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GUARANTEED EIGENVALUE BOUNDS FOR THE STEKLOV EIGENVALUE PROBLEM*

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Abstract. To provide mathematically rigorous eigenvalue bounds for the Steklov eigenvalue problem, an enhanced version of the eigenvalue estimation algorithm developed by the third author is proposed, which removes the requirements of the positive definiteness of bilinear forms in the formulation of eigenvalue problems. In practical eigenvalue estimation, the Crouzeix-Raviart finite element method (FEM) along with quantitative error estimation is adopted. Numerical experiments for eigenvalue problems defined on a square domain and an L-shaped domain are provided to validate the precision of computed eigenvalue bounds.

Keywords. the Steklov eigenvalue problem, eigenvalue bounds, the Crouzeix-Raviart finite element method, verified computing

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1. Introduction. We aim to provide explicit eigenvalue bounds for Steklov-type eigenvalue problems, such as

(1.1)
$$-\Delta u + u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \vec{n}} = \lambda u \text{ on } \partial \Omega$$

Here, Ω is a bounded domain with Lipschitz boundary and \vec{n} is the unit outward normal on the boundary $\partial \Omega$. Such problems have increasing sequences of eigenvalues (see, for example, [2]):

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

Eigenvalue problems with eigenvalue parameters in the boundary conditions appear in many practical applications. For example, they can be found when modeling antiplane shearing in a system of collinear faults under a slip-dependent friction law [7], or the vibration modes of a linear elastic structure containing an inviscid fluid [4].

It is important to obtain concrete values or exact eigenvalue bounds. For example, in the error analysis when verifying solutions to nonlinear partial differential equations, explicit values of many error constants are desired [24, 21, 27]. Such constants are often determined by solving differential eigenvalue problems; see [14, 17]. As a concrete example, the constant in the trace theorem is directly related to the Steklov eigenvalue problem. The trace theorem states that for all H^1 functions defined on a domain Ω with Lipschitz boundary, there exists a constant C that makes the following inequality hold:

$$\|u\|_{L^2(\partial\Omega)} \le C \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega)$$

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The constant C here is determined by the first eigenvalue of the eigenvalue problem (1.1), that is, $C = 1/\sqrt{\lambda_1}$.

The Steklov eigenvalue problem belongs to the class of eigenvalue problems involving self-adjoint differential operators, such as the Laplacian eigenvalue problem. It is known that upper eigenvalue bounds can easily be obtained using the Rayleigh– Ritz method with trial functions, e.g., using polynomial trigonometric functions and finite element methods (FEMs). However, finding lower eigenvalue bounds remains a difficult problem and has drawn the interest of many researchers. In the literature, various techniques have been developed for providing lower eigenvalue bounds; see, for example, the survey in [19] and the papers cited therein. Note that most of the existing methods only work for special domains and the computed results cannot be guaranteed to be mathematically correct.

To give guaranteed eigenvalue bounds, the rounding error in the floating-point number computing should also be estimated. Early work to provide mathematically rigorous eigenvalue bounds can be found in Plum [23], where the homotopy method is developed, and Nakao, et. al. [22], which provides eigenvalue bounds by identifying ranges where eigenvalues can and cannot exist.

In research on FEMs, bounding eigenvalues from two sides is an important topic. The approximate eigenvalue given by the mass lumping method itself is a lower bound, but the approximate eigenvalue can only be shown to be an exact lower bound for special domains with well-constructed meshes [13]. Many non-conforming FEMs also provide lower eigenvalue bounds asymptotically, i.e., when the mesh is fine enough, the computed eigenvalues converge to the exact values from below; see the work surveyed in [28, 20] and an efficiency improvement using a multilevel correction scheme [12]. However, the precondition required for these asymptotic lower bounds, that the mesh size be small enough, cannot be verified in solving practical problems.

Birkhoff, et al. [5] proposed a method of finding eigenvalue bounds for smooth Sturm-Liouville systems using piecewise-cubic polynomials. Inspired by the idea of [5] and using the hypercircle equation technique along with the linear conforming FEM and the lowest-order Raviart-Thomas FEM, in [19, 18], Liu and Oishi developed an algorithm to provide guaranteed two-sided bounds for the Laplacian eigenvalue problem, which can naturally handle eigenvalue problems over bounded polygonal domains of arbitrary shapes. In [16], Liu extends such an algorithm to create an abstract framework for general self-adjoint differential operators. Carstensen et al. [8, 9] also developed explicit eigenvalue bounds for Laplace and biharmonic operators. As presented, these eigenvalue bounds require a so called "separation condition," but in reality, this is not needed, as shown in [16].

To deal with the Steklov eigenvalue problem, the framework of [16] must be further extended since its bilinear forms (3.1) and (3.6) are defined on different domains, i.e., the interior of Ω and the boundary of Ω . In this paper, we extend the framework to more general variationally formulated eigenvalue problems and successfully obtain lower eigenvalue bounds for the Steklov eigenvalue problem along with the Crouzeix– Raviart FEM. The result in Theorem 3.9 shows that the lower bound for the *i*th eigenvalue is obtained as

(1.2)
$$\lambda_i \ge \frac{\lambda_{h,i}}{1 + C_h^2 \lambda_{h,i}}$$

Here, $\lambda_{h,i}$ is the *i*th approximate eigenvalue computed using the Crouzeix–Raviart FEM (see §3.1 for details) and C_h is a quantity for which we have worked hard to provide an explicit value. The Steklov eigenvalue problem is also considered by Ivana

and Tomáš in [26], where the *a prior*–*a posteriori* inequalities and a complementarity technique have been applied to calculate two side eigenvalue bounds. However, the proposed method needs a priori information about exact eigenvalues, which is usually unknown. To the best of the authors' knowledge, our paper is the first report on rigorous eigenvalue bounds for Steklov eigenvalue problems.

The explicit lower eigenvalue bound (1.2) has a convergence rate of O(h) (where h is the mesh size) even for convex domains, which is not optimal compared to the convergence rate of $\lambda_{h,i}$. In [15], it is proved that assuming the eigenfunctions are H^2 -regular, $\lambda_{h,i}$ itself is an exact lower bound when the mesh size is *small enough* and the convergence rate is $O(h^2)$.

The rest of the paper is organized as follows. In $\S2$, we introduce the abstractly formulated eigenvalue problem along with the main theorem, which provides the lower eigenvalue bounds. In $\S3$, results from the previous section are applied to the Steklov eigenvalue problem to obtain lower eigenvalue bounds, taking care to give explicit error estimates for the projection operator. In $\S4$, computation results are presented to demonstrate the efficiency of our proposed method for bounding eigenvalues. Finally, in $\S5$, we summarize the results of this paper and discuss the issues with the current algorithm.

2. Variationally formulated eigenvalue problem and lower eigenvalue bounds. First, we formulate the assumptions for the eigenvalue problem in this paper, which can be regarded as an extension of Liu [16].

- (A1) V is a Hilbert space with inner product $M(\cdot, \cdot)$ and norm $\|\cdot\|_M$.
- (A2) $N(\cdot, \cdot)$ is a symmetric positive semi-definite bilinear form of \widetilde{V} and the corresponding semi-norm is denoted by $\|\cdot\|_N$.
- (A3) $\|\cdot\|_N$ is compact with respect to $\|\cdot\|_M$, i.e., every sequence of \widetilde{V} bounded under $\|\cdot\|_M$ has a subsequence that is Cauchy under $\|\cdot\|_N$.
- (A4) V and V_h are closed linear subspaces of V, and V_h is finite-dimensional.

REMARK 2.1. In the case where $N(\cdot, \cdot)$ is a positive definite bilinear form, the assumptions here are the same as the ones in [16].

REMARK 2.2. The assumptions (A1)-(A4) are designed to ease the theoretical analysis. For practical problems, the target eigenvalue problems will be configured in the space V and solved numerically with the introduction of \tilde{V} and V_h ; see §3 for the case of the Steklov eigenvalue problem.

Let \mathcal{K} be the operator that maps $f \in V$ to the solution $\mathcal{K}f \in V$ of the variational equation

(2.1)
$$M(\mathcal{K}f, v) = N(f, v) \quad \forall v \in V.$$

Assumptions (A1)–(A4) then assert that $\mathcal{K}: V \mapsto V$ is a compact self-adjoint operator.

Spectrum of \mathcal{K} **.** Let Ker(\mathcal{K}) be the kernel space of \mathcal{K} . Thus, from the definition of \mathcal{K} in (2.1), we have

$$\operatorname{Ker}(\mathcal{K}) = \left\{ v \in V \mid N(v, v) = 0 \right\}.$$

Let $\operatorname{Ker}(\mathcal{K})^{\perp}$ denote the orthogonal complement subspace of $\operatorname{Ker}(\mathcal{K})$ in V with respect to $M(\cdot, \cdot)$. Let

$$d = \dim(\operatorname{Ker}(\mathcal{K})^{\perp}).$$

Note that d can be ∞ . From the theory of compact self-adjoint operators, it is well-known that \mathcal{K} has the spectrum $\{\mu_k\}_{k=1}^d$, and 0 if $\operatorname{Ker}(\mathcal{K}) \neq \emptyset$. Here, the sequence $\{\mu_k\}$ is monotonically decreasing, i.e., $\mu_k \geq \mu_{k+1}$; each μ_k is positive and the number of times it occurs is given by its geometric multiplicity. Let $\{u_k\}_{k=1}^d$ be the orthonormal eigenfunctions associated with μ_k 's. Then V has the following orthonormal decomposition:

$$V = \operatorname{Ker}(\mathcal{K}) \oplus \operatorname{Ker}(\mathcal{K})^{\perp}, \quad \operatorname{Ker}(\mathcal{K})^{\perp} = \operatorname{span}\{u_k\}_{k=1}^d$$

Eigenvalue problem. Consider the following variational eigenvalue problem¹: Find $(\lambda, u) \in \mathbb{R} \times \text{Ker}(\mathcal{K})^{\perp}$, s.t.

(2.2)
$$M(u,v) = \lambda N(u,v) \quad \forall v \in \operatorname{Ker}(\mathcal{K})^{\perp}$$

Let $\lambda_k := \mu_k^{-1}$. The eigenpair of (2.2) is given by $\{\lambda_k, u_k\}$ $(k = 1, 2, \dots, d)$.

Discrete eigenvalue problem. Analogously, introduce $\mathcal{K}_h : V_h \mapsto V_h$ for $f \in V_h$,

(2.3)
$$M(\mathcal{K}_h f, v_h) = N(f, v_h) \quad \forall v_h \in V_h .$$

Then, the kernel space of \mathcal{K}_h is given by

$$\operatorname{Ker}(\mathcal{K}_h) = \{ v_h \in V_h \mid N(v_h, v_h) = 0 \},\$$

and denote its *M*-orthogonal complement in V_h by $\operatorname{Ker}(\mathcal{K}_h)^{\perp}$.

Now let us consider the following eigenvalue problem: Find $(\lambda_h, u_h) \in \mathbb{R} \times \text{Ker}(\mathcal{K}_h)^{\perp}$, s.t.,

(2.4)
$$M(u_h, v_h) = \lambda_h N(u_h, v_h) \quad \forall v_h \in \operatorname{Ker}(\mathcal{K}_h)^{\perp}$$

Let $n = \dim(\operatorname{Ker}(\mathcal{K}_h)^{\perp})$ and $\{(\lambda_{h,k}, u_{h,k})\}_{k=1}^n$ be the eigenpairs of (2.4) with

$$0 < \lambda_{h,1} \le \lambda_{h,2} \le \ldots \le \lambda_{h,n}$$

and $M(u_{h,i}, u_{h,j}) = \delta_{ij}$ (δ_{ij} is the Kronecker delta).

REMARK 2.3. In practice, construction of the kernel space can be avoided by defining the eigenvalue problem over V_h as follows: Find $(\mu_h, u_h) \in \mathbb{R} \times V_h$, s.t.

(2.5)
$$N(u_h, v_h) = \mu_h M(u_h, v_h) \quad \forall v_h \in V_h.$$

Let $\mu_{h,k}$ (k = 1, ..., n) be the non-zero eigenvalues of (2.5). The eigenvalues of (2.4) are simply the inverses of $\mu_{h,k}$'s, i.e., $\lambda_{h,k} = \mu_{h,k}^{-1}$ (k = 1, ..., n). Moreover, The eigenvalues of (2.5) can be rigorously calculated by solving a matrix eigenvalue problem with verified numerical methods.

¹Another formulation of the eigenvalue problem is as follows: Find $(\lambda, u) \in \mathbb{R} \times V$, s.t. $||v||_N = 1$ and $M(u, v) = \lambda N(u, v) \quad \forall v \in V$.

Denote the Rayleigh quotient over \widetilde{V} by $R(\cdot)$ as follows: for any $v \in \widetilde{V}$, $||v||_N \neq 0$,

(2.6)
$$R(v) := \frac{M(v,v)}{N(v,v)}.$$

The stationary values and points of $R(\cdot)$ over V and V_h thus correspond to the eigenpairs of the eigenvalue problems (2.2) and (2.4), respectively. For all λ_k and $\lambda_{h,k}$, the min-max principle asserts that

(2.7)

$$\lambda_{k} = \min_{S_{k}} \max_{\substack{v \in S_{k} \\ \|v\|_{N} \neq 0}} R(v) = \max_{\substack{v \in E_{k} \\ \|v\|_{N} \neq 0}} R(v), \ \lambda_{h,k} = \min_{S_{h,k}} \max_{\substack{v_{h} \in S_{h,k} \\ \|v_{h}\|_{N} \neq 0}} R(v_{h}) = \max_{\substack{v_{h} \in E_{h,k} \\ \|v_{h}\|_{N} \neq 0}} R(v_{h}),$$

where S_k (resp. $S_{h,k}$) is a k-dimensional subspace of $\operatorname{Ker}(\mathcal{K})^{\perp}$ (resp. $\operatorname{Ker}(\mathcal{K}_h)^{\perp}$) and E_k (resp. $E_{h,k}$) is the space spanned by the eigenfunctions $\{u_i\}_{i=1}^k$ (resp. $\{u_{h,i}\}_{i=1}^k$).

Let $P_h : \widetilde{V} \mapsto V_h$ be the projection with respect to $M(\cdot, \cdot)$: given $u \in \widetilde{V}$, $P_h u \in V_h$ satisfies

(2.8)
$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h.$$

Next, we present a theorem that provides the lower eigenvalue bounds.

THEOREM 2.4 (Lower eigenvalue bounds). Suppose that there exists a positive constant C_h such that

(2.9)
$$||u - P_h u||_N \le C_h ||u - P_h u||_M \quad \forall u \in V.$$

Let λ_k and $\lambda_{h,k}$ be as defined in (2.2) and (2.4). Lower eigenvalue bounds are then given by

(2.10)
$$\lambda_k \ge \frac{\lambda_{h,k}}{1 + C_h^2 \lambda_{h,k}}, \quad k = 1, 2, \dots, \min(n, d)$$

Proof. Since $\|\cdot\|_N$ is compact in \widetilde{V} with respect to $\|\cdot\|_M$, resulting from the argument of compactness (see §8 of [1]), there exists $(0 <) \overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \ldots$ such that

(2.11)
$$\overline{\lambda}_k = \min_{S^k \subset \widetilde{V}} \max_{v \in S^k} R(v) = \max_{W \subset \widetilde{V}, \dim(W) \le k-1} \min_{v \in W^\perp} R(v) ,$$

where S^k denotes any k-dimensional subspace of \widetilde{V} ; W^{\perp} denotes the orthogonal complement of W in \widetilde{V} respect to $M(\cdot, \cdot)$. Since $V \subset \widetilde{V}$, we have $\lambda_k \geq \overline{\lambda}_k$ due to the minmax principle. Further, by choosing W in (2.11) as $E_{h,k-1} := \operatorname{span}\{u_{h,1}, \ldots, u_{h,k-1}\}$, a lower bound for λ_k is obtained:

(2.12)
$$\lambda_k \ge \overline{\lambda}_k \ge \min_{v \in E_{h,k-1}^{\perp}} R(v)$$

Let $E_{h,k-1}^{\perp,h}$ denote the orthogonal complement of $E_{h,k-1}$ in $\operatorname{Ker}(\mathcal{K}_h)^{\perp}$ with respect to $M(\cdot,\cdot)$. Hence, $V_h = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h} \oplus \operatorname{Ker}(\mathcal{K}_h)$. Then \widetilde{V} can be decomposed by:

$$\widetilde{V} = V_h \oplus V_h^{\perp} = E_{h,k-1} \oplus E_{h,k-1}^{\perp,h} \oplus \operatorname{Ker}(\mathcal{K}_h) \oplus V_h^{\perp}.$$

Moreover, we have $E_{h,k-1}^{\perp} = E_{h,k-1}^{\perp,h} \oplus \operatorname{Ker}(\mathcal{K}_h) \oplus V_h^{\perp}$. For any $v \in E_{h,k-1}^{\perp}$, we have

$$v = P_h v + (I - P_h)v, \quad P_h v \in E_{h,k-1}^{\perp,h} \oplus \operatorname{Ker}(\mathcal{K}_h), \quad (I - P_h)v \in V_h^{\perp}.$$

Further, decompose $P_h v$ by $P_h v = v^{\perp} + v_0$, where $v^{\perp} \in E_{h,k-1}^{\perp,h}$, $v_0 \in \text{Ker}(\mathcal{K}_h)$. Therefore, we have $\|v^{\perp}\|_N \leq \lambda_{h,k}^{-1/2} \|v^{\perp}\|_M$ by noticing that

$$\lambda_{h,k} = \min_{v \in E_{h,k-1}^{\perp,h}} R(v) \,.$$

Since $||P_h v||_N = ||v^{\perp}||_N$, $||v^{\perp}||_M \le ||P_h v||_M$, from condition (2.9), we have

$$||v||_N \le ||P_h v||_N + ||v - P_h v||_N \le \lambda_{h,k}^{-1/2} ||P_h v||_M + C_h ||v - P_h v||_M.$$

which leads to

$$\|v\|_N^2 \le \left(\lambda_{h,k}^{-1} + C_h^2\right) \left(\|P_h v\|_M^2 + \|v - P_h v\|_M^2\right) = \left(\lambda_{h,k}^{-1} + C_h^2\right) \|v\|_M^2.$$

Hence, we obtain

$$R(v) \ge \lambda_{h,k} / \left(1 + C_h^2 \lambda_{h,k}\right) \quad \text{for any } v \in E_{h,k-1}^{\perp}.$$

Using (2.12), we can draw the conclusion in Theorem 2.4.

REMARK 2.5. Theorem 2.4 provides the same lower bound as in [16] if the function space V in [16] is taken as $Ker(\mathcal{K})^{\perp}$ here. However, one cannot give the proof of Theorem 2.4 by just simply replacing \widetilde{V} with $Ker(\mathcal{K})^{\perp}$ in the proof of [16]. This is because that generally,

(2.13)
$$P_h\left(\operatorname{Ker}(\mathcal{K})^{\perp}\right) \neq \operatorname{Ker}(\mathcal{K}_h)^{\perp}, \quad P_h\left(\operatorname{Ker}(\mathcal{K})\right) \neq \operatorname{Ker}(\mathcal{K}_h).$$

A concrete example is the Steklov eigenvalue problem to be discussed in next section; see Remark 3.1.

3. The Steklov eigenvalue problem. In the rest of this paper, we consider eigenvalue bounds for the Steklov eigenvalue problem (1.1), where Ω is taken as an \mathbb{R}^2 domain. As a remark, the method to be introduced here can be applied to the one with mixed boundary conditions and there is essentially no difficulty to deal with more general Steklov-type eigenvalue problems.

3.1. Preliminaries. The analysis is undertaken within the framework of Sobolev spaces. Let Ω be a connected bounded domain in \mathbb{R}^d (d = 1, 2). The $L^2(\Omega)$ function space is the set of real square integrable functions over Ω , for which the inner product is denoted by $(\cdot, \cdot)_{\Omega}$. We shall use the standard notation for the Sobolev spaces $W^{k,p}(\Omega)$ and their associated norms $\|\cdot\|_{k,p,\Omega}$ and seminorms $|\cdot|_{k,p,\Omega}$ (see, e.g., Chapter 1 of [6] and Chapter 1 of [10]). For p = 2, we define $H^k(\Omega) = W^{k,2}(\Omega)$, $\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}$ and $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$.

To bound the eigenvalues of (1.1) by applying the results in §2, we take V = $H^1(\Omega)$ and define

(3.1)
$$M(u,v) := \int_{\Omega} \nabla u \cdot \nabla v + uv \, \mathrm{d}\Omega, \quad N(u,v) := \int_{\partial \Omega} (\gamma u) \, (\gamma v) \, \mathrm{d}s \quad \forall u, v \in V,$$

where $\gamma: H^1(\Omega) \mapsto L^2(\partial \Omega)$ is the trace operator. Under the current domain boundary assumption, γ is a compact operator; see, e.g., [11]. 6

Let us restrict the domain Ω of (1.1) to be a bounded polygonal domain in \mathbb{R}^2 , and select the finite-dimensional spaces V_h as finite element spaces. Let K^h be a triangular subdivision of Ω , K_b^h be the set of elements of K^h having an edge on $\partial\Omega$, ε^h be the set of all edges of K^h , and ε_b^h be the set of all boundary edges of K^h . Given an element $K \in K^h$, h_K denotes the length of the longest edge of K. In addition, to make the *a priori* error estimate concise (see the proof of Corollary 3.4), we further require that all elements K of K^h have at most one edge on the boundary of the domain.

Now, let us introduce the Crouzeix–Raviart finite element space V_h on K^h :

(3.2)
$$V_h := \{ v \mid v \text{ is a piecewise-linear function on } K^h \text{ and continuous at} \\ \text{the mid-points of interior edges} \}.$$

Since $V_h \not\subset H^1(\Omega)$, we introduce the discrete gradient operator ∇_h , which takes the gradient element-wise for $v_h \in V_h$. The seminorm $\|\nabla_h v_h\|_{(L^2(\Omega))^2}$ is still denoted by $|v_h|_{1,\Omega}$.

The restriction of the trace operator γ to V_h , denoted by γ_h , is well-defined for $v_h \in V_h$, if we regard γ as an element-wise operator on the boundary elements.

The extension of M to V_h is defined by

(3.3)
$$M(u_h, v_h) := \sum_{K \in K^h} \int_K (\nabla u_h \cdot \nabla v_h + u_h v_h) \, \mathrm{d}K, \quad \forall u_h, v_h \in V_h$$

Let $\tilde{V} := V + V_h$. The above settings thus satisfy the assumptions (A1)–(A4). The kernel spaces $\text{Ker}(\mathcal{K})$ and $\text{Ker}(\mathcal{K}_h)$ are determined by the trace operator γ as follows:

(3.4)
$$\operatorname{Ker}(\mathcal{K}) = \operatorname{Ker}(\gamma) := \{ v \in H^1(\Omega), v = 0 \text{ on } \partial\Omega \}$$

(3.5)
$$\operatorname{Ker}(\mathcal{K}_h) = \operatorname{Ker}(\gamma_h) := \{ v_h \in V_h, v_h = 0 \text{ on } \partial\Omega \}.$$

With the above definitions in place, we can follow $\S2$ and define the variational eigenvalue problem for (1.1) and the discrete problem in finite element space.

Variational form of the Steklov eigenvalue problem. Find $(\lambda, u) \in \mathbb{R} \times \text{Ker}(\gamma)^{\perp}$, s.t.,

(3.6)
$$M(u,v) = \lambda N(u,v), \quad \forall v \in \operatorname{Ker}(\gamma)^{\perp}.$$

Eigenvalue problem in finite element space. Find $(\lambda_h, u_h) \in \mathbb{R} \times \text{Ker}(\gamma_h)^{\perp}$, s.t.,

(3.7)
$$M(u_h, v_h) = \lambda_h N(u_h, v_h), \quad \forall v_h \in \operatorname{Ker}(\gamma_h)^{\perp}.$$

We use the same notation for the eigenpairs of (3.6) and (3.7) as in §2.

Let $P_h: V + V_h \mapsto V_h$ be the projection operator with respect to the inner product $M(\cdot, \cdot)$. For $u \in V + V_h$, $P_h u$ satisfies

(3.8)
$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V_h.$$

In §3.2 and §3.3, we show how to obtain the following explicit bound for the projection error constant C_h required by Theorem 2.4:

(3.9)
$$\|u - P_h u\|_{L^2(\partial\Omega)} \le C_h \|u - P_h u\|_{1,\Omega} \quad \forall u \in V .$$

In preparation for evaluating C_h , let us introduce the Crouzeix–Raviart interpolation operator $\Pi_h : V \mapsto V_h$, which is defined element-wise. For any given K, whose edges are denoted by e_1, e_2 and e_3 , $(\Pi_h u)|_K$ is a linear polynomial satisfying

(3.10)
$$\int_{e_i} (\Pi_h u)|_K \, \mathrm{d}s = \int_{e_i} u \, \mathrm{d}s = 0, \ i = 1, 2, 3$$

Orthogonality of interpolation Π_h . The interpolation operator Π_h has an important orthogonality property:

(3.11)
$$(\nabla_h (\Pi_h u - u), \nabla_h v_h)_{\Omega} = 0 \quad \forall v_h \text{ in } V_h .$$

To prove the above equation, it is sufficient to show that the following equation holds for each element K of K^h ,

$$\int_{K} \nabla(\Pi_{h} u - u) \cdot \nabla v_{h} dK = \int_{\partial K} (\Pi_{h} u - u) \nabla v_{h} \cdot \vec{n} ds - \int_{K} (\Pi_{h} u - u) \Delta v_{h} dK = 0.$$

The last equality holds because of the definition of $\Pi_h u$ and the fact that $v_h|_K$ is a linear function.

REMARK 3.1. The projection operator $P_h(=\Pi_h)$ does not mapping $Ker(\mathcal{K})$ to $Ker(\mathcal{K}_h)$. A counterexample is to consider the triangulation of the unit square domain with two triangle elements; see Fig. 1. Let K be the element with vertices (0,0), (0,1) and (1,0). Denote the only interior edge by e. Suppose u = 0 on $\partial\Omega$ and $\int_e u \, ds = 1$. Then, $(\Pi_h u)|_K = \sqrt{2}(x + y - 1/2)$ and $\Pi_h u \notin Ker(\mathcal{K}_h)$.

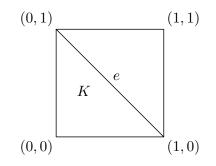


FIG. 1. Special triangulation of unit square domain

In next subsection, we focus on error estimation for Π_h , which helps to obtain the explicit bound (3.9) for C_h .

3.2. Error estimate for interpolation operator. First, we quote a result of the restriction of Π_h to an element K, which is still denoted by Π_h .

LEMMA 3.2 (Liu [16]). For any triangle element K, whose longest edge length is denoted by h_K , we have

(3.12)
$$\|u - \Pi_h u\|_{0,K} \le 0.1893 h_K |u - \Pi_h u|_{1,K} \quad \forall u \in H^1(K) .$$

Next, we estimate $||u - \Pi_h u||_N$ using information on the boundary elements.

Consider the triangle K (see Figure 2), whose nodes are denoted by \mathbf{P}_1 , \mathbf{P}_2 , and \mathbf{P}_3 . The edge $\mathbf{P}_1\mathbf{P}_2$ is denoted by e. Define the height of triangle K respect to edge e by H_K . Thus,

(3.13)
$$H_K = 2|K|/|e|.$$

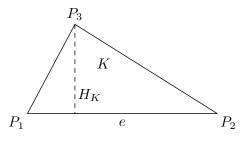


FIG. 2. Parametrization of the triangle ${\cal K}$

THEOREM 3.3 (Interpolation error estimate). For a given element K, configured as in Figure 2, the following error estimate holds for any $u \in H^1(K)$:

(3.14)
$$\|u - \Pi_h u\|_{0,e} \le 0.6711 \frac{h_K}{\sqrt{H_K}} |u - \Pi_h u|_{1,K}$$

Proof. For any $w \in H^1(K)$, the Green theorem leads to

$$\int_{K} ((x,y) - \mathbf{P_3}) \cdot \nabla(w^2) \mathrm{d}K = \int_{\partial K} ((x,y) - \mathbf{P_3}) \cdot \mathbf{n}w^2 \mathrm{d}s - \int_{K} 2w^2 \mathrm{d}K$$

For the term $((x, y) - \mathbf{P_3}) \cdot \mathbf{n}$, we have

(3.15)
$$((x,y) - \mathbf{P_3}) \cdot \boldsymbol{n} = \begin{cases} 0, & \text{on } \mathbf{P_1P_3}, \ \mathbf{P_2P_3}, \\ 2|K|/|e| & \text{on } e. \end{cases}$$

Thus,

$$2\frac{|K|}{|e|} \int_{e} w^{2} ds = \int_{K} 2w^{2} dK + \int_{K} ((x, y) - \mathbf{P_{3}}) \cdot \nabla(w^{2}) dK$$
$$\leq \int_{K} 2w^{2} dK + 2h_{K} \int_{K} |w| |\nabla w| dK$$
$$\leq 2||w||_{0,K}^{2} + 2h_{K} ||w||_{0,K} ||\nabla w||_{0,K} .$$

Taking $w = u - \prod_h u$ and applying of estimate (3.12), we have,

$$\|w\|_{0,e} \le \sqrt{0.1893^2 + 0.1893} \sqrt{\frac{|e|}{|K|}} h_K \|\nabla w\|_{0,K} \le 0.6711 \frac{h_K}{\sqrt{H_K}} \|\nabla w\|_{0,K}$$

The above inequality gives the desired result.

Next, let us apply Theorem 3.3 to show the result related to trace theorem.

COROLLARY 3.4. Given $u \in H^1(\Omega)$, the following error estimate holds:

(3.16)
$$\|u - \Pi_h u\|_N \le 0.6711 \Big(\max_{K \in K_b^h} \frac{h_K}{\sqrt{H_K}}\Big) |u - \Pi_h u|_{1,\Omega}.$$

Here, h_K is the length of the longest edge of K; H_K is the height of K defined in (3.13), where edge $\mathbf{P}_1\mathbf{P}_2$ is aligned on the boundary of the domain.

Proof. The conclusion follows straightforwardly from Theorem 3.3 by noticing that

$$\|u - \Pi_h u\|_N^2 = \sum_{e \in \varepsilon_b^h} \|u - \Pi_h u\|_{0,e}^2 \le 0.6711^2 \max_{K \in K_b^h} \frac{h_K^2}{H_K} \sum_{K \in K_b^h} |u - \Pi_h u|_{1,K}^2.$$

Owing to the assumption that all elements K of K^h have at most one edge on the boundary, the term $|u - \prod_h u|_{1,K}$ in the above inequality only needs to be counted at most once.

REMARK 3.5. Numerial computations indicate that, when the height of triangle is fixed, the constant C in the estimate $||u - \Pi_h u||_{0,e} \leq C||\nabla(u - \Pi_h u)||_{0,K}$ decreases to zero with rate $C = O(|e|^{1/2})$ as |e| tends to 0. However, this behavior of the constant C cannot be deduced from Theorem 3.3. Below is a sketch of the proof for this property.

Define constants C(K) and $C_e(K)$ by

$$C(K) = \sup_{v \in H^1(K)} \frac{\|u - \Pi_h u\|_{0,e}}{\|\nabla (u - \Pi_h u)\|_{0,K}}, \quad C_e(K) = \sup_{v \in H^1(K), \int_e v ds = 0} \frac{\|u\|_{0,e}}{\|\nabla u\|_{0,K}}$$

Then, it is easy to see $C(K) \leq C_e(K)$. For the purpose of simplicity in the argument, assume K to be an acute triangle. Let \widehat{K} be a reference triangle with the length of base of being unit. Suppose K is obtained by scaling \widehat{K} by h (h = |e|) along x direction. Let \widetilde{K} be the scaled \widehat{K} by h in both x and y directions; see Figure 3. Noticing that for any $u \in H^1(K)$, $u|_{\widetilde{K}} \in H^1(\widetilde{K})$,

$$\|\nabla(u|_{\widetilde{K}})\|_{0,\widetilde{K}} \le \|\nabla u\|_{0,K}$$

Thus, one can easily show that

(3.17)
$$C_e(K) \le C_e(\widetilde{K}), \quad C_e(\widetilde{K}) = \sqrt{|e|}C_e(\widehat{K}).$$

Hence,

$$C_e(K) \le \sqrt{|e|} C_e(\widehat{K}) = O(|e|^{1/2}).$$

For K being an obtuse triangle, the relation $\tilde{K} \subset K$ does not hold any more and transformation of triangles is needed. The argument in this case is little complicated and omitted here.

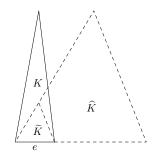


FIG. 3. Scaling of reference triangle

3.3. Error estimate for projection P_h . In this subsection, we give an explicit bound of C_h required in the error estimate of P_h in (3.9).

First, we estimate the mapping \mathcal{K}_h defined in (2.3). That is, for $\phi_h \in V_h$, $\mathcal{K}_h \phi_h \in V_h$ satisfies

(3.18)
$$M(\mathcal{K}_h\phi_h, v_h) = N(\phi_h, v_h) \quad \forall v_h \in V_h.$$

LEMMA 3.6. For all $\phi_h \in W_h$, we have

(3.19)
$$\|\mathcal{K}_h\phi_h\|_M \le \frac{1}{\sqrt{\lambda_{h,1}}} \|\phi_h\|_N,$$

where $\lambda_{h,1}$ is the smallest eigenvalue for the discrete Steklov eigenvalue problem (3.7). Proof. From the definition of \mathcal{K}_h in (3.18), by selecting $v_h := \mathcal{K}_h \phi_h$, we obtain

(3.20)
$$\|\mathcal{K}_h\phi_h\|_M^2 = N(\phi_h, \mathcal{K}_h\phi_h) \le \|\phi_h\|_N \|\mathcal{K}_h\phi_h\|_N.$$

From the definition of $\lambda_{h,1}$ in (3.7) and the min-max principle, we also have

(3.21)
$$\lambda_{h,1} \leq \frac{\|\mathcal{K}_h \phi_h\|_M^2}{\|\mathcal{K}_h \phi_h\|_N^2}, \text{ which implies } \|\mathcal{K}_h \phi_h\|_N \leq \frac{1}{\sqrt{\lambda_{h,1}}} \|\mathcal{K}_h \phi_h\|_M.$$

We can now complete the proof using (3.20) and (3.21).

The following lemma gives an estimate of the difference between the interpolation and projection operators.

LEMMA 3.7. For all $u \in H^1(\Omega)$, we have

(3.22)
$$\|\Pi_h u - P_h u\|_N \le C_1 |u - \Pi_h u|_{1,\Omega},$$

where

$$C_1 := \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K \in K^h} h_K.$$

Proof. Take $v_h := \prod_h u - P_h u$ and let $\psi_h := \mathcal{K}_h \cdot v_h \in V_h$. Then

$$\begin{aligned} \|v_h\|_N^2 &= N(v_h, v_h) = M(\psi_h, v_h) \\ &= M(\psi_h, \Pi_h u - u + u - P_h u) = M(\psi_h, \Pi_h u - u) \,. \end{aligned}$$

Noting the orthogonality of $\Pi_h u$, as shown in (3.11), we have

$$\begin{aligned} \|v_h\|_N^2 &= (\psi_h, \Pi_h u - u)_\Omega \le \|\psi_h\|_{0,\Omega} \|\Pi_h u - u\|_{0,\Omega} \\ &\le \|\psi_h\|_M \ \|\Pi_h u - u\|_{0,\Omega} \le \frac{1}{\sqrt{\lambda_{h,1}}} \|v_h\|_N \ \|\Pi_h u - u\|_{0,\Omega}. \end{aligned}$$

As a consequence,

$$\|\Pi_h u - P_h u\|_N \le \frac{1}{\sqrt{\lambda_{h,1}}} \|\Pi_h u - u\|_{0,\Omega}$$

By (3.12), we have

$$\|\Pi_h u - u\|_{0,\Omega} \le 0.1893 \max_{K \in K^h} h_K |\Pi_h u - u|_{1,\Omega}$$

Followed by the above two estimates, we obtain

$$\|\Pi_h u - P_h u\|_N \le \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K \in K^h} h_K |u - \Pi_h u|_{1,\Omega}.$$

From the definition of C_1 , we get the error estimation (3.22).

THEOREM 3.8 (Projection error estimate). The following error estimate holds:

$$(3.23) ||u - P_h u||_N \le C_h ||u - P_h u||_M \quad \forall u \in V,$$

where

(3.24)
$$C_h := 0.6711 \max_{K \in K_b^h} \frac{h_K}{\sqrt{H_K}} + \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K \in K^h} h_K.$$

Proof. For all $u \in V$, by using (3.16) and (3.22), we obtain

$$\begin{split} \|u - P_h u\|_N &\leq \|u - \Pi_h u\|_N + \|\Pi_h u - P_h u\|_N \\ &\leq 0.6711 \max_{K \in K_b^h} \frac{h_K}{\sqrt{H_K}} \|u - \Pi_h u\|_{1,\Omega} \\ &+ \frac{0.1893}{\sqrt{\lambda_{h,1}}} \max_{K \in K^h} h_K \|u - \Pi_h u\|_{1,\Omega} \\ &= C_h \|u - \Pi_h u\|_{1,\Omega} \leq C_h \|u - P_h u\|_{1,\Omega} \leq C_h \|u - P_h u\|_M. \end{split}$$

The second-last inequality holds because of the orthogonality of Π_h in (3.11).

3.4. Explicit lower eigenvalue bounds. With Theorem 2.4 and the explicit error estimate for P_h in Theorem 3.8, we can now obtain explicit lower bounds for Steklov eigenvalues.

THEOREM 3.9 (Explicit lower bounds for Steklov eigenvalues). Let $\lambda_{h,i}$ be the approximate eigenvalues of (3.7). We have the following lower bounds for eigenvalues of the Steklov eigenvalue problem (3.6),

(3.25)
$$\lambda_i \ge \frac{\lambda_{h,i}}{1 + C_h^2 \lambda_{h,i}}, \quad i = 1, 2, \dots, n$$

Here, $n = \dim(Ker(\gamma_h)^{\perp})$ and C_h is the quantity defined in Theorem 3.8.

REMARK 3.10. Note that since $C_h = O(\sqrt{h})$ as $h \to 0$, the lower eigenvalue bound obtained in (3.25) only converges at a rate of O(h). This is not optimal when compared with the approximate eigenvalues themselves, which have a convergence rate of $O(h^2)$ for solutions with H^2 -regularity; see, e.g., [28]. An idea to recover the convergence rate is to utilized the property of constant described in Remark 3.5 and refine the mesh for boundary elements.

4. Computation Results. Two example Steklov eigenvalue problems are considered here, one on the unit square $\Omega = (0, 1) \times (0, 1)$ and the other on the L-shaped domain $\Omega = (0, 2) \times (0, 2) \setminus [1, 2] \times [1, 2]$. For each example, explicit lower eigenvalue bounds are obtained by applying Theorem 3.9.

In order to estimate the floating-point rounding errors, interval arithmetic is utilized for the numerical computation to guarantee that the results are mathematically correct. The method of Behnke [3] is used, along with the INTLAB toolbox, developed by Rump [25], to give verified eigenvalue bounds for the generalized matrix eigenvalue problems.

4.1. Unit square domain. We uniformly triangulate the domain $\Omega = (0, 1) \times (0, 1)$; see Figure 4 for a sample mesh with mesh size h = 1/8. Here, the mesh size is defined by the length of the triangle side adjacent to the right angle. Note that the maximum edge length for each element is $h_K = \sqrt{2}/8$.

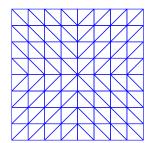


FIG. 4. Sample uniform triangulation for the unit square (h = 1/8)

Over a quite refined mesh, the approximate eigenvalues $\tilde{\lambda}_i$ (i = 1, 2, ..., 5) with better precision are calculated; see Table 1. However, these results are not guaranteed to be strictly correct. In Table 2, we show verified eigenvalue bounds of the leading 5 eigenvalues for different mesh sizes. For example, (0.231, 0.241) in the 1/8 column and λ_1 row means that $0.231 < \lambda_1 < 0.241$ in the case h = 1/8. The lower bounds are obtained using Theorem 3.9 together with the Crouzeix–Raviart FEM, while the upper bounds are obtained using the first-order Lagrange FEM.

TABLE 1 Approximate eigenvalue over refined mesh (h = 1/256).

$\widetilde{\lambda}_1$	$\widetilde{\lambda}_2$	$\widetilde{\lambda}_3$	$\widetilde{\lambda}_4$	$\widetilde{\lambda}_5$
0.240079	1.492293	1.492293	2.082616	4.733516

To investigate the convergence rate of the approximate eigenvalues, we consider the total errors for the lower and upper eigenvalue bounds, denoted by $\lambda_{i,\text{lower}}$ and $\lambda_{i,\text{upper}}$, respectively. The total errors Err_{lower} and Err_{upper} are defined by

$$Err_{\text{lower}} := \sum_{i=1}^{5} |\widetilde{\lambda}_i - \lambda_{i,\text{lower}}|, \quad Err_{\text{upper}} := \sum_{i=1}^{5} |\widetilde{\lambda}_i - \lambda_{i,\text{upper}}|.$$

The convergence rates of the total errors Err_{lower} and Err_{upper} are denoted by σ_{upper} and σ_{lower} , respectively, in Table 2. It can be seen that the upper bound has much better convergence than the lower bound.

TABLE 2

 $\label{eq:Verified eigenvalue bounds for the unit square domain. (Only 4 significant digits are displayed due to space limitations)$

h	1/8	1/16	1/32	1/64
C_h	0.4039	0.2715	0.1849	0.1272
λ_1	(0.231, 0.241)	(0.235, 0.241)	(0.238, 0.241)	(0.239, 0.241)
λ_2	(1.195, 1.503)	(1.342, 1.496)	(1.419, 1.494)	(1.456, 1.493)
λ_3	(1.195, 1.503)	(1.342, 1.496)	(1.419, 1.494)	(1.456, 1.493)
λ_4	(1.541, 2.148)	(1.800, 2.099)	(1.942, 2.087)	(2.014, 2.084)
λ_5	(2.570, 4.897)	(3.456, 4.779)	(4.054, 4.746)	(4.391, 4.737)
$\sigma_{\rm lower}$	-	0.83	0.95	1.00
$\sigma_{ m upper}$	-	1.90	1.97	1.99

4.2. Domain with re-entrant corner (L-shaped). Here, we consider the eigenvalue bounds for a Steklov eigenvalue problem on the L-shaped domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$. Non-uniform meshes are used in the FEM computations. Figure 5 shows a sample non-uniform mesh with a geometrically graded triangular subdivision, where $h_K = O(r^{1/3})$ and r is the distance from the element K to the corner.

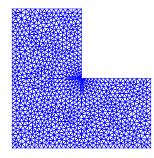


FIG. 5. Non-uniform mesh for the L-shaped domain

Mathematically rigorous lower and upper bounds for the leading 5 eigenvalues are listed in Table 3. The lower bounds are obtained using Theorem 3.9 together with 4916 elements and $C_h = 0.2224$. The upper bounds are acquired using a linear Lagrange finite element space.

			1	
i	Lower Bound	CR Element $(\lambda_{h,i})$	$\widetilde{\lambda}_i$	Upper Bound
1	0.33575	0.34141	0.34141	0.34143
2	0.59833	0.61673	0.61686	0.61717
3	0.93844	0.98421	0.98427	0.98448
4	1.56047	1.69159	1.69206	1.69332
5	1.56791	1.70041	1.70092	1.70230

 TABLE 3

 Verified eigenvalue bounds for the L-shaped domain.

5. Summary. In this paper, we propose an abstract framework that provides computable lower eigenvalue bounds for variationally formulated eigenvalue problems. The framework is successfully applied to the Steklov eigenvalue problem and explicit lower eigenvalue bounds are obtained in conjunction with the Crouzeix–Raviart FEM. Due to the error term related to the trace theorem in Theorem 3.8, the guaranteed lower bound obtained in Theorem 3.9 cannot achieve an optimal convergence rate even for convex domain, compared with the asymptotic theoretical analysis. As a future work, we will try to apply the Lehmann–Goerisch method to improve the convergence rate of rigorous eigenvalue bounds.

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