma_1/2}^{\sigma_1/2} |D^j w \circ \alpha(t + t_0, s)|^2 dt &\leq C \sum_{j=0}^2 \int_{-\sigma_1/2}^{\sigma_1/2} |\nabla^j w_0(t, -t)|^2 dt \\ &\leq C(\|w_0\|_{\mathbb{W}^{2,2}(E(\gamma))}^2 + \|f \circ \psi^{-1}\|_{\mathbb{W}^{1,2}(E(\gamma))}^2) \leq C(\|w\|_{\mathbb{W}^{2,2}(\Omega)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}^2). \end{aligned}$$

Thus the estimates (4.51) follows from the finitely covering theorem. By a similar argument, we have

$$\sum_{j=0}^2 \int_{\varepsilon}^{b-\varepsilon} |D^j w \circ \alpha(t_k, s)|^2 ds \leq C_{\varepsilon}(\|w\|_{\mathbb{W}^{2,2}(\Omega)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}^2), \quad k = 1, 2,$$

where  $t_1 = 0$  and  $t_2 = a$ , which particularly imply that

$$\int_{\varepsilon}^{b-\varepsilon} |p'_k(s)|^2 (b-s) ds \leq C_{\varepsilon}(\|w\|_{\mathbb{W}^{2,2}(\Omega)}^2 + \|f\|_{\mathbb{W}^{1,2}(\Omega)}^2), \quad k = 1, 2, . \quad (4.52)$$

**Step 2** We treat the estimates at the angular points  $\alpha(0, 0)$ ,  $\alpha(0, b)$ ,  $\alpha(a, 0)$ , and  $\alpha(a, b)$ , respectively.

Consider the angular  $\alpha(0, b)$  first. Let  $\varepsilon > 0$  be given small. From Lemma 4.3, there is an asymptotic coordinate system  $\psi : B(\alpha(0, b), \sigma_0) \rightarrow \mathbb{R}^2$  with  $\psi(\alpha(0, b)) = (0, 0)$  such that

$$\gamma(t) = \psi(\alpha(t, b)) = (t, -t) \quad \text{for } t \in [0, \varepsilon],$$

$$\beta(s) = \psi(\alpha(0, b-s)) = (\beta_1(s), \beta_2(s)), \quad \beta'_1(s) > 0, \quad \beta'_2(s) > 0 \quad \text{for } s \in [0, \varepsilon].$$

Consider the region  $\Omega_{\alpha(0,b)} = \Omega \cap \psi^{-1}[\Xi_1(\beta, \gamma)]$ . From (4.26), we have

$$\mathcal{T}_2 \alpha_s(0, b-s) = -\beta'_2(s) \partial x_2 \quad \text{for } s \in (0, \varepsilon).$$

It follows from Proposition 3.11 that

$$\begin{aligned} \int_{b-\varepsilon_1}^b |p'_1(s)|^2 (b-s) ds &\leq C(\|f \circ \psi^{-1}\|_{\mathbb{W}^{1,2}(\Xi_1(\beta, \gamma))}^2 + \|w \circ \psi_1^{-1}\|_{\mathbb{W}^{1,2}(\Xi_1(\beta, \gamma))}^2) \\ &\leq C(\|f\|_{\mathbb{W}^{1,2}}^2 + \|w\|_{\mathbb{W}^{2,2}}^2). \end{aligned}$$

Similarly, we can treat the estimates at the other angular points. Thus estimate (4.50) follows by combing the above estimates with those in Step 1.  $\square$

**Proof of Theorem 4.3** Let  $\mathcal{R}$  be the set of all  $0 < \omega \leq b$  such that estimate (4.48) is true. Set  $\omega_0 = \sup_{\omega \in \mathcal{R}} \omega$ . By Lemmas 4.6 and 4.7, it is sufficient to prove

$$\omega_0 \in \mathcal{R}.$$

By following the proof of Theorem 4.1, we obtain a  $\varepsilon > 0$  small such that

$$\|w\|_{\mathbb{W}^{2,2}(\Omega(\omega_0-\varepsilon,\omega_0))}^2 \leq C \left[ \int_{\omega_0-\varepsilon}^{\omega_0} (|p_1'(s)|^2 + |p_2'(s)|^2)(\omega_0 - s) ds + \Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w) + \|f\|_{\mathbb{W}^{1,2}}^2 \right],$$

where  $\Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w)$  is given in (4.14). On the other hand, we fix  $0 < \varepsilon_1 < \varepsilon$  and apply Lemma 4.7 to the region  $\Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1)$  to obtain

$$\begin{aligned} \Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w) &\leq C [\|w\|_{\mathbb{W}^{2,2}(\Omega(\omega_0-\varepsilon,\omega_0-\varepsilon_1))}^2 + \|f\|_{\mathbb{W}^{2,2}(\Omega(\omega_0-\varepsilon,\omega_0-\varepsilon_1))}^2] \\ &\leq C [\|w\|_{\mathbb{W}^{2,2}(\Omega(0,\omega_0-\varepsilon_1))}^2 + \|f\|_{\mathbb{W}^{2,2}(\Omega(0,\omega_0-\varepsilon_1))}^2] \quad (\text{by (4.48)}) \\ &\leq C [\|f\|_{\mathbb{W}^{1,2}(\Omega)}^2 + \Gamma(\Omega, w)]. \end{aligned}$$

By Lemma 4.6, we have  $\omega_0 \in \mathcal{R}$ . By Lemma 4.7, we obtain  $\omega_0 = b$ .  $\square$

**Proof of Theorem 4.4** Let  $w \in \Upsilon(\Omega)$  and  $\varepsilon > 0$  be given. We shall find a  $\hat{w} \in \mathcal{H}_2(\Omega)$  such that

$$\|w - \hat{w}\|_{\mathbb{W}^{2,2}(\Omega)}^2 < \varepsilon.$$

Let  $p_1, p_2$ , and  $q_0, q_1$  be given in (4.4) and (4.5), respectively. Towards approximating  $w$  by  $\mathcal{H}(\Omega)$  functions, we first approximate its traces  $q_0, q_1, p_1$ , and  $p_2$ . From Theorem 4.3, those traces are regular except for the angular points  $\alpha(0, 0)$ ,  $\alpha(0, b)$ ,  $\alpha(a, 0)$ , and  $\alpha(a, b)$ . Next, we change their values near those angular points to make them regular and to let the 1th order compatibility conditions hold at  $\alpha(0, 0)$  and  $\alpha(a, 0)$ .

**Step 1** Consider the point  $\alpha(0, 0)$ . Let  $\sigma > 0$  be given small by Lemma 4.3 such that there is an asymptotic coordinate system  $\psi : B(\alpha(0, 0), \sigma) \rightarrow \mathbb{R}^2$  with  $\psi(\alpha(0, 0)) = (0, 0)$  such that

$$\psi(\alpha(t, 0)) = (t, -t) \quad \text{for } t \in [0, t_0],$$

$$\beta(s) = \psi(\alpha(0, s)) = (\beta_1(s), \beta_2(s)), \quad \beta_1'(0) = \beta_2'(0), \quad \beta_1'(s) > 0, \quad \beta_2'(s) > 0 \quad \text{for } s \in [0, t_0],$$

for some  $0 < t_0 < \min\{a, b\}/4$  small.

From Lemma 4.3, we have

$$p_1(s) = w_{x_2} \circ \beta(s) \beta_2'(s) \quad \text{for } s \in [0, t_0],$$

$$q_0'(t) = w_{x_1}(t, -t) - w_{x_2}(t, -t), \quad -\sqrt{2}q_1'(t) = w_{x_1}(t, -t) + w_{x_2}(t, -t) \quad (4.53)$$

for  $t \in (0, t_0)$ . where  $w(x) = w \circ \psi^{-1}(x)$ . Moreover, we have

$$D^2w(\mathcal{T}_1\alpha_t, \mathcal{T}_1\alpha_t) = D^2w(\partial x_1, \partial x_1) = w_{x_1x_1}(t, -t) - D_{\partial x_1}\partial x_1(w)(t, -t) = \varphi_{11} + \phi_1,$$

and

$$D^2w(\mathcal{T}_2\alpha_t, \mathcal{T}_2\alpha_t) = \varphi_{22} + \phi_2.$$

By differential the equations in (4.53) in  $t \in (0, t_0)$  and using the formulas (4.1) and (4.18), we obtain

$$\varphi_{11} = \frac{1}{2}[q_0''(t) - \sqrt{2}q_1'(t)], \quad \varphi_{22} = \frac{1}{2}[q_0''(t) + \sqrt{2}q_1'(t)] \quad \text{for } t \in (0, t_0),$$

$\phi_1 =$  some first order terms of  $w$ ,  $\phi_2 =$  some first order terms of  $w$ .

By Theorem 4.3

$$p_1 \in W^{1,2}(0, t_0), \quad q_0 \in W^{1,2}(0, t_0), \quad q_1, \varphi_{11}t^{1/2}, \varphi_{22}, \phi_1, \phi_2 \in L^2(0, t_0).$$

Thus

$$q_0''t^{1/2} = (\varphi_{11} + \varphi_{22})t^{1/2} \in L^2(0, t_0), \quad q_1't^{1/2} = \frac{1}{\sqrt{2}}(\varphi_{22} - \varphi_{11})t^{1/2} \in L^2(0, t_0).$$

We also need the following.

**Lemma 4.8** *Let*

$$z(t) = \frac{1}{2}[q_0'(t) + \sqrt{2}q_1(t)] \quad \text{for } t \in (0, t_0).$$

Then  $z \in C[0, t_0]$  and

$$p_1(0) + z(0)\beta_2'(0) = 0. \tag{4.54}$$

**Proof of Lemma 4.8** It follows from  $z' = \varphi_{22} \in L^2(0, t_0)$  that  $z \in C[0, t_0]$ .

We have

$$w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t) = \int_{-t}^{\beta_2 \circ \beta_1^{-1}(t)} w_{x_2 x_2}(t, s) ds,$$

from which we obtain

$$|w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t)|^2 \leq [\beta_2 \circ \beta_1^{-1}(t) + t] \int_{-t}^{\beta_2 \circ \beta_1^{-1}(t)} |w_{x_2 x_2}(t, s)|^2 ds.$$

For  $\varepsilon > 0$  given, let  $\vartheta \in [\varepsilon/2, \varepsilon]$  be fixed such that

$$|w_{x_2} \circ \beta \circ \beta_1^{-1}(\vartheta) - w_{x_2}(\vartheta, -\vartheta)|^2 = \inf_{t \in [\varepsilon/2, \varepsilon]} |w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t)|^2.$$

Then

$$\begin{aligned} |w_{x_2} \circ \beta \circ \beta_1^{-1}(\vartheta) - w_{x_2}(\vartheta, -\vartheta)|^2 &\leq \frac{2}{\varepsilon} [\beta_2 \circ \beta_1^{-1}(\varepsilon) + \varepsilon] \int_{\varepsilon/2}^{\varepsilon} \int_{-t}^{\beta_2 \circ \beta_1^{-1}(t)} |w_{x_2 x_2}(t, s)|^2 ds \\ &\leq \sigma \int_0^{\varepsilon} \int_{-t}^{\beta_2 \circ \beta_1^{-1}(t)} |w_{x_2 x_2}(t, s)|^2 ds \quad \text{for } t \in [\varepsilon/2, \varepsilon]. \end{aligned}$$

Thus,  $w \in W^{2,2}(\Omega)$  implies, by (4.53), that (4.54) holds.

Let  $0 < \varepsilon < t_0$  given small. We shall construct  $\hat{q}_0$  and  $\hat{q}_1$  to satisfy the following.

- (1)  $\hat{q}_0(t) = q_0(t)$ ,  $\hat{q}_1(t) = q_1(t)$  for  $t \in [\varepsilon, a]$ ;
- (2)  $\hat{q}_0 \in W^{2,2}(0, a)$  and  $\hat{q}_1 \in W^{1,2}(0, a)$ ;
- (3) The following 1th order compatibility conditions hold at the point  $\alpha(0, 0)$ ,

$$2p_1(0) + [\hat{q}_0'(0) + \sqrt{2}\hat{q}_1(0)]\beta_2'(0) = 0;$$

(4) If  $\hat{w} \in \Upsilon(\Omega)$  is such that

$$\begin{aligned} \hat{w} \circ \alpha(t, 0) &= \hat{q}_0(t), \quad \frac{1}{2} \langle D\hat{w}, (\mathcal{T}_2 - \mathcal{T}_1)\alpha_t \rangle \circ \alpha(t, 0) = \hat{q}_1(t) \quad \text{for } t \in (0, a), \\ \langle D\hat{w}, \mathcal{T}_1\alpha_s \rangle \circ \alpha(0, s) &= p_1(s), \quad \langle D\hat{w}, \mathcal{T}_2\alpha_s \rangle \circ \alpha(0, s) = p_2(s) \quad \text{for } s \in (0, b), \end{aligned}$$

then

$$\Gamma(\Omega, \hat{w} - w) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

For the above purposes, we define

$$\hat{q}_0(t) = \begin{cases} \sigma_0(\varepsilon) + [q'_0(t_0) - \int_\varepsilon^{t_0} \varphi_{11} ds - \int_0^{t_0} \varphi_{22} ds]t + \int_0^t (t-s)\varphi_{22}(s)ds, & t \in [0, \varepsilon], \\ q_0(t) & t \in [\varepsilon, a], \end{cases}$$

and

$$\hat{q}_1(t) = \begin{cases} q_1(t_0) + \frac{1}{\sqrt{2}} \int_\varepsilon^{t_0} \varphi_{11} ds - \frac{1}{\sqrt{2}} \int_t^{t_0} \varphi_{22} ds, & t \in (0, \varepsilon), \\ q_1(t), & t \in [\varepsilon, a], \end{cases}$$

where

$$\sigma_0(\varepsilon) = q_0(\varepsilon) - q'_0(\varepsilon)\varepsilon + \int_0^\varepsilon s\varphi_{22}(s)ds.$$

Clearly, (1) and (2) hold for the above  $\hat{q}_0$  and  $\hat{q}_1$ . Since

$$\begin{aligned} q'_0(t_0) - \int_\varepsilon^{t_0} \varphi_{11} ds - \int_0^{t_0} \varphi_{22} ds &= q'_0(\varepsilon) - \int_0^\varepsilon \varphi_{22}(s)ds = q'_0(\varepsilon) - z(\varepsilon) + z(0), \quad \text{for } t \in (0, \varepsilon), \\ \frac{1}{\sqrt{2}} \int_\varepsilon^{t_0} \varphi_{11} ds - \frac{1}{\sqrt{2}} \int_t^{t_0} \varphi_{22} ds &= q_1(\varepsilon) - q_1(t_0) + \frac{1}{\sqrt{2}} [z(t) - z(\varepsilon)], \end{aligned}$$

using (4.54), we have

$$2p_1(0) + [\hat{q}'_0(0) + \sqrt{2}\hat{q}_1(0)]\beta'_2(0) = q'_0(\varepsilon) + \sqrt{2}q_1(\varepsilon) - 2z(\varepsilon) = 0.$$

Next, we check (4). It follows that

$$\begin{aligned} |q_0(t) - \hat{q}_0(t)|^2 &= \left| \int_t^\varepsilon \int_s^\varepsilon q''_0(\tau) d\tau ds + \int_t^\varepsilon (t-s)\varphi_{22}(s)ds \right|^2 \\ &\leq 2(\varepsilon - t + t \ln \frac{t}{\varepsilon}) \int_0^\varepsilon |q''_0(\tau)|^2 \tau d\tau + \frac{2}{3}\varepsilon^3 \int_0^\varepsilon |\varphi_{22}(s)|^2 ds \quad \text{for } t \in (0, \varepsilon). \end{aligned}$$

In addition,

$$|q'_0(t) - \hat{q}'_0(t)|^2 = \left| \int_t^\varepsilon \varphi_{11}(s)ds \right|^2 \leq (\ln \frac{\varepsilon}{t}) \int_0^\varepsilon |\varphi_{11}(s)|^2 s ds \quad \text{for } t \in (0, \varepsilon).$$

Similarly, we have

$$|q_1(t) - \hat{q}_1(t)|^2 \leq (\ln \frac{\varepsilon}{t}) \int_0^\varepsilon |\varphi_{11}(s)|^2 s ds \quad \text{for } t \in (0, \varepsilon).$$

Using (4.14) and the above estimates, we have

$$\begin{aligned}
\Gamma(\alpha(\cdot, 0), w - \hat{w}) &= \sum_{j=0}^1 \|D(w - \hat{w}) \circ \alpha(\cdot, 0)\|_{L^2(0, \varepsilon)}^2 + \int_0^\varepsilon [|D^2(w - \hat{w})(\mathcal{T}_1\alpha_t, \mathcal{T}_1\alpha_t)|^2 t \\
&\quad + |D^2(w - \hat{w})(\mathcal{T}_2\alpha_t, \mathcal{T}_2\alpha_t)|^2 (a - t)] dt \\
&\leq C \int_0^\varepsilon (|q_0(s) - \hat{q}_0(s)|^2 + |q'_0(t) - \hat{q}'_0(t)|^2 + |q_1(t) - \hat{q}_1(t)|^2 \\
&\quad + |\varphi_{11} - \hat{\varphi}_{11}|^2 t + |\varphi_{22} - \hat{\varphi}_{22}|^2 t) dt \\
&\leq C \int_0^\varepsilon [(|q''_0(t)|^2 + |\varphi_{11}(t)|^2) t + |\varphi_{22}(t)|^2] dt, \tag{4.55}
\end{aligned}$$

where

$$\hat{\varphi}_{11} = \frac{1}{2}[\hat{q}''_0(t) - \sqrt{2}\hat{q}'_1(t)] = 0, \quad \hat{\varphi}_{22} = \frac{1}{2}[\hat{q}''_0(t) + \sqrt{2}\hat{q}'_1(t)] = \varphi_{22}.$$

Thus (4) follows.

**Step 2** As in Step 1, we change the values of  $q_0$  and  $q_1$  near the point  $\alpha(a, 0)$  to get  $\hat{q}_0$  and  $\hat{q}_1$  in  $W^{2,2}(0, a)$  and in  $W^{1,2}(0, a)$ , respectively, such that the 1th order compatibility conditions at  $\alpha(a, 0)$  hold to approximate  $q_0$  and  $q_1$ . Then we change the values of  $p_1$  and  $p_2$  near the points  $\alpha(0, b)$  and  $\alpha(a, b)$ , respectively, such that  $\hat{p}_1, \hat{p}_2 \in W^{1,2}(0, b)$  approximate  $p_1$  and  $p_2$ , respectively. Thus the proof completes from Theorem 4.2.  $\square$

## 5 Proofs of Main Results in Section 1

**Proof of Theorem 1.1** Let  $\Omega \subset M$  be a noncharacteristic region of class  $C^{2,1}$ . For  $U \in C^{1,1}(\Omega, T_{\text{sym}}^2)$  given, we consider problem

$$\text{sym } \nabla y = U \quad \text{on } \Omega. \tag{5.1}$$

(1) Consider problem

$$\langle D^2 v, Q^* \Pi \rangle = P(U) - 2v\kappa \text{tr}_g \Pi + X(v) \quad \text{for } x \in \Omega, \tag{5.2}$$

where  $P(U)$  and  $X$  are given in (2.26) and (2.27), respectively, with the boundary data

$$\langle Dv, \mathcal{T}_2\alpha_s \rangle \circ \alpha(0, s) = \langle Dv, \mathcal{T}_2\alpha_s \rangle \circ \alpha(a, s) = 0 \quad \text{for } s \in (0, b), \tag{5.3}$$

$$v \circ \alpha(t, 0) = \frac{1}{\sqrt{2}} \langle Dv, (\mathcal{T}_2 - \mathcal{T}_1)\alpha_t \rangle \circ \alpha(t, 0) = 0 \quad \text{for } t \in (0, a), \tag{5.4}$$

where  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are given in (4.2).

Since

$$P(U) \in L^\infty(\Omega), \quad X \in L^\infty(\Omega),$$

it follows from Theorem 4.1 that problem (5.2) with the data (5.3) and (5.4) has a unique solution  $v \in C^{0,1}(\Omega)$  with the bounds

$$\|v\|_{C^{0,1}(\Omega)} \leq C \|U\|_{C^{1,1}(\Omega, T_{\text{sym}}^2)}. \tag{5.5}$$

From Theorem 2.1, there is a solution  $y \in C^{0,1}(\Omega, \mathbb{R}^3)$  to (5.1). Let

$$w = \langle y, \vec{n} \rangle, \quad W = y - w\vec{n}.$$

Then  $w \in C^{0,1}(\Omega)$ . It follows from [12, lemma 4.3] that  $W \in C^{1,1}(\Omega, T)$  and (1.2) holds.

(2) Let  $\Omega \in C^{m+2,1}$  and  $U \in C^{m+1,1}(\Omega, T_{\text{sym}}^2)$  be given for some  $m \geq 1$ . Let

$$q_0(t) = q_1(t) = 0 \quad \text{for } t \in [0, a].$$

Let  $\mathcal{Q}_k(0, 0, P(U))(t)$  be given in the formula (4.8) for  $t \in [0, a]$  and  $1 \leq k \leq m-1$ . We define

$$\phi_j(s) = \begin{cases} 0, & m = 1, \\ \sum_{l=1}^{m-1} \frac{p_j^{(l)}(t_j)}{l!} s^l, & m \geq 2, \end{cases} \quad \text{for } s \in [0, b], \quad j = 1, 2, \quad (5.6)$$

where  $p_j^{(l)}(t_j)$  are given by the right hand sides of (4.10) for  $1 \leq l \leq m-1$  and  $1 \leq j \leq 2$ , where  $q_0 = q_1 = 0$  and  $f = P(U)$ . Clearly, the  $m$ th compatibility conditions hold true for the above  $q_0, q_1, \phi_1, \phi_2$ , and  $P(U)$ . From Theorem 4.1, there is a solution  $v \in C^{m,1}(\bar{\Omega})$  to problem (5.2) with the data

$$\langle Dv, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(0, s) = \phi_1(s), \quad \langle Dv, \mathcal{T}_2 \alpha_s \rangle \circ \alpha(a, s) = \phi_2(s) \quad \text{for } s \in (0, b),$$

$$v \circ \alpha(t, 0) = \frac{1}{\sqrt{2}} \langle Dv, (\mathcal{T}_2 - \mathcal{T}_1) \alpha_t \rangle \circ \alpha(t, 0) = 0 \quad \text{for } t \in (0, a).$$

Moreover, it follows from (4.11) and (2.26) that

$$\|v\|_{C^{m,1}(\bar{\Omega})} \leq C \|U\|_{C^{m+1,1}(\Omega, T^2)},$$

which implies the estimate (1.3) is true.  $\square$

**Proof of Theorem 1.2** Let

$$V = W + w\vec{n}, \quad w = \langle V, \vec{n} \rangle.$$

The regularity of

$$\text{sym } DW = -w\Pi \in W^{2,2}(\Omega, \mathbb{R}^3)$$

implies

$$W \in W^{3,2}(\Omega, T).$$

Let  $E_1, E_2$  be a frame field on  $\Omega$  with the positive orientation and let

$$v = \frac{1}{2} [\nabla V(E_2, E_1) - \nabla V(E_1, E_2)].$$

From Theorem 2.1  $v$  is a solution to problem

$$\langle D^2 v, Q^* \Pi \rangle = -2v\kappa \text{tr}_g \Pi + X(v) \quad \text{for } x \in \Omega, \quad (5.7)$$

where  $\kappa \operatorname{tr}_g \Pi \in C^{m,1}(\overline{\Omega})$  and  $X = (\nabla \vec{n})^{-1} D\kappa \in C^{m-1,1}(\overline{\Omega}, T)$ , where  $C^{-1,1}(\overline{\Omega}, T) = L^\infty(\Omega, T)$ .

It is easy to check that

$$\nabla_{E_i} V = D_{E_i} W + w \nabla_{E_i} \vec{n} + [E_i(w) - \Pi(W, E_i)] \vec{n} \quad \text{for } i = 1, 2.$$

Thus

$$v = DW(E_2, E_1) - DW(E_1, E_2) \in W^{2,2}(\Omega).$$

From Theorems 4.4, 4.1, and 4.2, there are solutions  $v_n \in C^{m,1}(\overline{\Omega})$  to problem (5.7) such that

$$\lim_{n \rightarrow \infty} \|v_n - v\|_{W^{2,2}(\Omega)} = 0.$$

Let

$$u_n = -Q(\nabla \vec{n})^{-1} Dv_n, \quad u = -Q(\nabla \vec{n})^{-1} Dv.$$

Then  $u_n \in C^{m-1,1}(\overline{\Omega})$ .

From Theorem 2.1 (see (2.11)), there exist  $\hat{V}_n \in C^{m,1}(\overline{\Omega}, \mathbb{R}^3)$  such that

$$\begin{cases} \nabla_{E_1} \hat{V}_n = v_n E_2 + \langle u_n, E_1 \rangle \vec{n}, \\ \nabla_{E_2} \hat{V}_n = -v_n E_1 + \langle u_n, E_2 \rangle \vec{n}, \end{cases} \quad \text{for } n = 1, 2, \dots \quad (5.8)$$

Define

$$V_n(\alpha(t, s)) = \hat{V}_n(\alpha(0, s)) - \hat{V}_n(\alpha(0, 0)) + V(\alpha(0, 0)) + \int_0^t \nabla_{\alpha_t} \hat{V}_n dt \quad \text{for } n = 1, 2, \dots$$

Thus  $V_n \in \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{m,1}(\Omega, \mathbb{R}^3)$  satisfy (1.4).  $\square$

**Proof of Theorem 1.3** As in [5] we conduct in  $2 \leq i \leq m$ . Let

$$u_\varepsilon = \sum_{j=0}^{i-1} \varepsilon^j w_j$$

be an  $(i-1)$ th order isometry of class  $C^{2(m-i)+1,1}(\overline{\Omega}, \mathbb{R}^3)$ , where  $w_0 = \operatorname{id}$  and  $w_1 = V$  for some  $i \geq 2$ . Then

$$\sum_{j=0}^k \nabla^T w_j \nabla w_{k-j} = 0 \quad \text{for } 0 \leq k \leq i-1.$$

Next, we shall find out  $w_i \in C^{2(m-i)+1,1}(\overline{\Omega}, \mathbb{R}^3)$  such that

$$\phi_\varepsilon = u_\varepsilon + \varepsilon^i w_i$$

is an  $i$ th order isometry. From Theorem 1.1 there exists a solution  $w_i \in C^{2(m-i)+1,1}(\overline{\Omega}, \mathbb{R}^3)$  to problem

$$\operatorname{sym} \nabla w_i = -\frac{1}{2} \operatorname{sym} \sum_{j=1}^{i-1} \operatorname{sym} \nabla^T w_j \nabla w_{i-j}$$

which satisfies

$$\begin{aligned} \|w_i\|_{C^{2(m-i)+1,1}(\bar{\Omega}, \mathbb{R}^3)} &\leq C \left\| \sum_{j=1}^{i-1} \text{sym } \nabla^T w_j \nabla w_{i-j} \right\|_{C^{2(m-i)+1,1}(\bar{\Omega}, \mathbb{R}^3)} \\ &\leq C \sum_{j=1}^{i-1} \|w_j\|_{C^{2(m-i+1)+1,1}(\bar{\Omega}, \mathbb{R}^3)} \|w_{i-j}\|_{C^{2(m-i+1)+1,1}(\bar{\Omega}, \mathbb{R}^3)}. \end{aligned}$$

The conduction completes.  $\square$

**Theorem 1.4** will follow from the density of the Sobolev space and Proposition 5.1 below.

**Proposition 5.1** *Let  $\Omega \subset M$  be a noncharacteristic region of class  $C^{2,1}$ . Then for  $U \in W^{3,2}(\Omega, T_{\text{sym}}^2)$  there exists a solution  $w \in W^{2,2}(\Omega, \mathbb{R}^3)$  to problem*

$$\text{sym } \nabla w = U.$$

**Proof** Consider problem (5.2) with the data (5.3) and (5.4). By (4.9) the first order compatibility conditions hold. Since  $P(U) \in W^{1,2}(\Omega)$ , the proposition follows from Theorems 4.2 and 2.1.  $\square$

**Proof of Theorem 1.7** A recovery sequence can be constructed, based on Theorems 1.2 and 1.3, as in the proof of [5, Theorem 6.2]. We present a skeleton of the proof. For the further details, see [5].

From the density of Theorem 1.2 and the continuity of the functional  $I$  with respect to the strong topology of  $W^{2,2}$ , we can assume  $V \in \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{2m-1,1}(\Omega, \mathbb{R}^3)$ .

**Step 1** Let  $\varepsilon = \frac{\sqrt{e^h}}{h}$  so  $\varepsilon \rightarrow 0$ , as  $h \rightarrow 0$ , by assumption (1.6). Therefore, by Theorem 1.3, there exists a sequence  $w_\varepsilon : \bar{\Omega} \rightarrow \mathbb{R}^3$ , equibounded in  $C^{1,1}(\Omega, \mathbb{R}^3)$ , for all  $h > 0$ ,

$$u_\varepsilon = \text{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$$

is a  $m$ th isometry of class  $C^{1,1}$ . Then

$$\varepsilon^{m+1} = o(\sqrt{e^h}).$$

Consider the sequence of deformations  $u_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$  defined by

$$u_h(x + t\vec{n}) = u_\varepsilon(x) + t\vec{n}_\varepsilon(x) + \frac{t^2}{2}\varepsilon d_h(x) \quad \text{for } x + t\vec{n}(x) \in \Omega_h,$$

where  $\vec{n}_\varepsilon(x)$  denotes the unit normal to  $u_\varepsilon(\Omega)$  at  $u_\varepsilon(x)$  and  $d_h \in W^{1,\infty}(\Omega, \mathbb{R}^3)$  is such that  $\lim_{h \rightarrow 0} h^{1/2} \|d_h\|_{W^{1,\infty}} = 0$  and

$$\lim_{h \rightarrow 0} d_h(x) = 2c(x, \text{sym}(\nabla(A\vec{n}) - A\Pi)_{\text{tan}}) \quad \text{for } x \in \Omega,$$

where  $c(x, F_{\tan})$  denotes the unique vector satisfying  $\mathcal{Q}_2(x, F_{\tan}) = \mathcal{Q}_3(F_{\tan} + c \otimes \vec{n}(x) + \vec{n}(x) \otimes c)$ . We have

$$\vec{n}_\varepsilon(x) = \vec{n}(x) + \varepsilon A\vec{n} + \mathcal{O}(\varepsilon^2).$$

**Step 2** We have

$$\frac{E_h(u_h)}{e^h} = \frac{1}{e^h h_0} \int_{-h_0/2}^{h_0/2} \int_{\Omega} W(\nabla_h y^h(x + t\vec{n}(x))) \left(1 + \frac{th}{h_0} \operatorname{tr}_g \Pi + \frac{t^2 h^2}{h_0^2} \kappa\right) dg dt,$$

where  $\nabla_h y^h(x + t\vec{n}(x)) = \nabla u_h(x + \frac{th}{h_0} \vec{n})$ . Let

$$K^h(x + t\vec{n}(x)) = (\nabla_h y^h)^T \nabla_h y^h - \operatorname{Id}.$$

Using the formulas  $\nabla^T u_\varepsilon \nabla u_\varepsilon = \operatorname{Id} + \mathcal{O}(\varepsilon^{m+1}) = \operatorname{Id} + o(\sqrt{e^h})$  and  $h\varepsilon = \sqrt{e^h}$ , we have

$$K_{\tan}^h = 2 \frac{t\sqrt{e^h}}{h_0} \left(\operatorname{Id} + \frac{th}{h_0} \Pi\right)^{-1} \operatorname{sym}(\nabla(A\vec{n}) - A\Pi) \left(\operatorname{Id} + \frac{th}{h_0} \Pi\right)^{-1} + o(\sqrt{e^h}),$$

$$\langle K^h \vec{n}, \vec{n} \rangle = 2 \frac{t\sqrt{e^h}}{h_0} \langle \vec{n}_\varepsilon, d_h \rangle + o(\sqrt{e^h}),$$

$$\langle K^h \alpha, \vec{n} \rangle = \frac{t\sqrt{e^h}}{h_0} \langle \nabla u_\varepsilon (\operatorname{Id} + \frac{th}{h_0} \Pi)^{-1} \alpha, d_h \rangle + o(\sqrt{e^h}) \quad \text{for } \alpha \in T_x \Omega.$$

Then

$$\lim_{h \rightarrow 0} \frac{K_{\tan}^h}{2\sqrt{e^h}} = \frac{t}{h_0} \operatorname{sym}(\nabla(A\vec{n}) - A\Pi) \quad \text{in } L^\infty(\Omega_{h_0}),$$

$$\lim_{h \rightarrow 0} \frac{K^h \vec{n}}{2\sqrt{e^h}} = \frac{2t}{h_0} c(x, \operatorname{sym}(\nabla(A\vec{n}) - A\Pi)_{\tan}) \quad \text{in } L^\infty(\Omega_{h_0}).$$

**Step 3** We have

$$\frac{W(\nabla_y y_h)}{e^h} = \frac{1}{2} \mathcal{Q}_3\left(\frac{K^h}{2\sqrt{e^h}} + \frac{1}{\sqrt{e^h}} \mathcal{O}(|K^h|^2)\right) + \frac{1}{e^h} o(|K^h|^2).$$

Then the limit (1.8) follows from Step 2.  $\square$

### Compliance with Ethical Standards

Conflict of Interest: The author declares that there is no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by the author.

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