# Square-free graphs with no six-vertex induced path 

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#### Abstract

We elucidate the structure of $\left(P_{6}, C_{4}\right)$-free graphs by showing that every such graph either has a clique cutset, or a universal vertex, or belongs to several special classes of graphs. Using this result, we show that for any $\left(P_{6}, C_{4}\right)$-free graph $G,\left\lceil\frac{5 \omega(G)}{4}\right\rceil$ and $\left\lceil\frac{\Delta(G)+\omega(G)+1}{2}\right\rceil$ are tight upper bounds for the chromatic number of $G$. Moreover, our structural results imply that every $\left(P_{6}, C_{4}\right)$-free graph with no clique cutset has bounded clique-width, and thus the existence of a polynomial-time algorithm that computes the chromatic number (or stability number) of any ( $P_{6}, C_{4}$ )-free graph.


Keywords: Square-free graphs; $P_{6}$-free graphs; Chromatic number; $\chi$ boundedness; Clique size; Degree.

## 1 Introduction

All our graphs are finite and have no loops or multiple edges. For any integer $k$, a $k$-coloring of a graph $G$ is a mapping $c: V(G) \rightarrow\{1, \ldots, k\}$ such that any two adjacent vertices $u, v$ in $G$ satisfy $c(u) \neq c(v)$. A graph is $k$-colorable if it admits a $k$-coloring. The chromatic number $\chi(G)$ of a graph $G$ is the smallest integer $k$ such that $G$ is $k$-colorable. In general, determining whether a graph is $k$-colorable or not is well-known to be $N P$-complete for every fixed $k \geq 3$. Thus designing algorithms for computing the chromatic number by putting restrictions on the input graph and obtaining bounds for the chromatic number are of interest.

A clique in a graph $G$ is a set of pairwise adjacent vertices. Let $\omega(G)$ denote the maximum clique size in a graph $G$. Clearly $\chi(H) \geq \omega(H)$ for every induced subgraph $H$ of $G$. A graph $G$ is perfect if every induced subgraph $H$ of $G$ satisfies $\chi(H)=\omega(H)$. The existence of triangle-free graphs with aribtrarily large chromatic number shows that for general graphs the chromatic number cannot be upper bounded by a function of the clique number. However, for restricted classes of graphs such a function may exist. Gyárfás [19] called such classes of graphs $\chi$-bounded classes. A family of graphs $\mathcal{G}$ is $\chi$-bounded with

[^0]$\chi$-bounding function $f$ if, for every induced subgraph $H$ of $G \in \mathcal{G}, \chi(H) \leq$ $f(\omega(H))$. For instance, the class of perfect graphs is $\chi$-bounded with $f(\omega)=\omega$.

Given a family of graphs $\mathcal{F}$, a graph $G$ is $\mathcal{F}$-free if no induced subgraph of $G$ is isomorphic to a member of $\mathcal{F}$; when $\mathcal{F}$ has only one element $F$ we say that $G$ is $F$-free. Several classes of graphs defined by forbidding certain families of graphs were shown to be $\chi$-bounded: even-hole-free graphs [1; odd-hole-free graphs [34; quasi-line graphs [10; claw-free graphs with stability number at least 3 [13]; see also [6, 8, 12, 22, 24] for more instances.

For any integer $\ell$ we let $P_{\ell}$ denote the path on $\ell$ vertices and $C_{\ell}$ denote the cycle on $\ell$ vertices. A cycle on 4 vertices is referred to as a square. It is well known that every $P_{4}$-free graph is perfect. Gyárfás 19 showed that the class of $P_{k}$-free graphs is $\chi$-bounded. Gravier et al. [18] improved Gyárfás's bound slightly by showing that every $P_{k}$-free graph $G$ satisfies $\chi(G) \leq(k-2)^{\omega(G)-1}$. In particular every $P_{6}$-free graph $G$ satisfies $\chi(G) \leq 4^{\omega(G)-1}$. Improving this exponential bound seems to be a difficult open problem. In fact the problem of determining whether the class of $P_{5}$-free graphs admits a polynomial $\chi$-bounding function remains open, and the known $\chi$-bounding function $f$ for such class of graphs satisfies $c\left(\omega^{2} / \log w\right) \leq f(\omega) \leq 2^{\omega}$ [23]. So the recent focus is on obtaining (linear) $\chi$-bounding functions for some classes of $P_{t}$-free graphs, where $t \geq 5$. It is shown in [8] that every $\left(P_{5}, C_{4}\right)$-free graph $G$ satisfies $\chi(G) \leq\left\lceil\frac{5 \omega(G)}{4}\right\rceil$, and in [7] that every $\left(P_{2} \cup P_{3}, C_{4}\right)$-free graph $G$ satisfies $\chi(G) \leq\left\lceil\frac{5 \omega(G)}{4}\right\rceil$. Gaspers and Huang [14] studied the class of $\left(P_{6}, C_{4}\right)$-free graphs (which generalizes the class of ( $P_{5}, C_{4}$ )-free graphs and the class of $\left(P_{2} \cup P_{3}, C_{4}\right)$-free graphs) and showed that every such graph $G$ satisfies $\chi(G) \leq \frac{3 \omega(G)}{2}$. We improve their result and establish the best possible bound, as follows.
Theorem 1.1 Let $G$ be any $\left(P_{6}, C_{4}\right)$-free graph. Then $\chi(G) \leq\left\lceil\frac{5 \omega(G)}{4}\right\rceil$. Moreover, this bound is tight.

The degree of a vertex in $G$ is the number of vertices adjacent to it. The maximum degree over all vertices in $G$ is denoted by $\Delta(G)$. For any graph $G$, we have $\chi(G) \leq \Delta(G)+1$. Brooks [5] showed that if $G$ is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$. Reed 33] conjectured that every graph $G$ satisfies $\chi(G) \leq\left\lceil\frac{\Delta(G)+\omega(G)+1}{2}\right\rceil$. Despite several partial results [25, 31, 33, Reed's conjecture is still open in general, even for triangle-free graphs. Using Theorem 1.1, we will show that Reed's conjecture holds for the class of $\left(P_{6}, C_{4}\right)$-free graphs:
Theorem 1.2 If $G$ is a $\left(P_{6}, C_{4}\right)$-free graph, then $\chi(G) \leq\left\lceil\frac{\Delta(G)+\omega(G)+1}{2}\right\rceil$.
One can readily see that the bounds in Theorem 1.1] and in Theorem 1.2 are tight on the following example. Let $G$ be a graph whose vertex-set is partitioned into five cliques $Q_{1}, \ldots, Q_{5}$ such that for each $i \bmod 5$, every vertex in $Q_{i}$ is adjacent to every vertex in $Q_{i+1} \cup Q_{i-1}$ and to no vertex in $Q_{i+2} \cup Q_{i-2}$, and $\left|Q_{i}\right|=q$ for all $i(q>0)$. Clearly $\omega(G)=2 q$ and $\Delta(G)=3 q-1$. Since $G$ has no stable set of size $3, G$ is $P_{6}$-free and $\chi(G) \geq\left\lceil\frac{5 q}{2}\right\rceil$. Moreover, since no two non-adjacent vertices in $G$ has a common neighbor in $G$, we also see that $G$ is $C_{4}$-free.

Finally, we also have the following result.
Theorem 1.3 There is a polynomial-time algorithm which computes the chromatic number of any $\left(P_{6}, C_{4}\right)$-free graph.

The proof of Theorem 1.3 is based on the concept of clique-width of a graph $G$, which was defined in 9 as the minimum number of labels which are necessary to generate $G$ using a certain type of operations. (We omit the details.) It is known from [26, 32] that if a class of graphs has bounded clique-width, then there is a polynomial-time algorithm that computes the chromatic number of every graph in this class. We are able to prove that every $\left(P_{6}, C_{4}\right)$-free graph that has no clique cutset has clique-width at most 36 , which implies the validity of Theorem 1.3. However a similar result, using similar techniques, was proved by Gaspers, Huang and Paulusma [15]. Hence we refer to [15], or to the extended version of our manuscript [21] for the detailed proof of Theorem 1.3

We finish on this theme by noting that the class of $\left(P_{6}, C_{4}\right)$-free graph itself does not have bounded clique-width, since the class of split graphs (which are all ( $P_{6}, C_{4}$ )-free) does not have bounded clique-width [2, 29]. The clique-width argument might also be used for solving other optimization problems in $\left(P_{6}, C_{4}\right)$ free graphs, in particular the stability number. However this problem was solved earlier by Mosca [30, and the weighted version was solved in 4, and both algorithms have reasonably low complexity.

Theorems 1.1 and 1.2 will be derived from the structural theorem below (Theorem 1.4). Before stating it we recall some definitions.

In a graph $G$, the neighborhood of a vertex $x$ is the set $N_{G}(x)=\{y \in$ $V(G) \backslash x \mid x y \in E(G)\}$; we drop the subscript $G$ when there is no ambiguity. The closed neighborhood is the set $N[x]=N(x) \cup\{x\}$. Two vertices $x, y$ are clones if $N[x]=N[y]$. For any $x \in V(G)$ and $A \subseteq V(G) \backslash x$, we let $N_{A}(x)=N(x) \cap A$. For any two subsets $X$ and $Y$ of $V(G)$, we denote by $[X, Y]$, the set of edges that has one end in $X$ and other end in $Y$. We say that $X$ is complete to $Y$ or [ $X, Y]$ is complete if every vertex in $X$ is adjacent to every vertex in $Y$; and $X$ is anticomplete to $Y$ if $[X, Y]=\emptyset$. If $X$ is singleton, say $\{v\}$, we simply write $v$ is complete (anticomplete) to $Y$ instead of writing $\{v\}$ is complete (anticomplete) to $Y$. If $S \subseteq V(G)$, then $G[S]$ denote the subgraph induced by $S$ in $G$. A vertex is universal if it is adjacent to all other vertices. A stable set is a set of pairwise non-adjacent vertices. A clique-cutset of a graph $G$ is a clique $K$ in $G$ such that $G \backslash K$ has more connected components than $G$. A matching is a set of pairwise non-adjacent edges. The union of two vertex-disjoint graphs $G$ and $H$ is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The union of $k$ copies of the same graph $G$ will be denoted by $k G$; for example $2 P_{3}$ denotes the graph that consists in two disjoint copies of $P_{3}$.

A vertex is simplicial if its neighborhood is a clique. It is easy to see that in any graph $G$ that has a simplicial vertex, letting $S$ denote the set of simplicial vertices, every component of $G[S]$ is a clique, and any two adjacent simplicial vertices are clones.

A hole is an induced cycle of length at least 4. A graph is chordal if it contains no hole as an induced subgraph. Chordal graphs have many interesting properties (see e.g. [17), in particular: every chordal graph has a simplicial vertex; every chordal graph that is not a clique has a clique-cutset; and every chordal graph that is not a clique has two non-adjacent simplicial vertices.

In a graph $G$, let $A, B$ be disjoint subsets of $V(G)$. It is easy to see that the following two conditions (i) and (ii) are equivalent: (i) any two vertices $a, a^{\prime} \in A$ satisfy either $N_{B}(a) \subseteq N_{B}\left(a^{\prime}\right)$ or $N_{B}\left(a^{\prime}\right) \subseteq N_{B}(a)$; (ii) any two vertices $b, b^{\prime} \in B$
satisfy either $N_{A}(b) \subseteq N_{A}\left(b^{\prime}\right)$ or $N_{A}\left(b^{\prime}\right) \subseteq N_{A}(b)$. If this condition holds we say that the pair $\{A, B\}$ is graded. Clearly in a $C_{4}$-free graph any two disjoint cliques form a graded pair. See also Lemma 2.3 below.

Some special graphs Let $F_{1}, F_{2}, F_{3}$ be three graphs (as in [14]), as shown in Figure 1


$F_{2}$


Figure 1: $F_{1}, F_{2}, F_{3}$
Let $H_{1}, H_{2}, H_{3}, H_{4}, H_{5}$ be five graphs, as shown in Figure 2 where $H_{1}$ is the Petersen graph.


Figure 2: $H_{1}, H_{2}, H_{3}, H_{4}, H_{5}$


Figure 3: (a) Schematic representation of the graph $F_{k, l}$. Here, the vertices in a shaded box form a clique, and an edge between a vertex and a box indicates that the vertex is adjacent to all the vertices in the box. For example, the vertex $x$ is adjacent to all the vertices in the boxes $A, U$, and $W$. (b) $F_{2,2}$.

Graphs $F_{k, \ell}$ For integers $k, \ell \geq 0$ let $F_{k, \ell}$ be the graph whose vertex-set can be partitioned into sets $A, B, U, W$ and $\{x, y, z\}$ such that:

- $A=\left\{a_{0}, a_{1}, \ldots, a_{k}\right\}$ is a clique of size $k+1$, and $U=\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}$ is a stable set of size $k$, and the edges between $A$ and $U$ form a matching of size $k$, namely, $[A, U]=\left\{a_{i} u_{i} \mid i \in\{1, \ldots, k\}\right\} ;$
- $B=\left\{b_{0}, b_{1}, \ldots, b_{\ell}\right\}$ is a clique of size $\ell+1$, and $W=\left\{w_{1}, \ldots, w_{\ell}\right\}$ is a stable set of size $\ell$, and the edges between $B$ and $W$ form a matching of size $\ell$, namely, $[B, W]=\left\{b_{j} w_{j} \mid j \in\{1, \ldots, \ell\}\right\} ;$
- The neighborhood of $x$ is $A \cup U \cup W \cup\{y\}$;
- The neighborhood of $y$ is $B \cup U \cup W \cup\{x\}$;
- The neighborhood of $z$ is $A \cup B$.

See Figure 3 for the schematic representation of the graph $F_{k, l}$ and for the graph $F_{2,2}$.

Blowups A blowup of a graph $H$ is any graph $G$ such that $V(G)$ can be partitioned into $|V(H)|$ (not necessarily non-empty) cliques $Q_{v}, v \in V(H)$, such that $\left[Q_{u}, Q_{v}\right]$ is complete if $u v \in E(H)$, and $\left[Q_{u}, Q_{v}\right]=\emptyset$ if $u v \notin E(H)$. See Figure 4 (a) for a blowup of a $C_{5}$.


Figure 4: Schematic representations of: (a) a blowup of a $C_{5}$, (b) a band, and (c) a belt. In (a), (b) and (c), the circles represent a collection of sets into which the vertex set of the graph is partitioned. Each shaded circle represents a nonempty clique, a solid line between two circles indicates that the two sets are complete to each other, and the absence of a line between two circles indicates that the two sets are anticomplete to each other. In (b), a dotted line between two circles means that the respective pair of sets is graded. For example, the pair $\left\{Q_{3}, Q_{4}\right\}$ is graded. In (c), the dashed lines between the sets $R_{2}, R_{3}, Q_{2}$ and $Q_{3}$ mean that the adjacency between these sets are subject to the fourth item of the definition of a belt.

Bands A band is any graph $G$ (see Figure $4(\mathrm{~b})$ ) whose vertex-set can be partitioned into seven sets $Q_{1}, \ldots, Q_{5}, R_{2}, R_{3}$ such that:

- Each of $Q_{1}, \ldots, Q_{5}, R_{2}, R_{3}$ is a clique.
- The sets $\left[Q_{5}, Q_{1} \cup Q_{4}\right],\left[R_{2}, Q_{1} \cup Q_{2} \cup Q_{3}\right],\left[R_{3}, Q_{2} \cup Q_{3} \cup Q_{4}\right]$ and $\left[Q_{2}, Q_{3}\right]$ are complete.
- The sets $\left[Q_{1}, Q_{3} \cup R_{3} \cup Q_{4}\right],\left[Q_{4}, Q_{1} \cup Q_{2} \cup R_{2}\right]$ and $\left[Q_{5}, Q_{2} \cup R_{2} \cup Q_{3} \cup R_{3}\right]$ are empty.
- The pairs $\left\{Q_{1}, Q_{2}\right\},\left\{Q_{3}, Q_{4}\right\}$ and $\left\{R_{2}, R_{3}\right\}$ are graded.


Figure 5: Partial structure of a boiler. Here, each shaded circle represents a nonempty clique, and ovals labelled $M$ and $B$ represents the union of the sets represented by the circles inside that oval. The sets in oval $B$ forms a clique, and the ovals $M$ and $L$ induces a $\left(P_{4}, 2 P_{3}\right)$-free graph. A solid line between two shapes indicates that the respective sets are complete to each other. The absence of a line between any two shapes indicates that the respective sets are anticomplete to each other. A dashed line between any two shapes means that the adjacency between these sets are subject to the definition of a boiler.

Belts A belt is any $\left(P_{6}, C_{4}, C_{6}\right)$-free graph $G$ (see Figure $\left.4(\mathrm{c})\right)$ whose vertexset can be partitioned into seven sets $Q_{1}, \ldots, Q_{5}, R_{2}, R_{3}$ such that:

- Each of $Q_{1}, \ldots, Q_{5}$ is a clique.
- The sets $\left[Q_{1}, Q_{2} \cup R_{2} \cup Q_{5}\right]$ and $\left[Q_{4}, Q_{3} \cup R_{3} \cup Q_{5}\right]$ are complete.
- The sets $\left[Q_{1}, Q_{3} \cup R_{3} \cup Q_{4}\right],\left[Q_{4}, Q_{2} \cup R_{2} \cup Q_{1}\right]$, $\left[Q_{5}, Q_{2} \cup R_{2} \cup Q_{3} \cup R_{3}\right]$ are empty.
- For each $j \in\{2,3\},\left[Q_{j}, R_{j}\right]$ is complete, every vertex in $Q_{j} \cup R_{j}$ has a neighbor in $Q_{5-j} \cup R_{5-j}$, and no vertex of $R_{j}$ is universal in $G\left[R_{j}\right]$.

Boilers A boiler is a $\left(P_{6}, C_{4}, C_{6}\right)$-free graph $G$ whose vertex-set can be partitioned into five sets $Q, A, B, L, M$ such that:

- The sets $Q, A, B$ and $M$ are non-empty, and $Q, A$ and $B$ are cliques.
- The sets $[Q, A],[Q, M]$, and $[B, L]$ are complete.
- The sets $[Q, B],[Q, L]$ and $[L, M]$ are empty.
- $G[L]$ and $G[M]$ are $\left(P_{4}, 2 P_{3}\right)$-free.
- Every vertex in $L$ has a neighbor in $A$.
- For some integer $k \geq 3, M$ is partitioned into $k$ non-empty sets $M_{1}, \ldots$, $M_{k}$, pairwise anticomplete, and $B$ is partitioned into $k$ non-empty sets $B_{1}, \ldots, B_{k}$, such that for each $i \in\{1, \ldots, k\}$ every vertex in $M_{i}$ has a neighbor in $B_{i}$ and no neighbor in $B \backslash B_{i}$; and every vertex in $B$ has a neighbor in $M$.
- $\left[A, M_{1} \cup B_{1} \cup M_{2} \cup B_{2}\right]$ is complete, and for each $i \in\{3, \ldots, k\}$ every vertex in $A$ is either complete or anticomplete to $M_{i} \cup B_{i}$, and no vertex in $A$ is complete to $B$.

See Figure 5 for the partial structure of a boiler.
We consider that the definition of blowups (of certain fixed graphs) and of bands (using Lemma 2.3) is also a complete description of the structure of such graphs. However this is not so for belts and boilers. Such graphs have additional properties, and a description of their structure is given in Section 4

Now we can state our main structural result. The existence of such a decomposition theorem was inspired to us by the results from [14 which go a long way in that direction.

Theorem 1.4 If $G$ is any $\left(P_{6}, C_{4}\right)$-free graph, then one of the following holds:

- G has a clique cutset.
- $G$ has a universal vertex.
- $G$ is a blowup of either $H_{1}, \ldots, H_{5}, F_{3}$ or $F_{k, \ell}$ (for some $k, \ell \geq 1$ ).
- $G$ is either a band, a belt, or a boiler.

Theorem 1.4 is derived from Theorem 1.5
Theorem 1.5 Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph that has no clique-cutset and no universal vertex. Then the following hold:

1. If $G$ contains an $F_{3}$, then $G$ is a blowup of $F_{3}$.
2. If $G$ contains an $F_{1}$ and no $F_{3}$, then $G$ is a band.
3. If $G$ is $F_{1}$-free, and $G$ contains an induced $C_{6}$, then $G$ is a blowup of one of the graphs $H_{1}, H_{2}, H_{3}, H_{4}$.
4. If $G$ is $C_{6}$-free, and $G$ contains an $F_{2}$, then $G$ is a blowup of either $H_{5}$ or $F_{k, \ell}$ for some integers $k, \ell \geq 1$.
5. If $G$ contains no $C_{6}$ and no $F_{2}$, and $G$ contains a $C_{5}$, then $G$ is either a belt or a boiler.

Proof. The proof of each of these items is given below in Theorems 3.4 3.5, 3.6, 3.7 and 3.8 respectively.

Proof of Theorem 1.4, assuming Theorem 1.5 ,
Let $G$ be any $\left(P_{6}, C_{4}\right)$-free graph. If $G$ is chordal, then either $G$ is a complete graph (so it has a universal vertex) or $G$ has a clique cutset. Now suppose that $G$ is not chordal. Then it contains an induced cycle of length either 5 or 6 . So it satisfies the hypothesis of one of the items of Theorem 1.5 and consequently it satisfies the conclusion of this item. This established Theorem 1.4 ,

## 2 Classes of square-free graphs

In this section, we study some classes of square-free graphs and prove some useful lemmas and theorems that are needed for the later sections. We first note that any blowup of a $P_{6}$-free chordal graph is $P_{6}$-free chordal.

Lemma 2.1 In a chordal graph $G$, every non-simplicial vertex lies on a chordless path between two simplicial vertices.

Proof. Let $x$ be a non-simplicial vertex in $G$, so it has two non-adjacent neighbors $y, z$. If both $y, z$ are simplicial, then $y-x-z$ is the desired path. Hence assume that $y$ is non-simplicial. Since $G$ is not a clique, it has two simplicial vertices, so it has a simplicial vertex $s$ different from $z$. So $s \notin\{y, z\}$. In $G \backslash s$, the vertex $x$ is non-simplicial, so, by induction, there is a chordless path $P=p_{0}-p_{1} \cdots-p_{k}$ in $G \backslash s$, with $k \geq 2$, such that $p_{0}$ and $p_{k}$ are simplicial in $G \backslash s$ and $x=p_{i}$ for some $i \in\{1, \ldots, k-1\}$. If $p_{0}$ and $p_{k}$ are simplicial in $G$, then $P$ is the desired path. So suppose that $p_{0}$ is not simplicial in $G$, so $s p_{0} \in E(G)$. Since $s$ is simplicial in $G$ we have $N_{P}(s) \subseteq\left\{p_{0}, p_{1}\right\}$. Then we see that either $s-p_{0}-p_{1} \cdots-p_{k}$ or $s-p_{1} \cdots-p_{k}$ is the desired path.

Lemma 2.2 In a chordal graph $G$, let $X$ and $A$ be disjoint subsets of $V(G)$ such that $A$ is a clique and every simplicial vertex of $G[X]$ has a neighbor in $A$. Then every vertex in $X$ has a neighbor in $A$.

Proof. Consider any non-simplicial vertex $x$ of $G[X]$. By Lemma 2.1 there is a chordless path $P=p_{0}-p_{1} \cdots-p_{k}$ in $G[X]$, with $k \geq 2$, such that $p_{0}$ and $p_{k}$ are simplicial in $G[X]$ and $x=p_{i}$ for some $i \in\{1, \ldots, k-1\}$. By the hypothesis $p_{0}$ has neighbor $a \in A$ and $p_{k}$ has a neighbor $a^{\prime}$ in $A$. Suppose that $x$ has no neighbor in $\left\{a, a^{\prime}\right\}$. Let $h$ be the largest integer in $\{0, \ldots, i-1\}$ such that $p_{h}$ has a neighbor in $\left\{a, a^{\prime}\right\}$, and let $g$ be the smallest integer in $\{i+1, \ldots, k\}$ such that $p_{g}$ has a neighbor in $\left\{a, a^{\prime}\right\}$. Then $\left\{p_{h}, p_{h+1}, \ldots, p_{g}, a, a^{\prime}\right\}$ contains a hole, a contradiction. So $x$ has a neighbor in $A$.

Lemma 2.3 In a $C_{4}$-free graph $G$, let $A, B$ be two disjoint cliques. Then:

- There is a labeling $a_{1}, \ldots, a_{|A|}$ of the vertices of $A$ such that $N_{B}\left(a_{1}\right) \supseteq$ $N_{B}\left(a_{2}\right) \supseteq \cdots \supseteq N_{B}\left(a_{|A|}\right)$. Similarly, there is a labeling $b_{1}, \ldots, b_{|B|}$ of the vertices of $B$ such that $N_{A}\left(b_{1}\right) \supseteq N_{A}\left(b_{2}\right) \supseteq \cdots \supseteq N_{A}\left(b_{|B|}\right)$.
- If every vertex in $A$ has a neighbor in $B$, then some vertex in $B$ is complete to $A$.
- If every vertex in $A$ has a non-neighbor in $B$, then some vertex in $B$ is anticomplete to $A$.
- If $[A, B]$ is not complete, there are indices $i \leq|A|$ and $j \leq|B|$ such $a_{i} b_{j} \notin E(G)$, and $a_{i} b_{h} \in E(G)$ for all $h<j$, and $a_{g} b_{j} \in E(G)$ for all $g<i$. Moreover, every maximal clique of $G$ contains one of $a_{i}, b_{j}$.

Proof. Consider any two vertices $a, a^{\prime} \in A$. If there are vertices $b \in N_{B}(a) \backslash$ $N_{B}\left(a^{\prime}\right)$ and $b^{\prime} \in N_{B}\left(a^{\prime}\right) \backslash N_{B}(a)$, then $\left\{a, a^{\prime}, b, b^{\prime}\right\}$ induces a $C_{4}$. Hence we have either $N_{B}(a) \subseteq N_{B}\left(a^{\prime}\right)$ or $N_{B}\left(a^{\prime}\right) \subseteq N_{B}(a)$. This inclusion relation for all $a, a^{\prime}$ implies the existence of a total ordering on $A$, which corresponds to a labeling as desired, and the same holds for $B$. This proves the first item of the lemma. The second and third item are immediate consequences of the first.

Now suppose that $A$ is not complete to $B$. Consider any vertex $a_{i^{\prime}} \in A$ that has a non-neighbor in $B$, and let $j$ be the smallest index such that $a_{i^{\prime}} b_{j} \notin$ $E(G)$. Let $i$ be the smallest index such that $a_{i} b_{j} \notin E(G)$. So $i \leq i^{\prime}$. We have
$a_{g} b_{j} \in E(G)$ for all $g<i$ by the choice of $i$. We also have $a_{i} b_{h} \in E(G)$ for all $h<j$, for otherwise, since $i \leq i^{\prime}$ we also have $a_{i^{\prime}} b_{h} \notin E(G)$, contradicting the definition of $j$. This proves the first part of the fourth item.

Finally, consider any maximal clique $K$ of $G$. Let $g$ be the largest index such that $a_{g} \in K$ and let $h$ be the largest index such that $b_{h} \in K$. By the properties of the labelings and the maximality of $K$ we have $K=\left\{a_{1}, \ldots, a_{g}\right\} \cup\left\{b_{1}, \ldots, b_{h}\right\}$. If both $g<i$ and $h<j$, then the properties of $a_{i}, b_{j}$ imply that $K \cup\left\{a_{i}\right\}$ (and also $K \cup\left\{b_{j}\right\}$ ) is a clique of $G$, contradicting the maximality of $K$. Hence we have either $g \geq i$ or $h \geq j$, and so $K$ contains one of $a_{i}, b_{j}$.

Lemma 2.4 In a $\left(P_{6}, C_{4}\right)$-free graph $G$, let $X, Y$ and $\{c\}$ be disjoint subsets of $V(G)$ such that:

- $Y$ is a clique, and every vertex in $X$ has a neighbor in $Y$,
- $c$ is complete to $X$ and anticomplete to $Y$;
- Either $G[X]$ is not connected, or there are vertices $c^{\prime}, c^{\prime \prime} \in V(G) \backslash(X \cup Y)$ such that $c^{\prime}$ is complete to $Y$ and anticomplete to $X$, and $c^{\prime \prime}$ is anticomplete to $X \cup Y$, and $c^{\prime} c^{\prime \prime} \in E(G)$.

Then $G[X]$ is $\left(P_{4}, 2 P_{3}\right)$-free.
Proof. First suppose that there is a $P_{4} p_{1}-p_{2}-p_{3}-p_{4}$ in $G[X]$. By the hypothesis $p_{1}$ has a neighbor $a \in Y$. Then $a p_{3} \notin E(G)$, for otherwise $\left\{p_{1}, a, p_{3}, c\right\}$ induces a $C_{4}$; and similarly $a p_{4} \notin E(G)$. If $G[X]$ is connected, then either $p_{3}-p_{2}-p_{1}-a-c^{\prime}-c^{\prime \prime}$ or $p_{4}-p_{3}-p_{2}-a-c^{\prime}-c^{\prime \prime}$ is a $P_{6}$. Now suppose that $G[X]$ is not connected. So $X$ contains a vertex $p$ that is anticomplete to $\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$. By the hypothesis $p$ has a neighbor $a^{\prime} \in Y$. As above we have $a p \notin E(G)$ and $a^{\prime} p_{i} \notin E(G)$ for all $i \in\{1, \ldots, 4\}$ for otherwise there is a $C_{4}$. But then either $p-a^{\prime}-a-p_{1}-p_{2}-p_{3}$ or $p-a^{\prime}-a-p_{2}-p_{3}-p_{4}$ is a $P_{6}$.

Now suppose that there is a $2 P_{3}$ in $G[X]$, with vertices $p_{1}, \ldots, p_{6}$ and edges $p_{1} p_{2}, p_{2} p_{3}, p_{4} p_{5}, p_{5} p_{6}$. We know that $p_{1}$ has a neighbor $a \in Y$, and as above we have $a p_{i} \notin E(G)$ for each $i \in\{3,4,5,6\}$, for otherwise there is a $C_{4}$. Likewise, $p_{6}$ has a neighbor $a^{\prime} \in Y$, and $a^{\prime} p_{j} \notin E(G)$ for each $j \in\{1,2,3,4\}$. Then $p_{h+1}-p_{h}-a-a^{\prime}-p_{g}-p_{g-1}$ is an induced $P_{6}$ for some $h \in\{1,2\}$ and $g \in\{5,6\}$.
$\left(P_{4}, C_{4}\right)$-free graphs We want to understand the structure of $\left(P_{4}, C_{4}, 2 P_{3}\right)$ free graphs as they play a major role in the structure of belts and boilers. Recall that $\left(P_{4}, C_{4}\right)$-free graphs were studied by Golumbic [16], who called them trivially perfect graphs. Clearly any such graph is chordal. It was proved in 16 that every connected $\left(P_{4}, C_{4}\right)$-free graph has a universal vertex. It follows that trivially perfect graphs are exactly the class $\mathcal{T}$ of graphs that can be built recursively as follows, starting from complete graphs:

- The disjoint union of any number of trivially perfect graphs is trivially perfect;
- If $G$ is any trivially perfect graph, then the graph obtained from $G$ by adding a universal vertex is trivially perfect.

As a consequence, any connected member $G$ of $\mathcal{T}$ can be represented by a rooted directed tree $T(G)$ defined as follows. If $G$ is a clique, let $T(G)$ have one node, which is the set $V(G)$. If $G$ is not a clique, then by Golumbic's result the
set $U(G)$ of universal vertices of $G$ is not empty, and $G \backslash U(G)$ has a number $k \geq 2$ of components $G_{1}, \ldots, G_{k}$. Let then $T(G)$ be the tree whose root is $U(G)$ and the children (out-neighbors) of $U(G)$ are the roots of $T\left(G_{1}\right), \ldots, T\left(G_{k}\right)$.

The following properties of $T(G)$ appear immediately. Every node of $T(G)$ is a non-empty clique of $G$, and every vertex $v$ of $G$ is in exactly one such clique, which we call $A_{v}$; moreover, $A_{v}$ is a homogeneous set (all member of $A_{v}$ are pairwise clones). For every vertex $v$ of $G$, the closed neighborhood of $v$ consists of $A_{v}$ and all the vertices in the cliques that are descendants and ancestors of $A_{v}$ in $T(G)$. Every maximal clique of $G$ is the union of the nodes of a directed path in $T(G)$. All vertices in any leaf of $T(G)$ are simplicial vertices of $G$, and every simplicial vertex of $G$ is in some leaf of $T(G)$.

We say that a member $G$ of $\mathcal{T}$ is basic if every node of $T(G)$ is a clique of size 1. (We can view $T(G)$ as a directed tree, where every edge is directed away from the root; and then $G$ is the underlying undirected graph of the transitive closure of $T(G)$.). It follows that every member of $\mathcal{T}$ is a blowup of a basic member of $\mathcal{T}$. In a basic member $G$ of $\mathcal{T}$, two vertices are adjacent if and only if one of them is an ancestor of the other in $T(G)$, and every clique of $G$ consists of the set of vertices of any directed path in $T(G)$.

A dart is the graph with vertex-set $\{a, b, c, d, e\}$ and edge-set $\{a b, b c, c d, d a$, $a c, c e\}$. Let $K_{1,3}^{+}$be the tree obtained from $K_{1,3}$ by subdividing one edge. Next we give the following useful lemma.

Lemma 2.5 Let $G$ be a $\left(P_{4}, C_{4}\right)$-free graph.
(a) If $G$ does not have three pairwise non-adjacent simplicial vertices, then $G$ is a blowup of $P_{3}$.
(b) If $G$ does not have four pairwise non-adjacent simplicial vertices, then $G$ is a blowup of a dart.

Proof. The hypothesis of (a) or (b) means that, if $H$ is a connected component of $G$, then $T(H)$ is a tree with at most three leaves. Since each internal vertex of $T(H)$ has at least two leaves, $T(H)$ is either $K_{1}, K_{2}, P_{3}$ (rooted at its vertex of degree 2 ), $K_{1,3}$ (rooted at its vertex of degree 3 ), or $K_{1,3}^{+}$(rooted at its vertex of degree 2). Then the conclusion follows directly from our assumption on $G$ and the preceding arguments.
$\left(P_{4}, C_{4}, 2 P_{3}\right)$-free graphs Let $\mathcal{C}$ be the class of $\left(P_{4}, C_{4}, 2 P_{3}\right)$-free graphs. So $\mathcal{C} \subset \mathcal{T}$. If $G$ is any member of $\mathcal{C}$, and $G$ is connected and not a clique, then since $G$ is $2 P_{3}$-free all components of $G \backslash U(G)$, except possibly one, are cliques. So all children of $U(G)$ in $T(G)$, except possibly one, are leaves. Applying this argument recursively we see that the tree $T(G)$ consists of a rooted directed path plus a positive number of leaves adjacent to every node of this path, with at least two leaves adjacent to the last node of this path. We call such a tree a bamboo. By the same argument as above, every member of $\mathcal{C}$ is a blowup of a basic member of $\mathcal{C}$.
$\mathcal{C}$-pairs A graph $G$ is a $\mathcal{C}$-pair if $G$ is $P_{6}$-free, chordal, and $V(G)$ can be partitioned into two sets $X$ and $A$ such that $A$ is a clique, $G[X] \in \mathcal{C}$, every vertex in $X$ has a neighbor in $A$, and any two non-adjacent vertices in $X$ have
no common neighbor in $A$. Depending on the context we may also write that $(X, A)$ is a $\mathcal{C}$-pair.

We say that $G$ is a basic $\mathcal{C}$-pair if the subgraph $G[X]$ is a basic member of $\mathcal{C}$, with vertices $x_{1}, \ldots, x_{k}$ for some integer $k$, and a clique $A=\left\{a_{0}, a_{1}, \ldots, a_{k}\right\}$; and for each $i \in\{1, \ldots, k\}$, if $x_{i}$ is simplicial in $G[X]$ then $N_{A}\left(x_{i}\right)=\left\{a_{i}\right\}$, else $N_{A}\left(x_{i}\right)$ consists of $\left\{a_{i}\right\}$ plus the union of $N_{A}(y)$ over all descendants $y$ of $x_{i}$ in $T(G[X])$.

Before describing how all $\mathcal{C}$-pairs can be obtained from basic $\mathcal{C}$-pairs we need to introduce another definition. Let $H$ be any graph and $M$ be a matching in $H$. An augmentation of $H$ along $M$ is any graph $G$ whose vertex-set can be partitioned into $|V(H)|$ cliques $Q_{v}, v \in V(H)$, such that $\left[Q_{u}, Q_{v}\right]$ is complete if $u v \in E(H) \backslash M$, and $\left[Q_{u}, Q_{v}\right]=\emptyset$ if $u v \notin E(H)$, and $\left\{Q_{u}, Q_{v}\right\}$ is a graded pair if $u v \in M$. (See 28 for a similar definition.)

In a basic $\mathcal{C}$-pair $G$, with the same notation as above, we say that a matching $M$ is acceptable if there is a clique $\left\{x_{i_{1}}, \ldots, x_{i_{h}}\right\}$ in $G[X]$ such that $M=$ $\left\{x_{i_{1}} a_{i_{1}}, \ldots, x_{i_{h}} a_{i_{h}}\right\}$.


Figure 6: Schematic representations of: (a) a basic $\mathcal{C}$-pair, (b) an acceptable matching in (a), and (c) an augmentation of the graph in (a) along an acceptable matching in (b). In (a) and (b), the vertices in a shaded box represents a clique. In (b), the dashed lines represent the matching edges. In (c), the circles represent a collection of sets into which the vertex set of the graph is partitioned, each shaded circle represents a clique, and the circles inside the oval form a clique, a solid line between two circles indicates that the two sets are complete to each other, the dotted line between two circles means that the respective pair of sets is graded, and the absence of a line between two circles indicates that the two sets are anticomplete to each other.

Theorem 2.1 A graph is a $\mathcal{C}$-pair then it is an augmentation of a basic $\mathcal{C}$-pair along an acceptable matching.

Proof. Let $G$ be any $\mathcal{C}$-pair, with the same notation as above. Since $G[X]$ is $\left(P_{4}, C_{4}, 2 P_{3}\right)$-free it admits a representative tree $T(G[X])$ which is a bamboo. We claim that:

If $Y, Z$ are two nodes of $T(G[X])$ such that $Z$ is a descendant of $Y$, then $Y$ is complete to $N_{A}(Z)$.

Proof: Consider any $y \in Y$ and $a \in N_{A}(Z)$; so there is a vertex $z \in Z$ with $z a \in E(G)$. Since $Y$ is not a leaf of $T(G[X])$, there is a child $Z^{\prime}$ of $Y$ in $T(G[X])$ such that $Z^{\prime}$ is not on the directed path from $Z$ to $Y$, and so $Z$ and $Z^{\prime}$ are not adjacent (they are anticomplete to each other). Pick any $z^{\prime} \in Z^{\prime}$. Then
$y z, y z^{\prime} \in E(G)$ and $z z^{\prime} \notin E(G)$. We know that $z^{\prime}$ has a neighbor $a^{\prime} \in A$. We have $a z^{\prime}, a^{\prime} z \notin E(G)$ by the definition of a $\mathcal{C}$-pair ( $z$ and $z^{\prime}$ have no common neighbor in $A$ ). Then $y a, y a^{\prime} \in E(G)$, for otherwise $G\left[y, z, z^{\prime}, a, a^{\prime}\right]$ contains an induced hole of length 4 or 5 , contradicting the fact that $G$ is chordal. So (1) holds.

Let $X_{1}, \ldots, X_{k}$ be the nodes of $T(G[X])$. For each $i \in\{1, \ldots, k\}$, let $U_{i}$ be the union of $N_{A}(Z)$ over all descendants $Z$ of $X_{i}$ in $T(G[X])$, and let $A_{i}=$ $N_{A}\left(X_{i}\right) \backslash U_{i}$. Let $A_{0}=A \backslash\left(A_{1} \cup \cdots \cup A_{k}\right)\left(\right.$ so $\left.\left[X, A_{0}\right]=\emptyset\right)$.

Let $X_{i_{1}}, \ldots, X_{i_{h}}$ be the nodes of $T(G[X])$ that are not homogeneous in $G$ (if any). Note that for each $i \in\left\{i_{1}, \ldots, i_{h}\right\}$ the pair $\left\{X_{i}, A_{i}\right\}$ is graded since $G$ is $C_{4}$-free. We claim that:

$$
\begin{equation*}
X_{i_{1}} \cup \cdots \cup X_{i_{h}} \text { is a clique. } \tag{2}
\end{equation*}
$$

Proof: Suppose, on the contrary, and up to symmetry, that $\left[X_{i_{1}}, X_{i_{2}}\right.$ ] is not complete, and so $\left[X_{i_{1}}, X_{i_{2}}\right]=\emptyset$. For each $t \in\{1,2\}$, since $X_{i_{t}}$ is not homogeneous in $G$, there are vertices $y_{t}, z_{t} \in X_{i_{t}}$ and a vertex $a_{t} \in A$ that is adjacent to $y_{t}$ and not to $z_{t}$. Since non-adjacent vertices in $X$ have no common neighbor in $A$, we have $a_{1} \neq a_{2}$ and $a_{1} y_{2}, a_{1} z_{2}, a_{2} y_{1}, a_{2} z_{1} \notin E(G)$. Then $z_{1}-y_{1}-a_{1}-a_{2}-y_{2}-z_{2}$ is a $P_{6}$. So (21) holds.

Let $H$ be the basic member of $\mathcal{C}$ of which $G[X]$ is a blowup. Let $H$ have vertices $x_{1}, \ldots, x_{k}$, where $x_{i}$ corresponds to the node $X_{i}$ of $T(G[X])$ for all $i$. Let $G_{0}$ be the graph obtained from $H$ by adding a set $A=\left\{a_{0}, a_{1}, \ldots, a_{k}\right\}$, disjoint from $V(H)$, and edges so that $A$ is a clique in $G_{0}$ and, for all $i \in\{1, \ldots, k\}$ and $j \in\{0,1, \ldots, k\}$, vertices $x_{i}$ and $a_{j}$ are adjacent in $G_{0}$ if and only if $\left[X_{i}, A_{j}\right] \neq \emptyset$ in $G$. By this construction and by (11) $G_{0}$ is a basic $\mathcal{C}$-pair. In $G_{0}$ let $M=$ $\left\{x_{i_{1}} a_{i_{1}}, \ldots, x_{i_{h}} a_{i_{h}}\right\}$. It follows from (21) that $M$ is an acceptable matching of $G_{0}$ and from all the points above that $G$ is an augmentation of $G_{0}$ along $M$.

## 3 Structure of ( $P_{6}, C_{4}$ )-free graphs

In this section, we give the proof of Theorem 1.5. We say that a subgraph $H$ of $G$ is dominating if every vertex in $V(G) \backslash V(H)$ is a adjacent to a vertex in $H$. We will use the following theorem of Brandstädt and Hoàng 4 .

Theorem 3.1 ([4]) Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph that has no clique cutset. Then the following statements hold.
(i) Every induced $C_{5}$ is dominating.
(ii) If $G$ contains an induced $C_{6}$ which is not dominating, then $G$ is the join of a complete graph and a blowup of the Petersen graph.

In the next two theorems we make some general observations about the situation when a $\left(P_{6}, C_{4}\right)$-free graph contains a hole (which must have length either 5 or 6 ). Observe that in a $C_{4}$-free graph $G$, if $u-v-w$ is a $P_{3}$, then any $x \in V(G) \backslash\{u, v, w\}$ which is adjacent to $u$ and $w$ is also adjacent to $v$.

Theorem 3．2 Let $G$ be any $\left(P_{6}, C_{4}\right)$－free graph that contains a $C_{5}$ with vertex－ set $C=\left\{v_{1}, \ldots, v_{5}\right\}$ and $\left\{v_{i} v_{i+1} \mid i \in\{1, \ldots, 5\}, i \bmod 5\right\}$ ．Let：

$$
\begin{aligned}
A & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=C\right\} . \\
T_{i} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i-1}, v_{i}, v_{i+1}\right\} .\right. \\
W_{i} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i}\right\} .\right. \\
X_{i, i+1} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i}, v_{i+1}\right\} .\right.
\end{aligned}
$$

Moreover，let $T=T_{1} \cup \cdots \cup T_{5}, W=W_{1} \cup \cdots \cup W_{5}$ ，and $X=X_{12} \cup X_{23} \cup$ $X_{34} \cup X_{45} \cup X_{51}$ ．Then the following properties hold for all $i$ ：
（a）$A \cup T_{i}$ is a clique．
（b）$\left[T_{i}, T_{i+2}\right],\left[X_{i, i+1}, X_{i+2, i+3}\right],\left[W_{i}, W_{i+1}\right],\left[T_{i}, W_{i-2} \cup W_{i+2}\right],\left[T_{i}, X_{i+2, i+3}\right]$ and $\left[X_{i, i+1}, W_{i} \cup W_{i+1}\right]$ are empty．
（c）$\left[X_{i, i+1}, X_{i+1, i+2}\right],\left[W_{i}, W_{i+2}\right]$ ，and $\left[X_{i, i+1}, W_{i-1} \cup W_{i+2}\right]$ are complete．
（d）If $G$ is $C_{6}$－free，then for each $i$ one of $X_{i, i+1}$ and $X_{i+1, i+2}$ is empty，and one of $W_{i}$ and $W_{i+2}$ is empty，and one of $X_{i, i+1}$ and $W_{i-1} \cup W_{i+2}$ is empty．
（e）If $G$ has no clique cutset，then the set $\left\{x \in V(G) \backslash C \mid N_{C}(x)=\emptyset\right\}$ is empty and $\left[T_{i}, W_{i}\right]$ is complete．
（f）If $G$ has no clique cutset，then $V(G)=V(C) \cup A \cup T \cup W \cup X$ ．
Proof．（回）If there are non－adjacent vertices $a, b \in A \cup T_{i}$ ，then $\left\{a, v_{i-1}, b, v_{i+1}\right\}$ induces a $C_{4}$ ．
（b）Let $i=1$ and suppose that there is an edge $x y$ in one of the listed sets． If $x \in T_{1}$ and $y \in T_{3}$ ，then $\left\{x, y, v_{4}, v_{5}\right\}$ induces a $C_{4}$ ．If $x \in X_{12}$ and $y \in X_{34}$ ， then $\left\{x, v_{2}, v_{3}, y\right\}$ induces a $C_{4}$ ．If $x \in T_{1}$ and $y \in W_{4}$ then $\left\{x, y, v_{4}, v_{5}\right\}$ induces a $C_{4}$ ．If $x \in W_{1}$ and $y \in W_{2}$ ，then $x-y-v_{2}-v_{3}-v_{4}-v_{5}$ is an induced $P_{6}$ ．If $x \in T_{1}$ and $y \in X_{34}$ ，then $\left\{x, v_{2}, v_{3}, y\right\}$ induces a $C_{4}$ ．If $x \in X_{12}$ and $y \in W_{1}$ ，then $y-x-v_{2}-v_{3}-v_{4}-v_{5}$ is a $P_{6}$ ．The other cases are symmetric．
（ㄷ）and（d）Let $i=1$ and suppose that there are vertices $x \in X_{12} \cup W_{1}$ and $y \in X_{23} \cup W_{3}$ ．If $x y \notin E(G)$ ，then $x-v_{1}-v_{5}-v_{4}-v_{3}-y$ is a $P_{6}$ ．This proves（ㄷ）．If $x y \in E(G)$ then the same vertices induce a $C_{6}$ ，which proves（d）．
（四）Follows from Theorem 3.1
（f）Follows by Theorem 3．1］and（®区）．

Theorem 3．3 Let $G$ be any $\left(P_{6}, C_{4}\right)$－free graph that contains a $C_{6}$ with vertex－ set $C=\left\{v_{1}, \ldots, v_{6}\right\}$ and $\left\{v_{i} v_{i+1} \mid i \in\{1, \ldots, 6\}, i \bmod 6\right\}$ ．Let：

$$
\begin{aligned}
S & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=C\right\} \\
A_{i} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i-1}, v_{i}, v_{i+1}\right\}\right\} \\
B_{i} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i-1}, v_{i}, v_{i+1}, v_{i+2}\right\} .\right. \\
D_{i} & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\left\{v_{i}, v_{i+3}\right\}\right\} . \\
L & =\left\{x \in V(G) \backslash C \mid N_{C}(x)=\emptyset\right\} .
\end{aligned}
$$

Moreover，let $A=A_{1} \cup \cdots \cup A_{6}, B=B_{1} \cup \cdots \cup B_{6}$ ，and $D=D_{1} \cup \cdots \cup D_{6}$ ． Then the following properties hold for all $i, i \bmod 6$ ：
（a）$V(G)=V(C) \cup A \cup B \cup D \cup S \cup L$ ．
（b）Each of $A_{i} \cup B_{i} \cup B_{i+5}, D_{i}$ and $S$ is a clique．
（c）$\left[A_{i}, A_{i+1} \cup A_{i+5} \cup D_{i}\right],\left[B_{i}, B_{i+1} \cup B_{i+3} \cup B_{i+5} \cup D_{i+2}\right]$ ，and $\left[S, A_{i} \cup B_{i} \cup D_{i}\right]$ are complete．
（d）$\left[A_{i}, A_{i+3} \cup B_{i+2} \cup B_{i+3} \cup D_{i+1} \cup D_{i+2}\right],\left[B_{i}, B_{i+2} \cup B_{i+4}\right]$ ，and $\left[D_{i}, D_{i+1}\right]$ are empty．
（e）If $B_{i} \neq \emptyset$ ，then $D_{i} \cup D_{i+1}=\emptyset$ ．
（f）If $B_{i} \neq \emptyset$ and $B_{i+1} \neq \emptyset$ ，then $B_{i+3} \cup B_{i+4}=\emptyset$ ．
Proof．We note that $D_{i}=D_{i+3}$ ，for all $i$ ．
（囵）Suppose that there is a vertex $x$ in $G$ ．We may assume that $x \in V(G) \backslash$ $V(C)$ ．If $x$ has no neighbor in $C$ ，then $x \in L$ ．So，suppose that $x$ has a neighbor in $C$ ．If $N_{C}(x)=\left\{v_{i}\right\}$（or $\left\{v_{i}, v_{i+1}\right\}$ ），for some $i$ ，then $\left(C \backslash\left\{v_{i+1}\right\}\right) \cup\{x\}$ induces a $P_{6}$ ．In all the remaining cases，we see that either $C \cup\{x\}$ contains an induced $C_{4}$ or $x \in A \cup B \cup D \cup S$ ．So（回）holds．
（b）If there are non－adjacent vertices $x$ and $y$ in one of the listed sets，then either $\left\{x, v_{i-1}, v_{i+1}, y\right\}$ or $\left\{x, v_{i}, v_{i+3}, y\right\}$ induces a $C_{4}$ ．
（C）Let $i=1$ and suppose that there are non－adjacent vertices $x$ and $y$ in one of the listed sets．If $x \in A_{1}$ and $y \in A_{2} \cup D_{1}$ ，then $\left\{v_{2}, x, v_{6}, v_{5}, v_{4}, y\right\}$ induces a $P_{6}$ ．If $x \in B_{1}$ and $y \in B_{2}$ ，then $\left\{x, v_{1}, y, v_{3}\right\}$ induces a $C_{4}$ ．If $x \in B_{1}$ and $y \in B_{4} \cup D_{3}$ ，then $\left\{x, v_{3}, y, v_{6}\right\}$ induces a $C_{4}$ ．If $x \in S$ and $y \in A_{1} \cup B_{1} \cup D_{1}$ ，then either $\left\{x, v_{6}, y, v_{2}\right\}$ or $\left\{x, v_{1}, y, v_{3}\right\}$ induces a $C_{4}$ ．The other cases are symmetric．
（d）Let $i=1$ and suppose that there is an edge $x y$ in one of the listed sets． If $x \in A_{1}$ and $y \in A_{4} \cup B_{3} \cup D_{2}$ ，then $\left\{x, v_{6}, v_{5}, y\right\}$ induces a $C_{4}$ ．If $x \in B_{1}$ and $y \in B_{3}$ ，then $\left\{x, v_{6}, v_{5}, y\right\}$ induces a $C_{4}$ ．If $x \in D_{1}$ and $y \in D_{2}$ ，then $\left\{x, v_{1}, v_{2}, y\right\}$ induces a $C_{4}$ ．The other cases are symmetric．
（目）Let $i=1$ and let $x \in B_{1}$ ．Up to symmetry，if there exists a vertex $y \in D_{1}$ ， then by（（G），$x y \in E(G)$ ．But then $\left\{x, y, v_{4}, v_{3}\right\}$ induces a $C_{4}$ ．So $D_{1}=\emptyset$ ．
（I）Let $i=1$ ．Let $x \in B_{1}$ and $y \in B_{2}$ ．Up to symmetry，if there exists a vertex $z \in B_{4}$ ，then by（CC），$x y, x z \in E(G)$ ，and by（d），$y z \notin E(G)$ ．But then $\left\{x, y, v_{3}, z\right\}$ induces a $C_{4}$ ．So $B_{4}=\emptyset$ ．

This shows Theorem 3．3．

## When there is an $F_{3}$

Now we can give the proof of the first item of Theorem 1.5 which we restate it as follows．

Theorem 3．4 Let $G$ be a $\left(P_{6}, C_{4}\right)$－free graph with no universal vertex and no clique cutset．Suppose that $G$ contains an $F_{3}$ ．Then $G$ is a blowup of $F_{3}$ ．

Proof．Consider the graph $F_{3}$ as shown in Figure 1 and let $C=\left\{v_{1}, \ldots, v_{6}\right\}$ ．By Theorem 3．3（a），and with the same notation，every vertex in $V(G) \backslash C$ belongs to $A_{i} \cup B_{i} \cup D_{i} \cup S \cup L$ for some $i$ ．Note that $x \in A_{2}, y \in A_{4}$ and $z \in A_{6}$ ．We first claim that：

$$
\begin{equation*}
B_{i} \cup D_{i}=\emptyset, \text { for all } i \tag{1}
\end{equation*}
$$

Proof：Suppose on the contrary，and up to symmetry，that there is a vertex $u \in B_{1} \cup D_{1}$ ．Suppose that $u \in B_{1}$ ．By Theorem 3．3（b）we have $u x \in E(G)$ ，and
by Theorem 3.3(d) we have $u y \notin E(G)$. Then either $\left\{u, x, z, v_{6}\right\}$ or $\left\{u, v_{3}, y, z\right\}$ induces a $C_{4}$. Now suppose that $u \in D_{1}$. By Theorem 3.3.d) we have $u x, u z \notin$ $E(G)$. Then $u-v_{4}-v_{5}-z-x-v_{2}$ is a $P_{6}$. So (1) holds.

Next, we claim that:

$$
\begin{equation*}
L \cup S=\emptyset \tag{2}
\end{equation*}
$$

Proof: Suppose that $L \neq \emptyset$. By Theorem 3.3(b), $S$ is a clique. Since $S$ is not a clique cutset, and by (11), some vertex $w$ in $L$ has a neighbor $a \in A$, say $a \in A_{1}$. But then $w-a-v_{6}-v_{5}-v_{4}-v_{3}$ is a $P_{6}$, a contradiction. Hence $L=\emptyset$. Now if $S \neq \emptyset$, then by Theorem 3.3(b) and (ㄷ) , any vertex in $S$ is universal, a contradiction. So (2) holds.

We note that every vertex $a \in A_{2}$ is either complete or anticomplete to $\{y, z\}$, for otherwise $G\left[\left\{a, v_{3}, y, z, v_{1}\right\}\right]$ has an induced $C_{4}$. So let $A_{2}^{\prime}=\left\{v_{2}\right\} \cup\left\{u \in A_{2} \mid\right.$ $u$ is anticomplete to $\{y, z\}\}$ and $X=A_{2} \backslash A_{2}^{\prime}$. Note that $x \in X$. Define sets $A_{4}^{\prime}, Y, A_{6}^{\prime}, Z$ similarly.

By Theorem 3.3 (c) and (d), we know that $\left[A_{1}, A_{2}^{\prime} \cup X \cup A_{6}^{\prime} \cup Z\right]$ is complete and $\left[A_{1}, A_{4}^{\prime} \cup Y\right]=\emptyset$. Likewise, $\left[A_{3}, A_{2}^{\prime} \cup X \cup A_{4}^{\prime} \cup Y\right]$ is complete and $\left[A_{3}, A_{6}^{\prime} \cup Z\right]=\emptyset$, and $\left[A_{5}, A_{4}^{\prime} \cup Y \cup A_{6}^{\prime} \cup Z\right]$ is complete and $\left[A_{5}, A_{1}^{\prime} \cup X\right]=\emptyset$. Moreover there is no edge $a_{1} a_{3}$ with $a_{1} \in A_{1}$ and $a_{3} \in A_{3}$, for otherwise $\left\{a_{1}, a_{3}, y, z\right\}$ induces a $C_{4}$. So $\left[A_{1}, A_{3}\right]=\emptyset$, and similarly $\left[A_{3}, A_{5}\right]=\emptyset$ and $\left[A_{5}, A_{1}\right]=\emptyset$.

There is no edge $a_{2}^{\prime} a_{4}^{\prime}$ with $a_{2}^{\prime} \in A_{2}^{\prime}$ and $a_{4}^{\prime} \in A_{4}^{\prime}$, for otherwise $\left\{a_{2}^{\prime}, a_{4}^{\prime}, y, x\right\}$ induces a $C_{4}$. So $\left[A_{2}^{\prime}, A_{4}^{\prime}\right]=\emptyset$, and similarly $\left[A_{4}^{\prime}, A_{6}^{\prime}\right]=\emptyset$ and $\left[A_{6}^{\prime}, A_{2}^{\prime}\right]=\emptyset$.

There is no edge $a_{2}^{\prime} y^{\prime}$ with $a_{2}^{\prime} \in A_{2}^{\prime}$ and $y^{\prime} \in Y$, for otherwise $\left\{a_{2}^{\prime}, y^{\prime}, z, v_{1}\right\}$ induces a $C_{4}$. Hence, and by symmetry, $\left[A_{2}^{\prime}, Y \cup Z\right]=\emptyset$, and similarly $\left[A_{4}^{\prime}, Z \cup\right.$ $X]=\emptyset$ and $\left[A_{6}^{\prime}, X \cup Y\right]=\emptyset$.

Finally, any two vertices $x^{\prime} \in X$ and $y^{\prime} \in Y$ are adjacent, for otherwise $\left\{x^{\prime}, v_{3}, y^{\prime}, z\right\}$ induces a $C_{4}$. Hence $[X, Y]$ is complete, and similarly $[X, Z]$ and $[Y, Z]$ are complete. Now we exhibit the mapping $Q_{v} \rightarrow v, v \in V\left(F_{3}\right)$ of the definition of a blowup, as follows: $A_{i}^{\prime} \rightarrow v_{i}$, for $i$ even, and $A_{i} \rightarrow v_{i}$, for $i$ odd, $X \rightarrow x, Y \rightarrow y$, and $Z \rightarrow z$. Then the above properties mean that $G$ is a blowup of $F_{3}$. This completes the proof.

## When there is an $F_{1}$ and no $F_{3}$

Here we give the proof of the second item of Theorem 1.5.
Theorem 3.5 Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph with no universal vertex and no clique cutset. Suppose that $G$ contains an $F_{1}$ and no $F_{3}$. Then $G$ is a band.

Proof. Consider the graph $F_{1}$ as shown in Figure 1 and let $C=\left\{v_{1}, \ldots, v_{5}\right\}$. We use the same notation as in Theorem 3.2 So $x \in X_{12}, y \in X_{23}$ and $z \in X_{34}$. By Theorem [3.2(b) and (c), we know that $\left[X_{23}, X_{12} \cup X_{34}\right]$ is complete and $\left[X_{12}, X_{34}\right]=\emptyset$. Note that $X_{12}$ is a clique, for otherwise $v_{1}, y$ and two nonadjacent vertices from $X_{12}$ induce a $C_{4}$. Similarly, $X_{23}$ and $X_{34}$ are cliques. We claim that:

$$
\begin{equation*}
W=\emptyset, \text { and } X_{51} \cup X_{54}=\emptyset, \text { and } A=\emptyset \tag{1}
\end{equation*}
$$

Proof: Suppose the contrary. Up to symmetry, there is a vertex $u \in W_{1} \cup W_{2} \cup$ $W_{5} \cup X_{51} \cup A$. Suppose $u \in W_{1}$. By Theorem 3.2(b) we have $u x, u y \notin E(G)$. Then $u-v_{1}-x-y-v_{3}-v_{4}$ is a $P_{6}$. Now suppose $u \in W_{2}$. By Theorem 3.2 (b) we have
$u z \notin E(G)$. Then either $u-v_{2}-y-z-v_{4}-v_{5}$ or $u-y-z-v_{4}-v_{5}-v_{1}$ is a $P_{6}$. Now suppose $u \in W_{5}$. By Theorem 3.2(b) we have $u z \notin E(G)$. Then $u-v_{5}-v_{1}-v_{2}-v_{3}-z$ is a $P_{6}$. Now suppose $u \in X_{51}$. By Theorem 3.2(b) we have $u y, u z \notin E(G)$. Then $u-v_{1}-v_{2}-y-z-v_{4}$ is a $P_{6}$. Thus we have established that $W=\emptyset$ and $X_{51} \cup X_{54}=\emptyset$ so $X=X_{12} \cup X_{23} \cup X_{34}$. Finally, suppose that $u \in A$. We note that for any two vertices $x^{\prime} \in X_{12}$ and $y^{\prime} \in X_{23}$ the vertex $u$ is either complete or anticomplete to $\left\{x^{\prime}, y^{\prime}\right\}$, for otherwise $G\left[u, v_{1}, x^{\prime}, y^{\prime}, v_{3}\right]$ contains an induced $C_{4}$. The same holds for any two vertices in $X_{23}$ and $X_{34}$. It follows that $u$ is either complete or anticomplete to $X$. If $u$ is complete to $X$ then by Theorem 3.2(回), $u$ is a universal vertex, a contradiction. If $u$ is anticomplete to $X$ then $\left\{v_{1}, \ldots, v_{5}, x, y, z, u\right\}$ induces an $F_{3}$, a contradiction. Thus (1) holds.

By (11) we have $V(G)=C \cup T_{1} \cup \cdots \cup T_{5} \cup X_{12} \cup X_{23} \cup X_{34}$. By Theorem(3.2(b) we know that $\left[T_{5}, T_{2} \cup T_{3} \cup X_{23}\right]=\emptyset$. We claim that:

$$
\begin{equation*}
\left[T_{5}, X_{12} \cup X_{34}\right]=\emptyset, \text { and }\left[T_{5}, T_{1} \cup T_{4}\right] \text { is complete. } \tag{2}
\end{equation*}
$$

Proof: Pick any vertex $t_{5} \in T_{5}$. Suppose up to symmetry that $t_{5}$ has a neighbor $x^{\prime} \in X_{12}$. Then either $\left\{t_{5}, x^{\prime}, y, z\right\}$ induces a $C_{4}$ or $v_{5}-t_{5}-x^{\prime}-v_{2}-v_{3}-z$ is a $P_{6}$, a contradiction. Now suppose up to symmetry that $t_{5}$ has a non-neighbor $t_{1} \in T_{1}$. Then $t_{5}-v_{5}-t_{1}-v_{2}-v_{3}-z$ is a $P_{6}$ (since $t_{1} z \notin E(G)$ by Theorem 3.2(b)). Thus (2) holds.

By Theorem3.2(b) we have $\left[T_{1}, T_{3} \cup T_{4} \cup X_{34}\right]=\emptyset$ and $\left[T_{4}, T_{1} \cup T_{2} \cup X_{12}\right]=\emptyset$. We claim that:

$$
\begin{equation*}
\left[T_{1}, X_{12}\right] \text { and }\left[T_{4}, X_{34}\right] \text { are complete. } \tag{3}
\end{equation*}
$$

Proof: If, up to symmetry, there are non-adjacent vertices $t_{1} \in T_{1}$ and $x^{\prime} \in X_{12}$, then either $\left\{t_{1}, v_{1}, x^{\prime}, y\right\}$ induces a $C_{4}$ or $t_{1}-v_{1}-x^{\prime}-y-z-v_{4}$ is a $P_{6}$. Thus (3) holds.

By Theorem 3.2(b) we have $\left[T_{2}, T_{4}\right]=\emptyset$ and $\left[T_{3}, T_{1}\right]=\emptyset$. We claim that:
[ $T_{2}, X_{12} \cup X_{23}$ ] and [ $T_{3}, X_{23} \cup X_{34}$ ] are complete. Moreover, every vertex in $T_{2}$ is complete either to $T_{1}$ or to $T_{3}$, and every vertex in
$T_{3}$ is complete either to $T_{2}$ or to $T_{4}$.
Proof: Up to symmetry pick any $t_{2} \in T_{2}, x^{\prime} \in X_{12}$ and $y^{\prime} \in X_{23}$. Then $t_{2} y^{\prime} \in E(G)$, for otherwise either $\left\{t_{2}, v_{2}, y^{\prime}, z\right\}$ induces a $C_{4}$ or $t_{2}-v_{2}-y^{\prime}-z-v_{4}-v_{5}$ is a $P_{6}$. Then $t_{2} x^{\prime} \in E(G)$, for otherwise $\left\{t_{2}, y^{\prime}, x^{\prime}, v_{1}\right\}$ induces a $C_{4}$. This proves the first sentence of (4). Now suppose that some $t_{2} \in T_{2}$ has a nonneighbor $t_{1} \in T_{1}$ and a non-neighbor $t_{3} \in T_{3}$. Then either $\left\{t_{1}, v_{1}, t_{2}, y\right\}$ induces a $C_{4}$ or $t_{1}-v_{1}-t_{2}-y-t_{3}-v_{4}$ is a $P_{6}$. Thus (4) holds.

$$
\begin{equation*}
\text { Every vertex in } X_{23} \text { is anticomplete to } T_{1} \text { or } T_{4} . \tag{5}
\end{equation*}
$$

Proof: If any $y^{\prime} \in X_{23}$ has neighbors $t_{1} \in T_{1}$ and $t_{4} \in T_{4}$, then $\left\{y, t_{1}, v_{5}, t_{4}\right\}$ induces a $C_{4}$. Thus (5) holds.

By (5) there is a partition $Y_{1}, Y_{4}$ of $X_{23}$ such that $\left[Y_{1}, T_{4}\right]=\left[Y_{4}, T_{1}\right]=\emptyset$.
Now let $Q_{i}=\left\{v_{i}\right\} \cup T_{i}$ for each $i \in\{1,4,5\}$. We observe that the set $T_{2} \cup\left\{v_{2}\right\} \cup X_{12} \cup Y_{1}$ is a clique, because each of $T_{2} \cup\left\{v_{2}\right\}, X_{12}$ and $Y_{1}$ and they are pairwise complete as proved above. Likewise $T_{3} \cup\left\{v_{3}\right\} \cup X_{34} \cup Y_{4}$ is a clique. Let $R_{2}=\left\{u \in T_{2} \cup\left\{v_{2}\right\} \cup X_{12} \cup Y_{1} \mid u\right.$ is complete to $\left.Q_{1}\right\}$, and let $Q_{2}=\left(T_{2} \cup\left\{v_{2}\right\} \cup X_{12} \cup Y_{1}\right) \backslash R_{2}$. Likewise let $R_{3}=\left\{u \in T_{3} \cup\left\{v_{3}\right\} \cup X_{34} \cup Y_{4} \mid u\right.$
is complete to $\left.Q_{4}\right\}$, and let $Q_{3}=\left(T_{3} \cup\left\{v_{3}\right\} \cup X_{34} \cup Y_{4}\right) \backslash R_{3}$. Note that $\left\{v_{2}\right\} \cup X_{12} \subseteq R_{2}$ and $\left\{v_{3}\right\} \cup X_{34} \subseteq R_{3}$ by (3). So $Q_{2} \subseteq T_{2} \cup Y_{1}$ and $Q_{3} \subseteq T_{3} \cup Y_{4}$. We observe that $\left[Q_{2}, Q_{3}\right]$ is complete by (4) and because $X_{23}$ is a clique. Further, we claim that:

$$
\begin{equation*}
\left[Q_{3}, R_{2}\right] \text { and }\left[Q_{2}, R_{3}\right] \text { are complete. } \tag{6}
\end{equation*}
$$

Proof: Suppose that there are non-adjacent vertices $q \in Q_{3}$ and $r \in R_{2}$. Then $r \notin\left\{v_{2}\right\} \cup Y_{1}$, and so $r \in T_{2} \cup X_{12}$, and $q$ has a non-neighbor $t \in T_{4}$. If $r \in X_{12}$, then $q-v_{3}-t-v_{5}-v_{1}-r$ is a $P_{6}$ (since $r t \notin E(G)$, by Theorem 3.2(b)), a contradiction. So $r \in T_{2}$. Then since $q z \in E(G)$ (by (4i)) and $\left\{q, z, r, v_{2}\right\}$ does not induce a $C_{4}, r z \notin E(G)$. But then $v_{5}-t-z-q-v_{2}-r$ is a $P_{6}$, a contradiction. Thus (6) holds.

Moreover, by the definition of $Q_{1}, \ldots, Q_{4}, R_{2}$ and $R_{3}$, the pairs $\left\{Q_{1}, Q_{2}\right\}$, $\left\{Q_{2}, Q_{3}\right\}$ and $\left\{R_{2}, R_{3}\right\}$ are graded. Hence the sets $Q_{1}, \ldots, Q_{5}, R_{2}, R_{3}$ form a partition of $V(G)$ which shows that $G$ is a band.

## When there is a $C_{6}$ and no $F_{1}$

Here we give the proof of the third item of Theorem 1.5, which we restate as follows.

Theorem 3.6 Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph that has no clique-cutset and no universal vertex, and suppose that $G$ is $F_{1}$-free. If $G$ contains an induced $C_{6}$, then $G$ is a blowup of one of the graphs $H_{1}, H_{2}, H_{3}, H_{4}$.

Proof. Let $C=\left\{v_{1}, v_{2}, \ldots, v_{6}\right\}$ be the vertex-set of a $C_{6}$ in $G$, with edges $v_{i} v_{i+1}$ $(\bmod 6)$. We use Theorem 3.3 with the same notation. If $C$ is not dominating, then by Theorem 3.1 and since $G$ has no universal vertex, $G$ is a blowup of the Petersen graph. Therefore we may assume that $C$ is dominating. So $L=\emptyset$ and $V(G)=V(C) \cup A \cup B \cup D \cup S$. Moreover, since $G$ is $F_{1}$-free, we have $\left[A_{i}, A_{i+2}\right]=\emptyset$ and $\left[A_{i}, B_{i+1}\right]$ is complete. So, by Theorem 3.3, each of the sets $A_{1} \cup\left\{v_{1}\right\}, \ldots, A_{6} \cup\left\{v_{6}\right\}, B_{1}, \ldots, B_{6}, D_{1}, D_{2}, D_{3}, S$ is a clique and that any two of them are either complete or anticomplete to each other. So $G$ is a blowup of some graph. We now make this more precise. Since $G$ has no universal vertex, by Theorem 3.3(b) and (c), we have $S=\emptyset$. If $B=\emptyset$, then $G$ is a blowup of the Petersen graph. Now assume that $B \neq \emptyset$. First, suppose that two consecutive $B_{j}$ 's are non-empty, say $B_{i}, B_{i+1} \neq \emptyset$. Then by Theorem[3.3(fi), $B_{i+3} \cup B_{i+4}=\emptyset$, and by Theorem 3.3(可) $D=\emptyset$. So again by Theorem 3.3(f), $G$ is a blowup of $H_{4}$. Next, suppose that no two consecutive $B_{j}$ 's are non-empty and let $B_{i} \neq \emptyset$. Then $B_{i-1}=\emptyset=B_{i+1}$ and by Theorem 3.3(回), $D_{i}=\emptyset=D_{i+1}$. Now, if $B_{i+3} \neq \emptyset$ or $B_{i+2} \cup B_{i+4}=\emptyset$, then $G$ is a blowup of $H_{2}$, and if $B_{i+3}=\emptyset$ and $B_{i+2} \cup B_{i+4} \neq \emptyset$, then by Theorem 3.3(-®), $D=\emptyset$, and so $G$ is a blowup of $H_{3}$.

## When there is an $F_{2}$ and no $C_{6}$

Here we give the proof of the fourth item of Theorem 1.5. which we restate it as follows.

Theorem 3.7 Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph that has no clique-cutset and no universal vertex, and suppose that $G$ is $C_{6}$-free. If $G$ contains an $F_{2}$, then $G$ is a blowup of either $H_{5}$ or $F_{k, \ell}$ for some integers $k, \ell \geq 1$.

Proof. Consider the graph $F_{2}$ as shown in Figure 1 and let $C=\left\{v_{1}, \ldots, v_{5}\right\}$. We use the same notation as in Theorem 3.2. Note that $t \in T_{5}$, and $x \in X_{12}$ and $y \in X_{34}$, so Theorem 3.2(或) implies that the sets $X_{23}, X_{45}, X_{15}$ and $W_{2}$, $W_{3}, W_{5}$ are all empty, and one of $W_{1}, W_{4}$ is empty. So $V(G)=C \cup T_{1} \cup \cdots \cup$ $T_{5} \cup X_{12} \cup X_{34} \cup A \cup W_{1} \cup W_{4}$. We establish a number of properties. (Some of them were also proved in [14, Proof of Lemma 4].)
(i) Each vertex in $T_{5}$ is either complete or anticomplete to $X_{12} \cup X_{34}$. In particular, $t$ is complete to $X_{12} \cup X_{34}$.
Proof: Suppose that some vertex $t_{5} \in T_{5}$ is not complete and not anticomplete to $X_{12} \cup X_{34}$. It follows that $t_{5}$ has a neighbor $x^{\prime} \in X_{12}$ and a non-neighbor $y^{\prime} \in X_{34}$, or vice-versa. Then $v_{5}-t_{5}-x^{\prime}-v_{2}-v_{3}-y^{\prime}$ is a $P_{6}$.
(ii) $X_{12}$ and $X_{34}$ are cliques.

Proof: If, up to symmetry, $X_{12}$ contains two non-adjacent vertices $x^{\prime}, x^{\prime \prime}$, then by (ii), $\left\{t, x^{\prime}, x^{\prime \prime}, v_{2}\right\}$ induces a $C_{4}$.
(iii) Each vertex in $T_{2}$ is either complete or anticomplete to $X_{12}$, and each vertex in $T_{3}$ is either complete or anticomplete to $X_{34}$.
Proof: If, up to symmetry, some vertex $t_{2} \in T_{2}$ has a neighbor $x^{\prime}$ and a non-neighbor $x^{\prime \prime}$ in $X_{12}$, then, by (iii), $x^{\prime \prime}-x^{\prime}-t_{2}-v_{3}-v_{4}-v_{5}$ is a $P_{6}$.
(iv) $\left[T_{2}, X_{34}\right]=\emptyset$, and $\left[T_{3}, X_{12}\right]=\emptyset$.

Proof: Suppose, up to symmetry, that there are adjacent vertices $t_{2} \in T_{2}$ and $y^{\prime} \in X_{34}$. If $t_{2} t \in E(G)$ then $\left\{t_{2}, v_{3}, v_{4}, t\right\}$ induces a $C_{4}$. If $t_{2} t \notin E(G)$, then by (ii), $\left\{t_{2}, y^{\prime}, t, v_{1}\right\}$ induces a $C_{4}$.
(v) $\left[T_{1}, T_{2} \cup T_{5} \cup X_{12}\right]$ and $\left[T_{4}, T_{3} \cup T_{5} \cup X_{34}\right]$ are complete.

Proof: Suppose, up to symmetry, that some vertex $t_{1} \in T_{1}$ has a nonneighbor $u \in T_{2} \cup T_{5} \cup X_{12}$. Recall that $t_{1} y \notin E(G)$ by Theorem 3.2(b). Also, since $\left\{v_{5}, t, y, v_{3}, v_{2}, t_{1}\right\}$ does not induce a $C_{6}, t_{1} t \in E(G)$. Suppose that $u \in X_{12}$. Then $\left\{t_{1}, t, u, v_{2}\right\}$ induces a $C_{4}$, a contradiction. In particular $t_{1} x \in E(G)$. Now suppose that $u \in T_{5}$ and $u \neq t$. If $u x \in E(G)$, then $\left\{u, x, t_{1}, v_{5}\right\}$ induces a $C_{4}$. If $u x \notin E(G)$, then by (iil), $u y \notin E(G)$, and $u-v_{5}-t_{1}-v_{2}-v_{3}-y$ is a $P_{6}$. Finally, if $u \in T_{2}$, then by (iv), we have $u y \notin E(G)$, and $u-v_{2}-t_{1}-v_{5}-v_{4}-y$ is a $P_{6}$.
(vi) $\left[A, X_{12} \cup X_{34}\right]$ is complete.

Proof: If, up to symmetry, there are non-adjacent vertices $a \in A$ and $x^{\prime} \in X_{12}$, then by Theorem3.2(a) and (ii) the set $\left\{a, t, x^{\prime}, v_{2}\right\}$ induces a $C_{4}$.

Now let:

$$
\begin{aligned}
& Q_{i}=\left\{v_{i}\right\} \cup T_{i} \text { for } i \in\{1,4\}, \\
& Q_{2}=\left\{v_{2}\right\} \cup\left\{u \in T_{2} \mid u \text { is complete to } X_{12}\right\} \text { and } R_{2}=T_{2} \backslash Q_{2}, \\
& Q_{3}=\left\{v_{3}\right\} \cup\left\{u \in T_{3} \mid u \text { is complete to } X_{34}\right\} \text { and } R_{3}=T_{3} \backslash Q_{3}, \\
& Q_{5}=\left\{u \in T_{5} \mid u \text { is complete to } X_{12} \cup X_{34}\right\} \text { and } R_{5}=\left\{v_{5}\right\} \cup\left(T_{5} \backslash Q_{5}\right),
\end{aligned}
$$

Recall that, by Theorem 3.2(b), $\left[T_{i}, T_{i+2}\right]=\emptyset$, for all $i$. Then:
(vii) $\left[Q_{2}, R_{3}\right]$ and $\left[Q_{3}, R_{2}\right]$ are complete.

Proof: If there are non-adjacent vertices $u \in Q_{2}$ and $r \in R_{3}$, then $r-v_{3}-u$ -$x-t-v_{5}$ is a $P_{6}$. The proof is similar for $\left[Q_{3}, R_{2}\right]$.
(viii) $\left[R_{2}, R_{3}\right]=\emptyset$.

Proof: If $r_{2} \in R_{2}$ and $r_{3} \in R_{3}$ are adjacent then $x-v_{1}-r_{2}-r_{3}-v_{4}-y$ is a $P_{6}$.
Suppose that $W_{1} \cup W_{4}=\emptyset$. By (vil) and Theorem 3.2(国), $[A, V(G) \backslash A]$ is complete and $A$ is a clique; since $G$ has no universal vertex, we deduce that $A=\emptyset$. Then $V(G)$ is partitioned into the ten cliques $Q_{1}, Q_{2}, Q_{3}, Q_{4}, Q_{5}, R_{5}$, $X_{12}, R_{2}, R_{3}, X_{34}$, and any two of them are either complete or anticomplete to each other, and the adjacencies proved above show that $G$ is a blowup of $H_{5}$.

Therefore let us assume that $W_{1} \cup W_{4} \neq \emptyset$. By Theorem 3.2dd) one of $W_{1}$ and $W_{4}$ is empty. Up to symmetry, let us assume that $W_{1} \neq \emptyset$ and $W_{4}=\emptyset$. Hence $V(G)=Q_{1} \cup \cdots \cup Q_{5} \cup R_{2} \cup R_{3} \cup R_{5} \cup X_{12} \cup X_{34} \cup W_{1} \cup A$. Recall that every induced $C_{5}$ in $G$ is dominating, by Theorem 3.1. Then:
(ix) $\left[W_{1}, Q_{1}\right]$ is complete, and $\left[W_{1}, Q_{3} \cup R_{3} \cup Q_{4} \cup X_{12}\right]=\emptyset$. This follows directly from Theorem 3.2(b)-(目).
(x) $\left[W_{1}, Q_{5}\right]$ is complete.

Proof: If any $w \in W_{1}$ and $u \in Q_{5}$ are non-adjacent, then either $\left\{w, v_{1}, u, y\right\}$ induces a $C_{4}$ or $\left\{u, x, v_{2}, v_{3}, y\right\}$ is a non-dominating $C_{5}$ by (ix).
(xi) $\left[W_{1}, Q_{2} \cup R_{5}\right]=\emptyset$.

Proof: Suppose that $w \in W_{1}$ and $u \in Q_{2} \cup R_{5}$ are adjacent. If $u \in Q_{2}$, then, since $t \in Q_{5}$ and by (ix) and (ख) , $\{w, t, x, u\}$ induces a $C_{4}$. If $u \in R_{5}$, then $w-u-v_{4}-v_{3}-v_{2}-x$ is a $P_{6}$ by (ix).
(xii) $R_{3}=\emptyset$.

Proof: Pick any $w \in W_{1}$. If there is any vertex $r \in R_{3}$, then $\left\{w, v_{1}, v_{2}, r, v_{4}\right.$, $y\}$ induces a $P_{6}$ or a $C_{6}$ by (ix).
(xiii) Each component $Z$ of $W_{1}$ is homogeneous in $G \backslash A$.

Proof: Otherwise, there are adjacent vertices $z, z^{\prime} \in Z$ and a vertex $u \notin$ $W_{1} \cup A$ adjacent to $z$ and not to $z^{\prime}$. By the preceding points $u$ is in $R_{2} \cup X_{34}$. If $u \in R_{2}$, then $z^{\prime}-z-u-v_{3}-v_{4}-v_{5}$ is a $P_{6}$. If $u \in X_{34}$, then $z^{\prime}-z-u-v_{3}-v_{2}-x$ is a $P_{6}$ by (ix).
(xiv) Each component $Z$ of $W_{1}$ has either a neighbor in $R_{2}$ and no neighbor in $X_{34}$, or a neighbor in $X_{34}$ and no neighbor in $R_{2}$.
Proof: If $Z$ has no neighbor in $R_{2} \cup X_{34}$, then by the preceding points we have $N(Z)=Q_{1} \cup Q_{5} \cup A^{\prime}$ for some $A^{\prime} \subseteq A$, and so $N(Z)$ is a clique by Theorem 3.2, contradicting the hypothesis that $G$ has no clique cutset. On the other hand if $Z$ has a neighbor $r \in R_{2}$ and a neighbor $u \in X_{34}$, then by xiii) for any $z \in Z$ we see that $\left\{z, r, v_{3}, u\right\}$ induces a $C_{4}$.
(xv) Each component $Z$ of $W_{1}$ is a clique.

Proof: Suppose that $Z$ contains non-adjacent vertices $z, z^{\prime}$. By xiii) and (xiv) $z$ and $z^{\prime}$ have a common neighbor $u$ in $R_{2} \cup X_{34}$. Then $\left\{z, u, z^{\prime}, t\right\}$ or $\left\{z, u, z^{\prime}, v_{1}\right\}$ induces a $C_{4}$.
(xvi) If $Z, Z^{\prime}$ are distinct components of $W_{1}$, then $N(Z) \cap N\left(Z^{\prime}\right) \cap\left(R_{2} \cup X_{34}\right)=\emptyset$. (Otherwise there is a $C_{4}$ as in the proof of (XV).)
(xvii) $A=\emptyset$.

Proof: Suppose that there exists $a \in A$. Since $G$ has no universal vertex,
there is a non-neighbor $z$ of $a$. By Theorem 3.2(a) and by (vi) we have $z \in W_{1}$. By (xiii) and (xiv) $z$ has a neighbor $u \in R_{2} \cup X_{34}$. But then $\{a, u, z, t\}$ or $\left\{a, u, z, v_{1}\right\}$ induces a $C_{4}$.

By (xiil) and xviil) we have $V(G)=Q_{1} \cup \cdots \cup Q_{5} \cup R_{2} \cup R_{5} \cup X_{12} \cup X_{34} \cup W_{1}$. Now it is a routine matter to check that $G$ is a blowup of $F_{k, \ell}$ for some $k, \ell \geq 1$. We clarify this point by exhibiting the mapping $Q_{v} \rightarrow v$ of the definition of a blowup, as follows. If $Z$ is any component of $W_{1}$, we say that it is an $R_{2^{-}}$ component (resp. $X_{34}$-component) if it has a neighbor in $R_{2}$ (resp. in $X_{34}$ ), and we call the set $N(Z) \cap R_{2}$ (resp. $N(Z) \cap X_{34}$ ) the support of $Z$. By xiv) and (Xvi) the supports are non-empty and pairwise disjoint. Let $Z_{1}, Z_{2}, \ldots, Z_{p}$ be the $R_{2}$-components of $W_{1}$, and let $Z_{1}^{\prime}, Z_{2}^{\prime}, \ldots, Z_{q}^{\prime}$ be the $X_{34}$-components of $W_{1}$. Let $k=p+1$ and $\ell=q+1$. Then:

- $Z_{i} \rightarrow u_{i}$ and $N\left(Z_{i}\right) \cap R_{2} \rightarrow a_{i}$ for each $i \in\{1,2 \ldots, p\}$, and $X_{12} \rightarrow u_{p+1}$ and $Q_{2} \rightarrow a_{p+1}$, and $R_{2} \backslash \cup_{i=1}^{p}\left(N\left(Z_{i}\right) \cap R_{2}\right) \rightarrow a_{0}$.
- $Z_{j}^{\prime} \rightarrow w_{j}$ and $N\left(Z_{j}^{\prime}\right) \cap X_{34} \rightarrow b_{j}$ for each $j \in\{1,2 \ldots, q\}$, and $R_{5} \rightarrow w_{q+1}$ and $Q_{4} \rightarrow b_{q+1}$, and $X_{34} \backslash \cup_{j=1}^{q}\left(N\left(Z_{j}^{\prime}\right) \cap X_{34}\right) \rightarrow b_{0}$.
- $Q_{1} \rightarrow x, Q_{5} \rightarrow y$, and $Q_{3} \rightarrow z$.

Since the components of $W_{1}$ and their supports are cliques, we see that $G$ is a blowup of $F_{k, \ell}$. This completes the proof of the theorem.

## When there is a $C_{5}$, no $C_{6}$ and no $F_{2}$

Here we give the proof of the last item of Theorem 1.5.
Theorem 3.8 Let $G$ be a $\left(P_{6}, C_{4}\right)$-free graph that has no clique-cutset and no universal vertex, and suppose that $G$ is $C_{6}$-free and $F_{2}$-free. If $G$ contains a $C_{5}$, then $G$ is either a belt or a boiler.

Proof. Let $C=\left\{v_{1}, \ldots, v_{5}\right\}$ be the vertex-set of a $C_{5}$ in $G$ with edges $v_{i} v_{i+1}$ $(\bmod 5)$. We use the same notation as in Theorem 3.2. We choose $C$ such that $|T|$ is minimized. Remark that since $G$ is $\left(P_{6}, C_{4}, C_{6}\right)$-free every hole in $G$ has length 5 and is dominating by Theorem 3.1. We establish a number of properties. (Some of them were also proved in [14, Lemma 5].)
(i) If $X_{i-2, i-1} \cup X_{i+1, i+2}=\emptyset$, then $T_{i}$ is complete to $T_{i-1} \cup T_{i+1}$.

Proof: Up to symmetry let $i=1$ and suppose that $X_{23} \cup X_{45}=\emptyset$ and that some vertex $t_{1} \in T_{1}$ has a non-neighbor $t_{2} \in T_{2}$. Let $C^{\prime}=$ $\left\{t_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$. So $C^{\prime}$ induces a $C_{5}$, and $t_{2}$ has only two neighbors on it, so the choice of $C$ (minimizing $|T|$ ) implies the existence of a vertex that has three neighbors on $C^{\prime}$ and two on $C$. Such a vertex must be in $X_{23} \cup X_{45}$, a contradiction.
(ii) Every component $Z$ of $W_{i}$ is anticomplete to one of $T_{i-1}, T_{i+1}$.

Proof: Let $i=1$ and suppose that there are vertices $z, z^{\prime} \in Z$ such that $z$ has a neighbor $t_{2} \in T_{2}$ and $z^{\prime}$ has a neighbor $t_{5} \in T_{5}$. If we can choose $z=z^{\prime}$, then $C \cup\left\{z, t_{2}, t_{5}\right\}$ induces an $F_{2}$. Otherwise let $P$ be a shortest path between $z$ and $z^{\prime}$ in $G[Z]$. Then $V(P) \cup\left\{t_{2}, v_{2}, t_{5}, v_{5}\right\}$ contains an induced $P_{6}$.
(iii) For every component $Z$ of $W_{i}$, every vertex of $T_{i-1} \cup T_{i+1}$ is either complete or anticomplete to $Z$.
Proof: Let $i=1$. Suppose that $y, z$ are adjacent vertices in $Z$ and that some vertex $t_{2} \in T_{2}$ is adjacent to $y$ and not to $z$. By (iii) $\left[Z, T_{5}\right]=\emptyset$. Then $z-y-t_{2}-v_{3}-v_{4}-v_{5}$ is a $P_{6}$.
(iv) Every vertex in $W_{i}$ has a neighbor in $X_{i-2, i+2}$. In particular if $W_{i} \neq \emptyset$, then $X_{i-2, i+2} \neq \emptyset$.
Proof: Let $i=1$ and suppose that some vertex of $W_{1}$ has no neighbor in $X_{34}$. Let $Z$ be the component of $W_{1}$ that contains this vertex. By (iil) we may assume that $\left[Z, T_{5}\right]=\emptyset$. Let $Z_{0}=\left\{z \in Z \mid z\right.$ has no neighbor in $\left.X_{34}\right\}$, so $Z_{0} \neq \emptyset$. Let $Y_{0}$ be a component of $G\left[Z_{0}\right]$, and let $Y_{1}=N\left(Y_{0}\right) \cap\left(Z \backslash Z_{0}\right)$ and $Y_{2}=N\left(Y_{0}\right) \cap T_{2}$, and $A_{0}=N\left(Y_{0}\right) \cap A$. By Theorem 3.2 and since $\left[Z, T_{5}\right]=\emptyset$ we have $N\left(Y_{0}\right)=\left\{v_{1}\right\} \cup T_{1} \cup Y_{1} \cup Y_{2} \cup A_{0}$ and $Y_{0}$ is complete to $\left\{v_{1}\right\} \cup T_{1}$, and by (iiil) $Y_{2}$ is complete to $Y_{0}$. Suppose that some vertex $y \in Y_{1}$ is not complete to $Y_{0}$. Then there are adjacent vertices $y_{0}, z_{0} \in Y_{0}$ and a vertex $x \in X_{34}$ such that $z_{0}-y_{0}-y-x-v_{4}-v_{5}$ is a $P_{6}$. Hence $Y_{0}$ is complete to $N\left(Y_{0}\right) \backslash A_{0}$. Since $G$ has no clique-cutset, there are nonadjacent vertices $u, v \in N\left(Y_{0}\right)$. By Theorem 3.2 and (iiil) we know that $\left[Y_{1} \cup A_{0},\left\{v_{1}\right\} \cup T_{1} \cup Y_{2}\right]$ is complete, so we have either (a) $u, v \in Y_{1}$, or (b) $u \in Y_{1}$ and $v \in A_{0}$, or (c) $u \in T_{1}$ and $v \in Y_{2}$. Pick any $y_{0} \in Y_{0}$. In case (a), by the definition of $Z_{0}$ there are vertices $x, x^{\prime} \in X_{34}$ such that $x u, x^{\prime} v \in E(G)$. If we can choose $x=x^{\prime}$, then $\left\{x, u, y_{0}, v\right\}$ induces a $C_{4}$; and in the opposite case either $\left\{x, x^{\prime}, u, v, y_{0}\right\}$ induces a non-dominating $C_{5}$ (if $x x^{\prime} \in E(G)$ ), because $v_{5}$ has no neighbor in it, or $\left\{y_{0}, u, v, x, x^{\prime}, v_{4}\right\}$ induces a $C_{6}$, a contradiction. In case (b) we may choose $y_{0}$ adjacent to $v$. By the definition of $Z_{0}, u$ has a neighbor $x \in X_{34}$. Then $v x \notin E(G)$, for otherwise $\left\{v, x, u, y_{0}\right\}$ induces a $C_{4}$. But then $\left\{v_{1}, v_{3}, v_{4}, v_{5}, x, y_{0}, u, v\right\}$ induces an $F_{2}$. In case (c), $\left\{y_{0}, u, v_{2}, v\right\}$ induces a $C_{4}$.
(v) $X_{i+2, i-2}$ is anticomplete to one of $T_{i-1}, T_{i+1}$.

Proof: Let $i=1$ and suppose that there are vertices $x, y \in X_{34}$ such that $x$ has a neighbor $t_{2} \in T_{2}$ and $y$ has a neighbor $t_{5} \in T_{5}$. Then $x t_{5} \notin$ $E(G)$, for otherwise $\left\{v_{1}, t_{2}, x, t_{5}\right\}$ induces a $C_{4}$; and similarly $y t_{2} \notin E(G)$. Moreover $x y \notin E(G)$, for otherwise $v_{2}-t_{2}-x-y-t_{5}-v_{5}$ is a $P_{6}$. But then $\left\{v_{1}, v_{2}, v_{3}, v_{4}, x, y, t_{2}, t_{5}\right\}$ induces an $F_{2}$.
(vi) If $\left[X_{i+2, i-2}, T_{i+1}\right] \neq \emptyset$ then $X_{i-1, i}=\emptyset$. Likewise if $\left[X_{i+2, i-2}, T_{i-1}\right] \neq \emptyset$ then $X_{i, i+1}=\emptyset$.
Proof: Let $i=1$, and suppose that some vertex $x \in X_{34}$ has a neighbor $t \in T_{2}$ and that there is a vertex $y \in X_{51}$. Then $x y \notin E(G)$, for otherwise $\left\{x, v_{4}, v_{5}, y\right\}$ induces a $C_{4}$, and $t y \in E(G)$, for otherwise $v_{2}-t-x-v_{4}-v_{5}-y$ is a $P_{6}$; but then $C \cup\{t, x, y\}$ induces an $F_{2}$.
(vii) Every vertex in $X_{i+2, i-2}$ that has a neighbor in $T_{i+1}$ is complete to $T_{i-2}$. Proof: Let $i=1$, and suppose that some vertex $x \in X_{34}$ has a neighbor $t \in$ $T_{2}$ and that $x$ is not adjacent to a vertex $y \in T_{4}$. Then by Theorem 3.2 (b), $t y \notin E(G)$. But then $C \cup\{t, x, y\}$ induces an $F_{2}$.
(viii) If $W_{i} \neq \emptyset$, then $\left[X_{i+2, i-2}, T_{i+2} \cup T_{i-2}\right]$ is complete.

Proof: Let $i=1$, and suppose that, up to symmetry, there are nonadjacent vertices $x \in X_{34}$ and $t \in T_{3}$ and that there is a vertex $w \in W_{1}$. Then $\left\{w, v_{1}, v_{2}, t, v_{4}, x\right\}$ induces a $P_{6}$ or a $C_{6}$.

Suppose that $X=\emptyset$. Then (iv) implies that $W=\emptyset$, so $V(G)=C \cup$ $T \cup A$. Moreover $A=\emptyset$, for otherwise any vertex in $A$ is universal in $G$, by Theorem 3.2 (a); and (ii) implies that $\left[T_{i}, T_{i+1}\right]$ is complete for all $i$. So $G$ is a blowup of $C_{5}$, which is a special case of a belt.

Now assume that $X \neq \emptyset$, say $X_{34} \neq \emptyset$. By Theorem 3.2(d) and by symmetry, we may assume that $X_{23} \cup X_{45} \cup X_{51}=\emptyset$, so $X=X_{12} \cup X_{34}$, and consequently, by ( $\mathbf{\nabla}$ ) and (vil) and up to symmetry, that $\left[X_{34}, T_{5}\right]=\emptyset$ and $\left[X_{12}, T_{5}\right]=\emptyset$. By (iv) we have $W=W_{1} \cup W_{4}$, and by Theorem 3.2(e) one of $W_{1}, W_{4}$ is empty, so, still up to symmetry, we may assume that $W_{4}=\emptyset$. Let:
$W_{1}^{T}=\left\{w \in W_{1} \mid w\right.$ has a neighbor in $\left.T_{2}\right\}$,
$W_{1}^{N}=\left\{w \in W_{1} \mid w\right.$ has no neighbor in $\left.T_{2}\right\}$,
$X_{34}^{T}=\left\{x \in X_{34} \mid x\right.$ has a neighbor in $\left.T_{2}\right\}$,
$X_{34}^{N}=\left\{x \in X_{34} \mid x\right.$ has no neighbor in $T_{2}$ and has a neighbor in $\left.W_{1}\right\}$,
$X_{34}^{0}=\left\{x \in X_{34} \mid x\right.$ has no neighbor in $\left.T_{2} \cup W_{1}\right\}$,
$X_{34}^{W}=\left\{x \in X_{34} \mid x\right.$ has a neighbor in $\left.W_{1}\right\}$.
Clearly $W_{1}=W_{1}^{T} \cup W_{1}^{N}$ and $X_{34}=X_{34}^{T} \cup X_{34}^{N} \cup X_{34}^{0}$. Moreover we have $X_{34}^{N} \subseteq X_{34}^{W} \subseteq X_{34}^{N} \cup X_{34}^{T}$. Recall that $\left[W_{1}, T_{1}\right]$ is complete and that [ $W_{1}, T_{3} \cup$ $\left.T_{4} \cup X_{12}\right]=\emptyset$ by Theorem [3.2(b)-(®). By (iil), $\left[T_{1}, T_{2} \cup T_{5}\right]$ and $\left[T_{4}, T_{3} \cup T_{5}\right]$ are complete. We establish some additional facts.
(ix) $\left[W_{1}, T_{5}\right]=\emptyset$.

Proof: Suppose that $w \in W_{1}$ and $t \in T_{5}$ are adjacent. By (iv) $w$ has a neighbor $x \in X_{34}$. Since $\left[X_{34}, T_{5}\right]=\emptyset$, we see that $v_{5}-t-w-x-v_{3}-v_{2}$ is a $P_{6}$.
(x) $\left[W_{1}^{T}, W_{1}^{N}\right]=\emptyset$.

This follows directly from (iiii).
(xi) For every edge $w x$ with $w \in W_{1}$ and $x \in X_{34}$, every vertex $u$ in $T_{2}$ is either complete or anticomplete to $\{w, x\}$. Hence $\left[W_{1}^{T}, X_{34}^{N}\right]=\emptyset$ and $\left[W_{1}^{N}, X_{34}^{T}\right]=\emptyset$. Also every vertex $u$ in $A$ is either complete or anticomplete to $\{w, x\}$.
Proof: In the opposite case there is a $C_{4}$ in $G\left[\left\{v_{1}, w, x, v_{3}, u\right\}\right]$.
(xii) $\left[A, X_{34}^{T} \cup W_{1}^{T}\right]$ is complete.

Proof: Consider any $a \in A$. First pick any $x \in X_{34}^{T}$, so $x$ has a neighbor $t \in T_{2}$. Then $a t \in E(G)$ by Theorem 3.2(a), and $a x \in E(G)$, for otherwise $\left\{a, t, x, v_{4}\right\}$ induces a $C_{4}$. Now pick any $w \in W_{1}^{T}$. So $w$ has a neighbor $t \in T_{2}$ and, by (iv), a neighbor $x \in X_{34}$. Then $x t \in E(G)$ by xil), so $x \in X_{34}^{T}$, and $a x \in E(G)$ by the preceding point of this claim. Then $a w \in E(G)$, for otherwise $\left\{a, v_{1}, w, x\right\}$ induces a $C_{4}$.
(xiii) Any vertex $x \in X_{34}^{W}$ is complete to $\left(X_{34} \backslash x\right) \cup T_{3} \cup T_{4}$.

Proof: Suppose up to symmetry that $x$ has a non-neighbor $y \in\left(X_{34} \backslash x\right) \cup$ $T_{3}$. Let $w \in W_{1}$ be any neighbor of $x$. Then either $\left\{w, x, y, v_{3}\right\}$ induces a $C_{4}$ (if $w y \in E(G)$ ) or $v_{5}-v_{1}-w-x-v_{3}-y$ is a $P_{6}$.
(xiv) Every vertex in $X_{12}$ has a neighbor in $T_{3}$.

Proof: Suppose on the contrary that the set $Z=\left\{z \in X_{12} \mid z\right.$ has no neighbor in $\left.T_{3}\right\}$ is non-empty, and let $Y$ be the vertex-set of a component of $G[Z]$. Let $Y^{\prime}=N(Y) \cap X_{12}, T_{1}^{\prime}=N(Y) \cap T_{1}, T_{2}^{\prime}=N(Y) \cap T_{2}$,
and $A^{\prime}=N(Y) \cap A$. By Theorem 3.2 and the current assumption we have $N(Y)=\left\{v_{1}, v_{2}\right\} \cup Y^{\prime} \cup T_{1}^{\prime} \cup T_{2}^{\prime} \cup A^{\prime}$. Suppose that some vertex $u \in Y^{\prime} \cup T_{1}^{\prime} \cup T_{2}^{\prime}$ is not complete to $Y$; so there are adjacent vertices $y, z \in Y$ with $u y \in E(G)$ and $u z \notin E(G)$. If $u \in Y^{\prime}$, then $u \in X_{12} \backslash Z$, so $u$ has a neighbor $t \in T_{3}$, and then $z-y-u-t-v_{4}-v_{5}$ is a $P_{6}$. If $u \in T_{1}^{\prime}$, then $z-y-u-v_{5}-v_{4}-v_{3}$ is a $P_{6}$. The proof is similar if $u \in T_{2}^{\prime}$. Hence $Y$ is complete to $\left\{v_{1}, v_{2}\right\} \cup Y^{\prime} \cup T_{1}^{\prime} \cup T_{2}^{\prime}$. Since $G$ has no clique cutset, the set $N(Y)$ contains two non-adjacent vertices $u, v$. By Theorem 3.2, and since $\left[T_{1}, T_{2}\right]$ is complete, and up to symmetry, we have $u \in Y^{\prime}$ and $v \in Y^{\prime} \cup T_{1}^{\prime} \cup T_{2}^{\prime} \cup A^{\prime}$. So $u$ has a neighbor $t \in T_{3}$. Pick any $y \in Y$. If $v \in Y^{\prime}$, then $v$ has a neighbor $s \in T_{3}$, and either $\{y, u, v, s\}$ induces a $C_{4}$ (if we can choose $s=t)$ or $\{y, u, v, s, t\}$ induces $C_{5}$ that does not dominate $v_{5}$. If $v \in T_{1}^{\prime}$, then $\left\{y, u, t, v_{4}, v_{5}, v\right\}$ induces a $C_{6}$. If $v \in T_{2}^{\prime}$, then either $\{y, u, t, v\}$ induces a $C_{4}$, or $\left\{y, u, t, v_{3}, v\right\}$ induces a $C_{5}$ that does not dominate $v_{5}$. If $v \in A^{\prime}$, then we can choose $y$ adjacent to $v$, and then $\{y, u, t, v\}$ induces a $C_{4}$.
$(\mathrm{xv})\left[X_{12}, T_{1}\right]$ is complete.
Proof: This follows from (xiv) and (vii).
(xvi) $\left[X_{12}, A\right]$ is complete.

Proof: Pick any $a \in A$ and $x \in X_{12}$. By xiv) $x$ has a neighbor $t \in T_{3}$. We have at $\in E(G)$ by Theorem 3.2 and $a x \in E(G)$, for otherwise $\left\{a, t, x, v_{1}\right\}$ induces a $C_{4}$.
(xvii) For any component $Z$ of $G\left[X_{34}^{0}\right]$ the set $[Z, N(Z) \backslash A]$ is complete and $N(Z) \backslash A$ is a clique.
Proof: Let $Z$ be (the vertex-set of) a component of $G\left[X_{34}^{0}\right]$. Then $N(Z) \backslash$ $A \subseteq\left\{v_{3}, v_{4}\right\} \cup T_{3} \cup T_{4} \cup X_{34}^{N} \cup X_{34}^{T}$. First suppose that $[Z, N(Z) \backslash A]$ is not complete. So there are adjacent vertices $y, z \in Z$ and a vertex $u \in N(Z) \backslash A$ with $u y \in E(G)$ and $u z \notin E(G)$. Clearly $u \notin\left\{v_{3}, v_{4}\right\}$. If $u \in X_{34}^{N} \cup X_{34}^{T}$, then, by xiiil $u$ has no neighbor in $W_{1}$, so $u$ has a neighbor $t \in T_{2}$, and then $z-y-u-t-v_{1}-v_{5}$ is a $P_{6}$. If $u \in T_{3}$, then $z-y-u-v_{2}-v_{1}-v_{5}$ is a $P_{6}$. If $u \in T_{4}$ then $z-y-u-v_{5}-v_{1}-v_{2}$ is a $P_{6}$, a contradiction. Now suppose that $N(Z) \backslash A$ is not a clique, so it contains two non-adjacent vertices $u, v$. Pick any $z \in Z$. By Theorem 3.2 and since $\left[T_{4}, T_{3}\right]$ is complete we have either (a) $u, v \in X_{34}^{N} \cup X_{34}^{T}$ or (b) $u \in X_{34}^{N} \cup X_{34}^{T}$ and $v \in T_{3} \cup T_{4}$. In case (a), by (xiii) $u$ and $v$ have no neighbor in $W_{1}$, so they have neighbors respectively $t$ and $t^{\prime}$ in $T_{2}$; then $\left\{z, u, v, t, t^{\prime}\right\}$ induces either a $C_{4}$ or a non-dominating $C_{5}$ (because $v_{5}$ has no neighbor in it), a contradiction. In case (b), item (xiii) implies that $u$ has no neighbor in $W_{1}$, so $u$ has a neighbor $t \in T_{2}$. If $v \in T_{3}$, then $\left\{z, u, t, v_{2}, v\right\}$ induces a non-dominating $C_{5}$ (because of $v_{5}$ ). If $v \in T_{4}$, then $v_{2}-t-u-z-v-v_{5}$ is a $P_{6}$.
(xviii) For each component $Z$ of $G\left[X_{34}^{0}\right]$ there are vertices $a \in A, z \in Z, w \in W_{1}$ and $x \in X_{34}^{N}$ such that $a z, w x \in E(G)$ and aw, ax $\notin E(G)$.
Proof: We have $N(Z) \subseteq X_{34}^{N} \cup X_{34}^{T} \cup\left\{v_{3}, v_{4}\right\} \cup T_{3} \cup T_{4} \cup A$. Since $G$ has no clique cutset there are two non-adjacent vertices $u, v \in N(Z)$. By (xvii) and Theorem 3.2, and since $T_{3} \cup T_{4}$ is a clique, we have $u \in A$ and consequently $v \in X_{34}^{N} \cup X_{34}^{T}$, and by (xii) $v \in X_{34}^{N}$. So $v$ has a neighbor $w \in W_{1}$, and $u w \notin E(G)$ by xil.

Suppose that $X_{34}^{N}=\emptyset$. Then $X_{34}^{0}=\emptyset$ by xviii), and $W_{1}^{N}=\emptyset$ by (iv) and (xi). So $X_{34}=X_{34}^{T}$ and $W_{1}=W_{1}^{T}$. Now $[A, V(G) \backslash A]$ is complete by

Theorem 3.2 (xiil) and xvil), and since $G$ has no universal vertex it follows that $A=\emptyset$. So $V(G)=C \cup T_{1} \cup \cdots \cup T_{5} \cup W_{1}^{T} \cup X_{12} \cup X_{34}^{T}$. Let:

$$
\begin{aligned}
Q_{i} & =\left\{v_{i}\right\} \cup T_{i} \text { for each } i \in\{1,4,5\} . \\
Q_{2} & =\left\{v \mid v \text { is universal in } G\left[\left\{v_{2}\right\} \cup T_{2} \cup X_{12} \cup W_{1}\right]\right\} . \\
R_{2} & =\left(\left\{v_{2}\right\} \cup T_{2} \cup X_{12} \cup W_{1}\right) \backslash Q_{2} . \\
Q_{3} & =\left\{v \mid v \text { is universal in } G\left[\left\{v_{3}\right\} \cup T_{3} \cup X_{34}\right]\right\} . \\
R_{3} & =\left(\left\{v_{3}\right\} \cup T_{3} \cup X_{34}\right) \backslash Q_{3} .
\end{aligned}
$$

Hence $V(G)=Q_{1} \cup \cdots \cup Q_{5} \cup R_{2} \cup R_{3}$. We claim that $Q_{2} \neq \emptyset$. Indeed, if $W_{1}=\emptyset$ then $v_{2} \in Q_{2}$. So suppose that $W_{1} \neq \emptyset$. By (iv) and (xiii) the set $Y_{34}=\left\{x \in X_{34} \mid x\right.$ has a neighbor in $\left.W_{1}\right\}$ is non-empty and is a clique. Since $X_{34}^{N} \cup X_{34}^{0}=\emptyset$, every vertex of $Y_{34}$ has a neighbor in $T_{2}$, and it follows that some vertex $t$ in $T_{2}$ is complete to $Y_{34}$ (otherwise there are vertices $y^{\prime}, y^{\prime \prime} \in Y_{34}$ and $t^{\prime}, t^{\prime \prime} \in T_{2}$ that induce a $C_{4}$ ). Let us verify that $t \in Q_{2}$. We know that $t$ is complete to $T_{2} \backslash t$ by Theorem 3.2. Any $w \in W_{1}$ has a neighbor $x \in X_{34}$ by (iv), and so $t w \in E(G)$ for otherwise $\left\{t, x, w, v_{1}\right\}$ induces a $C_{4}$. Now consider any $y \in X_{12}$. Pick any $w \in W_{1}$ and $x \in X_{34} \cap N(w)$. Then $t y \in E(G)$, for otherwise $y-v_{2}-t-x-v_{4}-v_{5}$ is a $P_{6}$. So $t \in Q_{2}$, and the claim that $Q_{2} \neq \emptyset$ is established. Now the properties of the nine sets $Q_{1}, \ldots, Q_{5}, R_{2}, R_{3}$ satisfy all the axioms of the belt. We make this more precise as follows:

- By Theorem 3.2 and by (ii), we know that $Q_{1}, Q_{4}$ and $Q_{5}$ are non-empty cliques, $\left[Q_{1} \cup Q_{4}, Q_{5}\right]$ is complete and $\left[Q_{1}, Q_{4}\right]=\emptyset$.
- Clearly $Q_{2}$ and $Q_{3}$ are cliques, with $v_{3} \in Q_{3}$, and $Q_{2} \neq \emptyset$ as seen above.
- By (il), (vii) and Theorem[3.2, $\left[Q_{1}, Q_{2} \cup R_{2}\right]$ and $\left[Q_{4}, Q_{3} \cup R_{3}\right]$ are complete.
- By the definition of $Q_{2}$ and $Q_{3}$, Theorem 3.2 , and since $\left[X_{12} \cup X_{34}, T_{5}\right]=\emptyset$, we have $\left[Q_{2}, Q_{4} \cup Q_{5}\right]=\emptyset$ and $\left[Q_{3}, Q_{1} \cup Q_{5}\right]=\emptyset$.
- By the definition of $Q_{2}, Q_{3}, R_{2}$ and $R_{3}$, we have: for each $j \in 2,3,\left[Q_{j}, R_{j}\right]$ is complete, every vertex in $R_{j}$ has a non-neighbor in $R_{j}$, every vertex in $Q_{2} \cup R_{2}$ has a neighbor in $Q_{3} \cup R_{3}$ (by (iv) and (xiv)), and every vertex in $Q_{3} \cup R_{3}$ has a neighbor in $Q_{2} \cup R_{2}$ (by the definition of $\left.X_{34}^{T}\right)$ ).

Thus $G$ is a belt.
Therefore we may assume that $X_{34}^{N} \neq \emptyset$. So, $W_{1} \neq \emptyset$. Then:
(xix) $X_{34}^{N} \cup X_{34}^{T} \cup T_{3} \cup T_{4}$ is a clique.

Proof: By viiii) and by Theorem 3.2, it is enough to show that $X_{34}^{N} \cup X_{34}^{T}$ is a clique. Suppose that there are non-adjacent vertices $x, x^{\prime} \in X_{34}^{N} \cup X_{34}^{T}$. Pick any $y \in X_{34}^{N}$. By xiiil), $y \notin\left\{x, x^{\prime}\right\}$ and $y x, y x^{\prime} \in E(G)$, and $x, x^{\prime} \in$ $X_{34}^{T}$. So $x$ has a neighbor $t \in T_{2}$, and $x^{\prime}$ has a neighbor $t^{\prime} \in T_{2}$. Then $\left\{y, x, x^{\prime}, t, t^{\prime}\right\}$ induces a cycle of length either 4 (if $t=t^{\prime}$ ) or 5 and not dominating (because $v_{5}$ has no neighbor in it), a contradiction.
(xx) $G\left[X_{34}^{0}\right]$ is chordal.

Proof: If $G\left[X_{34}\right]$ contains a hole $C$, then $C$ either has length 4 or at least 6 or is a non-dominating $C_{5}$ (because of $v_{5}$ ).
(xxi) For every simplicial vertex $s$ of $G\left[X_{34}^{0}\right]$, there are vertices $a \in A$ and $u \in X_{34}^{0} \cup X_{34}^{N}$ with $s a, s u \in E(G)$ and $a u \notin E(G)$.

Proof: Let $Z$ be the vertex-set of the component of $G\left[X_{34}^{0}\right]$ that contains $s$. So $N_{Z}(s)$ is a clique. We have $N(s) \subseteq N_{Z}(s) \cup(N(Z) \backslash A) \cup A$. By xviil the set $N_{Z}(s) \cup(N(Z) \backslash A)$ is a clique. Since $G$ has no clique cutset there are two non-adjacent vertices $u, v$ in $N(s)$, and so $u \in N_{Z}(s) \cup(N(Z) \backslash A)$ and $v \in A$. Since $N(Z) \backslash A \subseteq X_{34}^{N} \cup X_{34}^{T} \cup\left\{v_{3}, v_{4}\right\} \cup T_{3} \cup T_{4}$ and $A$ is complete to $X_{34}^{T} \cup\left\{v_{3}, v_{4}\right\} \cup T_{3} \cup T_{4}$ by Theorem 3.2 and (xii), we have $u \in X_{34}^{0} \cup X_{34}^{N}$.

Let $A_{0}=\left\{a \in A \mid a\right.$ has a neighbor in $X_{34}^{0}$ and a non-neighbor in $\left.X_{34}^{N}\right\}$. By xviii) we have $A_{0} \neq \emptyset$. Since $A_{0}$ and $X_{34}^{N}$ are cliques (by Theorem 3.2 and (xix)) and by the third item of Lemma [2.3, there is a vertex $x_{0}$ in $X_{34}^{N}$ that is anticomplete to $A_{0}$. Let $w_{0}$ be a neighbor of $x_{0}$ in $W_{1}$. Then $w_{0}$ is anticomplete to $A_{0}$ by (xi).
(xxii) $x_{0}$ is complete to $X_{34}^{0}$.

Proof: If there is a vertex $x \in X_{34}^{0}$ that is non-adjacent to $x_{0}$, then $v_{2}-v_{1}-$ $w_{0}-x_{0}-v_{4}-x$ is a $P_{6}$.
(xxiii) $G\left[X_{34}^{0} \cup A_{0}\right]$ is chordal.

Proof: If $G\left[X_{34}^{0} \cup A_{0}\right]$ contains a hole $C$, then $C$ either has length 4 or at least 6 or is a non-dominating $C_{5}$ (because of $w_{0}$ ).
(xxiv) Every vertex in $X_{34}^{0}$ has a neighbor in $A_{0}$.

Proof: Let $Z$ be the vertex-set of any component of $G\left[X_{34}^{0}\right]$, and let $Z_{A}=$ $\left\{z \in Z \mid z\right.$ has a neighbor in $\left.A_{0}\right\}$, and suppose that $Z \neq Z_{A}$. By xxiiil and Lemma 2.2 applied to $G\left[Z \cup A_{0}\right], Z$ and $A_{0}$, some simplicial vertex $s^{\prime}$ of $G[Z]$ has no neighbor in $A_{0}$. Let $S=\left\{s^{\prime \prime} \in Z \mid N_{Z}\left[s^{\prime \prime}\right]=N_{Z}\left[s^{\prime}\right]\right\}$; so the vertices in $S$ are simplicial in $G[Z]$ and pairwise clones, and $S$ is a clique. Let $s$ be a vertex in $S$ with the smallest number of neighbors in $A$. If $s$ has any neighbor $a \in A_{0}$, then, since $\left\{S, A_{0}\right\}$ is a graded pair of cliques, by Lemma 2.3 all vertices in $S$ are adjacent to $a$, a contradiction. So $s$ has no neighbor in $A_{0}$. By (xxil) there are vertices $b \in A$ and $u \in Z \cup X_{34}^{N}$ with $s b, s u \in E(G)$ and $b u \notin E(G)$. We know that $b \notin A_{0}$, so $b$ is complete to $X_{34}^{N}$, and so $u \in Z$. Moreover $u \notin S$, for otherwise the choice of $s$ is contradicted (since $b \in A$, and the pair $\{A, S\}$ is graded). Hence $u$ is not a simplicial vertex of $G[Z]$, and so it has a neighbor $v \in Z \backslash N[s]$. Consider any $a \in A_{0}$. We know that as $\notin E(G)$; then also $a u \notin E(G)$, for otherwise $\{a, b, s, u\}$ induces a $C_{4}$; and $a v \notin E(G)$, for otherwise $s-u-v-a-v_{1}-w_{0}$ is a $P_{6}$. Hence $\{s, u, v\}$ is anticomplete to $A_{0}$. Let $Y$ be the component of $G\left[Z \backslash Z_{A}\right]$ that contains $s, u, v$. Let $Z_{Y}=\left\{z \in Z_{A} \mid z\right.$ has a neighbor in $Y\}$, and let $A_{Y}=\left\{b^{\prime} \in A \mid b^{\prime}\right.$ has a neighbor in $\left.Y\right\}$. Note that $\left[A_{Y}, X_{34}^{N}\right]$ is complete. Since $Z_{A} \neq \emptyset$ and $G[Z]$ is connected, $Z_{Y} \neq \emptyset$. Then $\left[Y, Z_{Y}\right]$ is complete, for otherwise there are adjacent vertices $y, y^{\prime} \in Y$, a vertex $z \in Z_{Y}$, and a vertex $a \in A_{0}$ such that $y^{\prime}-y-z-a-v_{1}-w_{0}$ is a $P_{6}$. Then $Z_{Y}$ is a clique, for otherwise $\left\{s, v, z, z^{\prime}\right\}$ induces a $C_{4}$ for any two non-adjacent vertices $z, z^{\prime}$ in $Z_{Y}$. Moreover, for any $b^{\prime} \in A_{Y}$ and $z \in Z_{Y}$, we have $b^{\prime} z \in E(G)$, for otherwise $\left\{b^{\prime}, y, z, a\right\}$ induces a $C_{4}$ for any $y \in Y \cap N(b)$ and $a \in A_{0} \cap N(z)$. We have $N(Y) \subseteq Z_{Y} \cup A_{Y} \cup(N(Z) \backslash A)$, and by Theorem [3.2, items (xii) and (xviil) and the fact that $\left[A_{Y}, X_{34}^{N}\right]$ is complete, this set is a clique, a contradiction.
(xxv) $\left[X_{34}^{0}, X_{34}^{T}\right]$ is complete.

Proof: Suppose that some $z \in X_{34}^{0}$ and $x \in X_{34}^{T}$ are non-adjacent. By (xxiv) $x$ has a neighbor $a \in A_{0}$. Then, by (xiil), $\left\{a, x, x_{0}, z\right\}$ induces a $C_{4}$.
(xxvi) For any two components $Z, Z^{\prime}$ of $G\left[W_{1}\right]$, the sets $N(Z) \cap X_{34}$ and $N\left(Z^{\prime}\right) \cap$ $X_{34}$ are disjoint.
Proof: Otherwise $\left\{v_{1}, x, z, z^{\prime}\right\}$ induces a $C_{4}$ for some $z \in Z, z^{\prime} \in Z^{\prime}$ and $x \in N(Z) \cap N\left(Z^{\prime}\right) \cap X_{34}$.

Let:

$$
\begin{aligned}
Q & =\left\{v_{1}\right\} \cup T_{1}, \\
B & =\left\{v_{3}, v_{4}\right\} \cup T_{3} \cup T_{4} \cup X_{34}^{T} \cup X_{34}^{N}, \\
M & =\left\{v_{2}, v_{5}\right\} \cup T_{2} \cup T_{5} \cup X_{12} \cup W_{1}, \\
L & =X_{34}^{0} .
\end{aligned}
$$

We know that $A$ and $Q$ are cliques, and $B$ is a clique by (xix). Every vertex in $L$ has a neighbor in $A$ by xxiv, and every vertex in $M$ has a neighbor in $B$ by (iv) and xiv). The subgraph $G[L]$ is $\left(P_{4}, 2 P_{3}\right)$-free by Lemma 2.4, using $A_{0}$ in the role of $Y, x_{0}$ in the role of $c$, and $v_{1}$ and $w_{0}$, respectively, in the role of $c^{\prime}$ and $c^{\prime \prime}$. The subgraph $G[M]$ has at least three components because $\left\{v_{2}\right\} \cup X_{12}$, $\left\{v_{5}\right\} \cup T_{5}$ and $W_{1}$ are pairwise anticomplete to each other and non-empty, and $G[M]$ is $\left(P_{4}, 2 P_{3}\right)$-free by Lemma 2.4 using $B$ in the role of $Y, v_{1}$ in the role of $c$ and the fact that $G[M]$ is not connected. Hence the sets $Q, A, B, L, M$ form a partition of $V(G)$ that shows that $G$ is a boiler.

## 4 Additional properties of belts and boilers

Belts and boilers have some additional and useful properties that we give below.

### 4.1 Belts

Theorem 4.1 Let $G$ be a belt, with the same notation as in Section 1, Then:
(a) For each $j \in\{2,3\}$, any two non-adjacent vertices in $R_{j}$ have no common neighbor in $Q_{5-j}$.
(b) $\left[R_{2}, R_{3}\right]=\emptyset$.
(c) For each $j \in\{2,3\}$, every vertex of $Q_{j}$ that has a neighbor in $R_{5-j}$ is complete to $Q_{5-j}$.
(d) The graphs $G\left[R_{2}\right]$ and $G\left[R_{3}\right]$ are $\left(P_{4}, 2 P_{3}\right)$-free.

Proof. (国) If two non-adjacent vertices $r, r^{\prime} \in R_{2}$ have a common neighbor $v$ in $Q_{3}$, then $\left\{v_{1}, r, r^{\prime}, v\right\}$ induces a $C_{4}$.
(b) Suppose that any $r_{2} \in R_{2}$ and $r_{3} \in R_{3}$ are adjacent. By the definition of a belt, for each $j \in\{2,3\}$ the vertex $r_{j}$ has a non-neighbor $r_{j}^{\prime} \in R_{j}$. Then $r_{2} r_{3}^{\prime} \notin E(G)$, for otherwise $\left\{r_{2}, r_{3}^{\prime}, v_{4}, r_{3}\right\}$ induces a $C_{4}$, and similarly $r_{3} r_{2}^{\prime} \notin$ $E(G)$. Then $\left\{r_{2}^{\prime}, v_{1}, r_{2}, r_{3}, v_{4}, r_{3}^{\prime}\right\}$ induces a $P_{6}$ or $C_{6}$.
(c) Consider any $u \in Q_{3}$ which has a neighbor $r_{2} \in R_{2}$, and suppose that $u$ has a non-neighbor $v \in Q_{2}$. By the definition of a belt $r_{2}$ has a non-neighbor
$r_{2}^{\prime} \in R_{2}$. Then $u r_{2}^{\prime} \notin E(G)$, for otherwise $\left\{u, r_{2}^{\prime}, v_{1}, r_{2}\right\}$ induces a $C_{4}$. But then $r_{2}^{\prime}-v-r_{2}-u-v_{4}-v_{5}$ is a $P_{6}$. The proof is similar when $j=2$.
(d) Pick a vertex $q_{i} \in Q_{i}$ for each $i \in\{1,4,5\}$. Lemma 2.4, using vertices $q_{1}, q_{4}$ and $q_{5}$ in the role of $c, c^{\prime}$ and $c^{\prime \prime}$, implies that $G\left[R_{2}\right]$ is $\left(P_{4}, 2 P_{3}\right)$-free. The proof is similar for $G\left[R_{3}\right]$.

Note that Theorem 4.1 means that $\left(R_{2}, Q_{3}\right)$ and $\left(R_{3}, Q_{2}\right)$ are $\mathcal{C}$-pairs.

### 4.2 Boilers

Let $G$ be a boiler, with the same notation as in the definition. Since every vertex in $A$ has a non-neighbor in $B$, Lemma 2.3 implies that some vertex $b^{*}$ in $B$ is anticomplete to $A$. Let $m^{*}$ be any neighbor of $b^{*}$ in $M$. Then $m^{*}$ too is anticomplete to $A$ (for otherwise $\left\{m^{*}, a, b, b^{*}\right\}$ induces a $C_{4}$ for some $a \in A$ and $\left.b \in B_{1} \cup B_{2}\right)$. Pick a vertex $z \in Q$.

If $L$ is a clique, then $(A \cup M, B \cup L)$ is a $\mathcal{C}$-pair, so the structure of $G$ is completely determined by Theorem[2.1] and the fact that $Q$ is complete to $A \cup M$ and anticomplete to $B \cup L$.

Therefore let us assume that $L$ is not a clique. Let $U$ be the set of universal vertices of $L$. (Possibly $U=\emptyset$.) Let $A_{L}=\{a \in A \mid a$ has a neighbor in $L\}$ and $A_{L}^{\prime}=\{a \in A \mid a$ has a neighbor in $L \backslash U\}$.

Theorem 4.2 Let $G$ be a boiler, with the same notation as above, and assume that $L$ is not a clique. Then, up to a permutation of the set $\{3, \ldots, k\}$, there is an integer $j \in\{3, \ldots, k\}$ such that the following hold:
(i) For each $a \in A_{L} \backslash A_{L}^{\prime}$, there is an integer $i \in\{j, \ldots, k\}$ such that $a$ is complete to $M_{1} \cup B_{1} \cup \cdots \cup M_{i-1} \cup B_{i-1}$ and anticomplete to $M_{i} \cup B_{i} \cup$ $\cdots \cup M_{k} \cup B_{k} ;$
(ii) $A_{L}^{\prime}$ is complete to $(M \cup B) \backslash\left(M_{k} \cup B_{k}\right)$ and anticomplete to $M_{k} \cup B_{k}$;
(iii) $A \backslash A_{L}$ is complete to $M_{1} \cup B_{1} \cup \cdots \cup M_{j-1} \cup B_{j-1}$ and anticomplete to $M_{j} \cup B_{j} \cup \cdots \cup M_{k} \cup B_{k}$.

Proof. Since $A$ and $B$ are disjoint cliques and $G$ is $C_{4}$-free, $\left[A, B_{1} \cup B_{2}\right]$ is complete, and $b^{*}$ is anticomplete to $A$, Lemma 2.3 implies that there is a permutation of $\{3, . ., k\}$ such that for every vertex $a \in A$ there is an integer $i \in\{3, \ldots, k\}$ such that $a$ is complete to $M_{1} \cup B_{1} \cup \cdots \cup M_{i-1} \cup B_{i-1}$ and anticomplete to $M_{i} \cup B_{i} \cup \cdots \cup M_{k} \cup B_{k}$. We may assume that $b^{*} \in B_{k}$ and $m^{*} \in M_{k}$.

Let $J=\left\{i \in\{3, \ldots, k\} \mid\right.$ some vertex in $A$ is anticomplete to $\left.M_{i} \cup B_{i}\right\}$. By the preceding paragraph there is an integer $j$ such that $J=\{j, \ldots, k\}$. In particular this implies the validity of item (ii) of the lemma.

Now consider any vertex $a \in A_{L}^{\prime}$. So $a$ has a neighbor $x \in L \backslash U$, so $x$ has a non-neighbor $x^{\prime} \in L$, and by the definition of a boiler we have $a x^{\prime} \notin E(G)$. Suppose that $a$ is not complete to $M_{i} \cup B_{i}$ for some $i<k$, so $a$ is anticomplete to $M_{i} \cup B_{i}$, and pick any $m \in M_{i}$. Then $m-z-a-x-b^{*}-x^{\prime}$ is a $P_{6}$. So $a$ is complete to $(M \cup B) \backslash\left(M_{k} \cup B_{k}\right)$, which proves (iii).

Finally, consider any vertex $d \in A \backslash A_{L}$. So $d$ is anticomplete to $L$. Pick any $i \in J$ and $b \in B_{i}$. So there is a vertex $a \in A_{L}$ that is anticomplete to $B_{i} \cup M_{i}$. By the definition of $A_{L}$ the vertex $a$ has a neighbor $x \in L$. Then $d b$ is not an
edge, for otherwise $\{d, b, x, a\}$ induces a $C_{4}$. It follows that $d$ is anticomplete to $B_{i} \cup M_{i}$ which proves (iiii).

## 5 Bounding the chromatic number

In this section, we give a proof for Theorem 1.1 and Theorem 1.2 .
We say that a stable set of a graph $G$ is good if it meets every clique of size $\omega(G)$ in $G$; and that it is very good if it meets every (inclusionwise) maximal clique of $G$. Moreover, we say that a clique $K$ in $G$ is a $t$-clique of $G$ if $|K|=t$.

We will use the following theorem as a tool in proving Theorem 1.1
Theorem 5.1 Let $G$ be a graph such that every proper induced subgraph $G^{\prime}$ of $G$ satisfies $\chi\left(G^{\prime}\right) \leq\left\lceil\frac{5}{4} \omega\left(G^{\prime}\right)\right\rceil$. Suppose that one of the following occurs:
(i) $G$ has a vertex of degree at most $\left\lceil\frac{5}{4} \omega(G)\right\rceil-1$.
(ii) G has a (very) good stable set;
(iii) $G$ has a stable set $S$ such that $G \backslash S$ is perfect.
(iv) For some integer $t \geq 5$ the graph $G$ has $t$ stable sets $S_{1}, \ldots, S_{t}$ such that $\omega\left(G \backslash\left(S_{1} \cup \cdots \cup S_{t}\right)\right) \leq \omega(G)-(t-1)$.

Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. (ii) Suppose that $G$ has a vertex $u$ with $d(u) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil-1$. By the hypothesis we have $\chi(G \backslash u) \leq\left\lceil\frac{5}{4} \omega(G \backslash u)\right\rceil$. So we can take any $\chi(G \backslash u)$-coloring of $G \backslash u$ and extend it to a $\left\lceil\frac{5}{4} \omega(G)\right\rceil$-coloring of $G$, using for $u$ a (possibly new) color that does not appear in its neighborhood.
(iii) Suppose that $G$ has a (very) good stable set $S$. Then $\omega(G \backslash S)=\omega(G)-1$. By the hypothesis we have $\chi(G \backslash S) \leq\left\lceil\frac{5}{4} \omega(G \backslash S)\right\rceil=\left\lceil\frac{5}{4}(\omega(G)-1)\right\rceil \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil-$ 1. We can take any $\chi(G \backslash S)$-coloring of $G \backslash S$ and add $S$ as a new color class, and we obtain a coloring of $G$. Hence $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
(iiii) Suppose that $G$ has a stable set $S$ such that $G \backslash S$ is perfect. Then $\chi(G \backslash S)=\omega(G \backslash S) \leq \omega(G)$. We can take any $\chi(G \backslash S)$-coloring of $G \backslash S$ and add $S$ as a new color class. Hence $\chi(G) \leq \omega(G)+1 \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
(iv) Note that $\frac{t}{t-1} \leq \frac{5}{4}$ because $t \geq 5$. We take any $\chi\left(G \backslash\left(S_{1} \cup \cdots \cup S_{t}\right)\right)$ coloring of $G \backslash\left(S_{1} \cup \cdots \cup S_{t}\right)$ and use $S_{1}, \ldots, S_{t}$ as $t$ new colors and we get a coloring of $G$. Then $\chi(G) \leq \chi\left(G \backslash\left(S_{1} \cup \cdots \cup S_{t}\right)\right)+t \leq\left\lceil\frac{5}{4}(\omega(G)-(t-1))\right\rceil+t \leq$ $\left\lceil\frac{5}{4} \omega(G)\right\rceil$ because $\frac{t}{t-1} \leq \frac{5}{4}$.

### 5.1 Chromatic bound for blowups

We first note that by a result of Lovász [27, any blowup of a perfect graph is a perfect graph.

For any integer $t \geq 2$ we say that $G$ is a $t$-blowup of $H$ if $\left|Q_{u}\right|=t$ for all $u \in V(H)$. Remark that, for an integer $k$, a $k$-coloring of the $t$-blowup of $H$ is equivalent to a collection of $k$ stable sets of $H$ such that every vertex of $H$ belongs to at least $t$ of them.

## Blowups of Petersen graph

Let $H_{1}$ be the Petersen graph as shown in Figure 2,

Lemma 5.1 Let $G$ be the 2-blowup of the Petersen graph $H_{1}$. Then $\chi(G)=5$.
Proof. The five sets $\left\{a, b, w_{3}, w_{6}\right\},\left\{b, c, w_{1}, w_{4}\right\},\left\{a, c, w_{2}, w_{5}\right\},\left\{z, w_{1}, w_{3}, w_{5}\right\}$ and $\left\{z, w_{2}, w_{4}, w_{6}\right\}$ are five stable sets, and every vertex of $H_{1}$ belongs to two of them. As observed above this is equivalent to a 5 -coloring of $G$. This is optimal because $G$ has 20 vertices and every stable set in $G$ has size at most 4.

Theorem 5.2 If $G$ is any blowup of the Petersen graph $H_{1}$, then $\chi(G) \leq$ $\left\lceil\frac{5}{4} \omega(G)\right\rceil$.

Proof. Let $q=\omega(G)$. We prove the theorem by induction on $|V(G)|$. We may assume that $G$ is connected (otherwise we consider each component separately) and that $G$ is not a clique. Moreover, the theorem holds easily if $G$ is any induced subgraph of $H_{1}$. Now suppose that $G$ is not an induced subgraph of $H_{1}$. So there is $x \in V\left(H_{1}\right)$ such that $\left|Q_{x}\right| \geq 2$. Since $G$ is connected and not a clique there exists $y \in N_{H_{1}}(x)$ such that $Q_{y} \neq \emptyset$, and so $q \geq 3$. By Theorem [5.1 (iii) we may assume that $G$ has no good stable set.

Note that every maximal clique of $G$ consists of $Q_{u} \cup Q_{v}$ for some edge $u v \in E\left(H_{1}\right)$ with $Q_{u} \neq \emptyset$ and $Q_{v} \neq \emptyset$, and we denote it as $Q_{u v}$. We say that such a maximal clique is balanced if $\left|Q_{u}\right| \geq 2$ and $\left|Q_{v}\right| \geq 2$.

Suppose that every $q$-clique of $G$ is balanced. So $q \geq 4$. Let $X$ be a subset of $V(G)$ obtained by taking $\min \left\{2,\left|Q_{v}\right|\right\}$ vertices from $Q_{v}$ for each $v \in V\left(H_{1}\right)$. We claim that:

$$
\begin{equation*}
\omega(G \backslash X)=q-4 \tag{1}
\end{equation*}
$$

Proof: Consider any maximal clique $K$ in $G$. As observed above we have $K=$ $Q_{u} \cup Q_{v}$ for some edge $u v \in E(G)$ with $Q_{u} \neq \emptyset$ and $Q_{v} \neq \emptyset$. Suppose that $|K|=q$. The hypothesis that every $q$-clique is balanced implies that $X$ contains exactly four vertices from $K$, so $|K \backslash X|=|K|-4=q-4$. Now suppose that $|K| \leq q-1$. The definition of $X$ implies that either $|K| \geq 3$ and $X$ contains at least two vertices from $Q_{u}$ and one from $Q_{v}$, or vice-versa, or $|K|=2$ and $X$ contains one vertex from each of $Q_{u}, Q_{v}$, and in any case we have $|K \backslash X| \leq q-4$. Thus (1) holds.

By (11) and the induction hypothesis we have $\chi(G \backslash X) \leq\left\lceil\frac{5}{4} \omega(G \backslash X)\right\rceil=$ $\left\lceil\frac{5}{4}(q-4)\right\rceil=\left\lceil\frac{5}{4} q\right\rceil-5$. By Lemma 5.1 we know that $G[X]$ is 5 -colorable. We can take any $\chi(G \backslash X)$-coloring of $G \backslash X$ and use five new colors for the vertices of $X$, and we obtain a coloring of $G$. It follows that $\chi(G) \leq\left\lceil\frac{5}{4} q\right\rceil$ as desired.

Therefore we may assume that some $q$-clique of $G$ is not balanced, say, up to symmetry, the clique $Q_{z a}$, with $\left|Q_{z}\right| \geq q-1$ and $\left|Q_{a}\right| \leq 1$. So we also have $\left|Q_{b}\right| \leq 1$ and $\left|Q_{c}\right| \leq 1$.

Suppose that both $Q_{a w_{1}}$ and $Q_{a w_{4}}$ are $q$-cliques. So $\left|Q_{w_{1}}\right| \geq q-1$ and $\left|Q_{w_{4}}\right| \geq q-1$. This implies $\left|Q_{w_{j}}\right| \leq 1$ for each $j \in\{2,3,5,6\}$. It follows that each of the cliques $Q_{b w_{2}}, Q_{b w_{5}}, Q_{c w_{3}}, Q_{c w_{6}}, Q_{w_{2} w_{3}}, Q_{w_{5} w_{6}}$ has size at most 2, so they are not $q$-cliques. Then $\left\{z, w_{1}, w_{4}\right\}$ is a good stable set.

Therefore we may assume that one of $Q_{a w_{1}}$ and $Q_{a w_{4}}$ is not a $q$-clique. Likewise, one of $Q_{b w_{2}}$ and $Q_{b w_{5}}$ is not a $q$-clique, and one of $Q_{c w_{3}}$ and $Q_{c w_{6}}$ is not a $q$-clique. This implies, up to symmetry, that we have either: (a) each of $Q_{a w_{1}}, Q_{b w_{5}}, Q_{c w_{3}}$ is not a $q$-clique, or (b) each of $Q_{a w_{1}}, Q_{b w_{2}}, Q_{c w_{3}}$ is not a $q$-clique. In case (a), we see that $\left\{z, w_{2}, w_{4}, w_{6}\right\}$ is a good stable set of $G$. Hence assume that we are in case (b) and not in case (a), and so $Q_{b w_{5}}$ is a $q$-clique,
and so $\left|Q_{w_{5}}\right| \geq q-1$. Hence $\left|Q_{w_{4}}\right| \leq 1$ and $\left|Q_{w_{6}}\right| \leq 1$. It follows that $Q_{a w_{4}}$ and $Q_{c w_{6}}$ are cliques of size at most 2 , so they are not $q$-cliques. Now $Q_{a w_{4}}, Q_{b w_{2}}$, and $Q_{c w_{6}}$ are not $q$-cliques, so we are in a situation similar to case (a). This completes the proof.

We immediately have the following.
Corollary 5.1 If $G$ is any blowup of $C_{5}$, then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.

## Blowups of $F_{3}$

Consider the graph $F_{3}$ as shown in Figure 1
Lemma 5.2 Let $G$ be the 2-blowup of $F_{3}$. Then $\chi(G)=7$.
Proof. For each $v \in V\left(F_{3}\right)$ we call $v$ and $v^{\prime}$ the two vertices of $Q_{v}$ in $G$. The seven sets $\left\{x, v_{4}, v_{6}\right\},\left\{y, v_{2}, v_{6}^{\prime}\right\},\left\{z, v_{2}^{\prime}, v_{4}^{\prime}\right\},\left\{x^{\prime}, v_{5}\right\},\left\{y^{\prime}, v_{1}\right\},\left\{z^{\prime}, v_{3}\right\}$ and $\left\{v_{1}^{\prime}, v_{3}^{\prime}, v_{5}^{\prime}\right\}$ form a 7 -coloring of $G$. Hence $\chi(G) \leq 7$. On the other hand we see that $\chi\left(G\left[Q_{v_{1}} \cup Q_{v_{2}} \cup Q_{v_{3}} \cup Q_{y} \cup Q_{z}\right]\right) \geq 5$ since that subgraph has 10 vertices and no stable set of size 3, and consequently $\chi\left(G\left[Q_{x} \cup Q_{1} \cup Q_{2} \cup Q_{3} \cup Q_{y} \cup Q_{z}\right]\right) \geq 7$. Hence $\chi(G) \geq 7$.

We say that $G$ is a special blowup of $F_{3}$ if (up to symmetry) we have $\left|Q_{u}\right| \leq 1$ for each $u \in\left\{x, v_{4}, v_{5}, v_{6}\right\}$ and $\left|Q_{v}\right|=t$ for each $v \in\left\{v_{1}, v_{2}, v_{3}, y, z\right\}$, for some integer $t \geq 2$.

Lemma 5.3 Let $G$ be a special blowup of $F_{3}$. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. We prove the theorem by induction on $|V(G)|$. If $Q_{x} \cup Q_{v_{4}} \cup Q_{v_{5}} \cup Q_{v_{6}}=\emptyset$, then $G$ is a blowup of $C_{5}$, so the lemma holds by Corollary 5.1 Hence assume that $Q_{x} \cup Q_{v_{4}} \cup Q_{v_{5}} \cup Q_{v_{6}} \neq \emptyset$. It follows that $\omega(G)=2 t+1$. Let $X$ be a subset of $V(G)$ obtained by taking two vertices from $Q_{v}$ for each $v \in\left\{v_{1}, v_{2}, v_{3}, y, z\right\}$ and the set $Q_{x} \cup Q_{v_{4}} \cup Q_{v_{5}} \cup Q_{v_{6}}$. Then $\omega(G \backslash X)=2 t-4=\omega(G)-5$. In $F_{3}$ the six sets $\left\{v_{1}, v_{3}, v_{5}\right\},\left\{v_{2}, y\right\},\left\{v_{2}, z\right\},\left\{v_{1}, y\right\},\left\{v_{3}, z\right\}$ and $\left\{x, v_{4}, v_{6}\right\}$ are such that every vertex from $\left\{v_{1}, v_{2}, v_{3}, y, z\right\}$ belongs to two of them and every vertex from $\left\{x, v_{4}, v_{5}, v_{6}\right\}$ belongs to one of them; hence they are equivalent to a 6 -coloring of $G[X]$. We can take any $\chi(G \backslash X)$-coloring of $G \backslash X$ and use six new colors for $X$, and we obtain a coloring of $G$. Hence $\chi(G) \leq \chi(G \backslash X)+6 \leq$ $\left\lceil\frac{5}{4}(\omega(G)-5)\right\rceil+6=\left\lceil\frac{5}{4} \omega(G)-\frac{25}{4}\right\rceil+6 \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.

Theorem 5.3 If $G$ is any blowup of $F_{3}$, then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. Let $q=\omega(G)$. We prove the theorem by induction on $|V(G)|$. Obviously the theorem holds if $G$ is any induced subgraph of $F_{3}$. Now suppose that $G$ is not an induced subgraph of $F_{3}$. By Theorem5.1(iii) we may assume that $G$ has no good stable set.

Note that every maximal clique of $G$ consists of $Q_{u} \cup Q_{v} \cup Q_{w}$ for some triangle $\{u, v, w\}$ in $F_{3}$, and we denote it as $Q_{u v w}$. We say that such a maximal clique is balanced if $\left|Q_{u}\right| \geq 2,\left|Q_{v}\right| \geq 2$, and $\left|Q_{w}\right| \geq 2$.

Suppose that every $q$-clique of $G$ is balanced. Let $X$ be a subset of $V(G)$ obtained by taking $\min \left\{2,\left|Q_{v}\right|\right\}$ vertices from $Q_{v}$ for each $v \in V\left(F_{3}\right)$. The hypothesis that every $q$-clique is balanced implies that $X$ contains exactly six
vertices from each $q$-clique of $G$, so $\omega(G \backslash X)=\omega(G)-6$. By the induction hypothesis we have $\chi(G \backslash X) \leq\left\lceil\frac{5}{4} \omega(G \backslash X)\right\rceil=\left\lceil\frac{5}{4}(q-6)\right\rceil=\left\lceil\frac{5}{4} q-\frac{30}{4}\right\rceil \leq\left\lceil\frac{5}{4} q\right\rceil-7$. By Lemma 5.2 we know that $G[X]$ is 7 -colorable. We can take any $\chi(G \backslash X)$ coloring of $G \backslash X$ and use seven new colors for the vertices of $X$, and we obtain a coloring of $G$. It follows that $\chi(G) \leq\left\lceil\frac{5}{4} q\right\rceil$ as desired. Therefore we may assume that some $q$-clique of $G$ is not balanced.

For each $v \in V\left(F_{3}\right)$, let $R_{v}$ consist of one vertex from $Q_{v}$ if $Q_{v} \neq \emptyset$, otherwise let $R_{v}=\emptyset$. We claim that we may assume that:

$$
\begin{equation*}
\text { Each of } Q_{x}, Q_{y} \text { and } Q_{z} \text { is non-empty. } \tag{1}
\end{equation*}
$$

Proof: Suppose up to symmetry that $Q_{x}=\emptyset$. If also $Q_{v_{2}}=\emptyset$, then $G$ is a blowup of $F_{3} \backslash\left\{x, v_{2}\right\}$, which is a chordal graph, so $\chi(G)=\omega(G)$ and the theorem holds. Therefore $Q_{v_{2}} \neq \emptyset$. Likewise, $Q_{v_{1}} \neq \emptyset$ and $Q_{v_{3}} \neq \emptyset$. Since $R_{v_{1}} \cup R_{v_{3}} \cup R_{v_{5}}$ is not a good stable set, we have $Q_{v_{5}}=\emptyset$. Moreover, if $Q_{v_{4}} \cup Q_{v_{6}}=\emptyset$, then $G$ is a blowup of $C_{5}$, and the theorem holds by Corollary 5.1. So up to symmetry we may assume that $Q_{v_{4}} \neq \emptyset$. Now if $Q_{z}=\emptyset$, then $G$ is a blowup of $F_{3} \backslash\left\{x, z, v_{5}\right\}$, which is a chordal graph, so $\chi(G)=\omega(G)$ and the theorem holds. So suppose that $Q_{z} \neq \emptyset$. Then $R_{v_{2}} \cup R_{v_{4}} \cup R_{z}$ is a good stable set of $G$. Hence we may assume that (11) holds.

We claim that we may assume that:
Each of $Q_{x y z}, Q_{x y v_{3}}, Q_{y z v_{5}}, Q_{z x v_{1}}, Q_{x v_{1} v_{2}}, Q_{y v_{3} v_{4}}$ is a $q$-clique, and either $Q_{z v_{5} v_{6}}$ or $Q_{x v_{2} v_{3}}$ is a $q$-clique.

Proof: If two of $R_{v_{1}}, R_{v_{3}}, R_{v_{5}}$ are empty, say $R_{v_{1}} \cup R_{v_{3}}=\emptyset$, then $G$ is a blowup of $F_{3} \backslash\left\{v_{1}, v_{3}\right\}$, which is a chordal graph, so $\chi(G)=\omega(G)$. So at least two of $R_{v_{1}}, R_{v_{3}}, R_{v_{5}}$ are non-empty. Since $R_{v_{1}} \cup R_{v_{3}} \cup R_{v_{5}}$ is not a good stable set, there is a $q$-clique in $G \backslash\left(R_{v_{1}} \cup R_{v_{3}} \cup R_{v_{5}}\right)$, and this clique can only be $Q_{x y z}$. Now consider the stable set $R_{x 46}=R_{x} \cup R_{v_{4}} \cup R_{v_{6}}$, which is not empty by (11). Since it is not a good stable set, there is a $q$-clique in $G \backslash R_{x 46}$, and so $Q_{y z v_{5}}$ is a $q$-clique. Likewise, $Q_{x y v_{3}}$ and $Q_{z x v_{1}}$ are $q$-cliques. Now consider the stable set $R_{x} \cup R_{v_{5}}$. Since it is not a good stable set, we deduce that one of $Q_{y v_{3} v_{4}}$ and $Q_{z v_{6} v_{1}}$ is a $q$-clique. Likewise, one of $Q_{z v_{5} v_{6}}$ and $Q_{x v_{2} v_{3}}$ is a $q$-clique, and one of $Q_{x v_{1} v_{2}}$ and $Q_{y v_{4} v_{5}}$ is a $q$-clique. Up to symmetry this yields the possibilities described in (2). Thus we may assume that (2) holds.

Next we claim that we may assume that:

$$
\begin{equation*}
Q_{z v_{5} v_{6}} \text { is not a } q \text {-clique. } \tag{3}
\end{equation*}
$$

Proof: Suppose not.
First we show that we may assume that $\left|Q_{v_{1}}\right| \geq 2$. Suppose that $\left|Q_{v_{1}}\right|=\varepsilon \leq$ 1. Let $a=\left|Q_{v_{2}}\right|$ and $b=\left|Q_{x}\right|$. Since $Q_{x v_{1} v_{2}}$ is a $q$-clique, we have $a+b+\varepsilon=q$. Then, using the $q$-cliques given by (2), we deduce successively that $\left|Q_{z}\right|=a$, $\left|Q_{y}\right|=\varepsilon,\left|Q_{v_{5}}\right|=b,\left|Q_{v_{6}}\right|=\varepsilon,\left|Q_{v_{3}}\right|=a$, and $\left|Q_{v_{4}}\right|=b$. We have $\left|Q_{x v_{2} v_{3}}\right|=$ $b+2 a \leq q=a+b+\varepsilon$, so $a \leq \varepsilon$. Also we have $\left|Q_{y v_{4} v_{5}}\right|=2 b+\varepsilon \leq q=a+b+\varepsilon$, so $b \leq a$. Hence $b \leq a \leq \varepsilon \leq 1$, which means that $G$ is isomorphic to an induced subgraph of $F_{3}$, so the theorem holds. So we may assume that $\left|Q_{v_{1}}\right| \geq 2$. Likewise, we may assume that $\left|Q_{v_{3}}\right| \geq 2$, and $\left|Q_{v_{5}}\right| \geq 2$.

Next we may assume that $\left|Q_{x}\right| \geq 2$ (otherwise since $Q_{x y z}$ and $Q_{y z v_{5}}$ are $q$ cliques (by (2)), we have $\left|Q_{v_{5}}\right| \leq 1$, a contradiction). Likewise, we have $\left|Q_{y}\right| \geq 2$ and $\left|Q_{z}\right| \geq 2$.

Further, we may assume that $\left|Q_{v_{6}}\right| \geq 2$ (otherwise since by (21) and by our assumption, $Q_{y z v_{5}}$ and $Q_{z v_{5} v_{6}}$ are $q$-cliques, we have $\left|Q_{y}\right| \leq 1$, a contradiction). Likewise, we have $\left|Q_{v_{2}}\right| \geq 2$ and $\left|Q_{v_{4}}\right| \geq 2$.

Hence the above analysis shows that every $q$-clique in $G$ is balanced, and the theorem holds as above. Thus we may assume that (3) holds.

Now by (2) and (3), we may assume that $Q_{x v_{2} v_{3}}$ is a $q$-clique. Let $a=\left|Q_{v_{5}}\right|$, $b=\left|Q_{z}\right|$ and $t=\left|Q_{y}\right|$. Then by (2), $a+b+t=q$, and by using the $q$-cliques given by (2), we deduce successively that $\left|Q_{x}\right|=a,\left|Q_{v_{1}}\right|=t$ and $\left|Q_{v_{2}}\right|=b$. Then again by (21) and by our assumption, since $Q_{x v_{2} v_{3}}$ and $Q_{x y v_{3}}$ are $q$-cliques, we see that $\left|Q_{v_{3}}\right|=b=t$. So, $q=a+2 t$. Since $Q_{y v_{3} v_{4}}$ is a $q$-clique (by (2)), we have $\left|Q_{v_{4}}\right|=a$. Thus $\left|Q_{y v_{4} v_{5}}\right|=2 a+t \leq q=a+2 t$, so $a \leq t$. First suppose that $t \leq 1$. Then $a \leq 1$ and hence $q \leq 3$. This implies that, we may assume that $\left|Q_{v_{6}}\right| \leq 1$ (otherwise since $Q_{z v_{5} v_{6}}$ is not a $q$-clique (by (3)), $a+2 t>a+t+\left|Q_{v_{6}}\right|$, and hence $t \geq 2$ which is a contradiction.). Thus $G$ is an induced subgraph of $F_{3}$ and the theorem holds. So suppose that $t \geq 2$. Since some $q$-clique of $G$ is not balanced, there is a vertex $w \in\left\{x, v_{4}, v_{5}\right\}$ such that $\left|Q_{w}\right| \leq 1$. In any case, we have $a \leq 1$, and hence $q \leq 2 t+1$. Now $\left|Q_{v_{6} v_{1} z}\right|=\left|Q_{v_{6}}\right|+2 t \leq q \leq 2 t+1$, so $\left|Q_{v_{6}}\right| \leq 1$. Hence the above analysis shows that $G$ is a special blowup of $F_{3}$, so the theorem holds as a consequence of Lemma 5.3 .

## Blowups of $H_{2}, H_{3}, H_{4}$ and $H_{5}$

Let $H_{2}, \ldots, H_{5}$ be the graphs as shown in Figure 2
Theorem 5.4 Let $G$ be any blowup of $H_{2}$. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. By the definition of a blowup, $V(G)$ is partitioned into cliques $Q_{v}$, $v \in V\left(H_{2}\right)$. If $Q_{v} \neq \emptyset$ we call $v$ one vertex of $Q_{v}$, and if $\left|Q_{v}\right| \geq 2$ we call $v^{\prime}$ a second vertex of $Q_{v}$. We denote, e.g., the clique $Q_{a} \cup Q_{v_{1}} \cup Q_{v_{2}}$ by $Q_{a v_{1} v_{2}}$, etc. Let $q=\omega(G)$. We prove the theorem by induction on $|V(G)|$. By Theorem 5.1 we may assume that every vertex $x \in V(G)$ satisfies $d(x) \geq\left\lceil\frac{5}{4} q\right\rceil$ and that $G$ has no good stable set.

Suppose that $Q_{v_{1}} \cup Q_{v_{2}}=\emptyset$. If $Q_{b} \neq \emptyset$, then $\{b\}$ is a good stable set. If $Q_{b}=\emptyset$, then $G$ is a blowup of $C_{5}$, and the result follows from Corollary 5.1. Hence we may assume that $Q_{v_{1}} \cup Q_{v_{2}} \neq \emptyset$. Then both $Q_{v_{1}}$ and $Q_{v_{2}}$ are nonempty, for otherwise some vertex in $Q_{v_{1}} \cup Q_{v_{2}}$ is simplicial (and so has degree less than $q$ ). Since $N\left[v_{1}\right]$ is partitioned into the two cliques $Q_{v_{6}}$ and $Q_{a v_{1} v_{2}}$, and $d\left(v_{1}\right) \geq\left\lceil\frac{5}{4} q\right\rceil$, we deduce that $\left|Q_{v_{6}}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1 \geq 2$; and similarly (since $N\left[v_{1}\right\rceil$ is also partitioned into cliques $Q_{a v_{1} v_{6}}$ and $Q_{v_{2}}$ ) we have $\left|Q_{v_{2}}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1 \geq 2$. Likewise $\left|Q_{v_{3}}\right| \geq 2$ and $\left|Q_{v_{1}}\right| \geq 2$. By the same argument we may assume that both $Q_{v_{4}}$ and $Q_{v_{5}}$ are non-empty, and consequently $\left|Q_{v_{4}}\right| \geq 2$ and $\left|Q_{v_{5}}\right| \geq 2$.

If $Q_{c}=\emptyset$, then $G$ is a blow-up of $F_{3}$, and the theorem follows from Theorem 5.3. So we may assume that $\left|Q_{c}\right| \geq 1$. Then the set of maximal cliques of $G$ is $\left\{Q_{a v_{1} v_{6}}, Q_{a v_{1} v_{2}}, Q_{a v_{2} v_{3}}, Q_{b v_{3} v_{4}}, Q_{b v_{4} v_{5}}, Q_{b v_{5} v_{6}}, Q_{a b c v_{3}}, Q_{a b c v_{6}}\right\}$.

Suppose that $\left|Q_{c}\right| \geq 2$. Consider the five stable sets $\left\{v_{1}, v_{3}, v_{5}\right\},\left\{v_{2}, v_{4}, v_{6}\right\}$, $\left\{c, v_{1}^{\prime}, v_{5}^{\prime}\right\},\left\{c^{\prime}, v_{2}^{\prime}, v_{4}^{\prime}\right\}$, and $\left\{v_{3}^{\prime}, v_{6}^{\prime}\right\}$. Then every maximal clique of $G$ contains
four vertices from their union; so the result follows from Theorem 5.1 (iv) (with $t=5)$. Therefore let us assume that $\left|Q_{c}\right|=1$.

Suppose that both $Q_{a}$ and $Q_{b}$ are non-empty. Consider the six stable sets $\left\{v_{1}, v_{3}, v_{5}\right\},\left\{v_{2}, v_{4}, v_{6}\right\},\left\{v_{3}^{\prime}, v_{6}^{\prime}\right\},\left\{a, v_{5}^{\prime}\right\},\left\{b, v_{2}^{\prime}\right\}$ and $\left\{c, v_{1}^{\prime}, v_{4}^{\prime}\right\}$. Then every maximal clique of $G$ contains five vertices from their union; so the result follows from Theorem 5.1 (iv) (with $t=6$ ).

Therefore we may assume up to symmetry that $Q_{a}=\emptyset$. Note that $Q_{b c v_{3}}$ is not a $q$-clique of $G$, because $Q_{b v_{3} v_{4}}$ is a clique and $\left|Q_{v_{4}}\right|>\left|Q_{c}\right|$. Likewise, $Q_{b c v_{6}}$ is not a $q$-clique of $G$. Consider the five stable sets $\left\{v_{1}, v_{3}, v_{5}\right\},\left\{v_{2}, v_{4}, v_{6}\right\}$, $\left\{v_{2}^{\prime}, v_{4}^{\prime}\right\},\left\{v_{3}^{\prime}, v_{6}^{\prime}\right\}$ and $\left\{c, v_{1}^{\prime}, v_{5}^{\prime}\right\}$. Then every maximal clique of $G$ contains four vertices from their union, except for $Q_{b c v_{3}}$ and $Q_{b c v_{6}}$, which contain only three vertices from their union, but we know that these two are not $q$-cliques. It follows that $\omega(G \backslash X) \leq q-4$, so the result follows from Theorem 5.1 (iv).

Theorem 5.5 Let $G$ be any blowup of $H_{3}$. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. By the definition of a blowup, $V(G)$ is partitioned into nine cliques $Q_{i}$, $v_{i} \in V\left(H_{3}\right)$. If $Q_{i} \neq \emptyset$ we call $v_{i}$ one vertex of $Q_{i}$. Note that every maximal clique of $G$ consists of $Q_{u} \cup Q_{v} \cup Q_{w}$ for some triangle $\{u, v, w\}$ in $H_{3}$. If each of $Q_{1}, Q_{4}, Q_{7}$ is non-empty, then $\left\{v_{1}, v_{4}, v_{7}\right\}$ is a good stable set of $G$, and the result follows from Theorem 5.1 (iii). Hence we may assume that one of $Q_{1}, Q_{4}, Q_{7}$ is empty. Likewise we may assume that one of $Q_{2}, Q_{5}, Q_{8}$ is empty, and that one of $Q_{3}, Q_{6}, Q_{9}$ is empty. Up to symmetry and relabelling, this yields the following two cases.
(i) $Q_{i} \cup Q_{i+1}=\emptyset$ for some $i$. Then $G$ is a chordal graph, so $\chi(G)=\omega(G)$.
(ii) $Q_{i} \cup Q_{i+2} \cup Q_{i+4}=\emptyset$ for some $i$. Then $G$ is a blowup of $C_{5}$, and the result follows from Corollary 5.1.

Theorem 5.6 Let $G$ be a blowup of $H_{5}$. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. By the definition of a blowup, $V(G)$ is partitioned into ten cliques $Q_{v}$, $v \in V\left(H_{5}\right)$. Note that if $Q_{t_{i-1}} \cup Q_{t_{i+1}}=\emptyset$ for some $i$, then the vertices of $Q_{t_{i}}$ can be moved to $Q_{v_{i}}$, so we may assume in that case that $Q_{t_{i}}=\emptyset$ too. Let $q=\omega(G)$. We prove the theorem by induction on $|V(G)|$.

If $Q_{v_{i}} \cup Q_{t_{i}}=\emptyset$ for some $i$, then $G$ is a chordal graph (as it is a blowup of a chordal graph), so $\chi(G)=\omega(G)$. Hence let us assume that $Q_{v_{i}} \cup Q_{t_{i}} \neq \emptyset$ for all $i$. For each $i$ let $x_{i}=t_{i}$ if $Q_{t_{i}} \neq \emptyset$, else let $x_{i}=v_{i}$. In any case if $d\left(x_{i}\right)<\left\lceil\frac{5}{4} q\right\rceil$ then we can conclude using Theorem5.1(i) and induction. Hence assume that $d\left(x_{i}\right) \geq\left\lceil\frac{5}{4} q\right\rceil$ for all $i$. If $x_{i}=t_{i}$, then $N\left[x_{i}\right]$ is partitioned into the two sets $Q_{v_{i-1}}$ and $Q_{v_{i}} \cup Q_{t_{i}} \cup Q_{v_{i+1}}$, and the latter set is a clique (of size at most $q$ ), so the inequality $d\left(x_{i}\right) \geq\left\lceil\frac{5}{4} q\right\rceil$ implies $\left|Q_{v_{i-1}}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1 \geq 2$. Similarly $\left|Q_{v_{i+1}}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1 \geq 2$. On the other hand suppose that $x_{i}=v_{i}$ (i.e., $Q_{t_{i}}=\emptyset$ ). If $Q_{t_{i-2}} \neq \emptyset$ then the same argument implies $\left|Q_{v_{i-1}}\right| \geq 2$; while if $Q_{t_{i-2}}=\emptyset$, then, as observed above, we have $Q_{t_{i-1}}=\emptyset$, so the same argument (about $v_{i}$ ), implies $\left|Q_{v_{i-1}}\right| \geq 2$ again. Hence in all cases we have $\left|Q_{v_{j}}\right| \geq 2$ for all $j$.

For each $i$ let $u_{i}, v_{i}$ be two vertices in $Q_{v_{i}}$. Consider the five stable sets $\left\{u_{i}, v_{i+2}\right\}(i=1, \ldots, 5)$, and let $X$ be their union. Any maximal clique $K$ of $G$ is included in $Q_{v_{i}} \cup Q_{v_{i+1}}$ for some $i$, and so $K$ contains $u_{i}, v_{i}, u_{i+1}, v_{i+1}$. So $\omega(G \backslash X)=q-4$ and we can conclude using Theorem 5.1 (iv) (with $t=5$ ) and the induction hypothesis.

Theorem 5.7 Let $G$ be any blowup of $H_{4}$. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. By the definition of a blowup, $V(G)$ is partitioned into nine cliques $Q_{v}$, $v \in V\left(H_{4}\right)$. If $Q_{v} \neq \emptyset$ we call $v$ one vertex of $Q$. If $Q_{v_{5}} \cup Q_{v_{6}}=\emptyset$, then $G$ is a chordal graph, so $\chi(G)=\omega(G)$. Hence let us assume up to symmetry that $Q_{v_{5}} \neq \emptyset$. If $Q_{v_{1}}=\emptyset$, then $G$ is a blowup of $H_{5}$, so the result follows from Theorem5.6. Hence let us assume that $Q_{v_{1}} \neq \emptyset$. If $Q_{v_{3}}=\emptyset$, then $G$ is a blowup of $H_{5}$ again. Hence let us assume that $Q_{v_{3}} \neq \emptyset$. Now it is easy to see that $\left\{v_{1}, v_{3}, v_{5}\right\}$ is a good stable set, so the result follows from Theorem 5.1 (iii).

Blowups of $F_{k, \ell}$
Theorem 5.8 For integers $k, \ell \geq 0$, let $G$ be any blowup of $F_{k, \ell}$. Then $\chi(G) \leq$ $\left\lceil\frac{5}{4} \omega(G)\right\rceil$.

Proof. We use the same notation as in the definition of $F_{k, \ell}$. By the definition of a blowup $V(G)$ is partitioned into cliques $Q_{v}, v \in V\left(F_{k, \ell}\right)$, such that $\left[Q_{u}, Q_{v}\right]$ is complete if $u v \in E\left(F_{k, \ell}\right)$ and otherwise $\left[Q_{u}, Q_{v}\right]=\emptyset$. Let $Q_{A}=\bigcup_{i=0}^{k} Q_{a_{i}}$ and $Q_{B}=\bigcup_{j=0}^{\ell} Q_{b_{j}}$. Let $D=\bigcup_{v \in U \cup W} Q_{v}$. As a convention it is convenient, for any $u \in V\left(F_{k, \ell}\right)$ such that $Q_{u} \neq \emptyset$, to use the name $u$ for one vertex of $Q_{u}$; moreover if $\left|Q_{u}\right| \geq 2$ we call $u^{\prime}$ another vertex from $Q_{u}$, and if $\left|Q_{u}\right| \geq 3$ we call $u^{\prime \prime}$ a third vertex from $Q_{u}$. We denote, e.g., the clique $Q_{x} \cup Q_{y} \cup Q_{u_{i}}$ by $Q_{x y u_{i}}$, etc. Let $q=\omega(G)$. We prove the lemma by induction on $|V(G)|+k+\ell$. We may assume that $G$ does not satisfy any of the hypotheses (ii)-(iiii) of Theorem 5.1, for otherwise we can find a $\left\lceil\frac{5}{4} q\right\rceil$-coloring of $G$ using induction.

We remark that if $k>0$ and $Q_{u_{i}}=\emptyset$ for some $i \in\{1, \ldots, k\}$, then the vertices of $Q_{a_{i}}$ can be moved to $Q_{a_{0}}$, and so $G$ is a blowup of $F_{k-1, \ell}$ and the result holds by induction. Moreover, if $k>0$ and either $\left|Q_{a_{i}}\right| \leq\left\lceil\frac{q}{4}\right\rceil$ for some $i$, or $\left|Q_{y}\right| \leq\left\lceil\frac{q}{4}\right\rceil$, then, since $N\left[u_{i}\right]=Q_{a_{i}} \cup Q_{u_{i}} \cup Q_{x} \cup Q_{y}$ and $Q_{a_{i} u_{i} x}$ and $Q_{u_{i} x y}$ are cliques that contain $u_{i}$, we have $d\left(u_{i}\right) \leq q-1+\left\lceil\frac{q}{4}\right\rceil<\left\lceil\frac{5}{4} q\right\rceil$, so the result holds by Theorem 5.1 (ii). In summary, we may assume that:

$$
\begin{align*}
& \text { If } k>0 \text { then } Q_{u_{i}} \neq \emptyset \text { and }\left|Q_{a_{i}}\right|>\left\lceil\frac{q}{4}\right\rceil \text { for all } i \text {, and }\left|Q_{y}\right|>\left\lceil\frac{q}{4}\right\rceil \text {. }  \tag{1}\\
& \text { Also if } \ell>0 \text { then } Q_{w_{j}} \neq \emptyset \text { and }\left|Q_{b_{j}}\right|>\left\lceil\frac{q}{4}\right\rceil \text { for all } j \text { and }\left|Q_{x}\right|>\left\lceil\frac{q}{4}\right\rceil .
\end{align*}
$$

It follows from (11) that $k \leq 3$, for otherwise $\left|Q_{A}\right|>q$; and similarly $\ell \leq 3$. Moreover, if $\ell>0$ then $k \leq 2$, for otherwise $\left|Q_{A} \cup Q_{x}\right|>q$; and similarly if $k>0$ then $\ell \leq 2$. We assume up to symmetry that $k \leq \ell$. Consequently we have either $k=0$ and $\ell \leq 3$, or $k=1$ and $\ell \in\{1,2\}$, or $k=\ell=2$. In any case $k \leq 2$. If $k \leq 1$ and $\ell \leq 1$, then $G$ is a blowup of (an induced subgraph of) $H_{5}$, so the result follows from Theorem 5.6. So we may assume that $\ell \geq 2$. Consequently we have either $k=0$ and $\ell \in\{2,3\}$, or $k=1$ and $\ell=2$, or $k=\ell=2$.

Suppose that $Q_{A}=\emptyset$. Then $Q_{z}=\emptyset$, for otherwise $d(z) \leq q-1$, and $Q_{y}=\emptyset$, for otherwise $\{y\}$ is a good stable set. Then we can view $G$ as a blowup of $F_{0, \ell-1}$ (putting $Q_{b_{\ell}}$ and $Q_{w_{\ell}}$ in the role of $Q_{z}$ and $Q_{a_{0}}$ respectively) and use induction. Therefore we may assume that $Q_{A} \neq \emptyset$. If $k \geq 1$, then $\left|Q_{a_{1}}\right| \geq 2$ by (11), and if $k=0$ then $\left|Q_{a_{0}}\right| \geq 2$, for otherwise either $d(z) \leq q$ (if $Q_{z} \neq \emptyset$ ) or $d\left(a_{0}\right) \leq q$ (if $Q_{z}=\emptyset$ ). Hence in any case we have $\left|Q_{A}\right| \geq 2$. Let $a, a^{\prime}$ be two vertices from $Q_{A}$, chosen as follows: if $k=0$, let $a, a^{\prime} \in Q_{a_{0}}$. If $k=1$, let $a, a^{\prime} \in Q_{a_{1}}$. If $k=2$, let $a \in Q_{a_{1}}$ and $a^{\prime} \in Q_{a_{2}}$.

Let $p=\max \left\{\left|Q_{v}\right|, v \in U \cup W\right\}$. So $p \geq 1$. We claim that:
We may assume that $p \geq 2$.
Proof: Suppose that $p=1$; so $Q_{v}=\{v\}$ for all $v \in U \cup W$. If $\left|Q_{z}\right| \leq 1$, then $U \cup W \cup Q_{z}$ is a stable set, and $G \backslash\left(U \cup W \cup Q_{z}\right)$ is perfect (a blowup of $P_{4}$ ), so the result follows from Theorem 5.1 (iiii). Hence $\left|Q_{z}\right| \geq 2$. Define five stable sets as follows: Let $T_{1}=\left\{a, b_{1}\right\}, T_{2}=\left\{b_{2}, x\right\}, T_{3}=\left\{z, x^{\prime}\right\}, T_{4}=\left\{a^{\prime}, y\right\}$, and $T_{5}=\left\{z^{\prime}, y^{\prime}\right\}$, where $y, y^{\prime} \in Q_{y}$, with the convention that $y^{\prime}$ vanishes if $\left|Q_{y}\right|=1$, and in that case if $\left|Q_{x}\right| \geq 3$ then $T_{5}=\left\{z^{\prime}, x^{\prime \prime}\right\}$ for some $x^{\prime \prime} \in Q_{x} \backslash\left\{x, x^{\prime}\right\}$, and $y$ too vanishes if $Q_{y}=\emptyset$. Let $T^{*}=T_{1} \cup \cdots \cup T_{5}$. We claim that every maximal clique $K$ of $G$ satisfies $\left|K \backslash T^{*}\right| \leq q-4$. The following cases (i)-(vii) occur:
(i) $K=Q_{z} \cup Q_{A}$. Then $K$ contains four vertices $\left(z, z^{\prime}, a, a^{\prime}\right)$ from $T^{*}$, so $\left|K \backslash T^{*}\right| \leq q-4$. Likewise, if $K=Q_{z} \cup Q_{B}$, then $K$ contains $z, z^{\prime}, b_{1}, b_{2}$.
(ii) $K=Q_{x} \cup Q_{A}$. Then $K$ contains $a, a^{\prime}, x, x^{\prime}$ from $T^{*}$.
(iii) $K=Q_{y} \cup Q_{B}$. Then $Q_{y} \neq \emptyset$ because $Q_{B}$ is not a maximal clique (since $Q_{z} \neq \emptyset$ ). If $\left|Q_{y}\right| \geq 2$, then $K$ contains four vertices $b_{1}, b_{2}, y, y^{\prime}$ from $T^{*}$. If $\left|Q_{y}\right|=1$ then (since $\left|Q_{z}\right| \geq 2$ ) $|K|<\left|Q_{z} \cup Q_{A}\right| \leq q$, and $K$ contains three vertices $b_{1}, b_{2}, y$ from $T^{*}$, so $\left|K \backslash T^{*}\right| \leq q-4$.
(iv) $k \geq 1$ and $K=Q_{x y u_{i}}$ for some $i \in\{1, \ldots, k\}$. Then $Q_{y} \neq \emptyset$ because $Q_{x u_{i}}$ is not a maximal clique (since $Q_{a_{i}} \neq \emptyset$ ). If $\left|Q_{y}\right| \geq 2$, then $K$ contains four vertices $\left(x, y, x^{\prime}, y^{\prime}\right)$ from $T^{*}$. If $\left|Q_{y}\right|=1$, then (since $\left.\left|Q_{a_{i}}\right| \geq 2\right)|K|<\left|Q_{x u_{i} a_{i}}\right| \leq q$ and $K$ contains three vertices $x, x^{\prime}, y$ from $T^{*}$.
(v) $k \geq 1$ and $K=Q_{x a_{i} u_{i}}$ for some $i \in\{1, \ldots, k\}$, say $i=1$. If $k=1$ then $K$ contains $x, x^{\prime}, a, a^{\prime}$. Suppose $k=2$. Since $q \geq\left|Q_{x a_{1} a_{2}}\right|$, and $\left|Q_{a_{2}}\right| \geq 2$, we have $|K| \leq q-1$. Then $K$ contains three vertices $x, x^{\prime}, a$ from $T^{*}$, so $\left|K \backslash T^{*}\right| \leq q-4$. (vi) $K=Q_{x y w_{j}}$ for some $j \in\{1, \ldots, \ell\}$. If $\left|Q_{y}\right| \geq 2$ then $K$ contains four vertices ( $x, y, x^{\prime}, y^{\prime}$ ) from $T^{*}$. If $\left|Q_{y}\right| \leq 1$, then $K$ contains at least two vertices from $T^{*}$, so if $|K| \leq q-2$ we are done. If $|K| \geq q-1$, then $\left|Q_{x}\right|+2 \geq|K| \geq q-1 \geq$ $\left|Q_{z} \cup Q_{B}\right|-1 \geq 2(\ell+1)-1 \geq 5$, so $\left|Q_{x}\right| \geq 3$, so the vertex $x^{\prime \prime}$ exists and $K$ contains three vertices $x, x^{\prime}, x^{\prime \prime}$ from $T^{*}$.
(vii) $K=Q_{y b_{j} w_{j}}$ for some $j \in\{1, \ldots, \ell\}$. Since $q \geq\left|Q_{z} \cup Q_{B}\right|$, we have $\left|Q_{b_{j}}\right| \leq q-2 \ell$. If $\ell=3$, then either $|K| \leq q-4$, or $|K|=q-3$ and $Q_{y} \neq \emptyset$ and $K$ contains $y$ from $T^{*}$. Hence suppose that $\ell=2$. So $b_{j} \in K$. Then either $|K| \leq q-3$, or $|K|=q-2$ and $K$ also contains $y$ from $T^{*}$. So $\left|K \backslash T^{*}\right| \leq q-4$. In either case Theorem [5.1) (iv) implies the desired result. Thus (2) holds.

Suppose that $k \leq 1$. We know that $\ell \in\{2,3\}$. By (11) we have $\left|Q_{b_{j}}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1$ for all $j \in\{1, \ldots, \ell\}$. Recall that $Q_{A} \neq \emptyset$. Let $a^{*}=a_{0}$ if $Q_{a_{0}} \neq \emptyset$ and $a^{*}=a_{1}$ otherwise. In either case the set $N\left(a^{*}\right)$ can be partitioned into two cliques such that $Q_{z}$ is one of them. By Theorem 5.1(1i) we may assume that $d(a) \geq\left\lceil\frac{5}{4} q\right\rceil$, so $\left|Q_{z}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1$. Consequently $q \geq\left|Q_{z} \cup Q_{B}\right| \geq(\ell+1)\left(\left\lceil\frac{q}{4}\right\rceil+1\right)$. The inequality $q \geq(\ell+1)\left(\left\lceil\frac{q}{4}\right\rceil+1\right)$ is violated if $\ell \geq 3$, so $\ell=2$. Moreover, the inequality with $\ell=2$ implies $q \geq 12$. Hence (11) yields that $\left|Q_{x}\right| \geq 3$, and $\left|Q_{b_{j}}\right| \geq 3$ for each $j \in\{1,2\}$, and $\left|Q_{A}\right| \geq 3$, and similarly $\left|Q_{z}\right| \geq 3$.

Suppose that $k=0$. We may assume that $Q_{x y w_{j}}$ is a $q$-clique for each $j \in$ $\{1,2\}$, for otherwise the set $\left\{a_{0}, b_{j}, w_{3-j}\right\}$ is a good stable set. Hence $\left|Q_{w_{1}}\right|=$ $\left|Q_{w_{2}}\right|=p$. Note that the set of maximal cliques of $G$ is $\left\{Q_{z a_{0}}, Q_{x a_{0}}, Q_{x y w_{1}}\right.$, $\left.Q_{x y w_{2}}, Q_{y w_{1} b_{1}}, Q_{y w_{2} b_{2}}, Q_{z b_{0} b_{1} b_{2}}\right\}$ plus $Q_{y b_{0} b_{1} b_{2}}$ if $Q_{y} \neq \emptyset$. Let $S_{1}=\left\{b_{1}, w_{2}, a_{0}\right\}$, $S_{2}=\left\{b_{2}, w_{1}, a_{0}^{\prime}\right\}, S_{3}=\left\{z, w_{1}^{\prime}, w_{2}^{\prime}\right\}$, and $S_{4}=\left\{b_{1}^{\prime}, x\right\}$. If $Q_{y} \neq \emptyset$, let $S_{5}=$ $\left\{a_{0}^{\prime \prime}, y\right\}$. If $Q_{y}=\emptyset$, then one of $Q_{w_{1} b_{1}}, Q_{w_{2} b_{2}}$ is a $q$-clique, for otherwise $\{x, z\}$ is
a good stable set; so for some $j \in\{1,2\}$ we have $\left|Q_{w_{j} b_{j}}\right|=q \geq\left|Q_{b_{1} b_{2}}\right|$, whence $p=\left|Q_{w_{j}}\right| \geq\left|Q_{b_{3-j}}\right| \geq 3$; so we let $S_{5}=\left\{a_{0}^{\prime \prime}, w_{1}^{\prime \prime}, w_{2}^{\prime \prime}\right\}$. In either case, $S_{1}, \ldots, S_{5}$ are stable sets and it is easy to see that every maximal clique of $G$ contains at least four vertices from their union, so the result follows from Theorem 5.1 (iv).

Now suppose that $k=1$, and so $\ell=2$. By (11), we have $\left|Q_{y}\right| \geq 2$. Note that the set of maximal cliques of $G$ is $\left\{Q_{z a_{0} a_{1}}, Q_{x a_{0} a_{1}}, Q_{x a_{1} u_{1}}, Q_{x y u_{1}}, Q_{x y w_{1}}, Q_{x y w_{2}}\right.$, $\left.Q_{y w_{1} b_{1}}, Q_{y w_{2} b_{2}}, Q_{y b_{0} b_{1} b_{2}}, Q_{z b_{0} b_{1} b_{2}}\right\}$. Let $S_{1}=\left\{b_{1}, w_{2}, u_{1}\right\}$ plus $a_{0}$ if $Q_{a_{0}} \neq \emptyset$. Let $S_{2}=\left\{b_{2}, w_{1}, a_{1}\right\}, S_{3}=\{x, z\}, S_{4}=\left\{y^{\prime}, z^{\prime}\right\}$, and $S_{5}=\left\{a_{1}^{\prime}, y\right\}$. In either case, $S_{1}, \ldots, S_{5}$ are stable sets and that every maximal clique of $G$ contains at least four vertices from their union, so the result follows from Theorem 5.1(iv).

Finally suppose that $k=2$ and $\ell=2$. Let $S_{1}=\left\{a_{1}, b_{1}, u_{2}, w_{2}\right\}, S_{2}=$ $\left\{a_{2}, b_{2}, u_{1}, w_{1}\right\}, S_{3}=\left\{x, b_{1}^{\prime}\right\}, S_{4}=\left\{y, a_{1}^{\prime}\right\}$, and let $S_{5}$ consist of one vertex from each component of $Q_{z} \cup\left(D \backslash\left\{u_{1}, u_{2}, w_{1}, w_{2}\right\}\right)$. Let $S^{*}=S_{1} \cup \cdots \cup S_{5}$. We claim that every maximal clique $K$ of $G$ satisfies $\left|K \backslash S^{*}\right| \leq q-4$. Indeed if $K=Q_{x} \cup Q_{A}$ then $K$ contains $x, a_{1}, a_{1}^{\prime}, a_{2}$ from $S^{*}$. If $K=Q_{z} \cup Q_{A}$ then $Q_{z} \neq \emptyset$ and $K$ contains $z, a_{1}, a_{1}^{\prime}, a_{2}$. If $K=Q_{x a_{1} u_{1}}$ then $K$ contains $x, a_{1}, a_{1}^{\prime}, u_{1}$. If $K=Q_{x a_{2} u_{2}}$ then $K$ contains $x, a_{2}, u_{2}$ from $S^{*}$, so if $|K| \leq q-1$ we are done; and if $|K|=q$ then $\left|Q_{x a_{2} u_{2}}\right|=q \geq\left|Q_{x a_{1} a_{2}}\right|$ so $\left|Q_{u_{2}}\right| \geq 2$, so $Q_{u_{2}}$ contains a vertex $u_{2}^{\prime}$ from $S_{5}$. If $K=Q_{x y u_{1}}$ then $K$ contains $x, y, u_{1}$ from $S^{*}$, so if $|K| \leq q-1$ we are done; and if $|K|=q$ then since $p \geq 2$ we have $\left|Q_{u_{1}}\right| \geq 2$, so $Q_{u_{1}}$ contains a vertex $u_{1}^{\prime}$ from $S^{*}$. The other cases are symmetric. Hence the result follows from Theorem 5.1 (iv). This completes the proof.

### 5.2 Chromatic bound for bands, belts and boilers

Theorem 5.9 Let $G$ be a band. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. We use the same notation as in the definition of a band (see also Figure $4(\mathrm{~b})$ ), and we prove the theorem by induction on $|V(G)|$. First suppose that $\left[R_{2}, R_{3}\right]$ is not complete. By Lemma 2.3 there exist non-adjacent vertices $u \in R_{2}$ and $v \in R_{3}$ such that every maximal clique in $G\left[R_{2} \cup R_{3}\right]$ contains $u$ or $v$. If $Q_{5} \neq \emptyset$, pick any $w \in Q_{5}$ and let $S=\{u, v, w\}$; else let $S=\{u, v\}$. Then $S$ is a very good stable set of $G$, so the result follows from Theorem 5.1 (iii). Therefore we may assume that $\left[R_{2}, R_{3}\right]$ is complete. Now suppose that $\left[Q_{1}, Q_{2}\right]$ is not complete. By Lemma 2.3 there exist non-adjacent vertices $u \in Q_{1}$ and $v \in Q_{2}$ such that every maximal clique in $G\left[Q_{1} \cup Q_{2}\right]$ contains $u$ or $v$. If $Q_{4} \neq \emptyset$, pick any $w \in Q_{4}$ and let $S=\{u, v, w\}$; else let $S=\{u, v\}$. Then $S$ is a very good stable set of $G$, so the result follows from Theorem 5.1 (iii). Therefore we may assume that $\left[Q_{1}, Q_{2}\right]$ is complete, and similarly that $\left[Q_{3}, Q_{4}\right]$ is complete. Now $G$ is a blowup of $C_{5}$, so the result follows from Corollary 5.1.

We say that a graph $G$ is an extended $\mathcal{C}$-pair if $V(G)$ can be partitioned into three sets $Q, X, A$ such that $(X, A)$ is a $\mathcal{C}$-pair, $Q$ is a clique, $[Q, X]$ is complete and $[Q, A]=\emptyset$.

Lemma 5.4 Let $G$ be an extended $\mathcal{C}$-pair. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. We prove the lemma by induction on $|V(G)|$. Let $V(G)$ be partitioned into $Q, X, A$ as in the definition above. Let $q=\omega(G)$. If some vertex $a \in A$ has no neighbor in $X$, then $a$ is simplicial, so $d(a)<q$, and we can conclude using

Theorem 5.1(ii) and by the induction hypothesis. Therefore we may assume that every vertex in $A$ has a neighbor in $X$.

Suppose that $G[X]$ has four pairwise non-adjacent simplicial vertices $s_{1}, s_{2}$, $s_{3}, s_{4}$. If $d\left(s_{i}\right) \leq\left\lceil\frac{5}{4} q\right\rceil-1$, then we can conclude using Theorem(5.1(ii). So assume that $d\left(s_{i}\right) \geq\left\lceil\frac{5}{4} q\right\rceil$. We have $N\left(s_{i}\right)=Q \cup N_{X}\left(s_{i}\right) \cup N_{A}\left(s_{i}\right)$, and $Q \cup N_{X}\left(s_{i}\right)$ is a clique, so we must have $\left|N_{A}\left(s_{i}\right)\right| \geq\left\lceil\frac{q}{4}\right\rceil+1$. By the definition of a $\mathcal{C}$-pair the sets $N_{A}\left(s_{1}\right), \ldots, N_{A}\left(s_{4}\right)$ are pairwise disjoint. It follows that $|A| \geq 4\left(\left\lceil\frac{q}{4}\right\rceil+1\right)>q$, a contradiction. Hence $G[X]$ has at most three pairwise non-adjacent simplicial vertices. If $X$ is a clique then $G$ is a chordal graph, so $\chi(G)=\omega(G)$ and the theorem holds trivially. Therefore we may assume that $G[X]$ has exactly $k$ pairwise non-adjacent simplicial vertices with $k \in\{2,3\}$. Since $G[X] \in \mathcal{C}$ and by Lemma 2.5, we have the following two cases (a) and (b).
(a) $k=2$, so $X$ is partitioned into three cliques $X_{1}, X_{2}$ and $U$ such that $X_{1}, X_{2}$ are non-empty, $\left[U, X_{1} \cup X_{2}\right]$ is complete and $\left[X_{1}, X_{2}\right]=\emptyset$. Suppose that $U \neq \emptyset$. Then Theorem 2.1 and the fact that every vertex in $A$ has a neighbor in $X$ implies that some vertex $u$ in $U$ is universal in $G$, so $\{u\}$ is a very good stable set and we conclude using Theorem 5.1(iii). Hence $U=\emptyset$. Then $G$ is a band, and we conclude with Theorem 5.9 .


Figure 7: Schematic representation of the graph in Case (b) of Lemma 5.4 where $U=\emptyset$. Here, each shaded circle represents a clique, and the circles inside the oval form a clique, a solid line between two circles indicates that the two sets are complete to each other, the absence of line between any two circles indicates that the sets are anticomplete to each other, and a dashed line between two circles indicates that the adjacency between the two sets are arbitrary.
(b) $k=3$, so $X$ is partitioned into five cliques $X_{1}, X_{2}, X_{3}, W$ and $U$ such that $X_{1}, X_{2}, X_{3}$ are non-empty and pairwise anticomplete, [ $W, X_{1} \cup X_{2}$ ] is complete, $\left[W, X_{3}\right]=\emptyset$, and $[U, X \backslash U]$ is complete. As in case (a) we may assume that $U=\emptyset$. By Theorem 2.1 and the fact that every vertex in $A$ has a neighbor in $X$, the set $A$ is partitioned into four sets $A_{1}, A_{2}, A_{3}, B$ such that $N_{A}\left(X_{i}\right)=A_{i}$ for each $i \in\{1,2,3\}, N_{A}(W)=A_{1} \cup A_{2} \cup B$, and $\left[W, A_{1} \cup A_{2}\right]$ is complete, and there is no other edge between $X$ and $A$. Moreover, if one of $\left[X_{j}, A_{j}\right](j \in\{1,2,3\})$ is not complete, then $\left[X_{t}, A_{t}\right]$ is complete for each $t \in\{1,2,3\} \backslash\{j\}$. See Figure 7 .

Suppose that $B \neq \emptyset$. Since every vertex of $A$ has a neighbor in $X$, every vertex of $B$ has a neighbor in $W$. So by Lemma 2.3 there exists a vertex $w \in W$ such that $[w, B]$ is complete. Hence $w$ is universal in $G\left[V(G) \backslash\left(X_{3} \cup A_{3}\right)\right]$. We may assume that $\left[X_{3}, A_{3}\right]$ is not complete (otherwise $\left\{w, x_{3}\right\}$, for any $x_{3} \in X_{3}$, is a very good stable set of $G$, and we can conclude by using Theorem 5.1). Then by Lemma 2.3, there exist non-adjacent vertices $x_{3} \in X_{3}$ and $a_{3} \in A_{3}$ such that every maximal clique in $G\left[X_{3} \cup A_{3}\right]$ contains $x_{3}$ or $a_{3}$. Then $\left\{w, x_{3}, a_{3}\right\}$ is a very good stable set of $G$, and we can conclude by using Theorem 5.1. So we
may assume that $B=\emptyset$.
Suppose that $\left[X_{1}, A_{1}\right]$ is not complete. Then, as remarked earlier, $\left[X_{2}, A_{2}\right]$ and $\left[X_{3}, A_{3}\right]$ are complete. Also by Lemma 2.3, there exist non-adjacent vertices $x_{1} \in X_{1}$ and $a_{1} \in A_{1}$ such that every maximal clique in $G\left[X_{1} \cup A_{1}\right]$ contains $x_{1}$ or $a_{1}$. Pick any $x_{2} \in X_{2}$ and $x_{3} \in X_{3}$. Then $\left\{a_{1}, x_{1}, x_{2}, x_{3}\right\}$ is a very good stable set of $G$, and we can conclude by using Theorem 5.1. Therefore assume that $\left[X_{1}, A_{1}\right]$ is complete, and, similarly, that $\left[X_{2}, A_{2}\right]$ is complete.

Suppose that $\left[X_{3}, A_{3}\right]$ is not complete. Then by Lemma 2.3, there are nonadjacent vertices $x_{3} \in X_{3}$ and $a_{3} \in A_{3}$ such that every maximal clique in $G\left[X_{3} \cup A_{3}\right]$ contains $x_{3}$ or $a_{3}$. If $W \neq \emptyset$, then any $w \in W$ is universal in $G\left[V(G) \backslash\left(X_{3} \cup A_{3}\right)\right]$. But now $\left\{w, x_{3}, a_{3}\right\}$ is a very good stable set of $G$, and we can conclude by using Theorem 5.1 So $W=\emptyset$. Now pick any $x_{1} \in X_{1}$ and $x_{2} \in X_{2}$. Then $\left\{x_{1}, x_{2}, x_{3}, a_{3}\right\}$ is a very good stable set of $G$, and we can conclude by Theorem 5.1 Therefore assume that $\left[X_{3}, A_{3}\right]$ is complete.

Now $G$ is a blowup of $F_{2,0}$ (with $A_{1} \cup A_{2}$ is the role of $Q_{A}$, and $X_{3}$ in the role of $Q_{B}$, and $A_{3}$ in the role of $Q_{z}$, and $Q$ in the role of $Q_{y}$, and $W$ in the role of $Q_{x}$, and $X_{1}, X_{2}$ in the role of $Q_{u_{1}}, Q_{u_{2}}$ ), so we can conclude using Theorem 5.8. This completes the proof.

Theorem 5.10 Let $G$ be a belt. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. We use the same notation as in the definition of a belt, and we will also use the properties listed in Theorem 4.1. We prove the theorem by induction on $\omega(G)$. If $\omega(G)=2$ then $G$ is a $C_{5}$ and the theorem holds obviously. Now assume that $\omega(G) \geq 3$. Let $q=\omega(G)$.

Suppose that both $R_{2}, R_{3}$ are non-empty. Recall from Theorem 4.1 that $G\left[R_{2}\right]$ is $\left(P_{4}, C_{4}, 2 P_{3}\right)$-free, hence chordal. Moreover, the axiom that $G\left[R_{2}\right]$ has no universal vertex implies that $R_{2}$ is not a clique, so it has two non-adjacent simplicial vertices $r_{1}, r_{2}$. For each $h \in\{1,2\}$ let $X_{h}$ be the closed neighborhood of $r_{h}$ in $R_{2}$; so $X_{h}$ is a clique. Let $Y_{h}=N\left(r_{h}\right) \cap Q_{3}$. If $d\left(r_{h}\right)<\left\lceil\frac{5}{4} q\right\rceil$ then we can conclude using Theorem5.1(ii) and induction. Hence assume that $d\left(r_{h}\right) \geq\left\lceil\frac{5}{4} q\right\rceil$ for each $h \in\{1,2\}$. By the definition of a belt, we have $N\left[r_{h}\right]=Q_{1} \cup Q_{2} \cup X_{h} \cup Y_{h}$, and $Q_{1} \cup Q_{2} \cup X_{h}$ is a clique, so we must have $\left|Y_{h}\right| \geq\left\lceil\frac{q}{4}\right\rceil+1$. By Theorem 4.1((a)), the sets $Y_{1}, Y_{2}$ are pairwise disjoint. By the same argument $G\left[R_{3}\right]$ has two non-adjacent simplicial vertices and consequently there are two disjoint subsets $Z_{1}, Z_{2}$ of $Q_{2}$ with size at least $\left\lceil\frac{q}{4}\right\rceil+1$. By Theorem4.1(c) the set $Y_{1} \cup Y_{2} \cup Z_{1} \cup Z_{2}$ is a clique, and its size is strictly larger than $q$, a contradiction.

Therefore we may assume that $R_{3}=\emptyset$. Let $X=Q_{2} \cup R_{2} \cup Q_{5}$ and $A=$ $Q_{3} \cup Q_{4}$. Then the partition of $V(G)$ into $Q_{1}, X$ and $A$ shows that $G$ is an extended $\mathcal{C}$-pair, so the result follows from Lemma 5.4.

Theorem 5.11 Let $G$ be a boiler. Then $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$.
Proof. We use the same definition as in the definition of a boiler. Let $q=\omega(G)$. By Theorem[5.1] we may assume that every vertex in $G$ has degree at least $\left\lceil\frac{5}{4} q\right\rceil$. If $L$ is a clique, then the partition of $V(G)$ into $Q, M \cup A$ and $L \cup B$ shows that $G$ is an extended $\mathcal{C}$-pair, so the result follows from Lemma 5.4. Therefore assume that $L$ is not a clique. By the same argument as in the proof of Theorem 5.10 implies that there are two disjoint subsets $A_{1}, A_{2}$ of $A$ of size at least $\left\lceil\frac{q}{4}\right\rceil+1$. By
the same argument applied to $G\left[M_{1} \cup M_{2}\right]$, there are two disjoint sets $Y_{1} \subseteq B_{1}$ and $Y_{2} \subseteq B_{2}$ of size at least $\left\lceil\frac{q}{4}\right\rceil+1$. Then $A_{1} \cup A_{2} \cup B_{1} \cup B_{2}$ is a clique, with size strictly larger than $q$, a contradiction.

### 5.3 Chromatic bounds for $\left(P_{6}, C_{4}\right)$-free graphs

Proof of Theorem 1.1. Let $G$ be any $\left(P_{6}, C_{4}\right)$-free graph. We prove the theorem by induction on $|V(G)|$.

If $G$ has a universal vertex $u$, then $\omega(G)=\omega(G \backslash u)+1$, and by the induction hypothesis we have $\chi(G)=\chi(G \backslash u)+1 \leq\left\lceil\frac{5}{4}(\omega(G \backslash u)\rceil+1\right.$, which implies $\chi(G) \leq\left\lceil\frac{5}{4}(\omega(G)\rceil\right.$.

If $G$ has a clique cutset $K$, let $A, B$ be a partition of $V(G) \backslash K$ such that both $A, B$ are non-empty and $[A, B]=\emptyset$. Clearly $\chi(G)=\max \{\chi(G[K \cup A]), \chi(G[K \cup$ $B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

Finally, if $G$ has no universal vertex and no clique cutset, then the result follows from Theorem 1.5 and Theorems 5.2 5.11

Next we prove Theorem 1.2 by using the following theorem.
Theorem 5.12 ([20]) If a graph $G$ satisfies $\chi(G) \leq\left\lceil\frac{5}{4} \omega(G)\right\rceil$, then it satisfies $\chi(G) \leq\left\lceil\frac{\Delta(G)+\omega(G)+1}{2}\right\rceil$.

Proof of Theorem 1.2. This follows from Theorems 1.1 and 5.12

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