An $O(n \log n)$ -Time Algorithm for the k-Center Problem in Trees*

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Abstract. We consider a classical k-center problem in trees. Let T be a tree of n vertices and every vertex has a nonnegative weight. The problem is to find k centers on the edges of T such that the maximum weighted distance from all vertices to their closest centers is minimized. Megiddo and Tamir (SIAM J. Comput., 1983) gave an algorithm that can solve the problem in $O(n \log^2 n)$ time by using Cole's parametric search. Since then it has been open for over three decades whether the problem can be solved in $O(n \log n)$ time. In this paper, we present an $O(n \log n)$ time algorithm for the problem and thus settle the open problem affirmatively.

1 Introduction

In this paper, we study a classical k-center problem in trees. Let T be a tree of n vertices. Each edge e(u,v) connecting two vertices u and v has a positive length d(u,v), and we consider the edge as a line segment of length d(u,v) so that we can talk about "points" on the edge. For any two points p and q of T, there is a unique path in T from p to q, denoted by $\pi(p,q)$, and by slightly abusing the notation, we use d(p,q) to denote the length of $\pi(p,q)$. Each vertex v of T is associated with a weight $w(v) \geq 0$. The k-center problem is to compute a set Q of k points on T, called centers, such that the maximum weighted distance from all vertices of T to their closest centers is minimized, or formally, the value $\max_{v \in V(T)} \min_{q \in Q} \{w(v) \cdot d(v,q)\}$ is minimized, where V(T) is the vertex set of T. Note that each center can be in the interior of an edge of T.

Kariv and Hakimi [20] first gave an $O(n^2 \log n)$ time algorithm for the problem. Jeger and Kariv [19] proposed an $O(kn \log n)$ time algorithm. Megiddo and Tamir [25] solved the problem in $O(n \log^2 n \log \log n)$ time, and the running time of their algorithm can be reduced to $O(n \log^2 n)$ by applying Cole's parametric search [12]. Some progress has been made very recently by Banik et al. [3] for small values of k, where an $O(n \log n + k \log^2 n \log(n/k))$ -time algorithm and another $O(n \log n + k^2 \log^2(n/k))$ -time algorithm were given.

Since Megiddo and Tamir's work [25], it has been open whether the problem can be solved in $O(n \log n)$ time. In this paper, we settle this three-decade long open problem affirmatively by presenting an $O(n \log n)$ -time algorithm. Note that the previous $O(n \log^2 n)$ -time algorithm [12,25] and the first algorithm in [3] both rely on Cole's parametric search, which involves a large constant in the time complexity due to the AKS sorting network [2]. Our algorithm, however, avoids Cole's parametric search.

If each center is required to be located at a vertex of T, then we call it the *discrete* case. The previously best-known algorithm for this case runs in $O(n \log^2 n)$ time [26]. Our techniques also solve the discrete case in $O(n \log n)$ time.

^{*} A preliminary version of this paper will appear in the Proceedings of the 34th International Symposium on Computational Geometry (SoCG 2018).

1.1 Related Work

Many variations of the k-center problem have been studied. If k = 1, then the problem is solvable in O(n) time [23]. If T is a path, the k-center problem was already solved in $O(n \log n)$ time [9,12,25], and Bhattacharya and Shi [4] also gave an algorithm whose running time is linear in n but exponential in k.

For the unweighted case where the vertices of T have the same weight, an $O(n^2 \log n)$ -time algorithm was given in [8] for the k-center problem. Later, Megiddo et al. [26] solved the problem in $O(n \log^2 n)$ time, and the algorithm was improved to $O(n \log n)$ time [17]. Finally, Frederickson [16] solved the problem in O(n) time. The above four papers also solve the discrete case and the following problem version in the same running times: All points of T are considered as demand points and the centers are required to be at vertices of T. Further, if all points of T are demand points and centers can be any points of T, Megiddo and Tamir solved the problem in $O(n \log^3 n)$ time [25], and the running time can be reduced to $O(n \log^2 n)$ by applying Cole's parametric search [12].

As related problems, Frederickson [15] presented O(n)-time algorithms for the following tree partitioning problems: remove k edges from T such that the maximum (resp., minimum) total weight of all connected subtrees is minimized (resp., maximized).

Finding k centers in a general graph is NP-hard [20]. The geometric version of the problem in the plane is also NP-hard [24], i.e., finding k centers for n demanding points in the plane. Some special cases, however, are solvable in polynomial time. For example, if k = 1, then the problem can be solved in O(n) time [23], and if k = 2, it can be solved in $O(n \log^2 n \log^2 \log n)$ time [7] (also refer to [1] for a faster randomized algorithm). If we require all centers to be on a given line, then the problem of finding k centers can be solved in polynomial time [5,21,28]. Recently, problems on uncertain data have been studied extensively and some k-center problem variations on uncertain data were also considered, e.g., [13,18,30,31,27,29].

1.2 Our Approach

We discuss our approach for the non-discrete problem, and that for the discrete case is similar (and even simpler). Let λ^* be the optimal objective value, i.e., $\lambda^* = \max_{v \in V(T)} \min_{q \in Q} \{w(v) \cdot d(v, q)\}$ for an optimal solution Q. A feasibility test is to determine whether $\lambda \geq \lambda^*$ for a given value λ , and if yes, we call λ a feasible value. Given any λ , the feasibility test can be done in O(n) time [20].

Our algorithm follows an algorithmic scheme in [16] for the unweighted case, which is similar to that in [15] for the tree partition problems. However, a big difference is that three schemes were proposed in [15,16] to gradually solve the problems in O(n) time, while our approach only follows the first scheme and this significantly simplifies the algorithm. One reason the first scheme is sufficient to us is that our algorithm runs in $O(n \log n)$ time, which has a logarithmic factor more than the feasibility test algorithm. In contrast, most efforts of the last two schemes of [15,16] are to reduce the running time of the algorithms to O(n), which is the same as their corresponding feasibility test algorithms.

More specifically, our algorithm consists of two phases. The first phase will gather information so that each feasibility test can be done faster in sub-linear time. By using the faster feasibility test algorithm, the second phase computes the optimal objective value λ^* . As in [16], we also use a stem-partition of the tree T. In addition to a matrix searching algorithm [14], we utilize some other techniques, such as the 2D sublist LP queries [10] and line arrangements searching [11], etc.

Remark. It might be tempting to see whether the techniques of [15,16] can be adapted to solving the problem in O(n) time. Unfortunately, we found several obstacles that prevent us from doing so. For example, a key ingredient of the techniques in [15,16] is to build sorted matrices implicitly in O(n) time so that each matrix element can be obtained in O(1) time. In our problem, however, since the vertices of the tree T have weights, it is elusive how to achieve the same goal. Indeed, this is also one main difficulty to solve the problem even in $O(n \log n)$ time (and thus makes the problem open for such a long time). As will be seen later, our $O(n \log n)$ time algorithm circumvents the difficulty by combining several techniques. Based on our study, although we do not have a proof, we suspect that $\Omega(n \log n)$ is a lower bound of the problem.

The rest of the paper is organized as follows. In Section 2, we review some previous techniques that will be used later. In Section 3, we describe our techniques for dealing with a so-called "stem". We finally solve the k-center problem on T in Section 4. By slightly modifying the techniques, we solve the discrete case in Section 5.

2 Preliminaries

In this section we review some techniques that will be used later in our algorithm.

2.1 The Feasibility Test FTEST0

Given any value λ , the feasibility test is to determine whether λ is feasible, i.e., whether $\lambda \geq \lambda^*$. We say that a vertex v of T is covered (under λ) by a center q if $w(v) \cdot d(v, q) \leq \lambda$. Note that λ is feasible if and only if we can place k centers in T such that all vertices are covered. In the following we describe a linear-time feasibility test algorithm, which is essentially the same as the one in [20] although our description is much simpler.

We pick a vertex of T as the root, denoted by γ . For each vertex v, we use T(v) to denote the subtree of T rooted at v. Following a post-order traversal on T, we place centers in a bottom-up and greedy manner. For each vertex v, we maintain two values sup(v) and dem(v), where sup(v) is the distance from v to the closest center that has been placed in T(v), and dem(v) is the maximum distance from v such that if we place a center q within such a distance from v then all uncovered vertices of T(v) can be covered by q. We also maintain a variable count to record the number of centers that have been placed so far. Refer to Algorithm 1 for the pseudocode.

Initially, count = 0, and for each vertex v, $sup(v) = \infty$ and $dem(v) = \frac{\lambda}{w(v)}$. Following a post-order traversal on T, suppose vertex v is being visited. For each child u of v, we update sup(v) and dem(v) as follows. If $sup(u) \leq dem(u)$, then we can use the center of T(u) closest to u to cover the uncovered vertices of T(u), and thus we reset $sup(v) = \min\{sup(v), sup(u) + d(u, v)\}$. Note that since u connects v by an edge, d(v, u) is the length of the edge. Otherwise, if dem(u) < d(u, v), then we place a center on the edge e(u, v) at distance dem(u) from u, so we update count = count + 1 and $sup(v) = \min\{sup(v), d(u, v) - dem(u)\}$. Otherwise (i.e., $dem(u) \geq d(u, v)$), we update $dem(v) = \min\{dem(v), dem(u) - d(u, v)\}$.

After the root γ is visited, if $sup(\gamma) > dem(\gamma)$, then we place a center at γ and update count = count + 1. Finally, λ is feasible if and only if $count \leq k$. The algorithm runs in O(n) time. We use FTEST0 to refer to the algorithm.

Remark. The algorithm FTEST0 actually partitions T into at most k disjoint connected subtrees such that the vertices in each subtree is covered by the same center that is located in the subtree. We will make use of this observation later.

To solve the k-center problem, the key is to compute λ^* , after which we can find k centers by applying FTEST0 with $\lambda = \lambda^*$.

Algorithm 1: The feasibility test algorithm *FTEST0*

```
Input: The tree T with root \gamma and a value \lambda
   Output: Determine whether \lambda is feasible
 1 count \leftarrow 0;
2 for each vertex v do 3 \int sup(v) \leftarrow \infty, dem(v) \leftarrow \frac{\lambda}{w(v)};
   for each vertex v in the post-order traversal of T do
        for each child u of v do
             if sup(u) \leq dem(u) then
 6
                 sup(v) = \min\{sup(v), sup(u) + d(u, v)\};
 7
             else
8
                 if dem(u) < d(u, v) then
 9
                                          /* place a center on the edge e(u,v) at distance dem(u) from u */
10
                      sup(v) = \min\{sup(v), d(u, v) - dem(u)\};
11
12
                      dem(v) = \min\{dem(v), dem(u) - d(u, v)\};
14 if sup(\gamma) > dem(\gamma) then
    count + +;
                                                                                   /* place a center at the root \gamma */
16 Return true if and only if count < k;
```

2.2 A Matrix Searching Algorithm

We review an algorithm MSEARCH, which was proposed in [14] and was widely used, e.g., [15,16,17]. A matrix is sorted if elements in every row and every column are in nonincreasing order. Given a set of sorted matrices, a searching range (λ_1, λ_2) such that λ_2 is feasible and λ_1 is not, and a stopping count c, MSEARCH will produce a sequence of values one at a time for feasibility tests, and after each test, some elements in the matrices will be discarded. Suppose a value λ is produced. If $\lambda \notin (\lambda_1, \lambda_2)$, we do not need to test λ . If λ is feasible, then λ_2 is updated to λ ; otherwise, λ_1 is updated to λ . MSEARCH will stop once the number of remaining elements in all matrices is at most c. Lemma 1 is proved in [14] and we slightly change the statement to accommodate our need.

Lemma 1. [14,15,16,17] Let \mathcal{M} be a set of N sorted matrices $\{M_1, M_2, \ldots, M_N\}$ such that M_j is of dimension $m_j \times n_j$ with $m_j \leq n_j$, and $\sum_{j=1}^N m_j = m$. Let $c \geq 0$. The number of feasibility tests needed by MSEARCH to discard all but at most c of the elements is $O(\max\{\log\max_j\{n_j\},\log(\frac{m}{c+1})\})$, and the total time of MSEARCH exclusive of feasibility tests is $O(\kappa \cdot \sum_{j=1}^N m_j \log(\frac{2n_j}{m_j}))$, where $O(\kappa)$ is the time for evaluating each matrix element (i.e., the number of matrix elements that need to be evaluated is $O(\sum_{j=1}^N m_j \log(\frac{2n_j}{m_j}))$.

2.3 The 2D Sublist LP Queries

Let $H = \{h_1, h_2, \ldots, h_m\}$ be a set of m upper half-planes in the plane. Given two indices i and j with $1 \le i \le j \le m$, a 2D sublist LP query asks for the lowest point in the common intersection of $h_i, h_{i+1}, \ldots, h_j$. The line-constrained version of the query is: Given a vertical line l and two indices i and j with $1 \le i \le j \le m$, the query asks for the lowest point on l in the common intersection of $h_i, h_{i+1}, \ldots, h_j$. Lemma 2 was proved in [10] (i.e., Lemma 8 and the discussion after it; the query algorithm for the line-constrained version is used as a procedure in the proof of Lemma 8).

Lemma 2. [10] We can build a data structure for H in $O(m \log m)$ time such that each 2D sublist LP query can be answered in $O(\log^2 m)$ time, and each line-constrained query can be answered in $O(\log m)$ time.

Remark. With $O(m \log m)$ preprocessing time, any poly($\log m$)-time algorithms for both query problems would be sufficient for our purpose, where poly(\cdot) is any polynomial function.

2.4 Line Arrangement Searching

Let L be a set of m lines in the plane. Denote by $\mathcal{A}(L)$ the arrangement of the lines of L, and let y(v) denote the y-coordinate of each vertex v of $\mathcal{A}(L)$. Let $v_1(L)$ be the lowest vertex of $\mathcal{A}(L)$ whose y-coordinate is a feasible value, and let $v_2(L)$ be the highest vertex of $\mathcal{A}(L)$ whose y-coordinate is smaller than that of $v_1(L)$. By their definitions, $y(v_2(L)) < \lambda^* \le y(v_1(L))$, and $\mathcal{A}(L)$ does not have a vertex v with $y(v_2(L)) < y(v) < y(v_1(L))$. Lemma 3 was proved in [11].

Lemma 3. [11] Both vertices $v_1(L)$ and $v_2(L)$ can be computed in $O((m+\tau)\log m)$ time, where τ is the time for a feasibility test.

Remark. Alternatively, we can use Cole's parametric search [12] to compute the two vertices. First, we sort the lines of L by their intersections with the horizontal line $y = \lambda^*$, and this can be done in $O((m + \tau) \log m)$ time by Cole's parametric search [12]. Then, $v_1(L)$ and $v_2(L)$ can be found in additional O(m) time because each of them is an intersection of two adjacent lines in the above sorted order. The line arrangement searching technique, which modified the slope selection algorithms [6,22], avoids Cole's parametric search [12].

In the following, we often talk about some problems in the plane \mathbb{R}^2 , and if the context is clear, for any point $p \in \mathbb{R}^2$, we use x(p) and y(p) to denote its x- and y-coordinates, respectively.

3 The Algorithms for Stems

In this section, we first define *stems*, which are similar in spirit to those proposed in [16] for the unweighted case. Then, we will present two algorithms for solving the k-center problem on a stem, and both techniques will be used later for solving the problem in the tree T.

Let \widehat{P} be a path of m vertices, denoted by v_1, v_2, \ldots, v_m , sorted from left to right. For each vertex v_i , other than its incident edges in \widehat{P} , v_i has at most two additional edges connecting two vertices u_i and w_i that are not in \widehat{P} . Either vertex may not exist. Let P denote the union of \widehat{P} and the above additional edges (e.g., see Fig. 1). For any two points p and q on P, we still use $\pi(p,q)$ to denote the unique path between p and q in P, and use d(p,q) to denote the length of the path.

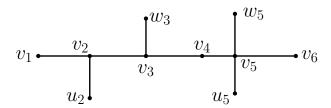


Fig. 1. Illustrating a stem.

With respect to a range (λ_1, λ_2) , we call P a *stem* if the following holds: For each $i \in [1, m]$, if u_i exists, then $w(u_i) \cdot d(u_i, v_i) \leq \lambda_1$; if w_i exists, then $w(w_i) \cdot d(w_i, v_i) \geq \lambda_2$.

Following the terminology in [16], we call $e(v_i, u_i)$ a thorn and $e(v_i, w_i)$ a twig. Each u_i is called a thorn vertex and each w_i is called a twig vertex. \widehat{P} is called the backbone of P, and each vertex of \widehat{P} is called a backbone vertex. We define m as the length of P. The total number of vertices of P is at most 3m.

Remark. Our algorithm in Section 4 will produce stems P as defined above, where \widehat{P} is a path in T and all vertices of P are also vertices of T. However, each thorn $e(u_i, v_i)$ may not be an original edge of T, but it corresponds to the path between u_i and v_i in T in the sense that the length of $e(u_i, v_i)$ is equal to the distance between u_i and v_i in T. This is also the case for each twig $e(w_i, v_i)$. Our algorithm in Section 4 will maintain a range (λ_1, λ_2) such that λ_1 is not feasible and λ_2 is feasible, i.e., $\lambda^* \in (\lambda_1, \lambda_2]$. Since any feasibility test will be made to a value $\lambda \in (\lambda_1, \lambda_2)$, the above definitions on thorns and twigs imply the following: For each thorn vertex u_i , we can place a center on the backbone to cover it (under λ), and for each twig vertex w_i , we need to place a center on the edge $e(w_i, v_i) \setminus \{v_i\}$ to cover it.

In the sequel we give two different techniques for solving the k-center problem on the stem P. In fact, in our algorithm for the k-center problem on T in Section 4, we use these techniques to process a stem P, rather than directly solve the k-center problem on P. Let λ^* temporarily refer to the optimal objective value of the k-center problem on P in the rest of this section, and we assume $\lambda^* \in (\lambda_1, \lambda_2]$.

3.1 The First Algorithm

This algorithm is motivated by the following easy observation: there exist two vertices v and v' in P such that a center q is located in the path $\pi(v, v')$ and $w(v) \cdot d(q, v) = w(v') \cdot d(q, v') = \lambda^*$ (since otherwise we could move q on P to achieve such a situation).

We assume that all backbone vertices of P are in the x-axis of an xy-coordinate system \mathbb{R}^2 where v_1 is at the origin and each v_i has x-coordinate $d(v_1, v_i)$. Each v_i defines two lines $l^+(v_i)$ and $l^-(v_i)$ both containing v_i and with slopes $w(v_i)$ and $-w(v_i)$, respectively (e.g., see Fig. 2). Each thorn u_i also defines two lines $l^+(u_i)$ and $l^-(u_i)$ as follows. Define u_i^l (resp., u_i^r) to be the point in \mathbb{R}^2 on the x-axis with x-coordinate $d(v_1, v_i) - d(u_i, v_i)$ (resp., $d(v_1, v_i) + d(u_i, v_i)$). Hence, u_i^l (resp., u_i^r) is to the left (resp., right) of v_i with distance $d(u_i, v_i)$ from v_i . Define $l^+(u_i)$ to be the line through u_i^l with slope $w(u_i)$ and $l^-(u_i)$ to be the line through u_i^r with slope $-w(u_i)$. Note that $l^+(u_i)$ and $l^-(u_i)$ intersect at the point whose x-coordinate is the same as that of v_i and whose y-coordinate is equal to $w(u_i) \cdot d(u_i, v_i)$. For each twig vertex w_i , we define points w_i^l and w_i^r , and lines $l^+(w_i)$ and $l^-(w_i)$, in the same way as those for u_i .

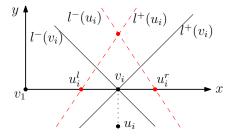


Fig. 2. Illustrating the definitions of the lines defined by a backbone vertex v_i and its thorn vertex u_i .

Consider a point q on the backbone of P to the right side of v_i . It can be verified that the weighted distance $w(v_i) \cdot d(v_i, q)$ from v_i to q is exactly equal to the y-coordinate of the intersection between $l^+(v_i)$ and the vertical line through q. If q is on the left side of v_i , we have a similar observation for $l^-(v_i)$. This is also true for u_i and w_i .

Let L denote the set of the lines in \mathbb{R}^2 defined by all vertices of P. Note that $|L| \leq 6m$. Based on the above observation, λ^* is equal to the y-coordinate of a vertex of the line arrangement $\mathcal{A}(L)$ of L. More precisely, λ^* is equal to the y-coordinate of the vertex $v_1(L)$, as defined in Section 2. By Lemma 3, we can compute λ^* in $O((m+\tau)\log m)$ time.

3.2 The Second Algorithm

This algorithm relies on the algorithm MSEARCH. We first form a set of sorted matrices.

For each $i \in [1, m]$, we define the two lines $l_i^+(v_i)$ and $l_i^-(v_i)$ in \mathbb{R}^2 as above in Section 3.1. If u_i exists, then we also define $l_i^+(u_i)$ and $l_i^-(u_i)$ as before; otherwise, both $l_i^+(u_i)$ and $l_i^-(u_i)$ refer to the x-axis. Let $h_{4(i-1)+j}$, $1 \le j \le 4$, denote respectively the four upper half-planes bounded by the above four lines (their index order is arbitrary). In this way, we have a set $H = \{h_1, h_2, \dots, h_{4m}\}$ of 4m upper half-planes.

For any i and j with $1 \le i \le j \le m$, we define $\alpha(i,j)$ as the y-coordinate of the lowest point in the common intersection of the upper half-planes of H from $h_{4(i-1)+1}$ to h_{4j} , i.e., all upper half-planes defined by u_t and v_t for $t \in [i,j]$. Observe that if we use one center to cover all backbone and thorn vertices u_t and v_t for $t \in [i,j]$, then $\alpha(i,j)$ is equal to the optimal objective value of this one-center problem.

We define a matrix M of dimension $m \times m$: For any i and j in [1, m], if $i + j \leq m + 1$, then $M[i, j] = \alpha[i, m + 1 - j]$; otherwise, M[i, j] = 0.

For each twig w_i , we define two arrays A_i^r and A_i^l of at most m elements each as follows. Let $h^+(w_i)$ and $h^-(w_i)$ denote respectively the upper half-planes bounded by the lines $l^+(w_i)$ and $l^-(w_i)$ defined in Section 3.1. The array A_i^r is defined on the vertices of P on the right side of v_i , as follows. For each $j \in [1, m-i+1]$, if we use a single center to cover w_i and all vertices u_t and v_t for $t \in [i, m+1-j]$, then $A_i^r[j]$ is defined to be the optimal objective value of this one-center problem, which is equal to the y-coordinate of the lowest point in the common intersection of $h^+(w_i)$ and the upper half-planes of H from $h_{4(i-1)+1}$ to $h_{4(m+1-j)}$. Symmetrically, array A_i^l is defined on the left side of v_i . Specifically, for each $j \in [1, i]$, if we use one center to cover w_i and all vertices u_t and v_t for $t \in [j, i]$, then $A^l[j]$ is defined to be the optimal objective value, which is equal to the y-coordinate of the lowest point in the common intersection of $h^-(w_i)$ and the upper half-planes of H from $h_{4(j-1)+1}$ to h_{4i} .

Let \mathcal{M} be the set of the matrices M and A_i^r and A_i^l for all $1 \leq i \leq m$. The following lemma implies that we can apply MSEARCH on \mathcal{M} to compute λ^* .

Lemma 4. Each matrix of \mathcal{M} is sorted, and λ^* is an element of a matrix in \mathcal{M} .

Proof. We first show that all matrices of \mathcal{M} are sorted.

Consider the matrix M. Consider two elements $M[i, j_1]$ and $M[i, j_2]$ in the same row with $j_1 < j_2$. Our goal is to show that $M[i, j_1] \ge M[i, j_2]$.

- 1. If $j_1 > m+1-i$, then both $M[i,j_1]$ and $M[i,j_2]$ are zero. Thus, $M[i,j_1] \geq M[i,j_2]$ trivially holds.
- 2. If $j_1 \le m+1-i < j_2$, then $M[i,j_2] = 0$ and $M[i,j_1] = \alpha(i,m+1-j_1)$. By our way of defining upper half-planes of H, one can verify that $\alpha(i,m+1-j_1) \ge 0$. Therefore, $M[i,j_1] \ge M[i,j_2]$.
- 3. If $j_2 \leq m+1-i$, then $M[i,j_1] = \alpha(i,m+1-j_1)$ and $M[i,j_2] = \alpha(i,m+1-j_2)$. Let H_1 (resp., H_2) be the set of the upper half-planes of H from $h_{4(i-1)+1}$ to $h_{4(m+1-j_1)}$ (resp., $h_{4(m+1-j_2)}$). Since $j_1 < j_2$, H_2 is a subset of H_1 , and thus the lowest point in the common intersection of the upper half-planes of H_2 is not higher than that of H_1 . Hence, $\alpha(i,m+1-j_1) \geq \alpha(i,m+1-j_2)$ and thus $M[i,j_1] \geq M[i,j_2]$.

The above proves $M[i, j_1] \ge M[i, j_2]$. Therefore, all elements in each row are sorted in non-increasing order. By the similar approach we can show that all elements in each column are also sorted in nonincreasing order. We omit the details. Hence, M is a sorted matrix.

Now consider an array A_i^l . Consider any two elements $A_i^l[j_1]$ and $A_i^l[j_2]$ with $j_1 < j_2$. Our goal is to show that $A_i^l[j_1] \ge A_i^l[j_2]$. The argument is similar as the above third case. Let H_1 (resp., H_2) be the set of $h^-(w_i)$ and the upper half-planes of H from $h_{4(j_1-1)+1}$ (resp., $h_{4(j_2-1)+1}$) to h_{4i} . Since $j_1 < j_2$, H_2 is a subset of H_1 and the lowest point in the common intersection of the upper half-planes of H_2 is not higher than that of H_1 . Hence, $A_i^l[j_1] \ge A_i^l[j_2]$.

We can show that A_i^r is also sorted in a similar way. We omit the details.

The above proves that every matrix of \mathcal{M} is sorted. In the following, we show that λ^* must be an element of one of these matrices.

Imagine that we apply our feasibility test algorithm FTEST0 on $\lambda = \lambda^*$ and the stem P by considering P as a tree with root v_m . Then, the algorithm will compute at most k centers in P. The algorithm actually partitions P into at most k disjoint connected subtrees such that the vertices in each subtree is covered by the same center that is located in the subtree. Further, there must be a subtree P_1 that has a center q and two vertices v' and v such that $w(v) \cdot d(v, q) = w(v') \cdot d(v', q) = \lambda^*$, since otherwise we could adjust the positions of the centers so that the maximum weighted distance from all vertices of P to their closest centers would be strictly smaller than λ^* . Since P_1 is connected and both v and v' are in P_1 , the path $\pi(v, v')$ is also in P_1 .

Depending on whether one of v and v' is a twig vertex, there are two cases.

If neither vertex is a twig vertex, then we claim that all thorn vertices connecting to the backbone vertices of $\pi(v, v')$ are covered by the center q. Indeed, suppose v_i is a backbone vertex in $\pi(v, v')$ and v_i connects to a thorn vertex u_i . Assume to the contrary that u_i is not covered by q. Recall that by the definition of thorns, $w(u_i) \cdot d(u_i, v_i) \leq \lambda_1$, and since $\lambda_1 < \lambda^*$, we have $w(u_i) \cdot d(u_i, v_i) < \lambda^*$. According to FTEST0, u_i is covered by a center q' that is not on $e(u_i, v_i)$. Hence, u_i and q' is in a connected subtree, denoted by P_2 , in the partition of P induced by FTEST0. Clearly, v_i is in $\pi(u_i, q')$. Since P_2 is connected and both q' and u_i are in P_2 , every vertex of $\pi(u_i, q')$ is in P_2 .

Because q' is not on $e(u_i, v_i)$, v_i must be in $\pi(u_i, q')$ and thus is in P_2 . However, since v_i is in $\pi(v, v')$, v_i is also in P_1 . This incurs contradiction since $P_1 \cap P_2 = \emptyset$. This proves the claim.

If v is a backbone vertex, then let i be its index, i.e., $v = v_i$; otherwise, v is a thorn vertex and let i be the index such that v connects the backbone vertex v_i . Similarly, define j for v'. Without loss of generality, assume $i \leq j$. The above claim implies that λ^* is equal to the y-coordinate of the lowest point in the common intersection of all upper half-planes defined by the backbone vertices v_t and thorn vertices u_t for all $t \in [i, j]$, and thus, $\lambda^* = \alpha(i, j)$, which is equal to M[i, m + 1 - j]. Therefore, λ^* is in the matrix M.

Next, we consider the case where at least one of v and v' is a twig vertex. For each twig vertex w_i of P, by definition, $w(w_i) \cdot d(w_i, v_i) \ge \lambda_2$, and since $\lambda^* \le \lambda_2$, the twig $e(w_i, v_i)$ must contain a center. Because both v and v' are covered by q, only one of them is a twig vertex (since otherwise we would need two centers to cover them since each twig must contain a center). Without loss of generality, we assume that v is a twig vertex, say, w_i . If v' is a backbone vertex, then let j be its index; otherwise, v' is a thorn vertex and let j be the index such that v' connects the backbone vertex v_j . Without loss of generality, we assume that $i \le j$.

By the same argument as the above, all thorn vertices u_t with $t \in [i, j]$ are covered by q. This implies that λ^* is the y-coordinate of the lowest point in the common intersection of $h^+(w_i)$ and all upper half-planes defined by the backbone vertices v_t and thorn vertices u_t for all $t \in [i, j]$. Thus, $\lambda^* = A_i^r[m+1-j]$. Therefore, λ^* is in the array A_i^r .

This proves that λ^* must be in a matrix of \mathcal{M} . The lemma thus follows.

Note that \mathcal{M} consists of a matrix M of dimension $m \times m$ and 2m arrays of lengths at most m. With the help of the 2D sublist LP query data structure in Lemma 2, the following lemma shows that the matrices of \mathcal{M} can be implicitly formed in $O(m \log m)$ time.

Lemma 5. With $O(m \log m)$ time preprocessing, each matrix element of \mathcal{M} can be evaluated in $O(\log^2 m)$ time.

Proof. We build a 2D sublist LP query data structure of Lemma 2 on the upper half-planes of H in $O(m \log m)$ time. Then, each element of M can be computed in $O(\log^2 m)$ time by a 2D sublist LP query.

Now consider an array A_i^l . Given any index j, to compute $A_i^l[j]$, recall that $A_i^l[j]$ is equal to the y-coordinate of the lowest point p^* of the common intersection of the upper half-plane $h^+(w_i)$ and those in H', where H' is the set of the upper half-planes of H from $h_{4(j-1)+1}$ to h_{4i} . The lowest point p' of the common intersection of the upper half-planes of H' can be computed in $O(\log^2 m)$ time by a 2D sublist LP query with query indices 4(j-1)+1 and 4i. Computing p^* can also be done in $O(\log^2 m)$ time by slightly modifying the query algorithm for computing p'. We briefly discuss it below and the interested reader should refer to [10] for details (the proof of Lemma 8 and the discussion after the lemma).

The query algorithm for computing p' is similar in spirit to the linear-time algorithm for the 2D linear programming problem in [23]. It is a binary search algorithm. In each iteration, the algorithm computes the highest intersection p'' between a vertical line l and the bounding lines of the half-planes of H', and based on the local information at the intersection, the algorithm will determine which side to proceed for the search. For computing p^* , we need to incorporate the additional half-plane $h^+(w_i)$. To this end, in each iteration of the binary search, after we compute the highest intersection p'', we compare it with the intersection of l and the bounding line of $h^+(w_i)$ and update

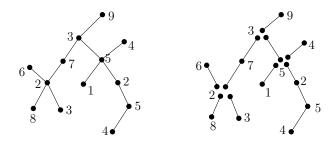


Fig. 3. Left: the tree T where the numbers are the weights of the vertices. Right: the path partition of T.

the highest intersection if needed. This costs only constant extra time for each iteration. Therefore, the total running time for computing p^* is still $O(\log^2 m)$.

Computing the elements of arrays A_i^r can be done similarly. The lemma thus follows.

By applying algorithm MSEARCH on \mathcal{M} with stopping count c=0 and $\kappa=O(\log^2 m)$, according to Lemma 1, MSEARCH produces $O(\log m)$ values for feasibility tests, and the total time exclusive of feasibility tests is $O(m\log^3 m)$ because we need to evaluate $O(m\log m)$ matrix elements of \mathcal{M} . Hence, the total time for computing λ^* is $O(m\log^3 m + \tau \cdot \log m)$.

Remark. Clearly, the first algorithm is better than the second one. However, later when we use the techniques of the second algorithm, m is often bounded by $O(\log^2 n)$ and thus $\log^3 m = O(\log n)$. In fact, we use the techniques of the second algorithm mainly because we need to set the stopping count c to some non-zero value.

4 Solving the k-Center Problem on T

In this section, we present our algorithm for solving the k-center problem on T. We will focus on computing the optimal objective value λ^* .

Frederickson [15] proposed a path-partition of T, which is a partition of the edges of T into paths where a vertex v is an endpoint of a path if and only if the degree of v in T is not equal to 2 (e.g., see Fig. 3). A path in a partition-partition of T is called a *leaf-path* if it contains a leaf of T.

As in [16], we generalize the path-partition to stem-partition as follows. During the course of our algorithm, a range $(\lambda_1, \lambda_2]$ that contains λ^* will be maintained and T will be modified by removing some edges and adding some thorns and twigs. At any point in our algorithm, let T' be T with all thorns and twigs removed. A *stem* of T is a path in the path-partition of T', along with all thorns and twigs that connect to vertices in the path. A *stem-partition* of T is to partition T into stems according to a path-partition of T'. A stem in a stem-partition of T is called a *leaf-stem* if it contains a leaf of T that is a backbone vertex of the stem.

Our algorithm follows the first algorithmic scheme in [16]. There are two main phases: Phase 1 and Phase 2. Let $r = \log^2 n$. Phase 1 gathers information so that the feasibility test can be made in sublinear $O(\frac{n}{r}\log^3 r)$ time. Phase 2 computes λ^* by using the faster feasibility test. If T has more than 2n/r leaves, then there is an additional phase, called Phase 0, which reduces the problem to a tree with at most 2n/r leaves. (Phase 0 is part of Phase 1 in [16], and we separates it from Phase 1 to make it clearer.) In the following, we consider the general case where T has more than 2n/r leaves. Algorithm 3 gives the pseudocode of the overall algorithm.

4.1 The Preprocessing and Computing the Vertex Ranks

We first perform some preprocessing. Recall that γ is the root of T. We compute the distances $d(v,\gamma)$ for all vertices v in O(n) time. Then, if u is an ancestor of v, $d(u,v) = d(\gamma,v) - d(\gamma,u)$, which can be computed in O(1) time. In the following, whenever we need to compute a distance d(u,v), it is always the case that one of u and v is an ancestor of the other, and thus d(u,v) can be obtained in O(1) time.

Next, we compute a "rank" rank(v) for each vertex v of T. These ranks will facilitate our algorithm later. For each vertex v, we define a point p(v) on the x-axis with x-coordinate equal to $d(\gamma, v)$ in an xy-coordinate system \mathbb{R}^2 , and define l(v) as the line through p(v) with slope equal to -w(v). Let L be the set of these n lines. Consider the line arrangement $\mathcal{A}(L)$ of L. Let $v_1(L)$ and $v_2(L)$ be the vertices as defined in Section 2. By Lemma 3, both vertices can be computed in $O(n \log n)$ time. Let l be a horizontal line strictly between $v_1(L)$ and $v_2(L)$. We sort all lines of L by their intersections with l from left to right, and for each vertex v, we define rank(v) = i if there are i-1 lines before l(v) in the above order. By the definitions of $v_1(L)$ and $v_2(L)$, the above order of L is also an order of L sorted by their intersections with the horizontal line $y = \lambda^*$.

4.2 Phase 0

Recall that T has more than 2n/r leaves. In this section, we reduce the problem to the problem of placing centers in a tree with at most 2n/r leaves. Our algorithm will maintain a range $(\lambda_1, \lambda_2]$ that contains λ^* . Initially, $\lambda_1 = y(v_2(L))$, the y-coordinate of $v_2(L)$, which is already computed in the preprocessing, and $\lambda_2 = y(v_1(L))$. We form a stem-partition of T, which is actually a path-partition since there are no thorns and twigs initially, and this can be done in O(n) time.

Recall that $r = \log^2 n$. While there are more than 2n/r leaves in T, we do the following.

Recall that the length of a stem is defined as the number of backbone vertices. Let S be the set of all leaf-stems of T whose lengths are at most r. Let n' be the number of all backbone vertices on the leaf-stems of S. For each leaf-stem of S, we form matrices by Lemma 5. Let \mathcal{M} denote the collection of matrices for all leaf-stems of S. We call MSEARCH on \mathcal{M} , with stopping count c = n'/(2r), by using the feasibility test algorithm FTESTO. After MSEARCH stops, we have an updated range (λ_1, λ_2) and matrix elements of \mathcal{M} in (λ_1, λ_2) are called active values. Since c = n'/(2r), at most n'/(2r) active values of \mathcal{M} remain, and thus at most n'/(2r) leaf-stems of S have active values.

For each leaf-stem $P \in S$ without active values, we perform the following post-processing procedure. The backbone vertex of P closest to the root is called the top vertex. We place centers on P, subtract their number from k, and replace P by either a thorn or a twig connected to the top vertex (P is thus removed from T except the top vertex), such that solving the k-center problem on the modified T also solves the problem on the original T. The post-processing procedure can be implemented in O(m) time, where m is the length of P. The details are given below.

The post-processing procedure on P. Let z be the top vertex of P. We run the feasibility test algorithm FTEST0 on P with z as the root and $\lambda = \lambda'$ that is an arbitrary value in (λ_1, λ_2) . After z is finally processed, depending on whether $sup(z) \leq dem(z)$, we do the following.

If $sup(z) \leq dem(z)$, then let q be the last center that has been placed. In this case, all vertices of P are covered and z is covered by q. According to algorithm FTEST0 and as discussed in the proof of Lemma 4, q covers a connected subtree of vertices, and let V(q) denote the set of these vertices excluding z. Note that V(q) can be easily identified during FTEST0. Let k' be the number

of centers excluding q that have been placed on P. Since $\lambda' \in (\lambda_1, \lambda_2)$ and the matrices formed based on P do not have any active values, we have the following key observation: if we run FTESTO with any $\lambda \in (\lambda_1, \lambda_2)$, the algorithm will also cover all vertices of $P \setminus (V(q) \cup \{z\})$ with k' centers and cover vertices of $V(q) \cup \{z\}$ with one center. Indeed, this is true because the way we form matrices for P is consistent with FTESTO, as discussed in the proof of Lemma 4. In this case, we replace P by attaching a twig e(u, z) to z with length equal to d(u, z), where u is a vertex of V(q) with the following property: For any $\lambda \in (\lambda_1, \lambda_2)$, if we place a center q' on the path $\pi(u, z)$ at distance $\lambda/w(u)$ from u, then q' will cover all vertices of V(q) under λ , i.e., u "dominates" all other vertices of V(q) and thus it is sufficient to keep u (since λ_2 is feasible, any subsequent feasibility test in the algorithm will use $\lambda \in (\lambda_1, \lambda_2)$). The following lemma shows that u is the vertex of V(q) with the largest rank.

Lemma 6. Let u be the vertex of V(q) with the largest rank. For any $\lambda \in (\lambda_1, \lambda_2)$, the following holds.

- 1. $\frac{\lambda}{w(u)} \le d(u, z)$.
- 2. If q' is the point on the path $\pi(u,z)$ with distance $\frac{\lambda}{w(u)}$ from u, then q' covers all vertices of V(q) under λ , i.e., $w(v) \cdot d(v,q') \leq \lambda$ for all $v \in V(q)$.

Proof. Before we prove $\frac{\lambda}{w(u)} \leq d(u, z)$, let q' be the point on $\pi(u, \gamma)$ with distance $\frac{\lambda}{w(u)}$ from u (if $\frac{\lambda}{w(u)} > d(u, \gamma)$, then we add a dummy edge e^* extended from the root γ long enough so that q' is on e^*). Later we will show that $\frac{\lambda}{w(u)} \leq d(u, z)$, which also proves that q' is on $\pi(u, z)$.

We first show that q' covers all vertices of V(q). Consider any vertex $v \in V(q)$. Our goal is to prove $w(v) \cdot d(v, q') \leq \lambda$. If v = u, this trivially holds. In the following, we assume $v \neq u$.

Note that q' may be on a twig of P. If q' is on a twig e(u,v) of P, then this means that q' is on P and $\frac{\lambda}{w(u)} \leq d(u,z)$ holds. In this case, if we run FTEST0 with λ , then q' will be a center placed by FTEST0 to cover u. On the other hand, according to the above key observation, FTEST0 with λ will use one center to cover all vertices of V(q). Hence, q' covers all vertices of V(q) and thus covers v. In the following, we assume that q' is not on a twig of P. Note that by the definition of thorns, q' cannot be in any thorn. Thus q' must be either on the backbone of P or outside P in $\pi(z,\gamma) \cup e^*$. We define $V_1(q)$ to be the set of vertices of V(q) in the subtree rooted at q' and let $V_2(q) = V(q) \setminus V_1(q)$. Depending on whether v is in $V_1(q)$ or $V_2(q)$, there are two cases.

The case $v \in V_1(q)$. Recall that in Section 4.1 each vertex v' defines a line l(v') in \mathbb{R}^2 . We consider the two lines l(v) and l(u). Let p_v and p_u denote the intersections of the horizontal line $y = \lambda$ with l(v) and l(u), respectively (e.g., see Fig. 4). Note that the point q' corresponding to the point $(x(p_u), 0)$ in the x-axis in the sense that $d(\gamma, q') = x(p_u)$. Since rank(v) < rank(u) and $\lambda \in (\lambda_1, \lambda_2)$, by the definition of ranks, it holds that $x(p_v) < x(p_u)$. Because the slope of l(v) is not positive, $y(p'_v) \leq \lambda$, where p'_v is the intersection of l(v) with the vertical line through p_u . On the other hand, since v is in $V_1(q)$, $y(p'_v)$ is exactly equal to $w(v) \cdot d(v, q')$. Therefore, we obtain $w(v) \cdot d(v, q') \leq \lambda$.

The case $v \in V_2(q)$. In this case, $V_2(q) \neq \emptyset$. By the definition of $V_2(q)$, q' must be in $\pi(u, v)$. According to the above key observation, we can use one center to cover all vertices of V(q) (under λ), and in particular, we can use one center to cover both u and v. By the definition of q', q' is the closest point to v on $\pi(u, v)$ that can cover u. Hence, q' must be able to cover v. Therefore, we obtain $w(v) \cdot d(v, q') \leq \lambda$.

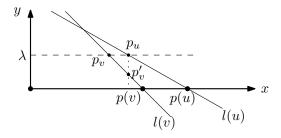


Fig. 4. Illustrating the proof of the case $v \in V_1(q)$. Note that p(v) and p(u) are the points defined respectively by v and u in Section 4.1.

The above proves $w(v) \cdot d(v, q') \leq \lambda$.

Finally, we argue that $\frac{\lambda}{w(u)} \leq d(u,z)$. Assume to the contrary that this is not true. Then, q' is outside P. This means that we can place a center outside P to cover all vertices of V(q) under λ . But this contradicts the above key observation that $FTEST\theta$ for λ' will place a center in P to cover the vertices of V(q). The lemma thus follows.

Due to the preprocessing in Section 4.1, we can find u from V(q) in O(m) time. This finishes our post-processing procedure for the case $\sup(z) \leq dem(z)$. Since $\frac{\lambda}{w(u)} \leq d(u,z)$ for any $\lambda \in (\lambda_1, \lambda_2)$, we have $w(u) \cdot d(u,z) \geq \lambda_2$, and thus, e(u,z) is indeed a twig.

Next, we consider the other case sup(z) > dem(z). In this case, P has some vertices other than z that are not covered yet, and we would need to place a center at z to cover them. Let V be the set of all uncovered vertices other than z, and V can be identified during FTEST0. In this case, we replace P by attaching a thorn e(u, z) to z with length equal to d(u, z), where u is a vertex of V with the following property: For any $\lambda \in (\lambda_1, \lambda_2)$, if there is a center q outside P covering u through z (by "through", we mean that $\pi(q, u)$ contains z) under distance λ , then q also covers all other vertices of V (intuitively u "dominates" all other vertices of V). Since later we will place centers outside P to cover the vertices of V through z under some $\lambda \in (\lambda_1, \lambda_2)$, it is sufficient to maintain u. The following lemma shows that u is the vertex of V with the largest rank.

Lemma 7. Let u be the vertex of V with the largest rank. Then, for any center q outside P that covers u through z under any distance $\lambda \in (\lambda_1, \lambda_2)$, q also covers all other vertices of V.

Proof. Let v be any vertex of V other than u. Our goal is to prove that $d(q, v) \cdot w(v) \leq \lambda$. The proof is similar to that for the case $v \in V_1(q)$ of Lemma 6 and we omit the details.

Since a center at z would cover u, it holds that $w(u) \cdot d(u, z) \leq \lambda$ for any $\lambda \in (\lambda_1, \lambda_2)$, which implies that $w(u) \cdot d(u, z) \leq \lambda_1$. Thus, e(u, z) is indeed a thorn.

The above replaces P by attaching to z either a thorn or a twig. We perform the following additional processing.

Suppose z is attached by a thorn e(z, u). If z already has another thorn e(z, u'), then we discard one of u' and u whose rank is smaller, because any center that covers the remaining vertex will cover the discarded one as well (the proof is similar to those in Lemma 6 and 7 and we omit it). This makes sure that z has at most one thorn.

Suppose z is attached by a twig e(z, u). If z already has another twig e(z, u'), then we can discard one of u and u' whose rank is larger (and subtract 1 from k). The reason is the following. Without loss of generality, assume rank(u) < rank(u'). Since both e(z, u) and e(z, u') are twigs,

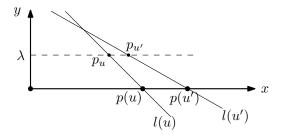


Fig. 5. Illustrating the proof of Lemma 8. Note that p(u) and p(u') are the points defined respectively by u and u' in Section 4.1.

if we apply FTEST0 on any $\lambda \in (\lambda_1, \lambda_2)$, then the algorithm will place a center q on e(z, u) with distance $\lambda/w(u)$ from u and place a center q' on e(z, u') with distance $\lambda/w(u')$ from u'. As rank(u) < rank(u'), we have the following lemma.

Lemma 8. $d(q, z) \le d(q', z)$.

Proof. The analysis is similar to those in Lemma 6 and 7. Consider the lines l(u) and l(u') in \mathbb{R}^2 defined by u and u', respectively, as discussed in Section 4.1. Let p_u and $p_{u'}$ be the intersections of the horizontal line $y = \lambda$ with l(u) and l(u'), respectively (e.g., see Fig. 5). Since rank(u) < rank(u') and $\lambda \in (\lambda_1, \lambda_2), x(p_u) \leq x(p_{u'})$. Note that $x(p_u) = d(\gamma, q)$ and $x(p_{u'}) = d(\gamma, q')$. Since $x(p_u) \leq x(p_{u'})$, we have $d(\gamma, q) \leq d(\gamma, q')$.

On the other hand, due to that $q \in e(z, u)$ and $q' \in e(z, u')$, $d(\gamma, z) \leq d(\gamma, q)$ and $d(\gamma, z) \leq d(\gamma, q')$. Thus, $d(q, z) = d(\gamma, q) - d(\gamma, z)$ and $d(q', z) = d(\gamma, q') - d(\gamma, z)$. Because $d(\gamma, q) \leq d(\gamma, q')$, we obtain $d(q, z) \leq d(q', z)$. The lemma thus follows.

Lemma 8 tells that any vertex that is covered by q' in the subsequent algorithm will also be covered by q. Thus, it is sufficient to maintain the twig e(z, u). Since we need to place a center at e(z, u'), we subtract 1 from k after removing e(z, u'). Hence, z has at most one twig.

This finishes the post-processing procedure for P. Due to the preprocessing in Section 4.1, the running time of the procedure is O(m).

Let T be the modified tree after the post-processing on each stem P without active values. If T still has more than 2n/r leaves, then we repeat the above. The algorithm stops once T has at most 2n/r leaves. This finishes Phase 0. The following lemma gives the time analysis, excluding the preprocessing in Section 4.1.

Lemma 9. Phase 0 runs in $O(n(\log \log n)^3)$ time.

Proof. We first argue that the number of iterations of the while loop is $O(\log r)$. The analysis is very similar to those in [15,16], and we include it here for completeness.

We consider an iteration of the while loop. Suppose the number of leaf-stems in T, denoted by m, is at least 2n/r. Then, at most n/r leaf-stems are of length larger than r. Hence, at least half of the leaf-stems are of length at most r. Thus, $|S| \geq m/2$. Recall that n' is the total number of backbone vertices in all leaf-stems of S. Because at most n'/(2r) leaf-stems have active values after MSEARCH, at least $|S| - n'/(2r) \geq m/2 - n'/(2r) \geq m/2 - n/(2r) \geq m/2 - m/4 = m/4$ leaf-stems will be removed. Note that removing two such leaf-stems may make an interior vertex become a new leaf in the modified tree. Hence, the tree resulting at the end of each iteration will have at

most 7/8 of the leaf-stems of the tree at the beginning of the iteration. Therefore, the number of iterations of the while loop needed to reduce the number of leaf-stems to at most 2n/r is $O(\log r)$.

We proceed to analyze the running time of Phase 0. In each iteration of the while loop, we call MSEARCH on the matrices for all leaf-stems of S. Since the length of each stem P of S is at most r, there are O(r) matrices formed for P. We perform the preprocessing of Lemma 5 on the matrices, so that each matrix element can be evaluated in $O(\log^2 r)$ time. The total time of the preprocessing on stems of S is $O(n'\log r)$. Since \mathcal{M} has O(n') matrices and the stopping account c is n'/(2r), each call to MSEARCH produces $O(\log r)$ values for feasibility tests in $O(n'\log^3 r)$ time (i.e., $O(n'\log r)$ matrix elements will be evaluated). For each leaf-stem without active values, the post-processing time for it is O(r). Hence, the total post-processing time in each iteration is O(n').

Since there are $O(\log r)$ iterations, the total number of feasibility tests is $O(\log^2 r)$, and thus the overall time for all feasibility tests in Phase 0 is $O(n\log^2 r)$. On the other hand, after each iteration, at most n'/(2r) leaf-stems of S have active values and other leaf-stems of S will be deleted. Since the length of each leaf-stem of S is at most r, the leaf-stems with active values have at most n'/2 backbone vertices, and thus at least n'/2 backbone vertices will be deleted in each iteration. Therefore, the total sum of such n' in all iterations is O(n). Hence, the total time for the preprocessing of Lemma 5 is $O(n\log r)$, the total time for MSEARCH is $O(n\log^3 r)$, and the total post-processing time for leaf-stems without active values is O(n).

In summary, the overall time of Phase 0 (excluding the preprocessing in Section 4.1) is $O(n \log^3 r)$, which is $O(n(\log \log n)^3)$ since $r = \log^2 n$.

4.3 Phase 1

We assume that the tree T now has at most 2n/r leaves and we want to place k centers in T to cover all vertices. Note that T may have some thorns and twigs. The main purpose of this phase is to gather information so that each feasibility test can be done in sublinear time, and specifically, $O(n/r \log^3 r)$ time. Recall that we have a range $(\lambda_1, \lambda_2]$ that contains λ^* .

We first form a stem-partition for T. Then, we further partition the stems into substems, each of length at most r, such that the lowest backbone vertex v in a substem is the highest backbone vertex in the next lower substem (if v has a thorn or/and a twig, then they are included in the upper substem). So this results in a partition of edges. Let S be the set of all substems. Let T_c be the tree in which each node represents a substem of S and node μ in T_c is the parent of node ν if the highest backbone vertex of the substem for ν is the lowest backbone vertex of the substem for μ , and we call T_c the stem tree. As in [15,16], since T has at most 2n/r leaves, |S| = O(n/r) and the number of nodes of T_c is O(n/r).

For each substem $P \in S$, we compute the set L_P of lines as in Section 3.1. Let L be the set of all the lines for all substems of S. We define the lines of L in the same xy-coordinate system \mathbb{R}^2 . Clearly, |L| = O(n). Consider the line arrangement A(L). Define vertices $v_1(L)$ and $v_2(L)$ of A(L) as in Section 2. With Lemma 3 and FTESTO, both vertices can be computed in $O(n \log n)$ time. We update $\lambda_1 = \max\{\lambda_1, y(v_2(L))\}$ and $\lambda_2 = \min\{\lambda_2, y(v_1(L))\}$. Hence, we still have $\lambda^* \in (\lambda_1, \lambda_2]$. We again call the values in (λ_1, λ_2) active values.

For each substem $P \in S$, observe that each element of the matrices formed based on P in Section 3.2 is equal to the y-coordinate of the intersection of two lines of L_P , and thus is equal to the y-coordinate of a vertex of $\mathcal{A}(L)$. By the definitions of $v_1(L)$ and $v_2(L)$, no matrix element of P is active.

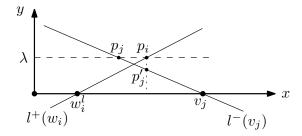


Fig. 6. Illustrating the proof of Lemma 11. Note that w_i^l is the point defined by w_i , as discussed in Section 3.1.

In the future algorithm, we will only need to test feasibilities for values $\lambda \in (\lambda_1, \lambda_2)$. In what follows, we compute a data structure on each substem P of S, so that it will help make the feasibility test faster. We will prove the following lemma and use FTEST1 to denote the feasibility test algorithm in the lemma.

Lemma 10. After $O(n \log n)$ time preprocessing, each feasibility test can be done in $O(n/r \cdot \log^3 r)$ time.

We first discuss the preprocessing and then present the algorithm FTEST1.

4.3.1 The Preprocessing for *FTEST1*

Consider a substem P of S. Let v_1, v_2, \ldots, v_m be the backbone vertices of P sorted from left to right, with v_m as the top vertex. Each vertex v_i may have a twig $e(w_i, v_i)$ and a thorn $e(u_i, v_i)$. Let λ be an arbitrary value in (λ_1, λ_2) . In the following, all statements made to λ is applicable to any $\lambda \in (\lambda_1, \lambda_2)$, and this is due to that none of the elements in the matrices produced by P is active.

By the definition of twigs, if we run FTEST0 with λ , the algorithm will place a center, denoted by q_i , on each twig $e(w_i, v_i)$ at distance $\lambda/w(w_i)$ from w_i We first run the following cleanup procedure to remove all vertices of P that can be covered by the centers on the twigs under λ .

The cleanup procedure. We first compute a rank rank'(l) for each line l of L, as follows.

Let L' be the sequence of the lines of L sorted by their intersections with the horizontal line $y = \lambda$ from left to right. By the definitions of λ_1 and λ_2 , L' is also the sequence of the lines of L sorted by their intersections with the horizontal line $y = \lambda^*$. In fact, the sequence L' is unique for any $\lambda \in (\lambda_1, \lambda_2)$. For any line $l \in L$, if there are i - 1 lines before l in L', then we define rank'(l) to be i. Clearly, rank'(l) for all lines $l \in L$ can be computed in O(n) time.

Consider a twig $e(w_i, v_i)$ and a backbone vertex v_j with $j \geq i$. Recall that w_i defines a line $l^+(w_i)$ of slope $w(w_i)$ and v_j defines a line $l^-(v_j)$ of slope $-w(v_j)$ in L_P (and thus are in L). We have the following lemma.

Lemma 11. For any $\lambda \in (\lambda_1, \lambda_2)$, the center q_i on $e(w_i, v_i)$ covers v_j if and only if $rank'(l^+(w_i)) > rank'(l^-(v_j))$.

Proof. Let p_i and p_j be the intersections of the horizontal line $y = \lambda$ with $l^+(w_i)$ and $l^-(v_j)$, respectively. Refer to Fig. 6. Let p'_j denote the intersection of $l^-(v_j)$ with the vertical line through p_i . Since q_i is located on $e(w_i, v_i)$, according to the definitions of $l^+(w_i)$ and $l^-(v_j)$, $y(p'_j)$ is exactly equal to $w(v_j) \cdot d(q_i, v_j)$. Hence, q_i covers v_j if and only if p'_j is below the line $y = \lambda$. On the other hand, p'_j is below the line $y = \lambda$ if and only if p_i is to the right of p_j , i.e., $rank'(l^+(w_i)) > rank'(l^-(v_j))$. The lemma thus follows.

Consider a thorn vertex u_j with $j \geq i$. Recall that u_j defines a line $l^-(u_j)$ in L_P with slope $-w(u_j)$. Similarly as above, q_i covers u_j if and only if $rank'(l^+(w_i)) > rank'(l^-(u_j))$.

Based on the above observations, we use the following algorithm to find all backbone and thorn vertices of P that can be covered by the centers on the twigs to their left sides. Let i' be the smallest index such that $v_{i'}$ has a twig $w_{i'}$. The algorithm maintains an index t. Initially, t = i'. For each i incrementally from i' to m, we do the following. If v_i has a twig-vertex w_i , then we reset t to i if $rank'(l^+(w_i)) > rank'(l^+(w_t))$. The reason we do so is that if $rank'(l^+(w_i)) > rank'(l^+(w_t))$, then for any $j \geq i$ such that v_j (resp., u_j) is covered by the center q_i on the twig $e(w_i, v_i)$, and thus it is sufficient to maintain the twig $e(w_i, v_i)$. Next, if $rank'(l^+(w_t)) > rank'(l^-(v_i))$, then we mark v_i as "covered". If u_i exits and $rank'(l^+(w_t)) > rank'(l^-(u_i))$, then we mark u_i as "covered".

The above algorithm runs in O(m) time and marks all vertices v_j (reps., u_j) such that there exists a twig $e(w_i, v_i)$ with $i \leq j$ whose center q_i covers v_j (reps., u_j). In a symmetric way by scanning the vertices from v_m to v_1 , we can mark in O(m) time all vertices v_j (reps., u_j) such that there exits a twig $e(w_i, v_i)$ with $i \geq j$ whose center q_i covers v_j (reps., u_j). We omit the details. This marks all vertices that are covered by centers on twigs.

Let V be the set of backbone and thorn vertices of P that are not marked, which are vertices of P that need to be covered by placing centers on the backbone of P or outside P. If a thorn vertex u_i is in V but its connected backbone vertex v_i is not in V, this means that v_i is covered by a center on a twig while u_i is not covered by any such center. Observe that any center on the backbone of P or outside P that covers u_i will cover v_i as well. For convenience of discussion, we include such v_i into V as well. Let v'_1, v'_2, \ldots, v'_t be the backbone vertices of V sorted from left to right (i.e., v'_t is closer to the root of T). Note that v'_1 may not be v_1 and v'_t may not be v_m . If v'_i has a thorn vertex in V, then we use u'_i to denote it.

This finishes the cleanup procedure.

Next, we compute a data structure for P to maintain some information for faster feasibility tests.

First of all, we maintain the index a of the twig vertex w_a such that $rank'(l^-(w_a)) < rank'(l^-(w_i))$ for any other twig vertex w_i of P. The reason we keep a is the following. Observe that for any vertex v that is a descent vertex of v_1 in T (so v is in another substem that is a descent substem of P in the stem tree T_c), if v can be covered by the center q_i on a twig $e(v_i, w_i)$ under any $\lambda \in (\lambda_1, \lambda_2)$, then v can also be covered by the center q_a on the twig $e(v_a, w_a)$ under λ . Symmetrically, we maintain the index b of the twig vertex w_b such that $rank'(l^+(w_b)) > rank'(l^+(w_i))$ for any other twig vertex w_i . Similarly, this is because for any vertex v that is not in any substem of the subtree rooted at P in T_c , if v can be covered by the center on the twig $e(v_i, w_i)$ under λ , then v can also be covered by the center on the twig $e(v_b, w_b)$ under λ . Both a and b can be computed in O(m) time.

For any two indices i and j with $1 \le i \le j \le t$, we use V[i,j] to denote the set of all backbone vertices v'_l and thorn vertices u'_l with $l \in [i,j]$.

For each index $i \in [1, t]$, we maintain an integer ncen(i) and a vertex v(i) of V, which we define below. Roughly speaking, ncen(i) is the minimum number of centers that are needed to cover all vertices of V[i, t], minus one, and if we use ncen(i) centers to cover as many vertices of V[i, t] as possible from left to right, v(i) is the vertex that is not covered but "dominates" all other uncovered vertices under λ . Their detailed definitions are given below.

Let j_i be the smallest index in [i, t] such that it is not possible to cover all vertices in $V[i, j_i]$ by one center under λ . If such a index j_i does not exist, we let $j_i = t + 1$.

If $j_i = t + 1$, then we define ncen(i) = 0. We define v(i) as the vertex v in V[i,t] such that $rank'(l^+(v)) < rank'(l^+(v'))$ for any other vertex $v' \in V[i,t]$. The reason we maintain such v(i) is as follows. Suppose during a feasibility test with λ , all vertices of V[1,i-1] have been covered and we need to place a new center to cover those in V[i,t]. According to the greedy strategy of FTEST0, we want to place a center as close to the root γ as possible. There are two cases.

In the first case, $w(v(i)) \cdot d(v(i), v_m) < \lambda$, and one can verify that if we place a center at the top vertex v_m , it can cover all vertices of V[i, t] under λ . In this case, we do not place a center on the backbone of P but will use a center outside P to cover them (more precisely, this center is outside the subtree of T_c rooted at substem P). We maintain v(i) because any center outside P covering v(i) will cover all other vertices of V[i, t] as well.

In the second case, $w(v(i)) \cdot d(v(i), v_m) \geq \lambda$, and we need to place a center on the backbone of P. Again, according to the greedy strategy of FTEST0, we want to place this center close to v_m as much as possible, and we use q^* to denote such a center. The following lemma shows that q^* is determined by v(i).

Lemma 12. q^* is on the path $\pi(v(i), v_m)$ of distance $\frac{\lambda}{w(v(i))}$ from v(i).

Proof. The proof is somewhat similar to that of Lemma 6.

Let p be the point on $\pi(v(i), v_m)$ of distance $\frac{\lambda}{w(v(i))}$ from v(i). Note that since $w(v(i)) \cdot d(v(i), v_m) \geq \lambda$, such a point p must exist on $\pi(v(i), v_m)$. By definition, p is the point on $\pi(v(i), v_m)$ closest to v_m that can cover v(i). If v(i) is a backbone vertex, then p is on the backbone of P. Otherwise, v(i) is a thorn vertex and $w(v(i)) \cdot d(v(i), v) \leq \lambda_1$, where v is the backbone vertex that connects v(i). Since $\lambda_1 < \lambda$, we obtain $w(v(i)) \cdot d(v(i), v) < \lambda$, and thus p must be on the backbone of P. Hence, in either case, p is on the backbone of P. Consider any vertex $v' \in V[i, t]$ with $v' \neq v(i)$. In the following, we show that v' is covered by p.

If v' is in the subtree rooted at p (i.e., $\pi(v', v_m)$ contains p), then since $rank'(l^+(v(i))) < rank'(l^+(v'))$, one can verify that v' is covered by p. Otherwise, assume to the contrary that p does not cover v'. Then, we would need to move p towards v' in order to cover v'. However, since p is the point on $\pi(v(i), v_m)$ closest to v_m that can cover v(i), p is also the point on $\pi(v(i), v')$ closest to v' that can cover v(i). Hence, moving p towards v' will make p not cover v(i) any more, which implies that no point on the backbone of P can cover both v' and v(i). This contradicts with the fact that it is possible to place a center on the backbone of P to cover all vertices in V[i,t]. Therefore, p covers v'.

The above shows that p is the point on the backbone of P closest to v_m that can cover all vertices of V[i,t]. Thus, p is q^* , and the lemma follows.

The above defines ncen(i) and v(i) for the case where $j_i = t + 1$. If $j_i \le t$, we define ncen(i) and v(i) recursively as $ncen(i) = ncen(j_i) + 1$ and $v(i) = v(j_i)$. Note that $i < j_i$, and thus this recursive definition is valid.

In the following, we present an algorithm to compute ncen(i) and v(i) for all $i \in [1, t]$. In fact, the above recursive definition implies a dynamic programming approach to scan the vertices v'_i backward from t to 1. The details are given in the following lemma.

Lemma 13. ncen(i) and v(i) for all $i \in [1, t]$ can be computed in $O(m \log^2 m)$ time.

Proof. For any i and j with $1 \le i \le j \le t$, consider the following one-center problem: find a center to cover all backbone and thorn vertices of V[i,j]. As discussed in Section 2, each backbone

or thorn vertex defines two upper half-planes such that the optimal objective value for the above one-center problem is equal to the y-coordinate of the lowest point in the common intersection of the at most 4(j-i+1) half-planes defined by the backbone and thorn vertices of V[i,j]. As in Section 2, as preprocessing, we first compute the upper half-planes defined by all vertices of V and order them by the indices of their corresponding vertices in V, and then compute the 2D sublist LP query data structure of Lemma 2 in $O(t \log t)$ time. As in Section 2, the lowest point of the common intersection of the upper half-planes defined by vertices of V[i,j] can be computed by a 2D sublist LP query in $O(\log^2 t)$ time. We use $\alpha(i,j)$ to denote the optimal objective value of the above one-center problem for V[i,j]. With the above preprocessing, given i and j, $\alpha(i,j)$ can be computed in $O(\log^2 t)$ time.

We proceed to compute ncen(i) and v(i) for all $i \in [1, t]$.

For each i from t downto 1, we do the following. We maintain the index j_i . Initially when i = t, we set $j_i = t + 1$, ncen(i) = 0, and $v(i) = v'_t$. We process index i as follows. We first compute $\alpha(i, j_i - 1)$ in $O(\log^2 t)$ time. Depending on whether $\alpha(i, j_i - 1) \leq \lambda$, there are two cases.

If $\alpha(i, j_i - 1) \leq \lambda$, then depending on whether $j_i \neq t + 1$, there are two subcases.

If $j_i \neq t+1$, then $ncen(i) = ncen(j_i) + 1$ and $v(i) = v(j_i)$. Otherwise, we first set ncen(i) = 0 and v(i) = v(i+1). If $rank'(l^+(v'_i)) < rank'(l^+(v(i)))$, then we reset v(i) to v'_i . Further, if v'_i has a thorn u'_i and $rank'(l^+(u'_i)) < rank'(l^+(v(i)))$, then we reset v(i) to u'_i .

If $\alpha(i, j_i - 1) > \lambda$, we keep decrementing j_i by one until $\alpha(i, j_i - 1) \leq \lambda$. Then, we reset $ncen(i) = ncen(j_i) + 1$ and v(i) to $v(j_i)$.

It is not difficult to see that the above algorithm runs in $O(t \log^2 t)$ time, which is $O(m \log^2 m)$ time since $t \leq m$.

Since $m \leq r$ and $r = \log^2 n$, we can compute the data structure for the substem P in $O(r(\log \log n)^2)$ time. The total time for computing the data structure for all substems of S is $O(n(\log \log n)^2)$. With these data structures, we show that a feasibility test can be done in $O(n/r\log^3 r)$ time.

4.3.2 The Faster Feasibility Test *FTEST1*

Given any $\lambda \in (\lambda_1, \lambda_2)$, the goal is to determine whether λ is feasible. We will work on the stem tree T_c , where each node represents a stem of S.

Initially, we set $sup(P) = \infty$ and dem(P) = sup(P) - 1 (so that dem(P) is an infinitely large value but still smaller than sup(P)) for every stem P of T_c . We perform a post-order traversal on T_c and maintain a variable count, which is the number of centers that have been placed so far. Suppose we are processing a stem P. For each child stem P' of P, we reset $sup(P) = \min\{sup(P), sup(P')\}$ and $dem(P) = \min\{dem(P), dem(P')\}$. After handling all children of P as above, we process P as follows.

First of all, we increase *count* by the number of twigs in P. Let V be the uncovered vertices of P as defined before, and v'_1, v'_2, \ldots, v'_t are backbone vertices of V. Note that $t \leq r$. Recall that we have maintained two twig indices a and b for P. Depending on whether $sup(P) \leq dem(P)$, there are two main cases.

The case $sup(P) \leq dem(P)$. If $sup(P) \leq dem(P)$, then the uncovered vertices in the children of P can be covered by the center q that determines the value sup(P) (i.e., q is in a child stem of P and $d(q, v_1) = sup(P)$, where v_1 is the lowest backbone vertex of P). Note that we do not need to

compute q and we use it only for the discussion. We do binary search on the list of V to find the largest index $i \in [1,t]$ such that q can cover the vertices V[1,i]. If no such i exists in [1,t], then let i=0. Such an index i can be found in $O(\log^2 r)$ time using the line-constrained 2D sublist LP queries of Lemma 2, as shown in the following lemma.

Lemma 14. Such an index i (i.e., the largest index $i \in [1,t]$ such that q can cover the vertices V[1,i]) can be found in $O(\log^2 r)$ time.

Proof. Given an index $i \in [1, t]$, we show below that we can determine whether q can cover all vertices of V[1, i] in $O(\log r)$ time.

Recall that in our preprocessing, each vertex of V defines two upper half-planes in \mathbb{R}^2 , and we have built a 2D sublist LP query data structure on all upper half-planes defined by the vertices of V. Let q' be the point on the x-axis of \mathbb{R}^2 with x-coordinate equal to -sup(P). Let l be the vertical line through q' and let p be the lowest point on l that is in the common intersection of all upper half-planes defined by the vertices of V[1,i]. An observation is that q can cover all vertices of V[1,i] if and only if the y-coordinate of p is at most λ , which can be determined in $O(\log r)$ time by a line-constrained 2D sublist LP query.

If q can cover all vertices of V[1,i], then we continue the search on the indices larger than i; otherwise, we continue the search on the indices smaller than i. If q cannot cover the vertices of V[1,i] for i=1, then we return i=0. The total time is $O(\log^2 r)$.

If i = t, then all vertices of V can be covered by q. In this case, we reset $sup(P) = \min\{sup(P) + d(v_1, v_m), d(w_b, v_m) - \lambda/w(w_b)\}$, where the latter value is the distance from v_m to the center at the twig $e(v_b, w_q)$.

If i < t (this includes the case i = 0), then we increase i by one and reset count = count + ncen(i). If $w(v(i)) \cdot d(v(i), v_m) < \lambda$, then we reset $dem(P) = \lambda/w(v(i)) - d(v(i), v_m)$ and $sup(P) = d(w_b, v_m) - \lambda/w(w_b)$. Otherwise, we need to place an additional center on P to cover the uncovered vertices of P including v(i), and thus we increase count by one and reset $sup(P) = \min\{d(v(i), v_m) - \lambda/w(v(i)), d(w_b, v_m) - \lambda/w(w_b)\}$ and $dem(P) = \infty$.

The case sup(P) > dem(P). In this case, we need to first deal with dem(P), i.e., covering the vertices in the children stems of P that are not covered.

If $dem(P) \geq d(v_1, w_a) - \lambda/w(w_a)$, then the center at the twig $e(v_a, w_a)$ can cover the uncovered vertices in the children stems of P. In this case, we increase count by ncen(1). If $w(v(1)) \cdot d(v(1), v_m) < \lambda$, then we postpone placing centers to the next stem and reset $dem(P) = \lambda/w(v(1)) - d(v(1), v_m)$ and $sup(P) = d(w_b, v_m) - \lambda/w(w_b)$. Otherwise, we increase count by one and reset $sup(P) = \min\{d(v(1), v_m) - \lambda/w(v(1)), d(w_b, v_m) - \lambda/w(w_b)\}$ and $dem(P) = \infty$.

If $dem(P) < d(v_1, w_a) - \lambda/w(w_a)$, then we do binary search to find the largest index $i \in [1, t]$ such that we can find a center q on the backbone of P to cover all vertices of V[1, i] with $d(v_1, q) \le dem(P)$. If such an index i does not exit, then we let i = 0. The following lemma shows that such an index i can be found in $O(\log^3 r)$ time.

Lemma 15. Such an index i can be found in $O(\log^3 r)$ time.

Proof. Given any index i, we first show that we can determine in $O(\log^2 r)$ time the answer to the following question: whether there exists a center q on the backbone of P that can cover all vertices of V[1,i] with $d(v_1,q) \leq dem(P)$?

By a 2D sublist LP query on the upper half-planes defined by the vertices of V[1,i], we compute the lowest point p in the common intersection of these half-planes. If $y(p) > \lambda$, then the answer to the question is no. Otherwise, if $x(p) \leq dem(P)$, then the answer to the question is yes. If x(p) > dem(P), then let l be the vertical line whose x-coordinate is dem(P). By a line-constrained 2D sublist LP query, we can compute the lowest point p' on l in the above common intersection of upper half-planes in $O(\log r)$ time. If $y(p') \leq \lambda$, then the answer to the above question is yes; otherwise the answer is no. The total time to determine the answer to the question is $O(\log^2 r)$.

If the answer is yes, then we continue the search on indices larger than i; otherwise we continue on indices smaller than i. If the answer to the question is no for i = 1, then we return i = 0. The total running time is $O(\log^3 r)$.

If i=t, then there are two subcases. If $w(v(1)) \cdot d(v(1), v_m) < \lambda$ and $dem(P) > d(v_1, v_m)$, then we postpone placing centers to the next stem by resetting $dem(P) = \min\{dem(P) - d(v_1, v_m), \lambda/w(v(1)) - d(v(1), v_m)\}$ and $sup(P) = d(w_b, v_m) - \lambda/w(w_b)$. Otherwise, we place a center on the backbone of P of distance $\delta = \max\{d(v(1), v_m) - dem(P), d(v(1), v_m) - \lambda/w(v(1))\}$ from v_m . Then, we increase count by one, and reset $sup(P) = \min\{\delta, d(w_b, v_m) - \lambda/w(w_b)\}$ and $dem(P) = \infty$.

If $i \neq t$ (this includes the case i = 0), then we place a center (at a location determined by the algorithm for Lemma 15) to cover dem(P) as well as the vertices of V[1,i] and increase count by one. Next, we increment i by one and increase count by ncen(i). If $w(v(i)) \cdot d(v(i), v_m) < \lambda$, then we reset $dem(P) = \lambda/w(v(i)) - d(v(i), v_m)$ and $sup(P) = d(w_b, v_m) - \lambda/w(w_b)$. Otherwise, we increase count by one and reset $sup(P) = \min\{d(v(i), v_m) - \lambda/w(v(i)), d(w_b, v_m) - \lambda/w(w_b)\}$ and $dem(P) = \infty$.

This finishes the processing of the stem P. After the stem P_{γ} that contains the root γ is processed, if $sup(P_{\gamma}) > dem(P_{\gamma})$, then we place a center at the root γ to cover the uncovered vertices and increase count by one. The value λ is feasible if and only if $count \leq k$. Since we spend $O(\log^3 r)$ time on each stem of T_c and T_c has O(n/r) stems, FTEST1 runs in $O(n/r \cdot \log^3 r)$ time. Refer to Algorithm 2 for the pseudocode of FTEST1.

4.4 Phase 2

In this phase, we will finally compute the optimal objective value λ^* , using the faster feasibility test *FTEST1*. Recall that we have computed a range $(\lambda_1, \lambda_2]$ that contains λ^* after Phase 1.

We first form a stem-partition of T. While there is more than one leaf-stem, we do the following. Let S be the set of all leaf-stems. For each stem $P \in S$, we compute the set of lines as in Section 3.1, and let L be the set of the lines for all stems of S. With Lemma 3 and FTEST1, we compute the two vertices $v_1(L)$ and $v_2(L)$ of the arrangement $\mathcal{A}(L)$ as defined in Section 2. We update $\lambda_1 = \max\{\lambda_1, y(v_2(L))\}$ and $\lambda_2 = \min\{\lambda_2, y(v_1(L))\}$. As discussed in Phase 1, each stem P of S does not have any active values (in the matrices defined by P). Next, for each stem P of S, we perform the post-processing procedure as in Section 4.2, i.e., place centers on P, subtract their number from k, and replace P by attaching a twig or a thorn to its top vertex. Let T be the modified tree.

After the while loop, T is a single stem. Then, we apply above algorithm on the only stem T, and the obtained value λ_2 is λ^* . The running time of Phase 2 is bounded by $O(n \log n)$, which is analyzed in the following theorem.

Theorem 1. The k-center problem on T can be solved in $O(n \log n)$ time.

Algorithm 2: The faster feasibility test algorithm *FTEST1*

```
Input: The stem-tree T_c, the original k from the input, and \lambda \in (\lambda_1, \lambda_2)
   Output: Determine whether \lambda is feasible
 1 count \leftarrow 0;
   for each stem P of the stem tree T_c do
    sup(P) \leftarrow \infty, dem(P) \leftarrow sup(P) - 1;
   for each stem P in the post-order traversal of T_c do
 4
        for each child P' of P do
 5
         sup(P) = \min\{sup(P), sup(P')\}, dem(P) = \min\{dem(P), dem(P')\};
 6
        Increase count by the number of twigs of P:
 7
        Let V be the set of the uncovered vertices of P, and v'_1, v'_2, \ldots, v'_t are the backbone vertices of V;
 8
        if sup(P) \leq dem(P) then
 9
             Let q be the center in a child stem of P that gives the value sup(P);
10
             Do binary search to find the largest index i \in [1, t] such that q can cover all vertices of V[1, i];
11
12
             if i = t then
                 sup(P) = min\{sup(P) + d(v_1, v_m), d(w_b, v_m) - \lambda/w(w_b)\};
13
             else
14
                  i + +, count = count + ncen(i);
15
                  if w(v(i)) \cdot d(v(i), v_m) < \lambda then
16
                     dem(P) = \lambda/w(v(i)) - d(v(i), v_m), sup(P) = d(w_b, v_m) - \lambda/w(w_b);
17
18
                   count + +, sup(P) = \min\{d(v(i), v_m) - \lambda/w(v(i)), d(w_b, v_m) - \lambda/w(w_b)\}, dem(P) = \infty;
19
20
        else
            if dem(P) \ge d(v_1, w_a) - \lambda/w(w_a) then
21
                  count = count + ncen(1);
22
                  if w(v(1)) \cdot d(v(1), v_m) < \lambda then
23
                      dem(P) = \lambda/w(v(1)) - d(v(1), v_m), sup(P) = d(w_b, v_m) - \lambda/w(w_b);
24
25
                   count + +, sup(P) = \min\{d(v(1), v_m) - \lambda/w(v(1)), d(w_b, v_m) - \lambda/w(w_b)\}, dem(P) = \infty;
26
27
             else
                  Do binary search to find the largest i \in [1, t] such that there exists a center q on the backbone of
28
                   P to cover all vertices of V[1,i] with d(v_1,q) \leq dem(P);
                  if i = t then
29
                      if w(v(1)) \cdot d(v(1), v_m) < \lambda and dem(P) > d(v_1, v_m) then
30
                           dem(P) = \min\{dem(P) - d(v_1, v_m), \lambda/w(v(1)) - d(v(1), v_m)\},
31
                             sup(P) = d(w_b, v_m) - \lambda/w(w_b);
                      else
32
                           count + +, \delta = \max\{d(v(1), v_m) - dem(P), d(v(1), v_m) - \lambda/w(v(1))\},\
33
                             sup(P) = min\{\delta, d(w_b, v_m) - \lambda/w(w_b)\}, dem(P) = \infty;
34
                  else
                      i + +, count = count + 1 + ncen(i);
35
                      if w(v(i)) \cdot d(v(i), v_m) < \lambda then
36
                           dem(P) = \lambda/w(v(i)) - d(v(i), v_m), sup(P) = d(w_b, v_m) - \lambda/w(w_b);
37
38
                           count + +, sup(P) = \min\{d(v(i), v_m) - \lambda/w(v(i)), d(w_b, v_m) - \lambda/w(w_b)\}, dem(P) = \infty;
39
40 if sup(P_{\gamma}) > dem(P_{\gamma}) then
    count + +;
42 Return true if and only if count < k;
```

Proof. As discussed before, Phases 0 and 1 run in $O(n \log n)$ time. Below we focus on Phase 2.

First of all, as in [16], the number of iterations of the while loop is $O(\log n)$ because the number of leaf-stems is halved after each iteration. In each iteration, let n' denote the total number of backbone vertices of all leaf-stems in S. Hence, |L| = O(n'). Thus, the call to Lemma 3 with FTEST1 takes $O((n'+n/r \cdot \log^3 r) \log n')$ time. The total time of the post-processing procedure for all leaf-stems of S is O(n'). Since all leaf-stems of S will be removed in the iteration, the total sum of all such n' is O(n) in Phase 2. Therefore, the total time of the algorithm in Lemma 3 in Phase 2 is $O(n \log n + n/r \cdot \log^3 r \log^2 n)$, which is $O(n \log n)$ since $r = \log^2 n$. Also, the overall time for the post-processing procedure in Phase 2 is O(n). Therefore, the total time of Phase 2 is $O(n \log n)$. This proves the theorem.

The pseudocode in Algorithm 3 summarizes the overall algorithm.

```
Algorithm 3: The k-center algorithm
   Input: A tree T and an integer k
   Output: The optimal objective values \lambda^* and k centers in T
 1 Perform the preprocessing in Section 4.1 and compute the "ranks" for all vertices of T;
    /* Phase 0
                                                                                                                  */
   r \leftarrow \log^2 n;
 3 Form a stem-partition of T;
   while there are more than 2n/r leaves in T do
       Let S be the set of all leaf-stems of lengths at most r;
       Form the set \mathcal{M} of matrices for all leaf-stems of S by Lemma 5;
       Let n' be the total number of all backbone vertices of the leaf-stems of S;
       Call MSEARCH on \mathcal{M} with stopping count c = n'/(2r), using FTEST0;
       for each leaf-stem P of S with no active values do
            Perform the post-processing on P, i.e., place centers on P, subtract their number from k, replace P
10
             by a thorn or a twig, and modify the stem-partition of T;
   /* Phase 1
                                                                                                                  */
11 For a stem-partition of T, and for each stem, partition it into substems of lengths at most r;
12 Let S be the set of all substems, and form the stem-tree T_c;
13 Compute the set L of lines for all stems of S in the way discussed in Section 3.1;
14 Compute the two vertices v_1(L) and v_2(L) of \mathcal{A}(L) by Lemma 3 and FTEST0, and update \lambda_1 and \lambda_2;
   for each substem P of S do
    Compute the data structure for the faster feasibility test FTEST1;
   /* Phase 2
                                                                                                                  */
17 Form a stem-partition of T;
   while there is more than one leaf-stem in T do
        Compute the set L of lines for all stems of S in the way discussed in Section 3.1;
19
        Compute the two vertices v_1(L) and v_2(L) of \mathcal{A}(L) by Lemma 3 and FTEST1, and update \lambda_1 and \lambda_2;
20
        for each leaf-stem of S do
21
            Perform the post-processing on P, i.e., place centers on P, subtract their number from k, replace P
22
             by a thorn or a twig, and modify the stem-partition of T;
23 Compute the set L of the lines for the only leaf-stem T;
24 Compute the two vertices v_1(L) and v_2(L) of A(L) by Lemma 3 and FTEST1, and update \lambda_1 and \lambda_2;
26 Apply FTEST0 on \lambda = \lambda^* to find k centers in the original tree T;
```

5 The Discrete k-Center Problem

In this section, we extend our techniques to solve in $O(n \log n)$ time the discrete k-center problem on T where centers must be located at the vertices of T. In fact, the problem becomes easier due to the following observation.

Observation 1 The optimal objective value λ^* is equal to $w(v) \cdot d(v, u)$ for two vertices u and v of T (i.e., a center is placed at u to cover v).

The previous $O(n \log^2 n)$ time algorithm in [26] relies on this observation. Megiddo et al. [26] first computed in $O(n \log^2 n)$ time a collection of $O(n \log n)$ sorted subsets that contain the intervertex distances d(u, v) for all pairs (u, v) of vertices of T. By multiplying the weight w(v) by the elements in the subsets corresponding to each vertex v, λ^* is contained in these new sorted subsets. Then, λ^* can be computed in $O(n \log^2 n)$ time by searching these sorted subsets, e.g., using MSEARCH. Frederickson and Johnson [17] later proposed an $O(n \log n)$ -time algorithm that computes a succinct representation of all intervertex distances of T by using sorted Cartesian matrices. With MSEARCH, their algorithm solves the unweighted case of the problem in $O(n \log n)$ time. However, their techniques may not be generalized to solving the weighted case because once we multiply the vertex weights by the elements of those Cartesian matrices, the new matrices are not sorted any more (i.e., we cannot guarantee that both columns and rows are sorted because different rows or columns are multiplied by weights of different vertices).

Our algorithm uses similar techniques as those for the previous non-directete k-center problem. In the following we briefly discuss it and mainly focus on pointing out the differences.

First of all, we need to modify the feasibility test algorithm FTEST0. The only difference is that when $sup(u) \leq dem(u)$ and dem(u) < d(u,v), instead of placing a center in the interior of the edge e(u,v), we place a center at u and update $sup(v) = \min\{sup(v), d(u,v)\}$ (i.e., use this to replace Line 11 in the pseudocode of Algorithm 1). The running time is still O(n). We use DFTEST0 to denote the new algorithm.

5.1 The Algorithm for Stems

The stem is defined slightly differently than before. Suppose we have a range $(\lambda_1, \lambda_2]$ that contains λ^* . Each backbone vertex v_i of a stem P still has at most one thorn and one twig. The thorn $e(u_i, v_i)$ is defined in the same way as before. However, a twig now consists of two edges $e(v_i, b_i)$ and $e(b_i, w_i)$ such that $w(w_i) \cdot d(w_i, v_i) \ge \lambda_2$ and $w(w_i) \cdot d(w_i, b_i) \le \lambda_1$, which means that we have to place a center at b_i to cover w_i under any $\lambda \in (\lambda_1, \lambda_2)$. We still call w_i a twig vertex, and following the terminology in [16], we call b_i a bud.

Next we give an algorithm to solve the k-center problem on a stem P of m backbone vertices. The algorithm is similar to that in Section 3.2 and uses MSEARCH, but we use a different way to form matrices based on Observation 1. (Note that we will not need a similar algorithm as that in Section 3.1.) Let λ^* temporarily refer to the optimal objective value for the k-center problem on P in this subsection, and we assume $\lambda^* \in (\lambda_1, \lambda_2]$.

Let v_1, v_2, \ldots, v_m be the backbone vertices of P. We again assume that all backbone vertices are in the x-axis such that v_1 is at the origin and v_i has x-coordinate $d(v_1, v_i)$. As in Section 3.1, for each thorn vertex u_i , we define two points u_i^l and u_i^r on the x-axis (whose weights are equal to $w(u_i)$), and we do the same for each bud and each twig vertex. Let \mathcal{P} be the set of all vertices on the x-axis. Hence, $|\mathcal{P}| \leq 7m$.

We sort all vertices of \mathcal{P} from left to right, and let the sorted list be z_1, z_2, \ldots, z_t for $t \leq 7m$. For any z_i , we use $x(z_i)$ to denote its x-coordinate. For each vertex z_i , we define two sorted arrays A_i^l and A_i^r of lengths at most t as follows. For each $j \in [1, t-i+1]$, define $A_i^r[j] = w(z_i) \cdot (x(z_{t+1-j}) - x(z_i))$. For each $j \in [1, i]$, define $A_i^l[j] = w(z_i) \cdot (x(z_i) - x(z_j))$. Both arrays are sorted.

Let \mathcal{M} denote the set of all O(m) sorted arrays defined above. By Observation 1, λ^* must be an element of an array in \mathcal{M} . With $O(m \log m)$ time preprocessing, each array element of \mathcal{M} can be computed in O(1) time. By applying MSEARCH on \mathcal{M} with stopping count c=0 and using DFTESTO, we can compute λ^* in $O(m \log m)$ time (i.e., MSEARCH produces $O(\log m)$ values for feasibility tests in $O(m \log m)$ time).

5.2 Solving the Problem in T

In the sequel, we solve the discrete k-center problem in T. First, we do the same preprocessing as in Section 4.1. Then, we have three phases as before. Let $r = \log^2 n$.

5.2.1 Phase 0

We assume that T has more than 2n/r leaves since otherwise we could skip this phase. Phase 0 is the same as before except the following changes. First, we use DFTEST0 to replace FTEST0. Second, we form the matrix set \mathcal{M} in the way discussed in Section 5.1. Third, for each leaf-stem P without active values, we modify the post-processing procedure as follows.

Let z be the top vertex of P. We run DFTEST0 on P with z as the root and $\lambda = \lambda'$ as any value in (λ_1, λ_2) . As before, after z is processed, depending on whether $sup(z) \leq dem(z)$, there are two main cases.

The case $sup(z) \leq dem(z)$. If $sup(z) \leq dem(z)$, we define q and V(q) in the same way as before but V(q) should not include bud vertices. A difference is that q now is a vertex of P. Note that $q \neq z$ because in this case we do not need to place a center at z.

Let w be the vertex that makes q as a center. Specifically, refer to the pseudocode in Algorithm 1, where we place a center q at vertex u at Line 10. Let w be the vertex that determines the value dem(u), i.e., $dem(u) = \lambda'/w(w) - d(w,u)$. Note that w must be a descendent of u = q. In this case, we replace P by a twig consisting of two edges e(z,q) and e(q,w) with lengths equal to d(z,q) and d(q,w), respectively. To find the vertex w, one way is to modify DFTEST0 so that the vertex that determines the value dem(v) for each vertex v of T is also maintained. Another way is that w is in fact the vertex of V(q) with the largest rank, which is proved in the following lemma (whose proof is similar to that of Lemma 6). Note that this is actually consistent with our way for creating twigs in the previous non-discrete case.

Lemma 16. w is the vertex of V(q) with the largest rank.

Proof. Let v be any vertex of $V(q) \setminus \{w\}$. Our goal is to show that rank(w) > rank(v). Note that q is either a backbone vertex or a bud.

We first discuss the case where q is a backbone vertex. We define $V_1(q)$ to be the set of vertices of V(q) in the subtree rooted at q and let $V_2(q) = V(q) \setminus V_1(q)$. Depending on whether v is in $V_1(q)$ or $V_2(q)$, there are two subcases.

If $v \in V_1(q)$, assume to the contrary that rank(w) < rank(v). Recall that v defines a line l(v) and w defines a line l(w) in \mathbb{R}^2 in the preprocessing (see Section 4.1). Refer to Fig. 7. Let p_v and

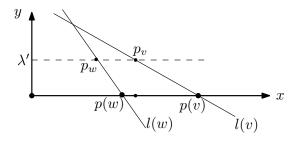
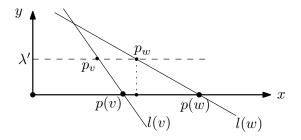


Fig. 7. Illustrating the proof of Lemma 17. Note that p(v) and p(w) are the points defined respectively by v and w in the preprocessing of Section 4.1.

 p_w denote the intersections of the horizontal line $y = \lambda'$ with l(v) and l(w), respectively. Since $\lambda' \in (\lambda_1, \lambda_2)$, by the definition of ranks, $x(p_w) < x(p_v)$. Note that $x(p_v)$ corresponds to a point q_v in $\pi(v, \gamma)$ in the sense that $x(p_v) = d(\gamma, q_v)$, and q_v is actually the point on $\pi(v, \gamma)$ closest to γ that can cover v. Similarly, $x(p_w)$ corresponds to a point q_w in $\pi(w, \gamma)$. Since $x(p_w) < x(p_v)$, $d(\gamma, q_w) < d(\gamma, q_v)$. One can verify that this contradicts with that w is the vertex that makes q as a center (i.e., w determines the value dem(u)), because both v and w are descendants of q.

If $v \in V_2(q)$, first note that v cannot be a twig vertex since otherwise q would need to be a bud in order to cover v. Depending on whether v is a backbone vertex or a thorn vertex, there are two subcases.



 $\lambda' \qquad p_w \qquad p_v \qquad$

Fig. 8. Illustrating the proof of Lemma 17.

Fig. 9. Illustrating the proof of Lemma 17.

- 1. If v is a backbone vertex, then v is an ancestor of q. Let p_v and p_w be the intersections of the horizontal line $y = \lambda'$ with l(v) and l_w , respectively (e.g., see Fig. 8). Since w determines the center q and v is an ancestor of q, according to DFTEST0, it holds that $w(w) \cdot d(w, v) < \lambda'$. Since q is an ancestor of w, v is also an ancestor of w. Therefore, the point p(v) (i.e., the intersection of l(v) with the x-axis) must be to the left of the vertical line through p_w . Since the slope of l(v) is not positive, $x(p_v) < x(p_w)$. Because $\lambda' \in (\lambda_1, \lambda_2)$, we obtain that rank(v) < rank(w).
- 2. If v is a thorn vertex, assume to the contrary that rank(w) < rank(v). Let v' be the backbone vertex that connects v. Since $v \in V_2(q)$ and q is a backbone vertex, v' must be an ancestor of q. Because rank(w) < rank(v), in the following we will prove $w(v) \cdot d(v,q) > \lambda'$, which implies that q cannot cover v and thus incurs contradiction.
 - By our preprocessing in Section 4.1, the vertex v' defines a point p(v') in the x-axis of \mathbb{R}^2 (e.g., see Fig. 9). With a little abuse of notation, we use x(v') to denote the x-coordinate of p(v'). We define p_v and p_w in the same way as before.
 - Since w determines q and v' is an ancestor of q, according to DFTEST0, it holds that $w(w) \cdot d(w, v') < \lambda'$. This implies that $x(v') < x(p_w)$. Since rank(w) < rank(v) and $\lambda' \in (\lambda_1, \lambda_2)$,

 $x(p_w) < x(p_v)$. Thus, $x(v') < x(p_v)$ and the y-coordinate of the intersection of the vertical line through p(v') and l(v) is larger than λ' . Note that the above y-coordinate is equal to $w(v) \cdot d(v, v')$. Therefore, we obtain $w(v) \cdot d(v, v') > \lambda'$. Since the path $\pi(q, v)$ contains the thorn e(v, v'), we have $d(v, q) \ge d(v, v')$. Thus, it holds that $w(v) \cdot d(v, q) > \lambda'$.

The above proves that rank(w) > rank(v) for the case where q is a backbone vertex.

Next we discuss the case where q is a bud. Since w is a descendent of q, w must be a twig vertex on the same twig as q. Let v' be the backbone vertex that connects q. We say that v is above v' if $\pi(z,v')$ either contains v (when v is a backbone vertex) or contains the backbone vertex that connects v (when v is a thorn vertex); otherwise, v is below v'. If v is above v', then the analysis is similar to the above subcase $v \in V_2(q)$. We omit the details. In the following, we analyze the case where v is below v'. Although v may be either a backbone vertex or a thorn vertex, we prove rank(w) > rank(v) in a uniform way.

Assume to the contrary that rank(w) < rank(v). Again, refer to Fig. 9. We define the points in the figure in the same way as before except that v' is now the backbone vertex that connects q. Our goal is to show that $w(v) \cdot d(v,q) > \lambda'$, which incurs contradiction since q covers v. To this end, since $d(v,v') \le d(v,q)$, it is sufficient to show that $w(v) \cdot d(v,v') > \lambda'$. The proof is similar to the above subcase where v is a thorn vertex, and we briefly discuss it below.

Since w determines q and v' is an ancestor of q, it holds that $w(w) \cdot d(w, v') < \lambda'$. This implies that $x(v') < x(p_w)$. Since rank(w) < rank(v) and $\lambda' \in (\lambda_1, \lambda_2), x(p_w) < x(p_v)$. Thus, $x(v') < x(p_v)$ and the y-coordinate of the intersection of the vertical line through p(v') and l(v) is larger than λ' . Again, the above y-coordinate is equal to $w(v) \cdot d(v, v')$. Therefore, $w(v) \cdot d(v, v') > \lambda'$.

This proves the lemma. \Box

The above replaces P by attaching a twig $e(z,q) \cup e(q,w)$ to z. In addition, if z already has another twig with bud q', then we discard the new twig if $d(q,z) \ge d(q',z)$ and discard the old twig otherwise. This guarantees that z has at most one twig.

The case $\sup(z) > \operatorname{dem}(z)$. If $\sup(z) > \operatorname{dem}(z)$, then we define V in the same way as before but excluding the buds. Let u be the vertex of V with the largest rank. As before in Lemma 7, u dominates all other vertices of V and thus we replace P by a thorn e(u,z) whose length is equal to d(u,z). In addition, if z already has another thorn e(z,u'), then as before (and for the same reason), we discard one of u and u' whose rank is smaller.

This finishes the post-processing procedure on P. The running time is still O(m).

By similar analysis as in Lemma 9, Phase 0 still runs in $O(n \log n)$ time and we omit the details.

5.2.2 Phase 1

First of all, we still form a stem-tree T_c as before and each node represents a substem of length at most r. Instead of using the line arrangement searching technique, we now resort to MSEARCH. Let S be the set of all substems. Let \mathcal{M} be the set of matrices of all these substems formed in the way described in Section 5.1. We apply MSEARCH on \mathcal{M} with stopping count c = 0 and using DFTESTO. Since \mathcal{M} has O(n) arrays of lengths O(r), MSEARCH will produce $O(\log n)$ values for feasibility tests in $O(n \log r)$ time. The total feasibility test time is $O(n \log n)$. Since c = 0, after MSEARCH stops, we have an updated range (λ_1, λ_2) and no matrix element of \mathcal{M} is active. Let λ be an arbitrary value in (λ_1, λ_2) .

We will compute a data structure for each stem P of S so that the feasibility test can be made in sublinear time. We will show that after $O(n \log n)$ time preprocessing, each feasibility test can be done in $O(n/r \log^3 r)$ time. Let P be a substem with backbone vertices v_1, v_2, \ldots, v_m , with v_m as the top vertex. The preprocessing algorithm works in a similar way as before.

The cleanup procedure. Again, we first perform a cleanup procedure to remove all vertices of P that can be covered by the centers at the buds of the twigs. Note that all buds and twig vertices can be automatically covered by the centers at buds. So we only need to find those backbone and thorn vertices that can be covered by buds. This is easier than before because the locations of the centers are now fixed at buds.

We use the following algorithm to find all backbone and thorn vertices that can be covered by buds to their left sides. Let i' be the smallest index such that $v_{i'}$ has a bud $b_{i'}$. The algorithm maintains an index t. Initially, t = i'. For each i incrementally from i' to m, we do the following. If v_i has a bud b_i , then we reset t to i if $d(b_i, v_i) \leq d(b_t, v_i)$, because for any $j \geq i$ such that v_j (resp., u_j) is covered by b_t , it is also covered by b_i . Next, if $w(v_i) \cdot d(v_i, b_t) \leq \lambda$, then we mark v_i as "covered". If u_i exits and $w(u_i) \cdot d(u_i, b_t) \leq \lambda$, then we mark u_i as "covered". Note that although u_i is not an ancestor or a descendent of b_t , we can still compute $d(u_i, b_t)$ in O(1) time because $d(u_i, b_t) = d(u_i, v_i) + d(v_i, b_t)$, and the latter two distances can be computed in O(1) time due to the preprocessing in Section 4.1.

The above algorithm runs in O(m) time. In a symmetric way by scanning P from right to left, we can also mark all backbone and thorn vertices that are covered by buds to their right sides. This finishes the cleanup procedure. Again, we define V in the same way as before, and let v'_1, v'_2, \ldots, v'_t be the backbone vertices in V. Also, define V[i,j] in the same way as before.

Computing the data structure. In the sequel, we compute a data structure for P.

As before, we maintain an index a of a twig such that $d(v_1, b_a) \leq d(v_1, b_i)$ for any other twig index i, and maintain an index b of a twig such that $d(v_m, b_b) \leq d(v_m, b_i)$ for any other twig index i. Both a and b can be found in O(m) time.

For each $i \in [1, t]$, we will also compute an integer ncen(i) and a vertex v(i) of V, whose definitions are similar as before. In addition, we maintain another vertex q(i). The details are given below. Define j_i similarly as before, i.e., it is the smallest index in [i, t] such that it is not possible to cover all vertices in $V[i, j_i]$ by a center located at a vertex of P under λ . Again, if such an index does not exist in [i, t], then let $j_i = t + 1$.

If $j_i = t + 1$, then we define ncen(i) = 0, and define v(i) as the vertex in V[i,t] with the largest rank. We should point out our definition on v(i) is consistent with before, which was based on $rank'(l^+(v))$ for $v \in V$, because for any two vertex v and v' in V, rank(v) > rank(v') if and only if $rank'(l^+(v)) < rank'(l^+(v'))$. With the same analysis as before, v(i) dominates all other vertices of V[i,t]. We define q(i) as follows. If $w(v(i)) \cdot d(v(i),v_m) < \lambda$, then q(i) is undefined. Otherwise, q(i) is the largest index $j \in [i,t]$ such that $w(v(i)) \cdot d(v(i),v_j) \le \lambda$. Therefore, q(i) refers to the index of the backbone vertex closest to v_m that can cover v(i), and by the definition of v(i), q(i) can also cover all other vertices of V[i,t] (the proof is similar to Lemma 12 and we omit it).

If $j_i < t+1$, then define $ncen(i) = ncen(j_i) + 1$, $v(i) = v(j_i)$, and $q(i) = q(j_i)$.

To compute ncen(i), v(i), and q(i) for all $i \in [1, t]$, observe that once v(i) is known, q(i) can be computed in additional $O(\log m)$ time by binary search on the backbone vertices of P. Therefore, we will focus on computing ncen(i) and v(i). We use a similar algorithm as that in Lemma 13. To this end, we need to solve the following subproblem: Given any two indices $i \leq j$ in [1, t], we want

to compute the optimal objective value, denoted by $\alpha'(i,j)$, of the discrete one-center problem on the vertices of V[i,j]. This can be done in $O(\log^2 m)$ time by using the 2D sublist LP query data structure, as shown in the following lemma.

Lemma 17. With $O(m \log m)$ time preprocessing, we can compute $\alpha'(i, j)$ in $O(\log^2 m)$ time for any two indices $i \leq j$ in [1, t].

Proof. As preprocessing, in $O(t \log t)$ time we build the 2D sublist LP query data structure on the upper half-planes defined by the vertices of V in the same way as before.

Given any indices $i \leq j$ in [1,t], by a 2D sublist LP query, we compute in $O(\log^2 t)$ time the lowest point p in the common intersection C of the upper half-planes defined by the vertices of V[i,j]. Let q be the point on the backbone of P corresponding to the x-coordinate of p (i.e., $d(v_1,q)=x(p)$). Note that q is essentially the optimal center for the non-discrete one-center problem on the vertices of V[i,j]. But since we are considering the discrete case, the optimal center q' for the discrete problem can be found as follows. If q is located at a vertex of P, then q'=q and $\alpha'(i,j)$ is equal to the y-coordinate of p. Otherwise, let v and v' be the two backbone vertices of P immediately on the left and right sides of q, respectively. Then, q' is either v or v', and this is because the boundary of C, which is the upper envelope of the bounding lines of the upper half-planes defined by the vertices of V[i,j], is convex. To compute $\alpha'(i,j)$, we do the following. Let l_v be the line in \mathbb{R}^2 whose x-coordinate is equal to $d(v_1,v)$. Defined $l_{v'}$ similarly. Let p_v be the lowest point of l_v in C. Define $p_{v'}$ similarly. Then, $\alpha'(i,j)$ is equal to the y-coordinate of the lower point of p_v and $p_{v'}$, and the center q' can also be determined correspondingly. Both v and v' can be found by binary search on the backbone vertices of V in $O(\log m)$ time. The point p_v (resp., $p_{v'}$) can be found by a line-constrained 2D sublist LP query in $O(\log t)$ time.

Since $t \leq m$, we can compute $\alpha'(i,j)$ in $O(\log^2 m)$ time.

With the preceding lemma, we can use a similar algorithm as in Lemma 13 to compute ncen(i) and v(i) for all $i \in [1, t]$ in $O(m \log^2 m)$ time. Again, q(i) for all $i \in [1, t]$ can be computed in additional $O(m \log m)$ time by binary search.

Recall that $m \leq r$. Hence, the preprocessing time for P is $O(r \log^2 r)$, which is $O(r(\log \log n)^2)$ time since $r = \log^2 n$. The total time for computing the data structure for all substems of S is $O(n(\log \log n)^2)$. With these data structures, we show that a feasibility test can be done in $O(n/r \log^3 r)$ time.

The faster feasibility test DFTEST1. The algorithm is similar as before, and we only explain the differences by referring to the pseudocode given in Algorithm 4 for DFTEST1.

First of all, we can still implement Line 11 in $O(\log^2 r)$ time by exactly the same algorithm in Lemma 14. For the binary search in Line 28, since now we have a constraint that q must be at a backbone vertex, we need to modify the algorithm in Lemma 15, as follows.

We show that given any index $i \in [1, t]$, we can determine in $O(\log^2 r)$ time whether there is a center at a backbone vertex of P that can cover all vertices of V[1, i] with $d(v_1, q) \leq dem(P)$. We first compute the center q and its optimal objective value $\alpha'(1, i)$ for the discrete one-center problem on the vertices of V[1, i], which can be done in $O(\log^2 r)$ time as shown the proof of Lemma 17. If $\alpha'(1, i) > \lambda$, then the answer is no. Otherwise, if $d(v_1, q) \leq dem(P)$, then the answer is yes. If $d(v_1, q) > dem(P)$, then let j be the largest index of [1, m] such that $dem(P) \geq d(v_1, v_j)$, and j can be found in $O(\log r)$ time by binary search on the backbone vertices of P. Let l be the vertical line of \mathbb{R}^2 whose x-coordinate is equal to $d(v_1, v_j)$. By a line-constrained 2D sublist LP query, we

Algorithm 4: The faster feasibility test algorithm *DFTEST1*

```
Input: The stem-tree T_c, the original k from the input, and \lambda \in (\lambda_1, \lambda_2)
   Output: Determine whether \lambda is feasible
 1 count \leftarrow 0:
 2 for each stem P of the stem tree T_c do
    sup(P) \leftarrow \infty, dem(P) \leftarrow sup(P) - 1;
   for each stem P in the post-order traversal of T_c do
        for each child P' of P do
 5
         sup(P) = \min\{sup(P), sup(P')\}, dem(P) = \min\{dem(P), dem(P')\};
 6
        Increase count by the number of twigs of P;
 7
        Let V be the set of the uncovered vertices of P, and v'_1, v'_2, \ldots, v'_t are the backbone vertices of V;
 8
        if sup(P) \leq dem(P) then
 9
            Let q be the center in a child stem of P that gives the value sup(P);
10
            Do binary search to find the largest index i \in [1, t] such that q can cover all vertices of V[1, i];
11
12
                 sup(P) = min\{sup(P) + d(v_1, v_m), d(b_b, v_m)\};
                                                                          /* b_b is the bud maintained for P */
13
14
            else
                 i + +, count = count + ncen(i);
15
16
                 if w(v(i)) \cdot d(v(i), v_m) < \lambda then
                  dem(P) = \lambda/w(v(i)) - d(v(i), v_m), sup(P) = d(b_b, v_m);
17
18
                  count + +, sup(P) = \min\{d(q(i), v_m), d(b_b, v_m)\}, dem(P) = \infty;
19
20
        else
            if dem(P) \geq d(v_1, b_a) then
                                                                            /* b_a is the bud maintained for P */
21
                 count = count + ncen(1);
22
                 if w(v(1)) \cdot d(v(1), v_m) < \lambda then
23
                     dem(P) = \lambda/w(v(1)) - d(v(1), v_m), sup(P) = d(b_b, v_m);
\mathbf{24}
25
                     count + +, sup(P) = min\{d(q(1), v_m), d(b_b, v_m)\}, dem(P) = \infty;
26
            else
27
                 Do binary search to find the largest i \in [1, t] such that there exists a center q at the backbone
28
                  vertex of P to cover all vertices of V[1,i] with d(v_1,q) \leq dem(P);
                 if i = t then
29
                     if w(v(1)) \cdot d(v(1), v_m) < \lambda and dem(P) > d(v_1, v_m) then
30
                          dem(P) = \min\{dem(P) - d(v_1, v_m), \lambda/w(v(1)) - d(v(1), v_m)\}, sup(P) = d(b_b, v_m);
31
                     else
32
                          Do binary search to find the largest j \in [1, m] such that dem(P) \ge d(v_1, v_j);
33
                          count + +, \delta = \max\{d(v_j, v_m), d(q(1), v_m)\}, sup(P) = \min\{\delta, d(b_b, v_m)\}, dem(P) = \infty;
34
                 else
35
                     i + +, count = count + 1 + ncen(i);
36
                     if w(v(i)) \cdot d(v(i), v_m) < \lambda then
37
                          dem(P) = \lambda/w(v(i)) - d(v(i), v_m), sup(P) = d(b_b, v_m);
38
                     else
39
                       40
41 if sup(P_{\gamma}) > dem(P_{\gamma}) then
       count + +;
43 Return true if and only if count \leq k;
```

compute the lowest point p' on l in the common intersection of the upper half-planes defined by the vertices of V[1,i]. The answer is yes if and only if the y-coordinate of p' is at most λ . Hence, the time to determine the answer to the above question is $O(\log^2 r)$. Therefore, the time for implementing Line 28 is $O(\log^3 r)$.

In addition, it is easy to see that the time of the binary search in Line 33 is $O(\log r)$.

Therefore, processing P takes $O(\log^3 r)$ time, and the total time of DFTEST1 is $O(n/r \log^3 r)$, the same as before.

5.2.3 Phase 2

This phase is the similar as before with the following changes. First, we use DFTEST1 to replace FTEST1. Second, we use the new post-processing procedure. Third, instead of using the line arrangement searching technique, we use MSEARCH. Specifically, in the pseudocode of Algorithm 3, we replace Lines 19 and 20 (and also Lines 23 and 24) by the following. For each leaf-stem of S, we form the matrices for P in the way discussed in Section 5.1, and let $\mathcal M$ denote the set of matrices for all leaf-stems of S. Then, we call MSEARCH on $\mathcal M$ with stopping count c=0 and DFTEST1.

The running time of all three phases is still $O(n \log n)$, as shown in Theorem 2.

Theorem 2. The discrete k-center problem for T can be solved in $O(n \log n)$ time.

Proof. The analysis is similar to that in Theorem 1, we briefly discuss it below. Since Phase 0 and Phase 1 run in $O(n \log n)$ time, we only discuss Phase 2.

Again, the number of iterations of the while loop is $O(\log n)$. Hence, there are $O(\log n)$ calls to MSEARCH. Each call to MSEARCH produces $O(\log n)$ values for feasibility tests. Therefore, the total number of feasibility tests is $O(n/r \cdot \log^3 r \cdot \log^2 n) = O(n \log n)$. In each iteration, let n' denote the total number of backbone vertices of all leaf-stems. According to our discussion in Section 5.1, the call to MSEARCH takes $O(n' \log n')$ time (excluding the time for feasibility tests) since each matrix element of \mathcal{M} can be obtained in O(1) time. As the total sum of all such n' is O(n) in Phase 2, the overall time of MSEARCH in Phase 2 is $O(n \log n)$. Also, the overall time for the post-processing procedure in Phase 2 is O(n). Therefore, the total time of Phase 2 is $O(n \log n)$. This proves the theorem. \square

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