Analysis of a time-stepping scheme for time fractional diffusion problems with nonsmooth data *

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Abstract

This paper establishes the convergence of a time-steeping scheme for time fractional diffusion problems with nonsmooth data. We first analyze the regularity of the model problem with nonsmooth data, and then prove that the time-steeping scheme possesses optimal convergence rates in $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H_0^1(\Omega))$-norm with respect to the regularity of the solution. Finally, numerical results are provided to verify the theoretical results.

Keywords: fractional diffusion problem, regularity, finite element, optimal a priori estimate.

1 Introduction

This paper considers the following time fractional diffusion problem:

$$\begin{cases} \mathrm{D}_{0+}^{\alpha}(u-u_0) - \Delta u = f & \text{in } \Omega\times(0,T), \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $0<\alpha<1$, D_{0+}^{α} is a Riemann-Liouville fractional differential operator, $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ is a convex polygonal domain, and u_0 and f are given functions.

A considerable amount of numerical algorithms for time fractional diffusion problems have been developed. Generally, these numerical algorithms can be divided into three types. The first type uses finite difference methods to approximate the time fractional derivatives. Despite their ease of implementation, the fractional difference methods are generally of temporal accuracy orders no greater than two; see [39, 15, 38, 3, 20, 46, 8, 44, 2, 7, 21, 11, 43, 36, 12, 18] and the references therein. The second type applies spectral methods to discretize the time fractional derivatives; see [19, 45, 16, 42, 40, 41, 37, 17]. The main

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advantage of these algorithms is that they possess high-order accuracy, provided the solution is sufficiently smooth. The third type adopts finite element methods to approximate the time fractional derivatives; see [22, 29, 26, 17, 28, 31, 30, 23, 32, 27]. These algorithms are easy to implement, like those in the first type, and possess high-order accuracy.

The convergence analysis of the aforementioned algorithms is generally carried out on the condition that the underlying solution is sufficiently smooth. So far, the works on the numerical analysis for nonsmooth data are very limited. By using the Laplace transformation, Mclean and Thomée [24] analyzed three fully discretizations for fractional order evolution equations, where the initial values are allowed to have only $L^2(\Omega)$-regularity. By using a growth estimation of the Mittag-Leffler function, Jin et al. [14, 13] analyzed the convergence of a spatial semi-discretization of problem (1). They derived the following results: if $f = 0$, then

$$\|u(t) - u_h(t)\|_{L^2(\Omega)} + h\,\|u(t) - u_h(t)\|_{H_0^1(\Omega)} \leqslant Ch^2\,|\ln h|\,t^{-\alpha}\,\|u_0\|_{L^2(\Omega)}\,;$$

if $u_0 = 0$ and $0 \leqslant \beta < 1$, then

$$\|u-u_h\|_{L^2(0,T;L^2(\Omega))} + h\,\|u-u_h\|_{L^2(0,T;H_0^1(\Omega))} \leqslant Ch^{2-\beta}\,\|f\|_{L^2(0,T;H^{-\beta}(\Omega))}\,,$$

$$\|u(t)-u_h(t)\|_{L^2(\Omega)} + h\,\|u(t)-u_h(t)\|_{H_0^1(\Omega)} \leqslant Ch^{2-\beta}\,|\ln h|^2\,\|f\|_{L^\infty(0,t;H^{-\beta}(\Omega))}\,.$$

Recently, McLean and Mustapha [23] derived that

$$\|u(t_n) - U^n\|_{L^2(\Omega)} \leqslant Ct_n^{-1}\Delta t\,\|u_0\|_{L^2(\Omega)}$$

for a piecewise constant DG scheme in temporal semi-discretization of fractional diffusion problems with $f = 0$. For more related work, we refer the reader to [6, 25].

In this paper, we present a rigorous analysis of the convergence of a time-stepping scheme for problem (1), which uses a space of continuous piecewise linear functions in the spatial discretization and a space of piecewise constant functions in the temporal discretization. We first apply the Galerkin method to investigate the regularity of problem (1) with non-smooth u_0 and f, and then we derive the following error estimates: if $0 < \alpha < 1/2$ and $0 \leqslant \beta < 1$, then

$$\begin{aligned}&(h + \tau^{\alpha/2})^{-1}\,\|u - U\|_{L^2(0,T;L^2(\Omega))} + \|u - U\|_{L^2(0,T;H_0^1(\Omega))}\\ &\leqslant C\left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{H^{-\beta}(\Omega)}\right);\end{aligned}$$

if $1/2 \leqslant \alpha < 1$ and $2 - 1/\alpha < \beta < 1$, then

$$\begin{aligned}&(h + \tau^{\alpha/2})^{-1}\,\|u - U\|_{L^2(0,T;L^2(\Omega))} + \|u - U\|_{L^2(0,T;H_0^1(\Omega))}\\ &\leqslant C\left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right);\end{aligned}$$

if $1/2 \leqslant \alpha < 1$ and $u_0 = 0$, then the above estimate also holds for all $0 \leqslant \beta < 1$. Furthermore, if $1/2 < \alpha < 1$ and $u_0 = 0$, then we derive the optimal error estimate

$$\|u - U\|_{L^2(0,T;L^2(\Omega))} \leqslant C(h^2 + \tau)\,\|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}\,.$$

By the techniques used in our analysis, we can also derive the error estimates under other conditions; for instance, u_0 and f are smoother than the aforementioned cases.

The rest of this paper is organized as follows. Section 2 introduces some Sobolev spaces, the Riemann-Liouville fractional calculus operators, the weak solution to problem (1), and a time-stepping scheme. Section 3 investigates the regularity of the weak solution, and Section 4 establishes the convergence of the time-steeping scheme. Finally, Section 5 provides some numerical experiments to verify the theoretical results.

2 Preliminaries

Sobolev Spaces. For a Lebesgue measurable subset ω of $\mathbb{R}^l$ $(l = 1, 2, 3)$, we use $H^\gamma(\omega)$ $(-\infty < \gamma < \infty)$ and $H_0^\gamma(\omega)$ $(0 < \gamma < \infty)$ to denote two standard Sobolev spaces [35]. Let X be a separable Hilbert space with an inner product $(\cdot,\cdot)_X$ and an orthonormal basis $\{e_i : i \in \mathbb{N}\}$. We use $H^\gamma(0, T; X)$ $(0 \leqslant \gamma < \infty)$ to denote an usual vector valued Sobolev space, and for $0 < \gamma < 1/2$, we also use the norm

$$|v|_{H^\gamma(0,T;X)} := \left(\sum_{i=0}^{\infty} |(v, e_i)_X|^2_{H^\gamma(0,T)} \right)^{1/2}, \quad \forall v \in H^\gamma(0, T; X).$$

Here, the norm $|\cdot|_{H^\gamma(0,T)}$ is given by

$$|w|_{H^\gamma(0,T)} := \left(\int_{\mathbb{R}} |\xi|^{2\gamma} \left| \mathcal{F}(w\chi_{(0,T)})(\xi) \right|^2 \, \mathrm{d}\xi \right)^{1/2}, \quad \forall w \in H^\gamma(0, T),$$

where $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fourier transform operator and $\chi_{(0,T)}$ is the indicator function of $(0, T)$.

Fractional Calculus Operators. Let X be a Banach space and let $-\infty \leqslant a < b \leqslant \infty$. For $0 < \gamma < \infty$, define

$$\left(\mathrm{I}_{a+}^\gamma v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} v(s) \, \mathrm{d}s, \quad t \in (a, b),$$
$$\left(\mathrm{I}_{b-}^\gamma v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} v(s) \, \mathrm{d}s, \quad t \in (a, b),$$

for all $v \in L^1(a, b; X)$, where $\Gamma(\cdot)$ is the gamma function. For $j - 1 < \gamma < j$ with $j \in \mathbb{N}_{>0}$, define

$$\mathrm{D}_{a+}^\gamma := \mathrm{D}^j \, \mathrm{I}_{a+}^{j-\gamma},$$
$$\mathrm{D}_{b-}^\gamma := (-1)^j \, \mathrm{D}^j \, \mathrm{I}_{b-}^{j-\gamma},$$

where D is the first-order differential operator in the distribution sense.

Eigenvectors of $-\Delta$. It is well known that there exists an orthonormal basis

$$\{\phi_i : i \in \mathbb{N}\} \subset H_0^1(\Omega) \cap H^2(\Omega)$$

of $L^2(\Omega)$ such that

$$-\Delta\phi_i = \lambda_i\phi_i,$$

where $\{\lambda_i : i \in \mathbb{N}\} \subset \mathbb{R}_{>0}$ is a non-decreasing sequence. For any $0 \leqslant \gamma < \infty$, define

$$\dot{H}^\gamma(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{i=0}^{\infty} \lambda_i^\gamma (v, \phi_i)^2_{L^2(\Omega)} < \infty \right\},$$

and equip this space with the norm

$$\|\cdot\|_{\dot{H}^\gamma(\Omega)} := \left(\sum_{i=0}^{\infty} \lambda_i^\gamma (\cdot, \phi_i)^2_{L^2(\Omega)} \right)^{1/2}.$$

For $\gamma \in [0,1] \setminus \{0.5\}$, the space $\dot{H}^\gamma(\Omega)$ coincides with $H_0^\gamma(\Omega)$ with equivalent norms, and for $1 < \gamma \leqslant 2$, the space $\dot{H}^\gamma(\Omega)$ is continuously embedded into $H^\gamma(\Omega)$.

Weak Solution. Define

$$W := H^{\alpha/2}(0,T; L^2(\Omega)) \cap L^2(0,T; \dot{H}^1(\Omega))$$

and endow this space with the norm

$$\|\cdot\|_W := |\cdot|_{H^{\alpha/2}(0,T;L^2(\Omega))} + \|\cdot\|_{L^2(0,T;\dot{H}^1(\Omega))}.$$

Assuming that

$$\mathrm{D}_{0+}^\alpha u_0 + f \in W^*, \tag{2}$$

we call $u \in W$ a weak solution to problem (1) if

$$\left\langle \mathrm{D}_{0+}^\alpha u, v \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla u, \nabla v \rangle_{\Omega\times(0,T)} = \left\langle \mathrm{D}_{0+}^\alpha u_0 + f, v \right\rangle_W \tag{3}$$

for all $v \in W$. Throughout the paper, if ω is a Lebesgue measurable set of $\mathbb{R}^l$ ($l = 1,2,3,4$) then the symbol $\langle p, q \rangle_\omega$ means $\int_\omega pq$, and if X is a Banach space then $\langle \cdot, \cdot \rangle_X$ means the duality pairing between X^* (the dual space of X) and X.

Remark 2.1. *The above weak solution is first introduced by Li and Xu [19]. Evidently, the well-known Lax-Milgram theorem indicates that problem* (1) *admits a unique weak solution by Lemma A.2. Moreover,*

$$\|u\|_W \leqslant C \left\| \mathrm{D}_{0+}^\alpha u_0 + f \right\|_{W^*},$$

where C is a positive constant that depends only on α.

Discretization. Let

$$0 = t_0 < t_1 < \ldots < t_J = T$$

be a partition of $[0,T]$. Set $I_j := (t_{j-1}, t_j)$ for each $1 \leqslant j \leqslant J$, and we use τ to denote the maximum length of these intervals. Let $\mathcal{K}_h$ be a conventional conforming and shape regular triangulation of Ω consisting of d-simplexes, and we use h to denote the maximum diameter of the elements in $\mathcal{K}_h$. Define

$$\mathcal{S}_h := \left\{ v_h \in H_0^1(\Omega) : v_h|_K \in P_1(K), \ \forall K \in \mathcal{K}_h \right\},$$
$$\mathcal{M}_{h,\tau} := \left\{ V \in L^2(0,T;\mathcal{S}_h) : V|_{I_j} \in P_0(I_j;\mathcal{S}_h), \ \forall 1 \leqslant j \leqslant J \right\},$$

where $P_1(K)$ is the set of all linear polynomials defined on K, and $P_0(I_j; \mathcal{S}_h)$ is the set of all constant $\mathcal{S}_h$-valued functions defined on I_j.

Naturally, the discretization of problem (3) reads as follows: seek $U \in \mathcal{M}_{h,\tau}$ such that

$$\left\langle \mathrm{D}_{0+}^{\alpha} U, V \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \left\langle \nabla U, \nabla V \right\rangle_{\Omega \times (0,T)} = \left\langle \mathrm{D}_{0+}^{\alpha} u_0 + f, V \right\rangle_{W} \tag{4}$$

for all $V \in \mathcal{M}_{h,\tau}$.

Remark 2.2. *Similarly to the stability estimate in Remark 2.1, we have*

$$\|U\|_W \leqslant C \left\| \mathrm{D}_{0+}^{\alpha} u_0 + f \right\|_{W^*},$$

where C is a positive constant depending only on α. Therefore, problem (4) is also stable under condition (2).

3 Regularity

Let us first consider the following problem: seek $y \in H^{\alpha/2}(0,T)$ such that

$$\left\langle \mathrm{D}_{0+}^{\alpha}(y - y_0), z \right\rangle_{H^{\alpha/2}(0,T)} + \lambda \left\langle y, z \right\rangle_{(0,T)} = \left\langle g, z \right\rangle_{(0,T)} \tag{5}$$

for all $z \in H^{\alpha/2}(0,T)$, where $g \in L^2(0,T)$, and y_0 and $\lambda > 1$ are two real constants. By Lemma A.2, the Lax-Milgram theorem indicates that the above problem admits a unique solution $y \in H^{\alpha/2}(0,T)$. Moreover, it is evident that

$$\mathrm{D}_{0+}^{\alpha}(y - y_0) = g - \lambda y \tag{6}$$

in $L^2(0,T)$.

For convenience, we use the following convention: if the symbol C has subscript(s), then it means a positive constant that depends only on its subscript(s), and its value may differ at each of its occurrence(s). Additionally, in this section we assume that u and y are the solutions to problems (3) and (5), respectively.

Lemma 3.1. *If $0 < \alpha < 1/2$ and $0 \leqslant \beta < 1$, then*

$$\begin{aligned} &\lambda^{\beta/2} |y|_{H^{\alpha(1-\beta/2)}(0,t)} + \lambda^{(1+\beta)/2} |y|_{H^{(1-\beta)\alpha/2}(0,t)} + \lambda \|y\|_{L^2(0,t)} \\ &\leqslant C_\alpha \left(\|g\|_{L^2(0,t)} + t^{1/2-\alpha} |y_0| \right) \end{aligned} \tag{7}$$

for all $0 < t < T$.

Proof. Let us first prove that $y \in H^\alpha(0,T)$. By the definition of D_{0+}^{α}, equality (6) implies

$$\left(\mathrm{I}_{0+}^{1-\alpha}(y - y_0) \right)' = g - \lambda y,$$

so that using integration by parts gives

$$\mathrm{I}_{0+}^{1-\alpha}(y - y_0) = \left(\mathrm{I}_{0+}^{1-\alpha}(y - y_0) \right)(0) + \mathrm{I}_{0+}(g - \lambda y).$$

In addition, since

$$\left| \left(\mathrm{I}_{0+}^{1-\alpha}(y - y_0) \right)(s) \right| \leqslant \frac{1}{\Gamma(1-\alpha)} \sqrt{\frac{s^{1-2\alpha}}{1-2\alpha}} \, \|y - y_0\|_{L^2(0,s)}, \quad 0 < s < T,$$

we have

$$\left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0)\right)(0)=\lim_{s\to 0+}\left(\mathrm{I}_{0+}^{1-\alpha}(y-y_0)\right)(s)=0.$$

Consequently,

$$\mathrm{I}_{0+}^{1-\alpha}(y-y_0)=\mathrm{I}_{0+}(g-\lambda y),$$

and hence a simple computation gives that

$$y=y_0+\mathrm{I}_{0+}^{\alpha}(g-\lambda y).$$

Therefore, Lemma A.4 indicates that $y\in H^\alpha(0,T)$.

Then let us prove that

$$|y|_{H^\alpha(0,t)}^2+\lambda\,|y|_{H^{\alpha/2}(0,t)}^2+\lambda^2\,\|y\|_{L^2(0,t)}^2\leqslant C_\alpha\left(\|g\|_{L^2(0,t)}^2+t^{1-2\alpha}\,|y_0|^2\right). \tag{8}$$

Multiplying both sides of (6) by y and integrating over $(0,t)$ yields

$$\left\langle \mathrm{D}_{0+}^{\alpha}\,y,y\right\rangle_{(0,t)}+\lambda\,\|y\|_{L^2(0,t)}^2=\langle g,y\rangle_{(0,t)}+\left\langle \mathrm{D}_{0+}^{\alpha}\,y_0,y\right\rangle_{(0,t)}.$$

Since

$$\langle g,y\rangle_{(0,t)}\leqslant\frac{1}{\lambda}\,\|g\|_{L^2(0,t)}^2+\frac{\lambda}{4}\,\|y\|_{L^2(0,t)}^2,$$
$$\left\langle \mathrm{D}_{0+}^{\alpha}\,y_0,y\right\rangle_{(0,t)}\leqslant\frac{1}{\lambda}\,\left\|\mathrm{D}_{0+}^{\alpha}\,y_0\right\|_{L^2(0,t)}^2+\frac{\lambda}{4}\,\|y\|_{L^2(0,t)}^2,$$

we have

$$\left\langle \mathrm{D}_{0+}^{\alpha}\,y,y\right\rangle_{(0,t)}+\lambda\,\|y\|_{L^2(0,t)}^2\leqslant C_\alpha\left(\lambda^{-1}\,\|g\|_{L^2(0,t)}^2+\lambda^{-1}t^{1-2\alpha}\,|y_0|^2\right).$$

From Lemma A.2 it follows that

$$\lambda\,|y|_{H^{\alpha/2}(0,t)}^2+\lambda^2\,\|y\|_{L^2(0,t)}^2\leqslant C_\alpha\left(\|g\|_{L^2(0,t)}^2+t^{1-2\alpha}\,|y_0|^2\right).$$

Analogously, multiplying both sides of (6) by $\mathrm{D}_{0+}^{\alpha}\,y$ and integrating over $(0,t)$, we obtain

$$|y|_{H^\alpha(0,t)}^2+\lambda\,|y|_{H^{\alpha/2}(0,t)}^2\leqslant C_\alpha\left(\|g\|_{L^2(0,t)}^2+t^{1-2\alpha}\,|y_0|^2\right).$$

Therefore, combining the above two estimates yields (8).

Now, let us prove that

$$\lambda^\beta\,|y|_{H^{\alpha(1-\beta/2)}(0,t)}^2+\lambda^{1+\beta}\,|y|_{H^{\alpha(1-\beta)/2}(0,t)}^2\leqslant C_\alpha\left(\|g\|_{L^2(0,t)}^2+t^{1-2\alpha}\,|y_0|^2\right). \tag{9}$$

Since

$$\alpha(1-\beta/2)=\beta\,\alpha/2+(1-\beta)\,\alpha,$$

applying [1, Proposition 1.32] yields

$$|y|_{H^{\alpha(1-\beta/2)}}(0,t)\leqslant|y|_{H^{\alpha/2}(0,t)}^{\beta}\,|y|_{H^\alpha(0,t)}^{1-\beta}.$$

Therefore, by (8) we obtain

$$\begin{aligned}
&\lambda^{\beta} \left| y \right|^2_{H^{\alpha(1-\beta/2)}(0,t)} \leqslant \left(\lambda \left| y \right|^2_{H^{\alpha/2}(0,T)} \right)^{\beta} \left(\left| y \right|^2_{H^{\alpha}(0,t)} \right)^{1-\beta} \\
&\leqslant \lambda \left| y \right|^2_{H^{\alpha/2}(0,t)} + \left| y \right|^2_{H^{\alpha}(0,t)} \\
&\leqslant C_\alpha \left(\left\| g \right\|^2_{L^2(0,t)} + t^{1-2\alpha} \left| y_0 \right|^2 \right),
\end{aligned}$$

by the Young's inequality. Analogously, we have

$$\lambda^{1+\beta} \left| y \right|^2_{H^{(1-\beta)\alpha/2}(0,t)} \leqslant C_\alpha \left(\left\| g \right\|^2_{L^2(0,t)} + t^{1-2\alpha} \left| y_0 \right|^2 \right),$$

and using the above two estimates then proves (9).

Finally, combing (8) and (9) yields (7) and thus concludes the proof. ■

Lemma 3.2. *If* $1/2 \leqslant \alpha < 1$ *and* $0 \leqslant \theta < 1/\alpha - 1$, *then*

$$\begin{aligned}
&\lambda^{(\theta-1)/2} \left\| y \right\|_{H^{\alpha}(0,T)} + \left\| y \right\|_{H^{\alpha(1+\theta)/2}(0,T)} + \lambda^{\theta/2} \left| y \right|_{H^{\alpha/2}(0,T)} + \lambda^{1/2} \left\| y \right\|_{H^{\alpha\theta/2}(0,T)} \\
&+ \lambda^{(1+\theta)/2} \left\| y \right\|_{L^2(0,T)} \leqslant C_{\alpha,\theta,T} \left(\lambda^{(\theta-1)/2} \left\| g \right\|_{L^2(0,T)} + \left| y_0 \right| \right).
\end{aligned} \tag{10}$$

Proof. Proceeding as in the proof of Lemma 3.1 yields

$$y = y_0 + \frac{c}{\Gamma(\alpha)} t^{\alpha-1} + \mathrm{I}^{\alpha}_{0+}(g - \lambda y),$$

where

$$c = \left(\mathrm{I}^{1-\alpha}_{0+}(y - y_0) \right)(0).$$

Since $y \in H^{\alpha/2}(0,T)$ and Lemma A.4 implies $\mathrm{I}^{\alpha}_{0+}(g - \lambda y) \in H^{\alpha}(0,T)$, it is evident that $c = 0$, and hence

$$y = y_0 + \mathrm{I}^{\alpha}_{0+}(g - \lambda y) \in H^{\alpha}(0,T).$$

Furthermore, Lemma A.4 indicates that

$$\left\| y \right\|_{H^{\alpha}(0,T)} \leqslant C_{\alpha,T} \left(\left| y_0 \right| + \left\| g - \lambda y \right\|_{L^2(0,T)} \right). \tag{11}$$

Now, we proceed to prove (10), and since the techniques used below are similar to that used in the proof of Lemma 3.1, the forthcoming proof will be brief. Firstly, let us prove that

$$\left| y \right|^2_{H^{\alpha/2}(0,T)} + \lambda \left\| y \right\|^2_{L^2(0,T)} \leqslant C_{\alpha,\theta,T} \left(\lambda^{-1} \left\| g \right\|^2_{L^2(0,T)} + \lambda^{-\theta} \left| y_0 \right|^2 \right). \tag{12}$$

Using the standard estimate that ([35, Lemma 16.3])

$$\int_0^T t^{-(1-\theta)\alpha} \left| y(t) \right|^2 \, \mathrm{d}t \leqslant C_{\alpha,\theta} \left| y \right|^2_{H^{(1-\theta)\alpha/2}(0,T)},$$

by the Cauchy-Schwarz inequality we obtain

$$\left\langle \mathrm{D}^{\alpha}_{0+} y_0, y \right\rangle_{H^{\alpha/2}(0,T)} = \frac{y_0}{\Gamma(1-\alpha)} \left\langle t^{-\alpha}, y \right\rangle_{(0,T)} \leqslant C_{\alpha,\theta,T} \left| y_0 \right| \left| y \right|_{H^{(1-\theta)\alpha/2}(0,T)}.$$

Since

$$|y|_{H^{(1-\theta)\alpha/2}(0,T)} \leqslant \|y\|_{L^2(0,T)}^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta},$$

it follows that

$$\begin{aligned}
\left\langle \mathrm{D}_{0+}^{\alpha} y_0, y \right\rangle_{H^{\alpha/2}(0,T)} &\leqslant C_{\alpha,\theta,T} |y_0| \|y\|_{L^2(0,T)}^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta} \\
&\leqslant C_{\alpha,\theta,T} |y_0| \lambda^{-\theta/2} \left(\lambda^{1/2} \|y\|_{L^2(0,T)}\right)^{\theta} |y|_{H^{\alpha/2}(0,T)}^{1-\theta} \\
&\leqslant C_{\alpha,\theta,T} |y_0| \lambda^{-\theta/2} \left(|y|_{H^{\alpha/2}(0,T)} + \lambda^{1/2} \|y\|_{L^2(0,T)}\right).
\end{aligned} \tag{13}$$

In addition, inserting $z = y$ into (5) yields

$$\begin{aligned}
&|y|_{H^{\alpha/2}(0,T)}^2 + \lambda \|y\|_{L^2(0,T)}^2 \\
\leqslant{}& C_\alpha \left(\lambda^{-1} \|g\|_{L^2(0,T)}^2 + \left\langle \mathrm{D}_{0+}^{\alpha} y_0, y \right\rangle_{H^{\alpha/2}(0,T)}\right).
\end{aligned}$$

Consequently, inserting (13) into the above inequality and applying the Young's inequality with ϵ, we obtain (12).

Secondly, let us prove that

$$\|y\|_{H^\alpha(0,T)}^2 \leqslant C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda^{1-\theta} |y_0|^2\right). \tag{14}$$

Multiplying both sides of (6) by $\mathrm{D}_{0+}^{\alpha}(y - y_0)$ and integrating over $(0,T)$, we obtain

$$\begin{aligned}
&\left\|\mathrm{D}_{0+}^{\alpha}(y - y_0)\right\|_{L^2(0,T)}^2 + \lambda |y|_{H^{\alpha/2}(0,T)}^2 \\
\leqslant{}& C_{\alpha,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{\alpha} y_0, y \right\rangle_{H^{\alpha/2}(0,T)}\right),
\end{aligned}$$

so that from (13) and (12) it follows that

$$\begin{aligned}
&\left\|\mathrm{D}_{0+}^{\alpha}(y - y_0)\right\|_{L^2(0,T)}^2 + \lambda |y|_{H^{\alpha/2}(0,T)}^2 \\
\leqslant{}& C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + |y_0| \lambda^{1-\theta/2} \left(\lambda^{-1/2} \|g\|_{L^2(0,T)} + \lambda^{-\theta/2} |y_0|\right)\right) \\
\leqslant{}& C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + |y_0| \lambda^{1/2-\theta/2} \|g\|_{L^2(0,T)} + \lambda^{1-\theta} |y_0|^2\right) \\
\leqslant{}& C_{\alpha,\theta,T} \left(\|g\|_{L^2(0,T)}^2 + \lambda^{1-\theta} |y_0|^2\right).
\end{aligned}$$

Therefore, combing (6) and (11) yields (14).

Finally, using the same technique as that used to derive (9), by (12) and (14) we conclude that

$$\begin{aligned}
&\|y\|_{H^{\alpha(1+\theta)/2}(0,T)}^2 + \lambda \|y\|_{H^{\alpha\theta/2}(0,T)}^2 \\
\leqslant{}& C_{\alpha,\theta,T} \left(\lambda^{\theta-1} \|g\|_{L^2(0,T)}^2 + |y_0|^2\right),
\end{aligned}$$

which, together with (12) and (14), yields inequality (10). This theorem is thus proved. ■

Lemma 3.3. *Assume that* $1/2 < \alpha < 1$. *If* $y_0 = 0$ *and* $g \in H^{1-\alpha}(0,T)$, *then*

$$\|y\|_{H^1(0,T)} + \lambda^{1/2} \|y\|_{H^{1-\alpha/2}(0,T)} + \lambda \|y\|_{L^2(0,T)} \leqslant C_{\alpha,T} \|g\|_{H^{1-\alpha}(0,T)}. \tag{15}$$

Proof. Let us first prove that

$$y' = \mathrm{D}_{0+}^{1-\alpha}(g - \lambda y). \tag{16}$$

Since we have already proved

$$y = \mathrm{I}_{0+}^{\alpha}(g - \lambda y)$$

in the proof of Lemma 3.2, by Lemma A.4 we obtain $y \in H^1(0,T)$. Moreover, because

$$\left|\mathrm{I}_{0+}^{\alpha}(g - \lambda y)(s)\right| \leqslant \frac{s^{\alpha-1/2}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \left\|g - \lambda y\right\|_{L^2(0,s)}, \quad 0 < s < T,$$

we have

$$\lim_{s \to 0+} \mathrm{I}_{0+}^{\alpha}(g - \lambda y)(s) = 0.$$

Consequently, we obtain $y(0) = 0$ and hence

$$\mathrm{D}_{0+}^{\alpha} y = \mathrm{D}_{0+}^{\alpha} \mathrm{I}_{0+} y' = \mathrm{I}_{0+}^{1-\alpha} y',$$

which, together with (6), yields

$$\mathrm{I}_{0+}^{1-\alpha} y' = g - \lambda y.$$

Therefore,

$$y' = \mathrm{D}\,\mathrm{I}_{0+} y' = \mathrm{D}\,\mathrm{I}_{0+}^{\alpha} \mathrm{I}_{0+}^{1-\alpha} y' = \mathrm{D}_{0+}^{1-\alpha} \mathrm{I}_{0+}^{1-\alpha} y' = \mathrm{D}_{0+}^{1-\alpha}(g - \lambda y).$$

This proves equality (16).

Then, let us prove (15). Multiplying both sides of (16) by y' and integrating over $(0,T)$ yields

$$\left\|y'\right\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{1-\alpha} y, y' \right\rangle_{(0,T)} = \left\langle \mathrm{D}_{0+}^{1-\alpha} g, y' \right\rangle_{(0,T)},$$

so that

$$\left\|y'\right\|_{L^2(0,T)}^2 + \lambda \left\langle \mathrm{D}_{0+}^{1-\alpha} y, y' \right\rangle_{(0,T)} \leqslant C_{\alpha,T} \left\|g\right\|_{H^{1-\alpha}(0,T)}^2,$$

by the Cauchy-Schwarz inequality, Lemma A.2 and the Young's inequality with ϵ. Additionally, using the fact that $y \in H^1(0,T)$ with $y(0) = 0$ gives

$$\mathrm{D}_{0+}^{1-\alpha} y = \mathrm{D}\,\mathrm{I}_{0+}^{\alpha} y = \mathrm{I}_{0+}^{\alpha} y',$$

so that

$$\left\langle \mathrm{D}_{0+}^{1-\alpha} y, y' \right\rangle_{(0,T)} \geqslant C_{\alpha,T} \left\|y\right\|_{H^{1-\alpha/2}(0,T)}^2,$$

by Lemmas A.3 and A.5. Therefore,

$$\left\|y'\right\|_{L^2(0,T)}^2 + \lambda \left\|y\right\|_{H^{1-\alpha/2}(0,T)}^2 \leqslant C_{\alpha,T} \left\|g\right\|_{H^{1-\alpha}(0,T)}^2,$$

and hence, as Lemma 3.2 implies

$$\lambda \left\|y\right\|_{L^2(0,T)} \leqslant C_{\alpha,T} \left\|g\right\|_{L^2(0,T)},$$

we readily obtain (15). This completes the proof. ■

It is clear that we can represent u in the following form

$$u(t) = \sum_{i=0}^{\infty} y_i(t)\phi_i, \quad 0 < t < T,$$

where y_i solves problem (5) with λ, g and y_0 replaced by λ_i, f_i and $u_{0,i}$, respectively. Here, note that f_i and $u_{0,i}$ are the coordinates of f and u_0 respectively under the orthonormal basis $\{\phi_i : i \in \mathbb{N}\}$. Therefore, by the above three lemmas we readily conclude the following regularity estimates for problem (3).

Theorem 3.1. *Assume that* $0 < \alpha < 1/2$. *If* $f \in L^2(0,T;H^{-\beta}(\Omega))$ *and* $u_0 \in H^{-\beta}(\Omega)$ *with* $0 \leqslant \beta < 1$, *then*

$$\begin{aligned} & |u|_{H^{\alpha(1-\beta/2)}(0,t;L^2(\Omega))} + |y|_{H^{\alpha/2}(0,t;\dot{H}^{1-\beta}(\Omega))} + |u|_{H^{\alpha(1-\beta)/2}(0,t;\dot{H}^1(\Omega))} \\ & + \|u\|_{L^2(0,t;\dot{H}^{2-\beta}(\Omega))} \leqslant C_{\alpha,\Omega}\left(\|f\|_{L^2(0,t;H^{-\beta}(\Omega))} + t^{1/2-\alpha}\|u_0\|_{H^{-\beta}(\Omega)}\right) \end{aligned}$$

for all $0 < t < T$.

Theorem 3.2. *Assume that* $1/2 \leqslant \alpha < 1$. *If* $f \in L^2(0,T;H^{-\beta}(\Omega))$ *with* $2 - 1/\alpha < \beta < 1$ *and* $u_0 \in L^2(\Omega)$, *then*

$$\begin{aligned} & \|u\|_{H^{\alpha(1-\beta/2)}(0,T;L^2(\Omega))} + |u|_{H^{\alpha/2}(0,T;\dot{H}^{1-\beta}(\Omega))} + \|u\|_{H^{\alpha(1-\beta)/2}(0,T;\dot{H}^1(\Omega))} \\ & + \|u\|_{L^2(0,T;\dot{H}^{2-\beta}(\Omega))} \leqslant C_{\alpha,\beta,T,\Omega}\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right). \end{aligned}$$

Moreover, if $u_0 = 0$ *and* $f \in L^2(0,T;H^{-\beta}(\Omega))$ *with* $0 \leqslant \beta < 1$, *then the above estimate also holds.*

Theorem 3.3. *Assume that* $1/2 < \alpha < 1$. *If* $u_0 = 0$ *and* $f \in H^{1-\alpha}(0,T;L^2(\Omega))$, *then*

$$\begin{aligned} & \|u\|_{H^1(0,T;L^2(\Omega))} + \|u\|_{H^{1-\alpha/2}(0,T;\dot{H}^1(\Omega))} + \|u\|_{L^2(0,T;\dot{H}^2(\Omega))} \\ & \leqslant C_{\alpha,T,\Omega}\|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}. \end{aligned}$$

4 Convergence

We assume that u and U are respectively the solutions to problems (3) and (4), and by $a \lesssim b$ we mean that there exists a generic positive constant C, independent of h, τ and u, such that $a \leqslant Cb$. The main task of this section is to prove the following a priori error estimates.

Theorem 4.1. *Assume that* $0 < \alpha < 1/2$ *and* $0 \leqslant \beta < 1$. *If* $u_0 \in H^{-\beta}(\Omega)$ *and* $f \in L^2(0,T;H^{-\beta}(\Omega))$, *then*

$$\begin{aligned} & \|u - U\|_{L^2(0,T;\dot{H}^1(\Omega))} \\ & \lesssim \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{H^{-\beta}(\Omega)}\right), \end{aligned} \tag{17}$$

$$\begin{aligned} & \|u - U\|_{L^2(0,T;L^2(\Omega))} \\ & \lesssim \left(h^{2-\beta} + \tau^{\alpha(1-\beta/2)}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{H^{-\beta}(\Omega)}\right). \end{aligned} \tag{18}$$

Theorem 4.2. *Assume that $1/2 \leqslant \alpha < 1$ and $2 - 1/\alpha < \beta \leqslant 1$. If $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;H^{-\beta}(\Omega))$, then*

$$
\begin{aligned}
&\|u-U\|_{L^2(0,T;\dot H^1(\Omega))} \\
\lesssim{}& \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right), \\
&\|u-U\|_{L^2(0,T;L^2(\Omega))} \\
\lesssim{}& \left(h^{2-\beta} + \tau^{\alpha(1-\beta/2)}\right)\left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{L^2(\Omega)}\right).
\end{aligned}
$$

Moreover, if $u_0 = 0$ and $f \in L^2(0,T;H^{-\beta}(\Omega))$, then the above two estimates also hold for all $0 \leqslant \beta < 1$.

Theorem 4.3. *Assume that $1/2 < \alpha < 1$. If $u_0 = 0$ and $f \in H^{1-\alpha}(0,T;L^2(\Omega))$, then*

$$
\begin{aligned}
\|u-U\|_{L^2(0,T;L^2(\Omega))} &\lesssim \left(h^2+\tau\right)\|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}, \\
\|u-U\|_{L^2(0,T;\dot H^1(\Omega))} &\lesssim \left(h+\tau^{1-\alpha/2}\right)\|f\|_{H^{1-\alpha}(0,T;L^2(\Omega))}.
\end{aligned}
$$

Since the proofs of Theorems 4.2 and 4.3 are similar to that of Theorem 4.1, below we only show the latter. To this end, we start by introducing two interpolation operators. For any $v \in L^1(0,T;X)$ with X being a separable Hilbert space, define $P_\tau v$ by

$$
(P_\tau v)\,|_{I_j} := \frac{1}{\tau_j}\int_{I_j} v(t)\,\mathrm{d}t, \quad 1 \leqslant j \leqslant J.
$$

Let $P_h : L^2(\Omega) \to \mathcal{S}_h$ be the well-known Clément interpolation operator. For the above two operators, we have the following standard estimates [5, 4]: if $0 \leqslant \beta \leqslant 1$ and $\beta \leqslant \gamma \leqslant 2$, then

$$
\|(I-P_h)v\|_{H^\beta(\Omega)} \lesssim h^{\gamma-\beta}\|v\|_{\dot H^\gamma(\Omega)}, \quad \forall v \in \dot H^\gamma(\Omega);
$$

if $0 \leqslant \beta < 1/2$ and $\beta \leqslant \gamma \leqslant 1$, then

$$
\|(I-P_\tau)w\|_{H^\beta(0,T)} \lesssim \tau^{\gamma-\beta}\|w\|_{H^\gamma(0,T)}, \quad \forall w \in H^\gamma(0,T).
$$

For clarity, below we shall use the above two estimates implicitly.

Proof of Theorem 4.1. Let us first prove (17). By Lemma A.2, a standard procedure yields that

$$
\|u-U\|_W \lesssim \|u-P_\tau P_h u\|_W,
$$

then using the triangle inequality gives

$$
\begin{aligned}
\|u-U\|_W \lesssim{}& |(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} + |(I-P_\tau)P_h u|_{H^{\alpha/2}(0,T;L^2(\Omega))} \\
&+ \|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} + \|(I-P_\tau)P_h u\|_{L^2(0,T;\dot H^1(\Omega))}.
\end{aligned}
$$

Since

$$
\begin{aligned}
|(I-P_\tau)P_h u|_{H^{\alpha/2}(0,T;L^2(\Omega))} &\leqslant |(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))}, \\
\|(I-P_\tau)P_h u\|_{L^2(0,T;\dot H^1(\Omega))} &\lesssim \|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))},
\end{aligned}
$$

it follows that

$$\begin{aligned} \|u-U\|_W \lesssim{} & |(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} + |(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} \\ & + \|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} + \|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))}. \end{aligned} \tag{19}$$

Therefore, (17) is a direct consequence of Theorem 3.1 and the following estimates:

$$\begin{aligned} \|(I-P_h)u\|_{L^2(0,T;\dot H^1(\Omega))} &\lesssim h^{1-\beta} \|u\|_{L^2(0,T;\dot H^{2-\beta}(\Omega))}, \\ |(I-P_h)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} &\lesssim h^{1-\beta} |u|_{H^{\alpha/2}(0,T;\dot H^{1-\beta}(\Omega))}, \\ |(I-P_\tau)u|_{H^{\alpha/2}(0,T;L^2(\Omega))} &\lesssim \tau^{\alpha(1-\beta)/2} |u|_{H^{\alpha(1-\beta/2)}(0,T;L^2(\Omega))}, \\ \|(I-P_\tau)u\|_{L^2(0,T;\dot H^1(\Omega))} &\lesssim \tau^{\alpha(1-\beta)/2} |u|_{H^{\alpha(1-\beta)/2}(0,T;\dot H^1(\Omega))}. \end{aligned}$$

Then let us prove (18). By Lemma A.2, the well known Lax-Milgram theorem implies that there exists a unique $z \in W$ such that

$$\left\langle \mathrm{D}_{T-}^\alpha z, v \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla z, \nabla v \rangle_{\Omega\times(0,T)} = \langle u-U, v\rangle_{\Omega\times(0,T)}$$

for all $v \in W$. Substituting $v = u - U$ into the above equation yields

$$\begin{aligned} \|u-U\|_{L^2(0,T;L^2(\Omega))}^2 &= \left\langle \mathrm{D}_{T-}^\alpha z, u-U \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla z, \nabla(u-U)\rangle_{\Omega\times(0,T)} \\ &= \left\langle \mathrm{D}_{0+}^\alpha (u-U), z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla(u-U), \nabla z\rangle_{\Omega\times(0,T)}, \end{aligned}$$

by Lemma A.2. Setting $Z = P_\tau P_h z$, as combining (3) and (4) gives

$$\left\langle \mathrm{D}_{0+}^\alpha (u-U), Z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla(u-U), \nabla Z\rangle_{\Omega\times(0,T)} = 0,$$

we obtain

$$\begin{aligned} & \|u-U\|_{L^2(0,T;L^2(\Omega))}^2 \\ ={} & \left\langle \mathrm{D}_{0+}^\alpha (u-U), z-Z \right\rangle_{H^{\alpha/2}(0,T;L^2(\Omega))} + \langle \nabla(u-U), \nabla(z-Z)\rangle_{\Omega\times(0,T)}. \end{aligned}$$

Then Lemma A.2 implies that

$$\begin{aligned} \|u-U\|_{L^2(0,T;L^2(\Omega))}^2 &\leqslant |u-U|_{H^{\alpha/2}(0,T;L^2(\Omega))} |z-Z|_{H^{\alpha/2}(0,T;L^2(\Omega))} \\ &\quad + \|u-U\|_{L^2(0,T;\dot H^1(\Omega))} \|z-Z\|_{L^2(0,T;\dot H^1(\Omega))} \\ &\leqslant \|u-U\|_W \|z-Z\|_W. \end{aligned} \tag{20}$$

Similarly to the regularity estimate in Theorem 3.1, we have

$$\|z\|_{H^\alpha(0,T;L^2(\Omega))} + |z|_{H^{\alpha/2}(0,T;\dot H^1(\Omega))} + \|z\|_{L^2(0,T;\dot H^2(\Omega))} \lesssim \|u-U\|_{L^2(0,T;L^2(\Omega))},$$

so that proceeding as in the proof of (17) yields

$$\|z-Z\|_W \lesssim (h+\tau^{\alpha/2}) \|u-U\|_{L^2(0,T;L^2(\Omega))}.$$

Collecting the above estimate, (19) and (20) gives

$$\begin{aligned} & \|u-U\|_{L^2(0,T;L^2(\Omega))} \\ \lesssim{} & (h+\tau^{\alpha/2}) \left(h^{1-\beta} + \tau^{\alpha(1-\beta)/2}\right) \left(\|f\|_{L^2(0,T;H^{-\beta}(\Omega))} + \|u_0\|_{H^{-\beta}(\Omega)}\right). \end{aligned}$$

Therefore, (18) is a direct consequence of the following two estimates:

$$
\begin{aligned}
h\tau^{\alpha(1-\beta)/2} &= \left(h^{2-\beta}\right)^{1/(2-\beta)} \left(\tau^{\alpha(1-\beta/2)}\right)^{1-1/(2-\beta)} \\
&\leqslant h^{2-\beta}/(2-\beta) + (1-1/(2-\beta))\tau^{\alpha(1-\beta/2)}, \\
h^{1-\beta}\tau^{\alpha/2} &= \left(h^{2-\beta}\right)^{(1-\beta)/(2-\beta)} \left(\tau^{\alpha(1-\beta/2)}\right)^{1-(1-\beta)/(2-\beta)} \\
&\leqslant (1-\beta)/(2-\beta)h^{2-\beta} + (1-(1-\beta)/(2-\beta))\tau^{\alpha(1-\beta/2)}.
\end{aligned}
$$

This completes the proof. ■

5 Numerical Results

This section performs some numerical experiments to verify our theoretical results in one-dimensional space. We set $\Omega = (0,1)$, $T = 1$ and

$$
\begin{aligned}
\mathcal{E}_1 &:= \|\widetilde{u} - U\|_{L^2(0,T;H_0^1(\Omega))}, \\
\mathcal{E}_2 &:= \|\widetilde{u} - U\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

where $\widetilde{u}$ is a reference solution.

Experiment 1. This experiment verifies Theorem 4.1 under the condition that

$$
\begin{aligned}
u_0(x) &:= x^r, & & 0 < x < 1, \\
f(x,t) &:= x^r t^{-0.49}, & & 0 < x < 1,\ 0 < t < T.
\end{aligned}
$$

We first summarize the numerical results in Table 1 as follows.

- If $r = -0.8$, then

  $$u_0 \in H^{-\beta}(\Omega) \quad \text{and} \quad f \in L^2(0,T;H^{-\beta}(\Omega))$$

  for all $\beta > 0.3$. Therefore, Theorem 4.1 indicates that the spatial convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(h^{0.7})$ and $\mathcal{O}(h^{1.7})$, respectively. This is confirmed by the numerical results.

- If $r = -0.99$, then

  $$u_0 \in H^{-\beta}(\Omega) \quad \text{and} \quad f \in L^2(0,T;H^{-\beta}(\Omega))$$

  for all $\beta > 0.49$. Therefore, Theorem 4.1 indicates that the spatial convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(h^{0.51})$ and $\mathcal{O}(h^{1.51})$, respectively. This agrees well with the numerical results.

In the case of $\alpha = 0.4$ and $r = -0.49$, Theorem 4.1 indicates that the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.2})$ and $\mathcal{O}(\tau^{0.4})$, respectively. In the case of $\alpha = 0.4$ and $r = 0.99$, Theorem 4.1 indicates that the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.1})$ and $\mathcal{O}(\tau^{0.3})$, respectively. These theoretical results coincide with the numerical results in Table 2.

	h	$\alpha = 0.2$ $\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\alpha = 0.4$ $\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
	2^{-3}	7.56e-1	–	1.15e-2	–	8.12e-1	–	2.87e-2	–
	2^{-4}	4.78e-1	0.66	3.64e-3	1.66	5.23e-1	0.64	9.42e-3	1.61
$r = -0.8$	2^{-5}	2.99e-1	0.68	1.14e-3	1.68	3.30e-1	0.66	3.02e-3	1.64
	2^{-6}	1.85e-1	0.69	3.53e-4	1.69	2.06e-1	0.68	9.51e-4	1.67
	2^{-3}	1.51e-0	–	5.10e-2	–	1.64e-0	–	5.45e-2	–
	2^{-4}	1.07e-0	0.49	1.84e-2	1.47	1.19e-0	0.47	2.01e-2	1.44
$r = -0.99$	2^{-5}	7.54e-1	0.41	6.53e-3	1.49	8.42e-1	0.49	7.25e-3	1.47
	2^{-6}	5.27e-1	0.52	2.31e-3	1.50	5.91e-1	0.51	2.58e-3	1.49

Table 1: *Convergence history with $\tau = 2^{-15}$ ($\widetilde{u}$ is the numerical solution at $h = 2^{-11}$).*

$r = -0.49$ τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$r = -0.99$ τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-5}	4.54e-1	–	1.20e-2	–	2^{-3}	1.80	–	3.49e-1	–
2^{-6}	3.77e-1	0.27	9.53e-2	0.33	2^{-4}	1.62	0.15	2.93e-1	0.25
2^{-7}	3.11e-1	0.28	7.39e-2	0.37	2^{-5}	1.45	0.16	2.42e-1	0.28
2^{-8}	2.56e-1	0.28	5.63e-2	0.39	2^{-6}	1.30	0.16	1.96e-1	0.30

Table 2: *Convergence history with $\alpha = 0.4$ and $h = 2^{-10}$ ($\widetilde{u}$ is the numerical solution at $\tau = 2^{-17}$).*

Experiment 2. This experiment verifies Theorem 4.2 under the condition that

$$\begin{aligned} u_0(x) &:= cx^{-0.49}, && 0 < x < 1, \\ f(x,t) &:= x^{-0.8}t^{-0.49}, && 0 < x < 1,\ 0 < t < T. \end{aligned}$$

For $\alpha = 0.7$, Theorem 4.2 implies the following results: if $c = 0$, then

$$\mathcal{E}_1 \approx \mathcal{O}(h^{0.7}) \quad \text{and} \quad \mathcal{E}_2 \approx \mathcal{O}(h^{1.7});$$

if $c = 1$, then

$$\mathcal{E}_1 \approx \mathcal{O}(h^{0.43}) \quad \text{and} \quad \mathcal{E}_2 \approx \mathcal{O}(h^{1.43}).$$

These theoretical results are confirmed by the numerical results in Table 3.

For $\alpha = 0.8$, Theorem 4.2 implies the following results: if $c = 0$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.28})$ and $\mathcal{O}(\tau^{0.68})$, respectively; if $c = 1$, then the temporal convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are close to $\mathcal{O}(\tau^{0.1})$ and $\mathcal{O}(\tau^{0.5})$, respectively. These theoretical results are verified by Table 4.

h	$c=0$				$c=1$			
	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-2}	7.50e-1	–	5.07e-2	–	1.76e-0	–	1.04e-1	–
2^{-3}	5.12e-1	0.55	1.77e-2	1.52	1.37e-0	0.36	4.19e-2	1.32
2^{-4}	3.42e-1	0.58	6.03e-3	1.55	1.04e-0	0.40	1.67e-2	1.33
2^{-5}	2.23e-1	0.62	2.00e-3	1.59	7.56e-1	0.46	6.35e-3	1.39
2^{-6}	1.42e-1	0.65	6.49e-4	1.63	5.18e-1	0.55	2.26e-3	1.49

Table 3: *Convergence history with $\alpha = 0.7$ and $\tau = 2^{-15}$ ($\widetilde{u}$ is the numerical solution at $h = 2^{-11}$).*

$\mathcal{E}_1$					$\mathcal{E}_2$				
τ	$c=0$	Order	$c=1$	Order	τ	$c=0$	Order	$c=1$	Order
2^{-4}	3.08e-1	–	8.32e-1	–	2^{-7}	1.53e-2	–	2.69e-2	–
2^{-5}	2.55e-1	0.27	7.34e-1	0.18	2^{-8}	1.05e-2	0.55	1.88e-2	0.52
2^{-6}	2.09e-1	0.29	6.50e-1	0.18	2^{-9}	6.91e-3	0.60	1.31e-2	0.53
2^{-7}	1.69e-1	0.30	5.75e-1	0.18	2^{-10}	4.47e-3	0.63	9.00e-2	0.54
2^{-8}	1.37e-1	0.31	5.06e-1	0.18	2^{-11}	2.84e-3	0.65	6.17e-2	0.55
2^{-9}	1.10e-1	0.31	4.44e-1	0.19	2^{-12}	1.78e-3	0.68	4.19e-2	0.56

Table 4: *Convergence history with $\alpha = 0.8$, $r = -0.8$, and $h = 2^{-10}$ ($\widetilde{u}$ is the numerical solution at $\tau = 2^{-17}$).*

Experiment 3. This experiment verifies Theorem 4.3. Here we set $\alpha = 0.8$ and

$$
\begin{aligned}
u_0(x) &:= 0, && 0 < x < 1, \\
f(x,t) &:= x^{-0.49} t^{-0.29}, && 0 < x < 1,\ 0 < t < T.
\end{aligned}
$$

Theorem 4.3 implies that the convergence orders of $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{O}(h + \tau^{0.6})$ and $\mathcal{O}(h^2 + \tau)$, respectively, which is confirmed by Tables 5 and 6.

h	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-3}	1.09e-2	–	4.08e-3	–
2^{-4}	5.87e-2	0.89	1.11e-3	1.88
2^{-5}	3.13e-2	0.91	2.98e-4	1.90
2^{-6}	1.66e-2	0.92	7.92e-5	1.91
2^{-7}	8.71e-3	0.93	2.09e-5	1.92
2^{-8}	4.55e-3	0.94	5.47e-6	1.93

Table 5: *Convergence history with $\tau = 2^{-15}$ ($\widetilde{u}$ is the numerical solution at $h = 2^{-12}$).*

τ	$\mathcal{E}_1$	Order	$\mathcal{E}_2$	Order
2^{-6}	2.32e-2	–	6.75e-3	–
2^{-7}	1.52e-2	0.62	4.19e-3	0.69
2^{-8}	9.73e-3	0.64	2.47e-3	0.76
2^{-9}	6.22e-3	0.65	1.41e-3	0.81
2^{-10}	3.97e-3	0.65	7.81e-4	0.85
2^{-11}	2.54e-3	0.65	4.27e-4	0.87

Table 6: *Convergence history with $h = 2^{-10}$ ($\widetilde{u}$ is the numerical solution at $\tau = 2^{-17}$).*

A Properties of Fractional Calculus Operators

Lemma A.1 ([34, 9, 33]). *Let $-\infty < a < b < \infty$. If $0 < \beta, \gamma < \infty$, then*

$$
\mathrm{I}_{a+}^{\beta}\, \mathrm{I}_{a+}^{\gamma} = \mathrm{I}_{a+}^{\beta+\gamma}, \quad \mathrm{I}_{b-}^{\beta}\, \mathrm{I}_{b-}^{\gamma} = \mathrm{I}_{b-}^{\beta+\gamma},
$$

and

$$\left\langle \mathrm{I}_{a+}^{\beta}\, v, w \right\rangle_{(a,b)} = \left\langle v, \mathrm{I}_{b-}^{\beta}\, w \right\rangle_{(a,b)}$$

for all $v, w \in L^2(a,b)$.

Lemma A.2 ([10]). *Assume that* $-\infty < a < b < \infty$ *and* $0 < \gamma < 1/2$. *If* $v \in H^\gamma(a,b)$, *then*

$$\begin{aligned}
&\left\| \mathrm{D}_{a+}^{\gamma}\, v \right\|_{L^2(a,b)} \leqslant |v|_{H^\gamma(a,b)}\,, \\
&\left\| \mathrm{D}_{b-}^{\gamma}\, v \right\|_{L^2(a,b)} \leqslant |v|_{H^\gamma(a,b)}\,, \\
&\left\langle \mathrm{D}_{a+}^{\gamma}\, v, \mathrm{D}_{b-}^{\gamma}\, v \right\rangle_{(a,b)} = \cos(\gamma\pi)\, |v|_{H^\gamma(a,b)}^2 .
\end{aligned}$$

Moreover, if $v, w \in H^\gamma(a,b)$, *then*

$$\begin{aligned}
&\left\langle \mathrm{D}_{a+}^{\gamma}\, v, \mathrm{D}_{b-}^{\gamma}\, w \right\rangle_{(a,b)} \leqslant |v|_{H^\gamma(a,b)}\, |w|_{H^\gamma(a,b)}\,, \\
&\left\langle \mathrm{D}_{a+}^{2\gamma}\, v, w \right\rangle_{H^\gamma(a,b)} = \left\langle \mathrm{D}_{a+}^{\gamma}\, v, \mathrm{D}_{b-}^{\gamma}\, w \right\rangle_{(a,b)} = \left\langle \mathrm{D}_{b-}^{2\gamma}\, w, v \right\rangle_{H^\gamma(a,b)} .
\end{aligned}$$

Lemma A.3. *If* $0 < \gamma < 1/2$ *and* $v \in L^2(0,1)$, *then*

$$C_1 \left\| \mathrm{I}_{0+}^{\gamma}\, v \right\|_{L^2(0,1)}^2 \leqslant \left(\mathrm{I}_{0+}^{\gamma}\, v, \mathrm{I}_{T-}^{\gamma}\, v \right)_{L^2(0,1)} \leqslant C_2 \left\| \mathrm{I}_{0+}^{\gamma}\, v \right\|_{L^2(0,1)}^2 , \tag{21}$$

where C_1 *and* C_2 *are two positive constants that depend only on* γ.

Proof. Extending v to $\mathbb{R}\backslash(0,1)$ by zero, we define

$$\begin{aligned}
w_+(t) &:= \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{t} (t-s)^{\gamma-1} v(s)\, \mathrm{d}s, \quad -\infty < t < \infty, \\
w_-(t) &:= \frac{1}{\Gamma(\gamma)} \int_{t}^{\infty} (s-t)^{\gamma-1} v(s)\, \mathrm{d}s, \quad -\infty < t < \infty.
\end{aligned}$$

Since $0 < \gamma < 1/2$, a routine calculation yields $w_+, w_- \in L^2(\mathbb{R})$, and [34, Theorem 7.1] implies that

$$\begin{aligned}
\mathcal{F}w_+(\xi) &= (\mathrm{i}\xi)^{-\gamma} \mathcal{F}v(\xi), \quad -\infty < \xi < \infty, \\
\mathcal{F}w_-(\xi) &= (-\mathrm{i}\xi)^{-\gamma} \mathcal{F}v(\xi), \quad -\infty < \xi < \infty.
\end{aligned}$$

By the Plancherel Theorem and the same technique as that used to prove [10, Lemma 2.4], it follows that

$$\begin{aligned}
&\left(\mathrm{I}_{0+}^{\gamma}\, v, \mathrm{I}_{1-}^{\gamma}\, v \right)_{L^2(0,1)} = (w_+, w_-)_{L^2(\mathbb{R})} = (\mathcal{F}w_+, \mathcal{F}w_-)_{L^2(\mathbb{R})} \\
={}& \cos\left(\gamma\pi\right) \int_{\mathbb{R}} |\xi|^{-2\gamma} \left| \mathcal{F}v(\xi) \right|^2 \, \mathrm{d}\xi \\
={}& \cos(\gamma\pi)\, \|w_+\|_{L^2(\mathbb{R})}^2 = \cos(\gamma\pi)\, \|w_-\|_{L^2(\mathbb{R})}^2 .
\end{aligned}$$

Therefore, by the Cauchy-Schwarz inequality, (21) follows from the following two estimates:

$$\left\| \mathrm{I}_{0+}^{\gamma}\, v \right\|_{L^2(0,1)} \leqslant \|w_+\|_{L^2(\mathbb{R})}\,, \quad \left\| \mathrm{I}_{1-}^{\gamma}\, v \right\|_{L^2(0,1)} \leqslant \|w_-\|_{L^2(\mathbb{R})} .$$

■

Lemma A.4. *If $\beta \in (0,1) \setminus \{0.5\}$ and $0 < \gamma < \infty$, then*

$$\left\| \mathrm{I}_{0+}^{\gamma} v \right\|_{H^{\beta+\gamma}(0,1)} \leqslant C_{\beta,\gamma} \left\| v \right\|_{H^{\beta}(0,1)} \tag{22}$$

for all $v \in H_0^{\beta}(0,1)$. Furthermore, if $0 < \gamma < 1/2$ and $v \in H^{1-\gamma}(0,1)$ with $v(0) = 0$, then

$$\left\| \mathrm{I}_{0+}^{\gamma} v \right\|_{H^{1}(0,1)} \leqslant C_{\gamma} \left\| v \right\|_{H^{1-\gamma}(0,1)} . \tag{23}$$

Proof. For the proof of (22), we refer the reader to [17] (Lemma A.4). Let us prove (23) as follows. Define $\widetilde{v} := v - g$, where

$$g(t) := tv(1), \quad 0 < t < 1.$$

It is clear that $\widetilde{v} \in H_0^{1-\gamma}(0,1)$, and hence (22) implies

$$\left\| \mathrm{I}_{0+}^{\gamma} \widetilde{v} \right\|_{H^{1}(0,1)} \leqslant C_{\gamma} \left\| \widetilde{v} \right\|_{H_0^{1-\gamma}(0,1)} .$$

Therefore, from the evident estimate

$$\left\| g \right\|_{H^{1-\gamma}(0,1)} + \left\| \mathrm{I}_{0+}^{\gamma} g \right\|_{H^{1}(0,1)} \leqslant C_{\gamma} \left| v(1) \right| ,$$

it follows that

$$\begin{aligned}
\left\| \mathrm{I}_{0+}^{\gamma} v \right\|_{H^{1}(0,1)} &\leqslant \left\| \mathrm{I}_{0+}^{\gamma} \widetilde{v} \right\|_{H^{1}(0,1)} + \left\| \mathrm{I}_{0+}^{\gamma} g \right\|_{H^{1}(0,1)} \\
&\leqslant C_{\gamma} \left\| \widetilde{v} \right\|_{H_0^{1-\gamma}(0,1)} + \left\| \mathrm{I}_{0+}^{\gamma} g \right\|_{H^{1}(0,1)} \\
&\leqslant C_{\gamma} \left(\left\| v \right\|_{H^{1-\gamma}(0,1)} + \left\| g \right\|_{H^{1-\gamma}(0,1)} \right) + \left\| \mathrm{I}_{0+}^{\gamma} g \right\|_{H^{1}(0,1)} \\
&\leqslant C_{\gamma} \left(\left\| v \right\|_{H^{1-\gamma}(0,1)} + \left| v(1) \right| \right) .
\end{aligned}$$

As $0 < \gamma < 1/2$ implies

$$\left\| v \right\|_{C[0,1]} \leqslant C_{\gamma} \left\| v \right\|_{H^{1-\gamma}(0,1)} ,$$

this indicates (23) and thus proves the lemma. ■

Lemma A.5. *If $0 < \gamma < 1/2$ and $v \in H^{1}(0,1)$, then*

$$C_1 \left\| v \right\|_{H^{1-\gamma}(0,1)} \leqslant \left| v(0) \right| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^{2}(0,1)} \leqslant C_2 \left\| v \right\|_{H^{1-\gamma}(0,1)} , \tag{24}$$

where C_1 and C_2 are two positive constants that depend only on γ.

Proof. Since a simple calculation gives

$$\mathrm{D}\, \mathrm{I}_{0+}^{\gamma} (v - v(0)) = \mathrm{D}\, \mathrm{I}_{0+}^{\gamma}\, \mathrm{I}_{0+}\, v' = \mathrm{I}_{0+}^{\gamma}\, v',$$

using Lemma A.4 yields

$$\begin{gathered}
\left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^{2}(0,1)} \leqslant \left\| \mathrm{I}_{0+}^{\gamma} (v - v(0)) \right\|_{H^{1}(0,1)} \\
\leqslant C_{\gamma} \left\| v - v(0) \right\|_{H^{1-\gamma}(0,1)} \leqslant C_{\gamma} \left(\left| v(0) \right| + \left\| v \right\|_{H^{1-\gamma}(0,1)} \right) ,
\end{gathered}$$

which, together with the estimate

$$\left| v(0) \right| \leqslant C_{\gamma} \left\| v \right\|_{H^{1-\gamma}(0,1)} \quad (\text{since } 1 - \gamma > 0.5),$$

indicates

$$|v(0)| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \leqslant C_\gamma \left\| v \right\|_{H^{1-\gamma}(0,1)}.$$

Conversely, by

$$v = \mathrm{I}_{0+}^{1-\gamma} \mathrm{I}_{0+}^{\gamma} v' + v(0),$$

using Lemma A.4 again yields

$$\left\| v \right\|_{H^{1-\gamma}(0,1)} \leqslant C_\gamma \left(|v(0)| + \left\| \mathrm{I}_{0+}^{\gamma} v' \right\|_{L^2(0,1)} \right).$$

This lemma is thus proved. ■

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