Square-free graphs with no six-vertex induced path

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Abstract

We elucidate the structure of (P_6, C_4) -free graphs by showing that every such graph either has a clique cutset, or a universal vertex, or belongs to several special classes of graphs. Using this result, we show that for any (P_6, C_4) -free graph G, $\lceil \frac{5\omega(G)}{4} \rceil$ and $\lceil \frac{\Delta(G)+\omega(G)+1}{2} \rceil$ are tight upper bounds for the chromatic number of G. Moreover, our structural results imply that every (P_6, C_4) -free graph with no clique cutset has bounded clique-width, and thus the existence of a polynomial-time algorithm that computes the chromatic number (or stability number) of any (P_6, C_4) -free graph.

Keywords: Square-free graphs; P_6 -free graphs; Chromatic number; χ -boundedness; Clique size; Degree.

1 Introduction

All our graphs are finite and have no loops or multiple edges. For any integer k, a k-coloring of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfy $c(u) \neq c(v)$. A graph is k-colorable if it admits a k-coloring. The chromatic number $\chi(G)$ of a graph G is the smallest integer k such that G is k-colorable. In general, determining whether a graph is k-colorable or not is well-known to be NP-complete for every fixed $k \geq 3$. Thus designing algorithms for computing the chromatic number by putting restrictions on the input graph and obtaining bounds for the chromatic number are of interest.

A clique in a graph G is a set of pairwise adjacent vertices. Let $\omega(G)$ denote the maximum clique size in a graph G. Clearly $\chi(H) \geq \omega(H)$ for every induced subgraph H of G. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. The existence of triangle-free graphs with aribtrarily large chromatic number shows that for general graphs the chromatic number cannot be upper bounded by a function of the clique number. However, for restricted classes of graphs such a function may exist. Gyárfás [19] called such classes of graphs χ -bounded classes. A family of graphs \mathcal{G} is χ -bounded with

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 χ -bounding function f if, for every induced subgraph H of $G \in \mathcal{G}$, $\chi(H) \leq f(\omega(H))$. For instance, the class of perfect graphs is χ -bounded with $f(\omega) = \omega$.

Given a family of graphs \mathcal{F} , a graph G is \mathcal{F} -free if no induced subgraph of G is isomorphic to a member of \mathcal{F} ; when \mathcal{F} has only one element F we say that G is F-free. Several classes of graphs defined by forbidding certain families of graphs were shown to be χ -bounded: even-hole-free graphs [1]; odd-hole-free graphs [34]; quasi-line graphs [10]; claw-free graphs with stability number at least 3 [13]; see also [6, 8, 12, 22, 24] for more instances.

For any integer ℓ we let P_{ℓ} denote the path on ℓ vertices and C_{ℓ} denote the cycle on ℓ vertices. A cycle on 4 vertices is referred to as a square. It is well known that every P_4 -free graph is perfect. Gyárfás [19] showed that the class of P_k -free graphs is χ -bounded. Gravier et al. [18] improved Gyárfás's bound slightly by showing that every P_k -free graph G satisfies $\chi(G) \leq (k-2)^{\omega(G)-1}$. In particular every P_6 -free graph G satisfies $\chi(G) \leq 4^{\omega(G)-1}$. Improving this exponential bound seems to be a difficult open problem. In fact the problem of determining whether the class of P_5 -free graphs admits a polynomial χ -bounding function remains open, and the known χ -bounding function f for such class of graphs satisfies $c(\omega^2/\log w) \leq f(\omega) \leq 2^{\omega}$ [23]. So the recent focus is on obtaining (linear) χ -bounding functions for some classes of P_t -free graphs, where $t \geq 5$. It is shown in [8] that every (P_5, C_4) -free graph G satisfies $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$, and in [7] that every $(P_2 \cup P_3, C_4)$ -free graph G satisfies $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$. Gaspers and Huang [14] studied the class of (P_6, C_4) -free graphs (which generalizes the class of (P_5, C_4) -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and the class of $(P_2 \cup P_3, C_4)$ -free graphs and showed that every such graph G satisfies $\chi(G) \leq \frac{3\omega(G)}{2}$. We improve their result and establish the best possible bound, as follows.

Theorem 1.1 Let G be any (P_6, C_4) -free graph. Then $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$. Moreover, this bound is tight.

The degree of a vertex in G is the number of vertices adjacent to it. The maximum degree over all vertices in G is denoted by $\Delta(G)$. For any graph G, we have $\chi(G) \leq \Delta(G) + 1$. Brooks [5] showed that if G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$. Reed [33] conjectured that every graph G satisfies $\chi(G) \leq \lceil \frac{\Delta(G) + \omega(G) + 1}{2} \rceil$. Despite several partial results [25, 31, 33], Reed's conjecture is still open in general, even for triangle-free graphs. Using Theorem 1.1, we will show that Reed's conjecture holds for the class of (P_6, C_4) -free graphs:

Theorem 1.2 If G is a (P_6, C_4) -free graph, then $\chi(G) \leq \lceil \frac{\Delta(G) + \omega(G) + 1}{2} \rceil$.

One can readily see that the bounds in Theorem 1.1 and in Theorem 1.2 are tight on the following example. Let G be a graph whose vertex-set is partitioned into five cliques Q_1, \ldots, Q_5 such that for each $i \mod 5$, every vertex in Q_i is adjacent to every vertex in $Q_{i+1} \cup Q_{i-1}$ and to no vertex in $Q_{i+2} \cup Q_{i-2}$, and $|Q_i| = q$ for all $i \ (q > 0)$. Clearly $\omega(G) = 2q$ and $\Delta(G) = 3q - 1$. Since G has no stable set of size 3, G is P_6 -free and $\chi(G) \ge \lceil \frac{5q}{2} \rceil$. Moreover, since no two non-adjacent vertices in G has a common neighbor in G, we also see that G is C_4 -free.

Finally, we also have the following result.

Theorem 1.3 There is a polynomial-time algorithm which computes the chromatic number of any (P_6, C_4) -free graph. The proof of Theorem 1.3 is based on the concept of clique-width of a graph G, which was defined in [9] as the minimum number of labels which are necessary to generate G using a certain type of operations. (We omit the details.) It is known from [26, 32] that if a class of graphs has bounded clique-width, then there is a polynomial-time algorithm that computes the chromatic number of every graph in this class. We are able to prove that every (P_6, C_4) -free graph that has no clique cutset has clique-width at most 36, which implies the validity of Theorem 1.3. However a similar result, using similar techniques, was proved by Gaspers, Huang and Paulusma [15]. Hence we refer to [15], or to the extended version of our manuscript [21] for the detailed proof of Theorem 1.3.

We finish on this theme by noting that the class of (P_6, C_4) -free graph itself does not have bounded clique-width, since the class of split graphs (which are all (P_6, C_4) -free) does not have bounded clique-width [2, 29]. The clique-width argument might also be used for solving other optimization problems in (P_6, C_4) free graphs, in particular the stability number. However this problem was solved earlier by Mosca [30], and the weighted version was solved in [4], and both algorithms have reasonably low complexity.

Theorems 1.1 and 1.2 will be derived from the structural theorem below (Theorem 1.4). Before stating it we recall some definitions.

In a graph G, the neighborhood of a vertex x is the set $N_G(x) = \{y \in$ $V(G) \setminus x \mid xy \in E(G)$; we drop the subscript G when there is no ambiguity. The closed neighborhood is the set $N[x] = N(x) \cup \{x\}$. Two vertices x, y are clones if N[x] = N[y]. For any $x \in V(G)$ and $A \subseteq V(G) \setminus x$, we let $N_A(x) = N(x) \cap A$. For any two subsets X and Y of V(G), we denote by [X, Y], the set of edges that has one end in X and other end in Y. We say that X is *complete* to Y or [X, Y] is complete if every vertex in X is adjacent to every vertex in Y; and X is anticomplete to Y if $[X, Y] = \emptyset$. If X is singleton, say $\{v\}$, we simply write v is complete (anticomplete) to Y instead of writing $\{v\}$ is complete (anticomplete) to Y. If $S \subseteq V(G)$, then G[S] denote the subgraph induced by S in G. A vertex is universal if it is adjacent to all other vertices. A stable set is a set of pairwise non-adjacent vertices. A *clique-cutset* of a graph G is a clique K in G such that $G \setminus K$ has more connected components than G. A matching is a set of pairwise non-adjacent edges. The union of two vertex-disjoint graphs G and H is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The union of k copies of the same graph G will be denoted by kG; for example $2P_3$ denotes the graph that consists in two disjoint copies of P_3 .

A vertex is *simplicial* if its neighborhood is a clique. It is easy to see that in any graph G that has a simplicial vertex, letting S denote the set of simplicial vertices, every component of G[S] is a clique, and any two adjacent simplicial vertices are clones.

A hole is an induced cycle of length at least 4. A graph is *chordal* if it contains no hole as an induced subgraph. Chordal graphs have many interesting properties (see e.g. [17]), in particular: every chordal graph has a simplicial vertex; every chordal graph that is not a clique has a clique-cutset; and every chordal graph that is not a clique has two non-adjacent simplicial vertices.

In a graph G, let A, B be disjoint subsets of V(G). It is easy to see that the following two conditions (i) and (ii) are equivalent: (i) any two vertices $a, a' \in A$ satisfy either $N_B(a) \subseteq N_B(a')$ or $N_B(a') \subseteq N_B(a)$; (ii) any two vertices $b, b' \in B$

satisfy either $N_A(b) \subseteq N_A(b')$ or $N_A(b') \subseteq N_A(b)$. If this condition holds we say that the pair $\{A, B\}$ is graded. Clearly in a C_4 -free graph any two disjoint cliques form a graded pair. See also Lemma 2.3 below.

Some special graphs Let F_1, F_2, F_3 be three graphs (as in [14]), as shown in Figure 1.



Figure 1: F_1 , F_2 , F_3

Let H_1, H_2, H_3, H_4, H_5 be five graphs, as shown in Figure 2, where H_1 is the Petersen graph.



Figure 2: H_1 , H_2 , H_3 , H_4 , H_5



Figure 3: (a) Schematic representation of the graph $F_{k,l}$. Here, the vertices in a shaded box form a clique, and an edge between a vertex and a box indicates that the vertex is adjacent to all the vertices in the box. For example, the vertex x is adjacent to all the vertices in the boxes A, U, and W. (b) $F_{2,2}$.

Graphs $F_{k,\ell}$ For integers $k, \ell \ge 0$ let $F_{k,\ell}$ be the graph whose vertex-set can be partitioned into sets A, B, U, W and $\{x, y, z\}$ such that:

- $A = \{a_0, a_1, \dots, a_k\}$ is a clique of size k + 1, and $U = \{u_1, u_2, \dots, u_k\}$ is a stable set of size k, and the edges between A and U form a matching of size k, namely, $[A, U] = \{a_i u_i \mid i \in \{1, \dots, k\}\};$
- $B = \{b_0, b_1, \dots, b_\ell\}$ is a clique of size $\ell + 1$, and $W = \{w_1, \dots, w_\ell\}$ is a stable set of size ℓ , and the edges between B and W form a matching of size ℓ , namely, $[B, W] = \{b_j w_j \mid j \in \{1, \dots, \ell\}\};$
- The neighborhood of x is $A \cup U \cup W \cup \{y\}$;
- The neighborhood of y is $B \cup U \cup W \cup \{x\}$;
- The neighborhood of z is $A \cup B$.

See Figure 3 for the schematic representation of the graph $F_{k,l}$ and for the graph $F_{2,2}$.

Blowups A blowup of a graph H is any graph G such that V(G) can be partitioned into |V(H)| (not necessarily non-empty) cliques $Q_v, v \in V(H)$, such that $[Q_u, Q_v]$ is complete if $uv \in E(H)$, and $[Q_u, Q_v] = \emptyset$ if $uv \notin E(H)$. See Figure 4:(a) for a blowup of a C_5 .



Figure 4: Schematic representations of: (a) a blowup of a C_5 , (b) a band, and (c) a belt. In (a), (b) and (c), the circles represent a collection of sets into which the vertex set of the graph is partitioned. Each shaded circle represents a nonempty clique, a solid line between two circles indicates that the two sets are complete to each other, and the absence of a line between two circles indicates that the two sets are anticomplete to each other. In (b), a dotted line between two circles means that the respective pair of sets is graded. For example, the pair $\{Q_3, Q_4\}$ is graded. In (c), the dashed lines between the sets R_2, R_3, Q_2 and Q_3 mean that the adjacency between these sets are subject to the fourth item of the definition of a belt.

Bands A *band* is any graph G (see Figure 4:(b)) whose vertex-set can be partitioned into seven sets $Q_1, \ldots, Q_5, R_2, R_3$ such that:

- Each of $Q_1, \ldots, Q_5, R_2, R_3$ is a clique.
- The sets $[Q_5, Q_1 \cup Q_4]$, $[R_2, Q_1 \cup Q_2 \cup Q_3]$, $[R_3, Q_2 \cup Q_3 \cup Q_4]$ and $[Q_2, Q_3]$ are complete.
- The sets $[Q_1, Q_3 \cup R_3 \cup Q_4]$, $[Q_4, Q_1 \cup Q_2 \cup R_2]$ and $[Q_5, Q_2 \cup R_2 \cup Q_3 \cup R_3]$ are empty.
- The pairs $\{Q_1, Q_2\}, \{Q_3, Q_4\}$ and $\{R_2, R_3\}$ are graded.



Figure 5: Partial structure of a boiler. Here, each shaded circle represents a nonempty clique, and ovals labelled M and B represents the union of the sets represented by the circles inside that oval. The sets in oval B forms a clique, and the ovals M and L induces a $(P_4, 2P_3)$ -free graph. A solid line between two shapes indicates that the respective sets are complete to each other. The absence of a line between any two shapes indicates that the respective sets are anticomplete to each other. A dashed line between any two shapes means that the adjacency between these sets are subject to the definition of a boiler.

Belts A *belt* is any (P_6, C_4, C_6) -free graph G (see Figure 4:(c)) whose vertexset can be partitioned into seven sets $Q_1, \ldots, Q_5, R_2, R_3$ such that:

- Each of Q_1, \ldots, Q_5 is a clique.
- The sets $[Q_1, Q_2 \cup R_2 \cup Q_5]$ and $[Q_4, Q_3 \cup R_3 \cup Q_5]$ are complete.
- The sets $[Q_1, Q_3 \cup R_3 \cup Q_4]$, $[Q_4, Q_2 \cup R_2 \cup Q_1]$, $[Q_5, Q_2 \cup R_2 \cup Q_3 \cup R_3]$ are empty.
- For each $j \in \{2,3\}$, $[Q_j, R_j]$ is complete, every vertex in $Q_j \cup R_j$ has a neighbor in $Q_{5-j} \cup R_{5-j}$, and no vertex of R_j is universal in $G[R_j]$.

Boilers A *boiler* is a (P_6, C_4, C_6) -free graph G whose vertex-set can be partitioned into five sets Q, A, B, L, M such that:

- The sets Q, A, B and M are non-empty, and Q, A and B are cliques.
- The sets [Q, A], [Q, M], and [B, L] are complete.
- The sets [Q, B], [Q, L] and [L, M] are empty.
- G[L] and G[M] are $(P_4, 2P_3)$ -free.
- Every vertex in L has a neighbor in A.
- For some integer $k \geq 3$, M is partitioned into k non-empty sets M_1, \ldots, M_k , pairwise anticomplete, and B is partitioned into k non-empty sets B_1, \ldots, B_k , such that for each $i \in \{1, \ldots, k\}$ every vertex in M_i has a neighbor in B_i and no neighbor in $B \setminus B_i$; and every vertex in B has a neighbor in M.

• $[A, M_1 \cup B_1 \cup M_2 \cup B_2]$ is complete, and for each $i \in \{3, \ldots, k\}$ every vertex in A is either complete or anticomplete to $M_i \cup B_i$, and no vertex in A is complete to B.

See Figure 5 for the partial structure of a boiler.

We consider that the definition of blowups (of certain fixed graphs) and of bands (using Lemma 2.3) is also a complete description of the structure of such graphs. However this is not so for belts and boilers. Such graphs have additional properties, and a description of their structure is given in Section 4.

Now we can state our main structural result. The existence of such a decomposition theorem was inspired to us by the results from [14] which go a long way in that direction.

Theorem 1.4 If G is any (P_6, C_4) -free graph, then one of the following holds:

- G has a clique cutset.
- G has a universal vertex.
- G is a blowup of either H_1, \ldots, H_5, F_3 or $F_{k,\ell}$ (for some $k, \ell \geq 1$).
- G is either a band, a belt, or a boiler.

Theorem 1.4 is derived from Theorem 1.5.

Theorem 1.5 Let G be a (P_6, C_4) -free graph that has no clique-cutset and no universal vertex. Then the following hold:

- 1. If G contains an F_3 , then G is a blowup of F_3 .
- 2. If G contains an F_1 and no F_3 , then G is a band.
- 3. If G is F₁-free, and G contains an induced C₆, then G is a blowup of one of the graphs H₁, H₂, H₃, H₄.
- 4. If G is C₆-free, and G contains an F_2 , then G is a blowup of either H_5 or $F_{k,\ell}$ for some integers $k, \ell \geq 1$.
- 5. If G contains no C_6 and no F_2 , and G contains a C_5 , then G is either a belt or a boiler.

Proof. The proof of each of these items is given below in Theorems 3.4, 3.5, 3.6, 3.7 and 3.8 respectively.

Proof of Theorem 1.4, assuming Theorem 1.5.

Let G be any (P_6, C_4) -free graph. If G is chordal, then either G is a complete graph (so it has a universal vertex) or G has a clique cutset. Now suppose that G is not chordal. Then it contains an induced cycle of length either 5 or 6. So it satisfies the hypothesis of one of the items of Theorem 1.5 and consequently it satisfies the conclusion of this item. This established Theorem 1.4. \Box

2 Classes of square-free graphs

In this section, we study some classes of square-free graphs and prove some useful lemmas and theorems that are needed for the later sections. We first note that any blowup of a P_6 -free chordal graph is P_6 -free chordal.

Lemma 2.1 In a chordal graph G, every non-simplicial vertex lies on a chordless path between two simplicial vertices.

Proof. Let x be a non-simplicial vertex in G, so it has two non-adjacent neighbors y, z. If both y, z are simplicial, then y-x-z is the desired path. Hence assume that y is non-simplicial. Since G is not a clique, it has two simplicial vertices, so it has a simplicial vertex s different from z. So $s \notin \{y, z\}$. In $G \setminus s$, the vertex x is non-simplicial, so, by induction, there is a chordless path $P = p_0 \cdot p_1 \cdots \cdot p_k$ in $G \setminus s$, with $k \ge 2$, such that p_0 and p_k are simplicial in $G \setminus s$ and $x = p_i$ for some $i \in \{1, \ldots, k-1\}$. If p_0 and p_k are simplicial in G, then P is the desired path. So suppose that p_0 is not simplicial in G, so $sp_0 \in E(G)$. Since s is simplicial in G we have $N_P(s) \subseteq \{p_0, p_1\}$. Then we see that either $s \cdot p_0 \cdot p_1 \cdots \cdot p_k$ or $s \cdot p_1 \cdots \cdot p_k$ is the desired path.

Lemma 2.2 In a chordal graph G, let X and A be disjoint subsets of V(G) such that A is a clique and every simplicial vertex of G[X] has a neighbor in A. Then every vertex in X has a neighbor in A.

Proof. Consider any non-simplicial vertex x of G[X]. By Lemma 2.1 there is a chordless path $P = p_0 \cdot p_1 \cdot \cdots \cdot p_k$ in G[X], with $k \ge 2$, such that p_0 and p_k are simplicial in G[X] and $x = p_i$ for some $i \in \{1, \ldots, k-1\}$. By the hypothesis p_0 has neighbor $a \in A$ and p_k has a neighbor a' in A. Suppose that x has no neighbor in $\{a, a'\}$. Let h be the largest integer in $\{0, \ldots, i-1\}$ such that p_h has a neighbor in $\{a, a'\}$, and let g be the smallest integer in $\{i + 1, \ldots, k\}$ such that p_g has a neighbor in $\{a, a'\}$. Then $\{p_h, p_{h+1}, \ldots, p_g, a, a'\}$ contains a hole, a contradiction. So x has a neighbor in A.

Lemma 2.3 In a C_4 -free graph G, let A, B be two disjoint cliques. Then:

- There is a labeling $a_1, \ldots, a_{|A|}$ of the vertices of A such that $N_B(a_1) \supseteq N_B(a_2) \supseteq \cdots \supseteq N_B(a_{|A|})$. Similarly, there is a labeling $b_1, \ldots, b_{|B|}$ of the vertices of B such that $N_A(b_1) \supseteq N_A(b_2) \supseteq \cdots \supseteq N_A(b_{|B|})$.
- If every vertex in A has a neighbor in B, then some vertex in B is complete to A.
- If every vertex in A has a non-neighbor in B, then some vertex in B is anticomplete to A.
- If [A, B] is not complete, there are indices $i \leq |A|$ and $j \leq |B|$ such $a_ib_j \notin E(G)$, and $a_ib_h \in E(G)$ for all h < j, and $a_gb_j \in E(G)$ for all g < i. Moreover, every maximal clique of G contains one of a_i, b_j .

Proof. Consider any two vertices $a, a' \in A$. If there are vertices $b \in N_B(a) \setminus N_B(a')$ and $b' \in N_B(a') \setminus N_B(a)$, then $\{a, a', b, b'\}$ induces a C_4 . Hence we have either $N_B(a) \subseteq N_B(a')$ or $N_B(a') \subseteq N_B(a)$. This inclusion relation for all a, a' implies the existence of a total ordering on A, which corresponds to a labeling as desired, and the same holds for B. This proves the first item of the lemma. The second and third item are immediate consequences of the first.

Now suppose that A is not complete to B. Consider any vertex $a_{i'} \in A$ that has a non-neighbor in B, and let j be the smallest index such that $a_{i'}b_j \notin E(G)$. Let i be the smallest index such that $a_ib_j \notin E(G)$. So $i \leq i'$. We have

 $a_g b_j \in E(G)$ for all g < i by the choice of i. We also have $a_i b_h \in E(G)$ for all h < j, for otherwise, since $i \le i'$ we also have $a_{i'} b_h \notin E(G)$, contradicting the definition of j. This proves the first part of the fourth item.

Finally, consider any maximal clique K of G. Let g be the largest index such that $a_g \in K$ and let h be the largest index such that $b_h \in K$. By the properties of the labelings and the maximality of K we have $K = \{a_1, \ldots, a_g\} \cup \{b_1, \ldots, b_h\}$. If both g < i and h < j, then the properties of a_i, b_j imply that $K \cup \{a_i\}$ (and also $K \cup \{b_j\}$) is a clique of G, contradicting the maximality of K. Hence we have either $g \ge i$ or $h \ge j$, and so K contains one of a_i, b_j .

Lemma 2.4 In a (P_6, C_4) -free graph G, let X, Y and $\{c\}$ be disjoint subsets of V(G) such that:

- Y is a clique, and every vertex in X has a neighbor in Y,
- c is complete to X and anticomplete to Y;
- Either G[X] is not connected, or there are vertices $c', c'' \in V(G) \setminus (X \cup Y)$ such that c' is complete to Y and anticomplete to X, and c'' is anticomplete to $X \cup Y$, and $c'c'' \in E(G)$.

Then G[X] is $(P_4, 2P_3)$ -free.

Proof. First suppose that there is a P_4 p_1 - p_2 - p_3 - p_4 in G[X]. By the hypothesis p_1 has a neighbor $a \in Y$. Then $ap_3 \notin E(G)$, for otherwise $\{p_1, a, p_3, c\}$ induces a C_4 ; and similarly $ap_4 \notin E(G)$. If G[X] is connected, then either p_3 - p_2 - p_1 -a-c'-c'' or p_4 - p_3 - p_2 -a-c'-c'' is a P_6 . Now suppose that G[X] is not connected. So X contains a vertex p that is anticomplete to $\{p_1, p_2, p_3, p_4\}$. By the hypothesis p has a neighbor $a' \in Y$. As above we have $ap \notin E(G)$ and $a'p_i \notin E(G)$ for all $i \in \{1, \ldots, 4\}$ for otherwise there is a C_4 . But then either p-a'-a- p_1 - p_2 - p_3 or p-a'-a- p_2 - p_3 - p_4 is a P_6 .

Now suppose that there is a $2P_3$ in G[X], with vertices p_1, \ldots, p_6 and edges $p_1p_2, p_2p_3, p_4p_5, p_5p_6$. We know that p_1 has a neighbor $a \in Y$, and as above we have $ap_i \notin E(G)$ for each $i \in \{3, 4, 5, 6\}$, for otherwise there is a C_4 . Likewise, p_6 has a neighbor $a' \in Y$, and $a'p_j \notin E(G)$ for each $j \in \{1, 2, 3, 4\}$. Then $p_{h+1}-p_h$ -a-a'- p_g - p_{g-1} is an induced P_6 for some $h \in \{1, 2\}$ and $g \in \{5, 6\}$.

 (P_4, C_4) -free graphs We want to understand the structure of $(P_4, C_4, 2P_3)$ -free graphs as they play a major role in the structure of belts and boilers. Recall that (P_4, C_4) -free graphs were studied by Golumbic [16], who called them trivially perfect graphs. Clearly any such graph is chordal. It was proved in [16] that every connected (P_4, C_4) -free graph has a universal vertex. It follows that trivially perfect graphs are exactly the class \mathcal{T} of graphs that can be built recursively as follows, starting from complete graphs:

- The disjoint union of any number of trivially perfect graphs is trivially perfect; - If G is any trivially perfect graph, then the graph obtained from G by adding a universal vertex is trivially perfect.

As a consequence, any connected member G of \mathcal{T} can be represented by a rooted directed tree T(G) defined as follows. If G is a clique, let T(G) have one node, which is the set V(G). If G is not a clique, then by Golumbic's result the

set U(G) of universal vertices of G is not empty, and $G \setminus U(G)$ has a number $k \geq 2$ of components G_1, \ldots, G_k . Let then T(G) be the tree whose root is U(G) and the children (out-neighbors) of U(G) are the roots of $T(G_1), \ldots, T(G_k)$.

The following properties of T(G) appear immediately. Every node of T(G) is a non-empty clique of G, and every vertex v of G is in exactly one such clique, which we call A_v ; moreover, A_v is a homogeneous set (all member of A_v are pairwise clones). For every vertex v of G, the closed neighborhood of v consists of A_v and all the vertices in the cliques that are descendants and ancestors of A_v in T(G). Every maximal clique of G is the union of the nodes of a directed path in T(G). All vertices in any leaf of T(G) are simplicial vertices of G, and every simplicial vertex of G is in some leaf of T(G).

We say that a member G of \mathcal{T} is *basic* if every node of T(G) is a clique of size 1. (We can view T(G) as a directed tree, where every edge is directed away from the root; and then G is the underlying undirected graph of the transitive closure of T(G).). It follows that every member of \mathcal{T} is a blowup of a basic member of \mathcal{T} . In a basic member G of \mathcal{T} , two vertices are adjacent if and only if one of them is an ancestor of the other in T(G), and every clique of G consists of the set of vertices of any directed path in T(G).

A dart is the graph with vertex-set $\{a, b, c, d, e\}$ and edge-set $\{ab, bc, cd, da, ac, ce\}$. Let $K_{1,3}^+$ be the tree obtained from $K_{1,3}$ by subdividing one edge. Next we give the following useful lemma.

Lemma 2.5 Let G be a (P_4, C_4) -free graph.

(a) If G does not have three pairwise non-adjacent simplicial vertices, then G is a blowup of P_3 .

(b) If G does not have four pairwise non-adjacent simplicial vertices, then G is a blowup of a dart.

Proof. The hypothesis of (a) or (b) means that, if H is a connected component of G, then T(H) is a tree with at most three leaves. Since each internal vertex of T(H) has at least two leaves, T(H) is either K_1 , K_2 , P_3 (rooted at its vertex of degree 2), $K_{1,3}$ (rooted at its vertex of degree 3), or $K_{1,3}^+$ (rooted at its vertex of degree 2). Then the conclusion follows directly from our assumption on G and the preceding arguments.

 $(P_4, C_4, 2P_3)$ -free graphs Let \mathcal{C} be the class of $(P_4, C_4, 2P_3)$ -free graphs. So $\mathcal{C} \subset \mathcal{T}$. If G is any member of \mathcal{C} , and G is connected and not a clique, then since G is $2P_3$ -free all components of $G \setminus U(G)$, except possibly one, are cliques. So all children of U(G) in T(G), except possibly one, are leaves. Applying this argument recursively we see that the tree T(G) consists of a rooted directed path plus a positive number of leaves adjacent to every node of this path, with at least two leaves adjacent to the last node of this path. We call such a tree a *bamboo*. By the same argument as above, every member of \mathcal{C} is a blowup of a basic member of \mathcal{C} .

C-pairs A graph G is a *C*-pair if G is P_6 -free, chordal, and V(G) can be partitioned into two sets X and A such that A is a clique, $G[X] \in C$, every vertex in X has a neighbor in A, and any two non-adjacent vertices in X have

no common neighbor in A. Depending on the context we may also write that (X, A) is a C-pair.

We say that G is a basic C-pair if the subgraph G[X] is a basic member of C, with vertices x_1, \ldots, x_k for some integer k, and a clique $A = \{a_0, a_1, \ldots, a_k\}$; and for each $i \in \{1, \ldots, k\}$, if x_i is simplicial in G[X] then $N_A(x_i) = \{a_i\}$, else $N_A(x_i)$ consists of $\{a_i\}$ plus the union of $N_A(y)$ over all descendants y of x_i in T(G[X]).

Before describing how all \mathcal{C} -pairs can be obtained from basic \mathcal{C} -pairs we need to introduce another definition. Let H be any graph and M be a matching in H. An *augmentation* of H along M is any graph G whose vertex-set can be partitioned into |V(H)| cliques $Q_v, v \in V(H)$, such that $[Q_u, Q_v]$ is complete if $uv \in E(H) \setminus M$, and $[Q_u, Q_v] = \emptyset$ if $uv \notin E(H)$, and $\{Q_u, Q_v\}$ is a graded pair if $uv \in M$. (See [28] for a similar definition.)

In a basic C-pair G, with the same notation as above, we say that a matching M is *acceptable* if there is a clique $\{x_{i_1}, \ldots, x_{i_h}\}$ in G[X] such that $M = \{x_{i_1}a_{i_1}, \ldots, x_{i_h}a_{i_h}\}$.



Figure 6: Schematic representations of: (a) a basic C-pair, (b) an acceptable matching in (a), and (c) an augmentation of the graph in (a) along an acceptable matching in (b). In (a) and (b), the vertices in a shaded box represents a clique. In (b), the dashed lines represent the matching edges. In (c), the circles represent a collection of sets into which the vertex set of the graph is partitioned, each shaded circle represents a clique, and the circles inside the oval form a clique, a solid line between two circles indicates that the two sets are complete to each other, the dotted line between two circles means that the respective pair of sets is graded, and the absence of a line between two circles indicates that the two sets are anticomplete to each other.

Theorem 2.1 A graph is a C-pair then it is an augmentation of a basic C-pair along an acceptable matching.

Proof. Let G be any C-pair, with the same notation as above. Since G[X] is $(P_4, C_4, 2P_3)$ -free it admits a representative tree T(G[X]) which is a bamboo. We claim that:

If Y, Z are two nodes of T(G[X]) such that Z is a descendant of Y, then Y is complete to $N_A(Z)$. (1)

Proof: Consider any $y \in Y$ and $a \in N_A(Z)$; so there is a vertex $z \in Z$ with $za \in E(G)$. Since Y is not a leaf of T(G[X]), there is a child Z' of Y in T(G[X]) such that Z' is not on the directed path from Z to Y, and so Z and Z' are not adjacent (they are anticomplete to each other). Pick any $z' \in Z'$. Then

 $yz, yz' \in E(G)$ and $zz' \notin E(G)$. We know that z' has a neighbor $a' \in A$. We have $az', a'z \notin E(G)$ by the definition of a C-pair (z and z' have no common neighbor in A). Then $ya, ya' \in E(G)$, for otherwise G[y, z, z', a, a'] contains an induced hole of length 4 or 5, contradicting the fact that G is chordal. So (1) holds.

Let X_1, \ldots, X_k be the nodes of T(G[X]). For each $i \in \{1, \ldots, k\}$, let U_i be the union of $N_A(Z)$ over all descendants Z of X_i in T(G[X]), and let $A_i = N_A(X_i) \setminus U_i$. Let $A_0 = A \setminus (A_1 \cup \cdots \cup A_k)$ (so $[X, A_0] = \emptyset$).

Let X_{i_1}, \ldots, X_{i_h} be the nodes of T(G[X]) that are not homogeneous in G (if any). Note that for each $i \in \{i_1, \ldots, i_h\}$ the pair $\{X_i, A_i\}$ is graded since G is C_4 -free. We claim that:

$$X_{i_1} \cup \dots \cup X_{i_h}$$
 is a clique. (2)

Proof: Suppose, on the contrary, and up to symmetry, that $[X_{i_1}, X_{i_2}]$ is not complete, and so $[X_{i_1}, X_{i_2}] = \emptyset$. For each $t \in \{1, 2\}$, since X_{i_t} is not homogeneous in G, there are vertices $y_t, z_t \in X_{i_t}$ and a vertex $a_t \in A$ that is adjacent to y_t and not to z_t . Since non-adjacent vertices in X have no common neighbor in A, we have $a_1 \neq a_2$ and $a_1y_2, a_1z_2, a_2y_1, a_2z_1 \notin E(G)$. Then $z_1 \cdot y_1 \cdot a_1 \cdot a_2 \cdot y_2 \cdot z_2$ is a P_6 . So (2) holds.

Let H be the basic member of \mathcal{C} of which G[X] is a blowup. Let H have vertices x_1, \ldots, x_k , where x_i corresponds to the node X_i of T(G[X]) for all i. Let G_0 be the graph obtained from H by adding a set $A = \{a_0, a_1, \ldots, a_k\}$, disjoint from V(H), and edges so that A is a clique in G_0 and, for all $i \in \{1, \ldots, k\}$ and $j \in \{0, 1, \ldots, k\}$, vertices x_i and a_j are adjacent in G_0 if and only if $[X_i, A_j] \neq \emptyset$ in G. By this construction and by (1) G_0 is a basic \mathcal{C} -pair. In G_0 let M = $\{x_{i_1}a_{i_1}, \ldots, x_{i_h}a_{i_h}\}$. It follows from (2) that M is an acceptable matching of G_0 and from all the points above that G is an augmentation of G_0 along M. \Box

3 Structure of (P_6, C_4) -free graphs

In this section, we give the proof of Theorem 1.5. We say that a subgraph H of G is *dominating* if every vertex in $V(G) \setminus V(H)$ is a adjacent to a vertex in H. We will use the following theorem of Brandstädt and Hoàng [4].

Theorem 3.1 ([4]) Let G be a (P_6, C_4) -free graph that has no clique cutset. Then the following statements hold.

(i) Every induced C_5 is dominating.

(ii) If G contains an induced C_6 which is not dominating, then G is the join of a complete graph and a blowup of the Petersen graph.

In the next two theorems we make some general observations about the situation when a (P_6, C_4) -free graph contains a hole (which must have length either 5 or 6). Observe that in a C_4 -free graph G, if u-v-w is a P_3 , then any $x \in V(G) \setminus \{u, v, w\}$ which is adjacent to u and w is also adjacent to v.

Theorem 3.2 Let G be any (P_6, C_4) -free graph that contains a C_5 with vertexset $C = \{v_1, ..., v_5\}$ and $\{v_i v_{i+1} \mid i \in \{1, ..., 5\}, i \mod 5\}$. Let:

$$A = \{x \in V(G) \setminus C \mid N_C(x) = C\}.$$

$$T_i = \{x \in V(G) \setminus C \mid N_C(x) = \{v_{i-1}, v_i, v_{i+1}\}.$$

$$W_i = \{x \in V(G) \setminus C \mid N_C(x) = \{v_i\}.$$

$$X_{i,i+1} = \{x \in V(G) \setminus C \mid N_C(x) = \{v_i, v_{i+1}\}.$$

Moreover, let $T = T_1 \cup \cdots \cup T_5$, $W = W_1 \cup \cdots \cup W_5$, and $X = X_{12} \cup X_{23} \cup X_{34} \cup X_{45} \cup X_{51}$. Then the following properties hold for all *i*:

- (a) $A \cup T_i$ is a clique.
- (b) $[T_i, T_{i+2}]$, $[X_{i,i+1}, X_{i+2,i+3}]$, $[W_i, W_{i+1}]$, $[T_i, W_{i-2} \cup W_{i+2}]$, $[T_i, X_{i+2,i+3}]$ and $[X_{i,i+1}, W_i \cup W_{i+1}]$ are empty.
- (c) $[X_{i,i+1}, X_{i+1,i+2}]$, $[W_i, W_{i+2}]$, and $[X_{i,i+1}, W_{i-1} \cup W_{i+2}]$ are complete.
- (d) If G is C₆-free, then for each i one of $X_{i,i+1}$ and $X_{i+1,i+2}$ is empty, and one of W_i and W_{i+2} is empty, and one of $X_{i,i+1}$ and $W_{i-1} \cup W_{i+2}$ is empty.
- (e) If G has no clique cutset, then the set $\{x \in V(G) \setminus C \mid N_C(x) = \emptyset\}$ is empty and $[T_i, W_i]$ is complete.
- (f) If G has no clique cutset, then $V(G) = V(C) \cup A \cup T \cup W \cup X$.

Proof. (a) If there are non-adjacent vertices $a, b \in A \cup T_i$, then $\{a, v_{i-1}, b, v_{i+1}\}$ induces a C_4 .

(b) Let i = 1 and suppose that there is an edge xy in one of the listed sets. If $x \in T_1$ and $y \in T_3$, then $\{x, y, v_4, v_5\}$ induces a C_4 . If $x \in X_{12}$ and $y \in X_{34}$, then $\{x, v_2, v_3, y\}$ induces a C_4 . If $x \in T_1$ and $y \in W_4$ then $\{x, y, v_4, v_5\}$ induces a C_4 . If $x \in W_1$ and $y \in W_2$, then $x \cdot y \cdot v_2 \cdot v_3 \cdot v_4 \cdot v_5$ is an induced P_6 . If $x \in T_1$ and $y \in X_{34}$, then $\{x, v_2, v_3, y\}$ induces a C_4 . If $x \in X_{12}$ and $y \in W_1$, then $y \cdot x \cdot v_2 \cdot v_3 \cdot v_4 \cdot v_5$ is a P_6 . The other cases are symmetric.

(c) and (d) Let i = 1 and suppose that there are vertices $x \in X_{12} \cup W_1$ and $y \in X_{23} \cup W_3$. If $xy \notin E(G)$, then $x \cdot v_1 \cdot v_5 \cdot v_4 \cdot v_3 \cdot y$ is a P_6 . This proves (c). If $xy \in E(G)$ then the same vertices induce a C_6 , which proves (d).

(e) Follows from Theorem 3.1.

(f) Follows by Theorem 3.1 and (e).

Theorem 3.3 Let G be any (P_6, C_4) -free graph that contains a C_6 with vertexset $C = \{v_1, \ldots, v_6\}$ and $\{v_i v_{i+1} \mid i \in \{1, \ldots, 6\}, i \mod 6\}$. Let:

$$S = \{x \in V(G) \setminus C \mid N_C(x) = C\}.$$

$$A_i = \{x \in V(G) \setminus C \mid N_C(x) = \{v_{i-1}, v_i, v_{i+1}\}\}.$$

$$B_i = \{x \in V(G) \setminus C \mid N_C(x) = \{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}.$$

$$D_i = \{x \in V(G) \setminus C \mid N_C(x) = \{v_i, v_{i+3}\}\}.$$

$$L = \{x \in V(G) \setminus C \mid N_C(x) = \emptyset\}.$$

Moreover, let $A = A_1 \cup \cdots \cup A_6$, $B = B_1 \cup \cdots \cup B_6$, and $D = D_1 \cup \cdots \cup D_6$. Then the following properties hold for all $i, i \mod 6$:

- (a) $V(G) = V(C) \cup A \cup B \cup D \cup S \cup L$.
- (b) Each of $A_i \cup B_i \cup B_{i+5}$, D_i and S is a clique.
- (c) $[A_i, A_{i+1} \cup A_{i+5} \cup D_i], [B_i, B_{i+1} \cup B_{i+3} \cup B_{i+5} \cup D_{i+2}], and [S, A_i \cup B_i \cup D_i]$ are complete.
- (d) $[A_i, A_{i+3} \cup B_{i+2} \cup B_{i+3} \cup D_{i+1} \cup D_{i+2}], [B_i, B_{i+2} \cup B_{i+4}], and [D_i, D_{i+1}]$ are empty.
- (e) If $B_i \neq \emptyset$, then $D_i \cup D_{i+1} = \emptyset$.
- (f) If $B_i \neq \emptyset$ and $B_{i+1} \neq \emptyset$, then $B_{i+3} \cup B_{i+4} = \emptyset$.

Proof. We note that $D_i = D_{i+3}$, for all *i*.

(a) Suppose that there is a vertex x in G. We may assume that $x \in V(G) \setminus V(C)$. If x has no neighbor in C, then $x \in L$. So, suppose that x has a neighbor in C. If $N_C(x) = \{v_i\}$ (or $\{v_i, v_{i+1}\}$), for some i, then $(C \setminus \{v_{i+1}\}) \cup \{x\}$ induces a P_6 . In all the remaining cases, we see that either $C \cup \{x\}$ contains an induced C_4 or $x \in A \cup B \cup D \cup S$. So (a) holds.

(b) If there are non-adjacent vertices x and y in one of the listed sets, then either $\{x, v_{i-1}, v_{i+1}, y\}$ or $\{x, v_i, v_{i+3}, y\}$ induces a C_4 .

(c) Let i = 1 and suppose that there are non-adjacent vertices x and y in one of the listed sets. If $x \in A_1$ and $y \in A_2 \cup D_1$, then $\{v_2, x, v_6, v_5, v_4, y\}$ induces a P_6 . If $x \in B_1$ and $y \in B_2$, then $\{x, v_1, y, v_3\}$ induces a C_4 . If $x \in B_1$ and $y \in B_4 \cup D_3$, then $\{x, v_3, y, v_6\}$ induces a C_4 . If $x \in S$ and $y \in A_1 \cup B_1 \cup D_1$, then either $\{x, v_6, y, v_2\}$ or $\{x, v_1, y, v_3\}$ induces a C_4 . The other cases are symmetric.

(d) Let i = 1 and suppose that there is an edge xy in one of the listed sets. If $x \in A_1$ and $y \in A_4 \cup B_3 \cup D_2$, then $\{x, v_6, v_5, y\}$ induces a C_4 . If $x \in B_1$ and $y \in B_3$, then $\{x, v_6, v_5, y\}$ induces a C_4 . If $x \in D_1$ and $y \in D_2$, then $\{x, v_1, v_2, y\}$ induces a C_4 . The other cases are symmetric.

(e) Let i = 1 and let $x \in B_1$. Up to symmetry, if there exists a vertex $y \in D_1$, then by (c), $xy \in E(G)$. But then $\{x, y, v_4, v_3\}$ induces a C_4 . So $D_1 = \emptyset$.

(f) Let i = 1. Let $x \in B_1$ and $y \in B_2$. Up to symmetry, if there exists a vertex $z \in B_4$, then by (c), $xy, xz \in E(G)$, and by (d), $yz \notin E(G)$. But then $\{x, y, v_3, z\}$ induces a C_4 . So $B_4 = \emptyset$.

This shows Theorem 3.3.

When there is an F_3

Now we can give the proof of the first item of Theorem 1.5 which we restate it as follows.

Theorem 3.4 Let G be a (P_6, C_4) -free graph with no universal vertex and no clique cutset. Suppose that G contains an F_3 . Then G is a blowup of F_3 .

Proof. Consider the graph F_3 as shown in Figure 1 and let $C = \{v_1, \ldots, v_6\}$. By Theorem 3.3(a), and with the same notation, every vertex in $V(G) \setminus C$ belongs to $A_i \cup B_i \cup D_i \cup S \cup L$ for some *i*. Note that $x \in A_2$, $y \in A_4$ and $z \in A_6$. We first claim that:

$$B_i \cup D_i = \emptyset$$
, for all *i*. (1)

Proof: Suppose on the contrary, and up to symmetry, that there is a vertex $u \in B_1 \cup D_1$. Suppose that $u \in B_1$. By Theorem 3.3(b) we have $ux \in E(G)$, and

by Theorem 3.3(d) we have $uy \notin E(G)$. Then either $\{u, x, z, v_6\}$ or $\{u, v_3, y, z\}$ induces a C_4 . Now suppose that $u \in D_1$. By Theorem 3.3(d) we have $ux, uz \notin E(G)$. Then $u \cdot v_4 \cdot v_5 \cdot z \cdot x \cdot v_2$ is a P_6 . So (1) holds.

Next, we claim that:

$$L \cup S = \emptyset. \tag{2}$$

Proof: Suppose that $L \neq \emptyset$. By Theorem 3.3(b), S is a clique. Since S is not a clique cutset, and by (1), some vertex w in L has a neighbor $a \in A$, say $a \in A_1$. But then w-a- v_6 - v_5 - v_4 - v_3 is a P_6 , a contradiction. Hence $L = \emptyset$. Now if $S \neq \emptyset$, then by Theorem 3.3(b) and (c), any vertex in S is universal, a contradiction. So (2) holds.

We note that every vertex $a \in A_2$ is either complete or anticomplete to $\{y, z\}$, for otherwise $G[\{a, v_3, y, z, v_1\}]$ has an induced C_4 . So let $A'_2 = \{v_2\} \cup \{u \in A_2 \mid u \text{ is anticomplete to } \{y, z\}\}$ and $X = A_2 \setminus A'_2$. Note that $x \in X$. Define sets A'_4 , Y, A'_6 , Z similarly.

By Theorem 3.3(c) and (d), we know that $[A_1, A'_2 \cup X \cup A'_6 \cup Z]$ is complete and $[A_1, A'_4 \cup Y] = \emptyset$. Likewise, $[A_3, A'_2 \cup X \cup A'_4 \cup Y]$ is complete and $[A_3, A'_6 \cup Z] = \emptyset$, and $[A_5, A'_4 \cup Y \cup A'_6 \cup Z]$ is complete and $[A_5, A'_1 \cup X] = \emptyset$. Moreover there is no edge a_1a_3 with $a_1 \in A_1$ and $a_3 \in A_3$, for otherwise $\{a_1, a_3, y, z\}$ induces a C_4 . So $[A_1, A_3] = \emptyset$, and similarly $[A_3, A_5] = \emptyset$ and $[A_5, A_1] = \emptyset$.

There is no edge $a'_2a'_4$ with $a'_2 \in A'_2$ and $a'_4 \in A'_4$, for otherwise $\{a'_2, a'_4, y, x\}$ induces a C_4 . So $[A'_2, A'_4] = \emptyset$, and similarly $[A'_4, A'_6] = \emptyset$ and $[A'_6, A'_2] = \emptyset$.

There is no edge $a'_2 y'$ with $a'_2 \in A'_2$ and $y' \in Y$, for otherwise $\{a'_2, y', z, v_1\}$ induces a C_4 . Hence, and by symmetry, $[A'_2, Y \cup Z] = \emptyset$, and similarly $[A'_4, Z \cup X] = \emptyset$ and $[A'_6, X \cup Y] = \emptyset$.

Finally, any two vertices $x' \in X$ and $y' \in Y$ are adjacent, for otherwise $\{x', v_3, y', z\}$ induces a C_4 . Hence [X, Y] is complete, and similarly [X, Z] and [Y, Z] are complete. Now we exhibit the mapping $Q_v \to v, v \in V(F_3)$ of the definition of a blowup, as follows: $A'_i \to v_i$, for i even, and $A_i \to v_i$, for i odd, $X \to x, Y \to y$, and $Z \to z$. Then the above properties mean that G is a blowup of F_3 . This completes the proof.

When there is an F_1 and no F_3

Here we give the proof of the second item of Theorem 1.5.

Theorem 3.5 Let G be a (P_6, C_4) -free graph with no universal vertex and no clique cutset. Suppose that G contains an F_1 and no F_3 . Then G is a band.

Proof. Consider the graph F_1 as shown in Figure 1 and let $C = \{v_1, \ldots, v_5\}$. We use the same notation as in Theorem 3.2. So $x \in X_{12}$, $y \in X_{23}$ and $z \in X_{34}$. By Theorem 3.2(b) and (c), we know that $[X_{23}, X_{12} \cup X_{34}]$ is complete and $[X_{12}, X_{34}] = \emptyset$. Note that X_{12} is a clique, for otherwise v_1, y and two nonadjacent vertices from X_{12} induce a C_4 . Similarly, X_{23} and X_{34} are cliques. We claim that:

$$W = \emptyset$$
, and $X_{51} \cup X_{54} = \emptyset$, and $A = \emptyset$. (1)

Proof: Suppose the contrary. Up to symmetry, there is a vertex $u \in W_1 \cup W_2 \cup W_5 \cup X_{51} \cup A$. Suppose $u \in W_1$. By Theorem 3.2(b) we have $ux, uy \notin E(G)$. Then $u \cdot v_1 \cdot x \cdot y \cdot v_3 \cdot v_4$ is a P_6 . Now suppose $u \in W_2$. By Theorem 3.2(b) we have $uz \notin E(G)$. Then either $u \cdot v_2 \cdot y \cdot z \cdot v_4 \cdot v_5$ or $u \cdot y \cdot z \cdot v_4 \cdot v_5 \cdot v_1$ is a P_6 . Now suppose $u \in W_5$. By Theorem 3.2(b) we have $uz \notin E(G)$. Then $u \cdot v_5 \cdot v_1 \cdot v_2 \cdot v_3 \cdot z$ is a P_6 . Now suppose $u \in X_{51}$. By Theorem 3.2(b) we have $uy, uz \notin E(G)$. Then $u \cdot v_1 \cdot v_2 \cdot y \cdot z \cdot v_4$ is a P_6 . Thus we have established that $W = \emptyset$ and $X_{51} \cup X_{54} = \emptyset$ so $X = X_{12} \cup X_{23} \cup X_{34}$. Finally, suppose that $u \in A$. We note that for any two vertices $x' \in X_{12}$ and $y' \in X_{23}$ the vertex u is either complete or anticomplete to $\{x', y'\}$, for otherwise $G[u, v_1, x', y', v_3]$ contains an induced C_4 . The same holds for any two vertices in X_{23} and X_{34} . It follows that u is either complete or anticomplete to X. If u is complete to X then by Theorem 3.2(a), u is a universal vertex, a contradiction. If u is anticomplete to X then $\{v_1, \ldots, v_5, x, y, z, u\}$ induces an F_3 , a contradiction. Thus (1) holds.

By (1) we have $V(G) = C \cup T_1 \cup \cdots \cup T_5 \cup X_{12} \cup X_{23} \cup X_{34}$. By Theorem 3.2(b) we know that $[T_5, T_2 \cup T_3 \cup X_{23}] = \emptyset$. We claim that:

$$[T_5, X_{12} \cup X_{34}] = \emptyset$$
, and $[T_5, T_1 \cup T_4]$ is complete. (2)

Proof: Pick any vertex $t_5 \in T_5$. Suppose up to symmetry that t_5 has a neighbor $x' \in X_{12}$. Then either $\{t_5, x', y, z\}$ induces a C_4 or v_5 - t_5 -x'- v_2 - v_3 -z is a P_6 , a contradiction. Now suppose up to symmetry that t_5 has a non-neighbor $t_1 \in T_1$. Then t_5 - v_5 - t_1 - v_2 - v_3 -z is a P_6 (since $t_1z \notin E(G)$ by Theorem 3.2(b)). Thus (2) holds.

By Theorem 3.2(b) we have $[T_1, T_3 \cup T_4 \cup X_{34}] = \emptyset$ and $[T_4, T_1 \cup T_2 \cup X_{12}] = \emptyset$. We claim that:

$$[T_1, X_{12}]$$
 and $[T_4, X_{34}]$ are complete. (3)

Proof: If, up to symmetry, there are non-adjacent vertices $t_1 \in T_1$ and $x' \in X_{12}$, then either $\{t_1, v_1, x', y\}$ induces a C_4 or t_1 - v_1 -x'-y-z- v_4 is a P_6 . Thus (3) holds.

By Theorem 3.2(b) we have $[T_2, T_4] = \emptyset$ and $[T_3, T_1] = \emptyset$. We claim that:

 $[T_2, X_{12} \cup X_{23}]$ and $[T_3, X_{23} \cup X_{34}]$ are complete. Moreover, every vertex in T_2 is complete either to T_1 or to T_3 , and every vertex in (4) T_3 is complete either to T_2 or to T_4 .

Proof: Up to symmetry pick any $t_2 \in T_2$, $x' \in X_{12}$ and $y' \in X_{23}$. Then $t_2y' \in E(G)$, for otherwise either $\{t_2, v_2, y', z\}$ induces a C_4 or $t_2 \cdot v_2 \cdot y' \cdot z \cdot v_4 \cdot v_5$ is a P_6 . Then $t_2x' \in E(G)$, for otherwise $\{t_2, y', x', v_1\}$ induces a C_4 . This proves the first sentence of (4). Now suppose that some $t_2 \in T_2$ has a non-neighbor $t_1 \in T_1$ and a non-neighbor $t_3 \in T_3$. Then either $\{t_1, v_1, t_2, y\}$ induces a C_4 or $t_1 \cdot v_1 \cdot t_2 \cdot y \cdot t_3 \cdot v_4$ is a P_6 . Thus (4) holds.

Every vertex in
$$X_{23}$$
 is anticomplete to T_1 or T_4 . (5)

Proof: If any $y' \in X_{23}$ has neighbors $t_1 \in T_1$ and $t_4 \in T_4$, then $\{y, t_1, v_5, t_4\}$ induces a C_4 . Thus (5) holds.

By (5) there is a partition Y_1, Y_4 of X_{23} such that $[Y_1, T_4] = [Y_4, T_1] = \emptyset$.

Now let $Q_i = \{v_i\} \cup T_i$ for each $i \in \{1, 4, 5\}$. We observe that the set $T_2 \cup \{v_2\} \cup X_{12} \cup Y_1$ is a clique, because each of $T_2 \cup \{v_2\}$, X_{12} and Y_1 and they are pairwise complete as proved above. Likewise $T_3 \cup \{v_3\} \cup X_{34} \cup Y_4$ is a clique. Let $R_2 = \{u \in T_2 \cup \{v_2\} \cup X_{12} \cup Y_1 \mid u$ is complete to $Q_1\}$, and let $Q_2 = (T_2 \cup \{v_2\} \cup X_{12} \cup Y_1) \setminus R_2$. Likewise let $R_3 = \{u \in T_3 \cup \{v_3\} \cup X_{34} \cup Y_4 \mid u\}$

is complete to Q_4 , and let $Q_3 = (T_3 \cup \{v_3\} \cup X_{34} \cup Y_4) \setminus R_3$. Note that $\{v_2\} \cup X_{12} \subseteq R_2$ and $\{v_3\} \cup X_{34} \subseteq R_3$ by (3). So $Q_2 \subseteq T_2 \cup Y_1$ and $Q_3 \subseteq T_3 \cup Y_4$. We observe that $[Q_2, Q_3]$ is complete by (4) and because X_{23} is a clique. Further, we claim that:

$$[Q_3, R_2]$$
 and $[Q_2, R_3]$ are complete. (6)

Proof: Suppose that there are non-adjacent vertices $q \in Q_3$ and $r \in R_2$. Then $r \notin \{v_2\} \cup Y_1$, and so $r \in T_2 \cup X_{12}$, and q has a non-neighbor $t \in T_4$. If $r \in X_{12}$, then $q \cdot v_3 \cdot t \cdot v_5 \cdot v_1 \cdot r$ is a P_6 (since $rt \notin E(G)$, by Theorem 3.2(b)), a contradiction. So $r \in T_2$. Then since $qz \in E(G)$ (by (4)) and $\{q, z, r, v_2\}$ does not induce a C_4 , $rz \notin E(G)$. But then $v_5 \cdot t \cdot z \cdot q \cdot v_2 \cdot r$ is a P_6 , a contradiction. Thus (6) holds.

Moreover, by the definition of Q_1, \ldots, Q_4, R_2 and R_3 , the pairs $\{Q_1, Q_2\}$, $\{Q_2, Q_3\}$ and $\{R_2, R_3\}$ are graded. Hence the sets $Q_1, \ldots, Q_5, R_2, R_3$ form a partition of V(G) which shows that G is a band.

When there is a C_6 and no F_1

Here we give the proof of the third item of Theorem 1.5, which we restate as follows.

Theorem 3.6 Let G be a (P_6, C_4) -free graph that has no clique-cutset and no universal vertex, and suppose that G is F_1 -free. If G contains an induced C_6 , then G is a blowup of one of the graphs H_1, H_2, H_3, H_4 .

Proof. Let $C = \{v_1, v_2, \ldots, v_6\}$ be the vertex-set of a C_6 in G, with edges $v_i v_{i+1}$ (mod 6). We use Theorem 3.3 with the same notation. If C is not dominating, then by Theorem 3.1 and since G has no universal vertex, G is a blowup of the Petersen graph. Therefore we may assume that C is dominating. So $L = \emptyset$ and $V(G) = V(C) \cup A \cup B \cup D \cup S$. Moreover, since G is F_1 -free, we have $[A_i, A_{i+2}] = \emptyset$ and $[A_i, B_{i+1}]$ is complete. So, by Theorem 3.3, each of the sets $A_1 \cup \{v_1\}, \ldots, A_6 \cup \{v_6\}, B_1, \ldots, B_6, D_1, D_2, D_3, S$ is a clique and that any two of them are either complete or anticomplete to each other. So G is a blowup of some graph. We now make this more precise. Since G has no universal vertex, by Theorem 3.3(b) and (c), we have $S = \emptyset$. If $B = \emptyset$, then G is a blowup of the Petersen graph. Now assume that $B \neq \emptyset$. First, suppose that two consecutive B_j 's are non-empty, say $B_i, B_{i+1} \neq \emptyset$. Then by Theorem 3.3(f), $B_{i+3} \cup B_{i+4} = \emptyset$, and by Theorem 3.3(e) $D = \emptyset$. So again by Theorem 3.3(f), G is a blowup of H_4 . Next, suppose that no two consecutive B_i 's are non-empty and let $B_i \neq \emptyset$. Then $B_{i-1} = \emptyset = B_{i+1}$ and by Theorem 3.3(e), $D_i = \emptyset = D_{i+1}$. Now, if $B_{i+3} \neq \emptyset$ or $B_{i+2} \cup B_{i+4} = \emptyset$, then G is a blowup of H_2 , and if $B_{i+3} = \emptyset$ and $B_{i+2} \cup B_{i+4} \neq \emptyset$, then by Theorem 3.3(e), $D = \emptyset$, and so G is a blowup of H_3 . \square

When there is an F_2 and no C_6

Here we give the proof of the fourth item of Theorem 1.5, which we restate it as follows.

Theorem 3.7 Let G be a (P_6, C_4) -free graph that has no clique-cutset and no universal vertex, and suppose that G is C_6 -free. If G contains an F_2 , then G is a blowup of either H_5 or $F_{k,\ell}$ for some integers $k, \ell \geq 1$.

Proof. Consider the graph F_2 as shown in Figure 1 and let $C = \{v_1, \ldots, v_5\}$. We use the same notation as in Theorem 3.2. Note that $t \in T_5$, and $x \in X_{12}$ and $y \in X_{34}$, so Theorem 3.2(e) implies that the sets X_{23} , X_{45} , X_{15} and W_2 , W_3 , W_5 are all empty, and one of W_1, W_4 is empty. So $V(G) = C \cup T_1 \cup \cdots \cup$ $T_5 \cup X_{12} \cup X_{34} \cup A \cup W_1 \cup W_4$. We establish a number of properties. (Some of them were also proved in [14, Proof of Lemma 4].)

- (i) Each vertex in T₅ is either complete or anticomplete to X₁₂ ∪ X₃₄. In particular, t is complete to X₁₂ ∪ X₃₄.
 Proof: Suppose that some vertex t₅ ∈ T₅ is not complete and not anticomplete to X₁₂ ∪ X₃₄. It follows that t₅ has a neighbor x' ∈ X₁₂ and a non-neighbor y' ∈ X₃₄, or vice-versa. Then v₅-t₅-x'-v₂-v₃-y' is a P₆.
- (ii) X_{12} and X_{34} are cliques. Proof: If, up to symmetry, X_{12} contains two non-adjacent vertices x', x'', then by (i), $\{t, x', x'', v_2\}$ induces a C_4 .
- (iii) Each vertex in T₂ is either complete or anticomplete to X₁₂, and each vertex in T₃ is either complete or anticomplete to X₃₄.
 Proof: If, up to symmetry, some vertex t₂ ∈ T₂ has a neighbor x' and a non-neighbor x'' in X₁₂, then, by (ii), x''-x'-t₂-v₃-v₄-v₅ is a P₆.
- (iv) $[T_2, X_{34}] = \emptyset$, and $[T_3, X_{12}] = \emptyset$. Proof: Suppose, up to symmetry, that there are adjacent vertices $t_2 \in T_2$ and $y' \in X_{34}$. If $t_2t \in E(G)$ then $\{t_2, v_3, v_4, t\}$ induces a C_4 . If $t_2t \notin E(G)$, then by (i), $\{t_2, y', t, v_1\}$ induces a C_4 .
- (v) $[T_1, T_2 \cup T_5 \cup X_{12}]$ and $[T_4, T_3 \cup T_5 \cup X_{34}]$ are complete. Proof: Suppose, up to symmetry, that some vertex $t_1 \in T_1$ has a nonneighbor $u \in T_2 \cup T_5 \cup X_{12}$. Recall that $t_1y \notin E(G)$ by Theorem 3.2(b). Also, since $\{v_5, t, y, v_3, v_2, t_1\}$ does not induce a C_6 , $t_1t \in E(G)$. Suppose that $u \in X_{12}$. Then $\{t_1, t, u, v_2\}$ induces a C_4 , a contradiction. In particular $t_1x \in E(G)$. Now suppose that $u \in T_5$ and $u \neq t$. If $ux \in E(G)$, then $\{u, x, t_1, v_5\}$ induces a C_4 . If $ux \notin E(G)$, then by (i), $uy \notin E(G)$, and $u - v_5 - t_1 - v_2 - v_3 - y$ is a P_6 .
- (vi) $[A, X_{12} \cup X_{34}]$ is complete. Proof: If, up to symmetry, there are non-adjacent vertices $a \in A$ and $x' \in X_{12}$, then by Theorem 3.2(a) and (i) the set $\{a, t, x', v_2\}$ induces a C_4 .

Now let:

$$\begin{array}{rcl} Q_i &=& \{v_i\} \cup T_i \text{ for } i \in \{1,4\}, \\ Q_2 &=& \{v_2\} \cup \{u \in T_2 \mid u \text{ is complete to } X_{12}\} \text{ and } R_2 = T_2 \setminus Q_2, \\ Q_3 &=& \{v_3\} \cup \{u \in T_3 \mid u \text{ is complete to } X_{34}\} \text{ and } R_3 = T_3 \setminus Q_3, \\ Q_5 &=& \{u \in T_5 \mid u \text{ is complete to } X_{12} \cup X_{34}\} \text{ and } R_5 = \{v_5\} \cup (T_5 \setminus Q_5), \end{array}$$

Recall that, by Theorem 3.2(b), $[T_i, T_{i+2}] = \emptyset$, for all *i*. Then:

(vii) $[Q_2, R_3]$ and $[Q_3, R_2]$ are complete.

Proof: If there are non-adjacent vertices $u \in Q_2$ and $r \in R_3$, then $r \cdot v_3 \cdot u \cdot x \cdot t \cdot v_5$ is a P_6 . The proof is similar for $[Q_3, R_2]$.

(viii) $[R_2, R_3] = \emptyset.$

Proof: If $r_2 \in R_2$ and $r_3 \in R_3$ are adjacent then $x \cdot v_1 \cdot r_2 \cdot r_3 \cdot v_4 \cdot y$ is a P_6 .

Suppose that $W_1 \cup W_4 = \emptyset$. By (vi) and Theorem 3.2(a), $[A, V(G) \setminus A]$ is complete and A is a clique; since G has no universal vertex, we deduce that $A = \emptyset$. Then V(G) is partitioned into the ten cliques $Q_1, Q_2, Q_3, Q_4, Q_5, R_5,$ X_{12}, R_2, R_3, X_{34} , and any two of them are either complete or anticomplete to each other, and the adjacencies proved above show that G is a blowup of H_5 .

Therefore let us assume that $W_1 \cup W_4 \neq \emptyset$. By Theorem 3.2(d) one of W_1 and W_4 is empty. Up to symmetry, let us assume that $W_1 \neq \emptyset$ and $W_4 = \emptyset$. Hence $V(G) = Q_1 \cup \cdots \cup Q_5 \cup R_2 \cup R_3 \cup R_5 \cup X_{12} \cup X_{34} \cup W_1 \cup A$. Recall that every induced C_5 in G is dominating, by Theorem 3.1. Then:

- (ix) $[W_1, Q_1]$ is complete, and $[W_1, Q_3 \cup R_3 \cup Q_4 \cup X_{12}] = \emptyset$. This follows directly from Theorem 3.2(b)–(e).
- (x) $[W_1, Q_5]$ is complete. Proof: If any $w \in W_1$ and $u \in Q_5$ are non-adjacent, then either $\{w, v_1, u, y\}$ induces a C_4 or $\{u, x, v_2, v_3, y\}$ is a non-dominating C_5 by (ix).
- (xi) $[W_1, Q_2 \cup R_5] = \emptyset$. Proof: Suppose that $w \in W_1$ and $u \in Q_2 \cup R_5$ are adjacent. If $u \in Q_2$, then, since $t \in Q_5$ and by (ix) and (x), $\{w, t, x, u\}$ induces a C_4 . If $u \in R_5$, then $w - u - v_4 - v_3 - v_2 - x$ is a P_6 by (ix).
- (xii) $R_3 = \emptyset$. Proof: Pick any $w \in W_1$. If there is any vertex $r \in R_3$, then $\{w, v_1, v_2, r, v_4, y\}$ induces a P_6 or a C_6 by (ix).
- (xiii) Each component Z of W_1 is homogeneous in $G \setminus A$. Proof: Otherwise, there are adjacent vertices $z, z' \in Z$ and a vertex $u \notin W_1 \cup A$ adjacent to z and not to z'. By the preceding points u is in $R_2 \cup X_{34}$. If $u \in R_2$, then z'-z-u- v_3 - v_4 - v_5 is a P_6 . If $u \in X_{34}$, then z'-z-u- v_3 - v_2 -x is a P_6 by (ix).
- (xiv) Each component Z of W_1 has either a neighbor in R_2 and no neighbor in X_{34} , or a neighbor in X_{34} and no neighbor in R_2 . Proof: If Z has no neighbor in $R_2 \cup X_{34}$, then by the preceding points we have $N(Z) = Q_1 \cup Q_5 \cup A'$ for some $A' \subseteq A$, and so N(Z) is a clique by Theorem 3.2, contradicting the hypothesis that G has no clique cutset. On the other hand if Z has a neighbor $r \in R_2$ and a neighbor $u \in X_{34}$, then by (xiii) for any $z \in Z$ we see that $\{z, r, v_3, u\}$ induces a C_4 .
- (xv) Each component Z of W_1 is a clique. Proof: Suppose that Z contains non-adjacent vertices z, z'. By (xiii) and (xiv) z and z' have a common neighbor u in $R_2 \cup X_{34}$. Then $\{z, u, z', t\}$ or $\{z, u, z', v_1\}$ induces a C_4 .
- (xvi) If Z, Z' are distinct components of W_1 , then $N(Z) \cap N(Z') \cap (R_2 \cup X_{34}) = \emptyset$. (Otherwise there is a C_4 as in the proof of (xv).)
- (xvii) $A = \emptyset$. Proof: Suppose that there exists $a \in A$. Since G has no universal vertex,

there is a non-neighbor z of a. By Theorem 3.2(a) and by (vi) we have $z \in W_1$. By (xiii) and (xiv) z has a neighbor $u \in R_2 \cup X_{34}$. But then $\{a, u, z, t\}$ or $\{a, u, z, v_1\}$ induces a C_4 .

By (xii) and (xvii) we have $V(G) = Q_1 \cup \cdots \cup Q_5 \cup R_2 \cup R_5 \cup X_{12} \cup X_{34} \cup W_1$. Now it is a routine matter to check that G is a blowup of $F_{k,\ell}$ for some $k, \ell \geq 1$. We clarify this point by exhibiting the mapping $Q_v \to v$ of the definition of a blowup, as follows. If Z is any component of W_1 , we say that it is an R_2 component (resp. X_{34} -component) if it has a neighbor in R_2 (resp. in X_{34}), and we call the set $N(Z) \cap R_2$ (resp. $N(Z) \cap X_{34}$) the support of Z. By (xiv) and (xvi) the supports are non-empty and pairwise disjoint. Let Z_1, Z_2, \ldots, Z_p be the R_2 -components of W_1 , and let Z'_1, Z'_2, \ldots, Z'_q be the X_{34} -components of W_1 . Let k = p + 1 and $\ell = q + 1$. Then:

- $Z_i \to u_i$ and $N(Z_i) \cap R_2 \to a_i$ for each $i \in \{1, 2..., p\}$, and $X_{12} \to u_{p+1}$ and $Q_2 \to a_{p+1}$, and $R_2 \setminus \bigcup_{i=1}^p (N(Z_i) \cap R_2) \to a_0$.
- $Z'_j \to w_j$ and $N(Z'_j) \cap X_{34} \to b_j$ for each $j \in \{1, 2..., q\}$, and $R_5 \to w_{q+1}$ and $Q_4 \to b_{q+1}$, and $X_{34} \setminus \bigcup_{j=1}^q (N(Z'_j) \cap X_{34}) \to b_0$.
- $Q_1 \to x, Q_5 \to y, \text{ and } Q_3 \to z.$

Since the components of W_1 and their supports are cliques, we see that G is a blowup of $F_{k,\ell}$. This completes the proof of the theorem.

When there is a C_5 , no C_6 and no F_2

Here we give the proof of the last item of Theorem 1.5.

Theorem 3.8 Let G be a (P_6, C_4) -free graph that has no clique-cutset and no universal vertex, and suppose that G is C_6 -free and F_2 -free. If G contains a C_5 , then G is either a belt or a boiler.

Proof. Let $C = \{v_1, \ldots, v_5\}$ be the vertex-set of a C_5 in G with edges $v_i v_{i+1}$ (mod 5). We use the same notation as in Theorem 3.2. We choose C such that |T| is minimized. Remark that since G is (P_6, C_4, C_6) -free every hole in G has length 5 and is dominating by Theorem 3.1. We establish a number of properties. (Some of them were also proved in [14, Lemma 5].)

- (i) If $X_{i-2,i-1} \cup X_{i+1,i+2} = \emptyset$, then T_i is complete to $T_{i-1} \cup T_{i+1}$. Proof: Up to symmetry let i = 1 and suppose that $X_{23} \cup X_{45} = \emptyset$ and that some vertex $t_1 \in T_1$ has a non-neighbor $t_2 \in T_2$. Let $C' = \{t_1, v_2, v_3, v_4, v_5\}$. So C' induces a C_5 , and t_2 has only two neighbors on it, so the choice of C (minimizing |T|) implies the existence of a vertex that has three neighbors on C' and two on C. Such a vertex must be in $X_{23} \cup X_{45}$, a contradiction.
- (ii) Every component Z of W_i is anticomplete to one of T_{i-1}, T_{i+1}. Proof: Let i = 1 and suppose that there are vertices z, z' ∈ Z such that z has a neighbor t₂ ∈ T₂ and z' has a neighbor t₅ ∈ T₅. If we can choose z = z', then C ∪ {z, t₂, t₅} induces an F₂. Otherwise let P be a shortest path between z and z' in G[Z]. Then V(P) ∪ {t₂, v₂, t₅, v₅} contains an induced P₆.

(iii) For every component Z of W_i , every vertex of $T_{i-1} \cup T_{i+1}$ is either complete or anticomplete to Z.

Proof: Let i = 1. Suppose that y, z are adjacent vertices in Z and that some vertex $t_2 \in T_2$ is adjacent to y and not to z. By (ii) $[Z, T_5] = \emptyset$. Then $z-y-t_2-v_3-v_4-v_5$ is a P_6 .

(iv) Every vertex in W_i has a neighbor in $X_{i-2,i+2}$. In particular if $W_i \neq \emptyset$, then $X_{i-2,i+2} \neq \emptyset$.

Proof: Let i = 1 and suppose that some vertex of W_1 has no neighbor in X_{34} . Let Z be the component of W_1 that contains this vertex. By (ii) we may assume that $[Z, T_5] = \emptyset$. Let $Z_0 = \{z \in Z \mid z \text{ has no neighbor in } X_{34}\},\$ so $Z_0 \neq \emptyset$. Let Y_0 be a component of $G[Z_0]$, and let $Y_1 = N(Y_0) \cap (Z \setminus Z_0)$ and $Y_2 = N(Y_0) \cap T_2$, and $A_0 = N(Y_0) \cap A$. By Theorem 3.2 and since $[Z, T_5] = \emptyset$ we have $N(Y_0) = \{v_1\} \cup T_1 \cup Y_1 \cup Y_2 \cup A_0$ and Y_0 is complete to $\{v_1\} \cup T_1$, and by (iii) Y_2 is complete to Y_0 . Suppose that some vertex $y \in Y_1$ is not complete to Y_0 . Then there are adjacent vertices $y_0, z_0 \in Y_0$ and a vertex $x \in X_{34}$ such that $z_0 - y_0 - y - x - v_4 - v_5$ is a P_6 . Hence Y_0 is complete to $N(Y_0) \setminus A_0$. Since G has no clique-cutset, there are nonadjacent vertices $u, v \in N(Y_0)$. By Theorem 3.2 and (iii) we know that $[Y_1 \cup A_0, \{v_1\} \cup T_1 \cup Y_2]$ is complete, so we have either (a) $u, v \in Y_1$, or (b) $u \in Y_1$ and $v \in A_0$, or (c) $u \in T_1$ and $v \in Y_2$. Pick any $y_0 \in Y_0$. In case (a), by the definition of Z_0 there are vertices $x, x' \in X_{34}$ such that $xu, x'v \in E(G)$. If we can choose x = x', then $\{x, u, y_0, v\}$ induces a C_4 ; and in the opposite case either $\{x, x', u, v, y_0\}$ induces a non-dominating C_5 (if $xx' \in E(G)$), because v_5 has no neighbor in it, or $\{y_0, u, v, x, x', v_4\}$ induces a C_6 , a contradiction. In case (b) we may choose y_0 adjacent to v. By the definition of Z_0 , u has a neighbor $x \in X_{34}$. Then $vx \notin E(G)$, for otherwise $\{v, x, u, y_0\}$ induces a C_4 . But then $\{v_1, v_3, v_4, v_5, x, y_0, u, v\}$ induces an F_2 . In case (c), $\{y_0, u, v_2, v\}$ induces a C_4 .

- (v) $X_{i+2,i-2}$ is anticomplete to one of T_{i-1}, T_{i+1} . Proof: Let i = 1 and suppose that there are vertices $x, y \in X_{34}$ such that x has a neighbor $t_2 \in T_2$ and y has a neighbor $t_5 \in T_5$. Then $xt_5 \notin E(G)$, for otherwise $\{v_1, t_2, x, t_5\}$ induces a C_4 ; and similarly $yt_2 \notin E(G)$. Moreover $xy \notin E(G)$, for otherwise v_2 - t_2 -x-y- t_5 - v_5 is a P_6 . But then $\{v_1, v_2, v_3, v_4, x, y, t_2, t_5\}$ induces an F_2 .
- (vi) If $[X_{i+2,i-2}, T_{i+1}] \neq \emptyset$ then $X_{i-1,i} = \emptyset$. Likewise if $[X_{i+2,i-2}, T_{i-1}] \neq \emptyset$ then $X_{i,i+1} = \emptyset$. Proof: Let i = 1, and suppose that some vertex $x \in X_{34}$ has a neighbor $t \in T_2$ and that there is a vertex $y \in X_{51}$. Then $xy \notin E(G)$, for otherwise $\{x, v_4, v_5, y\}$ induces a C_4 , and $ty \in E(G)$, for otherwise v_2 -t-x- v_4 - v_5 -y is a P_6 ; but then $C \cup \{t, x, y\}$ induces an F_2 .
- (vii) Every vertex in $X_{i+2,i-2}$ that has a neighbor in T_{i+1} is complete to T_{i-2} . Proof: Let i = 1, and suppose that some vertex $x \in X_{34}$ has a neighbor $t \in T_2$ and that x is not adjacent to a vertex $y \in T_4$. Then by Theorem 3.2(b), $ty \notin E(G)$. But then $C \cup \{t, x, y\}$ induces an F_2 .
- (viii) If $W_i \neq \emptyset$, then $[X_{i+2,i-2}, T_{i+2} \cup T_{i-2}]$ is complete. Proof: Let i = 1, and suppose that, up to symmetry, there are nonadjacent vertices $x \in X_{34}$ and $t \in T_3$ and that there is a vertex $w \in W_1$. Then $\{w, v_1, v_2, t, v_4, x\}$ induces a P_6 or a C_6 .

Suppose that $X = \emptyset$. Then (iv) implies that $W = \emptyset$, so $V(G) = C \cup T \cup A$. Moreover $A = \emptyset$, for otherwise any vertex in A is universal in G, by Theorem 3.2(a); and (i) implies that $[T_i, T_{i+1}]$ is complete for all *i*. So G is a blowup of C_5 , which is a special case of a belt.

Now assume that $X \neq \emptyset$, say $X_{34} \neq \emptyset$. By Theorem 3.2(d) and by symmetry, we may assume that $X_{23} \cup X_{45} \cup X_{51} = \emptyset$, so $X = X_{12} \cup X_{34}$, and consequently, by (v) and (vi) and up to symmetry, that $[X_{34}, T_5] = \emptyset$ and $[X_{12}, T_5] = \emptyset$. By (iv) we have $W = W_1 \cup W_4$, and by Theorem 3.2(e) one of W_1, W_4 is empty, so, still up to symmetry, we may assume that $W_4 = \emptyset$. Let:

Clearly $W_1 = W_1^T \cup W_1^N$ and $X_{34} = X_{34}^T \cup X_{34}^N \cup X_{34}^0$. Moreover we have $X_{34}^N \subseteq X_{34}^W \subseteq X_{34}^N \cup X_{34}^T$. Recall that $[W_1, T_1]$ is complete and that $[W_1, T_3 \cup T_4 \cup X_{12}] = \emptyset$ by Theorem 3.2(b)–(e). By (i), $[T_1, T_2 \cup T_5]$ and $[T_4, T_3 \cup T_5]$ are complete. We establish some additional facts.

(ix) $[W_1, T_5] = \emptyset$.

Proof: Suppose that $w \in W_1$ and $t \in T_5$ are adjacent. By (iv) w has a neighbor $x \in X_{34}$. Since $[X_{34}, T_5] = \emptyset$, we see that v_5 -t-w-x- v_3 - v_2 is a P_6 .

- (x) $[W_1^T, W_1^N] = \emptyset$. This follows directly from (iii).
- (xi) For every edge wx with $w \in W_1$ and $x \in X_{34}$, every vertex u in T_2 is either complete or anticomplete to $\{w, x\}$. Hence $[W_1^T, X_{34}^N] = \emptyset$ and $[W_1^N, X_{34}^T] = \emptyset$. Also every vertex u in A is either complete or anticomplete to $\{w, x\}$.

Proof: In the opposite case there is a C_4 in $G[\{v_1, w, x, v_3, u\}]$.

- (xii) $[A, X_{34}^T \cup W_1^T]$ is complete. Proof: Consider any $a \in A$. First pick any $x \in X_{34}^T$, so x has a neighbor $t \in T_2$. Then $at \in E(G)$ by Theorem 3.2(a), and $ax \in E(G)$, for otherwise $\{a, t, x, v_4\}$ induces a C_4 . Now pick any $w \in W_1^T$. So w has a neighbor $t \in T_2$ and, by (iv), a neighbor $x \in X_{34}$. Then $xt \in E(G)$ by (xi), so $x \in X_{34}^T$, and $ax \in E(G)$ by the preceding point of this claim. Then $aw \in E(G)$, for otherwise $\{a, v_1, w, x\}$ induces a C_4 .
- (xiii) Any vertex $x \in X_{34}^W$ is complete to $(X_{34} \setminus x) \cup T_3 \cup T_4$. Proof: Suppose up to symmetry that x has a non-neighbor $y \in (X_{34} \setminus x) \cup T_3$. Let $w \in W_1$ be any neighbor of x. Then either $\{w, x, y, v_3\}$ induces a C_4 (if $wy \in E(G)$) or v_5 - v_1 -w-x- v_3 -y is a P_6 .
- (xiv) Every vertex in X_{12} has a neighbor in T_3 . Proof: Suppose on the contrary that the set $Z = \{z \in X_{12} \mid z \text{ has no} neighbor in <math>T_3\}$ is non-empty, and let Y be the vertex-set of a component of G[Z]. Let $Y' = N(Y) \cap X_{12}$, $T'_1 = N(Y) \cap T_1$, $T'_2 = N(Y) \cap T_2$,

and $A' = N(Y) \cap A$. By Theorem 3.2 and the current assumption we have $N(Y) = \{v_1, v_2\} \cup Y' \cup T'_1 \cup T'_2 \cup A'$. Suppose that some vertex $u \in Y' \cup T'_1 \cup T'_2$ is not complete to Y; so there are adjacent vertices $y, z \in Y$ with $uy \in E(G)$ and $uz \notin E(G)$. If $u \in Y'$, then $u \in X_{12} \setminus Z$, so u has a neighbor $t \in T_3$, and then z-y-u-t-v_4-v_5 is a P_6 . If $u \in T'_1$, then z-y-u-v_5-v_4-v_3 is a P_6 . The proof is similar if $u \in T'_2$. Hence Y is complete to $\{v_1, v_2\} \cup Y' \cup T'_1 \cup T'_2$. Since G has no clique cutset, the set N(Y)contains two non-adjacent vertices u, v. By Theorem 3.2, and since $[T_1, T_2]$ is complete, and up to symmetry, we have $u \in Y'$ and $v \in Y' \cup T'_1 \cup T'_2 \cup A'$. So u has a neighbor $t \in T_3$. Pick any $y \in Y$. If $v \in Y'$, then v has a neighbor $s \in T_3$, and either $\{y, u, v, s\}$ induces a C_4 (if we can choose s = t) or $\{y, u, v, s, t\}$ induces C_5 that does not dominate v_5 . If $v \in T'_1$, then $\{y, u, t, v_4, v_5, v\}$ induces a C_6 . If $v \in T'_2$, then either $\{y, u, t, v\}$ induces a C_4 , or $\{y, u, t, v_3, v\}$ induces a C_5 that does not dominate v_5 . If $v \in A'$, then we can choose y adjacent to v, and then $\{y, u, t, v\}$ induces a C_4 .

- (xv) $[X_{12}, T_1]$ is complete. Proof: This follows from (xiv) and (vii).
- (xvi) $[X_{12}, A]$ is complete. Proof: Pick any $a \in A$ and $x \in X_{12}$. By (xiv) x has a neighbor $t \in T_3$. We have $at \in E(G)$ by Theorem 3.2, and $ax \in E(G)$, for otherwise $\{a, t, x, v_1\}$ induces a C_4 .
- (xvii) For any component Z of $G[X_{34}^0]$ the set $[Z, N(Z) \setminus A]$ is complete and $N(Z) \setminus A$ is a clique.

Proof: Let Z be (the vertex-set of) a component of $G[X_{34}^0]$. Then $N(Z) \setminus A \subseteq \{v_3, v_4\} \cup T_3 \cup T_4 \cup X_{34}^N \cup X_{34}^T$. First suppose that $[Z, N(Z) \setminus A]$ is not complete. So there are adjacent vertices $y, z \in Z$ and a vertex $u \in N(Z) \setminus A$ with $uy \in E(G)$ and $uz \notin E(G)$. Clearly $u \notin \{v_3, v_4\}$. If $u \in X_{34}^N \cup X_{34}^T$, then, by (xiii) u has no neighbor in W_1 , so u has a neighbor $t \in T_2$, and then z-y-u-t- v_1 - v_5 is a P_6 . If $u \in T_3$, then z-y-u- v_2 - v_1 - v_5 is a P_6 . If $u \in T_4$, then z-y-u- v_5 - v_1 - v_2 is a P_6 , a contradiction. Now suppose that $N(Z) \setminus A$ is not a clique, so it contains two non-adjacent vertices u, v. Pick any $z \in Z$. By Theorem 3.2 and since $[T_4, T_3]$ is complete we have either (a) $u, v \in X_{34}^N \cup X_{34}^T$ or (b) $u \in X_{34}^N \cup X_{34}^T$ and $v \in T_3 \cup T_4$. In case (a), by (xiii) u and v have no neighbor in W_1 , so they have neighbors respectively t and t' in T_2 ; then $\{z, u, v, t, t'\}$ induces either a C_4 or a non-dominating C_5 (because v_5 has no neighbor in W_1 , so u has a neighbor $t \in T_2$. If $v \in T_3$, then $\{z, u, t, v_2, v\}$ induces a non-dominating C_5 (because of v_5). If $v \in T_4$, then v_2 -t-u-z-v- v_5 is a P_6 .

(xviii) For each component Z of $G[X_{34}^0]$ there are vertices $a \in A, z \in Z, w \in W_1$ and $x \in X_{34}^N$ such that $az, wx \in E(G)$ and $aw, ax \notin E(G)$. Proof: We have $N(Z) \subseteq X_{34}^N \cup X_{34}^T \cup \{v_3, v_4\} \cup T_3 \cup T_4 \cup A$. Since G has no clique cutset there are two non-adjacent vertices $u, v \in N(Z)$. By (xvii) and Theorem 3.2, and since $T_3 \cup T_4$ is a clique, we have $u \in A$ and consequently $v \in X_{34}^N \cup X_{34}^T$, and by (xii) $v \in X_{34}^N$. So v has a neighbor $w \in W_1$, and $uw \notin E(G)$ by (xi).

Suppose that $X_{34}^N = \emptyset$. Then $X_{34}^0 = \emptyset$ by (xviii), and $W_1^N = \emptyset$ by (iv) and (xi). So $X_{34} = X_{34}^T$ and $W_1 = W_1^T$. Now $[A, V(G) \setminus A]$ is complete by

Theorem 3.2, (xii) and (xvi), and since G has no universal vertex it follows that $A = \emptyset$. So $V(G) = C \cup T_1 \cup \cdots \cup T_5 \cup W_1^T \cup X_{12} \cup X_{34}^T$. Let:

$$\begin{array}{rcl} Q_i &=& \{v_i\} \cup T_i \text{ for each } i \in \{1,4,5\}.\\ Q_2 &=& \{v \mid v \text{ is universal in } G[\{v_2\} \cup T_2 \cup X_{12} \cup W_1]\}\\ R_2 &=& (\{v_2\} \cup T_2 \cup X_{12} \cup W_1) \setminus Q_2.\\ Q_3 &=& \{v \mid v \text{ is universal in } G[\{v_3\} \cup T_3 \cup X_{34}]\}.\\ R_3 &=& (\{v_3\} \cup T_3 \cup X_{34}) \setminus Q_3. \end{array}$$

Hence $V(G) = Q_1 \cup \cdots \cup Q_5 \cup R_2 \cup R_3$. We claim that $Q_2 \neq \emptyset$. Indeed, if $W_1 = \emptyset$ then $v_2 \in Q_2$. So suppose that $W_1 \neq \emptyset$. By (iv) and (xiii) the set $Y_{34} = \{x \in X_{34} \mid x \text{ has a neighbor in } W_1\}$ is non-empty and is a clique. Since $X_{34}^N \cup X_{34}^0 = \emptyset$, every vertex of Y_{34} has a neighbor in T_2 , and it follows that some vertex t in T_2 is complete to Y_{34} (otherwise there are vertices $y', y'' \in Y_{34}$ and $t', t'' \in T_2$ that induce a C_4). Let us verify that $t \in Q_2$. We know that t is complete to $T_2 \setminus t$ by Theorem 3.2. Any $w \in W_1$ has a neighbor $x \in X_{34}$ by (iv), and so $tw \in E(G)$ for otherwise $\{t, x, w, v_1\}$ induces a C_4 . Now consider any $y \in X_{12}$. Pick any $w \in W_1$ and $x \in X_{34} \cap N(w)$. Then $ty \in E(G)$, for otherwise $y \cdot v_2 \cdot t \cdot x \cdot v_4 \cdot v_5$ is a P_6 . So $t \in Q_2$, and the claim that $Q_2 \neq \emptyset$ is established. Now the properties of the nine sets $Q_1, \ldots, Q_5, R_2, R_3$ satisfy all the axioms of the belt. We make this more precise as follows:

- By Theorem 3.2 and by (i), we know that Q_1, Q_4 and Q_5 are non-empty cliques, $[Q_1 \cup Q_4, Q_5]$ is complete and $[Q_1, Q_4] = \emptyset$.
- Clearly Q_2 and Q_3 are cliques, with $v_3 \in Q_3$, and $Q_2 \neq \emptyset$ as seen above.
- By (i), (vii) and Theorem 3.2, $[Q_1, Q_2 \cup R_2]$ and $[Q_4, Q_3 \cup R_3]$ are complete.
- By the definition of Q_2 and Q_3 , Theorem 3.2, and since $[X_{12} \cup X_{34}, T_5] = \emptyset$, we have $[Q_2, Q_4 \cup Q_5] = \emptyset$ and $[Q_3, Q_1 \cup Q_5] = \emptyset$.
- By the definition of Q_2 , Q_3 , R_2 and R_3 , we have: for each $j \in 2, 3$, $[Q_j, R_j]$ is complete, every vertex in R_j has a non-neighbor in R_j , every vertex in $Q_2 \cup R_2$ has a neighbor in $Q_3 \cup R_3$ (by (iv) and (xiv)), and every vertex in $Q_3 \cup R_3$ has a neighbor in $Q_2 \cup R_2$ (by the definition of X_{34}^T)).

Thus G is a belt.

Therefore we may assume that $X_{34}^N \neq \emptyset$. So, $W_1 \neq \emptyset$. Then:

- (xix) $X_{34}^N \cup X_{34}^T \cup T_3 \cup T_4$ is a clique.
 - Proof: By (viii) and by Theorem 3.2, it is enough to show that $X_{34}^N \cup X_{34}^T$ is a clique. Suppose that there are non-adjacent vertices $x, x' \in X_{34}^N \cup X_{34}^T$. Pick any $y \in X_{34}^N$. By (xiii), $y \notin \{x, x'\}$ and $yx, yx' \in E(G)$, and $x, x' \in X_{34}^T$. So x has a neighbor $t \in T_2$, and x' has a neighbor $t' \in T_2$. Then $\{y, x, x', t, t'\}$ induces a cycle of length either 4 (if t = t') or 5 and not dominating (because v_5 has no neighbor in it), a contradiction.
- (xx) $G[X_{34}^0]$ is chordal. Proof: If $G[X_{34}]$ contains a hole C, then C either has length 4 or at least 6 or is a non-dominating C_5 (because of v_5).
- (xxi) For every simplicial vertex s of $G[X_{34}^0]$, there are vertices $a \in A$ and $u \in X_{34}^0 \cup X_{34}^N$ with $sa, su \in E(G)$ and $au \notin E(G)$.

Proof: Let Z be the vertex-set of the component of $G[X_{34}^0]$ that contains s. So $N_Z(s)$ is a clique. We have $N(s) \subseteq N_Z(s) \cup (N(Z) \setminus A) \cup A$. By (xvii) the set $N_Z(s) \cup (N(Z) \setminus A)$ is a clique. Since G has no clique cutset there are two non-adjacent vertices u, v in N(s), and so $u \in N_Z(s) \cup (N(Z) \setminus A)$ and $v \in A$. Since $N(Z) \setminus A \subseteq X_{34}^N \cup X_{34}^T \cup \{v_3, v_4\} \cup T_3 \cup T_4$ and A is complete to $X_{34}^T \cup \{v_3, v_4\} \cup T_3 \cup T_4$ by Theorem 3.2 and (xii), we have $u \in X_{34}^0 \cup X_{34}^N$.

Let $A_0 = \{a \in A \mid a \text{ has a neighbor in } X_{34}^0 \text{ and a non-neighbor in } X_{34}^N \}$. By (xviii) we have $A_0 \neq \emptyset$. Since A_0 and X_{34}^N are cliques (by Theorem 3.2 and (xix)) and by the third item of Lemma 2.3, there is a vertex x_0 in X_{34}^N that is anticomplete to A_0 . Let w_0 be a neighbor of x_0 in W_1 . Then w_0 is anticomplete to A_0 by (xi).

- (xxii) x_0 is complete to X_{34}^0 . Proof: If there is a vertex $x \in X_{34}^0$ that is non-adjacent to x_0 , then v_2 - v_1 - w_0 - x_0 - v_4 -x is a P_6 .
- (xxiii) $G[X_{34}^0 \cup A_0]$ is chordal. Proof: If $G[X_{34}^0 \cup A_0]$ contains a hole C, then C either has length 4 or at least 6 or is a non-dominating C_5 (because of w_0).
- (xxiv) Every vertex in X_{34}^0 has a neighbor in A_0 .
 - Proof: Let Z be the vertex-set of any component of $G[X_{34}^0]$, and let $Z_A =$ $\{z \in Z \mid z \text{ has a neighbor in } A_0\}$, and suppose that $Z \neq Z_A$. By (xxiii) and Lemma 2.2 applied to $G[Z \cup A_0]$, Z and A_0 , some simplicial vertex s' of G[Z] has no neighbor in A_0 . Let $S = \{s'' \in Z \mid N_Z[s''] = N_Z[s']\}$; so the vertices in S are simplicial in G[Z] and pairwise clones, and S is a clique. Let s be a vertex in S with the smallest number of neighbors in A. If shas any neighbor $a \in A_0$, then, since $\{S, A_0\}$ is a graded pair of cliques, by Lemma 2.3 all vertices in S are adjacent to a, a contradiction. So s has no neighbor in A_0 . By (xxi) there are vertices $b \in A$ and $u \in Z \cup X_{34}^N$ with $sb, su \in E(G)$ and $bu \notin E(G)$. We know that $b \notin A_0$, so b is complete to X_{34}^N , and so $u \in \mathbb{Z}$. Moreover $u \notin S$, for otherwise the choice of s is contradicted (since $b \in A$, and the pair $\{A, S\}$ is graded). Hence u is not a simplicial vertex of G[Z], and so it has a neighbor $v \in Z \setminus N[s]$. Consider any $a \in A_0$. We know that $as \notin E(G)$; then also $au \notin E(G)$, for otherwise $\{a, b, s, u\}$ induces a C_4 ; and $av \notin E(G)$, for otherwise s-u-v-a- v_1 - w_0 is a P_6 . Hence $\{s, u, v\}$ is anticomplete to A_0 . Let Y be the component of $G[Z \setminus Z_A]$ that contains s, u, v. Let $Z_Y = \{z \in Z_A \mid z \text{ has a neighbor in}$ Y}, and let $A_Y = \{b' \in A \mid b' \text{ has a neighbor in } Y\}$. Note that $[A_Y, X_{34}^N]$ is complete. Since $Z_A \neq \emptyset$ and G[Z] is connected, $Z_Y \neq \emptyset$. Then $[Y, Z_Y]$ is complete, for otherwise there are adjacent vertices $y,y' \in Y,$ a vertex $z \in Z_Y$, and a vertex $a \in A_0$ such that $y'-y-z-a-v_1-w_0$ is a P_6 . Then Z_Y is a clique, for otherwise $\{s, v, z, z'\}$ induces a C_4 for any two non-adjacent vertices z, z' in Z_Y . Moreover, for any $b' \in A_Y$ and $z \in Z_Y$, we have $b'z \in E(G)$, for otherwise $\{b', y, z, a\}$ induces a C_4 for any $y \in Y \cap N(b)$ and $a \in A_0 \cap N(z)$. We have $N(Y) \subseteq Z_Y \cup A_Y \cup (N(Z) \setminus A)$, and by Theorem 3.2, items (xii) and (xvii) and the fact that $[A_Y, X_{34}^N]$ is complete, this set is a clique, a contradiction.
- (xxv) $[X_{34}^0, X_{34}^T]$ is complete.

Proof: Suppose that some $z \in X_{34}^0$ and $x \in X_{34}^T$ are non-adjacent. By (xxiv) x has a neighbor $a \in A_0$. Then, by (xii), $\{a, x, x_0, z\}$ induces a C_4 .

(xxvi) For any two components Z, Z' of $G[W_1]$, the sets $N(Z) \cap X_{34}$ and $N(Z') \cap X_{34}$ are disjoint.

Proof: Otherwise $\{v_1, x, z, z'\}$ induces a C_4 for some $z \in Z, z' \in Z'$ and $x \in N(Z) \cap N(Z') \cap X_{34}$.

Let:

$$Q = \{v_1\} \cup T_1, B = \{v_3, v_4\} \cup T_3 \cup T_4 \cup X_{34}^T \cup X_{34}^N, M = \{v_2, v_5\} \cup T_2 \cup T_5 \cup X_{12} \cup W_1, L = X_{34}^0.$$

We know that A and Q are cliques, and B is a clique by (xix). Every vertex in L has a neighbor in A by (xxiv), and every vertex in M has a neighbor in B by (iv) and (xiv). The subgraph G[L] is $(P_4, 2P_3)$ -free by Lemma 2.4, using A_0 in the role of Y, x_0 in the role of c, and v_1 and w_0 , respectively, in the role of c' and c''. The subgraph G[M] has at least three components because $\{v_2\} \cup X_{12}$, $\{v_5\} \cup T_5$ and W_1 are pairwise anticomplete to each other and non-empty, and G[M] is $(P_4, 2P_3)$ -free by Lemma 2.4, using B in the role of Y, v_1 in the role of c and the fact that G[M] is not connected. Hence the sets Q, A, B, L, M form a partition of V(G) that shows that G is a boiler.

4 Additional properties of belts and boilers

Belts and boilers have some additional and useful properties that we give below.

4.1 Belts

Theorem 4.1 Let G be a belt, with the same notation as in Section 1. Then:

- (a) For each $j \in \{2,3\}$, any two non-adjacent vertices in R_j have no common neighbor in Q_{5-j} .
- $(b) [R_2, R_3] = \emptyset.$
- (c) For each $j \in \{2,3\}$, every vertex of Q_j that has a neighbor in R_{5-j} is complete to Q_{5-j} .
- (d) The graphs $G[R_2]$ and $G[R_3]$ are $(P_4, 2P_3)$ -free.

Proof. (a) If two non-adjacent vertices $r, r' \in R_2$ have a common neighbor v in Q_3 , then $\{v_1, r, r', v\}$ induces a C_4 .

(b) Suppose that any $r_2 \in R_2$ and $r_3 \in R_3$ are adjacent. By the definition of a belt, for each $j \in \{2,3\}$ the vertex r_j has a non-neighbor $r'_j \in R_j$. Then $r_2r'_3 \notin E(G)$, for otherwise $\{r_2, r'_3, v_4, r_3\}$ induces a C_4 , and similarly $r_3r'_2 \notin E(G)$. Then $\{r'_2, v_1, r_2, r_3, v_4, r'_3\}$ induces a P_6 or C_6 .

(c) Consider any $u \in Q_3$ which has a neighbor $r_2 \in R_2$, and suppose that u has a non-neighbor $v \in Q_2$. By the definition of a belt r_2 has a non-neighbor

 $r'_2 \in R_2$. Then $ur'_2 \notin E(G)$, for otherwise $\{u, r'_2, v_1, r_2\}$ induces a C_4 . But then $r'_2 - v - r_2 - u - v_4 - v_5$ is a P_6 . The proof is similar when j = 2.

(d) Pick a vertex $q_i \in Q_i$ for each $i \in \{1, 4, 5\}$. Lemma 2.4, using vertices q_1, q_4 and q_5 in the role of c, c' and c'', implies that $G[R_2]$ is $(P_4, 2P_3)$ -free. The proof is similar for $G[R_3]$.

Note that Theorem 4.1(d) means that (R_2, Q_3) and (R_3, Q_2) are C-pairs.

4.2 Boilers

Let G be a boiler, with the same notation as in the definition. Since every vertex in A has a non-neighbor in B, Lemma 2.3 implies that some vertex b^* in B is anticomplete to A. Let m^* be any neighbor of b^* in M. Then m^* too is anticomplete to A (for otherwise $\{m^*, a, b, b^*\}$ induces a C_4 for some $a \in A$ and $b \in B_1 \cup B_2$). Pick a vertex $z \in Q$.

If L is a clique, then $(A \cup M, B \cup L)$ is a C-pair, so the structure of G is completely determined by Theorem 2.1 and the fact that Q is complete to $A \cup M$ and anticomplete to $B \cup L$.

Therefore let us assume that L is not a clique. Let U be the set of universal vertices of L. (Possibly $U = \emptyset$.) Let $A_L = \{a \in A \mid a \text{ has a neighbor in } L\}$ and $A'_L = \{a \in A \mid a \text{ has a neighbor in } L \setminus U\}$.

Theorem 4.2 Let G be a boiler, with the same notation as above, and assume that L is not a clique. Then, up to a permutation of the set $\{3, \ldots, k\}$, there is an integer $j \in \{3, \ldots, k\}$ such that the following hold:

- (i) For each $a \in A_L \setminus A'_L$, there is an integer $i \in \{j, \ldots, k\}$ such that a is complete to $M_1 \cup B_1 \cup \cdots \cup M_{i-1} \cup B_{i-1}$ and anticomplete to $M_i \cup B_i \cup \cdots \cup M_k \cup B_k$;
- (ii) A'_L is complete to $(M \cup B) \setminus (M_k \cup B_k)$ and anticomplete to $M_k \cup B_k$;
- (iii) $A \setminus A_L$ is complete to $M_1 \cup B_1 \cup \cdots \cup M_{j-1} \cup B_{j-1}$ and anticomplete to $M_j \cup B_j \cup \cdots \cup M_k \cup B_k$.

Proof. Since A and B are disjoint cliques and G is C_4 -free, $[A, B_1 \cup B_2]$ is complete, and b^* is anticomplete to A, Lemma 2.3 implies that there is a permutation of $\{3, ..., k\}$ such that for every vertex $a \in A$ there is an integer $i \in \{3, ..., k\}$ such that a is complete to $M_1 \cup B_1 \cup \cdots \cup M_{i-1} \cup B_{i-1}$ and anticomplete to $M_i \cup B_i \cup \cdots \cup M_k \cup B_k$. We may assume that $b^* \in B_k$ and $m^* \in M_k$.

Let $J = \{i \in \{3, ..., k\} \mid \text{some vertex in } A \text{ is anticomplete to } M_i \cup B_i\}$. By the preceding paragraph there is an integer j such that $J = \{j, ..., k\}$. In particular this implies the validity of item (i) of the lemma.

Now consider any vertex $a \in A'_L$. So a has a neighbor $x \in L \setminus U$, so x has a non-neighbor $x' \in L$, and by the definition of a boiler we have $ax' \notin E(G)$. Suppose that a is not complete to $M_i \cup B_i$ for some i < k, so a is anticomplete to $M_i \cup B_i$, and pick any $m \in M_i$. Then m-z-a-x-b*-x' is a P_6 . So a is complete to $(M \cup B) \setminus (M_k \cup B_k)$, which proves (ii).

Finally, consider any vertex $d \in A \setminus A_L$. So d is anticomplete to L. Pick any $i \in J$ and $b \in B_i$. So there is a vertex $a \in A_L$ that is anticomplete to $B_i \cup M_i$. By the definition of A_L the vertex a has a neighbor $x \in L$. Then db is not an edge, for otherwise $\{d, b, x, a\}$ induces a C_4 . It follows that d is anticomplete to $B_i \cup M_i$ which proves (iii).

5 Bounding the chromatic number

In this section, we give a proof for Theorem 1.1 and Theorem 1.2.

We say that a stable set of a graph G is good if it meets every clique of size $\omega(G)$ in G; and that it is very good if it meets every (inclusionwise) maximal clique of G. Moreover, we say that a clique K in G is a t-clique of G if |K| = t.

We will use the following theorem as a tool in proving Theorem 1.1.

Theorem 5.1 Let G be a graph such that every proper induced subgraph G' of G satisfies $\chi(G') \leq \lfloor \frac{5}{4}\omega(G') \rfloor$. Suppose that one of the following occurs:

- (i) G has a vertex of degree at most $\lfloor \frac{5}{4}\omega(G) \rfloor 1$.
- (ii) G has a (very) good stable set;
- (iii) G has a stable set S such that $G \setminus S$ is perfect.
- (iv) For some integer $t \ge 5$ the graph G has t stable sets S_1, \ldots, S_t such that $\omega(G \setminus (S_1 \cup \cdots \cup S_t)) \le \omega(G) (t-1).$

Then $\chi(G) \leq \left\lceil \frac{5}{4}\omega(G) \right\rceil$.

Proof. (i) Suppose that G has a vertex u with $d(u) \leq \lfloor \frac{5}{4}\omega(G) \rfloor - 1$. By the hypothesis we have $\chi(G \setminus u) \leq \lfloor \frac{5}{4}\omega(G \setminus u) \rfloor$. So we can take any $\chi(G \setminus u)$ -coloring of $G \setminus u$ and extend it to a $\lfloor \frac{5}{4}\omega(G) \rfloor$ -coloring of G, using for u a (possibly new) color that does not appear in its neighborhood.

(ii) Suppose that G has a (very) good stable set S. Then $\omega(G \setminus S) = \omega(G) - 1$. By the hypothesis we have $\chi(G \setminus S) \leq \lfloor \frac{5}{4}\omega(G \setminus S) \rfloor = \lfloor \frac{5}{4}(\omega(G) - 1) \rfloor \leq \lfloor \frac{5}{4}\omega(G) \rfloor - 1$. We can take any $\chi(G \setminus S)$ -coloring of $G \setminus S$ and add S as a new color class, and we obtain a coloring of G. Hence $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

(iii) Suppose that G has a stable set S such that $G \setminus S$ is perfect. Then $\chi(G \setminus S) = \omega(G \setminus S) \leq \omega(G)$. We can take any $\chi(G \setminus S)$ -coloring of $G \setminus S$ and add S as a new color class. Hence $\chi(G) \leq \omega(G) + 1 \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

(iv) Note that $\frac{t}{t-1} \leq \frac{5}{4}$ because $t \geq 5$. We take any $\chi(G \setminus (S_1 \cup \dots \cup S_t))$ coloring of $G \setminus (S_1 \cup \dots \cup S_t)$ and use S_1, \dots, S_t as t new colors and we get a
coloring of G. Then $\chi(G) \leq \chi(G \setminus (S_1 \cup \dots \cup S_t)) + t \leq \lceil \frac{5}{4}(\omega(G) - (t-1)) \rceil + t \leq \lceil \frac{5}{4}\omega(G) \rceil$ because $\frac{t}{t-1} \leq \frac{5}{4}$.

5.1 Chromatic bound for blowups

We first note that by a result of Lovász [27], any blowup of a perfect graph is a perfect graph.

For any integer $t \geq 2$ we say that G is a *t*-blowup of H if $|Q_u| = t$ for all $u \in V(H)$. Remark that, for an integer k, a k-coloring of the *t*-blowup of H is equivalent to a collection of k stable sets of H such that every vertex of H belongs to at least t of them.

Blowups of Petersen graph

Let H_1 be the Petersen graph as shown in Figure 2.

Lemma 5.1 Let G be the 2-blowup of the Petersen graph H_1 . Then $\chi(G) = 5$.

Proof. The five sets $\{a, b, w_3, w_6\}$, $\{b, c, w_1, w_4\}$, $\{a, c, w_2, w_5\}$, $\{z, w_1, w_3, w_5\}$ and $\{z, w_2, w_4, w_6\}$ are five stable sets, and every vertex of H_1 belongs to two of them. As observed above this is equivalent to a 5-coloring of G. This is optimal because G has 20 vertices and every stable set in G has size at most 4.

Theorem 5.2 If G is any blowup of the Petersen graph H_1 , then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. Let $q = \omega(G)$. We prove the theorem by induction on |V(G)|. We may assume that G is connected (otherwise we consider each component separately) and that G is not a clique. Moreover, the theorem holds easily if G is any induced subgraph of H_1 . Now suppose that G is not an induced subgraph of H_1 . So there is $x \in V(H_1)$ such that $|Q_x| \ge 2$. Since G is connected and not a clique there exists $y \in N_{H_1}(x)$ such that $Q_y \neq \emptyset$, and so $q \ge 3$. By Theorem 5.1 (ii) we may assume that G has no good stable set.

Note that every maximal clique of G consists of $Q_u \cup Q_v$ for some edge $uv \in E(H_1)$ with $Q_u \neq \emptyset$ and $Q_v \neq \emptyset$, and we denote it as Q_{uv} . We say that such a maximal clique is *balanced* if $|Q_u| \geq 2$ and $|Q_v| \geq 2$.

Suppose that every q-clique of G is balanced. So $q \ge 4$. Let X be a subset of V(G) obtained by taking min $\{2, |Q_v|\}$ vertices from Q_v for each $v \in V(H_1)$. We claim that:

$$\omega(G \setminus X) = q - 4. \tag{1}$$

Proof: Consider any maximal clique K in G. As observed above we have $K = Q_u \cup Q_v$ for some edge $uv \in E(G)$ with $Q_u \neq \emptyset$ and $Q_v \neq \emptyset$. Suppose that |K| = q. The hypothesis that every q-clique is balanced implies that X contains exactly four vertices from K, so $|K \setminus X| = |K| - 4 = q - 4$. Now suppose that $|K| \leq q - 1$. The definition of X implies that either $|K| \geq 3$ and X contains at least two vertices from Q_u and one from Q_v , or vice-versa, or |K| = 2 and X contains one vertex from each of Q_u, Q_v , and in any case we have $|K \setminus X| \leq q - 4$. Thus (1) holds.

By (1) and the induction hypothesis we have $\chi(G \setminus X) \leq \lfloor \frac{5}{4}\omega(G \setminus X) \rfloor = \lfloor \frac{5}{4}(q-4) \rfloor = \lfloor \frac{5}{4}q \rfloor - 5$. By Lemma 5.1 we know that G[X] is 5-colorable. We can take any $\chi(G \setminus X)$ -coloring of $G \setminus X$ and use five new colors for the vertices of X, and we obtain a coloring of G. It follows that $\chi(G) \leq \lfloor \frac{5}{4}q \rfloor$ as desired.

Therefore we may assume that some q-clique of G is not balanced, say, up to symmetry, the clique Q_{za} , with $|Q_z| \ge q - 1$ and $|Q_a| \le 1$. So we also have $|Q_b| \le 1$ and $|Q_c| \le 1$.

Suppose that both Q_{aw_1} and Q_{aw_4} are q-cliques. So $|Q_{w_1}| \ge q-1$ and $|Q_{w_4}| \ge q-1$. This implies $|Q_{w_j}| \le 1$ for each $j \in \{2, 3, 5, 6\}$. It follows that each of the cliques Q_{bw_2} , Q_{bw_5} , Q_{cw_3} , Q_{cw_6} , $Q_{w_2w_3}$, $Q_{w_5w_6}$ has size at most 2, so they are not q-cliques. Then $\{z, w_1, w_4\}$ is a good stable set.

Therefore we may assume that one of Q_{aw_1} and Q_{aw_4} is not a q-clique. Likewise, one of Q_{bw_2} and Q_{bw_5} is not a q-clique, and one of Q_{cw_3} and Q_{cw_6} is not a q-clique. This implies, up to symmetry, that we have either: (a) each of Q_{aw_1} , Q_{bw_5} , Q_{cw_3} is not a q-clique, or (b) each of Q_{aw_1} , Q_{bw_2} , Q_{cw_3} is not a q-clique. In case (a), we see that $\{z, w_2, w_4, w_6\}$ is a good stable set of G. Hence assume that we are in case (b) and not in case (a), and so Q_{bw_5} is a q-clique, and so $|Q_{w_5}| \ge q-1$. Hence $|Q_{w_4}| \le 1$ and $|Q_{w_6}| \le 1$. It follows that Q_{aw_4} and Q_{cw_6} are cliques of size at most 2, so they are not q-cliques. Now Q_{aw_4} , Q_{bw_2} , and Q_{cw_6} are not q-cliques, so we are in a situation similar to case (a). This completes the proof.

We immediately have the following.

Corollary 5.1 If G is any blowup of C_5 , then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Blowups of
$$F_3$$

Consider the graph F_3 as shown in Figure 1.

Lemma 5.2 Let G be the 2-blowup of F_3 . Then $\chi(G) = 7$.

Proof. For each $v \in V(F_3)$ we call v and v' the two vertices of Q_v in G. The seven sets $\{x, v_4, v_6\}$, $\{y, v_2, v'_6\}$, $\{z, v'_2, v'_4\}$, $\{x', v_5\}$, $\{y', v_1\}$, $\{z', v_3\}$ and $\{v'_1, v'_3, v'_5\}$ form a 7-coloring of G. Hence $\chi(G) \leq 7$. On the other hand we see that $\chi(G[Q_{v_1} \cup Q_{v_2} \cup Q_{v_3} \cup Q_y \cup Q_z]) \geq 5$ since that subgraph has 10 vertices and no stable set of size 3, and consequently $\chi(G[Q_x \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_y \cup Q_z]) \geq 7$. Hence $\chi(G) \geq 7$.

We say that G is a special blowup of F_3 if (up to symmetry) we have $|Q_u| \leq 1$ for each $u \in \{x, v_4, v_5, v_6\}$ and $|Q_v| = t$ for each $v \in \{v_1, v_2, v_3, y, z\}$, for some integer $t \geq 2$.

Lemma 5.3 Let G be a special blowup of F_3 . Then $\chi(G) \leq \lfloor \frac{5}{4} \omega(G) \rfloor$.

Proof. We prove the theorem by induction on |V(G)|. If $Q_x \cup Q_{v_4} \cup Q_{v_5} \cup Q_{v_6} = \emptyset$, then G is a blowup of C_5 , so the lemma holds by Corollary 5.1. Hence assume that $Q_x \cup Q_{v_4} \cup Q_{v_5} \cup Q_{v_6} \neq \emptyset$. It follows that $\omega(G) = 2t + 1$. Let X be a subset of V(G) obtained by taking two vertices from Q_v for each $v \in \{v_1, v_2, v_3, y, z\}$ and the set $Q_x \cup Q_{v_4} \cup Q_{v_5} \cup Q_{v_6}$. Then $\omega(G \setminus X) = 2t - 4 = \omega(G) - 5$. In F_3 the six sets $\{v_1, v_3, v_5\}$, $\{v_2, y\}$, $\{v_2, z\}$, $\{v_1, y\}$, $\{v_3, z\}$ and $\{x, v_4, v_6\}$ are such that every vertex from $\{v_1, v_2, v_3, y, z\}$ belongs to two of them and every vertex from $\{x, v_4, v_5, v_6\}$ belongs to one of them; hence they are equivalent to a 6-coloring of G[X]. We can take any $\chi(G \setminus X)$ -coloring of $G \setminus X$ and use six new colors for X, and we obtain a coloring of G. Hence $\chi(G) \leq \chi(G \setminus X) + 6 \leq [\frac{5}{4}(\omega(G) - 5)] + 6 = [\frac{5}{4}\omega(G) - \frac{25}{4}] + 6 \leq [\frac{5}{4}\omega(G)]$.

Theorem 5.3 If G is any blowup of F_3 , then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. Let $q = \omega(G)$. We prove the theorem by induction on |V(G)|. Obviously the theorem holds if G is any induced subgraph of F_3 . Now suppose that G is not an induced subgraph of F_3 . By Theorem 5.1 (ii) we may assume that G has no good stable set.

Note that every maximal clique of G consists of $Q_u \cup Q_v \cup Q_w$ for some triangle $\{u, v, w\}$ in F_3 , and we denote it as Q_{uvw} . We say that such a maximal clique is *balanced* if $|Q_u| \ge 2$, $|Q_v| \ge 2$, and $|Q_w| \ge 2$.

Suppose that every q-clique of G is balanced. Let X be a subset of V(G) obtained by taking min $\{2, |Q_v|\}$ vertices from Q_v for each $v \in V(F_3)$. The hypothesis that every q-clique is balanced implies that X contains exactly six

vertices from each q-clique of G, so $\omega(G \setminus X) = \omega(G) - 6$. By the induction hypothesis we have $\chi(G \setminus X) \leq \lceil \frac{5}{4}\omega(G \setminus X) \rceil = \lceil \frac{5}{4}(q-6) \rceil = \lceil \frac{5}{4}q - \frac{30}{4} \rceil \leq \lceil \frac{5}{4}q \rceil - 7$. By Lemma 5.2 we know that G[X] is 7-colorable. We can take any $\chi(G \setminus X)$ coloring of $G \setminus X$ and use seven new colors for the vertices of X, and we obtain a coloring of G. It follows that $\chi(G) \leq \lceil \frac{5}{4}q \rceil$ as desired. Therefore we may assume that some q-clique of G is not balanced.

For each $v \in V(F_3)$, let R_v consist of one vertex from Q_v if $Q_v \neq \emptyset$, otherwise let $R_v = \emptyset$. We claim that we may assume that:

Each of
$$Q_x$$
, Q_y and Q_z is non-empty. (1)

Proof: Suppose up to symmetry that $Q_x = \emptyset$. If also $Q_{v_2} = \emptyset$, then G is a blowup of $F_3 \setminus \{x, v_2\}$, which is a chordal graph, so $\chi(G) = \omega(G)$ and the theorem holds. Therefore $Q_{v_2} \neq \emptyset$. Likewise, $Q_{v_1} \neq \emptyset$ and $Q_{v_3} \neq \emptyset$. Since $R_{v_1} \cup R_{v_3} \cup R_{v_5}$ is not a good stable set, we have $Q_{v_5} = \emptyset$. Moreover, if $Q_{v_4} \cup Q_{v_6} = \emptyset$, then G is a blowup of C_5 , and the theorem holds by Corollary 5.1. So up to symmetry we may assume that $Q_{v_4} \neq \emptyset$. Now if $Q_z = \emptyset$, then G is a blowup of $F_3 \setminus \{x, z, v_5\}$, which is a chordal graph, so $\chi(G) = \omega(G)$ and the theorem holds. So suppose that $Q_z \neq \emptyset$. Then $R_{v_2} \cup R_{v_4} \cup R_z$ is a good stable set of G. Hence we may assume that (1) holds.

We claim that we may assume that:

Each of Q_{xyz} , Q_{xyv_3} , Q_{yzv_5} , Q_{zxv_1} , $Q_{xv_1v_2}$, $Q_{yv_3v_4}$ is a *q*-clique, and either $Q_{zv_5v_6}$ or $Q_{xv_2v_3}$ is a *q*-clique. (2)

Proof: If two of $R_{v_1}, R_{v_3}, R_{v_5}$ are empty, say $R_{v_1} \cup R_{v_3} = \emptyset$, then G is a blowup of $F_3 \setminus \{v_1, v_3\}$, which is a chordal graph, so $\chi(G) = \omega(G)$. So at least two of $R_{v_1}, R_{v_3}, R_{v_5}$ are non-empty. Since $R_{v_1} \cup R_{v_3} \cup R_{v_5}$ is not a good stable set, there is a q-clique in $G \setminus (R_{v_1} \cup R_{v_3} \cup R_{v_5})$, and this clique can only be Q_{xyz} . Now consider the stable set $R_{x46} = R_x \cup R_{v_4} \cup R_{v_6}$, which is not empty by (1). Since it is not a good stable set, there is a q-clique in $G \setminus R_{x46}$, and so Q_{yzv_5} is a q-clique. Likewise, Q_{xyv_3} and Q_{zxv_1} are q-cliques. Now consider the stable set $R_x \cup R_{v_5}$. Since it is not a good stable set, we deduce that one of $Q_{yv_3v_4}$ and $Q_{zv_6v_1}$ is a q-clique. Likewise, one of $Q_{zv_5v_6}$ and $Q_{xv_2v_3}$ is a q-clique, and one of $Q_{xv_1v_2}$ and $Q_{yv_4v_5}$ is a q-clique. Up to symmetry this yields the possibilities described in (2). Thus we may assume that (2) holds.

Next we claim that we may assume that:

$$Q_{zv_5v_6}$$
 is not a *q*-clique. (3)

Proof: Suppose not.

First we show that we may assume that $|Q_{v_1}| \geq 2$. Suppose that $|Q_{v_1}| = \varepsilon \leq 1$. Let $a = |Q_{v_2}|$ and $b = |Q_x|$. Since $Q_{xv_1v_2}$ is a q-clique, we have $a + b + \varepsilon = q$. Then, using the q-cliques given by (2), we deduce successively that $|Q_z| = a$, $|Q_y| = \varepsilon$, $|Q_{v_5}| = b$, $|Q_{v_6}| = \varepsilon$, $|Q_{v_3}| = a$, and $|Q_{v_4}| = b$. We have $|Q_{xv_2v_3}| = b + 2a \leq q = a + b + \varepsilon$, so $a \leq \varepsilon$. Also we have $|Q_{yv_4v_5}| = 2b + \varepsilon \leq q = a + b + \varepsilon$, so $b \leq a$. Hence $b \leq a \leq \varepsilon \leq 1$, which means that G is isomorphic to an induced subgraph of F_3 , so the theorem holds. So we may assume that $|Q_{v_1}| \geq 2$. Likewise, we may assume that $|Q_{v_3}| \geq 2$, and $|Q_{v_5}| \geq 2$. Next we may assume that $|Q_x| \ge 2$ (otherwise since Q_{xyz} and Q_{yzv_5} are qcliques (by (2)), we have $|Q_{v_5}| \le 1$, a contradiction). Likewise, we have $|Q_y| \ge 2$ and $|Q_z| \ge 2$.

Further, we may assume that $|Q_{v_6}| \geq 2$ (otherwise since by (2) and by our assumption, Q_{yzv_5} and $Q_{zv_5v_6}$ are q-cliques, we have $|Q_y| \leq 1$, a contradiction). Likewise, we have $|Q_{v_2}| \geq 2$ and $|Q_{v_4}| \geq 2$.

Hence the above analysis shows that every q-clique in G is balanced, and the theorem holds as above. Thus we may assume that (3) holds.

Now by (2) and (3), we may assume that $Q_{xv_2v_3}$ is a q-clique. Let $a = |Q_{v_5}|$, $b = |Q_z|$ and $t = |Q_y|$. Then by (2), a + b + t = q, and by using the q-cliques given by (2), we deduce successively that $|Q_x| = a$, $|Q_{v_1}| = t$ and $|Q_{v_2}| = b$. Then again by (2) and by our assumption, since $Q_{xv_2v_3}$ and Q_{xyv_3} are q-cliques, we see that $|Q_{v_3}| = b = t$. So, q = a + 2t. Since $Q_{yv_3v_4}$ is a q-clique (by (2)), we have $|Q_{v_4}| = a$. Thus $|Q_{yv_4v_5}| = 2a + t \le q = a + 2t$, so $a \le t$. First suppose that $t \le 1$. Then $a \le 1$ and hence $q \le 3$. This implies that, we may assume that $|Q_{v_6}| \le 1$ (otherwise since $Q_{zv_5v_6}$ is not a q-clique (by (3)), $a + 2t > a + t + |Q_{v_6}|$, and hence $t \ge 2$ which is a contradiction.). Thus G is an induced subgraph of F_3 and the theorem holds. So suppose that $t \ge 2$. Since some q-clique of G is not balanced, there is a vertex $w \in \{x, v_4, v_5\}$ such that $|Q_w| \le 1$. In any case, we have $a \le 1$, and hence $q \le 2t + 1$. Now $|Q_{v_6v_1z}| = |Q_{v_6}| + 2t \le q \le 2t + 1$, so $|Q_{v_6}| \le 1$. Hence the above analysis shows that G is a special blowup of F_3 , so the theorem holds as a consequence of Lemma 5.3.

Blowups of H_2, H_3, H_4 and H_5

Let H_2, \ldots, H_5 be the graphs as shown in Figure 2.

Theorem 5.4 Let G be any blowup of H_2 . Then $\chi(G) \leq \lfloor \frac{5}{4} \omega(G) \rfloor$.

Proof. By the definition of a blowup, V(G) is partitioned into cliques Q_v , $v \in V(H_2)$. If $Q_v \neq \emptyset$ we call v one vertex of Q_v , and if $|Q_v| \ge 2$ we call v' a second vertex of Q_v . We denote, e.g., the clique $Q_a \cup Q_{v_1} \cup Q_{v_2}$ by $Q_{av_1v_2}$, etc. Let $q = \omega(G)$. We prove the theorem by induction on |V(G)|. By Theorem 5.1 we may assume that every vertex $x \in V(G)$ satisfies $d(x) \ge \lceil \frac{5}{4}q \rceil$ and that G has no good stable set.

Suppose that $Q_{v_1} \cup Q_{v_2} = \emptyset$. If $Q_b \neq \emptyset$, then $\{b\}$ is a good stable set. If $Q_b = \emptyset$, then G is a blowup of C_5 , and the result follows from Corollary 5.1. Hence we may assume that $Q_{v_1} \cup Q_{v_2} \neq \emptyset$. Then both Q_{v_1} and Q_{v_2} are nonempty, for otherwise some vertex in $Q_{v_1} \cup Q_{v_2}$ is simplicial (and so has degree less than q). Since $N[v_1]$ is partitioned into the two cliques Q_{v_6} and $Q_{av_1v_2}$, and $d(v_1) \geq \lceil \frac{5}{4}q \rceil$, we deduce that $|Q_{v_6}| \geq \lceil \frac{q}{4} \rceil + 1 \geq 2$; and similarly (since $N[v_1]$ is also partitioned into cliques $Q_{av_1v_6}$ and Q_{v_2}) we have $|Q_{v_2}| \geq \lceil \frac{q}{4} \rceil + 1 \geq 2$. Likewise $|Q_{v_3}| \geq 2$ and $|Q_{v_1}| \geq 2$. By the same argument we may assume that both Q_{v_4} and Q_{v_5} are non-empty, and consequently $|Q_{v_4}| \geq 2$ and $|Q_{v_5}| \geq 2$.

If $Q_c = \emptyset$, then G is a blow-up of F_3 , and the theorem follows from Theorem 5.3. So we may assume that $|Q_c| \ge 1$. Then the set of maximal cliques of G is $\{Q_{av_1v_6}, Q_{av_1v_2}, Q_{av_2v_3}, Q_{bv_3v_4}, Q_{bv_4v_5}, Q_{bv_5v_6}, Q_{abcv_3}, Q_{abcv_6}\}$.

Suppose that $|Q_c| \ge 2$. Consider the five stable sets $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{c, v'_1, v'_5\}$, $\{c', v'_2, v'_4\}$, and $\{v'_3, v'_6\}$. Then every maximal clique of *G* contains

four vertices from their union; so the result follows from Theorem 5.1 (iv) (with t = 5). Therefore let us assume that $|Q_c| = 1$.

Suppose that both Q_a and Q_b are non-empty. Consider the six stable sets $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{v'_3, v'_6\}$, $\{a, v'_5\}$, $\{b, v'_2\}$ and $\{c, v'_1, v'_4\}$. Then every maximal clique of G contains five vertices from their union; so the result follows from Theorem 5.1 (iv) (with t = 6).

Therefore we may assume up to symmetry that $Q_a = \emptyset$. Note that Q_{bcv_3} is not a q-clique of G, because $Q_{bv_3v_4}$ is a clique and $|Q_{v_4}| > |Q_c|$. Likewise, Q_{bcv_6} is not a q-clique of G. Consider the five stable sets $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{v'_2, v'_4\}$, $\{v'_3, v'_6\}$ and $\{c, v'_1, v'_5\}$. Then every maximal clique of G contains four vertices from their union, except for Q_{bcv_3} and Q_{bcv_6} , which contain only three vertices from their union, but we know that these two are not q-cliques. It follows that $\omega(G \setminus X) \leq q - 4$, so the result follows from Theorem 5.1 (iv). \Box

Theorem 5.5 Let G be any blowup of H_3 . Then $\chi(G) \leq \lfloor \frac{5}{4} \omega(G) \rfloor$.

Proof. By the definition of a blowup, V(G) is partitioned into nine cliques Q_i , $v_i \in V(H_3)$. If $Q_i \neq \emptyset$ we call v_i one vertex of Q_i . Note that every maximal clique of G consists of $Q_u \cup Q_v \cup Q_w$ for some triangle $\{u, v, w\}$ in H_3 . If each of Q_1, Q_4, Q_7 is non-empty, then $\{v_1, v_4, v_7\}$ is a good stable set of G, and the result follows from Theorem 5.1 (ii). Hence we may assume that one of Q_1, Q_4, Q_7 is empty. Likewise we may assume that one of Q_2, Q_5, Q_8 is empty, and that one of Q_3, Q_6, Q_9 is empty. Up to symmetry and relabelling, this yields the following two cases.

(i) $Q_i \cup Q_{i+1} = \emptyset$ for some *i*. Then *G* is a chordal graph, so $\chi(G) = \omega(G)$. (ii) $Q_i \cup Q_{i+2} \cup Q_{i+4} = \emptyset$ for some *i*. Then *G* is a blowup of C_5 , and the result follows from Corollary 5.1.

Theorem 5.6 Let G be a blowup of H_5 . Then $\chi(G) \leq \left\lceil \frac{5}{4} \omega(G) \right\rceil$.

Proof. By the definition of a blowup, V(G) is partitioned into ten cliques Q_v , $v \in V(H_5)$. Note that if $Q_{t_{i-1}} \cup Q_{t_{i+1}} = \emptyset$ for some *i*, then the vertices of Q_{t_i} can be moved to Q_{v_i} , so we may assume in that case that $Q_{t_i} = \emptyset$ too. Let $q = \omega(G)$. We prove the theorem by induction on |V(G)|.

If $Q_{v_i} \cup Q_{t_i} = \emptyset$ for some *i*, then *G* is a chordal graph (as it is a blowup of a chordal graph), so $\chi(G) = \omega(G)$. Hence let us assume that $Q_{v_i} \cup Q_{t_i} \neq \emptyset$ for all *i*. For each *i* let $x_i = t_i$ if $Q_{t_i} \neq \emptyset$, else let $x_i = v_i$. In any case if $d(x_i) < \lceil \frac{5}{4}q \rceil$ then we can conclude using Theorem 5.1 (i) and induction. Hence assume that $d(x_i) \ge \lceil \frac{5}{4}q \rceil$ for all *i*. If $x_i = t_i$, then $N[x_i]$ is partitioned into the two sets $Q_{v_{i-1}}$ and $Q_{v_i} \cup Q_{t_i} \cup Q_{v_{i+1}}$, and the latter set is a clique (of size at most q), so the inequality $d(x_i) \ge \lceil \frac{5}{4}q \rceil$ implies $|Q_{v_{i-1}}| \ge \lceil \frac{q}{4} \rceil + 1 \ge 2$. Similarly $|Q_{v_{i+1}}| \ge \lceil \frac{q}{4} \rceil + 1 \ge 2$. On the other hand suppose that $x_i = v_i$ (i.e., $Q_{t_i} = \emptyset$). If $Q_{t_{i-2}} \neq \emptyset$ then the same argument implies $|Q_{v_{i-1}}| \ge 2$; while if $Q_{t_{i-2}} = \emptyset$, then, as observed above, we have $Q_{t_{i-1}} = \emptyset$, so the same argument (about v_i), implies $|Q_{v_{i-1}}| \ge 2$ again. Hence in all cases we have $|Q_{v_i}| \ge 2$ for all *j*.

For each *i* let u_i, v_i be two vertices in Q_{v_i} . Consider the five stable sets $\{u_i, v_{i+2}\}$ (i = 1, ..., 5), and let X be their union. Any maximal clique K of G is included in $Q_{v_i} \cup Q_{v_{i+1}}$ for some *i*, and so K contains $u_i, v_i, u_{i+1}, v_{i+1}$. So $\omega(G \setminus X) = q - 4$ and we can conclude using Theorem 5.1 (iv) (with t = 5) and the induction hypothesis.

Theorem 5.7 Let G be any blowup of H_4 . Then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. By the definition of a blowup, V(G) is partitioned into nine cliques Q_v , $v \in V(H_4)$. If $Q_v \neq \emptyset$ we call v one vertex of Q. If $Q_{v_5} \cup Q_{v_6} = \emptyset$, then G is a chordal graph, so $\chi(G) = \omega(G)$. Hence let us assume up to symmetry that $Q_{v_5} \neq \emptyset$. If $Q_{v_1} = \emptyset$, then G is a blowup of H_5 , so the result follows from Theorem 5.6. Hence let us assume that $Q_{v_3} \neq \emptyset$. If $Q_{v_3} = \emptyset$, then G is a blowup of H_5 again. Hence let us assume that $Q_{v_3} \neq \emptyset$. Now it is easy to see that $\{v_1, v_3, v_5\}$ is a good stable set, so the result follows from Theorem 5.1 (ii). \Box

Blowups of $F_{k,\ell}$

Theorem 5.8 For integers $k, \ell \ge 0$, let G be any blowup of $F_{k,\ell}$. Then $\chi(G) \le \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. We use the same notation as in the definition of $F_{k,\ell}$. By the definition of a blowup V(G) is partitioned into cliques $Q_v, v \in V(F_{k,\ell})$, such that $[Q_u, Q_v]$ is complete if $uv \in E(F_{k,\ell})$ and otherwise $[Q_u, Q_v] = \emptyset$. Let $Q_A = \bigcup_{i=0}^k Q_{a_i}$ and $Q_B = \bigcup_{j=0}^{\ell} Q_{b_j}$. Let $D = \bigcup_{v \in U \cup W} Q_v$. As a convention it is convenient, for any $u \in V(F_{k,\ell})$ such that $Q_u \neq \emptyset$, to use the name u for one vertex of Q_u ; moreover if $|Q_u| \ge 2$ we call u' another vertex from Q_u , and if $|Q_u| \ge 3$ we call u'' a third vertex from Q_u . We denote, e.g., the clique $Q_x \cup Q_y \cup Q_{u_i}$ by Q_{xyu_i} , etc. Let $q = \omega(G)$. We prove the lemma by induction on $|V(G)| + k + \ell$. We may assume that G does not satisfy any of the hypotheses (i)–(iii) of Theorem 5.1, for otherwise we can find a $\lceil \frac{5}{4}q \rceil$ -coloring of G using induction.

We remark that if k > 0 and $Q_{u_i} = \emptyset$ for some $i \in \{1, \ldots, k\}$, then the vertices of Q_{a_i} can be moved to Q_{a_0} , and so G is a blowup of $F_{k-1,\ell}$ and the result holds by induction. Moreover, if k > 0 and either $|Q_{a_i}| \leq \lceil \frac{q}{4} \rceil$ for some i, or $|Q_y| \leq \lceil \frac{q}{4} \rceil$, then, since $N[u_i] = Q_{a_i} \cup Q_{u_i} \cup Q_x \cup Q_y$ and $Q_{a_iu_ix}$ and Q_{u_ixy} are cliques that contain u_i , we have $d(u_i) \leq q - 1 + \lceil \frac{q}{4} \rceil < \lceil \frac{5}{4}q \rceil$, so the result holds by Theorem 5.1 (i). In summary, we may assume that:

If
$$k > 0$$
 then $Q_{u_i} \neq \emptyset$ and $|Q_{a_i}| > \lceil \frac{q}{4} \rceil$ for all i , and $|Q_y| > \lceil \frac{q}{4} \rceil$.
Also if $\ell > 0$ then $Q_{w_j} \neq \emptyset$ and $|Q_{b_j}| > \lceil \frac{q}{4} \rceil$ for all j and $|Q_x| > \lceil \frac{q}{4} \rceil$. (1)

It follows from (1) that $k \leq 3$, for otherwise $|Q_A| > q$; and similarly $\ell \leq 3$. Moreover, if $\ell > 0$ then $k \leq 2$, for otherwise $|Q_A \cup Q_x| > q$; and similarly if k > 0 then $\ell \leq 2$. We assume up to symmetry that $k \leq \ell$. Consequently we have either k = 0 and $\ell \leq 3$, or k = 1 and $\ell \in \{1, 2\}$, or $k = \ell = 2$. In any case $k \leq 2$. If $k \leq 1$ and $\ell \leq 1$, then G is a blowup of (an induced subgraph of) H_5 , so the result follows from Theorem 5.6. So we may assume that $\ell \geq 2$. Consequently we have either k = 0 and $\ell \in \{2, 3\}$, or k = 1 and $\ell = 2$, or $k = \ell = 2$.

Suppose that $Q_A = \emptyset$. Then $Q_z = \emptyset$, for otherwise $d(z) \leq q-1$, and $Q_y = \emptyset$, for otherwise $\{y\}$ is a good stable set. Then we can view G as a blowup of $F_{0,\ell-1}$ (putting Q_{b_ℓ} and Q_{w_ℓ} in the role of Q_z and Q_{a_0} respectively) and use induction. Therefore we may assume that $Q_A \neq \emptyset$. If $k \geq 1$, then $|Q_{a_1}| \geq 2$ by (1), and if k = 0 then $|Q_{a_0}| \geq 2$, for otherwise either $d(z) \leq q$ (if $Q_z \neq \emptyset$) or $d(a_0) \leq q$ (if $Q_z = \emptyset$). Hence in any case we have $|Q_A| \geq 2$. Let a, a' be two vertices from Q_A , chosen as follows: if k = 0, let $a, a' \in Q_{a_0}$. If k = 1, let $a, a' \in Q_{a_1}$. If k = 2, let $a \in Q_{a_1}$ and $a' \in Q_{a_2}$. Let $p = \max\{|Q_v|, v \in U \cup W\}$. So $p \ge 1$. We claim that:

We may assume that
$$p \ge 2$$
. (2)

Proof: Suppose that p = 1; so $Q_v = \{v\}$ for all $v \in U \cup W$. If $|Q_z| \leq 1$, then $U \cup W \cup Q_z$ is a stable set, and $G \setminus (U \cup W \cup Q_z)$ is perfect (a blowup of P_4), so the result follows from Theorem 5.1 (iii). Hence $|Q_z| \geq 2$. Define five stable sets as follows: Let $T_1 = \{a, b_1\}, T_2 = \{b_2, x\}, T_3 = \{z, x'\}, T_4 = \{a', y\},$ and $T_5 = \{z', y'\}$, where $y, y' \in Q_y$, with the convention that y' vanishes if $|Q_y| = 1$, and in that case if $|Q_x| \geq 3$ then $T_5 = \{z', x''\}$ for some $x'' \in Q_x \setminus \{x, x'\}$, and y too vanishes if $Q_y = \emptyset$. Let $T^* = T_1 \cup \cdots \cup T_5$. We claim that every maximal clique K of G satisfies $|K \setminus T^*| \leq q - 4$. The following cases (i)–(vii) occur: (i) $K = Q_z \cup Q_A$. Then K contains four vertices (z, z', a, a') from T^* , so

 $|K \setminus T^*| \le q - 4$. Likewise, if $K = Q_z \cup Q_B$, then K contains z, z', b_1, b_2 .

(ii) $K = Q_x \cup Q_A$. Then K contains a, a', x, x' from T^* .

(iii) $K = Q_y \cup Q_B$. Then $Q_y \neq \emptyset$ because Q_B is not a maximal clique (since $Q_z \neq \emptyset$). If $|Q_y| \ge 2$, then K contains four vertices b_1, b_2, y, y' from T^* . If $|Q_y| = 1$ then (since $|Q_z| \ge 2$) $|K| < |Q_z \cup Q_A| \le q$, and K contains three vertices b_1, b_2, y from T^* , so $|K \setminus T^*| \le q - 4$.

(iv) $k \ge 1$ and $K = Q_{xyu_i}$ for some $i \in \{1, \ldots, k\}$. Then $Q_y \ne \emptyset$ because Q_{xu_i} is not a maximal clique (since $Q_{a_i} \ne \emptyset$). If $|Q_y| \ge 2$, then K contains four vertices (x, y, x', y') from T^* . If $|Q_y| = 1$, then (since $|Q_{a_i}| \ge 2$) $|K| < |Q_{xu_ia_i}| \le q$ and K contains three vertices x, x', y from T^* .

(v) $k \ge 1$ and $K = Q_{xa_iu_i}$ for some $i \in \{1, \ldots, k\}$, say i = 1. If k = 1 then K contains x, x', a, a'. Suppose k = 2. Since $q \ge |Q_{xa_1a_2}|$, and $|Q_{a_2}| \ge 2$, we have $|K| \le q-1$. Then K contains three vertices x, x', a from T^* , so $|K \setminus T^*| \le q-4$. (vi) $K = Q_{xyw_j}$ for some $j \in \{1, \ldots, \ell\}$. If $|Q_y| \ge 2$ then K contains four vertices (x, y, x', y') from T^* . If $|Q_y| \le 1$, then K contains at least two vertices from T^* , so if $|K| \le q-2$ we are done. If $|K| \ge q-1$, then $|Q_x| + 2 \ge |K| \ge q-1 \ge |Q_z \cup Q_B| - 1 \ge 2(\ell + 1) - 1 \ge 5$, so $|Q_x| \ge 3$, so the vertex x'' exists and K contains three vertices x, x', x'' from T^* .

(vii) $K = Q_{yb_jw_j}$ for some $j \in \{1, \ldots, \ell\}$. Since $q \ge |Q_z \cup Q_B|$, we have $|Q_{b_j}| \le q - 2\ell$. If $\ell = 3$, then either $|K| \le q - 4$, or |K| = q - 3 and $Q_y \ne \emptyset$ and K contains y from T^* . Hence suppose that $\ell = 2$. So $b_j \in K$. Then either $|K| \le q - 3$, or |K| = q - 2 and K also contains y from T^* . So $|K \setminus T^*| \le q - 4$. In either case Theorem 5.1 (iv) implies the desired result. Thus (2) holds.

Suppose that $k \leq 1$. We know that $\ell \in \{2,3\}$. By (1) we have $|Q_{b_j}| \geq \lceil \frac{q}{4} \rceil + 1$ for all $j \in \{1, \ldots, \ell\}$. Recall that $Q_A \neq \emptyset$. Let $a^* = a_0$ if $Q_{a_0} \neq \emptyset$ and $a^* = a_1$ otherwise. In either case the set $N(a^*)$ can be partitioned into two cliques such that Q_z is one of them. By Theorem 5.1 (i) we may assume that $d(a) \geq \lceil \frac{5}{4}q \rceil$, so $|Q_z| \geq \lceil \frac{q}{4} \rceil + 1$. Consequently $q \geq |Q_z \cup Q_B| \geq (\ell + 1)(\lceil \frac{q}{4} \rceil + 1)$. The inequality $q \geq (\ell + 1)(\lceil \frac{q}{4} \rceil + 1)$ is violated if $\ell \geq 3$, so $\ell = 2$. Moreover, the inequality with $\ell = 2$ implies $q \geq 12$. Hence (1) yields that $|Q_x| \geq 3$, and $|Q_{b_j}| \geq 3$ for each $j \in \{1, 2\}$, and $|Q_A| \geq 3$, and similarly $|Q_z| \geq 3$.

Suppose that k = 0. We may assume that Q_{xyw_j} is a *q*-clique for each $j \in \{1, 2\}$, for otherwise the set $\{a_0, b_j, w_{3-j}\}$ is a good stable set. Hence $|Q_{w_1}| = |Q_{w_2}| = p$. Note that the set of maximal cliques of G is $\{Q_{za_0}, Q_{xa_0}, Q_{xyw_1}, Q_{xyw_2}, Q_{yw_1b_1}, Q_{yw_2b_2}, Q_{zb_0b_1b_2}\}$ plus $Q_{yb_0b_1b_2}$ if $Q_y \neq \emptyset$. Let $S_1 = \{b_1, w_2, a_0\}$, $S_2 = \{b_2, w_1, a'_0\}$, $S_3 = \{z, w'_1, w'_2\}$, and $S_4 = \{b'_1, x\}$. If $Q_y \neq \emptyset$, let $S_5 = \{a''_0, y\}$. If $Q_y = \emptyset$, then one of $Q_{w_1b_1}, Q_{w_2b_2}$ is a *q*-clique, for otherwise $\{x, z\}$ is

a good stable set; so for some $j \in \{1, 2\}$ we have $|Q_{w_j b_j}| = q \ge |Q_{b_1 b_2}|$, whence $p = |Q_{w_j}| \ge |Q_{b_{3-j}}| \ge 3$; so we let $S_5 = \{a_0'', w_1'', w_2''\}$. In either case, S_1, \ldots, S_5 are stable sets and it is easy to see that every maximal clique of G contains at least four vertices from their union, so the result follows from Theorem 5.1 (iv).

Now suppose that k = 1, and so $\ell = 2$. By (1), we have $|Q_y| \ge 2$. Note that the set of maximal cliques of G is $\{Q_{za_0a_1}, Q_{xa_0a_1}, Q_{xa_1u_1}, Q_{xyu_1}, Q_{xyw_1}, Q_{xyw_2}, Q_{yw_1b_1}, Q_{yw_2b_2}, Q_{yb_0b_1b_2}, Q_{zb_0b_1b_2}\}$. Let $S_1 = \{b_1, w_2, u_1\}$ plus a_0 if $Q_{a_0} \ne \emptyset$. Let $S_2 = \{b_2, w_1, a_1\}, S_3 = \{x, z\}, S_4 = \{y', z'\}$, and $S_5 = \{a'_1, y\}$. In either case, S_1, \ldots, S_5 are stable sets and that every maximal clique of G contains at least four vertices from their union, so the result follows from Theorem 5.1 (iv).

Finally suppose that k = 2 and $\ell = 2$. Let $S_1 = \{a_1, b_1, u_2, w_2\}, S_2 = \{a_2, b_2, u_1, w_1\}, S_3 = \{x, b_1'\}, S_4 = \{y, a_1'\}, and let S_5 consist of one vertex from each component of <math>Q_z \cup (D \setminus \{u_1, u_2, w_1, w_2\})$. Let $S^* = S_1 \cup \cdots \cup S_5$. We claim that every maximal clique K of G satisfies $|K \setminus S^*| \leq q - 4$. Indeed if $K = Q_x \cup Q_A$ then K contains x, a_1, a_1', a_2 from S^* . If $K = Q_z \cup Q_A$ then $Q_z \neq \emptyset$ and K contains z, a_1, a_1', a_2 . If $K = Q_{xa_1u_1}$ then K contains x, a_1, a_1', a_2 from S^* , so if $|K| \leq q - 1$ we are done; and if |K| = q then $|Q_{xa_2u_2}| = q \geq |Q_{xa_1a_2}|$ so $|Q_{u_2}| \geq 2$, so Q_{u_2} contains a vertex u_2' from S_5 . If $K = Q_{xyu_1}$ then K contains x, y, u_1 from S^* , so if $|K| \leq q - 1$ we are done; and if |K| = q then since $p \geq 2$ we have $|Q_{u_1}| \geq 2$, so Q_{u_1} contains a vertex u_1' from S^* . The other cases are symmetric. Hence the result follows from Theorem 5.1 (iv). This completes the proof.

5.2 Chromatic bound for bands, belts and boilers

Theorem 5.9 Let G be a band. Then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. We use the same notation as in the definition of a band (see also Figure 4:(b)), and we prove the theorem by induction on |V(G)|. First suppose that $[R_2, R_3]$ is not complete. By Lemma 2.3 there exist non-adjacent vertices $u \in R_2$ and $v \in R_3$ such that every maximal clique in $G[R_2 \cup R_3]$ contains u or v. If $Q_5 \neq \emptyset$, pick any $w \in Q_5$ and let $S = \{u, v, w\}$; else let $S = \{u, v\}$. Then S is a very good stable set of G, so the result follows from Theorem 5.1 (ii). Therefore we may assume that $[R_2, R_3]$ is complete. Now suppose that $[Q_1, Q_2]$ is not complete. By Lemma 2.3 there exist non-adjacent vertices $u \in Q_1$ and $v \in Q_2$ such that every maximal clique in $G[Q_1 \cup Q_2]$ contains u or v. If $Q_4 \neq \emptyset$, pick any $w \in Q_4$ and let $S = \{u, v, w\}$; else let $S = \{u, v\}$. Then S is a very good stable set of G, so the result follows from Theorem 5.1 (ii). Therefore we may assume that $[Q_1, Q_2]$ is complete, and similarly that $[Q_3, Q_4]$ is complete. Now G is a blowup of C_5 , so the result follows from Corollary 5.1.

We say that a graph G is an *extended* C-pair if V(G) can be partitioned into three sets Q, X, A such that (X, A) is a C-pair, Q is a clique, [Q, X] is complete and $[Q, A] = \emptyset$.

Lemma 5.4 Let G be an extended C-pair. Then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. We prove the lemma by induction on |V(G)|. Let V(G) be partitioned into Q, X, A as in the definition above. Let $q = \omega(G)$. If some vertex $a \in A$ has no neighbor in X, then a is simplicial, so d(a) < q, and we can conclude using

Theorem 5.1(i) and by the induction hypothesis. Therefore we may assume that every vertex in A has a neighbor in X.

Suppose that G[X] has four pairwise non-adjacent simplicial vertices s_1, s_2 , s_3, s_4 . If $d(s_i) \leq \lfloor \frac{5}{4}q \rfloor - 1$, then we can conclude using Theorem 5.1(i). So assume that $d(s_i) \geq \lfloor \frac{5}{4}q \rfloor$. We have $N(s_i) = Q \cup N_X(s_i) \cup N_A(s_i)$, and $Q \cup N_X(s_i)$ is a clique, so we must have $|N_A(s_i)| \geq \lfloor \frac{q}{4} \rfloor + 1$. By the definition of a \mathcal{C} -pair the sets $N_A(s_1), \ldots, N_A(s_4)$ are pairwise disjoint. It follows that $|A| \geq 4(\lfloor \frac{q}{4} \rfloor + 1) > q$, a contradiction. Hence G[X] has at most three pairwise non-adjacent simplicial vertices. If X is a clique then G is a chordal graph, so $\chi(G) = \omega(G)$ and the theorem holds trivially. Therefore we may assume that G[X] has exactly k pairwise non-adjacent simplicial vertices with $k \in \{2,3\}$. Since $G[X] \in \mathcal{C}$ and by Lemma 2.5, we have the following two cases (a) and (b).

(a) k = 2, so X is partitioned into three cliques X_1, X_2 and U such that X_1, X_2 are non-empty, $[U, X_1 \cup X_2]$ is complete and $[X_1, X_2] = \emptyset$. Suppose that $U \neq \emptyset$. Then Theorem 2.1 and the fact that every vertex in A has a neighbor in X implies that some vertex u in U is universal in G, so $\{u\}$ is a very good stable set and we conclude using Theorem 5.1(ii). Hence $U = \emptyset$. Then G is a band, and we conclude with Theorem 5.9.



Figure 7: Schematic representation of the graph in Case (b) of Lemma 5.4 where $U = \emptyset$. Here, each shaded circle represents a clique, and the circles inside the oval form a clique, a solid line between two circles indicates that the two sets are complete to each other, the absence of line between any two circles indicates that the sets are anticomplete to each other, and a dashed line between two circles indicates that the adjacency between the two sets are arbitrary.

(b) k = 3, so X is partitioned into five cliques X_1, X_2, X_3, W and U such that X_1, X_2, X_3 are non-empty and pairwise anticomplete, $[W, X_1 \cup X_2]$ is complete, $[W, X_3] = \emptyset$, and $[U, X \setminus U]$ is complete. As in case (a) we may assume that $U = \emptyset$. By Theorem 2.1 and the fact that every vertex in A has a neighbor in X, the set A is partitioned into four sets A_1, A_2, A_3, B such that $N_A(X_i) = A_i$ for each $i \in \{1, 2, 3\}, N_A(W) = A_1 \cup A_2 \cup B$, and $[W, A_1 \cup A_2]$ is complete, and there is no other edge between X and A. Moreover, if one of $[X_j, A_j]$ $(j \in \{1, 2, 3\})$ is not complete, then $[X_t, A_t]$ is complete for each $t \in \{1, 2, 3\} \setminus \{j\}$. See Figure 7.

Suppose that $B \neq \emptyset$. Since every vertex of A has a neighbor in X, every vertex of B has a neighbor in W. So by Lemma 2.3, there exists a vertex $w \in W$ such that [w, B] is complete. Hence w is universal in $G[V(G) \setminus (X_3 \cup A_3)]$. We may assume that $[X_3, A_3]$ is not complete (otherwise $\{w, x_3\}$, for any $x_3 \in X_3$, is a very good stable set of G, and we can conclude by using Theorem 5.1.). Then by Lemma 2.3, there exist non-adjacent vertices $x_3 \in X_3$ and $a_3 \in A_3$ such that every maximal clique in $G[X_3 \cup A_3]$ contains x_3 or a_3 . Then $\{w, x_3, a_3\}$ is a very good stable set of G, and we can conclude by using Theorem 5.1. So we

may assume that $B = \emptyset$.

Suppose that $[X_1, A_1]$ is not complete. Then, as remarked earlier, $[X_2, A_2]$ and $[X_3, A_3]$ are complete. Also by Lemma 2.3, there exist non-adjacent vertices $x_1 \in X_1$ and $a_1 \in A_1$ such that every maximal clique in $G[X_1 \cup A_1]$ contains x_1 or a_1 . Pick any $x_2 \in X_2$ and $x_3 \in X_3$. Then $\{a_1, x_1, x_2, x_3\}$ is a very good stable set of G, and we can conclude by using Theorem 5.1. Therefore assume that $[X_1, A_1]$ is complete, and, similarly, that $[X_2, A_2]$ is complete.

Suppose that $[X_3, A_3]$ is not complete. Then by Lemma 2.3, there are nonadjacent vertices $x_3 \in X_3$ and $a_3 \in A_3$ such that every maximal clique in $G[X_3 \cup A_3]$ contains x_3 or a_3 . If $W \neq \emptyset$, then any $w \in W$ is universal in $G[V(G) \setminus (X_3 \cup A_3)]$. But now $\{w, x_3, a_3\}$ is a very good stable set of G, and we can conclude by using Theorem 5.1. So $W = \emptyset$. Now pick any $x_1 \in X_1$ and $x_2 \in X_2$. Then $\{x_1, x_2, x_3, a_3\}$ is a very good stable set of G, and we can conclude by Theorem 5.1. Therefore assume that $[X_3, A_3]$ is complete.

Now G is a blowup of $F_{2,0}$ (with $A_1 \cup A_2$ is the role of Q_A , and X_3 in the role of Q_B , and A_3 in the role of Q_z , and Q in the role of Q_y , and W in the role of Q_x , and X_1, X_2 in the role of Q_{u_1}, Q_{u_2}), so we can conclude using Theorem 5.8. This completes the proof.

Theorem 5.10 Let G be a belt. Then $\chi(G) \leq \lfloor \frac{5}{4}\omega(G) \rfloor$.

Proof. We use the same notation as in the definition of a belt, and we will also use the properties listed in Theorem 4.1. We prove the theorem by induction on $\omega(G)$. If $\omega(G) = 2$ then G is a C_5 and the theorem holds obviously. Now assume that $\omega(G) \ge 3$. Let $q = \omega(G)$.

Suppose that both R_2 , R_3 are non-empty. Recall from Theorem 4.1 that $G[R_2]$ is $(P_4, C_4, 2P_3)$ -free, hence chordal. Moreover, the axiom that $G[R_2]$ has no universal vertex implies that R_2 is not a clique, so it has two non-adjacent simplicial vertices r_1, r_2 . For each $h \in \{1, 2\}$ let X_h be the closed neighborhood of r_h in R_2 ; so X_h is a clique. Let $Y_h = N(r_h) \cap Q_3$. If $d(r_h) < \lceil \frac{5}{4}q \rceil$ then we can conclude using Theorem 5.1 (i) and induction. Hence assume that $d(r_h) \ge \lceil \frac{5}{4}q \rceil$ for each $h \in \{1, 2\}$. By the definition of a belt, we have $N[r_h] = Q_1 \cup Q_2 \cup X_h \cup Y_h$, and $Q_1 \cup Q_2 \cup X_h$ is a clique, so we must have $|Y_h| \ge \lceil \frac{q}{4} \rceil + 1$. By Theorem 4.1(a), the sets Y_1, Y_2 are pairwise disjoint. By the same argument $G[R_3]$ has two non-adjacent simplicial vertices and consequently there are two disjoint subsets Z_1, Z_2 of Q_2 with size at least $\lceil \frac{q}{4} \rceil + 1$. By Theorem 4.1(c) the set $Y_1 \cup Y_2 \cup Z_1 \cup Z_2$ is a clique, and its size is strictly larger than q, a contradiction.

Therefore we may assume that $R_3 = \emptyset$. Let $X = Q_2 \cup R_2 \cup Q_5$ and $A = Q_3 \cup Q_4$. Then the partition of V(G) into Q_1, X and A shows that G is an extended C-pair, so the result follows from Lemma 5.4.

Theorem 5.11 Let G be a boiler. Then $\chi(G) \leq \lfloor \frac{5}{4} \omega(G) \rfloor$.

Proof. We use the same definition as in the definition of a boiler. Let $q = \omega(G)$. By Theorem 5.1 we may assume that every vertex in G has degree at least $\lceil \frac{5}{4}q \rceil$. If L is a clique, then the partition of V(G) into $Q, M \cup A$ and $L \cup B$ shows that G is an extended C-pair, so the result follows from Lemma 5.4. Therefore assume that L is not a clique. By the same argument as in the proof of Theorem 5.10 implies that there are two disjoint subsets A_1, A_2 of A of size at least $\lceil \frac{q}{4} \rceil + 1$. By the same argument applied to $G[M_1 \cup M_2]$, there are two disjoint sets $Y_1 \subseteq B_1$ and $Y_2 \subseteq B_2$ of size at least $\lceil \frac{q}{4} \rceil + 1$. Then $A_1 \cup A_2 \cup B_1 \cup B_2$ is a clique, with size strictly larger than q, a contradiction.

5.3 Chromatic bounds for (P_6, C_4) -free graphs

Proof of Theorem 1.1. Let G be any (P_6, C_4) -free graph. We prove the theorem by induction on |V(G)|.

If G has a universal vertex u, then $\omega(G) = \omega(G \setminus u) + 1$, and by the induction hypothesis we have $\chi(G) = \chi(G \setminus u) + 1 \leq \lceil \frac{5}{4}(\omega(G \setminus u) \rceil + 1)$, which implies $\chi(G) \leq \lceil \frac{5}{4}(\omega(G) \rceil$.

If G has a clique cutset K, let A, B be a partition of $V(G) \setminus K$ such that both A, B are non-empty and $[A, B] = \emptyset$. Clearly $\chi(G) = \max\{\chi(G[K \cup A]), \chi(G[K \cup B])\}$, so the desired result follows from the induction hypothesis on $G[K \cup A]$ and $G[K \cup B]$.

Finally, if G has no universal vertex and no clique cutset, then the result follows from Theorem 1.5 and Theorems 5.2—5.11. \Box

Next we prove Theorem 1.2 by using the following theorem.

Theorem 5.12 ([20]) If a graph G satisfies $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$, then it satisfies $\chi(G) \leq \lceil \frac{\Delta(G) + \omega(G) + 1}{2} \rceil$.

Proof of Theorem 1.2. This follows from Theorems 1.1 and 5.12. \Box

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