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### Abstract

In this paper, we first devise an ensemble hybridizable discontinuous Galerkin (HDG) method to efficiently simulate a group of parameterized convection diffusion PDEs. These PDEs have different coefficients, initial conditions, source terms and boundary conditions. The ensemble HDG discrete system shares a common coefficient matrix with multiple right hand side (RHS) vectors; it reduces both computational cost and storage. We have two contributions in this paper. First, we derive an optimal  $L^2$  convergence rate for the ensemble solutions on a general polygonal domain, which is the first such result in the literature. Second, we obtain a superconvergent rate for the ensemble solutions after an element-by-element postprocessing under some assumptions on the domain and the coefficients of the PDEs. We present numerical experiments to confirm our theoretical results.

# 1 Introduction

A challenge in numerical simulations is to reduce computational cost while keeping accuracy. Toward this end, many fast algorithms have been proposed, which include domain decomposition methods [30], multigrid methods [38], interpolated coefficient methods [8,16,34], and so on. These methods are only suitable for a single simulation, not for a group of simulations with different coefficients, initial conditions, source terms and boundary conditions in many scenarios; for example, one needs repeated simulations to obtain accurate statistical information about the outputs of interest in some uncertainty quantification problems. A common way is to treat the simulations seperately; this requires computational effort and memory. Parallel computing is one method that can solve this problem if sufficient memory is available.

However, the computational effort and storage requirement is still a great challenge in real simulations. An ensemble method was proposed by Jiang and Layton [25] to address this issue. They studied a set of J solutions of the Navier-Stokes equations with different initial conditions and forcing terms. This algorithm uses the mean of the solutions to form a common coefficient matrix at each time step. Hence, the problem is reduced to solving one linear system with many right hand side (RHS) vectors, which can be efficiently computed by many existing algorithms,

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such as LU factorization, GMRES, etc. The ensemble scheme has been extended to many different models; see, e.g., [17, 19–24, 26, 27]. Recently, Luo and Wang [28] extended this idea to a stochastic parabolic PDE. It is worthwhile to mention that all the above works only obtained *suboptimal*  $L^2$  convergence rate for the ensemble solutions.

All the previous works have used continuous Galerkin (CG) methods; however, for high Reynolds number flows [23, 26, 36] using a modified CG method may still cause non-physical oscillations. The literature on discontinuous Galerkin (DG) methods for simulating a *single* convection diffusion PDE is already substantial and the research in this area is still active; see, e.g. [1, 15, 37]. However, there are *no* theoretical or numerical analysis works on DG methods for the spatial discretization of a group of parameterized convection diffusion equations.

However, the number of degrees of freedom for DG methods is much larger compared to CG methods; this is the main drawback of DG methods. Hybridizable discontinuous Galerkin (HDG) methods were originally proposed by Cockburn, Gopalakrishnan, and Lazarov in [9] to fix this issue. The HDG methods are based on a mixed formulation and introduce a numerical flux and a numerical trace to approximate the flux and the trace of the solution. The discrete HDG global system is only in terms of the numerical trace variable since we can element-by-element eliminate the numerical flux and solution. Therefore, HDG methods have a significantly smaller number of globally coupled degrees of freedom compareed to DG methods. Moreover, HDG methods keep the advantages of DG methods, which are suitable for convection diffusion problems; see, e.g., [4–6,18,29]. Also, HDG methods have been applied to flow problems [2,10,12,13,13,14,31,32] and hyperbolic equations [7,33,35].

In this work, we propose a new Ensemble HDG method to investigate a group of parameterized convection diffusion equations on a Lipschitz polyhedral domain  $\Omega \subset \mathbb{R}^d$   $(d \geq 2)$ . For  $j = 1, 2, \dots, J$ , find  $(q_j, u_j)$  satisfying

$$c_{j}\boldsymbol{q}_{j} + \nabla u_{j} = 0 \quad \text{in } \Omega \times (0,T],$$
  

$$\partial_{t}u_{j} + \nabla \cdot \boldsymbol{q}_{j} + \boldsymbol{\beta}_{j} \cdot \nabla u_{j} = f_{j} \quad \text{in } \Omega \times (0,T],$$
  

$$u_{j} = g_{j} \quad \text{on } \partial\Omega \times (0,T],$$
  

$$u_{j}(\cdot,0) = u_{j}^{0} \quad \text{in } \Omega,$$
(1.1)

where the vector vector fields  $\beta_j$  satisfy

$$\nabla \cdot \boldsymbol{\beta}_j = 0. \tag{1.2}$$

We make other smoothness assumptions on the data of system (1.1) for our analysis.

The HDG Method. To better describe the Ensemble HDG method, we first give the semidiscretization of the system (1.1) use an existing HDG method [11]. Let  $\mathcal{T}_h$  be a collection of disjoint simplexes K that partition  $\Omega$  and let  $\partial \mathcal{T}_h$  be the set  $\{\partial K : K \in \mathcal{T}_h\}$ . Let  $e \in \mathcal{E}_h^o$  be the interior face if the Lebesgue measure of  $e = \partial K^+ \cap \partial K^-$  is non-zero, similarly,  $e \in \mathcal{E}_h^\partial$  be the boundary face if the Lebesgue measure of  $e = \partial K \cap \partial \Omega$  is non-zero. Finally, we set

$$(w,v)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} (w,v)_K, \qquad \langle \zeta, \rho \rangle_{\partial \mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K},$$

where  $(\cdot, \cdot)_K$  denotes the  $L^2(K)$  inner product and  $\langle \cdot, \cdot \rangle_{\partial K}$  denotes the  $L^2$  inner product on  $\partial K$ .

Let  $\mathcal{P}^k(K)$  denote the set of polynomials of degree at most k on the element K. We define the

following discontinuous finite element spaces

We use the notation  $\nabla v_h$  and  $\nabla \cdot \boldsymbol{r}_h$  to denote the gradient of  $v_h \in W_h$  and the divergence of  $\boldsymbol{r}_h \in \boldsymbol{V}_h$  applied piecewise on each element  $K \in \mathcal{T}_h$ .

The semidiscrete HDG method finds  $(q_{jh}, u_{jh}, \hat{u}_{jh}) \in V_h \times W_h \times M_h$  such that for all  $j = 1, 2 \cdots, J$ 

$$(c_{j}\boldsymbol{q}_{jh},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - (u_{jh},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle \widehat{u}_{jh},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} = -\langle g_{j},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\varepsilon_{h}^{\partial}},$$
  

$$(\partial_{t}u_{jh},v_{h})_{\mathcal{T}_{h}} - (\boldsymbol{q}_{jh} + \boldsymbol{\beta}_{j}u_{jh},\nabla v_{h})_{\mathcal{T}_{h}} + \langle \widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n},v_{h}\rangle_{\partial\mathcal{T}_{h}}$$
  

$$+\langle \boldsymbol{\beta}_{j}\cdot\boldsymbol{n}\widehat{u}_{jh},v_{h}\rangle_{\partial\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j}\cdot\boldsymbol{n}g_{j},v_{h}\rangle_{\varepsilon_{h}^{\partial}} = (f_{j},v_{h})_{\mathcal{T}_{h}},$$
  

$$\langle \widehat{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n} + \boldsymbol{\beta}_{j}\cdot\boldsymbol{n}\widehat{u}_{jh},\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} = 0,$$
  
(1.3)

for all  $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$ . Here the numerical traces on  $\partial \mathcal{T}_h$  are defined as

$$\widehat{q}_{jh} \cdot \boldsymbol{n} = \boldsymbol{q}_{jh} \cdot \boldsymbol{n} + \tau_j (u_{jh} - \widehat{u}_{jh}) \qquad \text{on } \partial \mathcal{T}_h \backslash \varepsilon_h^\partial, \tag{1.4}$$

$$\widehat{q}_{jh} \cdot \boldsymbol{n} = \boldsymbol{q}_{jh} \cdot \boldsymbol{n} + \tau_j (u_{jh} - g_j) \qquad \text{on } \varepsilon_h^\partial, \tag{1.5}$$

where  $\tau_i$  are positive stabilization functions defined on  $\partial \mathcal{T}_h$  satisfying

$$\tau_j = \tau + \boldsymbol{\beta}_j \cdot \boldsymbol{n} \text{ on } \partial \mathcal{T}_h,$$

and the function  $\tau$  is a positive constant on each element  $K \in \mathcal{T}_h$ .

The Ensemble HDG Method. It is obvious to see that the system (1.3)-(1.5) has J different coefficient matrices. The idea of the Ensemble HDG method is to treat the system to share one common coefficient matrix by changing the variables  $c_j$  and  $\beta_j$  into their ensemble means. Before we define the Ensemble HDG method, we give some notation first.

Suppose the time domain [0, T] is uniformly partition into N steps with time step  $\Delta t$  and let  $t_n = n\Delta t$  for  $n = 1, 2 \cdots, N$ . Moreover,  $\bar{c}^n$  and  $\bar{\beta}^n$  stand for the ensemble means of the inverse coefficient of diffusion and convection coefficient at time  $t_n$ , respectively, defined by

$$\bar{c}^n = \frac{1}{J} \sum_{j=1}^J c_j^n \quad \text{and} \quad \bar{\beta}^n = \frac{1}{J} \sum_{j=1}^J \beta_j^n,$$
(1.6)

the superscript n denotes the function value at the time  $t_n$ .

Substitute (1.4)-(1.5) into (1.3), and use some simple algebraic manipulation, the ensemble mean (1.6), and the previous step to replace the current step to obtain the Ensemble HDG formulation: find  $(\boldsymbol{q}_{jh}^n, u_{jh}^n, \hat{u}_{jh}^n) \in \boldsymbol{V}_h \times W_h \times M_h$  such that for all  $j = 1, 2, \cdots, J$ 

$$(\bar{c}^{n}\boldsymbol{q}_{jh}^{n},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - (u_{jh}^{n},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle \hat{u}_{jh}^{n},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} = ((\bar{c}^{n}-c_{j}^{n})\boldsymbol{q}_{jh}^{n-1},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - \langle g_{j}^{n},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\mathcal{E}_{h}^{\partial}}, \qquad (1.7a)$$

$$(\partial_t^+ u_{jh}^n, v_h)_{\mathcal{T}_h} + (\nabla \cdot \boldsymbol{q}_{jh}^n, v_h)_{\mathcal{T}_h} - \langle \boldsymbol{q}_{jh}^n \cdot \boldsymbol{n}, \hat{v}_h \rangle_{\partial \mathcal{T}_h} + (\overline{\boldsymbol{\beta}}^n \cdot \nabla u_{jh}^n, v_h)_{\mathcal{T}_h} - \langle \overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}, u_{jh}^n \hat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau (u_{jh}^n - \hat{u}_{jh}^n), v_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h} = (f_j^n, v_h)_{\mathcal{T}_h} + \langle \tau g_j^n, v_h \rangle_{\mathcal{E}_h^\partial} + ((\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n) \cdot \nabla u_{jh}^{n-1}, v_h)_{\mathcal{T}_h} - \langle (\overline{\boldsymbol{\beta}}^n - \boldsymbol{\beta}_j^n) \cdot \boldsymbol{n}, u_{jh}^{n-1} \hat{v}_h \rangle_{\partial \mathcal{T}_h},$$
(1.7b)

for all  $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$ . The initial conditions  $u_{jh}^0$  and  $\mathbf{q}_{jh}^0$  will be specified later. Finally, we let

$$\partial_t^+ u_{jh}^n = \frac{u_{jh}^n - u_{jh}^{n-1}}{\Delta t}$$

It is easy to see that the system (1.7) shares one matrix with J RHS vectors, and it is more efficient to solve than performing J separate simulations. It is worth mentioning that this is the *first* time that an ensemble scheme has been derived incorporating HDG methods; it is even the *first* time for DG methods. We provide a rigorous error analysis to obtain an optimal  $L^2$  convergence rate for the flux  $q_j$  and the solution  $u_j$  on general polygonal domain  $\Omega$  in Section 3. To the best of our knowledge, this is the *first* time in the literature. One of the excellent features of HDG methods is that we can obtain superconvergence after an element-by-element postprocessing; we show that this result also holds in the Ensemble HDG algorithm under some conditions on the domain  $\Omega$ and the velocity vector fields  $\beta_j$ . This is also the *first superconvergent* ensemble algorithm in the literature. Finally, some numerical experiments are presented to confirm our theoretical results in Section 4. Furthermore, we also present numerical results for convection dominated problems with  $c_j^{-1} \ll 1$  to demonstrate the performance of the Ensemble HDG method in this difficult case. The results show that the Ensemble HDG method is able to capture sharp layers in the solution. A thorough error analysis of the Ensemble HDG method for the convection dominated case will be in another paper.

## 2 Stability

We begin with some notation. We use the standard notation  $W^{m,p}(D)$  for Sobolev spaces on Dwith norm  $\|\cdot\|_{m,p,D}$  and seminorm  $|\cdot|_{m,p,D}$ . We also write  $H^m(D)$  instead of  $W^{m,2}(D)$ , and we omit the index p in the corresponding norms and seminorms. Also, we omit the index m when m = 0 in the corresponding norms and seminorms. Moreover, we drop the subscript D if there is no ambiguity in the statement. We denote by  $C(0,T;W^{m,s}(\Omega))$  the Banach space of all continuous functions from [0,T] into  $W^{m,s}(\Omega)$ , and  $L^p(0,T;W^{m,s}(\Omega))$  for  $1 \le p \le \infty$  is similarly defined.

To obtain the stability of (1.7) in this section, we assume the data of (1.1) satisfies

- (A1):  $f_j \in C(0,T; L^2(\Omega)), g_j \in C(0,T; H^{1/2}(\partial\Omega)), u_j^0 \in L^2(\Omega), c_j \in C(0,T; L^{\infty}(\Omega))$  and the vector fields  $\beta_j \in C(0,T; W^{1,\infty}(\Omega))$ .
- (A2): There exists a postive constant  $c_0$  such that  $c_j^n \ge c_0$ , and the ensemble mean satisfies the condition

$$|\bar{c}^n - c_j^n| < \min\{\bar{c}^n, \bar{c}^{n-1}\}, \ \forall \boldsymbol{x} \in \overline{\Omega} \ \text{ and } 1 \le n \le N, 1 \le j \le J.$$
 (2.1)

It is worth mentioning that we don't assume any conditions like (2.1) on the functions  $\beta_j$ . The function  $\tau$  is a piecewise constant function independent of j satisfying

$$\min_{1 \le j \le J} (\tau + \frac{1}{2} \boldsymbol{\beta}_j \cdot \boldsymbol{n}) \ge \frac{1}{2} \max_{1 \le j \le J} \|\boldsymbol{\beta}_j\|_{0,\infty}, \forall \boldsymbol{x} \in \partial \mathcal{T}_h.$$
(2.2)

Next, let  $\Pi_{\ell}$  and  $P_M$  denote the standard  $L^2$  projection operators  $\Pi_{\ell} : L^2(K) \to \mathcal{P}^{\ell}(K)$  and  $P_M : L^2(e) \to \mathcal{P}^k(e)$  satisfying

$$(\Pi_{\ell} w, v_h)_K = (w, v_h)_K, \quad \forall v_h \in \mathcal{P}^{\ell}(K),$$
(2.3a)

$$\langle P_M w, \hat{v}_h \rangle_e = \langle w, \hat{v}_h \rangle_e, \quad \forall \hat{v}_h \in \mathcal{P}^k(e).$$
 (2.3b)

The following error estimates for the  $L^2$  projections are standard:

**Lemma 1.** Suppose  $k, \ell \geq 0$ . There exists a constant C independent of  $K \in \mathcal{T}_h$  such that

$$\|w - \Pi_{\ell} w\|_{K} \le Ch^{\ell+1} |w|_{\ell+1,K}, \qquad \forall w \in H^{\ell+1}(K), \qquad (2.4a)$$

$$\|w - P_M w\|_{\partial K} \le Ch^{k+1/2} |w|_{k+1,K} \qquad \forall w \in H^{k+1}(K).$$
(2.4b)

Moreover, the vector  $L^2$  projection  $\mathbf{\Pi}_{\ell}$  is defined similarly.

We choose the initial conditions  $u_{jh}^0 = \Pi_{k+1}u_0$ ,  $\boldsymbol{q}_{jh}^0 = -\nabla u_{jh}^0/c_j^0$ . To make the presentation simple for the stability, we assume  $g_j = 0$  for  $j = 1, 2 \cdots, J$  in this section.

Lemma 2. If condition (2.1) holds, then the Ensemble HDG formulation is unconditionally stable and we have the following estimate:

$$\begin{aligned} \max_{1 \le n \le N} \|u_{jh}^n\|_{\mathcal{T}_h}^2 + \sum_{n=1}^N \|u_{jh}^n - u_{jh}^{n-1}\|_{\mathcal{T}_h}^2 + \Delta t \sum_{n=1}^N \|\sqrt{\bar{c}^n} \boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(u_{jh}^n - \widehat{u}_{jh}^n)\|_{\partial \mathcal{T}_h}^2 \\ \le C \Delta t \sum_{n=1}^N \|f_j^n\|_{\mathcal{T}_h}^2 + C \|u_{jh}^0\|_{\mathcal{T}_h}^2 + C \|\boldsymbol{q}_{jh}^0\|_{\mathcal{T}_h}^2. \end{aligned}$$

*Proof.* Take  $(\mathbf{r}_h, v_h, \hat{v}_h) = (\mathbf{q}_{jh}^n, u_{jh}^n, \hat{u}_{jh}^n)$  in (1.7), use the polarization identity

$$(a-b)a = \frac{1}{2}(a^2 - b^2 + (a-b)^2), \qquad (2.5)$$

and add the Equation (1.7a) and Equation (1.7b) together to give

$$\frac{\|u_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} - \|u_{jh}^{n-1}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \frac{\|u_{jh}^{n} - u_{jh}^{n-1}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \|\sqrt{\overline{c}^{n}}\boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} + \|\sqrt{\tau}(u_{jh}^{n} - \widehat{u}_{jh}^{n})\|_{\partial\mathcal{T}_{h}}^{2}} 
= -(\overline{\boldsymbol{\beta}}^{n} \cdot \nabla u_{jh}^{n}, u_{jh}^{n})_{\mathcal{T}_{h}} + \langle(\overline{\boldsymbol{\beta}}^{n} \cdot \boldsymbol{n})u_{jh}^{n}, \widehat{u}_{jh}^{n}\rangle_{\partial\mathcal{T}_{h}} + ((\overline{c}^{n} - c_{j}^{n})\boldsymbol{q}_{jh}^{n-1}, \boldsymbol{q}_{jh}^{n})_{\mathcal{T}_{h}} 
+ ((\overline{\boldsymbol{\beta}}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle(\overline{\boldsymbol{\beta}}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \boldsymbol{n}, u_{jh}^{n-1}\widehat{u}_{jh}^{n}\rangle_{\partial\mathcal{T}_{h}} + (f_{j}^{n}, u_{jh}^{n})_{\mathcal{T}_{h}}.$$
(2.6)

By Green's formula and the fact  $\langle (\overline{\beta}^n \cdot \boldsymbol{n}) \hat{u}_{jh}^n, \hat{u}_{jh}^n \rangle_{\partial \mathcal{T}_h} = 0$ , we have

$$-(\overline{\boldsymbol{\beta}}^n \cdot \nabla u_{jh}^n, u_{jh}^n)_{\mathcal{T}_h} + \langle (\overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}) u_{jh}^n, \widehat{u}_{jh}^n \rangle_{\partial \mathcal{T}_h} \leq \frac{1}{2} \| \sqrt{|\overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}|} (u_{jh}^n - \widehat{u}_{jh}^n) \|_{\partial \mathcal{T}_h}^2.$$

Hence, condition (2.2) gives

$$\begin{aligned} \frac{\|u_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} - \|u_{jh}^{n-1}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \frac{\|u_{jh}^{n} - u_{jh}^{n-1}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \|\sqrt{c^{n}}\boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{2}\|\sqrt{\tau}(u_{jh}^{n} - \widehat{u}_{jh}^{n})\|_{\partial\mathcal{T}_{h}}^{2}}{\leq ((\overline{c}^{n} - c_{j}^{n})\boldsymbol{q}_{jh}^{n-1}, \boldsymbol{q}_{jh}^{n})_{\mathcal{T}_{h}} + ((\overline{\boldsymbol{\beta}}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}}} - \langle (\overline{\boldsymbol{\beta}}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \boldsymbol{n}, u_{jh}^{n-1}\widehat{u}_{h}^{n} \rangle_{\partial\mathcal{T}_{h}} + (f_{j}^{n}, u_{jh}^{n})_{\mathcal{T}_{h}}} \\ = R_{1} + R_{2} + R_{3} + R_{4}. \end{aligned}$$

Next, we estimate  $\{R_i\}_{i=1}^4$ . First, by the condition (2.1), there exist  $0 < \alpha < 1$  such that

$$R_{1} = ((\overline{c}^{n} - c_{j}^{n})\boldsymbol{q}_{jh}^{n-1}, \boldsymbol{q}_{jh}^{n})_{\mathcal{T}_{h}} \leq \frac{\alpha}{2} \|\sqrt{\overline{c}^{n}}\boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} + \frac{\alpha}{2} \|\sqrt{\overline{c}^{n-1}}\boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_{h}}^{2}.$$

The term  $R_2 + R_3$  needs a detailed argument. For simplicity, let  $\gamma = \overline{\beta}^n - \beta_j^n$ . We have

$$R_{2} + R_{3} = (\boldsymbol{\gamma} \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle \boldsymbol{\gamma} \cdot \boldsymbol{n}, u_{jh}^{n-1} \widehat{u}_{jh}^{n} \rangle_{\partial \mathcal{T}_{h}}$$

$$= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \boldsymbol{n}, u_{jh}^{n-1} \widehat{u}_{jh}^{n} \rangle_{\partial \mathcal{T}_{h}}$$

$$+ (\boldsymbol{\Pi}_{0}\boldsymbol{\gamma} \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle \boldsymbol{\Pi}_{0}\boldsymbol{\gamma} \cdot \boldsymbol{n}, u_{jh}^{n-1} \widehat{u}_{jh}^{n} \rangle_{\partial \mathcal{T}_{h}}$$

$$= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \boldsymbol{n}, u_{jh}^{n-1} \widehat{u}_{jh}^{n} \rangle_{\partial \mathcal{T}_{h}}$$

$$+ (\overline{c}^{n}\boldsymbol{q}_{jh}^{n}, \boldsymbol{\Pi}_{0}\boldsymbol{\gamma} u_{jh}^{n-1})_{\mathcal{T}_{h}} - ((\overline{c}^{n} - c_{j}^{n})\boldsymbol{q}_{jh}^{n-1}, \boldsymbol{\Pi}_{0}\boldsymbol{\gamma} u_{jh}^{n-1})_{\mathcal{T}_{h}},$$

where we used Equation (1.7a) in the last identity. Hence,

$$\begin{aligned} R_{2} + R_{3} &= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \nabla u_{jh}^{n-1}, u_{jh}^{n})_{\mathcal{T}_{h}} - \langle (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \boldsymbol{n}, u_{jh}^{n-1} \hat{u}_{jh}^{n} \rangle_{\partial \mathcal{T}_{h}} \\ &+ (\bar{c}^{n} \boldsymbol{q}_{jh}^{n}, \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} u_{jh}^{n-1})_{\mathcal{T}_{h}} - ((\bar{c}^{n} - c_{j}^{n}) \boldsymbol{q}_{jh}^{n-1}, \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} u_{jh}^{n-1})_{\mathcal{T}_{h}} \\ &\leq \sum_{K \in \mathcal{T}_{h}} \|\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}\|_{\infty,K} \|\nabla u_{jh}^{n-1}\|_{K} \|u_{jh}^{n}\|_{K} \\ &+ \sum_{K \in \mathcal{T}_{h}} \|\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}\|_{\infty,\partial K} \|u_{jh}^{n-1}\|_{\partial K} (\|\hat{u}_{jh}^{n} - u_{jh}^{n}\|_{\partial K} + \|u_{jh}^{n}\|_{\partial K}) \\ &+ \|\boldsymbol{\Pi}_{0}\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_{h}} \|\bar{c}^{n} \boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}} \|u_{jh}^{n-1}\|_{\mathcal{T}_{h}} \\ &+ \|(\bar{c}^{n} - c_{j}^{n})\boldsymbol{\Pi}_{0}\boldsymbol{\gamma}\|_{\infty,\mathcal{T}_{h}} \|\boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_{h}} \|u_{jh}^{n-1}\|_{\mathcal{T}_{h}} \\ &= R_{31} + R_{32} + R_{33} + R_{34}. \end{aligned}$$

For  $R_{31}$ , use the local inverse inequality:

$$R_{31} \leq C \sum_{K \in \mathcal{T}_h} h_K \| \boldsymbol{\gamma} \|_{1,\infty,K} h_K^{-1} \| u_{jh}^{n-1} \|_K \| u_{jh}^n \|_K \leq C(\| u_{jh}^{n-1} \|_{\mathcal{T}_h}^2 + \| u_{jh}^n \|_{\mathcal{T}_h}^2).$$

Apply the trace inequality and inverse inequality for the term  $R_{32}$  to give

$$R_{32} \leq C \sum_{K \in \mathcal{T}_h} h_K \| \boldsymbol{\gamma} \|_{1,\infty,K} h_K^{-1/2} \| u_{jh}^{n-1} \|_K (\| \widehat{u}_{jh}^n - u_{jh}^n \|_{\partial K} + h_K^{-1/2} \| u_{jh}^n \|_K)$$
  
$$\leq C (\| u_{jh}^{n-1} \|_{\mathcal{T}_h}^2 + \| u_{jh}^n \|_{\mathcal{T}_h}^2) + \frac{1}{4} \| \sqrt{\tau} (\widehat{u}_{jh}^n - u_{jh}^n) \|_{\partial \mathcal{T}_h}^2.$$

For the terms  $R_{33}$  and  $R_{34}$ , use Young's inequality to obtain

$$R_{33} \leq \frac{1-\alpha}{4} \|\sqrt{\overline{c}^n} \boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2 + C \|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2,$$
  
$$R_{34} \leq \frac{1-\alpha}{4} \|\sqrt{\overline{c}^{n-1}} \boldsymbol{q}_{jh}^{n-1}\|_{\mathcal{T}_h}^2 + C \|u_{jh}^{n-1}\|_{\mathcal{T}_h}^2$$

The Cauchy-Schwarz inequality for the term  $\mathbb{R}_4$  gives

$$R_4 = (f_j^n, u_{jh}^n)_{\mathcal{T}_h} \le \frac{1}{2} (\|f_j^n\|_{\mathcal{T}_h}^2 + \|u_{jh}^n\|_{\mathcal{T}_h}^2).$$

We add (2.6) from n = 1 to n = N, and use the above inequalities to get

$$\max_{1 \le n \le N} \|u_{jh}^n\|_{\mathcal{T}_h}^2 + \sum_{n=1}^N \|u_{jh}^n - u_{jh}^{n-1}\|_{\mathcal{T}_h}^2 + \Delta t \sum_{n=1}^N \|\sqrt{c^n} \boldsymbol{q}_{jh}^n\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(u_{jh}^n - \widehat{u}_{jh}^n)\|_{\partial \mathcal{T}_h}^2$$
$$\le C\Delta t \sum_{n=1}^N \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + C\Delta t \sum_{n=1}^N \|f_j^n\|_{\mathcal{T}_h}^2 + C \|u_{jh}^0\|_{\mathcal{T}_h}^2 + C \|\boldsymbol{q}_{jh}^0\|_{\mathcal{T}_h}^2.$$

Gronwall's inequality applied to the above inequality gives the desired result.

# **3** Error analysis

The strategy of the error analysis for the Ensemble HDG method is based on [3] and [5]. First, we define the HDG projections, and use an energy argument to obtain an optimal convergence rate for the ensemble solutions. Second, we define an HDG elliptic projection as in [3], which is a crucial step to get the superconvergence. Next, we give our main results, and in the end, we provide a rigorous error estimation for our Ensemble HDG method.

Throughout, we assume the data and the solution of (1.1) are smooth enough, and the initial conditions  $(\boldsymbol{q}_{ih}^0, u_{ih}^0)$  of the Ensemble HDG system (1.7) are chosen as in Section 2.

### 3.1 HDG projection

For any  $t \in [0, T]$ , let  $(\mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}, \Pi_{W}^{j} u_{j})$  be the HDG projection of  $(\boldsymbol{q}_{j}, u_{j})$ , where  $\mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}$  and  $\Pi_{W}^{j} u_{j}$ denote components of the HDG projection of  $\boldsymbol{q}_{j}$  and  $u_{j}$  into  $V_{h}$  and  $W_{h}$ , respectively. On each element  $K \in \mathcal{T}_{h}$ ,  $(\mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}, \Pi_{W}^{j} u_{j})$  satisfy the following equations

$$(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j} + \boldsymbol{\beta}_{j}\boldsymbol{\Pi}_{W}^{j}\boldsymbol{u}_{j}, \boldsymbol{r})_{K} = (\boldsymbol{q}_{j} + \boldsymbol{\beta}_{j}\boldsymbol{u}_{j}, \boldsymbol{r})_{K}, \qquad (3.1a)$$

$$(\Pi_W^j u_j, w)_K = (u_j, w)_K, \tag{3.1b}$$

$$\langle \boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}\cdot\boldsymbol{n}+\boldsymbol{\beta}_{j}\cdot\boldsymbol{n}\boldsymbol{\Pi}_{W}^{j}\boldsymbol{u}_{j}+\tau\boldsymbol{\Pi}_{W}^{j}\boldsymbol{u}_{j},\boldsymbol{\mu}\rangle_{e}=\langle\boldsymbol{q}_{j}\cdot\boldsymbol{n}+\boldsymbol{\beta}_{j}\cdot\boldsymbol{n}\boldsymbol{u}_{j}+\tau\boldsymbol{u}_{j},\boldsymbol{\mu}\rangle_{e},$$
(3.1c)

for all  $(\mathbf{r}, w, \mu) \in [\mathcal{P}^{k-1}(K)]^d \times \mathcal{P}^{k-1}(K) \times \mathcal{P}^k(e)$  and for all faces e of the simplex K. We notice the projections are only determined by (3.1c) when k = 0. The proof of the following lemma is similar to a result established in [5] and hence is omitted.

**Lemma 3.** Suppose the polynomial degree satisfies  $k \ge 0$  and also  $\tau > 0$ . Then the system (3.1) is uniquely solvable for  $\mathbf{\Pi}_{V}^{j} \mathbf{q}_{j}$  and  $\Pi_{W}^{j} u_{j}$ . Furthermore, there is a constant C independent of K and  $\tau$  such that for  $\ell_{\mathbf{q}_{i}}, \ell_{u_{i}}$  in [0, k]

$$\|\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}-\boldsymbol{q}_{j}\|_{K} \leq Ch_{K}^{\ell_{\boldsymbol{q}_{j}}+1}|\boldsymbol{q}_{j}|_{\boldsymbol{H}^{\ell_{\boldsymbol{q}_{j}}+1}(K)} + Ch_{K}^{\ell_{\boldsymbol{u}_{j}}+1}|\boldsymbol{u}_{j}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}_{j}}+1}(K)},$$
  
$$\|\boldsymbol{\Pi}_{W}^{j}\boldsymbol{u}_{j}-\boldsymbol{u}_{j}\|_{K} \leq Ch_{K}^{\ell_{\boldsymbol{u}_{j}}+1}|\boldsymbol{u}_{j}|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}_{j}}+1}(K)} + Ch_{K}^{\ell_{\boldsymbol{q}_{j}}+1}|\nabla \cdot \boldsymbol{q}_{j}|_{\boldsymbol{H}^{\ell_{\boldsymbol{q}_{j}}}(K)}.$$

### 3.2 Main results

We can now state our main result for the Ensemble HDG method.

**Theorem 1.** Let  $(q_j^n, u_j^n)$  and  $(q_{jh}^n, u_{jh}^n)$  be the solution of (1.1) at time  $t_n$  and (1.7), respectively. If the coefficients  $c_j$  satisfy (2.1), then we have

$$\max_{1 \le n \le N} \|u_j^n - u_{jh}^n\|_{\mathcal{T}_h} \le C(h^{k+1} + \Delta t),$$
(3.3a)

$$\sqrt{\Delta t \sum_{n=1}^{N} \|\boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2}} \le C(h^{k+1} + \Delta t).$$
(3.3b)

Moreover, if  $k \ge 1$ , the elliptic regularity inequality (6.4) holds and the coefficients of the PDEs are independent of time, then we have

$$\sqrt{\Delta t \sum_{n=1}^{N} \|u_j^n - u_{jh}^{n\star}\|_{\mathcal{T}_h}^2} \le C(h^{k+2} + \Delta t),$$
(3.4)

where  $u_{jh}^{n\star}$  is the postprocessed approximation defined in (3.17).

**Remark 1.** To the best of our knowledge, all previous works only contain suboptimal  $L^2$  convergence rate for the ensemble solutions  $u_j$ ; our result (3.3) is the first time to obtain the optimal  $L^{\infty}(0,T;L^2(\Omega))$  convergence rate on a general polygonal domain  $\Omega$ . Moreover, if the coefficients of the PDEs are independent of time, then after an element-by-element postprocessing, we obtain the superconvergent rate (3.4) under some conditions on the domain; for example, a convex domain is sufficient. This is also the first such result in the literature.

### **3.3** Proof of (3.3) in Theorem 1

**Lemma 4.** For all  $n = 1, 2, \dots, N$ , we have the following equalities:

$$(c_j^n \mathbf{\Pi}_V^j \boldsymbol{q}_j^n, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\boldsymbol{\Pi}_W^j u_j^n, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h},$$

and

$$\begin{aligned} (\nabla \cdot \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n}, v_{h})_{\mathcal{T}_{h}} &- \langle \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle \tau (\Pi_{W}^{j} u_{j}^{n} - P_{M} u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j}^{n} \cdot \nabla \Pi_{W}^{j} u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}, (\Pi_{W}^{j} u_{j}^{n}) \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &= (f_{j}^{n} - \partial_{t} u_{j}^{n}, v_{h})_{\mathcal{T}_{h}}, \end{aligned}$$

for all  $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$  and  $j = 1, 2, \cdots, J$ .

*Proof.* By the definitions of  $\Pi_W^j$  in (3.1b),  $P_M$  in (2.3b), and the first equation (1.1), we get

$$(c_j^n \mathbf{\Pi}_V^j \mathbf{q}_j^n, \mathbf{r}_h)_{\mathcal{T}_h} - (\mathbf{\Pi}_W^j u_j^n, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h}$$

$$= (c_j^n \mathbf{q}_j^n, \mathbf{r}_h)_{\mathcal{T}_h} - (\mathbf{\Pi}_W^j u_j^n, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle P_M u_j^n, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + (c_j^n (\mathbf{\Pi}_V^j \mathbf{q}_j^n - \mathbf{q}_j^n), \mathbf{r}_h)_{\mathcal{T}_h}$$

$$= (c_j^n \mathbf{q}_j^n, \mathbf{r}_h)_{\mathcal{T}_h} - (u_j^n, \nabla \cdot \mathbf{r}_h)_{\mathcal{T}_h} + \langle u_j^n, \mathbf{r}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} + (c_j^n (\mathbf{\Pi}_V^j \mathbf{q}_j^n - \mathbf{q}_j^n), \mathbf{r}_h)_{\mathcal{T}_h}$$

$$= (c_j^n \mathbf{q}_j^n + \nabla u_j^n, \mathbf{r}_h)_{\mathcal{T}_h} + (c_j^n (\mathbf{\Pi}_V^j \mathbf{q}_j^n - \mathbf{q}_j^n), \mathbf{r}_h)_{\mathcal{T}_h}$$

$$= (c_j^n (\mathbf{\Pi}_V^j \mathbf{q}_j^n - \mathbf{q}_j^n), \mathbf{r}_h)_{\mathcal{T}_h}.$$

This proves the first identity.

Next, we prove the second identity. First

$$\begin{aligned} (\nabla \cdot \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n}, v_{h})_{\mathcal{T}_{h}} &- \langle \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle \tau (\Pi_{W}^{j} u_{j}^{n} - P_{M} u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j} \cdot \nabla \Pi_{W}^{j} u_{j}, v_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}, (\Pi_{W}^{j} u_{j}^{n}) \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &= (\nabla \cdot \boldsymbol{q}_{j}^{n}, v_{h})_{\mathcal{T}_{h}} + (\nabla \cdot (\Pi_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}), v_{h})_{\mathcal{T}_{h}} - \langle \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &+ \langle \tau (\Pi_{W}^{j} u_{j}^{n} - P_{M} u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + (\boldsymbol{\beta}_{j}^{n} \cdot \nabla u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j}^{n} \cdot \nabla (\Pi_{W}^{j} u_{j}^{n} - u_{j}^{n}), v_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}, (\Pi_{W}^{j} u_{j}^{n}) \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}}. \end{aligned}$$

By the definition of  $\Pi_V^j$  and  $\Pi_W^j$  in (3.1a) and  $\nabla \cdot \beta_j^n = 0$ , we have

$$\begin{split} (\nabla \cdot (\boldsymbol{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}), v_{h})_{\mathcal{T}_{h}} &+ (\boldsymbol{\beta}_{j}^{n} \cdot \nabla (\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - u_{j}^{n}), v_{h})_{\mathcal{T}_{h}} \\ &= -(\boldsymbol{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}, \nabla v_{h})_{\mathcal{T}_{h}} + \langle (\boldsymbol{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}) \cdot \boldsymbol{n}, v_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &- (\boldsymbol{\beta}_{j}^{n} (\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - u_{j}^{n}), \nabla v_{h})_{\mathcal{T}_{h}} + \langle (\boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}) (\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - u_{j}^{n}), v_{h} \rangle_{\partial \mathcal{T}_{h}} \\ &= \langle (\boldsymbol{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}) \cdot \boldsymbol{n}, v_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle (\boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}) (\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - u_{j}^{n}), v_{h} \rangle_{\partial \mathcal{T}_{h}}. \end{split}$$

Using 
$$(\nabla \cdot \boldsymbol{q}_{j}^{n}, v_{h})_{\mathcal{T}_{h}} + (\boldsymbol{\beta}_{j}^{n} \cdot \nabla u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} = (f_{j}^{n} - \partial_{t}u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} \text{ and } (3.1c), \text{ we have}$$
  
 $(\nabla \cdot \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n}, v_{h})_{\mathcal{T}_{h}} - \langle \mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle \tau(\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - P_{M} u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}}$   
 $+ (\boldsymbol{\beta}_{j} \cdot \nabla \boldsymbol{\Pi}_{W}^{j} u_{j}, v_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n}, (\boldsymbol{\Pi}_{W}^{j} u_{j}^{n}) \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}}$   
 $= (f_{j}^{n} - \partial_{t} u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} + \langle (\mathbf{\Pi}_{V}^{j} \boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}) \cdot \boldsymbol{n}, v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}}$   
 $+ \langle \tau(\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - P_{M} u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle (\boldsymbol{\beta}_{j}^{n} \cdot \boldsymbol{n})(\boldsymbol{\Pi}_{W}^{j} u_{j}^{n} - u_{j}^{n}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}}$   
 $= (f_{j}^{n} - \partial_{t} u_{j}^{n}, v_{h})_{\mathcal{T}_{h}}.$ 

Then, substracting the result of Lemma 4 from the Ensemble HDG system (1.7) gives the following error equations.

**Lemma 5.** For  $\eta_{jh}^{u^n} = u_{jh}^n - \Pi_W^j u_j^n$ ,  $\eta_{jh}^{q^n} = q_{jh}^n - \Pi_V^j q_j^n$  and  $\eta_{jh}^{\widehat{u}^n} = \widehat{u}_{jh}^n - P_M u_j^n$ , for all  $j = 1, 2, \cdots, J$ , we have the following error equations:

$$\begin{aligned} & (\bar{c}^n \eta_{jh}^{\boldsymbol{q}^n}, \boldsymbol{r}_h)_{\mathcal{T}_h} - (\eta_{jh}^{\boldsymbol{u}^n}, \nabla \cdot \boldsymbol{r}_h)_{\mathcal{T}_h} + \langle \eta_{jh}^{\tilde{\boldsymbol{u}}^n}, \boldsymbol{r}_h \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} \\ &= ((\bar{c}^n - c_j^n)(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h} - (c_j^n (\boldsymbol{\Pi}_V^j \boldsymbol{q}_j^n - \boldsymbol{q}_j^n), \boldsymbol{r}_h)_{\mathcal{T}_h}, \end{aligned} \tag{3.6a}$$

and

$$(\partial_{t}^{+}\eta_{jh}^{u^{n}}, v_{h})_{\mathcal{T}_{h}} + (\nabla \cdot \eta_{jh}^{q^{n}}, v_{h})_{\mathcal{T}_{h}} - \langle \eta_{jh}^{q^{n}} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + (\overline{\boldsymbol{\beta}}^{n} \cdot \nabla \eta_{jh}^{u^{n}}, v_{h})_{\mathcal{T}_{h}} - \langle \overline{\boldsymbol{\beta}}^{n} \cdot \boldsymbol{n}, \eta_{jh}^{u^{n}} \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle \tau(\eta_{jh}^{u^{n}} - \eta_{jh}^{\widehat{u}}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} = (\partial_{t}u_{j}^{n} - \partial_{t}^{+}\Pi_{W}^{j}u_{j}^{n}, v_{h})_{\mathcal{T}_{h}} + ((\overline{\boldsymbol{\beta}}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \nabla(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n}), v_{h})_{\mathcal{T}_{h}} - \langle (\overline{\boldsymbol{\beta}}_{j}^{n} - \boldsymbol{\beta}_{j}^{n}) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n})\widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}},$$

$$(3.6b)$$

for all  $(\mathbf{r}_h, v_h, \widehat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$  and  $n = 1, 2, \cdots, N$ .

**Lemma 6.** If condition (2.1) holds, then we have the following error estimate:

$$\max_{1 \le n \le N} \|\eta_{jh}^{u^n}\|_{\mathcal{T}_h} + \sqrt{\Delta t \sum_{n=1}^N \|\sqrt{\overline{c^n}} \eta_{jh}^{q^n}\|_{\mathcal{T}_h}^2} \le C\left(h^{k+1} + \Delta t\right).$$
(3.7)

*Proof.* We take  $(\mathbf{r}_h, v_h, \hat{v}_h) = (\eta_{jh}^{\mathbf{q}^n}, \eta_{jh}^{u^n}, \eta_{jh}^{\hat{u}^n})$  in (3.6), use the identity (2.5) and add Equation (3.6a) and Equation (3.15) together to get

$$\frac{\|\eta_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} - \|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \frac{\|\eta_{jh}^{u^{n}} - \eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \|\sqrt{c^{n}}\eta_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + \|\sqrt{\tau}(\eta_{jh}^{u^{n}} - \eta_{jh}^{\widehat{u}^{n}})\|_{\partial\mathcal{T}_{h}}^{2}}$$

$$= -(\overline{\beta}^{n} \cdot \nabla \eta_{jh}^{u^{n}}, \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} + \langle \overline{\beta}^{n} \cdot \boldsymbol{n}, \eta_{jh}^{u^{n}}\eta_{jh}^{\widehat{u}^{n}}\rangle_{\partial\mathcal{T}_{h}} + ((\overline{c}^{n} - c_{j}^{n})(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}^{n}), \eta_{jh}^{q^{n}})_{\mathcal{T}_{h}}$$

$$+ (\partial_{t}u_{j}^{n} - \partial_{t}^{+}\Pi_{W}^{j}u_{j}^{n}, \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} + ((\overline{\beta}^{n} - \beta_{j}^{n}) \cdot \nabla(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n}), \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}}$$

$$- \langle (\overline{\beta}^{n} - \beta_{j}^{n}) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n})\eta_{jh}^{\widehat{u}^{n}}\rangle_{\partial\mathcal{T}_{h}}.$$
(3.8)

By Green's formula and the fact  $\langle (\overline{\beta}^n \cdot \boldsymbol{n}) \eta_{jh}^{\widehat{u}^n}, \eta_{jh}^{\widehat{u}^n} \rangle_{\partial \mathcal{T}_h} = 0$ , we have

$$(\overline{\boldsymbol{\beta}}^n \cdot \nabla \eta_{jh}^{u^n}, \eta_{jh}^{u^n})_{\mathcal{T}_h} - \langle \overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}, \eta_{jh}^{u^n} \eta_{jh}^{\widehat{u}^n} \rangle_{\partial \mathcal{T}_h} \leq \frac{1}{2} \| \sqrt{|\overline{\boldsymbol{\beta}}^n \cdot \boldsymbol{n}|} (\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}^n}) \|_{\partial \mathcal{T}_h}^2.$$

Condition (2.2) and equality (3.8) give

$$\frac{\|\eta_{jh}^{u^n}\|_{\mathcal{T}_h}^2 - \|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \frac{\|\eta_{jh}^{u^n} - \eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \|\sqrt{c^n}\eta_{jh}^{q^n}\|_{\mathcal{T}_h}^2 + \frac{1}{2}\|\sqrt{\tau}(\eta_{jh}^{u^n} - \eta_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2}{\leq ((\overline{c}^n - c_j^n)(q_{jh}^{n-1} - \mathbf{\Pi}_V^j q_j^n), \eta_{jh}^{q^n})_{\mathcal{T}_h} + (\partial_t u_j^n - \partial_t^+ \mathbf{\Pi}_W^j u_j^n, \eta_{jh}^{u^n})_{\mathcal{T}_h}} + \left[((\overline{\beta}^n - \beta_j^n) \cdot \nabla(u_{jh}^{n-1} - \mathbf{\Pi}_W^j u_j^n), \eta_{jh}^{u^n})_{\mathcal{T}_h} - ((\overline{\beta}^n - \beta_j^n) \cdot \mathbf{n}, (u_{jh}^{n-1} - \mathbf{\Pi}_W^j u_j^n) \eta_{jh}^{\widehat{u}^n})_{\partial\mathcal{T}_h}\right] - (c_j^n (\mathbf{\Pi}_V^j q_j^n - q_j^n), \eta_{jh}^{q^n})_{\mathcal{T}_h} = R_1 + R_2 + R_3 + R_4.$$
(3.9)

Next, we estimate  $\{R_i\}_{i=1}^4$ . By the condition (2.1), there exist  $0 < \alpha < 1$  such that

$$\begin{aligned} R_{1} &= ((\overline{c}^{n} - c_{j}^{n})(\eta_{jh}^{q^{n-1}} - \Delta t \partial_{t}^{+} \Pi_{V}^{j} q_{j}^{n}), \eta_{jh}^{q^{n}})_{\mathcal{T}_{h}} \\ &\leq \frac{\alpha}{2} \left( \|\sqrt{\overline{c}^{n}} \eta_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + \|\sqrt{\overline{c}^{n-1}} \eta_{jh}^{q^{n-1}}\|_{\mathcal{T}_{h}}^{2} \right) + C\Delta t^{2} \|\partial_{t}^{+} \Pi_{V}^{j} q_{j}^{n}\|_{\mathcal{T}_{h}}^{2}, \\ R_{2} &= (\partial_{t}^{+} (u_{j}^{n} - \Pi_{W}^{j} u_{j}^{n}) - \partial_{t}^{+} u_{j}^{n} + \partial_{t} u_{j}^{n}, \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} \\ &\leq C \left( \|\partial_{t}^{+} (u_{j}^{n} - \Pi_{W}^{j} u_{j}^{n})\|_{\mathcal{T}_{h}}^{2} + \|\partial_{t}^{+} u_{j}^{n} - \partial_{t} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} + \|\eta_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} \right), \\ R_{4} &\leq \frac{1 - \alpha}{8} \|\sqrt{\overline{c}^{n}} \eta_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + Ch^{2k+2} (|u_{j}^{n}|_{k+1}^{2} + |q_{j}^{n}|_{k+1}^{2}). \end{aligned}$$

If we directly estimate  $R_3$ , we will obtain only suboptimal convergence rates. Therefore, we need a refined analysis for this term. For simplicity, let  $\gamma = \overline{\beta}^n - \beta_j^n$ . The following argument is similar to the proof of the stability Section 2; to make the proof self-contained, we include these details here. First

$$\begin{split} R_{3} &= (\boldsymbol{\gamma} \cdot \nabla (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}), \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} - \langle \boldsymbol{\gamma} \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}) \eta_{jh}^{\widehat{u}^{n}} \rangle_{\partial \mathcal{T}_{h}} \\ &= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma}) \cdot \nabla (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}), \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} \\ &- \langle (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma}) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}) \eta_{jh}^{\widehat{u}^{n}} \rangle_{\partial \mathcal{T}_{h}} \\ &+ (\boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \cdot \nabla (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}), \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} - \langle \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n}) \eta_{jh}^{\widehat{u}^{n}} \rangle_{\partial \mathcal{T}_{h}}. \end{split}$$

By the error equation (3.6a), we have

$$\begin{aligned} (\mathbf{\Pi}_{0}\boldsymbol{\gamma}\cdot\nabla(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n}),\eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} &- \langle \mathbf{\Pi}_{0}\boldsymbol{\gamma}\cdot\boldsymbol{n},(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})\eta_{jh}^{\widehat{u}^{n}}\rangle_{\partial\mathcal{T}_{h}} \\ &= (\nabla\cdot[\mathbf{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})],\eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} - \langle [(\mathbf{\Pi}_{0}\boldsymbol{\gamma}\cdot\boldsymbol{n})(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})],\eta_{jh}^{\widehat{u}^{n}}\rangle_{\partial\mathcal{T}_{h}} \\ &= (\bar{c}^{n}\eta_{jh}^{\boldsymbol{q}^{n}},[\mathbf{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})])_{\mathcal{T}_{h}} + (c_{j}^{n}(\mathbf{\Pi}_{V}^{j}\boldsymbol{q}_{j}^{n}-\boldsymbol{q}_{j}^{n}),[\mathbf{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})])_{\mathcal{T}_{h}} \\ &- ((\bar{c}^{n}-c_{j}^{n})(\boldsymbol{q}_{jh}^{n-1}-\mathbf{\Pi}_{V}^{j}\boldsymbol{q}_{j}^{n}),[\mathbf{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1}-\Pi_{W}^{j}u_{j}^{n})])_{\mathcal{T}_{h}}. \end{aligned}$$

This gives

$$\begin{aligned} R_{3} &= ((\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \nabla(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n}), \eta_{jh}^{u^{n}})_{\mathcal{T}_{h}} \\ &- \langle (\boldsymbol{\gamma} - \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n})\eta_{jh}^{u^{n}}\rangle_{\partial\mathcal{T}_{h}} \\ &+ (\bar{c}^{n}\eta_{jh}^{\boldsymbol{q}^{n}}, \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n}))_{\mathcal{T}_{h}} + (c_{j}^{n}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{j}^{n}), [\boldsymbol{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n})])_{\mathcal{T}_{h}} \\ &- ((\bar{c}^{n} - c_{j}^{n})(\boldsymbol{q}_{jh}^{n-1} - \boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}^{n}), \boldsymbol{\Pi}_{0}\boldsymbol{\gamma}(u_{jh}^{n-1} - \Pi_{W}^{j}u_{j}^{n}))_{\mathcal{T}_{h}}. \end{aligned}$$

Hence,

$$\begin{split} R_{3} &\leq \sum_{K \in \mathcal{T}_{h}} \| \boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty,K} \| \nabla (u_{jh}^{n-1} - \boldsymbol{\Pi}_{W}^{j} u_{j}^{n}) \|_{K} \| \eta_{jh}^{u^{n}} \|_{K} \\ &+ \sum_{K \in \mathcal{T}_{h}} \| \boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty,\partial K} \| u_{jh}^{n-1} - \boldsymbol{\Pi}_{W}^{j} u_{j}^{n} \|_{\partial K} (\| \eta_{jh}^{\widehat{u}^{n}} - \eta_{jh}^{u^{n}} \|_{\partial K} + \| \eta_{jh}^{u^{n}} \|_{\partial K}) \\ &+ \| \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty,\mathcal{T}_{h}} \| \bar{c}^{n} \eta_{jh}^{q^{n}} \|_{\mathcal{T}_{h}} \| u_{jh}^{n-1} - \boldsymbol{\Pi}_{W}^{j} u_{j}^{n} \|_{\mathcal{T}_{h}} \\ &+ \| (\bar{c}^{n} - c_{j}^{n}) \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty,\mathcal{T}_{h}} \| q_{jh}^{n-1} - \boldsymbol{\Pi}_{V}^{j} q_{j}^{n} \|_{\mathcal{T}_{h}} \| u_{jh}^{n-1} - \boldsymbol{\Pi}_{W}^{j} u_{j}^{n} \|_{\mathcal{T}_{h}} \\ &+ \| c_{j}^{n} \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty,\mathcal{T}_{h}} \| \boldsymbol{\Pi}_{V}^{j} q_{j}^{n} - q_{j}^{n} \|_{\mathcal{T}_{h}} \| u_{jh}^{n-1} - \boldsymbol{\Pi}_{W}^{j} u_{j}^{n} \|_{\mathcal{T}_{h}} \\ &= R_{31} + R_{32} + R_{33} + R_{34} + R_{35}. \end{split}$$

For  $R_{31}$ , use the local inverse inequality:

$$R_{31} \leq C \sum_{K \in \mathcal{T}_h} h_K \| \boldsymbol{\gamma} \|_{1,\infty,K} h_K^{-1} \| u_{jh}^{n-1} - \Pi_W^j u_j^n \|_K \| \eta_{jh}^{u^n} \|_K$$
$$\leq C \sum_{K \in \mathcal{T}_h} \| u_{jh}^{n-1} - \Pi_W^j u_j^n \|_K \| \eta_{jh}^{u^n} \|_K$$
$$\leq C (\| \eta_{jh}^{u^{n-1}} \|_{\mathcal{T}_h}^2 + \Delta t^2 \| \partial_t^+ \Pi_W^j u_j^n \|_{\mathcal{T}_h}^2 + \| \eta_{jh}^{u^n} \|_{\mathcal{T}_h}^2).$$

Apply the trace inequality and inverse inequality for the term  $R_{32}$  to give

$$\begin{aligned} R_{32} &\leq C \sum_{K \in \mathcal{T}_{h}} h_{K} \| \boldsymbol{\gamma} \|_{1,\infty,K} h_{K}^{-1/2} \| u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n} \|_{K} (\| \eta_{jh}^{\widehat{u}^{n}} - \eta_{jh}^{u^{n}} \|_{\partial K} + h_{K}^{-1/2} \| \eta_{jh}^{u^{n}} \|_{K}) \\ &\leq C \sum_{K \in \mathcal{T}_{h}} \| u_{jh}^{n-1} - \Pi_{W}^{j} u_{j}^{n} \|_{K} (\| \eta_{jh}^{\widehat{u}^{n}} - \eta_{jh}^{u^{n}} \|_{\partial K} + \| \eta_{jh}^{u^{n}} \|_{K}) \\ &\leq C (\| \eta_{jh}^{u^{n-1}} \|_{\mathcal{T}_{h}}^{2} + \Delta t^{2} \| \partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n} \|_{\mathcal{T}_{h}}^{2} + \| \eta_{jh}^{u^{n}} \|_{\mathcal{T}_{h}}^{2}) + \frac{1}{4} \| \sqrt{\tau} (\eta_{jh}^{\widehat{u}^{n}} - \eta_{jh}^{u^{n}} ) \|_{\partial \mathcal{T}_{h}}^{2}. \end{aligned}$$

For the terms  $R_{33}$ ,  $R_{34}$ ,  $R_{35}$  and  $R_4$ , use Young's inequality to obtain

$$\begin{split} R_{33} &\leq \frac{1-\alpha}{8} \|\sqrt{\bar{c}^n} \eta_{jh}^{q^n}\|_{\mathcal{T}_h}^2 + C(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2), \\ R_{34} &\leq \frac{1-\alpha}{8} \|\sqrt{\bar{c}^n} \eta_{jh}^{q^{n-1}}\|_{\mathcal{T}_h}^2 + \frac{\Delta t^2}{4} \|\partial_t^+ \Pi_V^j q_j^n\|_{\mathcal{T}_h}^2 \\ &\quad + C(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2), \\ R_{35} &\leq Ch^{2k+2}(|u_j^n|_{k+1}^2 + |q_j^n|_{k+1}^2) + C(\|\eta_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2) \end{split}$$

We add (3.9) from n = 1 to n = N, use the above inequalities to get

$$\max_{1 \le n \le N} \|\eta_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} + \Delta t \sum_{n=1}^{N} \|\sqrt{c^{n}}\eta_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} 
\le C\Delta t \sum_{n=1}^{N} \|\eta_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} + C \sum_{n=1}^{N} (\Delta t^{3} \|\partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} + \Delta t^{3} \|\partial_{t}^{+} \Pi_{V}^{j} q_{j}^{n}\|_{\mathcal{T}_{h}}^{2}) 
+ C \sum_{n=1}^{N} (\Delta t \|\partial_{t}^{+} (u_{j}^{n} - \Pi_{W}^{j} u_{j}^{n})\|_{\mathcal{T}_{h}}^{2} + \Delta t \|\partial_{t}^{+} u_{j}^{n} - \partial_{t} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2}) 
+ C h^{2k+2} \sum_{n=1}^{N} \Delta t (|u_{j}^{n}|_{k+1}^{2} + |q_{j}^{n}|_{k+1}^{2}) + \|\eta_{jh}^{u^{0}}\|_{\mathcal{T}_{h}}^{2} + \|\eta_{jh}^{q^{0}}\|_{\mathcal{T}_{h}}^{2}.$$
(3.10)

.

Now we move to bound the terms on the right side of the above inequality as follows,

$$\begin{aligned} \Delta t^{3} \sum_{n=1}^{N} \|\partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} &= \Delta t \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} \partial_{t} \Pi_{W}^{j} u_{j}^{n} dt \right]^{2} \\ &\leq C \Delta t^{2} \|\partial_{t} \Pi_{W}^{j} u_{j}^{n}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \\ \Delta t^{3} \sum_{n=1}^{N} \|\partial_{t}^{+} \Pi_{V}^{j} \boldsymbol{q}_{j}^{n}\|_{\mathcal{T}_{h}}^{2} &= \Delta t \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} \partial_{t} \Pi_{V}^{j} \boldsymbol{q}_{j}^{n} dt \right]^{2} \\ &\leq C \Delta t^{2} \|\partial_{t} \Pi_{V}^{j} \boldsymbol{q}_{j}^{n}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \end{aligned}$$

and

$$\begin{aligned} \Delta t \sum_{n=1}^{N} \|\partial_{t}^{+}(u_{j}^{n} - \Pi_{W}^{j}u_{j}^{n})\|_{\mathcal{T}_{h}}^{2} &= \Delta t^{-1} \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} \partial_{t}(u_{j}^{n} - \Pi_{W}^{j}u_{j}dt) \right]^{2} \\ &\leq C \|\partial_{t}(u_{j}^{n} - \Pi_{W}^{j}u_{j})\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \\ \Delta t \sum_{n=1}^{N} \|\partial_{t}^{+}u_{j}^{n} - \partial_{t}u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} &= \Delta t^{-1} \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} (t - t^{n-1})\partial_{tt}u_{j}dt \right]^{2} \\ &\leq C \Delta t^{2} \|\partial_{tt}u_{j}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}. \end{aligned}$$

Gronwall's inequality and the estimates above applied to (3.10) give the result.

From Lemma 6 and the estimate in Lemma 3 we complete the proof of (3.3) in Theorem 1.

#### **Proof of** (3.4) in Theorem 1 $\mathbf{3.4}$

To prove (3.4) in Theorem 1, we follow a similar strategy taken by Chen, Cockburn, Singler and Zhang [3] and introduce an HDG elliptic projection in Section 3.4.1. We first bound the error between the solutions of the HDG elliptic projection and the exact solution of the system (1.1). Then we bound the error between the solutions of the HDG elliptic projection and the Ensemble HDG problem (1.7). A simple application of the triangle inequality then gives a bound on the error between the solutions of the Ensemble HDG problem and the system (1.1). We note that the coefficients of the PDEs are independent of time throughout this section. Hence, we drop the superscript n from  $c_j^n, \beta_j^n$  and the ensemble means  $\overline{c}^n, \overline{\beta}^n$ .

#### HDG elliptic projection 3.4.1

For any  $t \in [0,T]$ , let  $(\overline{q}_{jh}, \overline{u}_{jh}, \widehat{\overline{u}}_{jh}) \in V_h \times W_h \times M_h$  be the solutions of the following steady state problems

$$(c_{j}\overline{\boldsymbol{q}}_{jh},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - (\overline{u}_{jh},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle \widehat{\overline{u}}_{jh},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} = -\langle g_{j},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\mathcal{E}_{h}^{\partial}}, \qquad (3.11a)$$
$$(\nabla\cdot\overline{\boldsymbol{q}}_{jh},v_{h})_{\mathcal{T}_{h}} - \langle \overline{\boldsymbol{q}}_{jh}\cdot\boldsymbol{n},\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} + \langle \tau(\overline{u}_{jh}-\widehat{\overline{u}}_{jh}),v_{h}-\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}}$$

$$\cdot \overline{oldsymbol{q}}_{jh}, v_h)_{\mathcal{T}_h} - \langle \overline{oldsymbol{q}}_{jh} \cdot oldsymbol{n}, \widehat{v}_h 
angle_{\partial \mathcal{T}_h} + \langle au(\overline{u}_{jh} - \overline{u}_{jh}), v_h - \widehat{v}_h 
angle_{\partial \mathcal{T}_h}$$

$$+(\boldsymbol{\beta}_{j}\cdot\nabla\overline{u}_{jh},v_{h})_{\mathcal{T}_{h}}-\langle\boldsymbol{\beta}_{j}\cdot\boldsymbol{n},\overline{u}_{jh}\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}}=(f_{j}-\Pi_{W}^{j}\partial_{t}u_{j},v_{h})_{\mathcal{T}_{h}}+\langle\tau g_{j},v_{h}\rangle_{\mathcal{E}_{h}^{\partial}},\qquad(3.11\mathrm{b})$$

for all  $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$  and  $j = 1, 2, \cdots, J$ .

The proofs of the following estimates are given in Section 6.

**Theorem 2.** For any  $t \in [0, T]$  and for all  $j = 1, 2, \dots, J$ , we have

$$\|\mathbf{\Pi}_{V}^{j}\boldsymbol{q}_{j} - \overline{\boldsymbol{q}}_{jh}\|_{\mathcal{T}_{h}} \le C\mathcal{A}_{j}, \qquad (3.12a)$$

$$\|\Pi_W^j u_j - \overline{u}_{jh}\|_{\mathcal{T}_h} \le Ch^{\min\{k,1\}} \mathcal{A}_j, \tag{3.12b}$$

$$\|\partial_t (\mathbf{\Pi}_V^j \boldsymbol{q}_j - \overline{\boldsymbol{q}}_{jh})\|_{\mathcal{T}_h} \le C\mathcal{B}_j, \tag{3.12c}$$

$$\begin{split} \|\Pi_W^j u_j - \overline{u}_{jh}\|_{\mathcal{T}_h} &\leq Ch^{\min\{k,1\}} \mathcal{A}_j, \\ \|\partial_t (\Pi_V^j q_j - \overline{q}_{jh})\|_{\mathcal{T}_h} &\leq C \mathcal{B}_j, \\ \|\partial_t (\Pi_W^j u_j - \overline{u}_{jh})\|_{\mathcal{T}_h} &\leq Ch^{\min\{k,1\}} \mathcal{B}_j, \\ \|\partial_{tt} (\Pi_W^j u_j - \overline{u}_{jh})\|_{\mathcal{T}_h} &\leq Ch^{\min\{k,1\}} \mathcal{C}_j, \end{split}$$
(3.12c)

$$\|\partial_{tt}(\Pi^j_W u_j - \overline{u}_{jh})\|_{\mathcal{T}_h} \le Ch^{\min\{k,1\}} \mathcal{C}_j, \tag{3.12e}$$

where

$$\begin{aligned} \mathcal{A}_{j} &= \|u_{j} - \Pi_{W}^{j} u_{j}\|_{\mathcal{T}_{h}} + \|\boldsymbol{q}_{j} - \boldsymbol{\Pi}_{V}^{j} \boldsymbol{q}_{j}\|_{\mathcal{T}_{h}} + \|\partial_{t} u_{j} - \Pi_{W}^{j} \partial_{t} u_{j}\|_{\mathcal{T}_{h}}, \\ \mathcal{B}_{j} &= \|\partial_{t} u_{j} - \Pi_{W}^{j} \partial_{t} u_{j}\|_{\mathcal{T}_{h}} + \|\partial_{t} \boldsymbol{q}_{j} - \boldsymbol{\Pi}_{V}^{j} \partial_{t} \boldsymbol{q}_{j}\|_{\mathcal{T}_{h}} + \|\partial_{tt} u_{j} - \Pi_{W}^{j} \partial_{tt} u_{j}\|_{\mathcal{T}_{h}}, \\ \mathcal{C}_{j} &= \|\partial_{tt} u_{j} - \Pi_{W}^{j} \partial_{tt} u_{j}\|_{\mathcal{T}_{h}} + \|\partial_{tt} \boldsymbol{q}_{j} - \boldsymbol{\Pi}_{V}^{j} \partial_{tt} \boldsymbol{q}_{j}\|_{\mathcal{T}_{h}} + \|\partial_{ttt} u_{j} - \Pi_{W}^{j} \partial_{ttt} u_{j}\|_{\mathcal{T}_{h}}. \end{aligned}$$

Note that Theorem 2 bounds the error between the HDG elliptic projection of the solutions and the exact solutions of the system (1.1). In the next three steps, we are going to bound the error between the HDG elliptic projection of the ensemble solutions and the solutions of the Ensemble HDG problem (1.7).

#### 3.4.2The equations of the projection of the errors

**Lemma 7.** For  $e_{jh}^{u^n} = u_{jh}^n - \overline{u}_{jh}^n$ ,  $e_{jh}^{q^n} = q_{jh}^n - \overline{q}_{jh}^n$  and  $e_{jh}^{\widehat{u}^n} = \widehat{u}_{jh}^n - \widehat{\overline{u}}_{jh}^n$ , for all  $j = 1, 2, \dots, J$ , we have the following error equations

$$(\bar{c}e_{jh}^{\boldsymbol{q}^{n}},\boldsymbol{r}_{h})_{\mathcal{T}_{h}}-(e_{jh}^{\boldsymbol{u}^{n}},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}}+\langle e_{jh}^{\widehat{\boldsymbol{u}}^{n}},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}}=((\bar{c}-c_{j})(\boldsymbol{q}_{jh}^{n-1}-\overline{\boldsymbol{q}}_{jh}^{n}),\boldsymbol{r}_{h})_{\mathcal{T}_{h}},$$
(3.13a)

and

$$\begin{aligned} &(\partial_t^+ e_{jh}^{u^n}, v_h)_{\mathcal{T}_h} + (\nabla \cdot e_{jh}^{q^n}, v_h)_{\mathcal{T}_h} - \langle e_{jh}^{q^n} \cdot \boldsymbol{n}, \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + (\overline{\boldsymbol{\beta}} \cdot \nabla e_{jh}^{u^n}, v_h)_{\mathcal{T}_h} \\ &- \langle \overline{\boldsymbol{\beta}} \cdot \boldsymbol{n}, e_{jh}^{u^n} \widehat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \tau(e_{jh}^{u^n} - e_{jh}^{\widehat{u}}), v_h - \widehat{v}_h \rangle_{\partial \mathcal{T}_h} - (\partial_t^+ \overline{u}_{jh}^n - \partial_t \Pi_W^j u_j^n, v_h)_{\mathcal{T}_h} \\ &= ((\overline{\boldsymbol{\beta}} - \boldsymbol{\beta}_j) \cdot \nabla (u_{jh}^{n-1} - \overline{u}_{jh}^n), v_h)_{\mathcal{T}_h} - \langle (\overline{\boldsymbol{\beta}}_j - \boldsymbol{\beta}_j) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \overline{u}_{jh}^n) \widehat{v}_h \rangle_{\partial \mathcal{T}_h}, \end{aligned}$$
(3.13b)

for all  $(\mathbf{r}_h, v_h, \hat{v}_h) \in \mathbf{V}_h \times W_h \times M_h$  and  $n = 1, 2, \cdots, N$ .

The proof of Lemma 7 follows immediately by simply subtracting Equation (3.11) from Equation (1.7).

#### **Energy** argument 3.4.3

Lemma 8. If condition (2.1) and the elliptic regularity inequality (6.4) holds, then we have the following error estimate:

$$\max_{1 \le n \le N} \|e_{jh}^{u^n}\|_{\mathcal{T}_h} \le C\left(h^{k+1+\min\{k,1\}} + \Delta t\right).$$
(3.14)

*Proof.* The following proof is similar to the proof in Section 3.3; to make the proof self-contained, we include the details here. We take  $(\mathbf{r}_h, v_h, \hat{v}_h) = (e_{jh}^{\mathbf{q}^n}, e_{jh}^{u^n}, e_{jh}^{\hat{u}^n})$  in (3.13), use the identity (2.5) and add Equation (3.13a) - Equation (3.13b) together to get

$$\frac{\|e_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} - \|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \frac{\|e_{jh}^{u^{n}} - e_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2}}{2\Delta t} + \|\sqrt{c}e_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + \|\sqrt{\tau}(e_{jh}^{u^{n}} - e_{jh}^{\widehat{u}^{n}})\|_{\partial\mathcal{T}_{h}}^{2}} 
= -(\overline{\beta} \cdot \nabla e_{jh}^{u^{n}}, e_{jh}^{u^{n}})_{\mathcal{T}_{h}} + \langle \overline{\beta} \cdot \boldsymbol{n}, e_{jh}^{u^{n}} e_{jh}^{\widehat{u}^{n}} \rangle_{\partial\mathcal{T}_{h}} + ((\overline{c} - c_{j})(\boldsymbol{q}_{jh}^{n-1} - \overline{\boldsymbol{q}}_{jh}^{n}), e_{jh}^{q^{n}})_{\mathcal{T}_{h}} 
+ (\partial_{t}^{+} \overline{u}_{jh}^{n} - \partial_{t} \Pi_{W}^{j} u_{j}^{n}, e_{jh}^{u^{n}})_{\mathcal{T}_{h}} + ((\overline{\beta} - \beta_{j}) \cdot \nabla (u_{jh}^{n-1} - \overline{u}_{jh}^{n}), e_{jh}^{u^{n}})_{\mathcal{T}_{h}} 
- \langle (\overline{\beta} - \beta_{j}) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \overline{u}_{jh}^{n}) e_{jh}^{\widehat{u}^{n}} \rangle_{\partial\mathcal{T}_{h}}.$$
(3.15)

By Green's formula and the fact  $\langle (\overline{\beta} \cdot \boldsymbol{n}) e_{jh}^{\widehat{u}^n}, e_{jh}^{\widehat{u}^n} \rangle_{\partial \mathcal{T}_h} = 0$ , we have

$$(\overline{\boldsymbol{\beta}} \cdot \nabla e_{jh}^{u^n}, e_{jh}^{u^n})_{\mathcal{T}_h} - \langle \overline{\boldsymbol{\beta}} \cdot \boldsymbol{n}, e_{jh}^{u^n} e_{jh}^{\widehat{\boldsymbol{u}}^n} \rangle_{\partial \mathcal{T}_h} \leq \frac{1}{2} \| \sqrt{|\overline{\boldsymbol{\beta}} \cdot \boldsymbol{n}|} (e_{jh}^{u^n} - e_{jh}^{\widehat{\boldsymbol{u}}^n}) \|_{\partial \mathcal{T}_h}^2.$$

Condition (2.2) and equality (3.15) give

$$\begin{aligned} \frac{\|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 - \|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \frac{\|e_{jh}^{u^n} - e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2}{2\Delta t} + \|\sqrt{\bar{c}}e_{jh}^{q^n}\|_{\mathcal{T}_h}^2 + \frac{1}{2}\|\sqrt{\tau}(e_{jh}^{u^n} - e_{jh}^{\widehat{u}^n})\|_{\partial\mathcal{T}_h}^2} \\ &\leq ((\bar{c} - c_j)(q_{jh}^{n-1} - \overline{q}_{jh}^n), e_{jh}^{q^n})_{\mathcal{T}_h} + (\partial_t^+ \overline{u}_{jh}^n - \partial_t \Pi_W^j u_j^n, e_{jh}^{u^n})_{\mathcal{T}_h} \\ &+ ((\overline{\beta} - \beta_j) \cdot \nabla(u_{jh}^{n-1} - \overline{u}_{jh}^n), e_{jh}^{u^n})_{\mathcal{T}_h} - \langle (\overline{\beta} - \beta_j) \cdot \boldsymbol{n}, (u_{jh}^{n-1} - \overline{u}_{jh}^n) e_{jh}^{\widehat{u}^n} \rangle_{\partial\mathcal{T}_h} \\ &= T_1 + T_2 + T_3. \end{aligned}$$

Next, we estimate  $\{T_i\}_{i=1}^3$ . By the condition (2.1), there exist  $0 < \alpha < 1$  such that

$$\begin{split} T_{1} &= ((\overline{c} - c_{j})(e_{jh}^{q^{n-1}} - \Delta t \partial_{t}^{+} \overline{q}_{jh}^{n}), e_{jh}^{q^{n}})_{\mathcal{T}_{h}} \\ &\leq \frac{\alpha}{2} \left( \|\sqrt{\overline{c}} e_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + \|\sqrt{\overline{c}} e_{jh}^{q^{n-1}}\|_{\mathcal{T}_{h}}^{2} \right) + C\Delta t^{2} \|\partial_{t}^{+} \overline{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2}, \\ T_{2} &= (\partial_{t}^{+} (\overline{u}_{jh}^{n} - \Pi_{W}^{j} u_{j}^{n}) + \partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n} - \partial_{t} \Pi_{W}^{j} u_{j}^{n}, e_{jh}^{u^{n}})_{\mathcal{T}_{h}} \\ &\leq C \left( \|\partial_{t}^{+} (\overline{u}_{jh}^{n} - \Pi_{W}^{j} u_{j}^{n})\|_{\mathcal{T}_{h}}^{2} + \|\partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n} - \partial_{t} \Pi_{W}^{j} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} + \|e_{jh}^{u^{n}}\|_{\mathcal{T}_{h}}^{2} \right). \end{split}$$

To treat the term  $T_3$ , we use the technique in the proof of Lemma 6, where we treat the term  $R_3$ . For  $\gamma = \overline{\beta} - \beta_j$ , we have

$$T_{3} \leq \sum_{K \in \mathcal{T}_{h}} \| \boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty, K} \| \nabla (u_{jh}^{n-1} - \overline{u}_{jh}^{n}) \|_{K} \| e_{jh}^{u^{n}} \|_{K} + \sum_{K \in \mathcal{T}_{h}} \| \boldsymbol{\gamma} - \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty, \partial K} \| u_{jh}^{n-1} - \overline{u}_{jh}^{n} \|_{\partial K} (\| e_{jh}^{\widehat{u}^{n}} - e_{jh}^{u^{n}} \|_{\partial K} + \| e_{jh}^{u^{n}} \|_{\partial K}) + \| \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty, \mathcal{T}_{h}} \| \bar{c} e_{jh}^{q^{n}} \|_{\mathcal{T}_{h}} \| u_{jh}^{n-1} - \overline{u}_{jh}^{n} \|_{\mathcal{T}_{h}} + \| (\bar{c} - c_{j}) \boldsymbol{\Pi}_{0} \boldsymbol{\gamma} \|_{\infty, \mathcal{T}_{h}} \| q_{jh}^{n-1} - \overline{q}_{jh}^{n} \|_{\mathcal{T}_{h}} \| u_{jh}^{n-1} - \overline{u}_{jh}^{n} \|_{\mathcal{T}_{h}} = T_{31} + T_{32} + T_{33} + T_{34}.$$

For  $T_{31}$ , use the local inverse inequality:

$$T_{31} \le C(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t^2 \|\partial_t^+ \overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 + \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2).$$

Apply the trace inequality and inverse inequality for the term  $T_{32}$  to give

$$T_{32} \leq C \sum_{K \in \mathcal{T}_h} h_K \| \boldsymbol{\gamma} \|_{1,\infty,K} h_K^{-1/2} \| u_{jh}^{n-1} - \overline{u}_{jh}^n \|_K (\| e_{jh}^{\widehat{u}^n} - e_{jh}^{u^n} \|_{\partial K} + h_K^{-1/2} \| e_{jh}^{u^n} \|_K)$$
  
$$\leq C (\| e_{jh}^{u^{n-1}} \|_{\mathcal{T}_h}^2 + \Delta t^2 \| \partial_t^+ \overline{u}_{jh}^n \|_{\mathcal{T}_h}^2 + \| e_{jh}^{u^n} \|_{\mathcal{T}_h}^2) + \frac{1}{4} \| \sqrt{\tau} (e_{jh}^{\widehat{u}^n} - e_{jh}^{u^n}) \|_{\partial \mathcal{T}_h}^2.$$

For the terms  $T_{33}$  and  $T_{34}$ , use Young's inequality to obtain

$$T_{33} \leq \frac{1-\alpha}{8} \|\sqrt{\bar{c}}e_{jh}^{q^{n}}\|_{\mathcal{T}_{h}}^{2} + C(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2} + \Delta t^{2}\|\partial_{t}^{+}\overline{u}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2}),$$
  
$$T_{34} \leq \frac{1-\alpha}{8} \|\sqrt{\bar{c}}e_{jh}^{q^{n-1}}\|_{\mathcal{T}_{h}}^{2} + \frac{\Delta t^{2}}{4} \|\partial_{t}^{+}\overline{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} + C(\|e_{jh}^{u^{n-1}}\|_{\mathcal{T}_{h}}^{2} + \Delta t^{2}\|\partial_{t}^{+}\overline{u}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2}).$$

We add (3.15) from n = 1 to n = N, and use the above inequalities to get

$$\max_{1 \le n \le N} \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + \sum_{n=1}^N \|e_{jh}^{u^n} - e_{jh}^{u^{n-1}}\|_{\mathcal{T}_h}^2 + \Delta t \sum_{n=1}^N \|\sqrt{c}e_{jh}^{q^n}\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(e_{jh}^{u^n} - e_{jh}^{\widehat{u}^n})\|_{\mathcal{T}_h}^2 
\le C\Delta t \sum_{n=1}^N \|e_{jh}^{u^n}\|_{\mathcal{T}_h}^2 + C \sum_{n=1}^N (\Delta t^3 \|\partial_t^+ \overline{u}_{jh}^n\|_{\mathcal{T}_h}^2 + \Delta t^3 \|\partial_t^+ \overline{q}_{jh}^n\|_{\mathcal{T}_h}^2) 
+ C \sum_{n=1}^N (\Delta t \|\partial_t^+ (\overline{u}_{jh}^n - \Pi_W^j u_j^n)\|_{\mathcal{T}_h}^2 + \Delta t \|\partial_t^+ \Pi_W^j u_j^n - \partial_t \Pi_W^j u_j^n\|_{\mathcal{T}_h}^2) 
+ C \|e_{jh}^{q^0}\|_{\mathcal{T}_h}^2 + C \|e_{jh}^{u^0}\|_{\mathcal{T}_h}^2.$$
(3.16)

Now we move to bound the terms on the right side of the above inequality as follows,

$$\begin{split} \Delta t^{3} \sum_{n=1}^{N} \|\partial_{t}^{+} \overline{u}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} &= \Delta t \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} \partial_{t} \overline{u}_{jh}^{n} dt \right]^{2} \leq C \Delta t^{2} \|\partial_{t} \overline{u}_{jh}^{n}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \\ \Delta t^{3} \sum_{n=1}^{N} \|\partial_{t}^{+} \overline{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2} &= \Delta t \sum_{n=1}^{N} \int_{\Omega} \left[ \int_{t^{n-1}}^{t^{n}} \partial_{t} \overline{q}_{jh}^{n} dt \right]^{2} \leq C \Delta t^{2} \|\partial_{t} \overline{q}_{jh}^{n}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \\ \Delta t \sum_{n=1}^{N} \|\partial_{t}^{+} (\overline{u}_{jh}^{n} - \Pi_{W}^{j} u_{j}^{n})\|_{\mathcal{T}_{h}}^{2} \leq C \|\partial_{t} (\overline{u}_{jh} - \Pi_{W}^{j} u_{j})\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}, \\ \Delta t \sum_{n=1}^{N} \|\partial_{t}^{+} \Pi_{W}^{j} u_{j}^{n} - \partial_{t} \Pi_{W}^{j} u_{j}^{n}\|_{\mathcal{T}_{h}}^{2} \leq C \Delta t^{2} \|\partial_{tt} \Pi_{W}^{j} u_{j}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2}. \end{split}$$

Gronwall's inequality and the estimates above applied to (3.16) give the result.

### 3.4.4 Superconvergence error estimates by postprocessing

The following element-by-element postprocessing is defined in [11]: Find  $u_{jh}^{n\star} \in \mathcal{P}^{k+1}(K)$  such that for all  $(z_h, w_h) \in [\mathcal{P}^{k+1}(K)]^{\perp} \times \mathcal{P}^0(K)$ 

$$(\nabla u_{jh}^{n\star}, \nabla z_h)_K = -(c_j \boldsymbol{q}_{jh}^n, \nabla z_h)_K, \qquad (3.17a)$$

$$(u_{jh}^{n\star}, w_h)_K = (u_h, w_h)_K,$$
 (3.17b)

where  $[\mathcal{P}^{k+1}(K)]^{\perp} = \{z_h \in \mathcal{P}^{k+1}(K) | (z_h, 1)_K = 0\}.$ 

**Lemma 9.** For any  $t \in [0, T]$  and  $k \ge 1$ , we have the following error estimate for the postprocessed solution:

$$\begin{aligned} \|\Pi_{k+1}u_{j}^{n}-u_{jh}^{n\star}\|_{\mathcal{T}_{h}} &\leq C \|\Pi_{W}^{j}u_{j}^{n}-u_{jh}^{n}\|_{\mathcal{T}_{h}} + Ch\|c_{j}(\boldsymbol{q}_{jh}^{n}-\boldsymbol{q}_{j}^{n})\|_{\mathcal{T}_{h}} \\ &+ Ch\|\nabla(u_{j}^{n}-\Pi_{k+1}u_{j}^{n})\|_{\mathcal{T}_{h}}. \end{aligned}$$

*Proof.* By the properties of  $\Pi_W$  and  $\Pi_{k+1}$ , we obtain

$$(\Pi_{W}^{j} u_{j}^{n}, w_{0})_{K} = (u_{j}^{n}, w_{0})_{K}, \text{ for all } w_{0} \in \mathcal{P}^{0}(K),$$
  
$$(\Pi_{k+1} u_{j}^{n}, w_{0})_{K} = (u_{j}^{n}, w_{0})_{K}, \text{ for all } w_{0} \in \mathcal{P}^{0}(K).$$

Hence, for all  $w_0 \in \mathcal{P}^0(K)$ , we have

$$(\Pi_W u_j^n - \Pi_{k+1} u_j^n, w_0)_K = 0$$

Let  $e_{jh}^n = u_{jh}^{n\star} - u_{jh}^n + \Pi_W^j u_j^n - \Pi_{k+1} u_j^n$ . Equation (3.17) and an inverse inequality give

$$\begin{aligned} \|\nabla e_{jh}^{n}\|_{K}^{2} &= (\nabla (u_{jh}^{n\star} - u_{jh}), \nabla e_{j,h}^{n})_{K} + (\nabla (\Pi_{W}^{j} u_{j}^{n} - \Pi_{k+1} u_{j}^{n}), \nabla e_{jh}^{n})_{K} \\ &= (-\nabla u_{jh}^{n} - c_{j} \boldsymbol{q}_{jh}^{n}, \nabla e_{jh}^{n})_{K} + (\nabla (\Pi_{W}^{j} u_{j}^{n} - \Pi_{k+1} u_{j}^{n}), \nabla e_{jh}^{n})_{K} \\ &= (\nabla (\Pi_{W}^{j} u_{j}^{n} - u_{jh}^{n}) - (\boldsymbol{q}_{jh}^{n} - \boldsymbol{q}_{j}^{n}) + \nabla (u_{j}^{n} - \Pi_{k+1} u_{j}^{n}), \nabla e_{jh}^{n})_{K} \end{aligned}$$

This implies

$$\|\nabla e_{jh}^{n}\|_{K} \le C(h_{K}^{-1}\|\Pi_{W}^{j}u_{j}^{n} - u_{jh}^{n}\|_{K} + \|c_{j}(\boldsymbol{q}_{jh}^{n} - \boldsymbol{q}_{j}^{n})\|_{K} + \|\nabla(u_{j}^{n} - \Pi_{k+1}u_{j}^{n})\|_{K}).$$
(3.18)

Since  $(e_h, 1)_K = 0$ , apply the Poincaré inequality and the estimate (3.18) to give

$$\begin{aligned} \|e_{jh}^{n}\|_{K} &\leq Ch_{K} \|\nabla e_{jh}^{n}\|_{K} \\ &\leq C(\|\Pi_{W}^{j}u_{j}^{n}-u_{jh}^{n}\|_{K}+h_{K}\|c_{j}(\boldsymbol{q}_{jh}^{n}-\boldsymbol{q}_{j}^{n})\|_{K}+h_{K}\|\nabla(u_{j}^{n}-\Pi_{k+1}u_{j}^{n})\|_{K}). \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\Pi_{k+1}u_{j}^{n}-u_{jh}^{n\star}\|_{\mathcal{T}_{h}} &\leq \|\Pi_{k+1}u_{j}^{n}-\Pi_{W}^{j}u_{j}^{n}-u_{jh}^{n\star}+u_{jh}^{n}\|_{\mathcal{T}_{h}}+\|\Pi_{W}^{j}u_{j}^{n}-u_{jh}^{n}\|_{\mathcal{T}_{h}} \\ &\leq C\|\Pi_{W}^{j}u_{j}^{n}-u_{jh}^{n}\|_{\mathcal{T}_{h}}+Ch\|c_{j}(\boldsymbol{q}_{jh}^{n}-\boldsymbol{q}_{j}^{n})\|_{\mathcal{T}_{h}} \\ &+Ch\|\nabla(u_{j}^{n}-\Pi_{k+1}u_{j}^{n})\|_{\mathcal{T}_{h}}.\end{aligned}$$

From Lemma 9 and the estimate in (2.4a) we complete the proof of (3.4) in Theorem 1.

# 4 Numerical experiments

In this section, we present some numerical tests of the Ensemble HDG method for parameterized convection diffusion PDEs. Although we derived the a priori error estimates for diffusion dominated problems, we also present numerical results for the convection dominated case to show the performance of the Ensemble HDG method for the convection dominated diffusion problems. For all examples, we take  $\tau = 1 + \max_{1 \le j \le J} \|\beta_j\|_{0,\infty}$  so that (2.2) is satisfied, the coefficients  $c_j$  satisfy the condition (2.1), and a group of simulations are considered containing J = 3 members. Let  $Eu_j$ 

be the error betteen the exact solution  $u_j$  at the final time T = 1 and the Ensemble HDG solution  $u_{jh}^N$ , i.e.,  $Eu_j = \|u_j^N - u_{jh}^N\|_{\mathcal{T}_h}$ . Let

$$E \boldsymbol{q}_{j} = \sqrt{\Delta t \sum_{n=1}^{N} \|\boldsymbol{q}_{j}^{n} - \boldsymbol{q}_{jh}^{n}\|_{\mathcal{T}_{h}}^{2}}, \text{ and } E u_{j}^{\star} = \sqrt{\Delta t \sum_{n=1}^{N} \|u_{j}^{n} - u_{jh}^{n\star}\|_{\mathcal{T}_{h}}^{2}}.$$

**Example 1.** We first test the convergence rate of the Ensemble HDG method for diffusion dominated PDEs on a square domain  $\Omega = [0, 1] \times [0, 1]$ . The data is chosen as

$$c_1 = 0.26959, \ c_2 = 0.26633, \ c_3 = 0.30525,$$
  
$$\boldsymbol{\beta}_1 = 1.6797[y, x], \ \boldsymbol{\beta}_2 = 1.6551[y, x], \ \boldsymbol{\beta}_3 = 1.1626[y, x],$$
  
$$u_j = \sin(t)\sin(x)\sin(y)/j, \ \boldsymbol{q}_j = -1/c_j \nabla u_j, \ j = 1, 2, 3,$$

and the initial conditions, boundary conditions, and source terms are chosen to match the exact solution of Equation (1.1).

In order to confirm our theoretical results, we take  $\Delta t = h$  when k = 0 and  $\Delta t = h^3$  when k = 1. The approximation errors of the Ensemble HDG method are listed in Table 1 and the observed convergence rates match our theory.

$\begin{array}{ c c c c c c c c c c c c c c c c c c c$			Table 1: History of convergence for Example 1.					1.
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	Degree	h	$E \boldsymbol{q}_1$		$Eu_1$		$Eu_1^{\star}$	
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			Error	Rate	Error	Rate	Error	Rate
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	k = 0		8.5356E-01		8.0704E-02		1.3681E-01	
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		-	5.3683E-01	0.67	4.6752 E-02	0.79	5.7997E-02	1.24
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			2.9377E-01	0.87	2.4599 E-02	0.93	2.6288E-02	1.14
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$			1.5300E-01	0.94	1.2677 E-02	0.96	1.2902E-02	1.03
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			7.8021E-02	0.97	6.4474E-03	0.98	6.4760E-03	0.99
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	k = 1		2.6429E-01		4.2641E-02		4.3413E-02	
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			7.5086E-02	1.82	1.0472 E-02	2.03	6.1017E-03	2.83
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			1.9707 E-02	1.93	2.6345E-03	1.99	7.9146E-04	2.95
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			5.0211E-03	1.97	6.6870E-04	1.98	1.0026E-04	2.98
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$		$2^{-5}$	1.2653E-03	1.99	1.6896E-04	1.98	1.2598E-05	2.99
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Degree							
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$				Rate		Rate		Rate
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	k = 0		8.5466E-01		8.1522E-02		1.3739E-01	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			5.3907 E-01	0.66	4.8107E-02	0.76	5.9168E-02	1.22
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			2.9567 E-01	0.87	2.5614 E-02	0.91	2.7258E-02	1.12
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			1.5420E-01	0.94	1.3277 E-02	0.95	1.3495 E-02	1.01
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			7.8696E-02	0.97	6.7714 E-03	0.97	6.7992e-03	0.99
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	k = 1		2.6577E-01		4.2796E-02		4.3973E-02	
$\begin{array}{c c c c c c c c c c c c c c c c c c c $		$2^{-2}$	7.5666 E-02	1.81	1.0405E-02	2.04	6.2024E-03	2.83
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			1.9879E-02	1.93	2.6069E-03	2.00	8.0552E-04	2.94
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			5.0673E-03	1.97	6.6105E-04	1.98	1.0209E-04	2.98
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$		$2^{-5}$	1.2772 E-03	1.99	1.6699E-04	1.99	1.2832 E-05	2.99
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	Degree		$Eq_3$		$Eu_3$		$Eu_3^{\star}$	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$				Rate	Error	Rate	Error	Rate
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	k = 0		8.0839E-01		3.4525E-02		1.1145E-01	
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			5.0993E-01	0.66	2.2025 E-02	0.65	3.9756E-02	1.49
$\begin{array}{c c c c c c c c c c c c c c c c c c c $			2.7915 E-01	0.87	1.2282 E-02	0.84	1.5196E-02	1.39
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$			1.4529E-01	0.94	6.5117 E-03	0.92	6.9102E-03	1.14
$k = 1  \begin{array}{c ccccccccccccccccccccccccccccccccccc$			7.4042 E-02	0.97	3.3567 E-03	0.96	3.4076E-03	1.02
k = 1 2 <sup>-3</sup> 1.8087E-02 1.95 2.9375E-03 1.93 7.1138E-04 2.95	k = 1		2.4988E-01		4.0593E-02		3.9155E-02	
			6.9685 E-02	1.84	1.1221E-02	1.86	5.5006E-03	2.83
		$2^{-3}$	1.8087 E-02	1.95	2.9375 E-03	1.93	7.1138E-04	2.95
		$2^{-4}$	4.5831E-03	1.98	7.5247 E-04	1.96	8.9813E-05	2.99
$2^{-5}$ 1.1520E-03 1.99 1.9046E-04 1.98 1.1261E-05 3.00		$2^{-5}$	1.1520E-03	1.99	1.9046E-04	1.98	1.1261E-05	3.00

Table 1: History of convergence for Example 1.

**Example 2.** Next, we perform Ensemble HDG computations for the convection dominated case with exact solutions haveing interior layers. But we do not attempt to compute convergence rates here; instead for illustration we plot all the ensemble members  $\{u_{jh}\}_{j=1}^{3}$  at the final time T = 0.1 and also plot the exact solution for comparison. We can see that the Ensemble HDG method is able to capture the very sharp interior layers in the solution with almost no oscillatory behavior, see e.g. Figures 1 to 3.

The domain is  $\Omega = [0, 1] \times [0, 1]$  and it is uniformly partition into 131072 triangles  $(h = \sqrt{2}/256)$  and also  $\Delta t = 1/1000$ . The initial conditions, boundary conditions, and source terms are chosen

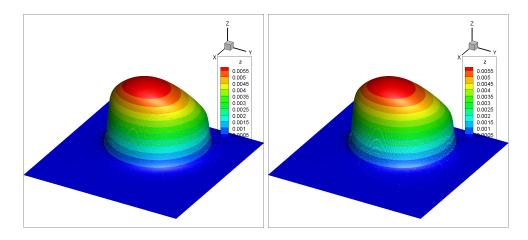


Figure 1: Left is the exact solution  $u_1$  and right is  $u_{1h}$  computed by Ensemble HDG.

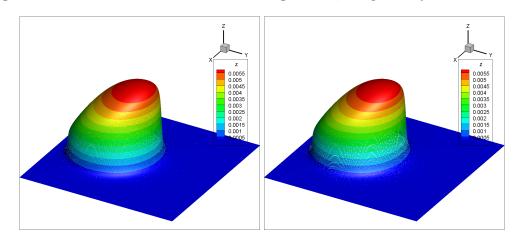


Figure 2: Left is the exact solution  $u_2$  and right is  $u_{2h}$  computed by Ensemble HDG.

to match Equation (1.1) for the data

$$c_1 = 10^4, \ c_2 = 2 \times 10^4, \ c_3 = 3 \times 10^4,$$
  
 $\boldsymbol{\beta}_1 = [2,3], \ \boldsymbol{\beta}_2 = [3,4], \ \boldsymbol{\beta}_3 = [4,5],$ 

and the exact solutions  $\{u_j\}_{j=1}^3$  are chosen as

$$u_{1} = \sin(t)x(1-x)y(1-y) \left[ \frac{1}{2} + \frac{\arctan 2\sqrt{c_{1}} \left(\frac{1}{12} - \left(x - \frac{1}{3}\right)^{2} - \left(y - \frac{1}{2}\right)^{2}\right)}{\pi} \right],$$
  

$$u_{2} = \sin(t)x(1-x)y(1-y) \left[ \frac{1}{2} + \frac{\arctan 2\sqrt{c_{2}} \left(\frac{1}{14} - \left(x - \frac{1}{2}\right)^{2} - \left(y - \frac{1}{3}\right)^{2}\right)}{\pi} \right],$$
  

$$u_{3} = \sin(t)x(1-x)y(1-y) \left[ \frac{1}{2} + \frac{\arctan 2\sqrt{c_{3}} \left(\frac{1}{16} - \left(x - \frac{1}{2}\right)^{2} - \left(y - \frac{1}{2}\right)^{2}\right)}{\pi} \right].$$

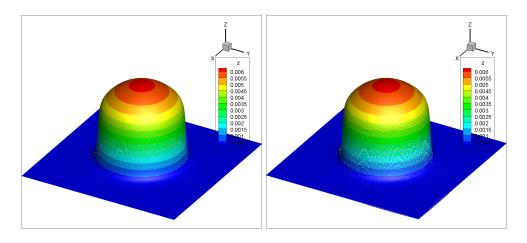


Figure 3: Left is the exact solution  $u_3$  and right is  $u_{3h}$  computed by Ensemble HDG.

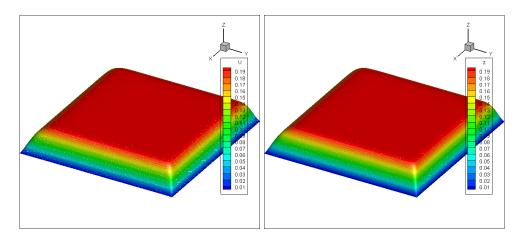


Figure 4: Left is solution  $u_{1h}$  and right is the postprocessed solution  $u_{1h}^{\star}$ .

**Example 3.** Fianlly, we perform the Ensemble HDG method for a group of convection dominated problems without exact solutions. In this example, the problems exhibit not interior layers but boundary layers. It is well known that the boundary layers are more difficult than interior layers for all numerical methods. Since in Example 2 the Ensemble HDG captured the interior layers without oscillations, we didn't plot the postprocessed solutions there. However, our numerical test shows that the postprocessed solutions  $u_{jh}^*$  are better than  $u_{jh}$  for solutions with boundary layers; see e.g. Figures 4 to 6. We note there is no superconvergent rate even for a single convection dominated diffusion problem PDE using HDG methods, see, e.g. [18].

We plot all the ensemble members  $u_{jh}$  and  $u_{jh}^{\star}$  at the final time T = 0.1 for comparison. The domain, the mesh, the time step, the bounday conditions and the initial conditions are the same with Example 2. For the other data, we take

$$c_1 = 60, \ c_2 = 120, \ c_3 = 180,$$
  
 $\boldsymbol{\beta}_1 = [2,3], \ \boldsymbol{\beta}_2 = [3,4], \ \boldsymbol{\beta}_3 = [4,5],$   
 $f_1 = 2, \ f_2 = 5, \ f_3 = 8.$ 

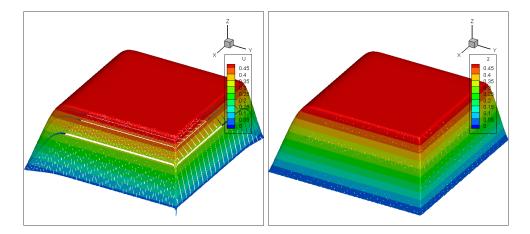


Figure 5: Left is solution  $u_{2h}$  and right is the postprocessed solution  $u_{2h}^{\star}$ .

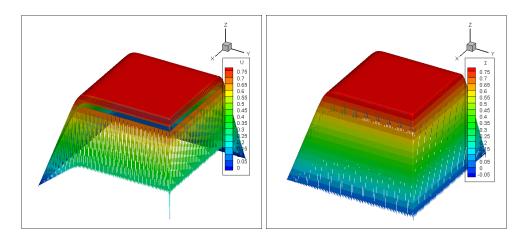


Figure 6: Left is solution  $u_{3h}$  and right is the postprocessed solution  $u_{3h}^{\star}$ .

# 5 Conclusion

In this work, we first devised a superconvergent Ensemble HDG method for parameterized convection diffusion PDEs. This Ensemble HDG method shares one common coefficient matrix and multiple RHS vectors, which is more efficient than performing separate simulations. We proved optimal error estimates for the flux  $q_j$  and the scalar variable  $u_j$ ; moreover, we obtained the superconvergent rate for  $u_j$ . As far as we are aware, this is the first time in the literature.

There are a number of topics that can be explored in the future, including devising high order time stepping methods, a group of convection dominated diffusion PDEs, and stochastic PDEs.

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# 6 Appendix

In this section we only give a proof for (3.12a) and (3.12b), since the rest are similar. To prove (3.12c)-(3.12e), we differentiate the error equations in Lemma 10 with respect to time t. It is easy to check that the operators  $\Pi_W^j$  commute with the time derivative, i.e.,  $\partial_t \Pi_W^j u_j = \Pi_W^j \partial_t u_j$ , since the velocity vector fields  $\beta_j$  are independent of time t.

### 6.1 The equations of the projection of the errors

Lemma 10. We have the following equalities

$$\begin{aligned} (c_{j}\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} &- (\boldsymbol{\Pi}_{W}^{j}u_{j},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle P_{M}u_{j},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} = (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}-\boldsymbol{q}_{j}),\boldsymbol{r}_{h})_{\mathcal{T}_{h}}, \\ (\nabla\cdot\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j},v_{h})_{\mathcal{T}_{h}} - \langle\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}\cdot\boldsymbol{n},\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} + \langle\boldsymbol{\tau}(\boldsymbol{\Pi}_{W}^{j}u_{j}-P_{M}u_{j}),v_{h}-\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} \\ + (\boldsymbol{\beta}_{j}\cdot\nabla\boldsymbol{\Pi}_{W}^{j}u_{j},v_{h})_{\mathcal{T}_{h}} - \langle\boldsymbol{\beta}_{j}\cdot\boldsymbol{n},\boldsymbol{\Pi}_{W}^{j}u_{j}\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} = (f_{j}-\partial_{t}u_{j},v_{h})_{\mathcal{T}_{h}}, \end{aligned}$$

for all  $(\boldsymbol{r}_h, v_h, \hat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ .

The proof is similar to the proof of Lemma 4, hence we omit it here. To simplify the notation, we set

$$\varepsilon_h^{u_j} = \overline{u}_{jh} - \Pi_W^j u_j, \ \varepsilon_h^{\boldsymbol{q}_j} = \overline{\boldsymbol{q}}_{jh} - \Pi_V^j \boldsymbol{q}_j, \ \varepsilon_h^{\widehat{u}_j} = \widehat{\overline{u}}_{jh} - P_M u_j.$$

Subtract (3.11) from (1.7) to get the following

Lemma 11. We have the error equations

$$(c_{j}\varepsilon_{jh}^{\boldsymbol{q}},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - (\varepsilon_{jh}^{\boldsymbol{u}},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle\varepsilon_{jh}^{\hat{\boldsymbol{u}}},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} = (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}-\boldsymbol{q}_{j}),\boldsymbol{r}_{h})_{\mathcal{T}_{h}},$$
(6.2a)

$$(\nabla \cdot \varepsilon_{jh}^{\boldsymbol{q}}, v_{h})_{\mathcal{T}_{h}} - \langle \varepsilon_{jh}^{\boldsymbol{q}} \cdot \boldsymbol{n}, \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} + (\boldsymbol{\beta}_{j} \cdot \nabla \varepsilon_{jh}^{u}, v_{h})_{\mathcal{T}_{h}} + \langle \tau(\varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}}), v_{h} - \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \varepsilon_{ih}^{u} \widehat{v}_{h} \rangle_{\partial \mathcal{T}_{h}} = (\partial_{t} u_{i} - \partial_{t} \Pi_{W}^{j} u_{i}, v_{h})_{\mathcal{T}_{h}}.$$
(6.2b)

for all  $(\boldsymbol{r}_h, v_h, \hat{v}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ .

# 6.2 Estimates for $q_i$

Lemma 12. We have

$$\begin{aligned} \|\sqrt{c_j}\varepsilon_{jh}^{\boldsymbol{q}}\|_{\mathcal{T}_h} + \|\sqrt{\tau}(\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\partial\mathcal{T}_h} \\ &\leq C\|\partial_t u_j - \partial_t \Pi_W^j u_j\|_{\mathcal{T}_h} + C\|\boldsymbol{q}_j - \boldsymbol{\Pi}_V^j \boldsymbol{q}_j\|_{\mathcal{T}_h} + C\|\varepsilon_{jh}^u\|_{\mathcal{T}_h} \end{aligned}$$

*Proof.* We take  $(\mathbf{r}_h, v_h, \hat{v}_h) = (\varepsilon_{jh}^{\mathbf{q}}, \varepsilon_{jh}^u, \varepsilon_{jh}^{\hat{u}})$  in (6.2), and add them together to get

$$\begin{aligned} \|\sqrt{c_j}\varepsilon_{jh}^{\boldsymbol{q}}\|_{\mathcal{T}_h}^2 + \|\sqrt{\tau}(\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\mathcal{T}_h}^2 + (\boldsymbol{\beta}_j \cdot \nabla \varepsilon_{jh}^u, \varepsilon_{jh}^u)_{\mathcal{T}_h} - \langle \boldsymbol{\beta}_j \cdot \boldsymbol{n}, \varepsilon_{jh}^u \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_h} \\ &= (c_j(\boldsymbol{\Pi}_V^j \boldsymbol{q}_j - \boldsymbol{q}_j), \varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_h} + (\partial_t u_j - \partial_t \boldsymbol{\Pi}_W^j u_j, \varepsilon_{jh}^u)_{\mathcal{T}_h}. \end{aligned}$$

By Green's formula and the fact  $\langle (\boldsymbol{\beta}_j \cdot \boldsymbol{n}) \varepsilon_{jh}^{\hat{u}}, \varepsilon_{jh}^{\hat{u}} \rangle_{\partial \mathcal{T}_h} = 0$  we have

$$(\boldsymbol{\beta}_{j} \cdot \nabla \varepsilon_{jh}^{u}, \varepsilon_{jh}^{u})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \varepsilon_{jh}^{u} \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_{h}} \leq \frac{1}{2} \| \sqrt{|\boldsymbol{\beta}_{j} \cdot \boldsymbol{n}} | (\varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}}) \|_{\partial \mathcal{T}_{h}}^{2}.$$
(6.3)

Then by condition (2.2), we get the desired result.

### 6.3 Dual arguments

The next step is the consideration of the dual problems:

$$c_{j} \boldsymbol{\Phi}_{j} + \nabla \Psi_{j} = 0 \qquad \text{in } \Omega,$$
  

$$\nabla \cdot \boldsymbol{\Phi}_{j} - \boldsymbol{\beta}_{j} \cdot \nabla \Psi_{j} = \Theta_{j} \qquad \text{in } \Omega,$$
  

$$\Psi_{j} = 0 \qquad \text{on } \partial\Omega.$$
(6.4)

Elliptic regularity. To obtain the superconvergent rate, we are going to assume that the domain  $\Omega$  is such that for any  $\Theta_j \in L^2(\Omega)$ , we have the regularity estimates for these boundary value problems (6.4):

$$\|\mathbf{\Phi}_{j}\|_{H^{1}(\Omega)} + \|\Psi_{j}\|_{H^{2}(\Omega)} \le C \|\Theta_{j}\|_{L^{2}(\Omega)}.$$
(6.5)

It is well known that this holds whenever  $\Omega$  is a convex polyhedral domain.

Lemma 13. If the elliptic regularity inequality (6.4) holds, then we have the error estimates

$$\begin{aligned} \|\sqrt{c_j}\varepsilon_{jh}^{\boldsymbol{q}}\|_{\mathcal{T}_h} + \|\sqrt{\tau}(\varepsilon_{jh}^u - \varepsilon_{jh}^{\widehat{u}})\|_{\partial\mathcal{T}_h} &\leq C\mathcal{A}_j, \\ \|\varepsilon_{jh}^u\|_{\mathcal{T}_h} &\leq Ch^{k+\min\{k,1\}}\mathcal{A}_j, \end{aligned}$$

where

$$\mathcal{A}_j = \|u_j - \Pi^j_W u_j\|_{\mathcal{T}_h} + \|\boldsymbol{q}_j - \boldsymbol{\Pi}^j_V \boldsymbol{q}_j\|_{\mathcal{T}_h} + \|\partial_t u_j - \Pi^j_W \partial_t u_j\|_{\mathcal{T}_h}.$$

*Proof.* Similar to Lemma 10, we have the following equations:

$$\begin{aligned} (c_{j}\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j},\boldsymbol{r}_{h})_{\mathcal{T}_{h}} - (\boldsymbol{\Pi}_{W}^{j}\boldsymbol{\Psi},\nabla\cdot\boldsymbol{r}_{h})_{\mathcal{T}_{h}} + \langle P_{M}\boldsymbol{\Psi}_{j},\boldsymbol{r}_{h}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} &= (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j}-\boldsymbol{\Phi}_{j}),\boldsymbol{r}_{h})_{\mathcal{T}_{h}}, \\ (\nabla\cdot\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j},v_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j}\cdot\boldsymbol{n},\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} + \langle \tau(\boldsymbol{\Pi}_{W}^{j}\boldsymbol{\Psi}-P_{M}\boldsymbol{\Psi}_{j}),v_{h}-\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} \\ - (\boldsymbol{\beta}_{j}\cdot\nabla\boldsymbol{\Pi}_{W}^{j}\boldsymbol{\Psi}_{j},v_{h})_{\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j}\cdot\boldsymbol{n},\boldsymbol{\Pi}_{W}^{j}\boldsymbol{u}_{j}\widehat{v}_{h}\rangle_{\partial\mathcal{T}_{h}} = (\Theta_{j},v_{h})_{\mathcal{T}_{h}}. \end{aligned}$$

Take  $(\mathbf{r}_h, v_h, \widehat{v}_h) = (\varepsilon_{jh}^{\mathbf{q}}, \varepsilon_{jh}^{u}, \varepsilon_{jh}^{\widehat{u}})$  and  $\Theta_j = \varepsilon_{jh}^{u}$  above to get,

$$\begin{split} \|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}}^{2} &= (\nabla \cdot \mathbf{\Pi}_{V}^{j} \mathbf{\Phi}_{j}, \varepsilon_{jh}^{u})_{\mathcal{T}_{h}} - \langle \mathbf{\Pi}_{V}^{j} \mathbf{\Phi}_{j} \cdot \mathbf{n}, \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_{h}} \\ &+ \langle \tau (\Pi_{W}^{j} \Psi_{j} - P_{M} \Psi_{j}), \varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_{h}} \\ &- (\boldsymbol{\beta}_{j} \cdot \nabla \Pi_{W}^{j} \Psi_{j}, \varepsilon_{jh}^{u})_{\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j} \cdot \mathbf{n}, \Pi_{W}^{j} \Psi_{j} \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial \mathcal{T}_{h}}. \end{split}$$

By (6.2a) one gets

$$\begin{split} \|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}}^{2} &= (c_{j}\varepsilon_{jh}^{q}, \mathbf{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j})_{\mathcal{T}_{h}} - (c_{j}(\mathbf{\Pi}_{V}^{j}\boldsymbol{q}_{j} - \boldsymbol{q}_{j}), \mathbf{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j})_{\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \mathbf{\Pi}_{W}^{j}\boldsymbol{\Psi}_{j}\varepsilon_{jh}^{\widehat{u}}\rangle_{\partial\mathcal{T}_{h}} \\ &+ \langle \tau(\mathbf{\Pi}_{W}^{j}\boldsymbol{\Psi}_{j} - P_{M}\boldsymbol{\Psi}_{j}), \varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}}\rangle_{\partial\mathcal{T}_{h}} - (\boldsymbol{\beta}_{j} \cdot \nabla \mathbf{\Pi}_{W}^{j}\boldsymbol{\Psi}_{j}, \varepsilon_{jh}^{u})_{\mathcal{T}_{h}}. \end{split}$$

Hence,

$$\begin{aligned} \|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}}^{2} &= (\Pi_{W}^{j}\Psi_{j}, \nabla \cdot \varepsilon_{jh}^{q})_{\mathcal{T}_{h}} - \langle P_{M}\Psi_{j}, \varepsilon_{jh}^{q} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_{h}} - (c_{j}(\Pi_{V}^{j}\Phi_{j} - \Phi_{j}), \varepsilon_{jh}^{j})_{\mathcal{T}_{h}} \\ &- (c_{j}(\Pi_{V}^{j}\boldsymbol{q}_{j} - \boldsymbol{q}_{j}), \Pi_{V}^{j}\Phi_{j})_{\mathcal{T}_{h}} + \langle \tau(\Pi_{W}^{j}\Psi_{j} - P_{M}\Psi_{j}), \varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}} \rangle_{\partial\mathcal{T}_{h}} \\ &- (\boldsymbol{\beta}_{j} \cdot \nabla \Pi_{W}^{j}\Psi_{j}, \varepsilon_{jh}^{u})_{\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \Pi_{W}^{j}\Psi_{j}\varepsilon_{jh}^{\widehat{u}} \rangle_{\partial\mathcal{T}_{h}}. \end{aligned}$$

By Green's formula one gets

$$\begin{split} \|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}}^{2} &= (\Pi_{W}^{j}\Psi, \nabla \cdot \varepsilon_{jh}^{q})_{\mathcal{T}_{h}} - \langle P_{M}\Psi_{j}, \varepsilon_{jh}^{q} \cdot \boldsymbol{n} \rangle_{\partial\mathcal{T}_{h}} - (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j} - \boldsymbol{\Phi}_{j}), \varepsilon_{jh}^{q})_{\mathcal{T}_{h}} \\ &- (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j} - \boldsymbol{q}_{j}), \boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j})_{\mathcal{T}_{h}} + \langle \boldsymbol{\tau}(\Pi_{W}^{j}\Psi - P_{M}\Psi), \varepsilon_{jh}^{u} - \varepsilon_{jh}^{\hat{u}}\rangle_{\partial\mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j} \cdot \nabla \varepsilon_{jh}^{u}, \Pi_{W}^{j}\Psi_{j})_{\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \Pi_{W}^{j}\Psi_{j}(\varepsilon_{jh}^{\hat{u}} - \varepsilon_{jh}^{u})\rangle_{\partial\mathcal{T}_{h}} \\ &= (\Pi_{W}^{j}\Psi_{j}, \nabla \cdot \varepsilon_{jh}^{q})_{\mathcal{T}_{h}} - \langle P_{M}\Psi_{j}, \varepsilon_{jh}^{q} \cdot \boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}} - (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j} - \boldsymbol{\Phi}_{j}), \varepsilon_{jh}^{q})_{\mathcal{T}_{h}} \\ &- (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j} - \boldsymbol{q}_{j}), \boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j})_{\mathcal{T}_{h}} + \langle \boldsymbol{\tau}(\Pi_{W}^{j}\Psi_{j} - P_{M}\Psi_{j}), \varepsilon_{jh}^{u} - \varepsilon_{jh}^{\hat{u}}\rangle_{\partial\mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j} \cdot \nabla \varepsilon_{jh}^{u}, \Pi_{W}^{j}\Psi_{j})_{\mathcal{T}_{h}} - \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \varepsilon_{jh}^{u}P_{M}\Psi_{j}\rangle_{\partial\mathcal{T}_{h}} + \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, \varepsilon_{jh}^{u}P_{M}\Psi_{j}\rangle_{\partial\mathcal{T}_{h}} \end{split}$$

By (6.2b) one gets

$$\begin{split} \|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}}^{2} &= -(c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j}-\boldsymbol{\Phi}_{j}),\varepsilon_{jh}^{\boldsymbol{q}})_{\mathcal{T}_{h}} - (c_{j}(\boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}-\boldsymbol{q}_{j}),\boldsymbol{\Pi}_{V}^{j}\boldsymbol{\Phi}_{j})_{\mathcal{T}_{h}} \\ &+ \langle \boldsymbol{\beta}_{j} \cdot \boldsymbol{n}, (\varepsilon_{jh}^{u}-\varepsilon_{jh}^{\widehat{u}})(P_{M}\Psi_{j}-\boldsymbol{\Pi}_{W}^{j}\Psi_{j}) \rangle_{\partial\mathcal{T}_{h}} \\ &+ (\boldsymbol{\beta}_{j} \cdot \nabla\boldsymbol{\Pi}_{W}^{j}\Psi_{j},\boldsymbol{\Pi}_{W}^{j}u_{j}-u_{j})_{\mathcal{T}_{h}} + (\partial_{t}u_{j}-\partial_{t}\boldsymbol{\Pi}_{W}^{j}u_{j},\boldsymbol{\Pi}_{W}^{j}\Psi_{j})_{\mathcal{T}_{h}} \\ &= \sum_{i=1}^{5} R_{i}. \end{split}$$

We estimate  $\{R_i\}_{i=1}^5$  term by term:

$$\begin{split} R_{1} &\leq Ch \| \mathbf{\Phi}_{j} \|_{1} \| \varepsilon_{jh}^{\mathbf{q}} \|_{\mathcal{T}_{h}} \\ &\leq Ch \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}}^{2} + Ch(\|u_{j} - \Pi_{W}^{j}u_{j}\|_{\mathcal{T}_{h}} + \| \mathbf{q}_{j} - \Pi_{V}^{j}\mathbf{q}_{j}\|_{\mathcal{T}_{h}}) \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}}, \\ R_{2} &\leq Ch^{\min\{k,1\}} \| \mathbf{\Phi}_{j} \|_{1} \| \mathbf{q}_{j} - \Pi_{V}^{j}\mathbf{q}_{j} \|_{\mathcal{T}_{h}} \leq Ch^{\min\{k,1\}} \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}} \| \mathbf{q}_{j} - \Pi_{V}^{j}\mathbf{q}_{j}\|_{\mathcal{T}_{h}}, \\ R_{3} &\leq Ch^{\frac{1}{2} + \min\{k,1\}} \| \Psi_{j} \|_{2} \| \sqrt{\tau} (\varepsilon_{jh}^{u} - \varepsilon_{jh}^{\widehat{u}}) \|_{\partial \mathcal{T}_{h}} \\ &\leq Ch^{\frac{1}{2} + \min\{k,1\}} (\| u_{j} - \Pi_{W}^{j}u_{j} \|_{\mathcal{T}_{h}} + \| \mathbf{q}_{j} - \Pi_{V}^{j}\mathbf{q}_{j} \|_{\mathcal{T}_{h}}) \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}} \\ &+ Ch^{\frac{1}{2} + \min\{k,1\}} (\| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}}^{2}, \\ R_{4} &= ((\beta_{j} - \Pi_{0}\beta_{j}) \cdot \nabla \Pi_{W}^{j}\Psi_{j}, \Pi_{W}^{j}u_{j} - u_{j})_{\mathcal{T}_{h}} \\ &\leq Ch \| \Psi_{j} \|_{1} \| u_{j} - \Pi_{W}^{j}u_{j} \|_{\mathcal{T}_{h}} \leq Ch \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}} \| u_{j} - \Pi_{W}^{j}u_{j} \|_{\mathcal{T}_{h}}, \\ R_{5} &\leq Ch^{\min\{k,1\}} \| \Psi_{j} \|_{1} \| \partial_{t}u_{j} - \Pi_{W}^{j}\partial_{t}u_{j} \|_{\mathcal{T}_{h}} \leq Ch^{\min\{k,1\}} \| \varepsilon_{jh}^{u} \|_{\mathcal{T}_{h}} \| \partial_{t}u_{j} - \Pi_{W}^{j}\partial_{t}u_{j} \|_{\mathcal{T}_{h}}. \end{split}$$

Hence, we have

$$\|\varepsilon_{jh}^{u}\|_{\mathcal{T}_{h}} \leq Ch^{\min\{k,1\}} \left( \|u_{j} - \Pi_{W}^{j}u_{j}\|_{\mathcal{T}_{h}} + \|\boldsymbol{q}_{j} - \boldsymbol{\Pi}_{V}^{j}\boldsymbol{q}_{j}\|_{\mathcal{T}_{h}} + \|\partial_{t}u_{j} - \Pi_{W}^{j}\partial_{t}u_{j}\|_{\mathcal{T}_{h}} \right).$$

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