A GENERALIZATION OF THE GORESKY-KLAPPER CONJECTURE, PART I

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ABSTRACT. For a fixed integer $n \ge 2$, we show that a permutation of the least residues mod p of the form $f(x) = Ax^k \mod p$ cannot map a residue class mod n to just one residue class mod n once p is sufficiently large, other than the maps $f(x) = \pm x \mod p$ when n is even and $f(x) = \pm x \text{ or } \pm x^{(p+1)/2} \mod p$ when n is odd.

1. INTRODUCTION

For an odd prime p we let I denote the reduced residues mod p,

$$I = \{1, 2, \dots, p - 1\},\$$

and A and k integers with

(1.1)
$$|A| < p/2, p \nmid A, 1 \le k < p-1, \gcd(k, p-1) = 1,$$

so that the map $f: I \to I$ given by

$$f(x) = Ax^k \mod p$$
.

is a permutation of I.

Goresky & Klapper [10] divided I into the even and odd residues

 $E = \{2, 4, \dots, p-1\}, \quad O = \{1, 3, \dots, p-2\},\$

and asked when f could also be a permutation of E (equivalently O). Originally the problem was phrased in terms of decimations of ℓ -sequences and was restricted to cases where 2 is a primitive root mod p, but this is the form that we are interested in here. Apart from the identity map (p; A, k) = (p; 1, 1) they found six cases

$$(p; A, k) = (5; -2, 3), (7; 1, 5), (11; -2, 3), (11; 3, 7), (11; 5, 9), (13; 1, 5),$$

and conjectured that there were no more for p > 13. This was proved for sufficiently large p in [3] and in full in [6], with asymptotic counts on $|f(E) \cap O|$ considered in [4]. Since $x \mapsto p - x$ switches elements of E and O, this is the same as asking when f(E) = O or f(O) = E on replacing A by -A.

Somewhat related is a question of Lehmer [12, Problem F12, p. 381] concerning the number of x mod p whose inverse, $f(x) = x^{-1} \mod p$, has opposite parity.

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Since k is defined mod (p-1) it is sometimes useful to allow negative exponents, |k| < (p-1)/2. This problem was solved by Zhang [25] using Kloosterman sums; see also the generalizations by Alkan, Stan and Zaharescu [1], Lu and Yi [16, 17], Shparlinski [19, 20], Xi and Yi [22], and Yi and Zhang [24].

Thinking of the evens and odds as a mod 2 restriction, we can ask a similar question for a general modulus n. Namely we can divide up I into the n congruence classes mod n

(1.2)
$$I_j = \{x : 1 \le x \le p-1, x \equiv j \mod n\}, j = 0, \dots, n-1,$$

and ask for examples of the following types.

Type (i): $f(I_i) = I_i$ for all j = 0, ..., n - 1.

Type (iia): $f(I_0), ..., f(I_{n-1})$ a permutation of $I_0, ..., I_{n-1}$.

Type (iib): $f(I_j) = I_j$ for some j.

Type (iii): There is a pair i, j with $f(I_i) \subseteq I_j$.

Type (iv): There is a pair i, j with $f(I_i) \cap I_j = \emptyset$.

In this paper we will be primarily be interested in the Type (i)-(iii) maps, though we will include some special cases of Type (iv), for example when

(1.3)
$$d := \gcd(k - 1, p - 1)$$

is small. We return to consider general Type (iv) in Part II.

Notice that for n = 2 determining Type (i) through Type (iv) are all the same problem, but for general n they can be quite different (indeed the I_j will not even have the same cardinality unless we restrict to $p \equiv 1 \mod n$). Note that these requirements become successively weaker (and the claim that there are no such examples for large enough p a successively stronger statement) as we move from (i) to (iia) or (iib), to (iii), to (iv). To make sense here we should probably think of pgrowing with n, for example we shall assume throughout that p > n + 1, otherwise all the residue classes have only 0 or 1 element and every permutation will be a Type (iia). Similarly if a permutation is not a Type (iii), or Type (iv), then we are demanding at least two, or at least n, values in each image of each residue class and so must have p > 2n, or $p > n^2$, for this to have any chance of being true.

If the map f randomly distributes the values mod n then we might expect to have $|f(I_i) \cap I_j| \sim p/n^2$ and so, for fixed n, no examples of Type (i) through (iv) once p is sufficiently large. However, as shown in [4] for n = 2, if the parameter $d = \gcd(k-1, p-1)$ is large we can't expect this equal distribution.

Indeed when n is odd it is not hard to see that we will have infinitely many examples of Type (iib) in addition to the identity map.

Example 1.1. Suppose that $p \equiv 1 \mod 4$ and that

$$f(x) = \pm x^{(p+1)/2} \mod p_1$$

If n is odd and $i \equiv 2^{-1}p \mod n$ then

 $f(I_i) = I_i.$

If n is even, or n is odd with $i \neq 2^{-1}p \mod n$, and $p > (n+1)^2$, then $f(I_i)$ hits exactly two residue classes, namely I_i and $I_{\overline{i}}$ where $\overline{i} \equiv p - i \mod n$.

The proof of Example 1.1 will be given in Section 7. At the expense of the explicit constant the condition $p > (n+1)^2$ could be replaced with $p \gg (n \log n)^{4/3}$ using the Burgess [5] bound $O(p^{1/4} \log p)$ for gaps between quadratic residues or nonresidues.

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A similar situation occurs for the map $f(x) = -x \mod p$; if p > n and n is even then the $f(I_j) = I_{\overline{j}}$ will be a derangement (i.e., a permutation fixing no element) of the I_j , while if n is odd this f will fix the I_i with $i \equiv 2^{-1}p \mod n$ and derange the remaining I_j .

Notice that in these examples the value of d is unusually large, namely d = p - 1 or (p - 1)/2. If d is not large then in fact each residue class does receive its fair share of values:

Theorem 1.1. For all i, j

(1.4)
$$|f(I_i) \cap I_j| = \frac{p}{n^2} + O(d\log^2 p) + O(p^{89/92}\log^2 p).$$

In particular, if n is fixed and $d = o(p/\log^2 p)$, then

$$|f(I_i) \cap I_j| \sim p/n^2.$$

This follows at once from the more numerically precise statement in Theorem 3.1 below, and relies on bounds for binomial exponential sums

(1.5)
$$\sum_{x=1}^{p-1} e_p(ax^k + bx).$$

As we show in Theorem 4.1 below, if we avoid those few cases in Example 1.1, then even for large d, for a given n there are at most finitely many cases of Type (iii); that is for all other mappings the image of each residue class $f(I_i)$ hits at least two different residue classes mod n.

Theorem 1.2. If n is even and $f(x) \neq \pm x \mod p$ or if n is odd and $f(x) \neq \pm x$ or $\pm x^{(p+1)/2} \mod p$, then there are no i, j with $f(I_i) \subseteq I_j$ once

$$p \ge 9 \cdot 10^{34} n^{92/3}$$
.

In the linear case we can be even more precise:

Theorem 1.3. Suppose that $f(x) = Ax \mod p$.

For p > 2n there are no Type (iii) linear maps $f(x) \neq \pm x \mod p$.

Similarly for the maps with k = (p+1)/2, but not of the form considered in Example 1.1, we can refine the bound in Theorem 1.2.

Theorem 1.4. Suppose that

$$f(x) = Ax^{(p+1)/2} \mod p, \quad A \neq \pm 1$$

If $n \ge 2$ and $p > (4n+1)^2$ then f(x) is not a Type (iii) map.

Theorems 1.3, 1.4 and Example 1.1, are the cases where the integer

$$L := (p - 1)/d$$

is 1 or 2. When $L \ge 3$ is small the argument in Theorem 4.1 similarly shows that there are no Type (iii) maps $f(x) = Ax^k \mod p$ once

$$p > 1214 n^2 (L-1)^2 \log^4(n(L-1))$$

At the other extreme, for the Lehmer type maps, k = -1, we have d = 2 and (1.4) certainly gives an asymptotic formula, but using the Kloosterman sum bound $2\sqrt{p}$ for (1.5) drastically improves the error term (see for example [1]). More generally

for small $|k|, k \neq 1$, one can use the Weil [21] bound $|k-1|\sqrt{p}$ to obtain (see also [20])

$$|f(I_i) \cap I_j| = \frac{p}{n^2} + O(|k|p^{1/2}\log^2 p).$$

Similarly when |k| is small we can obtain good bounds for both Type (iii) and (iv).

Theorem 1.5. Suppose that $f(x) = Ax^k \mod p$ with $k \neq 1$ positive or negative.

(a) If $p > 37|k-1|^2n^2$ then f(x) is not a Type (iii) map.

(b) If $p \ge 16.2|k-1|^2 n^4$ then f(x) is not a Type (iv) map.

If we want a stronger statement avoiding cases of Type (iv) even when d is large, that is, prove that the image of every residue class mod n hits every residue class mod n, then we will need to exclude more examples for n > 2. We explore this problem in [7]. The proofs of Theorems 1.2, 1.3 and 1.5 are given in Section 5 and Theorem 1.4 in Section 6.

For a given n we know from Theorem 1.2 that there are at most finitely many occurrences of Type (i), (iia) and (iib), but of course the bounds in this paper are far too large to obtain a complete determination as was done for n = 2 in [6]. We hope to employ the methods of [6] to complete this determination in a subsequent work.

2. Computations and Conjectures

Computations looking for maps of Type (i)-(iv) were performed for the primes p < 20,000, exponents k and moduli <math>n = 3 through 12. Of particular interest was obtaining Type (iii) examples with the ratio p/n as large as possible. This led to a more extensive investigation of the exponent k = (p + 1)/2. We quickly discovered Example 1.1 where for any odd n and prime $p \equiv 1 \mod 4$, the mapping $f(x) = \pm x^{(p+1)/2} \mod p$ is Type (iii). Further families with this exponent are given in Theorems 2.1 to 2.5. They were all discovered by looking at patterns in the data. Notice that a Type (iii) map of the form $f(x) = \pm x^{(p+1)/2} \mod p$ must produce a Type (iib) map for f(x) or -f(x); of course we are only interested maps of this type for n or even, or for n odd where the $f(I_i) = I_i$ has $2i \neq p \mod n$.

2.1. Type (iii) mappings: In Theorem 1.2 we verified the existence of a constant K(n) such that for p > K(n) and $f(x) \neq \pm x$ and when n is odd $f(x) \neq \pm x^{(p+1)/2}$ mod p, every residue class is mapped to at least two different residue classes, that is, f(x) is not Type (iii). The constant $K(n) = 9 \cdot 10^{34} n^{92/3}$ obtained there is undoubtedly far from the truth. Table 1 gives the five largest primes having an $f(x) = Ax^k \mod p$ with $f(I_i) \subseteq I_j$ for some i, j, found for each $3 \leq n \leq 12$ and $2n . Since <math>Ax^k$ has this property if and only if $-Ax^k$ does, we just consider positive A. From this data we make the following conjecture.

Conjecture 2.1. The optimal values for K(n) for n = 3 through 12 are

K(3) = 17, K(4) = 13, K(5) = 43, K(6) = 17, K(7) = 37,K(8) = 43, K(9) = 43, K(10) = 47, K(11) = 67, K(12) = 53.

The data suggests that one can take $K(n) = 2n^2$, although the correct bound is likely of order somewhere between $n \log n \log \log n$ and n^2 .

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p	Α	k	i		p	A	k	i	7	p	A	k	i
		n = 3					n = 8					n = 11	
7	3	5	$0,1,2^*$		23	2	3	0,7		41	6	3,23	4*
11	4	9	1*		23	3,10	5	0,7		41	17	7,27	4*
13	3	5,11	2^{*}		23	6,11	17	0,7		41	16	9,29	4*
17	4	5,13	1*		23	1	19	0,7		41	14	11,31	4*
		n = 4			23	10	21	0,7		41	18	13,33	4*
11	1	9	$0,1^*,2^*,3$		29	1	15	2,3		41	10	17,37	4*
13	2	5	0,1,2,3		29	7	19	6,7		41	19	19,39	4*
		n = 5		1	31	5	11	$3^{*}, 4^{*}$		41	18	29	1,7
19	5	17	2*		41	1	21	3,6		43	18	23	0,10
23	10	21	4*		43	2	13	0,3		43	6	41	5*
29	14	13	0,4				n = 9			47	15	45	7*
31	1	11	3*		29	5	9,23	4,7		53	2	25	10*
43	6	29	4*		29	9	11,25	4,7		53	2	51	10
		n = 6			29	10	13,27	1*		67	29	23	6*
13	1	5,11	$3^{*},4^{*}$		31	9	29	2*				n = 12	
13	1	7	2,3,4,5		37	4	$17,\!35$	5*		31	5	7	0,7
13	3	5	2,5		41	1	19	$1^{*}, 4^{*}$		31	7,14	7	8,11
13	3	11	$2^{*},5^{*}$		41	3	11,31	7^*		31	5	11	$0^{*},7^{*}$
13	6	11	0,1		41	4	13,33	7^*		31	6	11	$1,\!6,\!9,\!10$
17	1	9	0,5		41	10	9,29	7^*		31	10	13	0,7
17	2	5	2,3		41	11	19,39	7^*		31	6	17	0,7
17	4	7	0,5		41	12	3,23	7^*		31	8	19	$0^{*},7^{*}$
17	4	15	$0^{*},5^{*}$		41	13	7,27	7^*		31	15	19	$9^{*}, 10^{*}$
17	8	13	1,4		41	18	$17,\!37$	7^*		31	4,8	23	8,11
		n = 7			43	7	41	8*		31	12	23	0,7
19	2	17	6*				n = 10			31	3	29	$9^{*}, 10^{*}$
19	3	7	0,5		31	1	11	$3^{*}, 5^{*}, 6^{*}, 8^{*}$		31	5	29	$8^{*}, 11^{*}$
19	3	11	6*		37	8	7	8,9		31	9	29	$0^{*},7^{*}$
19	3	17	$0^{*},5^{*}$		37	14	31	0,7		37	1	19	4,5,8,9
19	5	5	6*		41	18	19	2,9		41	9	3, 13, 23, 33	8,9
19	6	7,11	0,5		41	2	21	3,8		41	1	11,21	8,9
19	7	7	6*		41	20	21	$5,\!6$		41	20	19,29	7,10
19	7	11	0,5		43	6	29	4*,9*		41	1	31	$7^{*}, 8, 9, 10^{*}$
19	8	13	6*		47	11	17	8,9		43	12	37	9,10
23	8	21	1*						-	53	1	27	1,4
23	9	21	$3^{*},6^{*}$										
29	14	13,27	4*										
31	2	29	5^{*}										
37	16	17	$4^{*},5^{*}$										

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TABLE 1. Type (iii): Five largest primes 2n having $an <math>f(x) = Ax^k \mod p$ with $f(I_i) \subseteq I_j$ for some i, j ($f(x) \neq x$ if nis even, and $f(x) \neq x$ or $x^{(p+1)/2}$ if n is odd). Type (iib): Cases of $f(I_i) = I_i$, are marked with a *.

Looking for larger ratios of p to n, we extended our computations to $13 \le n \le 86$ and $5n \le p \le 15n$. The values found with p/n > 9 are recorded in Table 2.

A large number of the Type (iii) maps with p/n large have k = (p+1)/2. The bounds in Example 1.1 and Theorem 1.4 enable a complete determination when k = (p+1)/2 and p > 2n for small n. There are no such Type (iii) mappings for n = 3, 4, 7 or 9, with a complete list of such maps for the remaining $5 \le n \le 12$ shown in Table 3.

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n	p	A	k	i	p/n
70	641	1	321	27,54	9.157142
84	773	1	387	$27,\!37,\!64,\!74$	9.202380
30	277	1	139	10,27	9.233333
39	367	1	245	8	9.410256
62	593	1	297	16, 19	9.564516
82	809	1	405	20,51	9.865853
60	593	1	297	21,32	9.883333
37	367	84	245	17	9.918918
85	853	221	143,569	44	10.035294
83	853	220	143,569	53	10.277108
35	367	83	245	26	10.485714
81	853	220	143,569	62	10.530864
76	809	1	405	$58,\!67$	10.644736
79	853	221	143,569	71	10.797468
86	941	1	471	83,84	10.941860
84	977	1	489	12,41	11.630952
86	1013	1	507	$2,\!65$	11.779069

TABLE 2. Type (iii) with $3 \le n \le 86$ and 9 < p/n < 15 $(f(x) = Ax^k \mod p$ with A > 0, and $f(x) \ne x^{(p+1)/2}$ if n is odd).

	p	Α	i]		p	Α	i	1		p	Α	i
n = 5	13	4,5	3,5		n = 10	29	2	2,7		n = 11	29	10	$7,\!11$
n = 6	13	1	$2,\!3,\!4,\!5$			29	9,10	9,10		n = 12	29	1	1,2,3,4
	17	1	5,6			29	14	4,5			29	12	8,9
n = 8	17	1	$1,\!3,\!6,\!8$			41	2	3,8			37	1	4,5,8,9
	29	1	2,3			41	20	5,6			41	1	8,9
	41	1	3,6						1		53	1	1,4

TABLE 3. All Type (iii) of the form $f(x) = Ax^{(p+1)/2} \mod p$ for $3 \le n \le 12$ (for A > 0 and excluding $f(x) = x^{(p+1)/2}$ if n is odd).

In the proof of Theorem 1.4 for k = (p+1)/2 we had to deal separately with the case A = 2, so additional computations were performed for $f(x) = 2x^{(p+1)/2} \mod p$ looking for examples with large ratio p/n. These corresponded to primes with a certain pattern of quadratic residues. Examining the corresponding n values led us to a family of Type (iii) mappings of the form $2x^{(p+1)/2}$, with arbitrarily large p/n, and requiring K(n) to be as large as $n \log n$.

Theorem 2.1. Let

$$f(x) = 2x^{(p+1)/2} \mod p.$$

Suppose that $p \equiv 1 \mod 4$ has

(2.1)
$$\begin{pmatrix} \frac{p}{q} \end{pmatrix} = \begin{cases} +1, & \text{if } q = 1 \mod 4 \\ -1, & \text{if } q = 3 \mod 4 \end{cases}$$

for all primes $3 \le q \le 4t - 1$, and that $n \equiv 2 \mod 4$ with

(2.2)
$$\frac{2p}{4t+1} \le n < \frac{2p}{4t-1}.$$

	p	t	n	p/n
	15461	9	838	18.449880
	23201	9	1258	18.442766
Theorem 2.1	40169	9	2174	18.477000
	70769	10	3454	20.488998
	75869	9	4102	18.495611

TABLE 4. Primes p < 100,000 with $p \equiv 1 \mod 4$ and $\left(\frac{-p}{q}\right) = 1$ for all odd $q \leq 4t - 1$ for some $t \geq 9$.

Then for both

(2.3)
$$i := \frac{1}{4}(2p - (4t - 1)n), \quad j := \begin{cases} 2i \mod n, & \text{if } \left(\frac{n}{p}\right) = -1, \\ p - 2i \mod n, & \text{if } \left(\frac{n}{p}\right) = 1, \end{cases}$$

and

(2.4)
$$i := \frac{1}{4}(2p - (4t - 3)n), \quad j := \begin{cases} p - 2i \mod n, & \text{if } \left(\frac{n}{p}\right) = -1, \\ 2i \mod n, & \text{if } \left(\frac{n}{p}\right) = 1, \end{cases}$$

we have $f(I_i) \subseteq I_j$.

The primes p < 100,000 with property (2.1) with $t \ge 9$, and the smallest n this gives in Theorem 2.1 are shown in Table 4

Using the Chinese remainder we can construct p with this property for arbitrarily large t. For example we could take $p \equiv 1 \mod 4Q_1$ and $-1 \mod Q_2$ where Q_1 and Q_2 are the products of the primes $q \leq 4t - 1$ that are 1 or $-1 \mod 4$ respectively (there are of course many other ways). Hence we can make Type (iii) examples with $p > (2t + \frac{1}{2} - \varepsilon)n$. In particular we can't take K(n) = Cn however large the C. Moreover, by the work of Heath-Brown [13] and Xylouris [23] on the smallest prime in an arithmetic progression, there exist such p with $p \ll Q^{5.18}$, with $Q = 4Q_1Q_2$, and hence examples of Type (iii) with $p > \frac{1}{11}n \log n$. Assuming GRH guarantees such $p < 2(Q \log Q)^2$ (see Bach [2] or Lamzouri, Li and Soundararajan [15]) and thus $p > (\frac{1}{4} - \varepsilon)n \log n$. The proofs of the theorems in this section are given in Section 8.

2.2. Type (iib) Mappings. The Type (iib) maps, where $f(I_i) = I_i$ for some *i*, are marked with an asterisk in Table 1; of course for such cases the iterates will also fix I_i and a number of these can be seen in the table. For example for n = 7, p = 19, the map $f(x) = 5x^5 \mod 19$ fixes I_6 , as does $f^2(x) = 7x^7 \mod 19$, $f^3(x) = -2x^{17} \mod 19$, $f^4(x) = -8x^{13} \mod 19$, $f^5(x) = -3x^{11} \mod 19$ and $f^6(x) = x \mod p$; in this case $p - 6 \equiv 6 \mod n$ so that the maps with negative A recorded in their positive guise also fix I_6 .

Many examples of Type (iib) mappings in our data with large ratio p/n are of the form $f(x) = \pm x^{(p+1)/2} \equiv \pm \left(\frac{x}{p}\right) x \mod p$. For n even, or n odd with $i \not\equiv 2^{-1}p \mod n$, it is readily seen that this requires p to have a string of roughly p/n consecutive quadratic residues or nonresidues. In Theorems 2.2 to 2.5 we explore how conversely long blocks of consecutive residues or nonresidues can produce large p/n values. We distinguish several cases frequently encountered in the data. Theorem 2.2 deals

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with consecutive quadratic residues starting at 1, Theorem 2.3 with an interval of consecutive residues or nonresidues around p/2, Theorem 2.4 with intervals around p/3 and 2p/3, and Theorem 2.5 with the remaining cases. Table 5 shows the primes p < 100,000 with a string of at least 25 consecutive residues or nonresidues, and examples arising from them when Theorems 2.2 through 2.5 are applied as appropriate. For the non-central interval we give the [a, a + t) with a < p/2, and omit the symmetric interval (p - a - t, p - a].

Theorem 2.2. Let t be a positive integer and p > t a prime with $p \equiv 1 \mod 8$ and $\left(\frac{p}{q}\right) = 1$ for all odd primes $q \leq t$. Then for any n > (p-1)/(t+1) the map $f(x) = \left(\frac{n}{p}\right) x^{(p+1)/2} \mod p$ is the identity map on I_0 .

Notice that i = 0 does not have $2i \equiv p \mod n$, so these examples are of interest for both odd and even n. Again, by the Chinese Remainder Theorem and Dirichlet's theorem, for any t, there exist infinitely many p satisfying the hypotheses of this theorem and so we get examples with p as large as $n \log n$, but for certain p we can push the size of K(n) a little bigger. By the work of Graham and Ringrose [11] we know there exist infinitely many primes $p \equiv 1 \mod 4$ having a least quadratic nonresidue of size at least $c \log p \log \log \log p$ for some constant c (with improvement to $c \log p \log \log p$ under GRH by Montgomery [18]). Taking $t = \lfloor c \log p \log \log \log p \rfloor$, the hypotheses of the theorem are satisfied by reciprocity, and thus with $n = \lceil (p-1)/t \rceil$, we obtain a Type (iib) mapping with $p \gg n \log n \log \log \log n$ (with improvement under GRH).

Since $p \equiv 1 \mod 4$, if our interval of consecutive quadratic residues or nonresidues contains (p-1)/2, then we have a symmetric interval around p/2 with $\left(\frac{x}{p}\right) = \left(\frac{(p-1)/2}{p}\right) = \left(\frac{2}{p}\right)$. For odd n we obtain examples with p/n close to the interval length t, but unfortunately (2.6) has $2i \equiv p \mod n$ which we know always gives a Type (iii) by Example 1.1, though additionally here f(x) is the identity map on I_i . If we restrict to even n then the ratio p/n is only close to t/2.

Theorem 2.3. Suppose that $p \equiv 1 \mod 4$ has t = 2T consecutive residues or nonresidues around p/2:

$$\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right), \quad a = \frac{p+1}{2} - T \le x \le \frac{p-1}{2} + T.$$

Equivalently suppose that $\left(\frac{q}{p}\right) = 1$ for all odd primes $q \leq 2T - 1$. Suppose n is even with

(2.5)
$$\frac{2p}{t+1} < n < \frac{2p}{t-1}, \quad i := an - \left(\frac{n}{2} - 1\right)p,$$

or n is odd with

(2.6)
$$\frac{p}{t+1} < n < \frac{p}{t-1}, \quad i := an - \left(\frac{n-1}{2}\right)p,$$

then $f(x) = \left(\frac{2n}{p}\right) x^{(p+1)/2} \mod p$ is the identity map on I_i .

A large interval of consecutive quadratic residues or nonresidues around p/3 (and hence under $x \mapsto p - x$ around 2p/3) will also lead to large p/n values, the size depending on $n \mod 3$.

A GENERALIZATION OF THE GORESKY-KLAPPER CONJECTURE, PART I

	p	t	a	n	i	p/n
Theorem 2.2	87481	$\overline{28}$	1	3017	0	29.115346
(n odd & even)	87481	28	1	3018	0	29.105699
	13381	28	6677	463	440	28.900647
	20749	28	10361	717	695	28.938633
	51349	28	25661	1771	1766	28.994353
Theorem 2.3	82021	30	40996	2647	2629	30.986399
(n odd)	87481	28	43727	3017	3011	28.996022
	89989	28	44981	3105	3077	28.981964
	92821	28	46397	3201	3197	28.997500
	99709	30	49840	3217	3208	30.994404
	13381	28	6677	924	907	14.481601
	20749	28	10361	1432	1417	14.489525
	51349	28	25661	3542	3532	14.497176
Theorem 2.3	82021	30	40996	5292	5287	15.499055
(n even)	87481	28	43727	6034	6022	14.498011
	89989	28	44981	6208	6181	14.495650
	92821	28	46397	6402	6394	14.498750
	99709	30	49840	6434	6416	15.497202

Theorem 2.4	p	t	a	T_1	T_2	n	i	p/n
$n \equiv 2 \mod 3$	52361	29	17437	17	12	1976	1974	26.498481
	65129	27	21693	17	10	2459	2436	26.485969
$n \equiv 1 \mod 3$	52361	29	17437	17	12	2833	2800	18.577832
	65129	27	21693	17	10	4204	4182	15.492150
$n \equiv 0 \mod 3$	52361	29	17437	17	12	2964	2961	17.665654
	65129	27	21693	17	10	3696	3529	17.621482
	p	t	a	u	n		i	p/n
Theorem 2.5	90313	$\overline{26}$	39556	$\overline{38}$	3386	2437	7,3226	26.672474

TABLE 5. Type (iii): Primes p < 100,000 with $t \ge 25$ consecutive quadratic residues or consecutive nonresidues, [a, a + t).

Theorem 2.4. Suppose that $p \equiv 1 \mod 4$ and set

$$\delta := \begin{cases} 1 & \text{if } p \equiv 1 \mod 3, \\ 2 & \text{if } p \equiv 2 \mod 3. \end{cases}$$

 $Suppose \ that$

$$\left(\frac{x}{p}\right) = \left(\frac{3}{p}\right),$$

 for

$$a_1 := \frac{1}{3}(p-\delta) - (T_1 - 1) \le x \le \frac{1}{3}(p-\delta) + T_2,$$

$$a_2 := \frac{1}{3}(2p+\delta) - T_2 \le x \le \frac{1}{3}(2p+\delta) + (T_1 - 1).$$

Equivalently

(2.7)
$$\left(\frac{3m-\delta}{p}\right) = 1, \ 1 \le m \le T_2, \quad \left(\frac{3m+\delta}{p}\right) = 1, \ 0 \le m < T_1.$$

Suppose that $n \equiv 0 \mod 3$ has

3T

$$\frac{3p}{3T_1 + \delta} < n < \frac{3p}{3T_1 + \delta - 3}, \quad i := a_1 n - \left(\frac{n}{3} - 1\right)p,$$

or

$$\frac{3p}{2+3-\delta} < n < \frac{3p}{3T_2-\delta}, \quad i := a_2n - \left(\frac{2n}{3}-1\right)p,$$

or $n \equiv 2 \mod 3$, when $T_1 \leq 2T_2 + 2 - \delta$ and

$$\frac{2p}{3T_1 + \delta} < n < \frac{2p}{3T_1 + \delta - 3}, \quad i := a_1 n - \left(\frac{n-2}{3}\right) p$$

or $n \equiv 2 \mod 3$, when $T_1 \ge 2T_2 + 2 - \delta$ and

$$\frac{p}{3T_2 + 3 - \delta} < n < \frac{p}{3T_2 - \delta}, \quad i := a_2 n - \left(\frac{2n - 1}{3}\right) p_i$$

or $n \equiv 1 \mod 3$, when $T_2 \geq 2T_1 - 1 + \delta$ and

$$\frac{p}{3T_1+\delta} < n < \frac{p}{3T_1+\delta-3}, \quad i := a_1n - \left(\frac{n-1}{3}\right)p,$$

or $n \equiv 1 \mod 3$, when $T_2 \leq 2T_1 - 1 + \delta$ and

$$\frac{2p}{3T_2 + 3 - \delta} < n < \frac{2p}{3T_2 - \delta}, \quad i := a_2 n - \left(\frac{2n - 2}{3}\right) p.$$

Then $f(x) = \left(\frac{3n}{p}\right) x^{(p+1)/2} \mod p$ is the identity map on I_i .

Note if $p \equiv 1 \mod 3$ then $\left(\frac{3}{p}\right) = 1$ and a large interval around p/3 leads to a long interval starting at a = 1 where one can use Theorem 2.2 to potentially produce a larger p/n. Similar theorems could no doubt be obtained for intervals around p/5 etc. Three p/3 type examples occur in Table 2; namely p = 277, $T_1 = 10$, $T_2 = 0$, where the largest p/n in Theorem 2.4 will be for $n \equiv 0 \mod 3$, with the smallest n = 27 (smallest even n = 30), p = 569, $T_1 = 3$, $T_2 = 6$ where the best choice is $n \equiv 1 \mod 3$, smallest n = 61 (smallest even n = 64), and p = 641, $T_1 = 5$, $T_2 = 6$ where the smallest $n \equiv 1 \mod 3$ is n = 70.

For general intervals of consecutive quadratic residues or nonresidues we have:

Theorem 2.5. Suppose that we have t consecutive quadratic residues or nonresidues mod p starting at an $a \ge 2$

$$\left(\frac{x}{p}\right) = \left(\frac{a}{p}\right), \ a \le x \le a + t - 1, \ a = st + r, \ 0 \le r < t.$$

If i = na - (s - u)p and n is an integer in

(2.8)
$$\max\left\{\frac{(s-u+1)p}{a+t}, \frac{(s-u)p}{a}\right\} \le n \le \frac{(s-u)p}{a-1}$$

or i = (s - u + 1)p - n(a + t - 1) and

(2.9)
$$\frac{(s-u+1)p}{a+t} \le n \le \min\left\{\frac{s-u}{a-1}, \frac{s-u+1}{a+t-1}\right\}p,$$

for some integer $\frac{(s+1-r)}{(t+1)} > u > s+1-\frac{1}{2}(a+t)$, then

(2.10)
$$f(x) = \left(\frac{an}{p}\right) x^{(p+1)/2} \mod p,$$

10

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is the identity map on I_i .

Notice that for very large a + t and a given u we are not guaranteed an n in the range (2.8), but for small a, for example $a + t < \sqrt{p}$, there will be at least one n, leading to an example with p/n close to t for small u. Computationally searching the primes p < 10,000 for Type (iii) mappings of the form $f(x) = x^{(p+1)/2} \mod p$, with $2i \neq p \mod n$ if n is odd, found 578 primes $p \equiv 1 \mod 4$ admitting an n with 7 < p/n < 20. In each case we checked the minimal n found against Theorem 2.5. Of these 545 corresponded to taking the longest string of consecutive quadratic residues or nonresidues in Theorem 2.5 (for a, t and a suitable u), with 16 to taking the second longest, and the remaining 17 cases found with shorter intervals.

Theorems 2.2 to 2.5 used runs of x where $x^{(p-1)/2} \mod p$ is constant. One could similarly consider intervals where $x^{(p-1)/3} \mod p$ is constant. We state the counterpart of Theorem 2.3, since several examples of an interval around p/2 appear in Table 2 (namely p = 853 and 367 with t = 10 and a = 422 or 179 respectively).

Theorem 2.6. Suppose that $p \equiv 1 \mod 3$ and that

(2.11)
$$x^{(p-1)/3} \equiv a^{(p-1)/3} \mod p, \quad a = \frac{p+1}{2} - T \le x \le \frac{p-1}{2} + T.$$

Then, with t = 2T and n and i as in (2.5) and (2.6), any map

(2.12) $f(x) = (an)^{2(k-1)}x^k \mod p, \ k = j(p-1)/3+1, \ j = 1, 2, \ \gcd(k, p-1) = 1,$

will be the identity map on I_i , and when n is odd any map

(2.13) $f(x) = (an)^{2(k-1)}x^k \mod p$, k = j(p-1)/6+1, j = 1, 5, gcd(k, p-1) = 1, will have $f(I_i) = I_i$.

This could be further generalized to intervals where $x^{(p-1)/q} \mod p$ is constant.

2.3. Type (i) and (iia) Mappings. Only a few cases were found where $Ax^k \mod p$ permutes every residue class:

Example 2.1. The only cases of Type (i), that is $f(I_j) = I_j$ for all j, found for $3 \le n \le 12$ and p < 20,000, were n = 3, (p; A, k) = (5; -1, 3) and (7; -3, 5).

The examples of Type (iia) found, that is where $f(I_1), \ldots, f(I_n)$ is a permutation of I_1, \ldots, I_n , are shown in Table 6. By symmetry we include only A > 0 and of course exclude f(x) = x.

The reoccurrence of p = n + 2 and p = n + 3 is easily explained; in these cases we have only one or two residue classes with two entries, $I_1 = \{1, -1\}$ fixed by any $f(x) = x^k$, or $I_1 = \{1, -2\}, I_2 = \{2, -1\}$ interchanged by $f(x) = 2x^{p-2}$, with the remaining singleton sets permuted. This leaves only a few examples with $p \ge n+4$. We searched for further Type (iia) examples with $n+4 \le p < 5n$ for $13 \le n \le 100$. The results are shown in Table 7. It turns out that all of these primes have the following property:

Theorem 2.7. If p = n + w with $2 \le w < n$, $p \equiv 1 \mod 4$ and

$$\left(\frac{y}{p}\right) = \left(\frac{w-y}{p}\right)$$
, for all $1 \le y < w/2$,

then $f(x) = x^{(p+1)/2} \mod p$ produces a Type (iia) permutation.

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	p	A	k	σ
n = 3	5	1	3	(02)
	$\overline{7}$	3	5	(13)
n = 4	7	2	5	(12)
	11	1	9	(03)
	13	2	5	(0312)
n = 5	7	1	5	(03) (24)
n = 8	11	2	9	(03)(12)(46)
	13	4	5	(06)(14)(23)(57)
n = 9	11	1	3	(0354)(2867)
	11	1	7	(0453)(2768)
	11	1	9	(05)(26)(34)(78)
	13	1	7	(58)(67)
n = 10	13	2	11	(08)(12)(35)(47)(69)
n = 11	13	1	5	$(07)(26)(39)(4\ 10)$
	13	1	7	(02)(58)(67)
	13	1	11	$(06)(27)(39)(4\ 10)(58)$

TABLE 6. Type (iia): $f(x) = Ax^k \mod p$ with $f(I_i) = I_{\sigma(i)}$ for some permutation σ , for $3 \le n \le 12$, n + 1 (with <math>A > 0 and $f(x) \ne x$).

n	p	A	k
24	29	1	15
33	37	1	19
35	41	1	21
48	53	1	27
57	61	1	31
68	73	1	37
69	73	1	37
83	89	1	45
91	101	1	51
92	97	1	$25,\!49,\!73$
93	97	1	49
96	101	1	51

TABLE 7. Type (iia): $f(x) \neq x$ for $13 \leq n \leq 100$ and $n+4 \leq p < 5n$.

The additional $f(x) = x^{(p+3)/4}$ and $x^{(3p+1)/4} \mod p$ for p = 97 are also predictable; for example for $p \equiv 1 \mod 24$ all three f(x) will occur for p = n + 4or p = n + 5 when $3^{(p-1)/4} \equiv 1 \mod p$ or $2^{(p-1)/4} \equiv 3^{(p-1)/4} \mod p$ respectively (where one might expect either of these to occur roughly half the time, 193 and 97 respectively being the first cases of these), with similar conditions for p = n + 6, n + 7 etc that should hold for a positive proportion of the time. The sparsity of Type (i) and (iia) examples suggests the following conjecture.

Conjecture 2.2. Suppose that p > 2n.

Excluding $f(x) = x \mod p$, there are only finitely many examples of Type (i), that is $f(I_i) = I_i$ for all i.

Excluding $f(x) = \pm x \mod p$ there are only finitely many examples of Type (iia), that is $f(I_i) = I_{\sigma(i)}$ for some permutation σ of $\{0, 1, \ldots, n-1\}$.

Indeed there may be no Type (i) or (iia) maps with p > 2n other than the few cases found above for n = 3 or 4. From the Graham & Ringrose [11] bound $g(p) \gg \log p \log \log \log p$ on the least quadratic nonresidue we see that we cannot replace the p > 2n in this conjecture by $p > n + C \log n$, however large the C.

3. Type (iii) and type (iv) intersections for small d

For $j = 0, 1, \ldots, n-1$, let I_j be the set of values in (1.2). Put

(3.1)
$$N_j := |I_j| = \begin{cases} \left\lfloor \frac{p-1+n-j}{n} \right\rfloor, & \text{if } j \neq 0; \\ \left\lfloor \frac{p}{n} \right\rfloor, & \text{if } j = 0. \end{cases}$$

Theorem 3.1. Suppose that p > 607. Then for any A and k satisfying (1.1), $2 \le n < p$, and $0 \le i, j \le n - 1$, we have

(3.2)
$$|f(I_i) \cap I_j| = p^{-1}|I_i||I_j| + E,$$

with

$$|E| \le \left(d + 1 + 2.293p^{89/92}\right) \left(\frac{4}{\pi^2} \log p + 0.381\right)^2,$$

and

$$\frac{p}{n^2} - 1 < p^{-1}|I_i||I_j| < \frac{p}{n^2} + 1.$$

For $7 \le p \le 607$, the same result holds with .381 replaced by 1/2.

Proof. For any $i, j \in \{0, 1, ..., n-1\}$ write

$$N_{ij} = |f(I_i) \cap I_j|.$$

We use \mathbb{Z}_p to denote the integers mod p, and view I_i, I_j as subsets of \mathbb{Z}_p . We write $\mathscr{I}_i(x), \mathscr{I}_j(x)$ for the characteristic functions for I_i, I_j so that

$$N_{ij} = \sum_{x \bmod p} \mathscr{I}_i(x) \mathscr{I}_j(Ax^k)$$

Since $\mathscr{I}_j(x)$ is a periodic function mod p we have a finite Fourier expansion

$$\mathscr{I}_j(x) = \sum_{u \mod p} a_j(u) e_p(ux)$$

where $e_p(x) = e^{2\pi i x/p}$, and for u = 0, ..., p - 1,

$$a_j(u) = \frac{1}{p} \sum_{y \mod p} \mathscr{I}_j(y) e_p(-yu) = \begin{cases} p^{-1} N_j, & \text{if } u = 0; \\ p^{-1} e_p(\xi_j u) \frac{\sin(\pi n u N_j/p)}{\sin(\pi n u/p)}, & \text{if } u \neq 0, \end{cases}$$

for some ξ_j in \mathbb{Z}_p . Hence, separating into zero and nonzero values of u and v, and observing that Ax^k is a permutation of \mathbb{Z}_p , we have

$$N_{ij} = \sum_{x=0}^{p-1} \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} a_i(u) e_p(ux) a_j(v) e_p(vAx^k) = M + T_1 + T_2 + E$$

where

$$M = p \ a_i(0)a_j(0) = p^{-1}N_iN_j,$$

$$T_{1} = a_{j}(0) \sum_{u=1}^{p-1} a_{i}(u) \sum_{x=0}^{p-1} e_{p}(ux) = 0,$$

$$T_{2} = a_{i}(0) \sum_{v=1}^{p-1} a_{j}(v) \sum_{x=0}^{p-1} e_{p}(vAx^{k}) = a_{i}(0) \sum_{v=1}^{p-1} a_{j}(v) \sum_{x=0}^{p-1} e_{p}(vx) = 0,$$

and

$$E = \sum_{u=1}^{p-1} \sum_{v=1}^{p-1} a_i(u) a_j(v) \sum_{x=0}^{p-1} e_p(ux + vAx^k).$$

Now from [8, Theorem 1.3] we have, with d = (k - 1, p - 1),

(3.3)
$$\left|\sum_{x=0}^{p-1} e_p(ux+vAx^k)\right| \le 1+d+2.292 \ p^{89/92},$$

and from [9, Theorem 1], observing that nx is a permutation of the $x \mod p$,

$$\sum_{u=1}^{p-1} |a_j(u)| \le \frac{1}{p} \sum_{x=1}^{p-1} \frac{|\sin(\pi x N_j/p)|}{|\sin(\pi x/p)|} \\ \le \frac{4}{\pi^2} \log p + .38 + \frac{0.608}{p} + \frac{0.116}{p^3} \\ \le \begin{cases} \frac{4}{\pi^2} \log p + .381, & \text{if } p > 607; \\ \frac{4}{\pi^2} \log p + \frac{1}{2}, & \text{if } p > 5. \end{cases}$$

Hence for p > 607,

$$|E| \le (d+1+2.292 \ p^{89/92}) \left(\sum_{u=1}^{p-1} |a_i(u)|\right) \left(\sum_{v=1}^{p-1} |a_j(v)|\right)$$
$$\le (d+1+2.292 \ p^{89/92}) \left(\frac{4}{\pi^2} \log p + .381\right)^2.$$

Since $p/n - 1 < N_j < p/n + 1$ we have $p^2/n - 1 < M < p^2/n + 1$.

Notice that if $d < 0.006p^{89/92}$ and $p \ge e^{333}(n \log n)^{184/3}$ then the main term in Theorem 3.1 exceeds the error term and we can say that $f(I_i) \cap I_j \ne \emptyset$ for all i, j. If our interest is just in proving that $f(I_i) \cap I_j$ is nonempty, rather than obtaining an asymptotic estimate of its cardinality, then as shown in the next theorem we do not need the $\log n$ term.

Theorem 3.2. Let p be an odd prime and A, k, n integers satisfying (1.1) with $2 \le n < p$, and

$$d = \gcd(k - 1, p - 1) \le 0.006p^{89/92}$$

(a) For any $i, j, 0 \leq i, j < n$, we have $f(I_i) \cap I_j \neq \emptyset$ provided that

$$p > 4 \cdot 10^{29} n^{\frac{184}{3}}$$

(b) For any $i, j, 0 \leq i, j < n$, we have $f(I_i) \not\subseteq I_j$ provided that

$$p > 9 \cdot 10^{34} n^{\frac{92}{3}}$$

Proof. (a) Recall, for $0 \le j \le n-1$, $I_j = \{x : x \equiv j \mod n, x \ne 0\} \subseteq \mathbb{Z}_p$, $N_j = |I_j|$. Let

$$J_j := \{j, j+n, \dots, j+(\lceil N_j/2\rceil - 1)n\} \subseteq \mathbb{Z}_p, \quad j \neq 0,$$

$$K_j := \{0, n, 2n, \dots, \lfloor N_j/2\rfloor n\} \subseteq \mathbb{Z}_p,$$

with $J_0 = \{n, 2n, \ldots, \lceil N_0/2 \rceil n\}$, so that $J_j + K_j \subseteq I_j$, and let $\alpha_j = \mathscr{I}_{J_j} * \mathscr{I}_{K_j}$, the convolution of the characteristic functions of J_j and K_j ,

$$\alpha_j(x) := \sum_{\substack{u \in J_j \\ u+v=x}} \sum_{\substack{v \in K_j \\ u+v=x}} 1,$$

with Fourier coefficients $b_j(y)$ say. Then α_j is supported on I_j , and so our goal is to show that for any i, j,

$$\sum_{\text{mod } p} \alpha_i(x) \alpha_j(Ax^k) > 0.$$

Expanding the sum as before we obtain

x

$$\sum_{\substack{x \mod p}} \alpha_i(x)\alpha_j(Ax^k)$$

= $b_i(0)b_j(0)p + \sum_{\substack{u \neq 0}} \sum_{\substack{v \neq 0}} b_i(u)b_j(v) \sum_{\substack{x \mod p}} e_p(ux + vAx^k)$
= $M' + E'$,

say, where M' is the main term and E' the error term. Plainly, we have

$$M' = |J_i||K_i||J_j||K_j|p^{-1}$$

Next, using the fact that $b_j(v) = pa'_j(v)a''_j(v)$ where $a'_j(v)$, $a''_j(v)$ are the Fourier coefficients of \mathcal{I}_{J_j} , \mathcal{I}_{K_j} , we obtain from the Cauchy-Schwarz inequality and Parseval identity that for any j,

$$\sum_{v \mod p} |b_j(v)| = p \sum_{v \mod p} |a'_j(v)| |a''_j(v)| \le p \left(\sum_{v \mod p} |a'_j(v)|^2\right)^{\frac{1}{2}} \left(\sum_{v \mod p} |a''_j(v)|^2\right)^{\frac{1}{2}} = |J_j|^{\frac{1}{2}} |K_j|^{\frac{1}{2}}.$$

Thus, since $d \leq 0.006p^{89/92}$ and $p > 10^{29}$,

$$|E'| \le (d+1+2.292\,p^{89/92})|J_j|^{\frac{1}{2}}|K_j|^{\frac{1}{2}}|J_i|^{\frac{1}{2}}|K_i|^{\frac{1}{2}} < 2.299\,p^{89/92}|J_j|^{\frac{1}{2}}|K_j|^{\frac{1}{2}}|J_i|^{\frac{1}{2}}|K_i|^{\frac{1}{2}},$$

and we see that $M' > |E'|$ provided that

and we see that
$$M' > |E'|$$
 provided that

(3.4)
$$|J_j||K_j||J_i||K_i| > p^2 \left(2.299 \ p^{89/92}\right)^2.$$

Since $|J_j| \ge N_j/2$, $|K_j| \ge N_j/2$ and $N_j \ge \frac{p}{n} - 1$ for all j, it suffices to have

$$(p-n)^4 > 2^4 n^4 p^2 \left(2.299 \ p^{89/92}\right)^2$$
,

and for this it suffices to have $p > 4 \cdot 10^{29} n^{\frac{184}{3}}$.

(b) We may assume that $n \ge 3$. For type (iii) intersections, we let $I_j^c = \mathbb{Z}_p \setminus I_j$, a set of cardinality $N_j^c = p - |I_j| \ge p(1 - \frac{1}{n}) - 1 \ge 2p/3 - 1$. For $j \ne 0$ we shall think of I_j^c as the arithmetic progression

$$I_j^c = \{j + nt : t = N_j, N_j + 1, \dots, p - 1\},\$$

on observing that the values corresponding to t = 0, 1, ..., p - 1 are distinct mod p, giving \mathbb{Z}_p , and that we have removed the $t = 0, 1, ..., N_j - 1$ constituting I_j . Similarly $I_0^c = \{nt : t = N_0 + 1, ..., p\}$. We define α_j as above with sets

$$J'_{j} = \{j + N_{j}n, j + (N_{j} + 1)n, \dots, j + (N_{j} + \lceil N_{j}^{c}/2 \rceil - 1)n\}, \ j \neq 0,$$

$$K'_{j} = \{0, n, 2n, \dots, \lfloor N_{j}^{c}/2 \rfloor n\},$$

with $J'_0 = \{(N_j + 1)n, \dots, (N_j + \lceil N_j^c/2 \rceil)n\}$, so that J'_j, K'_j have cardinalities at least $N_j^c/2$, and α_j is supported on I_j^c . Once again we succeed in obtaining $f(I_i) \cap I_j \neq \emptyset$, provided that

(3.5)
$$|J'_j||K'_j||J_i||K_i| > p^2(2.299p^{89/92})^2,$$

and for this it suffices to have

$$(3.6) p > 9 \cdot 10^{34} \ n^{\frac{92}{3}}.$$

4. Type (iii) intersections for large d

In this section we show that for large d we cannot have $f(I_i) \subseteq I_j$ for any i, j provided p is sufficiently large. Recall

$$I = \{1, 2, \dots, p-1\}, \qquad I_j : \{x \in I : x \equiv j \mod n\}.$$

Theorem 4.1. Suppose that $f(x) \neq \pm x \mod p$ when n is even, and $f(x) \neq \pm x$ or $\pm x^{\frac{1}{2}(p+1)} \mod p$ when n is odd. If $p > 10^6$ and $d \ge 0.66np^{1/2}\log^2 p$, then $f(I_i) \cap (I \setminus I_j) \neq \emptyset$ for all i, j.

The same conclusion holds if $p \ge 7$ and $d > 3n\sqrt{p}\log^2 p$.

Plainly $f(x) = \pm x \mod p$ maps I_i to I_i or to $I_{\overline{i}}$ where $\overline{i} \equiv p - i \mod n$ so must be excluded. The $f(x) = \pm x^{(p+1)/2}$ are dealt with in Example 1.1, where for $p > 4n^2$ and n even we always have $f(I_i) \cap (I \setminus I_j) \neq \emptyset$, but for odd n must be excluded.

Proof. Suppose that $(A, k) \neq (\pm 1, 1)$ or $(\pm 1, (p+1)/2)$. Observe that the set of absolute least residues

$$\mathscr{C} = \{ C = Ax^{k-1} \mod p : 1 \le x \le p-1, |C| < p/2 \}$$

must contain at least one element $C \neq \pm 1$. To see this observe that \mathscr{C} contains (p-1)/d elements and hence more than two unless d = (p-1) or (p-1)/2 and k = 1 or (p+1)/2. In these cases \mathscr{C} contains only A or $\pm A$ and we just need to avoid $A = \pm 1$. We need to prove that $f(I_i) \cap (I \setminus I_j) \neq \emptyset$. We shall suppose that our $C \equiv AB^{k-1} \mod p$ satisfies 1 < C < p/2; if all the potential C's are negative we replace A by -A and j by the least residue of $p - j \mod n$. We let

$$L := (p - 1)/d$$

and

 $\mathscr{U} = \{x \in I_i : Cx \mod p \in I \setminus I_j, x \equiv Bz^L \mod p \text{ for some } z\}.$ Notice that if x is in \mathscr{U} we have

$$Ax^k \equiv Cx(B^{-1}x)^{k-1} \equiv Cxz^{L(k-1)} = Cx(z^{p-1})^{(k-1)/d} \equiv Cx \mod p$$

and we have an f(x) in $f(I_i) \cap (I \setminus I_j)$. So it is enough to show that $|\mathscr{U}| > 0$. Let \hat{G} denote the set of Dirichlet (multiplicative) characters on \mathbb{Z}_p^* with principal character χ_0 and recall that

$$\sum_{\chi \in \hat{G}, \chi^L = \chi_0} \chi(y) = \begin{cases} L, & \text{if } y \text{ is an } L\text{th power mod } p, \\ 0, & \text{if } y \text{ is not an } L\text{th power mod } p. \end{cases}$$

Hence, writing $\mathscr{I}_j^c(x)$ for the characteristic function of $I \setminus I_j$, the complement of I_j , we have

$$L|\mathscr{U}| = \sum_{x \in \mathbb{Z}_p^*} \mathscr{I}_i(x) \mathscr{I}_j^c(Cx) \sum_{\chi \in \hat{G}, \chi^L = \chi_0} \chi(B^{-1}x).$$

Separating the principal character from the remaining L-1 characters with $\chi^L = \chi_0$

$$L|\mathscr{U}| = M + E,$$

where M is our 'main term'

$$M = \sum_{x \in \mathbb{Z}_p^*} \mathscr{I}_i(x) \mathscr{I}_j^c(Cx),$$

and E the 'error'

$$E = \sum_{\chi^L = \chi_0, \chi \neq \chi_0} \chi(B^{-1}) S(\chi),$$

with E = 0 when k = 1, where

$$S(\chi) = \sum_{x \in \mathbb{Z}_p} \chi(x) \mathscr{I}_i(x) \mathscr{I}_j^c(Cx)$$

Error Term. Taking the finite Fourier expansion for the intervals as in the proof of Theorem 3.1 we have as before

$$\mathscr{I}_{i}(x) = \sum_{y \in \mathbb{Z}_{p}} a_{i}(y)e_{p}(yx), \quad |a_{i}(y)| = \frac{1}{p} \begin{cases} N_{i}, & \text{if } y = 0; \\ \frac{|\sin(\pi N_{i}ny/p)|}{|\sin(\pi ny/p)|}, & \text{if } y \neq 0, \end{cases}$$

with $N_i = |I_i|$, and

$$\mathscr{I}_{j}^{c}(x) = \sum_{y \in \mathbb{Z}_{p}} a_{j}^{c}(y)e_{p}(yx), \quad a_{j}^{c}(y) = \begin{cases} 1 - a_{j}(0), & \text{if } y = 0; \\ -a_{j}(y), & \text{if } y \neq 0. \end{cases}$$

Again, separating the terms with u or v zero, we have

$$S(\chi) = \sum_{x \in \mathbb{Z}_p} \chi(x) \sum_{u=0}^{p-1} a_i(u) e_p(ux) \sum_{v=0}^{p-1} a_j^c(v) e_p(vCx) = T_1 + E_1 + E_2 + E_3$$

where

$$T_{1} = a_{i}(0)a_{j}^{c}(0)\sum_{x\in\mathbb{Z}_{p}}\chi(x) = 0,$$

$$E_{1} = a_{i}(0)\sum_{v=1}^{p-1}a_{j}^{c}(v)\sum_{x=0}^{p-1}\chi(x)e_{p}(Cvx),$$

$$E_{2} = a_{j}^{c}(0)\sum_{u=1}^{p-1}a_{i}(u)\sum_{x=0}^{p-1}\chi(x)e_{p}(ux),$$

and

$$E_3 = \sum_{u=1}^{p-1} \sum_{v=1}^{p-1} a_i(u) a_j^c(v) \sum_{x \in \mathbb{Z}_p} \chi(x) e_p((u+Cv)x).$$

Recalling that, for a non-principal character χ , the classic Gauss sums

$$G(\chi, A) = \sum_{x=0}^{p} \chi(x) e_p(Ax)$$

satisfy $|G(\chi, A)| = p^{1/2}$ if $p \nmid A$ and trivially $G(\chi, A) = 0$ if $p \mid A$, and again invoking [9, Theorem 1], we have for p > 607

$$\begin{aligned} |E_1| &\leq \frac{N_i}{p} \sum_{v=1}^{p-1} |a_j^c(v)| p^{1/2} \leq \frac{N_i}{p} \left(\frac{4}{\pi^2} \log p + 0.381\right) p^{1/2}, \\ |E_2| &\leq \frac{(p-1-N_j)}{p} \sum_{u=1}^{p-1} |a_i(u)| p^{1/2} \leq \frac{(p-1-N_j)}{p} \left(\frac{4}{\pi^2} \log p + 0.381\right) p^{1/2}, \\ |E_3| &\leq \left(\sum_{u=1}^{p-1} |a_i(u)|\right) \left(\sum_{v=1}^{p-1} |a_j^c(v)|\right) p^{1/2} \leq \left(\frac{4}{\pi^2} \log p + 0.381\right)^2 p^{1/2}, \end{aligned}$$

with $N_i + (p - 1 - N_j) \le p$. Hence, for $p > 10^6$,

$$|S(\chi)| \le \left(\frac{4}{\pi^2}\log p + 0.381\right) p^{1/2} + \left(\frac{4}{\pi^2}\log p + 0.381\right)^2 p^{1/2} < 0.22 \ p^{1/2}\log^2 p,$$

and

$$|E| < 0.22(L-1)p^{1/2}\log^2 p.$$

Main Term. We have

$$M = |I_i| - \sum_{x \in \mathbb{Z}_p^*} \mathscr{I}_i(x) \mathscr{I}_j(Cx) = N_i - M_{ij},$$

where N_i is as given in (3.1), and

$$M_{ij} = |\{x \in I_i : Cx \mod p \in I_j\}|.$$

So for a lower bound on M we need an upper bound on M_{ij} . Since for $1 \le x < p$ we have 0 < Cx < Cp we have

$$M_{ij} = \sum_{u=0}^{C-1} |\{x \in I_i : up \le Cx < (u+1)p, \ Cx - up \in I_j\}|.$$

Note, if $x \equiv i \mod n$ then $Cx - up \equiv j \mod n$ requires $u \equiv K := (Ci - j)p^{-1} \mod n$. Observing that the number of elements in a particular residue class mod n in an interval of cardinality B is at most $\lfloor (B-1)/n \rfloor + 1$ we have

(4.1)
$$M_{ij} = \sum_{\substack{u=0\\u\equiv K \bmod n}}^{C-1} \left| \left\{ x \in I_i : \frac{up}{C} \le x < \frac{up}{C} + \frac{p}{C} \right\} \right| \\ \le \left(\left\lfloor \frac{C-1}{n} \right\rfloor + 1 \right) \left(\left\lfloor \frac{p/C}{n} \right\rfloor + 1 \right).$$

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Plainly

(4.2)
$$M_{ij} \le \left(\frac{C}{n} + 1\right) \left(\frac{p}{Cn} + 1\right) = \frac{p}{n^2} + \frac{C}{n} + \frac{p}{Cn} + 1,$$

and so for $p/2n \ge C \ge 2n$,

$$M_{ij} \le \frac{2p}{n^2} + 1.$$

For $2n > C \ge n$, $\lfloor (C-1)/n \rfloor = 1$ and so by (4.1),

$$M_{ij} \le 2\left(\frac{p}{Cn}+1\right) \le 2\left(\frac{p}{n^2}+1\right) = \frac{2p}{n^2}+2,$$

while for $p/n \ge C > p/2n$, $\lfloor p/(Cn) \rfloor = 1$, and so

$$\leq 2\left(\frac{C}{n}+1\right) \leq 2\left(\frac{p}{n^2}+1\right) = \frac{2p}{n^2}+2$$

For C < n, since $2 \le C < p/2$, we have by (4.1),

$$M_{ij} \le \left(\left\lfloor \frac{C}{n} \right\rfloor + 1 \right) \left(\left\lfloor \frac{p/C}{n} \right\rfloor + 1 \right) \le 1 \cdot \left(\frac{p}{Cn} + 1 \right) \le \frac{p}{2n} + 1,$$

and when C > p/n

$$M_{ij} \le \left(\left\lfloor \frac{C}{n} \right\rfloor + 1 \right) \left(\left\lfloor \frac{p/C}{n} \right\rfloor + 1 \right) \le \left(\frac{C}{n} + 1 \right) \cdot 1 < \frac{p}{2n} + 1.$$

Hence, in all cases we have

(4.3)
$$M_{ij} \le \max\left\{\frac{2p}{n^2} + 2, \frac{p}{2n} + 1\right\}$$

and see that for $n \geq 3$,

$$M_{ij} \le \frac{2p}{3n} + 2$$

and

(4.4)
$$M \ge \left\lfloor \frac{p}{n} \right\rfloor - M_{ij} > \frac{p}{n} - 1 - M_{ij} \ge \frac{p}{3n} - 3.$$

Thus if $p/3n \ge (0.22p^{3/2}\log^2 p)/d$ we have |E| < M and $|\mathscr{U}| > 0$.

If instead, we use the bound $\sum_{u} |a_i(u)| \leq \frac{4}{\pi^2} \log p + \frac{1}{2}$, valid for $p \geq 7$, we obtain $|E| < L\sqrt{p} \log^2 p - 4$, and conclude that M > |E| for $d \geq 3n\sqrt{p} \log^2 p$.

5. Proofs of Theorems 1.2, 1.3, 1.5

Proof of Theorems 1.2. Suppose that $p > 9 \cdot 10^{34} n^{92/3}$. Then certainly $p > 6.7 \times 10^8$. If $d \le 0.006 p^{89/92}$ then Theorem 1.2 follows from Theorem 3.2, while if $d \ge 0.66 np^{1/2} \log^2 p$ it follows from Theorem 4.1. If neither of these occurs then $0.66 np^{1/2} \log^2 p > d > 0.006 p^{89/92}$ and so $p^{43/92}/\log^2 p < 110n$. But this does not occur for $p > 9 \cdot 10^{34} n^{92/3}$.

Proof of Theorem 1.3. We revisit the proof of Theorem 4.1. For k = 1 there is no error term E, and so we need only show that M > 0. This follows from (4.4) for p > 9n. Our computations, see Table 1, have checked $2n for <math>3 \le n \le 12$ so we can assume that n > 12. For $4n we plainly have <math>2p/n^2 + 2 < 18/n + 2 < 4 \le \lfloor p/n \rfloor$ and p/2n + 1 < p/n - 1 and thus by (4.3) $N_i > M_{ij}$. Finally for $2n by (4.3) we have <math>M_{ij} < 3$ and, since M_{ij} is a count, $M_{ij} \le 2$. Hence the result when p > 3n and $\lfloor p/n \rfloor \ge 3$, or when $2n and <math>\lfloor I_i \rfloor = 3$.

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It remains to check the case $2n when <math>|I_i| = 2$. Writing p = 2n + e with $1 \le e < n$, and I_n for I_0 , we have $I_j = \{j, j + n, j + 2n\}$ for $1 \le j \le e$, and $I_j = \{j, j + n\}$ for $e < j \le n$. Suppose $f(I_i) \subseteq I_j$ with $e < i \le n, 1 \le j \le n$. We assume that $A \ne \pm 1$ and that A > 0 (else replace A by -A and j by $\overline{j} = p - j \mod n$). Then

$$f(i) \equiv Ai \equiv j + un \mod p$$
, $f(i+n) \equiv Ai + An \equiv j + vn \mod p$,

for some $u \neq v \in \{0, 1, 2\}$. Subtracting, we get $A \equiv v - u \mod p$, and since $A \ge 2$, get A = 2, v = 2, u = 0. This yields $2i \equiv j \mod p$, meaning j = 2i > i since p > 2n. But, v = 2 implies that $|I_j| = 3$ and hence j < i.

Proof of Theorem 1.5. In the proof of Theorem 3.2 we use the Weil bound [21] $|k-1|\sqrt{p}$ in place of (3.3) and for (a) and (b) we just need

$$|J'_j||K'_j||J_i||K_i| > |k-1|^2 p^3$$
 and $|J_j||K_j||J_i||K_i| > |k-1|^2 p^3$,

in place of (3.5) and (3.4).

6. Proof of Theorem 1.4

Notice that $f(x) = Ax^{(p+1)/2} \mod p$ is related to two linear maps:

(6.1)
$$f(x) \equiv A\left(\frac{x}{p}\right) x \equiv \pm Ax \mod p,$$

and that the inverse mapping $f^{-1}(x)$ is given by

(6.2)
$$f^{-1}(x) = \left(\frac{A}{p}\right) A^{-1} x^{(p+1)/2} \mod p.$$

In order to prove f(x) is not a Type (iii) mapping we can replace f(x) with $\pm f(x)$ or $\pm f^{-1}(x)$ (with one exception), which amounts to changing A to $\pm A$ or $\pm A^{-1}$. Thus we define the quantity

(6.3)
$$C := \min\{|A|, |A^{-1}|\}$$

where A, A^{-1} are taken to be integers with $|A|, |A^{-1}| < p/2$. Note that if $|I_i| = |I_j|$ then $f(I_i) \subseteq I_j$ is the same as $f(I_i) = I_j$ or $f^{-1}(I_j) = I_i$. The one exception that needs special attention is if for some $i, j, f(I_i)$ is a proper subset I_j , that is, $|I_j| = |I_i| + 1$. Then $f(I_i) = I_j \setminus \{a\}$ for some a, with $f^{-1}(I_j \setminus \{a\}) = I_i$, and so in replacing f with f^{-1} we must remove one element from each of the larger I_i and still show that $f(I_i)$ hits at least two different residue classes.

To prove Theorem 1.4 we must show that for $p > (4n + 1)^2$, and $C \ge 2$, the mapping (6.1) is not a Type (iii) mapping. The theorem is an immediate consequence of the following two lemmas, the first dealing with the case C = 2, and the second all larger C.

Lemma 6.1. Suppose that $n \ge 2$, $f(x) = Ax^{(p+1)/2} \mod p$ with $A \ne \pm 1$, and let C be as given in (6.3). If $p > (2Cn + 1)^2$ then f(x) is not a Type (iii) mapping.

Proof. Suppose first that A = C and $0 \le i < n$. Consider the sets

$$U_1 = \{ u \in \mathbb{Z} : 0 \le u < (p/C - i)/n \}, U_2 = \{ u \in \mathbb{Z} : (p/C - i)/n \le u \le (2p/C - i)/n \},$$

with u = 0 excluded from U_1 when i = 0. Since $p > (2Cn + 1)^2$ we have

$$|U_i| \ge \frac{p}{Cn} - 1 > 2\sqrt{p},$$

and thus by the result of Hummel [14], any translate of these intervals must contain at least two quadratic residues and two nonresidues. Hence there will be u_1, u_2 in U_1 and u_3, u_4 in U_2 with

$$\left(\frac{in^{-1}+u_l}{p}\right) = \left(\frac{n}{p}\right), \qquad l = 1, 2, 3, 4,$$

and therefore

$$\left(\frac{i+u_ln}{p}\right) = 1, \qquad l = 1, 2, 3, 4.$$

Note that

$$0 < i + u_1 n < p/A, \qquad l = 1, 2,$$

$$p/A < i + u_l n < 2p/A \le p, \qquad l = 3, 4,$$

and thus $i + u_l n \in I_i$, $1 \le l \le 4$ with by (6.1),

$$f(i + u_l n) = C(i + u_l n) \equiv Ci \mod n, \qquad l = 1, 2$$

$$f(i + u_l n) = C(i + u_l n) - p \equiv Ci - p \mod n, \qquad l = 3, 4.$$

These two values must be distinct mod n.

Finally, if $A \neq C$ then we replace f(x) with -f(x), $f^{-1}(x)$ or $-f^{-1}(x)$ to make A = C, and note that passing to $f^{-1}(x)$ presents no new difficulties because each I_i had at least two quadratic residues and two quadratic nonresidues. \Box

Lemma 6.2. Suppose that $f(x) = Ax^{(p+1)/2} \mod p$, with $A \neq \pm 1$, $n \geq 2$ and $p > 9n^2$. Then f(x) is not a Type (iii) map.

Proof. Suppose first that $n \ge 4$ and that C is as given in (6.3). Lemma 6.1 dispenses with the case C = 2 and so we assume $3 \le C < p/2$, $p > 9n^2$. Writing $\overline{j} = p - j \mod n$, and

$$M_{ij} = |\{x \in I_i : Ax \mod p \in I_j\}|,$$

if $f(I_i) \subseteq I_j$ then by (6.1) the image of I_i under the linear map $Ax \mod p$ must lie in I_j or $I_{\overline{j}}$ and we must have $|I_i| \leq M_{ij} + M_{i\overline{j}}$. Hence to rule out a Type (iii) it will be enough to check that for all i, j

$$(6.4) 2M_{ij} < \frac{p}{n} - 1.$$

We proceed as in the proof of Theorem 4.1 considering various ranges for the size of C. In the cases where we need to replace f with f^{-1} and delete one element from the larger I_i , we have for some j, $|I_i| - 1 = |I_j| > p/n - 1$ and so it is still enough to show (6.4) for f^{-1} .

High C's. For $n \ge 5$ we claim that we cannot have C in the range

$$\frac{p}{2} - \frac{1}{2}\sqrt{p-4} < C \le \frac{p}{2}$$

Indeed, in this case either C^{-1} or $-C^{-1}$ is also in this range, say $C' := \pm C^{-1}$. Then

$$\pm 4 \equiv (p - 2C)(p - 2C') \mod p,$$

but $|(p-2C)(p-2C')\pm 4| < 4+\sqrt{p-4}^2 = p$ and parity rules out equality. Similarly, for n = 4 we claim that we cannot have C in either of the intervals

$$\frac{p}{s} - \frac{1}{4}\sqrt{p - 16} < C \le \frac{p}{s}, \qquad s = 2, 4.$$

Indeed, in this case with C' as defined before for some $s' \in \{2, 4\}$,

(6.5)
$$\pm 16 \equiv \left(\frac{4}{s}p - 4C\right) \left(\frac{4}{s'}p - 4C'\right) \mod p,$$
$$0 < \left(\frac{4}{s}p - 4C\right) \left(\frac{4}{s'}p - 4C'\right) < p - 16,$$

implying equality in (6.5) which cannot occur since $4 \nmid \frac{4}{s}p - 4C$ or $\frac{4}{s'}p - 4C$. For the remaining ranges we use the inequality in (4.1), (4.2),

(6.6)
$$M_{ij} \le \left(\left\lfloor \frac{C-1}{n} \right\rfloor + 1 \right) \left(\left\lfloor \frac{p/C}{n} \right\rfloor + 1 \right) \le \frac{p}{n^2} + \frac{C}{n} + \frac{p}{Cn} + 1.$$

Middle C values. Suppose that $2n \leq C \leq p/2n$. Since $p > 9n^2$ we have $2n \leq \sqrt{p} \leq p/2n$ and

$$M_{ij} < \frac{p}{n^2} + \frac{C}{n} + \frac{p}{Cn} + 1 \le \frac{3p}{2n^2} + 3 \le \frac{3p}{8n} + 3.$$

Hence,

$$2M_{ij} < \frac{3p}{4n} + 6 \le \frac{p}{n} - 1,$$

for $p \ge 28n$, which holds for $p > 9n^2$ and $n \ge 4$.

Small C values. Suppose that n < C < 2n. Then by (6.6),

$$M_{ij} < 2\left(\frac{p}{Cn} + 1\right) \le \frac{2p}{5n} + 2$$

and $2M_{ij} < p/n - 1$ for p > 25n, which holds as before.

Very Small C values. Suppose that $3 \le C \le n$. Then by (6.6),

$$M_{ij} < \left(\frac{p}{Cn} + 1\right) \le \frac{p}{3n} + 1$$

and $2M_{ij} < 2p/3n + 2 < p/n - 1$ as long as p > 9n.

Large C values. Suppose that $n \ge 5$ and $p/n < C < p/2 - \frac{1}{2}\sqrt{p-4}$, or n = 4 and $p/n < C < p/2 - \frac{1}{4}\sqrt{p-16}$. For $n \ge 5$ we get by (6.6),

$$M_{ij} \le \left(\frac{C-1}{n} + 1\right) \cdot 1 \le \frac{p}{2n} - \frac{\sqrt{p}}{2n} + 1$$

and $2M_{ij} < (p/n) - 1$ for $p > 9n^2$.

For n = 4 we have $2M_{ij} \leq \frac{p}{n} - \frac{\sqrt{p}}{2n} + 2 < \frac{p}{n} - 1$ for $p > 36n^2$, but there are no values giving Type (iii) with k = (p+1)/2 and p < 576.

Largish C values. Suppose that $n \ge 5$ and $p/2n < C \le p/n$, or n = 4 and $p/2n < C \le p/4 - \frac{1}{4}\sqrt{p-16}$. If $n \ge 5$ then by (6.6),

$$M_{ij} \le \left(\frac{C-1}{n} + 1\right) \cdot 2 \le 2\frac{p}{n^2} + 2 \le \frac{2p}{5n} + 2$$

and $2M_{ij} < (p/n) - 1$ for p > 25n.

For n = 4 we get from (6.6),

$$2M_{ij} \le \frac{p}{4} - \frac{1}{4}\sqrt{p} + 4 < \frac{p}{4} - 1$$

for $p > 20^2$. There are no examples with p < 400.

The Case n = 3. It remains to deal with n = 3. From our computations we know that there are no Type (iii) mappings with $6 . Replacing A by <math>A^{-1}$ as necessary we may assume that our C does not lie in any of the intervals

$$U_s = \left(\frac{p}{s} - \frac{1}{12}\sqrt{p - 144}, \frac{p}{s}\right), \quad s = 2, 4 \text{ or } 6$$

To see this observe that if $C = \pm A$ is in U_s and $C' = \pm A^{-1}$ is in $U_{s'}$ then

$$\pm 144 \equiv (12p/s - 12C)(12p/s' - 12C') \mod p, \quad 0 < (12p/s - 12C)(12p/s' - 12C') < p - 144,$$

where $2^2 \not\equiv 12p/s - 12C$ or $12p/s' - 12C'$ rules out equality

where $2^2 \nmid 12p/s - 12C$ or 12p/s' - 12C' rules out equality. For $2 \leq C \leq 9$ from Lemma 6.1 there are no such Type (iii) with p > 3025.

For $9 \le C \le p/9$ and p > 243 we have

$$2M_{ij} \le 2\left(\frac{p}{9} + \frac{C}{3} + \frac{p}{3C} + 1\right) \le \frac{8p}{27} + 8 < \frac{p}{3} - 1.$$

For $p/9 < C < p/6 - \frac{1}{12}\sqrt{p - 144}$ and p > 1764 we have

$$2M_{ij} \le 2\left(\frac{C-1}{3}+1\right) \cdot 3 \le \frac{p}{3} - \frac{\sqrt{p}}{6} + 6 < \frac{p}{3} - 1.$$

For $p/6 < C < p/4 - \frac{1}{12}\sqrt{p - 144}$ and p > 2025 we have

$$2M_{ij} \le 2\left(\frac{C-1}{3}+1\right) \cdot 2 \le \frac{p}{3} - \frac{\sqrt{p}}{9} + 4 < \frac{p}{3} - 1.$$

For $p/3 < C < p/2 - \frac{1}{12}\sqrt{p - 144}$ and p > 2916 we have

$$2M_{ij} \le 2\left(\frac{C-1}{3}+1\right) \cdot 1 \le \frac{p}{3} - \frac{\sqrt{p}}{18} + 2 < \frac{p}{3} - 1.$$

That just leaves the case where p/4 < C < p/3. We deal with the map

$$g(x) = Cx^{(p+1)/2} \equiv \left(\frac{x}{p}\right)Cx \mod p$$

directly on $I_0 = \{3, 6, 9, \ldots\}, I_1 = \{1, 4, 7, \ldots\}$ and $I_2 = \{2, 5, 8, \ldots\}.$

For I_0 observe that g(6) = 6C - p or $2p - 6C \equiv 2p \mod 3$ while $g(9) = 9C - 2p \equiv p \mod 3$ giving us an element in I_1 and an element in I_2 .

For I_1 we have g(1) = C and g(4) = 4C - p and these are distinct mod 3. For I_2 we have

$$\begin{pmatrix} \frac{2}{p} \end{pmatrix} = 1 \implies g(2) = 2C, \ g(8) = 8C - 2p,$$
$$\begin{pmatrix} \frac{2}{p} \end{pmatrix} = -1 \implies g(2) = p - 2C, \ g(8) = 2p - 8C,$$

and in either case these are distinct mod 3.

This deals with the case $C = \pm A$ and $f(x) = \pm g(x)$ or if $C = A^{-1}$ in the case when $f(I_i) \subseteq I_j$ and the $|I_i| = |I_j|$ as must happen when $p \equiv 1 \mod 3$. This just leaves the case where $p \equiv 2 \mod 3$ and $f(I_0)$ or $f(I_2)$ equals $I_1 \setminus \{a\}$ when the missing a = 1 or 4. But notice that when $p \equiv 2 \mod 3$ we have (p-1) and (p-4) in I_1 where $g(p-x) \equiv -g(x) \mod p$. Hence g(p-1) = p - C and g(p-4) = 2p - 4C and these two values are again distinct mod 3.

7. Proof of Example 1.1

Proof of Example 1.1. Suppose that $f(x) = \pm x^{(p+1)/2} \mod p$. We have

$$x^{(p+1)/2} = x \cdot x^{(p-1)/2} \equiv x\left(\frac{x}{p}\right) \equiv \pm x \mod p,$$

and f(x) = x or p - x, where $(p - x) \equiv x \mod n$ exactly when $x \equiv 2^{-1}p \mod n$ if n is odd and in no cases if n is even, and the first claim is plain.

If n is even, or n is odd and $i \neq 2^{-1}p \mod n$, then $x \not\equiv p-x \mod n$ for x in I_i , and $f(I_i)$ will hit two different residue classes as long as I_i contains both quadratic residues and nonresidues. Suppose that $\left(\frac{x}{p}\right)$ is constant on I_i , then $\left(\frac{n^{-1}i+y}{p}\right)$ is constant for y in an interval of length $|I_i|$. But by [14] there are less than \sqrt{p} consecutive residues or nonresidues, and for $p > (n+1)^2$ we have $|I_i| > p/n - 1 > \sqrt{p}$.

8. Proofs of Theorems 2.1, 2.2, 2.3, 2.4, 2.5 and 2.7

Proof of Theorem 2.1. Notice that

$$f(x) = 2x^{(p+1)/2} \mod p = 2\left(\frac{x}{p}\right)x \mod p.$$

Since $p = 1 \mod 4$ the quadratic residue property gives us

(8.1)
$$\left(\frac{m}{p}\right) = \begin{cases} +1, & \text{if } m = 1 \mod 4, \\ -1, & \text{if } m = 3 \mod 4, \end{cases}$$

for any integer m with $1 \le m \le 4t - 1$.

Suppose first that i = (2p - (4t - 1)n)/4. Since $n \equiv 2 \mod 4$ we are guaranteed that i = (2p - (4t - 1)n)/4 is an integer, with i > 0 from the upper bound in (2.8), and i < n for n > 2p/(4t + 3) which certainly follows from the lower bound. Plainly

$$(8.2) 4i \equiv 2p \mod n.$$

The lower bound in (2.8) is to ensure that $i + 2tn \ge p$ so that the elements x of I_i can be written $x = i + \ell n$ with $0 \le \ell \le 2t - 1$, where n < 2p/(4t - 3) from the upper bound ensures that i + (2t - 1)n < p. Writing $x = i + \ell n$ we have

$$\left(\frac{x}{p}\right) = \left(\frac{4x}{p}\right) = \left(\frac{-(4t-1)n + 4\ell n}{p}\right) = \left(\frac{n}{p}\right) \left(\frac{4\ell - (4t-1)}{p}\right).$$

Since $\left(\frac{-1}{p}\right) = 1$ we get from (8.1) that

$$\left(\frac{x}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{4(t-\ell)-1}{p}\right) = -\left(\frac{n}{p}\right), \ \ell = 0, \dots, t-1,$$

and

$$\left(\frac{x}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{4(\ell-t)+1}{p}\right) = \left(\frac{n}{p}\right), \ \ell = t, \dots, 2t-1.$$

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Notice that 2x < p iff $4\ell < 4t - 1$, that is 0 < 2x < p for $0 \le \ell \le t - 1$ and p < 2x < 2p for $\ell = t, \ldots, 2t - 1$. Hence

$$\left(\frac{n}{p}\right) = -1 \quad \Rightarrow \quad f(x) = \begin{cases} 2x, & \text{if } \ell = 0, \dots, t-1, \\ 2p - 2x, & \text{if } \ell = t, \dots 2t - 1, \end{cases}$$

while

$$\left(\frac{n}{p}\right) = 1 \quad \Rightarrow \quad f(x) = \begin{cases} p - 2x, & \text{if } \ell = 0, \dots, t - 1, \\ 2x - p, & \text{if } \ell = t, \dots, 2t - 1. \end{cases}$$

Condition (8.2) ensures that f(x) takes the value $j \mod n$ in (2.3) for all x in I_i . Similarly for i = (2p - (4t - 3)n)/4 we have 0 < i < n and $I_j = \{x = i + \ell n : 0 \le \ell \le 2t - 1\}$ for 2p/(4t + 1) < n < 2p/(4t - 3) with 2x < p iff $\ell \le t - 1$. This time

$$\left(\frac{x}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{4t-3-4\ell}{p}\right) = \left(\frac{n}{p}\right), \ \ell = 0, \dots, t-1,$$

and

$$\left(\frac{x}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{4\ell - 4t + 3}{p}\right) = -\left(\frac{n}{p}\right), \ \ell = t, \dots, 2t - 1,$$

giving the same forms for f(x), but with the role of $\left(\frac{n}{p}\right) = 1$ or -1 reversed. \Box

Proof of Theorem 2.2. Let $f(x) = \left(\frac{n}{p}\right) x^{(p+1)/2}$. Since $p \equiv 1 \mod 8$ the quadratic residue condition says that $\left(\frac{\ell}{p}\right) = 1$ for $1 \leq \ell \leq t$. We have $I_0 = \{\ell n : 1 \leq \ell \leq \lfloor (p-1)/n \rfloor\}$, where $\lfloor (p-1)/n \rfloor \leq t$. Hence for the $x = n\ell$ in I_0 we have $f(x) \equiv \left(\frac{n}{p}\right) \left(\frac{n\ell}{p}\right) x \mod p = x$.

Proof of Theorem 2.3. Observe that $\left(\frac{\frac{1}{2}(p-1)-i}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{2i+1}{p}\right)$, reducing the consecutive residues or nonresidues about p/2 condition to $\left(\frac{2i-1}{p}\right) = 1$ all $1 \le i \le T$.

For $i = an - (n/2 - 1 + \delta/2)p = p(1 - \delta/2) - n(t-1)/2$, where $\delta = 0$ for n even and 1 for n odd, we have i > 0 for $n < (2 - \delta)p/(t-1)$, and i < n for $n > (2 - \delta)p/(t+1)$. We also have $i + nt = p(1 - \delta/2) + n(t+1)/2 > p$ (immediately for n even and from n > p/(t+1) for n odd).

Hence x in I_i can be written $i + \ell n$ with $0 \le \ell < t$ and

(8.3)
$$\left(\frac{x}{p}\right) = \left(\frac{an+\ell n}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{a+\ell}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{2}{p}\right).$$

Proof of Theorem 2.4. Since $p \equiv 1 \mod 4$ we have $\left(\frac{\frac{1}{3}(p-\delta)\pm i}{p}\right) = \left(\frac{3}{p}\right) \left(\frac{3i\pm\delta}{p}\right)$ and the equivalent form (2.7) is plain. Writing $\varepsilon = 1, 2$ or 3 as $n \equiv \varepsilon \mod 3$, suppose that $i = na_1 - (n-\varepsilon)p/3 = \varepsilon p/3 - n(3T_1 - 3 + \delta)/3$ and i > 0 for $n < \varepsilon p/(3T_1 - 3 + \delta)$ and i < n for $n > \varepsilon p/(3T_1 + \delta)$. We also have $i + (T_1 + T_2)n = \varepsilon p/3 + (T_2 + 1 - \delta/3)n \ge p$ automatically for $\varepsilon = 3$, and for $\varepsilon = 1$ or 2 if $n \ge (3-\varepsilon)p/(3T_2 + 3 - \delta)$ which follows from $n > \varepsilon p/(3T_1 + \delta)$ for $\varepsilon T_2 + \varepsilon \ge \delta + (3 - \varepsilon)T_1$. Hence x in I_i can be written $x = i + \ell n$ with $0 \le \ell < T_1 + T_2$ and $\left(\frac{x}{p}\right) = \left(\frac{na_1 + \ell n}{p}\right) = \left(\frac{n}{p}\right) \left(\frac{a_1 + \ell}{p}\right) = \left(\frac{3n}{p}\right)$. The remaining cases are similar with $i = \varepsilon' p/3 - n(T_2 - \delta/3)$ where $\varepsilon' = 3, 2, 1$ as $n \equiv 0, 1$ or 2 mod 3. Proof of Theorem 2.5. For i = na - (s - u)p the upper bound $n \leq (s - u)p/(a - 1)$ in (2.8) ensures that $i \leq n$, and the lower bound n > (s-u)p/a that i > 0. From $n \ge (s - u + 1)p/(a + t)$ we also have $i + tn \ge p$, so that the elements of I_i can be written $x = i + \ell n$ with $0 \le \ell < t$ and

$$\left(\frac{x}{p}\right) = \left(\frac{i+\ell n}{p}\right) = \left(\frac{na+\ell n}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{a+\ell}{p}\right) = \left(\frac{na}{p}\right).$$

Since the gap between our upper and lower bounds in (2.8) is

$$\min\left\{\frac{(s+1-r-u(t+1))}{(a-1)(a+t)}, \frac{(s-u)}{a(a-1)}\right\} p > \frac{p}{(a+t)^2}$$

we are guaranteed an n if $(a + t) < \sqrt{p}$. The proof for i = (s - u + 1)p - n(a + t - 1) is similar.

Proof of Theorem 2.6. As in the proof of Theorem 2.3 we can write any x in I_i in the form $x = i + \ell n \equiv an + \ell n \mod p$ for $0 \le \ell < t$, and so by (2.11),

$$x^{(p-1)/3} \equiv (an + \ell n)^{(p-1)/3} = n^{(p-1)/3} (a + \ell)^{(p-1)/3} \equiv (an)^{(p-1)/3} \mod p.$$

So for (2.12) we have $f(x) \equiv (an)^{3(k-1)}x \equiv x \mod p$ on I_i . Similarly, for x in I_i ,

$$x^{(p-1)/6} \equiv n^{(p-1)/6} (a+\ell)^{(p-1)/6} \equiv \pm (an)^{(p-1)/6} \mod p,$$

and for (2.13) we have $f(x) \equiv \pm (an)^{3(k-1)}x \equiv \pm x \mod p$ on I_i , where $p - i \equiv i$ mod n for n odd. \square

Proof of Theorem 2.7. Suppose that $p \equiv 1 \mod 4$, p = n + w with $2 \leq w < n$. The residue classes with two elements consist of the pairs $I_y = \{y, -(w-y)\}, I_{w-y} =$ $\{w-y,-y\}, 1 \leq y \leq w/2$, with the remaining classes containing one element. If $\left(\frac{y}{p}\right), \left(\frac{w-y}{p}\right)$ both equal 1 then $f(x) = x^{(p+1)/2}$ fixes these sets; if both equal -1 it switches the pair.

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