# ANALYSIS OF A DEGENERATE AND SINGULAR VOLUME-FILLING CROSS-DIFFUSION SYSTEM MODELING BIOFILM GROWTH 

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#### Abstract

We analyze the mathematical properties of a multi-species biofilm crossdiffusion model together with very general reaction terms and mixed Dirichlet-Neumann boundary conditions on a bounded domain. This model belongs to the class of volumefilling type cross-diffusion systems which exhibit a porous medium-type degeneracy when the total biomass vanishes as well as a superdiffusion-type singularity when the biomass reaches its maximum cell capacity, which make the analysis extremely challenging. The equations also admit a very interesting non-standard entropy structure. We prove the existence of global-in-time weak solutions, study the asymptotic behavior and the uniqueness of the solutions, and complement the analysis by numerical simulations that illustrate the theoretically obtained results.


## 1. Introduction

In this paper we study the mathematical properties of a multi-species cross-diffusion biofilm model recently proposed by Rahman, Sudarsan and Eberl [22], which describes the local mixing effects between different components of multi-species biofilm colonies. These effects are extremely useful in wastewater engineering, where different processes (like aerobic and anoxic processes or simultaneous sulfate reduction and nitrogen removal) take place simultaneously. It has been pointed out [22] that when two colonies of different species merge, spatial biomass gradients can be observed, leading to a spatially heterogeneous distribution of biomass. These phenomena can be described by cross diffusion, which models how the diffusion of one species is influenced by the concentration gradient of the other species in diffusive multi-species systems. Recently, a cross-diffusion biofilm model (see (3)) was introduced by Rahman, Sudarsan and Eberl [22], which reflects the same properties as the single-species nonlinear diffusion model [9] (see (6)) constructed from experiments, namely a porous-medium type degeneracy when the local biomass vanishes, which leads to a finite speed of propagation of the interface, and a singularity when the

[^0]biomass reaches the maximum capacity, which guarantees the boundedness of the total mass. It can be formally derived from a space-discrete random-walk lattice model [20, 22, 26] (see Appendix). Due to the cross-diffusion structure, standard techniques like maximum principles and regularity theory cannot be used, and since the diffusion matrix is generally neither symmetric nor positive definite, even the local-in-time existence and boundedness of solutions is hard to prove. However, in recent years significant progress has been made in the analysis of cross-diffusion equations by using the entropy methods. These techniques are based on the identification of a structural condition, namely a formal gradient-flow or entropy structure, allowing for a mathematical treatment, see e.g. [2, 3, 7, 13, 14, 26].

In this article we prove the global-in-time existence of weak solutions to the multispecies cross-diffusion biofilm model [22], study its long-time behavior and prove uniqueness of solutions, and we complement our results by some numerical simulations with finite elements by using the free software DUNE [8]. For the analytical results, we significantly extend the entropy method in [13], which is based upon the idea of transforming the system into so called entropy variables such that the diffusion matrix in the new formulation is positive definite. This approach was adapted to a class of degenerate volume-filling type models] in [26]. However, compared to [26], we have to deal with an additional singularity, which significantly complicates the analysis, but surprisingly also helps to handle very general (even singular) reaction terms. The model we study also admits a non-standard entropy structure, which is an interesting mathematical issue by itself; see e.g. [6, [15, 16] for other works in this direction.

We are interested in a reaction-cross-diffusion system with volume-filling of the form

$$
\begin{equation*}
\partial_{t} u_{i}-\sum_{j=1}^{n} \operatorname{div}\left(A_{i j}(u) \nabla u_{j}\right)=r_{i}(u) \quad(i=1, \ldots, n) \quad \text { on } \Omega, t>0 \tag{1}
\end{equation*}
$$

Here, $\Omega \subseteq \mathbb{R}^{d}(d \geq 1)$ is a bounded domain with Lipschitz boundary, $A(u)=\left(A_{i j}(u)\right) \in$ $\mathbb{R}^{n \times n}$ is the diffusion matrix, $u=\left(u_{1}, \ldots, u_{n}\right): \Omega \times(0, \infty) \rightarrow \mathbb{R}^{n}$ is the vector of proportions of the species within the biofilm, where $u_{i}=u_{i}(t, x)$ depends on the time $t$ and the spatial variable $x$ on $\Omega$, and $r=\left(r_{1}, \ldots, r_{n}\right)$ is the vector of reactions. The diffusion coefficients $A_{i j}$ are derived under suitable modeling assumptions in a (formal) diffusive limit from a space-discrete lattice model [20, 21, 26, 22] (sketched in the Appendix). They have the form

$$
\begin{equation*}
A_{i j}(u)=\alpha_{i} \delta_{i j} p(M) q(M)+\alpha_{i} u_{i}\left(p(M) q^{\prime}(M)-p^{\prime}(M) q(M)\right) \tag{2}
\end{equation*}
$$

Here, $M=\sum_{i=1}^{n} u_{i}$ denotes the total biomass, $\delta_{i j}$ is the Kronecker delta symbol, while the functions $p$ and $q$ measure how favorable it is for species $u_{i}$ to leave or to arrive at a certain cell in the underlying discrete model. The constants $\alpha_{i}>0$ measure how fast biomass moves between neighboring sites of the underlying discrete lattice model (see Appendix).

[^1]We point out that eqs. (1)-(2) can be also written as

$$
\begin{equation*}
\partial_{t} u_{i}-\alpha_{i} \operatorname{div}\left(p^{2}(M) \nabla\left(\frac{u_{i} q(M)}{p(M)}\right)\right)=r_{i}(u), \quad i=1, \ldots, n . \tag{3}
\end{equation*}
$$

For simplicity, we have assumed that $p=p(M)$ and $q=q(M)$ only depend on the total biomass $M$ and are the same for all species $i=1, \ldots, n$. The total biomass $M$ cannot exceed a saturation value (normalized to 1 ), which depends on the maximum cell capacity (hence the denomination "volume filling"); i.e. $M \leq 1$ must hold during the time evolution of the system.

We supplement the model with initial conditions and mixed Dirichlet-Neumann boundary conditions on the bounded domain $\Omega \subset \mathbb{R}^{d}(d \geq 1)$

$$
\begin{align*}
& u(0, \cdot)=u^{0}(\cdot)>0 \quad \text { in } \Omega, \\
& u=u_{D}>0 \quad \text { on } \Gamma_{D}, \quad \nu \cdot \nabla u=0 \quad \text { on } \Gamma_{N}, \quad t>0, \tag{4}
\end{align*}
$$

where $\partial \Omega=\Gamma_{D} \cup \Gamma_{N}, \Gamma_{D} \cap \Gamma_{N}=\emptyset$ and $\left|\Gamma_{D}\right|>0$.
For simplicity we have assumed that $u_{D}=\left(u_{D, 1}, \ldots, u_{D, n}\right) \in(0, \infty)^{n}$ is a constant vector with positive components, but also $x$-dependent boundary data can be treated; see Remark 9 for details. For consistency with the constraint $M \leq 1$, we have to assume

$$
\begin{equation*}
M_{D}=\sum_{i=1}^{n} u_{D, i}<1, \quad \sup _{x \in \Omega} M_{0}(x)=\sum_{i=1}^{n} \sup _{x \in \Omega} u_{i, 0}(x)<1 . \tag{5}
\end{equation*}
$$

These boundary conditions describe well the behavior of the biofilm at its border, but also homogeneous Neumann boundary conditions can be handled, see Remarks 10, 11, 12,

In order to close the model, the functions $p$ and $q$ need to be chosen appropriately. Following the approach in [22], the idea is to derive $p$ and $q$ in (3) from the single-species biofilm model [9, 17] (here without reaction)

$$
\begin{equation*}
\partial_{t} M-\operatorname{div}\left(\frac{M^{a}}{(1-M)^{b}} \nabla M\right)=0, \quad a, b>1 \tag{6}
\end{equation*}
$$

under the natural assumption that the evolution of the multi-species model (3) reduces to the single-species model (6) if all species except one vanish or if all species are identical. Thus, we choose $p$ and $q$ such that

$$
\begin{equation*}
p^{2}(M) \nabla\left(\frac{M q(M)}{p(M)}\right)=\frac{M^{a}}{(1-M)^{b}} \nabla M, \quad a, b>1, \tag{7}
\end{equation*}
$$

and consequently

$$
\begin{equation*}
q(M)=\frac{p(M)}{M} \int_{0}^{M} \frac{s^{a}}{(1-s)^{b}} \frac{d s}{p(s)^{2}}, \quad M>0 \tag{8}
\end{equation*}
$$

For $p$ we assume that

$$
\begin{equation*}
p \text { is decreasing, } \quad p(1)=0, \quad \exists \kappa, c>0: \lim _{M \rightarrow 1}-(1-M)^{1+\kappa} \frac{p^{\prime}(M)}{p(M)}=c . \tag{9}
\end{equation*}
$$

These hypothesis are consistent with the modeling assumptions of the single-species model (6) in [21]. The last assumption quantifies how fast $p$ decreases to 0 for $M \rightarrow 1$; in fact, an integration yields the bound

$$
\begin{equation*}
p(M) \leq C_{1} \exp \left(-C_{2}(1-M)^{-\kappa}\right) \quad 0<M<1 \tag{10}
\end{equation*}
$$

This hypothesis on $p$ is not assumed in [22], and is needed here only for technical reasons in Lemma 7.

As mentioned before, due to the cross-diffusion structure, standard techniques like maximum principles and regularity theory cannot be used, but still, in recent years lots of progress has been made in the analysis of cross-diffusion equations by identifying a formal gradient-flow or entropy structure [13, 14]. Following the approach therein, we assume that there exists a convex function $h: \mathcal{D} \rightarrow \Omega$ called entropy density with $\mathcal{D} \subseteq \mathbb{R}^{n}$ such that the matrix $B=A(u) h^{\prime \prime}(u)^{-1}$ is positive semi-definite, and (1) can be written as

$$
\partial_{t} u-\operatorname{div}\left(B \nabla h^{\prime}(u)\right)=r(u)
$$

where $h^{\prime}$ and $h^{\prime \prime}$ are the Jacobian and the Hessian of $h$, respectively. This structural assumption has two very useful consequences. First, $H[u]=\int_{\Omega} h(u) d x$ is a Lyapunov functional along solutions to (1) and (2) if the reaction term vanishes, because

$$
\frac{d H}{d t}[u(t)]=\int_{\Omega} h^{\prime}(u) \cdot \partial_{t} u d x=-\int_{\Omega} \nabla u: h^{\prime \prime}(u) A(u) \nabla u d x=-\int_{\Omega} \nabla w: B \nabla w d x \leq 0
$$

were $w=h^{\prime}(u)$ are so called entropy variables. This often yields gradient-type estimates for $u$ if suitable lower bounds for the matrix $h^{\prime \prime}(u) A(u)$ are known. Second, if $h^{\prime}$ is invertible on $\mathcal{D}$, then it holds that $u=\left(h^{\prime}\right)^{-1}(w) \in \mathcal{D}$. Consequently we get that if $\mathcal{D}$ is a bounded domain, then we obtain lower and upper bound for $u$ without using a maximum principle. In our case, we require that any solution $u$ to (11)-(4) takes values in the set

$$
\begin{equation*}
\mathcal{D}=\left\{u \in(0, \infty)^{n}: \sum_{i=1}^{n} u_{i}<1\right\} \tag{11}
\end{equation*}
$$

Moreover, we define the (relative) entropy functional $H[u]$ of the system (1) as

$$
\begin{equation*}
H[u]=\int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x, \quad h^{*}\left(u \mid u_{D}\right)=h(u)-h\left(u_{D}\right)-h^{\prime}\left(u_{D}\right) \cdot\left(u-u_{D}\right) \tag{12}
\end{equation*}
$$

where the entropy density $h(u)$ is given by

$$
\begin{equation*}
h(u)=\sum_{i=1}^{n}\left(u_{i} \log u_{i}-u_{i}+1\right)+\int_{0}^{M} \log \left(\frac{q(s)}{p(s)}\right) d s \tag{13}
\end{equation*}
$$

and thus the entropy variables for $i=1, \ldots, n$ read as

$$
\begin{equation*}
w_{i}=\frac{\partial}{\partial u_{i}} h^{*}\left(u \mid u_{D}\right)=\frac{\partial h}{\partial u_{i}}(u)-\frac{\partial h}{\partial u_{i}}\left(u_{D}\right)=\log \left(\frac{u_{i} q(M)}{p(M)}\right)-\log \left(\frac{u_{D, i} q\left(M_{D}\right)}{p\left(M_{D}\right)}\right) \tag{14}
\end{equation*}
$$

We can show that the entropy dissipation leads to the following very interesting degeneratesingular entropy estimate

$$
\int_{0}^{T} \int_{\Omega} \frac{M^{a-1}|\nabla M|^{2}}{(1-M)^{1+b+\kappa}} d x d t+\sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x d t \leq C \quad t>0
$$

which has the same kind of singular-degenerate type structure like the nonlinear diffusion coefficients $M^{a} /(1-M)^{b}$ in the single species model (6), i.e. a degeneracy when $M \rightarrow 0$ and a singularity when $M \rightarrow 1$. Moreover, it leads to the following uniform bound for the singularity:

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}(1-M)^{1-b-\kappa} d x d t \leq C \tag{15}
\end{equation*}
$$

This estimate is essential in order to control the nonlinear terms in the equations; furthermore, it implies that $M<1$ a.e. in $Q_{T}$, i.e. saturation of the biofilm is excluded.
1.1. State of the art. In the literature, several model classes for biofilms can be found. The first class consists of deterministic continuous equations based on the one-dimensional Wanner-Gujer model [25]. An important assumption of this model is that the volume fractions occupied by the different species add up to unity. However, no mixing of initially separated species can occur under this assumption, which contradicts results of microscopic experiments, where spatially heterogeneous distributions of biomass could be observed. A second model class are stochastic discrete multi-species biofilm models, which do not need this problematic assumption for the volume fractions, and the amount of mixing can be decided by the user by formulating local interaction rules. However, these models have the drawback that mixing is often overemphasized and that the numerical solution is generally very time-consuming.

In order to compensate the disadvantages of the model classes described above, Rahman, Sudarsan, and Eberl [22] introduced a two-species diffusion model which captures the quantitative amount of local mixing effects between the species within a biofilm colony with the help of cross-diffusion terms. The derivations of this system from mass balances or from discrete lattice models do not (a priori) impose the condition that the volume fractions must add up to unity. Besides the new cross-diffusion effects, it has two additional difficulties: (i) a porous medium degeneracy when the total biomass $M=0$ is vanishing; (ii) a super-diffusion singularity when the total biomass equals one $M=1$. Due to property (i) the interface between the aqueous phase and biofilm region propagates with a finite speed. Property (ii) ensures that the solutions of (1) are bounded by the maximum cell density $M \leq 1$. Moreover, we are able to prove that similarly to the results in [10, 11] it holds that $M<1$ a.e. in $Q_{T}$ for all $t>0$ in the case of mixed Dirichlet-Neumann boundary conditions (even for nonzero Dirichlet data; see Step 4 in the proof of Theorem 1 for details). On the other hand, in the case of homogeneous Neumann boundary conditions on the whole $\partial \Omega$ we need to make sure that the total mass $\mathcal{M}(t)=|\Omega|^{-1} \int_{\Omega} M(x, t) d x$ remains strictly smaller than one for any time in order to prevent blowup, see Remark 10, which is again similar to the results in [10, 11]. While an existence analysis for the single-species biofilm model is
available in [11], the mathematical analysis of the multi-species model (3)-(4) has not been carried out so far (up to our knowledge). We note that Laoshen Li studied traveling wave solutions and instability conditions of a reaction-cross-diffusion biofilm model in [18], and recently Schulz and Knabner analyzed an effective model for biofilm growth in [23, 24]. Our technique is based on the boundedness-by-entropy method of A. Jüngel (see [13, Theorem 3] and [14]), which was refined for a general class of degenerate volume-filling type crossdiffusion models in [26]. However, the model considered in this article does not only exhibit a degeneracy at $M=0$, but also a very interesting singularity at $M=1$, which goes far beyond the framework of [26]. We point out that the mentioned singularity significantly complicates the analysis, but on the other hand it also yields an additional a priori bound of $(1-M)$ to a negative power (see (15)), which allows to handle very general (even singular) reaction terms (see assumptions (16)-(19) for the reaction terms). Note that in [26] no reaction terms were treated, and only no-flux boundary conditions were considered.
1.2. Mathematical assumptions on the reaction terms. We assume that

$$
\begin{equation*}
r(u)=r^{D}(u)+\tilde{r}(u) \tag{16}
\end{equation*}
$$

with $r^{D}, \tilde{r}$ continuous in the set $\{M<1\}$ satisfying the following conditions:

$$
\begin{align*}
& \exists \lambda_{r} \geq 0: \sum_{j=1}^{n} r_{j}^{D}(u)\left(\frac{\partial h}{\partial u_{j}}(u)-\frac{\partial h}{\partial u_{j}}\left(u_{D}\right)\right) \leq \lambda_{r}\left(1+h^{*}\left(u \mid u_{D}\right)\right),  \tag{17}\\
& \exists C_{r} \geq 0, \quad 0 \leq \mu<b-1, s>0:\left|\tilde{r}_{i}(u)\right| \leq \frac{C_{r} u_{i}^{s}}{(1-M)^{\mu}} \quad(i=1, \ldots, n),  \tag{18}\\
& \exists C_{r}^{\prime} \geq 0, \quad 0 \leq \eta<b+\kappa-1:\left|r_{i}^{D}(u)\right| \leq \frac{C_{r}^{\prime}}{(1-M)^{\eta}} \quad(i=1, \ldots, n) . \tag{19}
\end{align*}
$$

Note that $r(u)$ decomposes into a "dissipative" part $r^{D}$ (i.e. a part which can be controlled by the entropy density, see [13, assumption (H3)]), and a remainder $\tilde{r}$ which can be controlled by means of the entropy dissipation (see proof of Lemma 8). Let us point out that these reaction terms are rather general, in fact, even singular reaction terms are allowed.
1.3. Structure of the paper. The main results are given in Section 2. Proofs of some auxiliary results, like the asymptotic behavior of $p$ and $q$, the convexity of $h$, the invertibility of $h^{\prime}$ and the lower bound for the entropy dissipation are given in Section 3. Sections 4, 5 and 6 are devoted to proofs of the existence, long-time behavior and the uniqueness result, respectively. Finally, in Section 7 we discuss the formal derivation of the model and the underlying modeling assumptions (Subsection 7.1), we show some numerical simulations (Subsection 7.2), and we prove a non-standard version of the Poincaré inequality used within this paper (Subsection 7.3).

## 2. Main results

The first result we prove is about the global-in-time existence of weak solutions to (3)(4). In the following, $Q_{T} \equiv \Omega \times(0, T)$ for every $T>0$.

Theorem 1 (Existence theorem). Under the assumptions (8), (9), (16) -(19), eqs. (3) -(4) have a solution $u: \Omega \times(0, \infty) \rightarrow \mathbb{R}^{n}$ such that, for every $T>0$ and $1 \leq i \leq n$,

$$
\begin{aligned}
& u_{i} \geq 0, M=\sum_{j=1}^{n} u_{j}<1 \quad \text { a.e. in } Q_{T} \\
& M^{\frac{a+1}{2}} \in L^{2}\left(0, T ; H^{1}(\Omega)\right), \quad M^{\frac{a+1}{2}} \nabla u_{i} \in L^{2}\left(Q_{T}\right), \\
& (1-M)^{1-\kappa} \in L^{\infty}\left(0, T ; L^{1}(\Omega)\right), \quad(1-M)^{1-b-\kappa} \in L^{1}\left(Q_{T}\right), \\
& \partial_{t} u_{i} \in\left(L^{\frac{\rho+1}{\rho}}\left(0, T ; W^{1, \frac{\rho+1}{\rho}}(\Omega)\right) \cap L^{\frac{b+\kappa-1}{b+\kappa-1-\eta}}\left(Q_{T}\right)\right)^{\prime},
\end{aligned}
$$

where $a, b, \kappa, \eta$ are as in (8), (9), (19), and $\rho=\min \{1, \kappa /(b-1)\}$. Moreover, for any $t>0$ the following entropy inequality holds:

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+2 \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p(M)^{2}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x  \tag{20}\\
& \quad \leq C_{1} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+C_{2}
\end{align*}
$$

where $C_{1}$ and $C_{2}$ are some suitable nonnegative constants. Finally, $C_{1}=C_{2}=0$ if $\lambda_{r}=$ $C_{r}=0$ in (17) and (18).

The proof of Theorem 1 is based upon the semi-discretization in time of (1). The resulting elliptic problem reads as

$$
\frac{u_{i}^{j}-u_{i}^{j-1}}{\tau}+\operatorname{div}\left(p\left(M^{j}\right)^{2} \nabla\left(\frac{u_{i}^{j} q\left(M^{j}\right)}{p\left(M^{j}\right)}\right)\right)=r_{i}\left(u^{j}\right), \quad i=1, \ldots, n, x \in \Omega
$$

A higher order regularizing term is also added, which is needed in order to prove the well-posedness of the time-discretized equations. The key tool in the analysis is a discrete entropy inequality:

$$
\begin{aligned}
& \frac{1}{\tau} \int_{\Omega}\left(h^{*}\left(u^{j} \mid u_{D}\right)-h^{*}\left(u^{j-1} \mid u_{D}\right)\right) d x+2 \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p\left(M^{j}\right)^{2}\left|\nabla \sqrt{\frac{u_{i}^{j} q\left(M^{j}\right)}{p\left(M^{j}\right)}}\right|^{2} d x \\
& \quad \leq C_{1} \int_{\Omega} h^{*}\left(u^{j} \mid u_{D}\right) d x+C_{2}
\end{aligned}
$$

which yields crucial gradient estimates for the solution $u^{j}$ to the time-discretized problem. The entropy dissipation satisfies the bound (see Lemma 7)

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega} p^{2}(M)\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x d t \\
& \quad \geq C \int_{0}^{T} \int_{\Omega} \frac{M^{a-1}|\nabla M|^{2}}{(1-M)^{1+b+\kappa}} d x d t+\sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x d t
\end{aligned}
$$

which leads to (15). This estimate of the singularity is crucial to control the nonlinear terms, and implies that no saturation occurs in the biofilm.

The second result we prove concerns the long-time behavior of the solutions to (1)-(4).
Theorem 2 (Convergence to steady state). Let all the assumptions from Theorem 1 be fulfilled. In addition, assume that $\lambda_{r}=C_{r}=0$ in (17) and (18) and $b \geq 2$. Then there exists a constant $C>0$ such that for any $t>0$ it holds

$$
\sum_{i=1}^{n}\left\|u_{i}(t)-u_{D, i}\right\|_{L^{2}(\Omega)}^{2} \leq \frac{C}{1+t}
$$

This means that the solutions to (3)-(4) converge to the constant steady state $u_{D}$ as $t \rightarrow \infty$. The main idea of the large-time asymptotic analysis of $u_{i}(t):=u_{i}(\cdot, t)$ is to exploit the entropy inequality (20) in the case when $C_{1}=C_{2}=0$. We show that the entropy dissipation dominates the square of the entropy functional, i.e.
from where we deduce that the convergence is of order $1 / t$, as $t \rightarrow \infty$. Finally, strict convexity of the relative entropy density (see Lemma 5), gives the convergence in $L^{2}$-norm. We note that we were not able to prove an exponential decay rate due to the lack of suitable convex Sobolev inequalities for (11). However, we point out that our numerical simulations suggest that exponential decay should hold, see Subsection 7.2 ,

The third result we present is about the uniqueness of the solution to (11)-(4). Uniqueness of solutions is achieved provided that additional assumptions on the reaction term are made. Precisely, we assume that functions $r_{1}^{(0)}, \ldots, r_{n}^{(0)}, r^{(1)}, R:[0,1) \rightarrow \mathbb{R}$ exist such that

$$
\begin{align*}
& r_{i}(u)=r_{i}^{(0)}(M)+r^{(1)}(M) u_{i}, \quad i=1, \ldots, n, \quad u \in \mathcal{D},  \tag{21}\\
& \exists \varepsilon_{0}>0: r_{i}^{(0)}(M) \geq \max \left\{0, \varepsilon_{0} r^{(1)}(M)\right\}, \quad i=1, \ldots, n, \quad M \in[0,1),  \tag{22}\\
& \exists C_{R} \in \mathbb{R}: \sum_{j=1}^{n} r_{j}(u)=R(M)+C_{R} M, \quad u \in \mathcal{D},  \tag{23}\\
& \exists C_{R}^{\prime}>0: \frac{|R(M)|}{M}+\left|R^{\prime}(M)\right| \leq C_{R}^{\prime} M^{a / 2}, \quad M \in(0,1) \tag{24}
\end{align*}
$$

An example of reaction term satisfying both sets of assumptions (16) $-(19),(21)-(24)$ is

$$
\begin{equation*}
r_{i}(u)=u_{D, i}-u_{i}, \quad i=1, \ldots, n, \quad u \in \mathcal{D} \tag{25}
\end{equation*}
$$

Furthermore, we assume that the parameters $\alpha_{1}, \ldots, \alpha_{n}$ are all the same (and therefore without loss of generality we set $\left.\alpha_{i}=1, i=1, \ldots, n\right)$.
Theorem 3 (Uniqueness of solutions). Let the assumptions of Theorem 1 hold. Furthermore, we assume that $\alpha_{i}=1$ for all $i=1, \ldots, n$ and that the reaction terms satisfy the assumptions given by (21)-(24). Then there exists a unique weak solution to (3)-(4).

We point out that the proof of uniqueness of weak solutions for strongly coupled crossdiffusion systems is delicate. Similarly like in [4, 26], our uniqueness proof is based on a combination of the $H^{-1}$ method and the technique of Gajewski [12].

## 3. Auxiliary results

In this section we state technical results which are used for proving the main results of this paper: asymptotic behavior of functions $p(M)$ and $q(M)$ when $M \rightarrow 0$ and $M \rightarrow 1$, the convexity of the entropy density $h$, the invertibility of the gradient of the relative entropy density $h^{*}$ with respect to the variable $u$ and finally the lower bound for the entropy dissipation.

Lemma 4 (Asymptotic behavior of $p, q$ ). Let $a, b>1, \kappa>0$ and $0<M<1$. For functions $p$ and $q$ defined by (8) and (9), there exist positive constants $C_{1}, C_{2}$ and $C_{3}$ such that

$$
\begin{align*}
\lim _{M \rightarrow 1} \frac{p(M) q(M)}{(1-M)^{1+\kappa-b}} & =C_{1}  \tag{26}\\
\lim _{M \rightarrow 1} \frac{\log (q(M) / p(M))}{(1-M)^{-\kappa}} & =C_{2},  \tag{27}\\
\lim _{M \rightarrow 0} M^{-a} q(M) & =\frac{C_{3}}{p(0)} . \tag{28}
\end{align*}
$$

Proof. The proof of limits given by formulas (26) and (27) directly follows using l'Hôpital's rule. In order to show (28) we perform the change of variable $s=M \sigma$ in the integral appearing in the definition of $q$ (8).
Lemma 5 (Convexity of $h$ ). It holds that the matrix $\mathcal{H}(u) \equiv\left(\frac{\partial w_{i}}{\partial u_{j}}(u)\right)_{i, j=1}^{n}$ is positive definite and symmetric on $\mathcal{D}$, where $w_{i}$ is defined in (14) and $\mathcal{D}$ is given by (11).

Proof. Direct calculation using (8) gives

$$
\begin{equation*}
\mathcal{H}_{i j}=\frac{\partial w_{i}}{\partial u_{j}}=\left(\frac{\delta_{i j}}{u_{i}}-\frac{1}{M}\right)+\frac{M^{a}(1-M)^{-b} p(M)^{-2}}{\int_{0}^{M} s^{a}(1-s)^{-b} p(s)^{-2} d s} . \tag{29}
\end{equation*}
$$

Next, let us write the matrix $\mathcal{H}$ as the $\operatorname{sum} \mathcal{H}=\mathcal{A}+\mathcal{B}$, with $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n \times n}$ given by

$$
\mathcal{A} \equiv \operatorname{diag}\left(\frac{1}{u_{1}}, \ldots, \frac{1}{u_{n}}\right)-\frac{1}{M} \mathcal{C} \quad \text { and } \quad \mathcal{B} \equiv \frac{M^{a}(1-M)^{-b} p(M)^{-2}}{\int_{0}^{M} s^{a}(1-s)^{-b} p(s)^{-2} d s} \mathcal{C}
$$

and with $\mathcal{C} \in \mathbb{R}^{n \times n}, \mathcal{C}_{i j}:=1$ for all $i, j \in\{1, \ldots, n\}$. Clearly, the matrix $\mathcal{B}$ is positive semidefinite, since for any $v \in \mathbb{R}^{n}$ one has

$$
v \cdot \mathcal{B} v=\frac{M^{a}(1-M)^{-b} p(M)^{-2}}{\int_{0}^{M} s^{a}(1-s)^{-b} p(s)^{-2} d s}\left(\sum_{i=1}^{n} v_{i}\right)^{2} \geq 0
$$

On the other side, matrix $\mathcal{A}$ is also positive semidefinite. Namely, for any $v \in \mathbb{R}^{n}$ we have

$$
v \cdot \mathcal{A} v=\sum_{i=1}^{n} \frac{v_{i}^{2}}{u_{i}}-\frac{1}{M}\left(\sum_{i=1}^{n} v_{i}\right)^{2} \geq \sum_{i=1}^{n} z_{i}^{2}-\frac{1}{\sum_{j=1}^{n} u_{j}}\left(\sum_{i=1}^{n} u_{i}\right)\left(\sum_{i=1}^{n} z_{i}^{2}\right)=0 .
$$

where we used the notation $z_{i}:=v_{i} / \sqrt{u_{i}}$ and the Cauchy-Schwarz inequality. Consequently, it follows that $\mathcal{H}=\mathcal{A}+\mathcal{B}$ is positive semidefinite. It remains to show the strict positive definiteness of $\mathcal{H}$. For this, we take a vector $v \in \mathbb{R}^{n}$ and show that if $v \cdot \mathcal{H} v=0$, then it follows that $v_{i}=0$ for $i=1, \ldots, n$. Let be $v \cdot \mathcal{H} v=0$, then since matrices $\mathcal{A}$ and $\mathcal{B}$ are positive semidefinite, it holds $v \cdot \mathcal{B} v=0$, and $v \cdot \mathcal{A} v=0$. Now, from $v \cdot \mathcal{B} v=0$ follows directly that $\sum_{i=1}^{n} v_{i}=0$. On the other side, from

$$
0=v \cdot \mathcal{A} v=\sum_{i=1}^{n} \frac{v_{i}^{2}}{u_{i}}-\frac{1}{M}\left(\sum_{i=1}^{n} v_{i}\right)^{2}=\sum_{i=1}^{n} \frac{v_{i}^{2}}{u_{i}}
$$

we get directly that $v_{i}=0$ for all $i=1, \ldots, n$. Therefore $\mathcal{H}$ is positive definite in $\mathcal{D}$.
Lemma 6 (Invertibility of $\left.\left(h^{*}\right)^{\prime}\right)$. The function $\left(h^{*}\right)^{\prime}: \mathcal{D} \rightarrow \mathbb{R}^{n}$ is invertible, where $\mathcal{D}$ is defined in (11).
Proof. First, note that due to (14) we have (slight change)

$$
\begin{equation*}
\frac{u_{D, i}}{M_{D}} e^{w_{i}}=\frac{u_{i}}{M_{D}} \frac{q(M) / p(M)}{q\left(M_{D}\right) / p\left(M_{D}\right)} \tag{30}
\end{equation*}
$$

Now we define the auxiliary function

$$
\begin{equation*}
\Phi(M):=\frac{M q(M)}{p(M)} \tag{31}
\end{equation*}
$$

After summing the relation (30) for $i=1, \ldots n$, one gets

$$
\begin{equation*}
\Phi(M)=\Phi\left(M_{D}\right) \sum_{i=1}^{n} \frac{u_{D, i}}{M_{D}} e^{w_{i}} \tag{32}
\end{equation*}
$$

Note that function $\Phi(M)$ is strictly increasing with $\Phi(0)=0$ and $\lim _{M \rightarrow 1} \Phi(M)=+\infty$. Thus, there exists a unique solution $M=M[w] \in(0,1)$ to the nonlinear equation (32). Replacing $M=M[w]$ into relation (30) and then solving the resulting equation for $u_{i}$ yields the statement.
Lemma 7 (Lower bound for the entropy dissipation). For any sufficiently smooth function $u: Q_{T} \rightarrow \mathcal{D}$ it holds that

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega} p^{2}(M)\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x d t \\
& \quad \geq C \int_{0}^{T} \int_{\Omega} \frac{M^{a-1}|\nabla M|^{2}}{(1-M)^{1+b+\kappa}} d x d t+\sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x d t
\end{aligned}
$$

where $\kappa>0$ is defined in (9).

Proof. Let us define $f(M):=\sqrt{q(M) / p(M)}$. Direct calculation gives:
$p^{2}(M) \sum_{i=1}^{n}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2}=p(M) q(M) \sum_{i=1}^{n}\left|\nabla \sqrt{u_{i}}\right|^{2}+p(M)^{2} f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right)|\nabla M|^{2}$.
Let us first show that for $0<M<1$ the function $f^{\prime}(M)$ is strictly positive. Note that

$$
2 \frac{f^{\prime}(M)}{f(M)}=\frac{d}{d M} \log \left(\frac{q(M)}{p(M)}\right)
$$

Using the definition (8) we have

$$
\begin{equation*}
2 \frac{f^{\prime}(M)}{f(M)}=\frac{M^{a}(1-M)^{-b} p^{-2}(M)}{\int_{0}^{M} s^{a}(1-s)^{-b} p^{-2}(s) d s}-\frac{1}{M} . \tag{33}
\end{equation*}
$$

Since $p$ is decreasing it holds $(1-s)^{-b} p^{-2}(s) \leq(1-M)^{-b} p^{-2}(M)$ for $0 \leq s \leq M$. Therefore

$$
2 \frac{f^{\prime}(M)}{f(M)} \geq \frac{M^{a}}{\int_{0}^{M} s^{a} d s}-\frac{1}{M}=\frac{a}{M}>0, \quad 0<M<1
$$

from where it follows that $f^{\prime}(M)>0$ for $0<M<1$. Using this result, we get
$p^{2}(M) f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right) \geq p^{2}(M) M\left(f^{\prime}(M)\right)^{2} \geq \frac{a^{2} p^{2}(M)}{4 M}(f(M))^{2}=\frac{a^{2} q(M) p(M)}{4 M}$.
Since $p(s) \leq p(0)$ and $p(0)>0$ for $s>0$, one has

$$
p(M) q(M)=\frac{p^{2}(M)}{M} \int_{0}^{M} \frac{s^{a}}{(1-s)^{b}} \frac{d s}{p(s)^{2}} \geq \frac{p^{2}(M)}{p^{2}(0)} \frac{1}{M} \int_{0}^{M} \frac{s^{a}}{(1-s)^{b}} d s
$$

Since $p(M) \geq p(1 / 2)$ for $0 \leq M \leq 1 / 2$ and $(1-s)^{-b} \geq 1$ we have

$$
p(M) q(M) \geq \frac{p^{2}(1 / 2)}{p^{2}(0)} \frac{M^{a}}{a+1}
$$

Therefore we get

$$
\begin{equation*}
p^{2}(M) f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right) \geq C M^{a-1}, \quad 0 \leq M \leq \frac{1}{2} \tag{34}
\end{equation*}
$$

On the other side, let us find the lower bound of term $p^{2}(M) f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right)$ for $1 / 2 \leq M<1$. For that purpose, we can make the following estimate:

$$
p(M)^{2} f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right) \geq \frac{1}{2} p^{2}(M)\left(f^{\prime}(M)\right)^{2}=\frac{1}{2} p(M) q(M)\left(\frac{f^{\prime}(M)}{f(M)}\right)^{2}
$$

It remains to bound the term $f^{\prime}(M) / f(M)$ from below for $1 / 2 \leq M<1$. For the moment, we go back to (33). Note that for $s \leq M$ it holds $s^{a} \leq M^{a}$, so the following estimate holds:

$$
2 \frac{f^{\prime}(M)}{f(M)} \geq \frac{(1-M)^{-b} p^{-2}(M)}{\int_{0}^{M}(1-s)^{-b} p^{-2}(s) d s}-\frac{1}{M}
$$

We want to find a lower bound for the right-hand side of the above inequality for $M$ close to 1 . Thanks to (10) we have

$$
\lim _{M \rightarrow 1}(1-M)^{1+\kappa-b} p^{-2}(M)=+\infty
$$

Applying the l'Hôpital's rule and using (19), one gets

$$
\lim _{M \rightarrow 1} \frac{(1-M)^{1+\kappa-b} p^{-2}(M)}{\int_{0}^{M}(1-s)^{-b} p^{-2}(s) d s}=\lim _{M \rightarrow 1}\left((1+\kappa-b)(1-M)^{\kappa}-2(1-M)^{1+\kappa} \frac{p^{\prime}(M)}{p(M)}\right)=2 c
$$

It follows that there exists a constant $c_{1}>0$ such that

$$
\frac{f^{\prime}(M)}{f(M)} \geq c_{1}(1-M)^{-(1+\kappa)}, \quad \text { for } \quad \frac{1}{2} \leq M<1
$$

From the above estimate and (26) we deduce

$$
\begin{equation*}
p(M)^{2} f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right) \geq C(1-M)^{-1-b-\kappa}, \quad \frac{1}{2} \leq M<1 \tag{35}
\end{equation*}
$$

Putting (34), (35) together yields that there exists a constant $C>0$ such that

$$
p(M)^{2} f^{\prime}(M)\left(M f^{\prime}(M)+f(M)\right) \geq \frac{C M^{a-1}}{(1-M)^{1+b+\kappa}}, \quad 0 \leq M<1
$$

which finishes the proof of this Lemma.

## 4. Proof of Theorem 1

For $m \in \mathbb{Z}, m \geq 1$ we define the space

$$
H_{D}^{m}(\Omega)=\left\{u \in H^{m}(\Omega): u \equiv 0 \text { on } \Gamma_{D}\right\} .
$$

The proof is divided into several steps.
Step 1: discretization. Fix $T>0$. For $N \in \mathbb{N}$ we define $\tau=T / N, t_{j}=\tau j(j=$ $0, \ldots, N), u_{i}^{0}=u_{i, 0}(i=1, \ldots, n)$. In order to have a compact embedding $H^{m}(\Omega) \hookrightarrow$ $L^{\infty}(\Omega)$ we choose $m$ to be the smallest integer such that $m>d / 2$. For $j \geq 1$ consider the problem:

$$
\begin{align*}
& \text { given } w^{j-1} \in H_{D}^{m}(\Omega), \text { find } w^{j} \in H_{D}^{m}(\Omega) \text { such that } \\
& \sum_{i=1}^{n} \int_{\Omega}\left(\frac{u_{i}^{j}-u_{i}^{j-1}}{\tau} \phi_{i}+\alpha_{i} p\left(M^{j}\right)^{2} \nabla\left(\frac{u_{i}^{j} q\left(M^{j}\right)}{p\left(M^{j}\right)}\right) \cdot \nabla \phi_{i}-r_{i}\left(u^{j}\right) \phi_{i}\right) d x  \tag{36}\\
& \quad=-\tau \sum_{i=1}^{n}\left(w_{i}^{j}, \phi_{i}\right)_{H^{m}(\Omega)}, \quad \forall \phi=\left(\phi_{1}, \ldots, \phi_{n}\right) \in H_{D}^{m}(\Omega)^{n},
\end{align*}
$$

where $u^{j-1}, u^{j}: \Omega \times(0, T) \rightarrow \mathbb{R}^{n}$ are defined by

$$
h^{\prime}\left(u^{j-1}\right)-h^{\prime}\left(u_{D}\right)=w^{j-1}, \quad h^{\prime}\left(u^{j}\right)-h^{\prime}\left(u_{D}\right)=w^{j},
$$

and $M^{j} \equiv \sum_{i=1}^{n} u_{i}^{j}$, while $(\cdot, \cdot)_{H^{m}(\Omega)}$ denotes the standard scalar product in $H^{m}(\Omega)$. We point out that, since $h^{\prime}: \mathcal{D} \rightarrow \mathbb{R}^{n}$ is invertible, then $u^{j-1}, u^{j}$ are well defined.

Step 2: fixed point. We solve (36) via Leray-Schauder fixed point theorem. Let us define the mapping

$$
\begin{array}{cccc}
F: \quad L^{\infty}(\Omega) \times[0,1] & \rightarrow & L^{\infty}(\Omega) \\
\left(w^{*}, \sigma\right) & \mapsto & w
\end{array}
$$

where $w$ is the solution of the linearized approximated problem

$$
\begin{align*}
& \tau \sum_{i=1}^{n}\left(w_{i}, \phi_{i}\right)_{H^{m}(\Omega)}  \tag{37}\\
& =-\sigma \sum_{i=1}^{n} \int_{\Omega}\left(\frac{u_{i}^{*}-u_{i}^{j-1}}{\tau} \phi_{i}+\alpha_{i} p\left(M^{*}\right)^{2} \nabla\left(\frac{u_{i}^{*} q\left(M^{*}\right)}{p\left(M^{*}\right)}\right) \cdot \nabla \phi_{i}-r_{i}\left(u^{*}\right) \phi_{i}\right) d x \\
& \quad \forall \phi \in H_{D}^{m}\left(\Omega ; \mathbb{R}^{n}\right),
\end{align*}
$$

with $u^{*}: \Omega \times(0, T) \rightarrow \mathbb{R}^{n}$ defined by $h^{\prime}\left(u^{*}\right)-h^{\prime}\left(u_{D}\right)=w^{*}$ and $M^{*} \equiv \sum_{i=1}^{n} u_{i}^{*}$.
We first point out that $F$ is well defined. In fact, assumption $m>d / 2$ implies that $H^{m}(\Omega) \hookrightarrow L^{\infty}(\Omega)$. Since $w^{*} \in H^{m}\left(\Omega ; \mathbb{R}^{n}\right)$, this means that $\inf _{\Omega} M^{*}>0, \sup _{\Omega} M^{*}<1$, and $u_{i}^{*}, M^{*} \in H^{m}(\Omega)$. These properties ensure that the right-hand side of (37) defines a continuous linear functional $f: \phi \in H^{m}\left(\Omega ; \mathbb{R}^{n}\right) \mapsto f(\phi) \in \mathbb{R}$. Therefore we can deduce by Lax-Milgram Lemma the existence of a unique solution $w \in H_{D}^{m}\left(\Omega ; \mathbb{R}^{n}\right) \subset L^{\infty}\left(\Omega ; \mathbb{R}^{n}\right)$ to (37).

Next, we observe that $F(\cdot, 0) \equiv 0$ (trivial). Choosing $\phi=w$ in (37) allows us to easily deduce that $\|w\|_{H^{m}(\Omega)} \leq C$ for some constant $C=C\left[w^{*}, u^{j-1}\right]>0$. This bound and the compact embedding $H_{D}^{m}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ imply that $F$ is compact. By standard arguments we can prove that $F$ is continuous.

Let us now assume that $w \in H^{m}\left(\Omega ; \mathbb{R}^{n}\right)$ is a fixed point of $F(\cdot, \sigma)$ for some $\sigma \in[0,1]$, and rename $u \equiv u^{*}, M \equiv M^{*}$ for better readability. Define $q_{D} \equiv q\left(\sum_{i=1}^{n} u_{D, i}\right), p_{D} \equiv$ $p\left(\sum_{i=1}^{n} u_{D, i}\right)$. By choosing $\phi_{i}=w_{i}$ in (37) and exploiting (14) we obtain

$$
\sigma \sum_{i=1}^{n} \int_{\Omega}\left(\left.\frac{u_{i}-u_{i}^{j-1}}{\tau} w_{i}+4 \alpha_{i} p(M)^{2} \right\rvert\, \nabla{\sqrt{\frac{u_{i} q(M)}{p(M)}}}^{2}-r_{i}(u) w_{i}\right) d x+\tau \sum_{i=1}^{n}\left\|w_{i}\right\|_{H^{m}(\Omega)}^{2}=0 .
$$

However, since $w=h^{\prime}(u)-h^{\prime}\left(u_{D}\right)$ and $h$ is convex, it follows

$$
\sum_{i=1}^{n}\left(u_{i}-u_{i}^{j-1}\right) w_{i} \geq h^{*}\left(u \mid u_{D}\right)-h^{*}\left(u^{j-1} \mid u_{D}\right)
$$

and therefore

$$
\begin{aligned}
\frac{\sigma}{\tau} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x & +4 \sigma \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p(M)^{2}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x \\
& +\tau \sum_{i=1}^{n}\left\|w_{i}\right\|_{H^{m}(\Omega)}^{2} \leq \frac{\sigma}{\tau} \int_{\Omega} h^{*}\left(u^{j-1} \mid u_{D}\right) d x+\sigma \sum_{i=1}^{n} \int_{\Omega} r_{i}(u) w_{i} d x
\end{aligned}
$$

By applying Lemma 7 we deduce

$$
\begin{align*}
& \frac{\sigma}{\tau} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+2 \sigma \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p(M)^{2}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x+\tau \sum_{i=1}^{n}\left\|w_{i}\right\|_{H^{m}(\Omega)}^{2}  \tag{38}\\
& \quad \leq \frac{\sigma}{\tau} \int_{\Omega} h^{*}\left(u^{j-1} \mid u_{D}\right) d x+\sigma \sum_{i=1}^{n} \int_{\Omega} r_{i}(u) w_{i} d x-C_{0} \sigma \int_{\Omega} \frac{M^{a-1}|\nabla M|^{2}}{(1-M)^{1+b+\kappa}} d x .
\end{align*}
$$

We are going to show that the right-hand side of the above inequality can be bound by the entropy. Let $M_{D}=\sum_{i=1}^{n} u_{D, i} \in(0,1)$. It holds that

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x d t \\
& \geq \int_{\left\{M \geq M_{D}\right\}} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x d t \\
& \quad \geq M_{D}^{a-1} \int_{0}^{T} \int_{\Omega} \frac{\chi_{\left\{M \geq M_{D}\right\}}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x d t \\
& \quad=C \int_{0}^{T} \int_{\Omega}\left|\nabla\left((1-M)^{\frac{1-b-\kappa}{2}}-\left(1-M_{D}\right)^{\frac{1-b-\kappa}{2}}\right)_{+}\right|^{2} d x d t \\
& \quad \geq C_{P} \int_{0}^{T} \int_{\Omega}\left((1-M)^{\frac{1-b-\kappa}{2}}-\left(1-M_{D}\right)^{\frac{1-b-\kappa}{2}}\right)_{+}^{2} d x d t
\end{aligned}
$$

where we used the Poincaré inequality in the last line. Thus we obtain

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \frac{M^{a-1}|\nabla M|^{2}}{(1-M)^{1+b+\kappa}} d x d t \geq c \int_{0}^{T} \int_{\Omega}(1-M)^{1-b-\kappa} d x d t-C \tag{39}
\end{equation*}
$$

Thanks to (16)-(18) the reaction term can be bounded as follows:

$$
\begin{aligned}
& \sum_{i=1}^{n} \int_{\Omega} r_{i}(u) w_{i} d x \leq \lambda_{r} \int_{\Omega}\left(1+h^{*}\left(u \mid u_{D}\right)\right) d x \\
& \quad+C_{r} \sum_{i=1}^{n} \int_{\Omega} \frac{u_{i}^{s}}{(1-M)^{\mu}}\left|\log \left(\frac{u_{i} q(M)}{p(M)}\right)-\log \left(\frac{u_{D, i} q_{D}}{p_{D}}\right)\right| d x \\
& \leq \lambda_{r} \int_{\Omega}\left(1+h^{*}\left(u \mid u_{D}\right)\right) d x+C_{r} c_{1} \int_{\Omega}(1-M)^{-\mu} d x \\
& \quad+C_{r} c_{2} \int_{\Omega} \frac{M^{s}|\log (q(M) / p(M))|}{(1-M)^{\mu}} d x
\end{aligned}
$$

By using (27) we deduce

$$
\sum_{i=1}^{n} \int_{\Omega} r_{i}(u) w_{i} d x \leq \lambda_{r} \int_{\Omega}\left(1+h^{*}\left(u \mid u_{D}\right)\right) d x+C_{r} C \int_{\Omega}(1-M)^{-\kappa-\mu} d x
$$

Due to assumption (18) we have $\mu<b-1$, so we can apply Young inequality to the right-hand side of the above estimate and conclude

$$
\sum_{i=1}^{n} \int_{\Omega} r_{i}(u) w_{i} d x \leq \lambda_{r} \int_{\Omega}(1+h(u)) d x+C_{r} \varepsilon \int_{\Omega}(1-M)^{1-b-\kappa} d x+C_{r} C(\varepsilon)
$$

By choosing $\varepsilon>0$ small enough in the above estimate and exploiting (39), from (38) we get

$$
\begin{align*}
\frac{\sigma}{\tau} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x & +2 \sigma \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p(M)^{2}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} d x  \tag{40}\\
& +\tau \sum_{i=1}^{n}\left\|w_{i}\right\|_{H^{m}(\Omega)}^{2} \leq\left(\frac{\sigma}{\tau}+C_{1}\right) \int_{\Omega} h^{*}\left(u^{j-1} \mid u_{D}\right) d x+C_{2}
\end{align*}
$$

for some suitable constants $C_{1}, C_{2} \geq 0$, which are independent of both $\sigma$ and $\tau$. Moreover, the constants $C_{1}, C_{2}$ can be chosen to be equal to zero in the case that $\lambda_{r}=C_{r}=0$ in (17), (18). In particular, (40) yields a $\sigma$-uniform bound for $w$ in $H^{m}(\Omega)$, and a fortiori in $L^{\infty}(\Omega)$.

Thanks to Leray-Schauder's fixed point theorem we infer the existence of a fixed point $w^{j} \in H^{m}\left(\Omega ; \mathbb{R}^{n}\right)$ for $F(\cdot, 1)$, that is, a solution to (36).

Step 3: uniform in $\tau$ a-priori estimates. Let us define the piecewise constant-in-time functions

$$
u^{(\tau)}(t)=u^{0} \chi_{\{0\}}(t)+\sum_{j=1}^{N} u^{j} \chi_{\left(t_{j-1}, t_{j}\right]}(t), \quad w^{(\tau)}(t)=w^{0} \chi_{\{0\}}(t)+\sum_{j=1}^{N} w^{j} \chi_{\left(t_{j-1}, t_{j}\right]}(t),
$$

and let $M^{(\tau)}=\sum_{i=1}^{n} u_{i}^{(\tau)}$. We also define the discrete backward time derivative operator $D_{\tau}$ as follows: for every function $f: Q_{T} \rightarrow \mathbb{R}$,

$$
D_{\tau} f(x, t)=\frac{f(x, t)-f(x, t-\tau)}{\tau} \quad x \in \Omega, \quad t \in[\tau, T] .
$$

Now (40) can be written in the form of the discrete entropy inequality given by following:
Lemma 8 (Discrete entropy inequality). For all $t \in[0, T]$ it holds

$$
\begin{align*}
D_{\tau} \int_{\Omega} h^{*}\left(u^{(\tau)} \mid u_{D}\right) d x & +2 \sum_{i=1}^{n} \alpha_{i} \int_{\Omega} p\left(M^{(\tau)}\right)^{2}\left|\nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right|^{2} d x  \tag{41}\\
& +\tau \sum_{i=1}^{n}\left\|w_{i}^{(\tau)}\right\|_{H^{m}(\Omega)}^{2} \leq C_{1} \int_{\Omega} h^{*}\left(u^{(\tau)} \mid u_{D}\right) d x+C_{2}
\end{align*}
$$

for some suitable constants $C_{1}, C_{2} \geq 0$. Moreover, the constants $C_{1}, C_{2}$ can be chosen to be equal to zero in the case that $\lambda_{r}=C_{r}=0$ in (17) and (18).

From Lemma 7, the entropy inequality (41) and estimate (39) we deduce (via a discrete Gronwall argument) the following bounds, which are uniform with respect to $\tau$ :

$$
\begin{align*}
\left\|h\left(u^{(\tau)}\right)\right\|_{L^{\infty}\left(0, T ; L^{1}(\Omega)\right)} & \leq C  \tag{42}\\
\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} & \leq C \quad(i=1, \ldots, n),  \tag{43}\\
\left\|\sqrt{p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} \nabla \sqrt{u_{i}^{(\tau)}}\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} & \leq C \quad(i=1, \ldots, n),  \tag{44}\\
\left\|\nabla\left(M^{(\tau)}\right)^{\frac{a+1}{2}}\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} & \leq C  \tag{45}\\
\left\|\left(1-M^{(\tau)}\right)^{1-b-\kappa}\right\|_{L^{1}\left(0, T ; L^{1}(\Omega)\right)} & \leq C  \tag{46}\\
\tau^{1 / 2}\left\|w^{(\tau)}\right\|_{L^{2}\left(0, T ; H^{m}(\Omega)\right)} & \leq C \tag{47}
\end{align*}
$$

Moreover we recall that (by construction) $u^{(\tau)}(x, t) \in \mathcal{D}$ a.e. $(x, t) \in \Omega \times(0, T)$, where $\mathcal{D}$ is defined by (11). Therefore

$$
\begin{equation*}
\left\|u_{i}^{(\tau)}\right\|_{L^{\infty}\left(0, T ; L^{\infty}(\Omega)\right)} \leq C \quad(i=1, \ldots, n) \tag{48}
\end{equation*}
$$

Furthermore, from (28), (44) we get

$$
\begin{equation*}
\left\|\left(M^{(\tau)}\right)^{\frac{a-1}{2}} \nabla u_{i}^{(\tau)}\right\|_{L^{2}\left(Q_{T}\right)} \leq C \quad(i=1, \ldots, n) \tag{49}
\end{equation*}
$$

We also point out that (27), (42) imply

$$
\begin{equation*}
\left(1-M^{(\tau)}\right)^{1-\kappa} \in L^{\infty}\left(0, T ; L^{1}(\Omega)\right) \tag{50}
\end{equation*}
$$

The discretized-regularized system (361) can be rewritten, in the new notation, as

$$
\begin{align*}
& \sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega}\left(\left(D_{\tau} u_{i}^{(\tau)}\right) \phi_{i}+\alpha_{i} p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right) \cdot \nabla \phi_{i}-r_{i}\left(u^{(\tau)}\right) \phi_{i}\right) d x d t  \tag{51}\\
& \quad+\tau \sum_{i=1}^{n} \int_{0}^{T}\left(w_{i}^{(\tau)}, \phi_{i}\right)_{H^{m}} d t=0
\end{align*}
$$

for piecewise constant-in-time functions $\phi:[0, T] \rightarrow H_{D}^{m}\left(\Omega ; \mathbb{R}^{n}\right)$. However, thanks to a standard density argument, (51) holds for all $\phi \in L^{2}\left(0, T ; H_{D}^{m}\left(\Omega ; \mathbb{R}^{n}\right)\right)$.

Next, we wish to find a $\tau$-uniform bound for $D_{\tau} u^{(\tau)}$. We first estimate the term $p^{2}\left(M^{(\tau)}\right) \nabla\left(u_{i}^{(\tau)} q\left(M^{(\tau)}\right) / p\left(M^{(\tau)}\right)\right)$. We distinguish two cases.
Case 1: when $\kappa \geq b-1$, then $p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)$ is bounded in $L^{\infty}\left(Q_{T}\right)$ thanks to (26). It
follows

$$
\begin{aligned}
& \left\|p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right)\right\|_{L^{2}\left(Q_{T}\right)}\| \|_{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\left\|_{L^{\infty}\left(Q_{T}\right)}\right\| p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \|_{L^{2}\left(Q_{T}\right)} \\
& \leq 2 \\
& \leq 2\left\|p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)\right\|_{L^{\infty}\left(Q_{T}\right)}^{1 / 2}\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(Q_{T}\right)}
\end{aligned}
$$

Case 2: if $\kappa<b-1$, then $p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)$ is bounded in $L^{\frac{b+\kappa-1}{b-\kappa-1}}\left(Q_{T}\right)$ due to (26) and (46). This leads to

$$
\begin{aligned}
& \left\|p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right)\right\|_{L^{1+\frac{\kappa}{b-1}}\left(Q_{T}\right)}\| \|_{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\left\|_{L^{\frac{2(b+\kappa-1)}{b-\kappa-1}}\left(Q_{T}\right)}\right\| p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \|_{L^{2}\left(Q_{T}\right)} \\
& \leq 2\| \|\left\|p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)\right\|_{L^{\frac{b+\kappa-1}{b-\kappa-1}\left(Q_{T}\right)}}^{1 / 2}\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(Q_{T}\right)} .
\end{aligned}
$$

The above estimates and (43) allow us to deduce

$$
\begin{equation*}
\left\|p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right)\right\|_{L^{1+\rho}\left(Q_{T}\right)} \leq C \quad(i=1, \ldots, n), \quad \rho=\min \left\{1, \frac{\kappa}{b-1}\right\} \tag{52}
\end{equation*}
$$

As a byproduct of the above calculations we also get the following uniform bound:

$$
\left\{\begin{array}{l}
\left\|\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq C, \quad \kappa \geq b-1  \tag{53}\\
\left\|\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\right\|_{L^{\frac{2(b+\kappa-1)}{b-\kappa-1}}\left(Q_{T}\right)} \leq C, \quad \kappa<b-1
\end{array}\right.
$$

Let us now deal with the reaction term. From (18), (19) we deduce in particular that $\left|r_{i}\left(u^{(\tau)}\right)\right| \leq C\left(1-M^{(\tau)}\right)^{-\eta}$ with $\eta<b+\kappa-1$. Therefore (46) leads to

$$
\begin{equation*}
\left\|r_{i}\left(u^{(\tau)}\right)\right\|_{L^{\frac{b+\kappa-1}{\eta}}\left(Q_{T}\right)} \leq C . \tag{54}
\end{equation*}
$$

From (47), (52), (54) it follows

$$
\begin{align*}
& \int_{0}^{T} \int_{\Omega}\left(D_{\tau} u_{i}^{(\tau)}\right) \phi d x d t  \tag{55}\\
& \leq C\left(\|\phi\|_{L^{\frac{1+\rho}{\rho}}\left(0, T ; W^{1, \frac{1+\rho}{\rho}}(\Omega)\right)}+\|\phi\|_{L^{\frac{b+\kappa-1}{b+\kappa-1-\eta}\left(Q_{T}\right)}}+\tau^{1 / 2}\|\phi\|_{L^{2}\left(0, T ; H^{m}(\Omega)\right)}\right)
\end{align*}
$$

for $i=1, \ldots, n$. This estimate, together with the Sobolev embedding $H^{m}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ and the trivial relation $L^{\infty}(\Omega) \hookrightarrow L^{\frac{b+\kappa-1}{b+\kappa-1-\eta}}(\Omega)$, implies that $D_{\tau} u_{i}^{(\tau)}$ (and also $D_{\tau} M^{(\tau)}$ ) is uniformly bounded in $L^{1+\varepsilon}\left(0, T ;\left(W^{1, \frac{1+\rho}{\rho}}(\Omega) \cap H^{m}(\Omega)\right)^{\prime}\right)$, for $i=1, \ldots, n$ and some $\varepsilon>0$.
Step 4: Limit $\tau \rightarrow 0$. The uniform bound for $D_{\tau} M^{(\tau)}$ in $L^{2}\left(0, T ; H^{m}(\Omega)^{\prime}\right)$, together with (45) and (48), allows us to apply [5, Theorem 3] with $Q(s)=s^{\frac{a+3}{2}}$ and deduce

$$
\begin{equation*}
M^{(\tau)} \rightarrow M \quad \text { strongly in } L^{s}(\Omega \times(0, T)), \text { for all } s<\infty \tag{56}
\end{equation*}
$$

The strong convergence of $M^{(\tau)}$ and bound (46) yield via Fatou's lemma that $(1-M)^{1-b-\kappa} \in$ $L^{1}\left(Q_{T}\right)$, and therefore

$$
M<1 \quad \text { a.e. in } Q_{T}
$$

Moreover, thanks to the uniform bound for $D_{\tau} u_{i}^{(\tau)}$ in $L^{1+\varepsilon}\left(0, T ;\left(W^{1, \frac{1+\rho}{\rho}}(\Omega) \cap H^{m}(\Omega)\right)^{\prime}\right)$ and estimates (45), (48), (49), we can apply [26, Lemma 7] to deduce that

$$
\begin{equation*}
\left(M^{(\tau)}\right)^{\frac{a+1}{2}} u_{i}^{(\tau)} \rightarrow M^{\frac{a+1}{2}} u_{i} \quad \text { strongly in } L^{s}\left(Q_{T}\right), \text { for every } s<\infty, i=1, \ldots, n \tag{57}
\end{equation*}
$$

In particular, $\left(M^{(\tau)}\right)^{\frac{a+1}{2}} u_{i}^{(\tau)} \rightarrow M^{\frac{a+1}{2}} u_{i}$ a.e. in $Q_{T}$, which implies

$$
u_{i}^{(\tau)}=\frac{\left(M^{(\tau)}\right)^{\frac{a+1}{2}} u_{i}^{(\tau)}}{\left(M^{(\tau)}\right)^{\frac{a+1}{2}}} \rightarrow \frac{M^{\frac{a+1}{2}} u_{i}}{M^{\frac{a+1}{2}}}=u_{i} \quad \text { a.e. in } Q_{T} \cap\{M>0\}
$$

Moreover, since $0 \leq u_{i}^{(\tau)} \leq M^{(\tau)}$ for $1 \leq i \leq n$, clearly

$$
u_{i}^{(\tau)} \rightarrow 0 \quad \text { a.e. in } Q_{T} \cap\{M=0\} \quad(1 \leq i \leq n)
$$

However, $u_{i}=0$ on $Q_{T} \cap\{M=0\}$. In fact, given any nonnegative $\phi \in L^{2}\left(Q_{T}\right)$ having support contained in $\{M=0\}$, it holds

$$
0 \leq \int_{Q_{T}} u_{i}^{(\tau)} \phi d x d t \leq \int_{Q_{T}} M^{(\tau)} \phi d x d t \rightarrow \int_{Q_{T}} M \phi d x d t=0
$$

implying that the weak limit $u_{i}$ of $u_{i}^{(\tau)}$ vanishes on $Q_{T} \cap\{M=0\}$. Summarizing up, by dominated convergence,

$$
\begin{equation*}
u_{i}^{(\tau)} \rightarrow u_{i} \quad \text { strongly in } L^{s}\left(Q_{T}\right) \quad \forall s<\infty, \quad 1 \leq i \leq n \tag{58}
\end{equation*}
$$

From (52) it follows that $p\left(M^{(\tau)}\right)^{2} \nabla\left(u_{i}^{(\tau)} q\left(M^{(\tau)}\right) / p\left(M^{(\tau)}\right)\right)$ is weakly convergent in $L^{1+\rho}\left(Q_{T}\right)$, where $\rho=\min \{1, \kappa /(b-1)\}$. However,

$$
\begin{equation*}
p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right)=2 \sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} . \tag{59}
\end{equation*}
$$

Let us consider the first factor on the right-hand side of (59), i.e. $\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}$. The a.e. convergence of $u_{i}^{(\tau)}, M^{(\tau)}$ and the fact that $M<1$ a.e. in $Q_{T}$ imply that $\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} \rightarrow \sqrt{u_{i} p(M) q(M)}$ a.e. in $Q_{T}$. Bound (53) allows us to conclude that

$$
\begin{equation*}
\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} \rightarrow \sqrt{u_{i} p(M) q(M)} \quad \text { strongly in } L^{2}\left(Q_{T}\right) \tag{60}
\end{equation*}
$$

From (43) it follows

$$
\begin{equation*}
p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \rightharpoonup \psi_{i} \quad \text { weakly in } L^{2}\left(Q_{T}\right), \quad i=1, \ldots, n \tag{61}
\end{equation*}
$$

for some function $\psi \in L^{2}\left(Q_{T} ; \mathbb{R}^{n}\right)$. From (43), (60), (61) we deduce

$$
\begin{equation*}
p\left(M^{(\tau)}\right)^{2} \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right) \rightharpoonup 2 \sqrt{u_{i} p(M) q(M)} \psi_{i} \quad \text { weakly in } L^{1}\left(Q_{T}\right) \tag{62}
\end{equation*}
$$

We wish to identify the function $\psi_{i}$. For an arbitrary $\phi \in C_{c}^{\infty}\left(Q_{T}\right)$ let us consider

$$
\begin{align*}
& -\int_{Q_{T}} \phi p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} d x d t=J_{1}^{(\tau)}+J_{2}^{(\tau)}  \tag{63}\\
& J_{1}^{(\tau)}=\int_{Q_{T}} \sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} \nabla \phi d x d t \\
& J_{2}^{(\tau)}=\int_{Q_{T}} \phi p^{\prime}\left(M^{(\tau)}\right) \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \nabla M^{(\tau)} d x d t
\end{align*}
$$

From (60) it follows immediately that

$$
\begin{equation*}
J_{1}^{(\tau)} \rightarrow \int_{Q_{T}} \sqrt{u_{i} p(M) q(M)} \nabla \phi d x d t \tag{64}
\end{equation*}
$$

Let us now consider $J_{2}^{(\tau)}$ :

$$
\begin{aligned}
J_{2}^{(\tau)} & =\int_{Q_{T}} \phi p^{\prime}\left(M^{(\tau)}\right) \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \nabla M^{(\tau)} d x d t \\
& =\int_{Q_{T}} \phi \sqrt{u_{i}^{(\tau)}} g\left(M^{(\tau)}\right) \frac{\left(M^{(\tau)}\right)^{\frac{a-1}{2}} \nabla M^{(\tau)}}{\left(1-M^{(\tau)}\right)^{\frac{1+b+\kappa}{2}}} d x d t, \\
g(M) & \equiv M^{-\frac{a-1}{2}}(1-M)^{\frac{1+b+\kappa}{2}} \frac{p^{\prime}(M)}{p(M)} \sqrt{p(M) q(M)} \quad 0<M<1 .
\end{aligned}
$$

From (28) it follows that $g$ in continuous in $[0,1)$. From the a.e. convergence of $M^{(\tau)}$ and the fact that $M<1$ a.e. in $Q_{T}$ it follows that $g\left(M^{(\tau)}\right) \rightarrow g(M)$ a.e. in $Q_{T}$. On the other
hand, by exploiting (9), (26) the function $g$ can be estimated for $M \rightarrow 1$ as follows

$$
\begin{aligned}
|g(M)| & \leq C \sqrt{q(M) p(M)} \frac{\left|p^{\prime}(M)\right|}{p(M)}(1-M)^{\frac{1+b+\kappa}{2}} \\
& \leq C(1-M)^{\frac{1+\kappa-b}{2}}(1-M)^{-1-\kappa}(1-M)^{\frac{1+b+\kappa}{2}} \\
& \leq C \quad \text { as } M \rightarrow 1
\end{aligned}
$$

This means that $g$ is bounded in $[0,1]$. This fact, together with the a.e. convergence of $g\left(M^{(\tau)}\right)$, implies that $g\left(M^{(\tau)}\right) \rightarrow g(M)$ strongly in $L^{s}\left(Q_{T}\right)$ for every $s<\infty$.

Let us now consider the term

$$
\frac{\left(M^{(\tau)}\right)^{\frac{a-1}{2}} \nabla M^{(\tau)}}{\left(1-M^{(\tau)}\right)^{\frac{1+b+\kappa}{2}}}=\nabla \Phi\left(M^{(\tau)}\right), \quad \Phi(M) \equiv \int_{0}^{M} \frac{s^{\frac{a-1}{2}}}{(1-s)^{\frac{1+b+\kappa}{2}}} d s
$$

Lemma 7 and the entropy inequality (41) ensure that $\nabla \Phi\left(M^{(\tau)}\right)$ is uniformly bounded in $L^{2}\left(Q_{T}\right)$. Moreover we know that $\Phi\left(M^{(\tau)}\right) \rightarrow \Phi(M)$ a.e. in $Q_{T}$. Furthermore,

$$
\Phi(M) \leq \int_{0}^{M} \frac{d s}{(1-s)^{\frac{1+b+\kappa}{2}}}=\frac{(1-M)^{\frac{1-b-\kappa}{2}}-1}{(b+\kappa-1) / 2}
$$

which, thanks to (46), implies that $\Phi\left(M^{(\tau)}\right)$ is uniformly bounded in $L^{2}\left(Q_{T}\right)$. This means that $\Phi\left(M^{(\tau)}\right) \rightarrow \Phi(M)$ strongly in $L^{2-\delta}\left(Q_{T}\right)$ for every $\delta>0$. As a consequence, we get that $\nabla \Phi\left(M^{(\tau)}\right) \rightharpoonup \nabla \Phi(M)$ weakly in $L^{2}\left(Q_{T}\right)$. The weak convergence of $\nabla \Phi\left(M^{(\tau)}\right)$, the strong convergence of $g\left(M^{(\tau)}\right)$, and the strong convergence of $u^{(\tau)}$ (58) allow us to conclude

$$
\begin{equation*}
J_{2}^{(\tau)} \rightarrow \int_{Q_{T}} \phi p^{\prime}(M) \sqrt{\frac{u_{i} q(M)}{p(M)}} \nabla M d x d t \tag{65}
\end{equation*}
$$

By putting (61)-(65) together we conclude

$$
\begin{gather*}
p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} \rightharpoonup p(M) \nabla \sqrt{\frac{u_{i} q(M)}{p(M)}} \quad \text { weakly in } L^{2}\left(Q_{T}\right),  \tag{66}\\
p\left(M^{(\tau)}\right)^{2} \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right)  \tag{67}\\
\rightharpoonup p(M)^{2} \nabla\left(\frac{u_{i} q(M)}{p(M)}\right) \quad \text { weakly in } L^{1}\left(Q_{T}\right),
\end{gather*}
$$

for $i=1, \ldots, n$. Estimate (55) yields

$$
\begin{equation*}
D_{\tau} u_{i}^{(\tau)} \rightharpoonup \partial_{t} u_{i} \quad \text { weakly in } L^{1+\varepsilon}\left(0, T ;\left(W^{1, \frac{1+\rho}{\rho}}(\Omega) \cap H^{m}(\Omega)\right)^{\prime}\right), i=1, \ldots, n \tag{68}
\end{equation*}
$$

Let us now study the convergence of the reaction term. From (54) and the fact that $(b+\kappa-1) / \eta>1$ (by assumption (19)), if we can prove the a.e. convergence of $r_{i}\left(u^{(\tau)}\right)$ in $Q_{T}$ then strong convergence in a suitable space will follow. However, we know that $u^{(\tau)} \rightarrow u$ a.e. in $Q_{T}$ and $r$ is continuous in $\{M<1\}$; therefore $r\left(u^{(\tau)}\right) \rightarrow r(u)$ a.e. in $Q_{T}$. We conclude that $r\left(u^{(\tau)}\right) \rightarrow r(u)$ strongly in $L^{1}\left(Q_{T}\right)$.

Finally, $\tau w^{(\tau)} \rightarrow 0$ strongly in $L^{2}\left(0, T ; H^{m}(\Omega)\right)$ thanks to (47). We conclude that we can take the limit $\tau \rightarrow 0$ in (51) and obtain

$$
\begin{equation*}
\sum_{i=1}^{n}\left\langle\partial_{t} u_{i}, \phi_{i}\right\rangle+\sum_{i=1}^{n} \int_{\Omega}\left(\alpha_{i} p^{2}(M) \nabla\left(\frac{u_{i} q(M)}{p(M)}\right) \cdot \nabla \phi_{i}-r_{i}(u) \phi_{i}\right) d x=0 \tag{69}
\end{equation*}
$$

for every $\phi=\left(\phi_{1}, \ldots, \phi_{n}\right) \in C_{c}^{\infty}\left(Q_{T} ; \mathbb{R}^{n}\right)$. However, (52) and (54) allow us to deduce via a density argument that (69) holds for all $\phi \in L^{\frac{\rho+1}{\rho}}\left(0, T ; W^{1, \frac{\rho+1}{\rho}}(\Omega)\right) \cap L^{\frac{b+\kappa-1}{b+\kappa-1-\eta}}\left(Q_{T}\right)$.
Step 5: Entropy inequality. Testing (41) against an arbitrary test function $\phi \in C_{c}^{1}(0, T)$ (and performing a "discrete integration by parts") leads to

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega} h^{*}\left(u^{(\tau)} \mid u_{D}\right) D_{-\tau} \phi d x d t & +2 \sum_{i=1}^{n} \alpha_{i} \int_{0}^{T} \int_{\Omega} p\left(M^{(\tau)}\right)^{2}\left|\nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right|^{2} \phi d x d t  \tag{70}\\
& \leq \int_{0}^{T}\left(C_{1} \int_{\Omega} h^{*}\left(u^{(\tau)} \mid u_{D}\right) d x+C_{2}\right) \phi d t
\end{align*}
$$

From (66) and the weakly lower semicontinuity of the $L^{2}$ norm it follows

$$
\begin{aligned}
& \liminf _{\tau \rightarrow 0} \sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p\left(M^{(\tau)}\right)^{2}\left|\nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right|^{2} \phi d x d t \\
& \quad \geq \sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p(M)^{2} \left\lvert\, \nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}{ }^{2} \phi d x d t\right.
\end{aligned}
$$

On the other hand, (27) and the definition of $h$ implies (via l'Hôpital's rule) that $h(u) \leq$ $C(1-M)^{1-\kappa}$. Since (46) holds, we deduce that $h\left(u^{(\tau)}\right)$ is bounded in $L^{1+\delta}\left(Q_{T}\right)$ for some $\delta>0$. The a.e. convergence of $u^{(\tau)}$, the fact that $M<1$ a.e. in $Q_{T}$, and the continuity of $h$ in $\{M<1\}$ implies the a.e. convergence of $h\left(u^{(\tau)}\right)$ towards $h(u)$. We conclude that

$$
\begin{equation*}
h^{*}\left(u^{(\tau)} \mid u_{D}\right) \rightarrow h^{*}\left(u \mid u_{D}\right) \quad \text { strongly in } L^{1+\delta / 2}\left(Q_{T}\right) \tag{71}
\end{equation*}
$$

Since $D_{-\tau} \phi=-\tau^{-1}(\phi(\cdot+\tau)-\phi) \rightarrow-\partial_{t} \phi$ strongly in $L^{s}(0, T)$ for all $s<\infty$, from (70) -(71) we get

$$
\begin{align*}
& -\int_{0}^{T} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) \partial_{t} \phi d x d t+2 \alpha_{i} \sum_{i=1}^{n} \int_{0}^{T} \int_{\Omega} p(M)^{2}\left|\nabla \sqrt{\frac{u_{i} q(M)}{p(M)}}\right|^{2} \phi d x d t  \tag{72}\\
& \quad \leq \int_{0}^{T}\left(C_{1} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+C_{2}\right) \phi d t, \quad \forall \phi \in C_{c}^{1}(0, T): \phi \geq 0
\end{align*}
$$

Therefore (20) holds. This finishes the proof of the existence Theorem.
Remark 9. Theorem $⿴$ can be proved also in the case of nonconstant, $x$-dependent Dirichlet boundary data $u_{D, i}=u_{D, i}(x)(i=1, \ldots, n)$. We can assume for the sake of simplicity that $u_{D, i} \in W^{1, \infty}(\Omega), i=1, \ldots, n$, and $\sup _{\Omega} M_{D}<1$. The only relevant
difference with the case $u_{D}=$ constant lies in the proof of Lemma 8 (i.e. the discrete entropy inequality). In fact, the following additional term appears on the right-hand side of (38):

$$
\Xi=\sum_{i=1}^{n} \alpha_{i} \int_{0}^{T} \int_{\Omega} \nabla \log \left(\frac{u_{D, i} q\left(M_{D}\right)}{p\left(M_{D}\right)}\right) \cdot p^{2}\left(M^{(\tau)}\right) \nabla\left(\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}\right) d x d t
$$

This term can be estimated as follows:

$$
\begin{aligned}
\Xi= & 2 \sum_{i=1}^{n} \alpha_{i} \int_{0}^{T} \int_{\Omega} \nabla \log \left(\frac{u_{D, i} q\left(M_{D}\right)}{p\left(M_{D}\right)}\right) \cdot \sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)} p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}} d x d t \\
\leq & 2 \sum_{i=1}^{n} \alpha_{i}\left\|\nabla \log \left(\frac{u_{D, i} q\left(M_{D}\right)}{p\left(M_{D}\right)}\right)\right\|_{L^{\infty}\left(Q_{T}\right)}\left\|\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\right\|_{L^{2}\left(Q_{T}\right)} \\
& \times\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(Q_{T}\right)} \\
\leq & \frac{C}{\varepsilon}\left\|\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\right\|_{L^{2}\left(Q_{T}\right)}^{2}+C \varepsilon\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(Q_{T}\right)}^{2}
\end{aligned}
$$

for some $\varepsilon>0$. However, thanks to (26), it holds

$$
\begin{aligned}
\left\|\sqrt{u_{i}^{(\tau)} p\left(M^{(\tau)}\right) q\left(M^{(\tau)}\right)}\right\|_{L^{2}\left(Q_{T}\right)}^{2} & \leq C\left\|\left(1-M^{(\tau)}\right)^{(1+\kappa-b) / 2}\right\|_{L^{2}\left(Q_{T}\right)}^{2} \\
& =C \int_{0}^{T} \int_{\Omega}\left(1-M^{(\tau)}\right)^{1+\kappa-b} d x d t
\end{aligned}
$$

We wish to estimate the last integral in the above inequality. It holds

$$
\begin{aligned}
& \int_{0}^{T} \int_{\Omega}\left(1-M^{(\tau)}\right)^{1+\kappa-b} d x d t \\
& \quad=\iint_{\left\{1-M^{(\tau)} \geq \varepsilon^{\left.\frac{1}{\kappa}\right\}}\right.}\left(1-M^{(\tau)}\right)^{1+\kappa-b} d x d t+\iint_{\left\{1-M^{(\tau)}<\varepsilon^{\left.\frac{1}{\kappa}\right\}}\right.}\left(1-M^{(\tau)}\right)^{1+\kappa-b} d x d t \\
& \quad \leq C(\varepsilon)+\varepsilon^{2} \iint_{\left\{1-M^{(\tau)}<\varepsilon^{\left.\frac{1}{\kappa}\right\}}\right.}\left(1-M^{(\tau)}\right)^{1-\kappa-b} d x d t \\
& \quad \leq C(\varepsilon)+\varepsilon^{2} \int_{0}^{T} \int_{\Omega}\left(1-M^{(\tau)}\right)^{1-\kappa-b} d x d t
\end{aligned}
$$

Thus, we deduce that

$$
\Xi \leq C(\varepsilon)+C \varepsilon \int_{0}^{T} \int_{\Omega}\left(1-M^{(\tau)}\right)^{1-\kappa-b} d x d t+C \varepsilon\left\|p\left(M^{(\tau)}\right) \nabla \sqrt{\frac{u_{i}^{(\tau)} q\left(M^{(\tau)}\right)}{p\left(M^{(\tau)}\right)}}\right\|_{L^{2}\left(Q_{T}\right)}^{2}
$$

Lemma 7 allows us to bound $\Xi$ by means of the entropy dissipation by choosing $\varepsilon>0$ small enough, thereby yielding Lemma 8. The rest of the proof is analogue to the case $u_{D}=$ constant.

Remark 10. The existence proof works also in the case (4) is replaced by homogeneous Neumann boundary conditions on the whole $\partial \Omega$. In this case the relative entropy (12) cannot be employed (as $u_{D}$ is obviously not defined), and the absolute entropy $H[u]=$ $\int_{\Omega} h(u) d x$ can be used instead. However, the assumptions on the reaction term need to be modified, as the total mass

$$
\begin{equation*}
\mathcal{M}(t)=|\Omega|^{-1} \int_{\Omega} M(x, t) d x \tag{73}
\end{equation*}
$$

needs to remain strictly smaller then 1 for any finite time in order to prevent blowup. A sufficient condition for this reads as

$$
\begin{equation*}
\exists C \in \mathbb{R}: \quad \sum_{i=1}^{n} r_{i}(u) \leq C(1-M) \quad \forall u \in \mathcal{D} \tag{74}
\end{equation*}
$$

By integrating (1) in $\Omega$, summing from $i=1, \ldots, n$, and exploiting (74), one can easily show that

$$
1-\mathcal{M}(t) \geq e^{-\lambda t}(1-\mathcal{M}(0)) \quad t>0, \quad \lambda \equiv \max \{C, 0\}
$$

where $\mathcal{M}(t)$ is the total mass defined in (73). This control on the total mass $\mathcal{M}(t)$ allows us to apply Lemma 13 with $M=M^{(\tau)}$ and obtain bound (46) for the singularity, which is the only delicate point in the existence proof for the case of homogeneous Neumann boundary conditions; the rest of the argument works in a completely analogue way to the case of mixed Dirichlet-Neumann boundary conditions.

## 5. Proof of Theorem 2

From the definition (12), (13) of the relative entropy density it follows

$$
h^{*}\left(u \mid u_{D}\right)=\sum_{i=1}^{n}\left(u_{i} \log \frac{u_{i}}{u_{D, i}}+u_{D, i}-u_{i}\right)+\int_{M_{D}}^{M} \log \left(\frac{q(s) / p(s)}{q\left(M_{D}\right) / p\left(M_{D}\right)}\right) d s
$$

Now we split the above written relative entropy density in two parts,

$$
h^{*}\left(u \mid u_{D}\right)=\sum_{i=1}^{n} h_{1}^{*}\left(u_{i} \mid u_{D, i}\right)+h_{2}^{*}\left(M \mid M_{D}\right),
$$

where

$$
\begin{aligned}
h_{1}^{*}\left(u \mid u_{D}\right) & =u \log \frac{u}{u_{D}}+u_{D}-u \\
h_{2}^{*}\left(M \mid M_{D}\right) & =\int_{M_{D}}^{M} \log \left(\frac{q(s) / p(s)}{q\left(M_{D}\right) / p\left(M_{D}\right)}\right) d s
\end{aligned}
$$

The entropy inequality (20) and Lemma 7 yield, since $\lambda_{r}=C_{r}=0$,

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+C_{1} \sum_{i=1}^{n} \int_{\Omega} p(M) q(M)\left|\nabla{\sqrt{u_{i}}}^{2}\right|^{2} d x \\
&+C_{2} \int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x \leq 0 \tag{75}
\end{align*}
$$

We want to estimate the integral

$$
\int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x
$$

from below using the term $\int_{\Omega} h_{1}^{*}\left(u_{i} \mid u_{D, i}\right) d x$. We start by observing that, thanks to (26) and (28), it holds

$$
p(M) q(M) \geq C M^{a}(1-M)^{1+\kappa-b}
$$

It follows

$$
\begin{align*}
\int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x & \geq C \int_{\Omega} M^{a}(1-M)^{1+\kappa-b}\left|\nabla \sqrt{u_{i}}\right|^{2} d x \\
& \geq C \int_{\Omega} u_{i}^{a}(1-M)^{1+\kappa-b}\left|\nabla \sqrt{u_{i}}\right|^{2} d x \\
& \geq C \int_{\Omega} u_{i}^{a-1}(1-M)^{1+\kappa-b}\left|\nabla u_{i}\right|^{2} d x \\
& =C \int_{\Omega}(1-M)^{1+\kappa-b}\left|\nabla\left(u_{i}^{\frac{a+1}{2}}\right)\right|^{2} d x \tag{76}
\end{align*}
$$

On the other side, we notice that the term $\int_{\Omega}\left|\nabla\left(u_{i}^{\frac{a+1}{2}}\right)\right| d x$ can be estimated through the Cauchy-Schwarz inequality as follows

$$
\begin{align*}
\int_{\Omega}\left|\nabla\left(u_{i}^{\frac{a+1}{2}}\right)\right| d x & =\int_{\Omega} \frac{(1-M)^{(1+\kappa-b) / 2}\left|\nabla u_{i}^{(a+1) / 2}\right|}{(1-M)^{(1+\kappa-b) / 2}} d x \\
& \leq\left(\int_{\Omega}(1-M)^{-1-\kappa+b} d x\right)^{1 / 2}\left(\int_{\Omega}(1-M)^{1+\kappa-b}\left|\nabla u_{i}^{(a+1) / 2}\right|^{2} d x\right)^{1 / 2} \tag{77}
\end{align*}
$$

The first integral in the last row of (77) can be controlled by means of the $L^{\infty}\left(L^{1}\right)$ bound on $(1-M)^{1-\kappa}$ :

$$
\int_{\Omega}(1-M)^{-1-\kappa+b} d x \leq \int_{\Omega}(1-M)^{1-\kappa} d x \leq C
$$

where we used the fact that $-1-\kappa+b \geq 1-\kappa$ which holds true for $b \geq 2$. In this way we get

$$
\begin{equation*}
\int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x \geq C\left(\int_{\Omega}\left|\nabla u_{i}^{(a+1) / 2}\right| d x\right)^{2} . \tag{78}
\end{equation*}
$$

Finally, by using the Poincaré inequality we get

$$
\begin{equation*}
\int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x \geq C\left(\int_{\Omega}\left|u_{i}^{(a+1) / 2}-\left(u_{D, i}\right)^{(a+1) / 2}\right| d x\right)^{2} \tag{79}
\end{equation*}
$$

Next step is to bound term $\left|u_{i}^{(a+1) / 2}-\left(u_{D, i}\right)^{(a+1) / 2}\right|$ from below by $h_{1}^{*}\left(u_{i} \mid u_{D, i}\right)$. For $u_{i} \leq$ $u_{D, i} / 2$ it holds

$$
\left|u_{i}^{(a+1) / 2}-\left(u_{D, i}\right)^{(a+1) / 2}\right|=\left(u_{D, i}\right)^{(a+1) / 2}-u_{i}^{(a+1) / 2} \geq C \geq C\left|u_{i}-u_{D, i}\right|
$$

Moreover,

$$
h_{1}^{*}\left(u_{i} \mid u_{D, i}\right)=u_{i} \log \frac{u_{i}}{u_{D, i}}+u_{D, i}-u_{i} \leq u_{D, i}-u_{i} \leq\left|u_{i}-u_{D, i}\right|
$$

On the other hand, for $u_{i} \geq u_{D, i} / 2$, by the mean-value theorem and the Taylor's formula, there exist $\xi_{i}^{(1)}, \xi_{i}^{(2)}$ intermediate between $u_{i}$ and $u_{D, i}$ such that,

$$
\begin{aligned}
\left|u_{i}^{(a+1) / 2}-\left(u_{D, i}\right)^{(a+1) / 2}\right| & =\frac{a+1}{2}\left(\xi_{i}^{(1)}\right)^{(a-1) / 2}\left|u_{i}-u_{D, i}\right| \geq C\left|u_{i}-u_{D, i}\right| \\
h_{1}^{*}\left(u_{i} \mid u_{D, i}\right) & =\frac{1}{2 \xi_{i}^{(2)}}\left(u_{i}-u_{D, i}\right)^{2} \leq C\left|u_{i}-u_{D, i}\right|
\end{aligned}
$$

So in both cases

$$
\begin{equation*}
\left|u_{i}^{(a+1) / 2}-\left(u_{D, i}\right)^{(a+1) / 2}\right| \geq C\left|u_{i}-u_{D, i}\right| \geq C h_{1}^{*}\left(u_{i} \mid u_{D, i}\right) \quad(i=1, \ldots, n) \tag{80}
\end{equation*}
$$

From (79), (80) we deduce

$$
\begin{equation*}
\int_{\Omega} p(M) q(M)\left|\nabla \sqrt{u_{i}}\right|^{2} d x \geq C\left(\int_{\Omega} h_{1}^{*}\left(u \mid u_{D}\right) d x\right)^{2} \tag{81}
\end{equation*}
$$

Next step is to estimate the third integral in (75) from below by the integral

$$
\int_{\Omega} h_{2}^{*}\left(M \mid M_{D}\right) d x
$$

For that purpose we define

$$
\begin{equation*}
\Phi(M):=\int_{M_{D}}^{M} \frac{s^{(a-1) / 2}}{(1-s)^{(1+b+\kappa) / 2}} d s, \quad 0 \leq M<1 \tag{82}
\end{equation*}
$$

Now, one gets

$$
\begin{equation*}
\int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x=\int_{\Omega}|\nabla \Phi(M)|^{2} d x \geq C \int_{\Omega}|\Phi(M)|^{2} d x \tag{83}
\end{equation*}
$$

where we used the Poincaré inequality in order to make the last estimate.

Further, we claim that

$$
\begin{equation*}
h_{2}^{*}\left(M \mid M_{D}\right) \leq C \Phi(M)^{2} \tag{84}
\end{equation*}
$$

i.e.

$$
\begin{equation*}
\int_{M_{D}}^{M} \log \left(\frac{q(s) / p(s)}{q\left(M_{D}\right) / p\left(M_{D}\right)}\right) d s \leq C\left(\int_{M_{D}}^{M} \frac{s^{(a-1) / 2}}{(1-s)^{(1+b+\kappa) / 2}} d s\right)^{2} \tag{85}
\end{equation*}
$$

For checking the claim (84) we need to show that the function

$$
\begin{equation*}
F(M):=\frac{h_{2}^{*}\left(M \mid M_{D}\right)}{\Phi(M)^{2}} \tag{86}
\end{equation*}
$$

is bounded. Note that for $0 \leq M<1$ and $M \neq M_{D}$ it is clear that $F \in C\left([0,1) \backslash\left\{M_{D}\right\}\right)$. It remains to check the behavior of function $F$ near $M=1$ and $M=M_{D}$. By using the estimate $s^{(a-1) / 2} \geq M_{D}^{(a-1) / 2}$ inside the integral defining $\Phi$ and noticing that $\Phi^{\prime}\left(M_{D}\right) \neq 0$ we deduce

$$
\Phi(M) \sim\left|M-M_{D}\right| \quad\left(M \rightarrow M_{D}\right), \quad \Phi(M) \sim(1-M)^{\frac{1-b-\kappa}{2}} \quad(M \rightarrow 1)
$$

These relations, estimate (27) and the fact that $h_{2}^{*}\left(M_{D} \mid M_{D}\right)=\left.\frac{\partial}{\partial M} h_{2}^{*}\left(M \mid M_{D}\right)\right|_{M=M_{D}}=0$ allow us to obtain

$$
F(M) \sim \frac{(1-M)^{1-\kappa}}{(1-M)^{1-\kappa-b}} \sim(1-M)^{b} \rightarrow 0 \quad(M \rightarrow 1)
$$

and that $F$ is bounded as $M \rightarrow M_{D}$. It follows that function $F$ is bounded which proves our claim (84).

In this way we get

$$
\begin{equation*}
\int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x \geq C \int_{\Omega} h_{2}^{*}\left(M \mid M_{D}\right) d x \tag{87}
\end{equation*}
$$

On the other side, one has

$$
\begin{equation*}
\int_{\Omega} h_{2}^{*}\left(M \mid M_{D}\right) d x \leq \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x \leq \int_{\Omega} h^{*}\left(u_{0} \mid u_{D}\right) d x=C>0 \tag{88}
\end{equation*}
$$

so (87) implies a fortiori that

$$
\begin{equation*}
\int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x \geq C\left(\int_{\Omega} h_{2}^{*}\left(M \mid M_{D}\right) d x\right)^{2} \tag{89}
\end{equation*}
$$

Finally, we collect estimates (81) and (89) and we combine them with the entropy inequality (75). We get

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x+C\left(\int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x\right)^{2} \leq 0 \tag{90}
\end{equation*}
$$

Let us denote

$$
w(t)=\int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x
$$

Now, equation (90) can be written as

$$
\begin{equation*}
\frac{d}{d t} w(t)+C w(t)^{2} \leq 0 \tag{91}
\end{equation*}
$$

By integrating (91) with respect to time, from 0 to $t(t>0)$, one gets

$$
\frac{1}{H_{0}}-\frac{1}{w(t)} \leq-C t
$$

where $H_{0}=\int_{\Omega} h^{*}\left(u_{0} \mid u_{D}\right) d x$. Now, direct calculations give

$$
\begin{equation*}
\int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x \leq \frac{H_{0}}{1+t C H_{0}} . \tag{92}
\end{equation*}
$$

Since $h^{*}\left(u \mid u_{D}\right) \geq \sum_{i=1}^{n} h_{1}^{*}\left(u_{i} \mid u_{D, i}\right)$ and the Hessian of $u \mapsto h_{1}^{*}\left(u \mid u_{D}\right)$ is uniformly positive definite for $0<u \leq 1$, by Taylor-expanding $h_{1}^{*}\left(u_{i} \mid u_{D, i}\right)$ around $u_{D, i}$ we conclude

$$
\int_{\Omega} h^{*}\left(u \mid u_{D}\right) d x \geq \sum_{i=1}^{n} \int_{\Omega} h_{1}^{*}\left(u_{i} \mid u_{D, i}\right) d x \geq C \sum_{i=1}^{n} \int_{\Omega}\left|u_{i}-u_{D, i}\right|^{2} d x .
$$

This finishes the proof of Theorem 2.
Remark 11. A similar (albeit weaker) result on the long-time behavior of the solutions to (3) holds in the case of homogeneous Neumann boundary conditions with vanishing reactions. Indeed, one can show that, if $a \leq 1, \kappa<1, b \geq 2$, and $r(u) \equiv 0$, then $C>0$, $\theta \in(0,1)$ exist such that

$$
\|u-\langle u\rangle\|_{L^{2}(\Omega)}^{2} \leq \frac{C}{(1+t)^{\theta}} \quad t>0
$$

where $\langle u\rangle=|\Omega|^{-1} \int_{\Omega} u d x=|\Omega|^{-1} \int_{\Omega} u_{0} d x$ is the steady state of the system. The assumption $r=0$ is made for the sake of simplicity; as a matter of fact, the result could be generalized to the case of nonzero reaction terms with zero space average and suitable dissipative properties (like e.g. $r(u) \cdot \frac{\partial}{\partial u} h(u \mid\langle u\rangle) \leq 0$ ). However, such a case seems quite artificial.

The main differences with the case of mixed Dirichlet-Neumann boundary conditions appear in two points. The first one is the proof that the right-hand side of (78) dominates $\int_{\Omega} h_{1}^{*}(u \mid\langle u\rangle) d x$. In the case of homogeneous Neumann boundary conditions, Poincaré inequality yields (thanks to the assumption $a \leq 1$ ): ${ }^{2}$

$$
\int_{\Omega}\left|\nabla u_{i}^{(a+1) / 2}\right| d x \geq c \int_{\Omega}\left|\nabla u_{i}\right| d x \geq c^{\prime} \int_{\Omega}\left|u_{i}-\left\langle u_{i}\right\rangle\right| d x
$$

The above inequality, together with (78), (80), implies (81) with $u_{D}$ replaced by $\langle u\rangle$.
The other delicate point in the proof is to relate the second integral on the right-hand side of (75) with $\int_{\Omega} h_{2}^{*}(M \mid\langle M\rangle) d x$. We start by noticing that the relation $a \leq 1$ and

[^2]Poincaré inequality yield

$$
\begin{equation*}
\int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x \geq \int_{\Omega}|\nabla M|^{2} d x \geq C \int_{\Omega}|M-\langle M\rangle|^{2} d x \tag{93}
\end{equation*}
$$

Furthermore, since $\kappa<1$, then (thanks to (27)) $s \mapsto \log (q(s) / p(s))$ is integrable in [0, 1]. A straightforward consequence of this fact is the property

$$
h_{2}^{*}(M \mid\langle M\rangle) \leq C|M-\langle M\rangle|^{1-\kappa},
$$

for some constant $C>0$. By Jensen's inequality it follows

$$
\begin{equation*}
\left(\int_{\Omega} h_{2}^{*}(M \mid\langle M\rangle) d x\right)^{\frac{2}{1-\kappa}} \leq C\left(\int_{\Omega}|M-\langle M\rangle|^{1-\kappa} d x\right)^{\frac{2}{1-\kappa}} \leq C\|M-\langle M\rangle\|_{L^{2}(\Omega)}^{2} \tag{94}
\end{equation*}
$$

Putting (931), (94) together yields

$$
\int_{\Omega} \frac{M^{a-1}}{(1-M)^{1+b+\kappa}}|\nabla M|^{2} d x \geq C\left(\int_{\Omega} h_{2}^{*}(M \mid\langle M\rangle) d x\right)^{\frac{2}{1-\kappa}}
$$

The rest of the proof is completely analogue to the case of mixed Dirichlet-Neumann boundary conditions.

## 6. Proof of Theorem 3

The uniqueness proof is organized in two parts. First, using the $H^{-1}$-method, we prove the uniqueness of the total mass $M$. Consequently, in order to show the uniqueness of the solution $\left.u=\left(u_{1}, \ldots, u_{n}\right)\right)$ we apply the $E$-monotonicity technique of Gajewski [12]. By summing equations (3), taking into account the assumption that $\alpha_{i}=1$, and denoting $f(M)=M q(M) / p(M)$, one gets

$$
\begin{equation*}
\partial_{t} M=\operatorname{div}\left(p^{2}(M) \nabla f(M)\right)+\sum_{i=1}^{n} r_{i}(u) \tag{95}
\end{equation*}
$$

Direct calculation gives

$$
p^{2}(M) \nabla f(M)=\left(p(M) q(M)+M p(M) q^{\prime}(M)-M q(M) p^{\prime}(M)\right) \nabla M
$$

Let us define the function $Q$ as follows:

$$
\begin{equation*}
Q(M)=\int_{0}^{M}\left(p(s) q(s)+p(s) q^{\prime}(s) s-p^{\prime}(s) q(s) s\right) d s \tag{96}
\end{equation*}
$$

In this way, equation (95) can be written as

$$
\begin{equation*}
\partial_{t} M=\Delta Q(M)+\sum_{i=1}^{n} r_{i}(u) \tag{97}
\end{equation*}
$$

Let us assume that equation (97) has two different solutions $M_{1}$ and $M_{2}$. By substracting equations, one gets

$$
\begin{equation*}
\partial_{t}\left(M_{1}-M_{2}\right)=\Delta\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)+\sum_{i=1}^{n}\left(r_{i}\left(u_{1}\right)-r_{i}\left(u_{2}\right)\right) \tag{98}
\end{equation*}
$$

We test equation (98) using the test-function $\varphi(t)$ which we choose in a special way. More precisely, let $\varphi(t) \in H^{1}(\Omega)$ be the unique solution to the following initial-boundary value problem:

$$
\begin{align*}
-\Delta \varphi(t) & =\left(M_{1}-M_{2}\right)(t) \text { in } \Omega \\
\varphi & =0 \text { on } \Gamma_{D} \\
\frac{\partial \varphi}{\partial \nu} & =0 \text { on } \Gamma_{N}  \tag{99}\\
\varphi(0) & =\varphi_{0} \text { in } \Omega
\end{align*}
$$

By exploiting (98) and the fact that $\nabla \varphi(0)=0$ we deduce

$$
\begin{aligned}
& \frac{1}{2}\|\nabla \varphi(t)\|_{L^{2}(\Omega)}^{2}=\int_{0}^{t} \frac{1}{2} \frac{d}{d t}\|\nabla \varphi(s)\|_{L^{2}(\Omega)}^{2} d s=-\int_{0}^{t}\left\langle\partial_{t} \Delta \varphi, \varphi\right\rangle d s=\int_{0}^{t}\left\langle\partial_{t}\left(M_{1}-M_{2}\right), \varphi\right\rangle d s \\
& =\int_{0}^{t}\left\langle\Delta\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right), \varphi\right\rangle d s+\int_{0}^{t}\left\langle\sum_{i=1}^{n}\left(r_{i}\left(u_{1}\right)-r_{i}\left(u_{2}\right), \varphi\right\rangle d s\right.
\end{aligned}
$$

By integrating by parts and applying (99) we get

$$
\begin{align*}
\frac{1}{2}\|\nabla \varphi(t)\|_{L^{2}(\Omega)}^{2}=- & \int_{0}^{t} \int_{\Omega}\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right) d x d s \\
& +\int_{0}^{t} \int_{\Omega} \sum_{i=1}^{n}\left(r_{i}\left(u_{1}\right)-r_{i}\left(u_{2}\right)\right) \varphi d x d s \tag{100}
\end{align*}
$$

Let us first consider the second integral on the right-hand side of (100). Using the assumption (23) we have

$$
\begin{align*}
& \int_{0}^{t} \int_{\Omega} \sum_{i=1}^{n}\left(r_{i}\left(u_{1}\right)-r_{i}\left(u_{2}\right)\right) \varphi d x d s=C J_{1}+J_{2}  \tag{101}\\
& J_{1}=\int_{0}^{t} \int_{\Omega}\left(M_{1}-M_{2}\right) \varphi d x d s, \quad J_{2}=\int_{0}^{t} \int_{\Omega}\left(R\left(M_{1}\right)-R\left(M_{2}\right)\right) \varphi d x d s
\end{align*}
$$

We calculate:

$$
J_{1}=-\int_{0}^{t} \int_{\Omega}(\Delta \varphi) \varphi d x d s=\int_{0}^{t} \int_{\Omega}|\nabla \varphi|^{2} d x d s
$$

and using the mean-value theorem and assumption (24) we get

$$
J_{2} \leq \int_{0}^{t} \int_{\Omega}\left|R^{\prime}(\bar{M})\right|\left|M_{1}-M_{2}\right||\varphi| d x d s \leq C \int_{0}^{t} \int_{\Omega} \bar{M}^{a / 2}\left|M_{1}-M_{2}\right||\varphi| d x d s
$$

where $\bar{M}=\Theta M_{1}+(1-\Theta) M_{2}$, for some $\Theta \in[0,1]$. Next, Young inequality gives

$$
\begin{equation*}
J_{2} \leq C(\varepsilon) \int_{0}^{t} \int_{\Omega} \varphi^{2}(s) d x d s+\varepsilon \int_{0}^{t} \int_{\Omega} \bar{M}^{a}\left(M_{1}-M_{2}\right)^{2} d x d s \tag{102}
\end{equation*}
$$

Further on, using the Poincaré inequality in the first integral of (102) and the estimate $\bar{M} \leq \max \left\{M_{1}, M_{2}\right\}$, one has

$$
\begin{equation*}
J_{2} \leq C(\varepsilon) \int_{0}^{t} \int_{\Omega}|\nabla \varphi|^{2} d x d s+\varepsilon \int_{0}^{t} \int_{\Omega}\left(\max \left\{M_{1}, M_{2}\right\}\right)^{a}\left(M_{1}-M_{2}\right)^{2} d x d s \tag{103}
\end{equation*}
$$

At this point we go back to the equation (100) where we consider the first integral on the right-hand side. We claim that there exists a constant $D>0$ such that

$$
\begin{equation*}
D\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right) \geq\left(\max \left\{M_{1}, M_{2}\right\}\right)^{a}\left(M_{1}-M_{2}\right)^{2} \tag{104}
\end{equation*}
$$

In order to show (104), first we note that from (8), (96) follows that

$$
Q^{\prime}(M)=p^{2}(M)\left(\frac{M q(M)}{p(M)}\right)^{\prime}=\frac{M^{a}}{(1-M)^{b}}
$$

Now, after integrating the previous expression from 0 to $M$ we get

$$
\begin{equation*}
Q(M)=\int_{0}^{M} \frac{s^{a}}{(1-s)^{b}} d s+Q(0) \tag{105}
\end{equation*}
$$

Using (105), we calculate the term on the left-hand side of (104). In this way we have

$$
\begin{align*}
&\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right)=\left(M_{1}-M_{2}\right) \int_{M_{2}}^{M_{1}} \frac{s^{a}}{(1-s)^{b}} d s \\
& \geq\left(M_{1}-M_{2}\right) \int_{M_{2}}^{M_{1}} s^{a} d s=\frac{\left(M_{1}-M_{2}\right)^{2}}{a+1} M_{1}^{a} \frac{1-\left(M_{2} / M_{1}\right)^{a+1}}{1-\left(M_{2} / M_{1}\right)} \tag{106}
\end{align*}
$$

A straightforward computation yields

$$
\frac{1-x^{a+1}}{1-x} \geq C \max \left\{1, x^{a}\right\}=C(\max \{1, x\})^{a}, \quad x \geq 0, \quad x \neq 1
$$

therefore (106) leads to

$$
\begin{gather*}
\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right) \geq C\left(M_{1}-M_{2}\right)^{2} M_{1}^{a}\left(\max \left\{1, \frac{M_{2}}{M_{1}}\right\}\right)^{a} \\
=C\left(M_{1}-M_{2}\right)^{2}\left(\max \left\{M_{1}, M_{2}\right\}\right)^{a} \tag{107}
\end{gather*}
$$

proving in this way the claim (104). Inserting estimate (107) in (103), we get

$$
\begin{align*}
& J_{2} \leq C(\varepsilon) \int_{0}^{t} \int_{\Omega}|\nabla \varphi|^{2} d x d s \\
&+\varepsilon D \int_{0}^{t} \int_{\Omega}\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right) d x d s \tag{108}
\end{align*}
$$

Now we go back to (100). Using the estimate (108) in (101) we get

$$
\begin{align*}
\frac{1}{2}\|\nabla \varphi(t)\|_{L^{2}(\Omega)}^{2} & \leq(\varepsilon D-1) \int_{0}^{t} \int_{\Omega}\left(Q\left(M_{1}\right)-Q\left(M_{2}\right)\right)\left(M_{1}-M_{2}\right) d x d s \\
& +C(\varepsilon) \int_{0}^{t} \int_{\Omega}|\nabla \varphi|^{2} d x d s \tag{109}
\end{align*}
$$

Let us take $\varepsilon=1 / D$ in (109). By applying Gronwall's inequality and using the fact that $\nabla \varphi(0)=0$, one gets $\nabla \varphi(t)=0$ for all $t>0$. Finally, from $M_{1}(t)-M_{2}(t)=-\Delta \varphi(t)=0$ it follows directly that $\forall t \in[0, T]$ one has $M_{1}(t)=M_{2}(t)$. In this way the uniqueness of the total mass $M$ is proven.

In the second part of the proof, in order to prove the uniqueness of solution, we will follow the approach from [26] where the E-monotonicity technique of Gajewski [12] has been applied. This method is based on the convexity of the logarithmic entropy. For this purpose let us define the distance

$$
\begin{equation*}
\mathbf{d}(u, v)=\sum_{i=1}^{n} \int_{\Omega}\left(\xi\left(u_{i}\right)+\xi\left(v_{i}\right)-2 \xi\left(\frac{u_{i}+v_{i}}{2}\right)\right) d x \tag{110}
\end{equation*}
$$

where $\xi(s)=s \log s$. Notice that $\mathbf{d}(u, v) \geq 0$ due to the convexity of the function $\xi$.
Since $u_{i}$ and $v_{i}$ are only nonnegative and expressions like $\log u_{i}, \log v_{i}$ or $\log \left(\left(u_{i}+v_{i}\right) / 2\right.$ may be undefined, we need the regularization of distance given by (110). For that purpose let $0<\varepsilon<1$. We introduce the regularized distance

$$
\begin{equation*}
\mathbf{d}_{\varepsilon}(u, v)=\sum_{i=1}^{n} \int_{\Omega}\left(\xi_{\varepsilon}\left(u_{i}\right)+\xi_{\varepsilon}\left(v_{i}\right)-2 \xi_{\varepsilon}\left(\frac{u_{i}+v_{i}}{2}\right)\right) d x \tag{111}
\end{equation*}
$$

where $\xi_{\varepsilon}(s)=(s+\varepsilon) \log (s+\varepsilon)$.
Next, we observe that $\mathbf{d}_{\varepsilon}(u(0), v(0))=0$ as $u$ and $v$ have the same initial data. Using equation (1) written in the form

$$
\begin{equation*}
\partial_{t} u_{i}=\operatorname{div}\left(p q \nabla u_{i}+u_{i} p^{2} \nabla\left(\frac{q}{p}\right)\right)+r_{i}(u) \tag{112}
\end{equation*}
$$

we compute

$$
\begin{aligned}
& \frac{d}{d t} \mathbf{d}_{\varepsilon}(u, v)=\sum_{i=1}^{n}\left(I_{1, i}+I_{2, i}-I_{3, i}\right), \quad I_{1, i}=\left\langle\partial_{t} u_{i}, \log \left(u_{i}+\varepsilon\right)\right\rangle \\
& I_{2, i}=\left\langle\partial_{t} v_{i}, \log \left(v_{i}+\varepsilon\right)\right\rangle, \quad I_{3, i}=\left\langle\partial_{t}\left(u_{i}+v_{i}\right), \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right)\right\rangle .
\end{aligned}
$$

Taking into account the equation (112) and performing partial integration gives

$$
\begin{aligned}
I_{1, i}= & -\int_{\Omega} \frac{\nabla u_{i}}{u_{i}+\varepsilon}\left(p q \nabla u_{i}+u_{i} p^{2} \nabla\left(\frac{q}{p}\right)\right) d x+\int_{\Omega} r_{i}(u) \log \left(u_{i}+\varepsilon\right) d x \\
I_{2, i}= & -\int_{\Omega} \frac{\nabla v_{i}}{v_{i}+\varepsilon}\left(p q \nabla v_{i}+v_{i} p^{2} \nabla\left(\frac{q}{p}\right)\right) d x+\int_{\Omega} r_{i}(v) \log \left(v_{i}+\varepsilon\right) d x \\
I_{3, i}= & -\int_{\Omega} \frac{\nabla\left(u_{i}+v_{i}\right)}{u_{i}+v_{i}+2 \varepsilon}\left(p q \nabla\left(u_{i}+v_{i}\right)+\left(u_{i}+v_{i}\right) p^{2} \nabla\left(\frac{q}{p}\right)\right) d x \\
& +\int_{\Omega}\left(r_{i}(u)+r_{i}(v)\right) \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right) d x
\end{aligned}
$$

By rearranging the terms we get

$$
\begin{align*}
& \frac{d}{d t} \mathbf{d}_{\varepsilon}(u, v)=\mathcal{F}+S+S_{r}  \tag{113}\\
& \mathcal{F}=-\sum_{i=1}^{n} 4 \int_{\Omega}\left(\left|\nabla \sqrt{u_{i}+\varepsilon}\right|^{2}+\left|\nabla \sqrt{v_{i}+\varepsilon}\right|^{2}-\left|\nabla \sqrt{u_{i}+v_{i}+2 \varepsilon}\right|^{2}\right) p q d x \\
& S=-\sum_{i=1}^{n} \int_{\Omega}\left(\frac{u_{i}}{u_{i}+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon}\right) p^{2} \nabla u_{i} \cdot \nabla\left(\frac{q}{p}\right) d x \\
&-\sum_{i=1}^{n} \int_{\Omega}\left(\frac{v_{i}}{v_{i}+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon}\right) p^{2} \nabla v_{i} \cdot \nabla\left(\frac{q}{p}\right) d x \\
& S_{r}= \sum_{i=1}^{n} \int_{\Omega} r_{i}(u) \log \left(u_{i}+\varepsilon\right) d x+\sum_{i=1}^{n} \int_{\Omega} r_{i}(v) \log \left(v_{i}+\varepsilon\right) d x \\
&-\sum_{i=1}^{n} \int_{\Omega}\left(r_{i}(u)+r_{i}(v)\right) \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right) d x .
\end{align*}
$$

Using the fact that the Fisher information $\int_{\Omega}\left|\nabla u^{1 / 2}\right|^{2} d x$ is subadditive (see [26], Lemma $9)$, we get that $\mathcal{F} \leq 0$. Therefore, integrating (113) in time leads to

$$
\begin{equation*}
\mathbf{d}_{\varepsilon}(u(t), v(t)) \leq \int_{0}^{t} S(s) d s+\int_{0}^{t} S_{r}(s) d s \tag{114}
\end{equation*}
$$

Firstly, we treat the second integral in (114). We want to prove that

$$
\begin{equation*}
\int_{0}^{t} S_{r}(s) d s \leq C \int_{0}^{t} \mathbf{d}_{\varepsilon}(u(s), v(s)) d s \tag{115}
\end{equation*}
$$

By taking into account the assumptions on the reaction terms given by (21)-(24) and the fact that $M_{1}=M_{2}$, the left-hand side of (115) can be rewritten as

$$
\begin{aligned}
& \int_{0}^{t} S_{r}(s) d s=\sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega} r_{i}^{(0)}(M)\left(\log \left(u_{i}+\varepsilon\right)+\log \left(v_{i}+\varepsilon\right)-2 \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right)\right) d x d s \\
& +\sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega} r^{(1)}(M)\left(u_{i} \log \left(u_{i}+\varepsilon\right)+v_{i} \log \left(v_{i}+\varepsilon\right)-\left(u_{i}+v_{i}\right) \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right)\right) d x d s
\end{aligned}
$$

Furthermore, from the definition of $\mathbf{d}_{\varepsilon}(u, v)$ it follows

$$
\begin{aligned}
& \int_{0}^{t} S_{r}(s) d s \\
& =\sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega}\left(r_{i}^{(0)}(M)-\varepsilon r^{(1)}(M)\right)\left(\log \left(u_{i}+\varepsilon\right)+\log \left(v_{i}+\varepsilon\right)-2 \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right)\right) d x d s \\
& +\sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega} r^{(1)}(M)\left(\xi_{\varepsilon}\left(u_{i}\right)+\xi_{\varepsilon}\left(v_{i}\right)-2 \xi_{\varepsilon}\left(\frac{u_{i}+v_{i}}{2}\right)\right) d x d s \\
& \leq \sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega}\left(r_{i}^{(0)}(M)-\varepsilon r^{(1)}(M)\right)\left(\log \left(u_{i}+\varepsilon\right)+\log \left(v_{i}+\varepsilon\right)-2 \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right)\right) d x d s \\
& +C \sum_{i=1}^{n} \int_{0}^{t} \mathbf{d}_{\varepsilon}(u(s), v(s)) d s
\end{aligned}
$$

where the last inequality comes from the fact that $r^{(1)}$ is upper bounded, which is a straightforward consequence of (21)-(24). The convexity of $x \mapsto-\log (x)$ implies that

$$
\log \left(u_{i}+\varepsilon\right)+\log \left(v_{i}+\varepsilon\right)-2 \log \left(\frac{u_{i}+v_{i}}{2}+\varepsilon\right) \leq 0
$$

Together with the assumptions on the reaction terms, we deduce that (115) holds.
From (114), (115) it follows

$$
\mathbf{d}_{\varepsilon}(u(t), v(t)) \leq \int_{0}^{t} S(s) d s+C \sum_{i=1}^{n} \int_{0}^{t} \mathbf{d}_{\varepsilon}(u(s), v(s)) d s
$$

Using Gronwall's Lemma yields

$$
\begin{align*}
\mathbf{d}_{\varepsilon}(u(t), v(t)) & \leq e^{C t} \sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega}\left|\frac{u_{i}}{u_{i}+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon}\right| p^{2}\left|\nabla u_{i}\right|\left|\nabla\left(\frac{q}{p}\right)\right| d x d s \\
& +e^{C t} \sum_{i=1}^{n} \int_{0}^{t} \int_{\Omega}\left|\frac{v_{i}}{v_{i}+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon}\right| p^{2}\left|\nabla v_{i}\right|\left|\nabla\left(\frac{q}{p}\right)\right| d x d s . \tag{116}
\end{align*}
$$

Next, we want to apply the dominated convergence theorem to show that the right-hand side of (116) converges to zero as $\varepsilon \rightarrow 0$. Let $g \in\left\{u_{i}, v_{i}\right\}$. It is obvious that

$$
\begin{aligned}
& \frac{g}{g+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon} \rightarrow 0 \quad \text { a.e. in }\{g>0\} \text { as } \varepsilon \rightarrow 0 \\
& \left|\frac{g}{g+\varepsilon}-\frac{u_{i}+v_{i}}{u_{i}+v_{i}+2 \varepsilon}\right| \leq 2 \quad \text { a.e. in } \Omega \times(0, \infty)
\end{aligned}
$$

Therefore, in order to apply the dominated convergence theorem we need to show that

$$
p^{2} \nabla g \cdot \nabla\left(\frac{q}{p}\right) \in L^{1}(\Omega \times(0, T))
$$

We make the calculation for $g=u_{i}$. For $g=v_{i}$ the calculation is completely the same. The term of interest can be rewritten as follows

$$
\begin{equation*}
p^{2} \nabla u_{i} \cdot \nabla\left(\frac{q}{p}\right)=2 \sqrt{p q} \nabla \sqrt{u_{i}} \cdot \frac{\sqrt{u_{i}}}{\sqrt{p q}} p^{2} \nabla\left(\frac{q}{p}\right) \tag{117}
\end{equation*}
$$

Since $2 \sqrt{p q} \nabla \sqrt{u_{i}} \in L^{2}(\Omega \times(0, T))$ due to (444), it remains to show that $\frac{\sqrt{u_{i}}}{\sqrt{p q}} p^{2} \nabla\left(\frac{q}{p}\right) \in$ $L^{2}(\Omega \times(0, T))$. For $u_{i}>0$ (and $v_{i}>0$ respectively), we have

$$
\begin{equation*}
\frac{\sqrt{u_{i}}}{\sqrt{p q}} p^{2} \nabla\left(\frac{q}{p}\right)=2 \sqrt{u_{i}} p \nabla \sqrt{\frac{q}{p}}=2 p \nabla \sqrt{\frac{u_{i} q}{p}}-2 p \sqrt{\frac{q}{p}} \nabla \sqrt{u_{i}} . \tag{118}
\end{equation*}
$$

From Lemma 7 and (20) we conclude that

$$
\frac{\sqrt{u_{i}}}{\sqrt{p q}} p^{2} \nabla\left(\frac{q}{p}\right) \in L^{2}(\Omega \times(0, T))
$$

obtaining in this way finally that for $u_{i}, v_{i}>0$

$$
p^{2} \nabla u_{i} \cdot \nabla\left(\frac{q}{p}\right) \in L^{1}(\Omega \times(0, T))
$$

It remains to treat the case when $u_{i}=0$ (or respectively $v_{i}=0$ ). We want to show that

$$
p^{2} \nabla u_{i} \cdot \nabla\left(\frac{q}{p}\right)=0 \quad \text { a.e. in the set }\left\{u_{i}=0\right\}
$$

For this, we make the following estimate

$$
p^{2}\left|\nabla u_{i}\right|\left|\nabla\left(\frac{q}{p}\right)\right| \leq 4\left(\left|\nabla \sqrt{u_{i} p q}\right|+\left|\sqrt{u_{i}} \nabla \sqrt{p q}\right|\right)\left(\left|p \nabla \sqrt{\frac{u_{i} q}{p}}\right|+\sqrt{p q}\left|\nabla \sqrt{u_{i}}\right|\right) .
$$

Now, due to [19, p.153, 6.18 Corollary], it holds that

$$
\nabla \sqrt{u_{i} p q}=0 \text { where } \sqrt{u_{i} p q}=0
$$

It follows that $\nabla \sqrt{u_{i} p q}=\sqrt{u_{i}} \nabla \sqrt{p q}=0$ (and a fortiori $p^{2}\left|\nabla u_{i}\right||\nabla(q / p)|=0$ ) a.e. in the set $\left\{u_{i}=0\right\}$. In this way, since $d_{\varepsilon}$ is nonnegative, we get that $d_{\varepsilon}(u(t), v(t)) \rightarrow 0$ as $\varepsilon \rightarrow 0$, which implies that

$$
\begin{equation*}
\xi_{\varepsilon}\left(u_{i}\right)+\xi_{\varepsilon}\left(v_{i}\right)-2 \xi_{\varepsilon}\left(\frac{u_{i}+v_{i}}{2}\right) \rightarrow 0 \text { as } \varepsilon \rightarrow 0 \text { a.e. in } \Omega \times(0, \infty) \tag{119}
\end{equation*}
$$

Same calculation like in [26, p.26] gives the estimate

$$
\begin{equation*}
\xi_{\varepsilon}\left(u_{i}\right)+\xi_{\varepsilon}\left(v_{i}\right)-2 \xi_{\varepsilon}\left(\frac{u_{i}+v_{i}}{2}\right) \geq \frac{1}{8}\left(u_{i}-v_{i}\right)^{2} . \tag{120}
\end{equation*}
$$

Finally, (119) and (120) give that $u_{i}=v_{i}$ in $\Omega \times(0, \infty)$ for $i=1, \ldots, n$. This concludes the proof of Theorem 3.

Remark 12. The uniqueness result holds trivially (under the same assumptions) also if nonconstant Dirichlet data or homogeneous Neumann boundary conditions are considered.

## 7. Appendix

### 7.1. Formal derivation of the multi-species biofilm model from a spatially dis-

 crete lattice model. Here we discuss the modeling assumptions, which are strongly connected to the derivation of the multi-species biofilm model from a spatially discrete lattice ODE. More details on the derivation can be found in [21, 26, 22]. For simplicity, we sketch the derivation in 1D. Given a one-dimensional spatial lattice containing equidistant cells $x_{j}$ with cell distance $h=x_{j}-x_{j-1}$ of a finite interval, we consider the variables $u_{i}^{j}:=u_{i}\left(x_{j}\right)$, which model the density of the $i$ th species at the $j$ th grid cell. Moreover, transition rates $T_{i}^{j \pm}$ describe how species $u_{i}$ moves from cell $x_{j}$ to the neighboring cells $x_{j \pm 1}$. Biofilm movement into neighboring cells is driven by two principles: volume filling and quenching [17]. Volume filling means that the movement depends on the available space in the local site, and since the site's capacity of accommodation mass is limited, we can normalize the population densities with respect to their maximum densities, which means that $u_{i}^{j} \leq 1$. Thus we can interpret $u_{i}^{j}$ as the volume fraction of site $j$ occupied by the species $u_{i}$. The discrete master equation, which describes the balance between the density of populations which leave the site to move into the neighboring sites, and the density of populations which arrive from neighboring sites, reads as$$
\begin{equation*}
\partial_{t} u_{i}^{j}=T_{i}^{(j-1)+} u_{i}^{j-1}+T_{i}^{(j+1)-} u_{i}^{j+1}-\left(T_{i}^{j+}+T_{i}^{j-}\right)+r_{i}^{j} \tag{121}
\end{equation*}
$$

where $r_{i}^{j}=r_{i}^{j}\left(u^{j}\right)$ is the net growth rate of the $i$ th species, and the transition rates have the form

$$
T_{i}^{j \pm}=\alpha_{i} q_{i}\left(u_{1}^{j}, \ldots, u_{n}^{j}\right) p_{i}\left(u_{1}^{j \pm 1}, \ldots, u_{n}^{j \pm 1}\right)
$$

where $\alpha_{i}=\alpha_{i}(h)$ measure how fast populations move between neighboring cells, and the nonnegative continuous transition functions $q_{i}$ and $p_{i}$ describe the local movement of the species from one cell to the other. The transition function $q_{i}(u)$ measures the incentive of the density of species $i$ at grid cell $x_{j}$ to leave the cell $x_{j}$, and $p_{i}\left(u_{1}^{j \pm 1}, \ldots, u_{n}^{j \pm 1}\right)$ models the attractivity of the cell population $u_{i}^{j}$ for the incoming individuals $u_{i}^{j \pm 1}$ from the neighboring sites $j \pm 1$. The second principle for biofilm movement is that as long as there is capacity to accommodate new biomass locally in that cell, the incentive to move to a neighboring cell is small, which is called quenching. The transition from the spatially discretized to the
continuous model is now performed in a formal diffusive limit. First, we interpolate the grid functions by setting

$$
u_{i}(t, x)=u_{i}^{j}(t), \quad \text { for } \quad x_{j} \leq x \leq x_{j+1}
$$

Now, assuming sufficient smoothness of the functions $u_{1}, \ldots, u_{n}$, we can approximate $u_{i}\left(t, x^{j \pm 1}\right), q\left(u\left(t, x^{j \pm 1}\right)\right), p\left(u\left(t, x^{j \pm 1}\right)\right)$ by second order Taylor polynomials around $u\left(t, x_{j}\right)$. By substituting all these expressions into the master equation (121) and performing the (formal) diffusive limit $h \rightarrow 0$ under the assumption that $\lim _{h \rightarrow 0} \alpha_{i} h^{2}=\alpha_{i 0}>0$, we get for $u=\left(u_{1}, \ldots, u_{n}\right)$ the equation

$$
\partial_{t} u_{i}=\alpha_{i 0} \frac{\partial}{\partial x}\left(\sum_{j=1}^{n} A_{i j}(u) \frac{\partial u_{j}}{\partial x}\right)+r_{i}(u), \quad i=1, \ldots, n
$$

where the diffusion coefficients have the form

$$
\begin{equation*}
A_{i j}(u)=\delta_{i j} p_{i}(u) q_{i}(u)+u_{i}\left(p_{i}(u) \frac{\partial q_{i}}{\partial u_{j}}(u)-q_{i}(u) \frac{\partial p_{i}}{\partial u_{j}}(u)\right) . \tag{122}
\end{equation*}
$$

In more than one dimension, the same procedure can be applied, leading to the system

$$
\partial_{t} u_{i}=\alpha_{i 0} \operatorname{div}\left(\sum_{j=1}^{n} A_{i j}(u) \nabla u_{j}\right)+r_{i}(u), \quad i=1, \ldots, n
$$

where $A$ is defined in (122).
7.2. Numerical illustration of the relative entropy. In this subsection we present numerical simulations of the relative entropy given by (12) with the respect to time. We take a three-species model (11) on the rectangular domain $\Omega=[0,1] \times[0,1]$. The solution is calculated using the Distributed and Unified Numerics Environment is a modular toolbox for solving partial differential equations (PDEs) with grid-based methods DUNE [1]. For calculating the solution we use standard FEM on the rectangular grid with $Q_{1}$-elements. More precisely, we used the following Dune modules: 1. Core modules (dune-grid, dunegeometry, dune-localfunctions, dune-common, dune-istl); 2. Discretization modules (dunefem, dune-pdelab). The initial conditions are:

$$
\begin{aligned}
& u_{1}(x, y ; 0)=\left\{\begin{array}{l}
u_{D, 1}+\varepsilon, \quad 0.2 \leq x \leq 0.5 \text { and } 0 \leq y \leq 0.2, \\
u_{D, 1}, \quad \text { otherwise }
\end{array}\right. \\
& u_{2}(x, y ; 0)= \begin{cases}u_{D, 2}+\varepsilon, \quad 0.5 \leq x \leq 0.8 \text { and } 0 \leq y \leq 0.2, \\
u_{D, 2}, & \text { otherwise }\end{cases} \\
& u_{3}(x, y ; 0)= \begin{cases}u_{D, 3}+\varepsilon, & 0.2 \leq x \leq 0.8 \text { and } 0 \leq y \leq 0.2 \\
u_{D, 3}, & \text { otherwise }\end{cases}
\end{aligned}
$$



Figure 1. Logarithm of the relative entropy vs time

Here, $u_{D, i}=\varepsilon, i=1,2,3$ where $\varepsilon=0.1$, the time-step is $d t=10^{-4}$ and the final time $T_{\mathrm{fin}}=10$. For the function $p(M)$ we chose:

$$
\begin{equation*}
p(M)=\exp \left(-\frac{1}{(1-M)^{\kappa}}\right), \quad M=u_{1}+u_{2}+u_{3} . \tag{123}
\end{equation*}
$$

Concerning different choices of parameters $a, b$ from (7) and $\kappa$ in (123) as well as different boundary conditions (mixed Dirichlet-Neumann or homogeneous Neumann), we performed the following two tests.

Test 1. With this test we wanted to illustrate numerically our analytical result given by Theorem 2. For that purpose here we considered our model with mixed DirichletNeumann boundary conditions. More precisely, we took the Dirichlet boundary conditions $u_{D_{i}}, i=1,2,3$ on the upper-side of the rectangle and homogeneous Neumann on other three sides. For parameters we took $a=2, b=2$, and $\kappa=1$ and we used $r_{i}^{D}=u_{D, i}-u_{i}$ as the reaction terms. We note that Figure 7.2 (left) shows the exponential convergence of the solution to the steady state, which we were not able to obtain with our analytical tools.
Test 2. This test corresponds to Remark 11. Here we considered the homogeneous Neumann boundary conditions without any reaction terms. For parameters we took $a=1$, $b=2$ and $\kappa=0.9$. In Figure 7.2 (right), we observe very fast stabilization of the solution to the constant steady-state.

Summarizing up, it seems that in the situation described by Theorem 2 the convergence to the steady state is actually exponential. The algebraic decay result proved in the Theorem might be not optimal, most likely due to limitations in the analytical methods employed in the proof. Furthermore, convergence to the steady state appears to be helped by a reaction term with a suitable (dissipative) structure (as it is to be expected): the rate of convergence is higher in Test 1 than in Test 2, and the relative entropy reaches much smaller values in Test 1 than in Test 2. In fact, the relative entropy seems to stabilize around $10^{-6}$ in Test 2, which might be a symptom of numerical instability.

### 7.3. Additional auxiliary result.

Lemma 13 (Variant of the Poincaré Inequality). Let $\Omega \subset \mathbb{R}^{d}$ bounded open and connected. Let $\lambda \in(0,1), \alpha>0$. There exists a constant $C>0$ such that

$$
\left\|(1-M)^{-\alpha}\right\|_{L^{2}(\Omega)} \leq C\left(1+\left\|\nabla(1-M)^{-\alpha}\right\|_{L^{2}(\Omega)}\right)
$$

for every function $M \in H^{1}(\Omega)$ such that $0 \leq M<1$ a.e. in $\Omega$ and $|\Omega|^{-1} \int_{\Omega} M d x \leq \lambda$.
Proof. By contradiction. Assume $\forall k \geq 1$ there exists $M_{k} \in H^{1}(\Omega)$ such that $0 \leq M_{k}<1$ a.e. in $\Omega,|\Omega|^{-1} \int_{\Omega} M_{k} d x \leq \lambda$, and

$$
\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)}>k\left(1+\left\|\nabla\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)}\right)
$$

for all $k \geq 1$. Let us define $f_{k}=\frac{\left(1-M_{k}\right)^{-\alpha}}{\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)}}$. It follows that $\left\|f_{k}\right\|_{L^{2}(\Omega)}=1$,

$$
\left\|\nabla f_{k}\right\|_{L^{2}(\Omega)}<\frac{1}{k}-\frac{1}{\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)}}
$$

Clearly $\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)} \rightarrow \infty$ as $k \rightarrow \infty$, which in particular implies that $\left\|\nabla f_{k}\right\|_{L^{2}(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$. Moreover $f_{k}$ is bounded in $H^{1}(\Omega)$, which by compact Sobolev embedding implies that, up to subsequences, $f_{k} \rightarrow f$ strongly in $L^{2}(\Omega)$. Since $\left\|\nabla f_{k}\right\|_{L^{2}(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ it follows that

$$
\int_{\Omega} f \partial_{x_{i}} \phi d x=\lim _{k \rightarrow \infty} \int_{\Omega} f_{k} \partial_{x_{i}} \phi d x=-\lim _{k \rightarrow \infty} \int_{\Omega} \partial_{x_{i}} f_{k} \phi d x=0 \quad \forall \phi \in H_{0}^{1}(\Omega)
$$

This means that $f$ is constant. Moreover, the fact that $\left\|f_{k}\right\|_{L^{2}(\Omega)}=1$ for all $k$, together with the strong convergence of $f_{k}$, implies that $f>0$. However, up to subsequences, $f_{k} \rightarrow f$ a.e. in $\Omega$, i.e.

$$
\frac{\left(1-M_{k}\right)^{-\alpha}}{\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)}} \rightarrow f \quad \text { a.e. in } \Omega
$$

Since $\left\|\left(1-M_{k}\right)^{-\alpha}\right\|_{L^{2}(\Omega)} \rightarrow \infty$ as $k \rightarrow \infty$, we deduce that $M_{k} \rightarrow 1$ a.e. in $\Omega$, and therefore (by dominated convergence) also strongly in $L^{1}(\Omega)$. As a consequence $|\Omega|^{-1} \int_{\Omega} M_{k} d x \rightarrow 1$. This is a contradiction to the fact that $|\Omega|^{-1} \int_{\Omega} M_{k} d x \leq \lambda<1$ for all $k \geq 1$. This finishes the proof.

## References

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[^1]:    ${ }^{1}$ Volume-filling models take into account the fact that concentration may saturate.

[^2]:    ${ }^{2}$ We point out that, if $a>1$, the term $\left\langle u_{i}^{(a+1) / 2}\right\rangle$ would be present in the inequality in place of $\left\langle u_{i}\right\rangle$; it is not clear how the argument would work in this case.

