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A DEGREE SEQUENCE KOMLÓS THEOREM*

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Abstract. An important result of Komlós [Tiling Turán theorems, *Combinatorica*, 2000] yields the asymptotically exact minimum degree threshold that ensures a graph G contains an H-tiling covering an xth proportion of the vertices of G (for any fixed $x \in (0,1)$ and graph H). We give a degree sequence strengthening of this result which allows for a large proportion of the vertices in the host graph G to have degree substantially smaller than that required by Komlós's theorem. We also demonstrate that for certain graphs H, the degree sequence condition is essentially best possible in more than one sense.

Key words. graph tilings, regularity method, degree sequence

AMS subject classifications. 05C35, 05C70

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1. Introduction.

A central branch of extremal combinatorics concerns the study of conditions that force a graph G to contain some given substructure. For example, Turán's famous theorem determines the number of edges required to force a graph G to contain a copy of a fixed clique K_r on r vertices. Tutte's theorem characterizes all those graphs G that contain a perfect matching.

The study of *graph tilings* has proven to be a rich topic within this area: Given two graphs H and G, an *H-tiling* in G is a collection of vertex-disjoint copies of H in G. An H-tiling is called *perfect* if it covers all the vertices of G. Perfect H-tilings are also often referred to as *H-factors*, *perfect H-packings*, or *perfect H-matchings*. H-tilings can be viewed as generalizations of both the notion of a matching (which corresponds to the case when H is a single edge) and the Turán problem (i.e., a copy of H in G is simply an H-tiling of size one).

A cornerstone result in the area is the Hajnal–Szemerédi theorem [6] from 1970, which characterizes the minimum degree that ensures a graph contains a perfect K_r-tiling.

THEOREM 1.1 (Hajnal and Szemerédi [6]). *Every graph G whose order n is divisible by r and whose minimum degree satisfies $\delta(G) \geq (1-1/r)n$ contains a perfect K_r-tiling. Moreover, there are n-vertex graphs G with $\delta(G) = (1-1/r)n - 1$ that do not contain a perfect K_r-tiling.*

Although the minimum degree condition in the Hajnal–Szemerédi theorem is tight, this does not mean one cannot strengthen this result. Indeed, Kierstead and Kostochka [7] proved an *Ore-type* generalization of Theorem 1.1 where now one replaces the minimum degree condition with the condition that the sum of the degrees of every pair of nonadjacent vertices in G is at least $2(1-1/r)n - 1$. A conjecture of Balogh, Kostochka, and Treglown [3, Conjecture 7] would, if true, give a *degree*

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sequence strengthening of the Hajnal–Szemerédi theorem; in this conjecture one allows for G to have almost n/r vertices of degree less than $(1-1/r)n$. An asymptotic version of this conjecture was proven in [16].

There has also been significant interest in the minimum degree threshold that ensures a perfect H-tiling for an *arbitrary* graph H. After earlier work on this topic (see, e.g., [2, 9]), Kühn and Osthus [11, 12] determined, up to an additive constant, the minimum degree that forces a perfect H-tiling for *any* fixed graph H.

The focus of this paper is not on *perfect* H-tilings but rather on H-tilings covering an xth proportion of a graph G (for some fixed $x \in (0,1)$). The focal result on this topic is a theorem of Komlós [8] which determines asymptotically the minimum degree that ensures a graph G contains an H-tiling covering an xth proportion of its vertices. Before we can state this result, we require two definitions. The *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as

$$\chi_{cr}(H) := (\chi(H)-1)\frac{|H|}{|H|-\sigma(H)},$$

where $\sigma(H)$ denotes the size of the smallest possible color class in any $\chi(H)$-coloring of H. For all $x \in (0,1)$, define

$$g_H(x) := x\left(1-\frac{1}{\chi_{cr}(H)}\right)+(1-x)\left(1-\frac{1}{r-1}\right).$$

THEOREM 1.2 (Komlós [8]). *Suppose H is a graph of chromatic number r. Given any $\eta > 0$, there exists an $n_0 = n_0(\eta, x, H) \in \mathbb{N}$ such that if G is a graph on $n \geq n_0$ vertices and*

$$\delta(G) \geq g_H(x)n,$$

then G contains an H-tiling covering at least $(x-\eta)n$ vertices.

Note that the minimum degree condition in Theorem 1.2 is best possible in the sense that given any fixed H and $x \in (0,1)$, one cannot replace $g_H(x)$ with any fixed $g'_H(x) < g_H(x)$ (see [8, Theorem 7] for a proof of this). A consequence of the Erdős–Stone theorem is that every n-vertex graph G with $\delta(G) \geq (1-1/(\chi(H)-1)+o(1))n$ contains a copy of H. So a way to interpret Theorem 1.2 is that, for very small $x > 0$, the minimum degree threshold is governed essentially by the value of $\chi(H)-1$; however, as one increases x, the value of $\chi_{cr}(H)$ plays an increasing role in the value of the threshold.

An attractive consequence of Theorem 1.2 is the following result concerning *almost* perfect H-tilings.

THEOREM 1.3 (Komlós [8]). *Let $\eta > 0$, and let H be a graph. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that every graph G on $n \geq n_0$ vertices with*

$$\delta(G) \geq \left(1-\frac{1}{\chi_{cr}(H)}\right)n$$

contains an H-tiling covering all but at most ηn vertices.

As with Theorem 1.2, the minimum degree condition in Theorem 1.3 is best possible in the sense that one cannot replace the $(1-1/\chi_{cr}(H))$ term here with any smaller fixed constant. Despite this, Shoukoufandeh and Zhao [14] proved that one can strengthen the conclusion of the theorem to ensure the H-tiling covers all but a *constant* number of vertices (this constant depends only on H).

The main result of this paper is to prove the following degree sequence strengthening of Theorem 1.3.

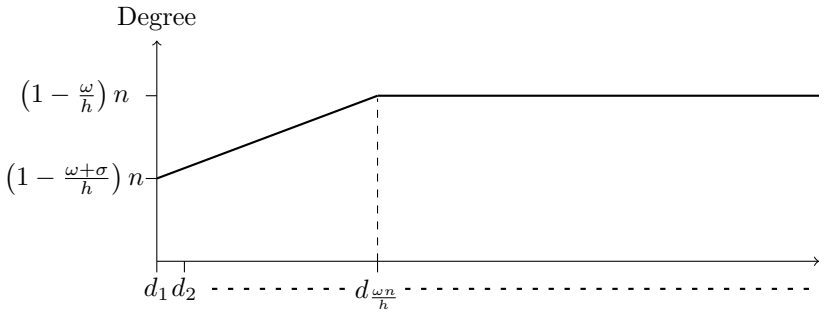

FIG. 1. *The degree sequence in Theorem* 1.4.

Graph	Bound on d_1	Bound on $d_{\frac{\omega n}{h}}$	Angle of slope
C_5	$2n/5$	$3n/5$	$1/2$
$K_{1,t}$	1	$n/(t+1)$	$1/t$
K_t	$(t-2)n/t$	$(t-1)n/t$	1
$K_{2,4,6}$	$5n/12$	$7n/12$	$2/5$

FIG. 2. *Values of the start points, endpoints, and angles of the slope in Theorem* 1.4 *for certain graphs.*

THEOREM 1.4. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$. Let $\sigma := \sigma(H)$, $h := |H|$, and $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega+\sigma}{h}\right) n + \frac{\sigma}{\omega} i \quad \text{for all} \quad 1 \leq i \leq \tfrac{\omega n}{h}.$$

Then G contains an H-tiling covering all but at most ηn vertices.

Note that if one considers an r-partition of H with smallest vertex class of size $\sigma = \sigma(H)$ and set $i = \omega n/h$, then we obtain that $(1-(\omega+\sigma)/h)\,n + \sigma i/\omega = 1 - 1/\chi_{cr}(H)$. Thus, Theorem 1.4 is a significant strengthening of Theorem 1.3. Indeed, Theorem 1.4 allows for up to $\omega n/h$ vertices to have degree below that in Theorem 1.3. In particular, when H is bipartite, the degree sequence condition in Theorem 1.4 starts at $d_1 \geq 1$ and allows for at least half of the vertices of H to have degree less than that required by Komlós's theorem. Figure 1 gives a visualization of the degree sequence in Theorem 1.4. Figure 2 presents some key properties of the degree sequence in Theorem 1.4 for several graphs. Here, "Angle of slope" refers to the value σ/ω.

The degree sequence in Theorem 1.4 is best possible in more than one sense for many graphs H. For all graphs H, one cannot allow significantly more than $\omega n/h$ vertices to have degree below the "Komlós threshold," so in this sense the bound on the number of "small degree" vertices in Theorem 1.4 is tight. Further, for many graphs H, we show that the degree sequence cannot start at a lower value and the angle of the "slope" in Figure 1 is best possible. This is discussed in more depth in section 3.

Theorem 1.4 deals with almost perfect tilings. A natural question now is whether such a degree sequence strengthening also exists for tilings covering an xth proportion of vertices, as in Theorem 1.2. Indeed, the following result is a straightforward consequence of Theorem 1.4.

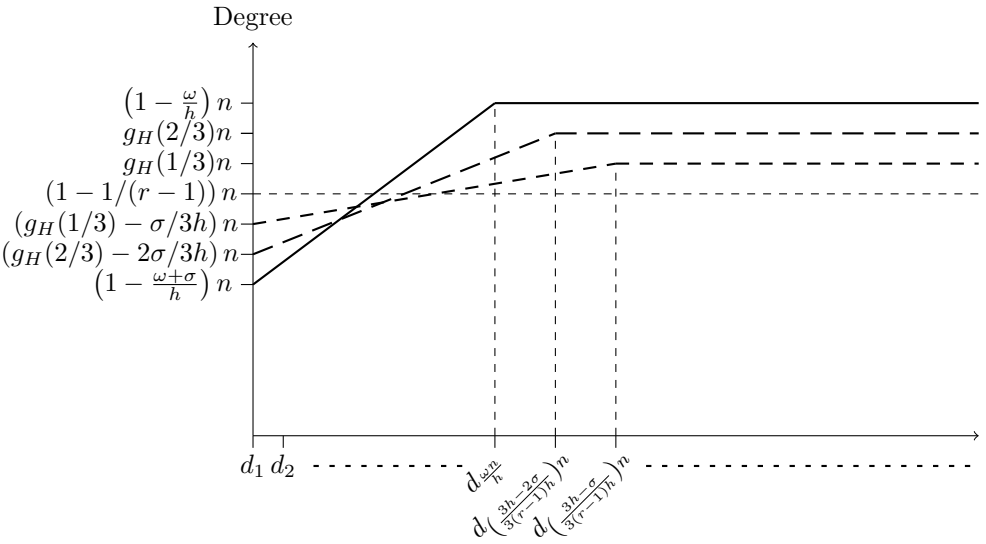

FIG. 3. *The degree sequence in Theorem* 1.5 *for* $x = 2/3$ *(long dashed),* $x = 1/3$ *(medium dashed).*

THEOREM 1.5. *Let* $x \in (0,1)$ *and* H *be a graph with* $\chi(H) = r$. *Set* $\eta > 0$. *Let* $\sigma := \sigma(H)$, $h := |H|$, *and* $\omega := (h-\sigma)/(r-1)$. *Then there exists an* $n_0 = n_0(\eta, x, H) \in \mathbb{N}$ *such that the following holds: Suppose* G *is a graph on* $n \geq n_0$ *vertices with degree sequence* $d_1 \leq d_2 \leq \cdots \leq d_n$ *such that*

$$d_i \geq \left(g_H(x) - \frac{x\sigma}{h}\right) n + \frac{(r-1)x\sigma}{h - x\sigma} i \quad \text{for all} \quad 1 \leq i \leq \left(\tfrac{h-x\sigma}{(r-1)h}\right) n.$$

Then G *contains an* H*-tiling covering at least* $(x-\eta)n$ *vertices.*

Theorem 1.5 is an improvement on Theorem 1.2. Indeed, Theorem 1.5 allows for almost $(h - x\sigma)n/(r-1)h$ vertices to have degree below $g_H(x)n$. Observe that as x approaches 0, the degree sequence condition in Theorem 1.5 tends toward the condition $\delta(G) \geq (1 - 1/(r-1))n$ and thus accords with the Erdős–Stone theorem (see Figure 3).

Piguet and Saumell [13, Theorem 1.3] recently proved another generalization of Theorem 1.2. In their result they only require a certain fraction of the vertices to satisfy the degree condition of Theorem 1.2, and all other vertices have *no* restriction on their degree (so some could even be isolated vertices). Note, though, that our result allows for more vertices to have small degree (i.e., smaller than the bound in Theorem 1.2) at the price of having some restriction of the degrees of these vertices. In the case of almost perfect H-tilings, Theorem 1.4 allows a large proportion of the vertices to have small degree, while in this case [13, Theorem 1.3] corresponds precisely to Komlós's theorem (Theorem 1.3 above).

As well as considering minimum degree and degree sequence conditions, it is also natural to seek conditions on the density of a graph G that forces an H-tiling covering a given fraction of the vertices of G. We remark, though, that only limited progress has been made on this question (though Allen et al. [1] did resolve this problem in the case of K_3-tilings).

Organization of the paper. The paper is organized as follows. In the next section we provide some essential notation and definitions. Then in section 3 we

give extremal examples for both Theorems 1.4 and 1.5. We introduce an "error term" version (Theorem 4.1) of Theorem 1.4 in section 4 and show that it implies Theorem 1.4. Szemerédi's regularity lemma and several auxiliary results are presented in section 5. Then in section 6 we provide the tools that we will need to prove Theorem 4.1. We prove a result that iteratively constructs an almost perfect H-tiling and then use this result to prove Theorem 4.1 in section 7. To conclude section 7 we show that Theorem 1.4 implies Theorem 1.5.

2. Notation and definitions. Let G be a graph. We define $V(G)$ to be the vertex set of G and $E(G)$ to be the edge set of G. Let $X \subseteq V(G)$. Then $G[X]$ is the *graph induced by X on G* and has vertex set X and edge set $E(G[X]) := \{xy \in E(G) : x, y \in X\}$. We also define $G \setminus X$ to be the graph with vertex set $V(G) \setminus X$ and edge set $E(G \setminus X) := \{xy \in E(G) : x, y \in V(G) \setminus X\}$. For each $x \in V(G)$, we define the *neighborhood of x in G* to be $N_G(x) := \{y \in V(G) : xy \in E(G)\}$ and define $d_G(x) := |N_G(x)|$. We drop the subscript G if it is clear from context which graph we are considering. We write $d_G(x, X)$ for the number of edges in G that x sends to vertices in X. Given a subgraph $G' \subseteq G$, we will write $d_G(x, G') := d_G(x, V(G'))$. Let $A, B \subseteq V(G)$ be disjoint. Then we define $e_G(A, B) := |\{xy \in E(G) : x \in A, y \in B\}|$.

Let $t \in \mathbb{N}$. We define the *blow-up* $G(t)$ to be the graph constructed by first replacing each vertex $x \in V(G)$ by a set V_x of t vertices and then replacing each edge $xy \in E(G)$ with the edges of the complete bipartite graph with vertex sets V_x and V_y.

Let $v \in \mathbb{N}$. We will refer to a vertex class of size v of G as a *v-class* of G. Set $r, \sigma, \omega \in \mathbb{N}$, and $\sigma < \omega$. We define the *r-partite bottle graph B with neck σ and width ω* to be the complete r-partite graph with one σ-class and $(r-1)$ ω-classes.

Let $i \in \mathbb{N}$ and $H_1, H_2, \dots, H_i$ be a collection of graphs. We define an *$(H_1, H_2, \dots, H_i)$-tiling in G* to be a collection of vertex-disjoint copies of graphs from the set $\{H_1, H_2, \dots, H_i\}$ in G. An $(H_1, H_2, \dots, H_i)$-tiling is called *perfect* if it covers all vertices in G.

We write $0 < a \ll b \ll c < 1$ to mean that we can choose the constants a, b, c from right to left. More precisely, there exist nondecreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that for all $a \le f(b)$ and $b \le g(c)$, our calculations and arguments in our proofs are correct. Larger hierarchies are defined similarly. Note that $a \ll b$ implies that we may assume, e.g., $a < b$ or $a < b^2$.

3. Extremal examples. In this section we present three extremal examples. The first demonstrates that the "slope" of the degree sequence in Theorem 1.4 is best possible for bottle graphs. The second shows that for many graphs H, the degree sequence in Theorem 1.4 "starts" at the correct place. The third shows that, for any graph H, to ensure an H-tiling covering at least $(x - \eta)n$ vertices we cannot have significantly more than $(h - x\sigma)n/(r-1)h$ vertices with degree below the "Komlós threshold" of $g_H(x)n$.

Extremal Example 1. Set $\eta \in \mathbb{R}$. Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. The following extremal example G on n vertices demonstrates that Theorem 1.4 is best possible for such graphs B in the sense that G satisfies the degree sequence of Theorem 1.4 except for a small linear part that only just fails the degree sequence but does not contain a B-tiling covering all but at most ηn vertices.

Proposition 3.1. *Set $\eta \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $0 < 1/n \ll \eta \ll 1$. Let B be a bottle graph with neck σ and width ω, where $b := |B|$. Additionally assume that b divides n. Then for any $1 \le k < \omega n/b - 2\eta n$, there exists a graph G on n vertices*

whose degree sequence $d_1 \le \cdots \le d_n$ *satisfies*

$$d_i \ge \left(1 - \frac{\omega+\sigma}{b}\right) n + \frac{\sigma}{\omega} i \quad \textit{for all} \quad i \in \{1, \dots, k-1, k+2\eta n+1, \dots, \omega n/b\},$$

$$d_i = \left(1 - \frac{\omega+\sigma}{b}\right) n + \left\lceil \frac{\sigma}{\omega} k \right\rceil \quad \textit{for all} \quad k \le i \le k + 2\eta n,$$

but such that G *does not contain a* B*-tiling covering all but at most* ηn *vertices.*

Proof. Let G be the graph on n vertices with r vertex classes $V_1, \dots, V_r$, where $|V_1| = \sigma n/b$ and $|V_2| = |V_3| = \cdots = |V_r| = \omega n/b$. Label the vertices of V_1 as $a_1, a_2, \dots, a_{\sigma n/b}$. Similarly, label the vertices of V_2 as $c_1, c_2, \dots, c_{\omega n/b}$. The edge set of G is constructed as follows.

First, let G have the following edges:

- All edges with an endpoint in V_1 and the other endpoint in $V(G) \setminus V_2$, in particular $G[V_1]$, is complete.
- All edges with an endpoint in V_i and the other endpoint in $V(G) \setminus (V_1 \cup V_i)$ for $2 \le i \le r$.
- Given any $1 \le i \le \omega n/b$ and $j \le \lceil \sigma i/\omega \rceil$ include all edges $c_i a_j$.

So at the moment G does satisfy the degree sequence in Theorem 1.4; we therefore modify G slightly. For all $k+1 \le i \le k+2\eta n$ and $\lceil \sigma k/\omega \rceil + 1 \le j \le \lceil \sigma(k+2\eta n)/\omega \rceil$ delete each edge between c_i and a_j (see Figure 4). One can easily check that G satisfies the degree sequence in the statement of the proposition. In particular, the vertices of degree $(1 - \frac{\omega+\sigma}{b})n + \lceil \frac{\sigma}{\omega} k \rceil$ are $c_k, \dots, c_{k+2\eta n}$.

Define $A := \{a_1, \dots, a_{\lceil \sigma k/\omega \rceil}\}$ and $C := \{c_1, \dots, c_{k+2\eta n}\}$. Note that there are no edges between C and $V_1 \setminus A$ in G.

Claim 3.2. *Let* T *be a* B*-tiling of* G*. Then* T *does not cover at least* $3\eta n/2$ *vertices in* C*.*

Consider any copy B' of B in G that contains an element of C. As C is an independent set in G, B' contains at most ω elements from C. Since there are no edges between C and $V_1 \setminus A$ in G, B' contains at least σ vertices in A. This implies that at most $\lceil \sigma k/\omega \rceil (\omega/\sigma) < k + \eta n/2$ vertices in C can be covered by T. Since $|C| = k + 2\eta n$, we have that T does not cover at least $3\eta n/2$ vertices in C. Therefore, Claim 3.2 holds. Hence, G does not have a B-tiling covering all but at most ηn vertices. ∎

Proposition 3.1 implies that for bottle graphs B, the degree sequence in Theorem 1.4 cannot be lowered significantly in a small part of the degree sequence and still ensure an almost perfect B-tiling, so the "slope" of the degree sequence in Theorem 1.4 cannot be improved on. It would be interesting to find other classes of graphs H for which the slope in Theorem 1.4 is also best possible; we suspect, though, that there are graphs H where the slope is not best possible.

Extremal Example 2. The next example shows that for many graphs H, Theorem 1.4 is best possible in the sense that we cannot start the degree sequence at a significantly lower value.

Proposition 3.3. *Let* H *be an* r*-partite graph so that, for every* $x \in V(H)$*,* $H[N(x)]$ *is* $(r-1)$*-partite. Let* $h := |H|$*,* $\sigma := \sigma(H)$*, and set* $\omega := (h-\sigma)/(r-1)$*. Additionally, suppose* $\sigma < \omega$*. Let* $0 < 1/n \ll \eta \ll (\omega - \sigma)/h$*, where* h *divides* n*. Then there is an* n*-vertex graph* G *with*

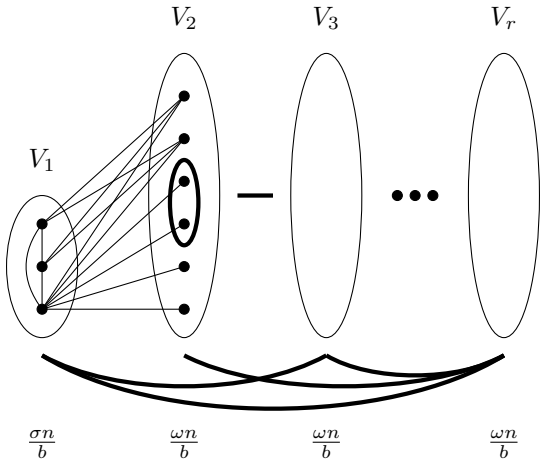

FIG. 4. *An example of a graph G in Proposition* 3.1*, where $\sigma = 1, \omega = 2$.*

(i) $\lfloor \eta n \rfloor + 1$ *vertices of degree* $(1 - \frac{\omega+\sigma}{h})n$*;*
(ii) *all other vertices have degree at least* $(1 - 1/\chi_{cr}(H))n = (1 - \omega/h)n$*,*
and G does not have an H-tiling covering all but at most ηn vertices.

Proof. Let G be the complete r-partite graph on n vertices with vertex classes $V_1, \dots, V_r$, where $|V_1| = \sigma n/h + \lfloor \eta n \rfloor + 1$, $|V_2| = \omega n/h - \lfloor \eta n \rfloor - 1$ and $|V_3| = \cdots = |V_r| = \omega n/h$. Let $V' \subseteq V_1$ be of size $\lfloor \eta n \rfloor + 1$. Delete from G all edges with one endpoint in V' and the other in V_2. By construction G satisfies (i) and (ii). Note that since the neighborhood of any $x \in V'$ induces an $(r-2)$-partite subgraph of G, no vertex in V' lies in a copy of H in G. So G does not have an H-tiling covering all but at most ηn vertices. ∎

EXTREMAL EXAMPLE 3. Set $\eta \in \mathbb{R}$ and $x \in (0, 1]$. Let H be a graph with $\chi(H) =: r$. Let $h := |H|$, $\sigma := \sigma(H)$, and set $\omega := (h - \sigma)/(r - 1)$. Define $g_H(1) := 1 - \omega/h$. We give an extremal example G on n vertices which satisfies the degree sequence of Theorem 1.5 except that $(h - x\sigma)n/(r-1)h + \eta n$ vertices have degree at most $(g_H(x) - \eta)n$ but does not contain an H-tiling covering at least $(x - \eta)n$ vertices.

PROPOSITION 3.4. *Set $\eta \in \mathbb{R}$ and $x \in (0, 1]$. Let H be a graph with $\chi(H) =: r$. Let $h := |H|$, $\sigma := \sigma(H)$, and set $\omega := (h - \sigma)/(r - 1)$. Then there exists a graph G on n vertices whose degree sequence $d_1 \le \cdots \le d_n$ satisfies*

$$d_i = (g_H(x) - \eta)\, n \quad \text{for all} \quad i \le \frac{h - x\sigma}{(r-1)h} n + \eta n,$$

$$d_i \ge g_H(x) n \quad \text{for all} \quad i > \frac{h - x\sigma}{(r-1)h} n + \eta n,$$

but such that G does not contain an H-tiling covering at least $(x - \eta)n$ vertices.

Proof. Let G be the complete r-partite graph on n vertices with vertex classes $V_1, \dots, V_r$ such that

- $|V_1| = \frac{x\sigma n}{h} - \eta n$;
- $|V_2| = \frac{(h - x\sigma)n}{(r-1)h} + \eta n$;
- $|V_3| = \cdots = |V_r| = \frac{(h - x\sigma)n}{(r-1)h}$.

Consider any H-tiling T of G. Observe that T can contain at most $xn/h - \eta n/\sigma$ copies of H. Indeed, to attain this bound one requires that all color classes of size σ in copies of H are placed into V_1. Hence, at most $x(r-1)\omega n/h - (r-1)\omega\eta n/\sigma$ vertices are covered by T in $V_2 \cup \cdots \cup V_r$. Thus, at most $(x-\eta)n - (r-1)\omega\eta n/\sigma$ vertices are covered by T. Hence, G does not contain an H-tiling covering at least $(x-\eta)n$ vertices. ∎

4. Deriving Theorem 1.4 from a weaker result. To prove Theorem 1.4 we will first prove the following "error term" version.

THEOREM 4.1. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$. Let $h := |H|$, $\sigma := \sigma(H)$, and set $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega+\sigma}{h}\right) n + \frac{\sigma}{\omega} i + \eta n \quad \text{for all} \quad 1 \leq i \leq \tfrac{\omega n}{h}.$$

Then G has an H-tiling covering all but at most ηn vertices.

Theorem 4.1 implies Theorem 1.4. Indeed, a simple argument (as in [8]) allows us to remove the error terms.

Proof of Theorem 1.4. Set $0 < \tau \ll \eta, 1/h$, and let $n \in \mathbb{N}$ be sufficiently large. Suppose G is an n-vertex graph as in the statement of Theorem 1.4. Let A be a set of τn vertices, and define G^* to be the graph with vertex set $V(G) \cup A$ and edge set $E(G^*) := E(G) \cup \{xy : x \in V(G) \cup A, y \in A, x \neq y\}$. Then G^* has degree sequence $d_{G^*,1} \leq d_{G^*,2} \leq \cdots \leq d_{G^*,(1+\tau)n}$, where

$$d_{G^*,i} \geq \left(1 - \frac{\omega+\sigma}{h}\right) n + \frac{\sigma}{\omega} i + \tau n \geq \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \left(\frac{\sigma}{\omega} i + \frac{\sigma\tau}{h} n\right) + \frac{\omega\tau}{h} n$$

for all $1 \leq i \leq \frac{\omega n}{h}$ and

$$d_{G^*,i} \geq \left(1 - \frac{\omega+\sigma}{h}\right)(1+\tau)n + \frac{\sigma}{\omega} i + \frac{\omega\tau}{2h}(1+\tau)n$$

for all $\frac{\omega n}{h} \leq i \leq \frac{\omega(1+\tau)n}{h}$. By Theorem 4.1 we have that G^* has an H-tiling T covering all but at most $\frac{\omega\tau}{2h}(1+\tau)n$ vertices.

Now, remove every copy of H from T that contains a vertex in A. Then we have removed at most $(h-1)\tau n$ vertices from $V(G) \subset V(G^*)$. Moreover, this implies that there exists an H-tiling in G covering all but at most $(h-1)\tau n + \frac{\omega\tau}{2h}(1+\tau)n$ vertices. Since $(h-1)\tau n + \frac{\omega\tau}{2h}(1+\tau)n < \eta n$, Theorem 1.4 holds. ∎

Outline of the proof of Theorem 4.1. The aim of the rest of the paper is to prove Theorem 4.1; we now outline the proof of this result. We first show that it suffices to prove Theorem 4.1 in the case when $H = B$, a bottle graph with neck σ and width ω (where $\sigma < \omega$). In particular, Theorem 4.1 is already known in the case when H is a balanced r-partite graph [16].

We then employ a variant of an idea of Komlós [8]. Roughly speaking, the idea is as follows: Let B^* be a suitably large blown-up copy of B. We apply the regularity lemma (Lemma 5.2) to obtain a reduced graph R of G. If R contains an almost perfect B^*-tiling, then one can rather straightforwardly conclude that G contains

an almost perfect B-tiling, as required (for this we apply Lemma 6.1). Otherwise, suppose that the largest B^*-tiling in R covers precisely $d \leq (1-o(1))|R|$ vertices. We then show that, for some $t \in \mathbb{N}$, there is a B^*-tiling in the blow-up $R(t)$ of R covering substantially more than dt vertices. Thus, crucially, the largest B^*-tiling in $R(t)$ covers a higher proportion of vertices than the largest B^*-tiling in R. By repeating this argument, we obtain a blow-up R' of R that contains an almost perfect B^*-tiling. We then show that this implies G contains an almost perfect B-tiling, as desired.

Other applications of this general method have been used in the past [4, 5, 16]. Note, however, our approach has different challenges. Indeed, the process of moving from a B^*-tiling $\mathcal{B}$ in R to a proportionally larger B^*-tiling in $R(t)$ is rather subtle. In particular, what we would like to do is conclude that one can find a tiling $\mathcal{B}_0$ (not necessarily of copies of B^*) in R that covers a larger proportion of the vertices in R *and* that when one takes a suitable blow-up $R(t)$ of R, then $\mathcal{B}_0$ corresponds to a B^*-tiling in $R(t)$. However, the vertices in R that are uncovered by $\mathcal{B}$ could perhaps all be "small degree" vertices (i.e., they do not have degree as large as that in Theorem 1.3). This is a barrier to finding such a special tiling $\mathcal{B}_0$. (Intuitively, one can imagine that if one has large degree vertices outside of $\mathcal{B}$, then one can glue such vertices onto $\mathcal{B}$ in such a way as to obtain our desired tiling $\mathcal{B}_0$.) In this case, one has to (through perhaps many steps) modify $\mathcal{B}$ and then blow-up R to obtain an intermediate blow-up $R(t')$ of R such that (i) there is a B^*-tiling $\mathcal{B}'$ in $R(t')$ that covers the same proportion of vertices compared to the tiling $\mathcal{B}$ in R and (ii) many of the vertices in $R(t')$ uncovered by $\mathcal{B}'$ are now such that they can be "glued" onto $\mathcal{B}'$ to obtain our desired larger tiling $\mathcal{B}_0$.

Despite these technicalities the proof of Theorem 4.1 is perhaps surprisingly short. The main work of the proof is encoded in Lemma 7.1, which ensures one can modify the tiling $\mathcal{B}$ as above.

5. Szemerédi's regularity lemma and auxiliary results. A key tool in the proof of Theorem 4.1 is Szemerédi's regularity lemma [15]. To state this lemma we will need the following notion of ε-regularity.

DEFINITION 5.1. Let $G = (A, B)$ be a bipartite graph with vertex classes A and B. We define the *density* of G to be

$$d_G(A,B) := \frac{e_G(A,B)}{|A||B|}.$$

Set $\varepsilon > 0$. We say that G is *ε-regular* if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$ we have that $|d_G(X,Y) - d_G(A,B)| < \varepsilon$.

LEMMA 5.2. (degree form of Szemerédi's regularity lemma). *Let $\varepsilon \in (0,1)$ and $M' \in \mathbb{N}$. Then there exist natural numbers M and n_0 such that for any graph G on $n \geq n_0$ vertices and any $d \in (0,1)$, there is a partition of the vertices of G into subsets $V_0, V_1, \ldots, V_k$ and a spanning subgraph G' of G such that the following hold:*

- *$M' \leq k \leq M$;*
- *$|V_0| \leq \varepsilon n$;*
- *$|V_1| = \cdots = |V_k| =: q$;*
- *$d_{G'}(x) > d_G(x) - (d+\varepsilon)n$ for all $x \in V(G)$;*
- *$e(G'[V_i]) = 0$ for all $i \geq 1$;*
- *for all $1 \leq i, j \leq k$ with $i \neq j$, the pair $(V_i, V_j)_{G'}$ is ε-regular and has density either 0 or at least d.*

We call $V_1, \ldots, V_k$ the *clusters* of our partition, V_0 the exceptional set, and G' the *pure graph*. We define the *reduced graph* R of G with parameters ε, d, and M' to

be the graph whose vertex set is $V_1, \dots, V_k$ and in which V_iV_j is an edge if and only if $(V_i, V_j)_{G'}$ is ε-regular with density at least d. Note also that $|R| = k$.

The proof of the next result is analogous to that of [16, Lemma 5.2]. It states that the degree sequence of G in Theorem 4.1 is "inherited" by its reduced graph R.

LEMMA 5.3. *Set $M', n_0 \in \mathbb{N}$, and $\varepsilon, d, \eta, b, \omega, \sigma$ to be positive constants such that $1/n_0 \ll 1/M' \ll \varepsilon \ll d \ll \eta \ll 1/b$ and where $\omega + \sigma \leq b$. Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \frac{b-\omega-\sigma}{b}n + \frac{\sigma}{\omega}i + \eta n \quad \text{for all } 1 \leq i \leq \tfrac{\omega n}{b}. \tag{5.1}$$

Let R be the reduced graph of G with parameters ε, d, and M', and set $k := |R|$. Then R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$ such that

$$d_{R,i} \geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \quad \text{for all } 1 \leq i \leq \tfrac{\omega k}{b}. \tag{5.2}$$

Proof. Let $V_1, \dots, V_k$ be the clusters of G and V_0 the exceptional set, and let G' be the pure graph of G. Set $q := |V_1| = \cdots = |V_k|$. Clearly we may assume $d_R(V_1) \leq d_R(V_2) \leq \cdots \leq d_R(V_k)$. Now consider any $i \leq \frac{\omega k}{b}$. Set $S := \cup_{1 \leq j \leq i} V_j$. Then $|S| = qi \leq \frac{\omega q k}{b} \leq \frac{\omega n}{b}$. Thus, by (5.1) there exists a vertex $x \in S$ such that $d_G(x) \geq d_{qi} \geq \frac{b-\omega-\sigma}{b}n + (\frac{\sigma}{\omega})qi + \eta n$. Suppose that $x \in V_j$, where $1 \leq j \leq i$. Since we have that $kq \leq n$, Lemma 5.2 implies that

$$\begin{aligned} d_R(V_j) \geq \frac{d_{G'}(x) - |V_0|}{q} &\geq \frac{1}{q}\left(\frac{b-\omega-\sigma}{b}n + \left(\frac{\sigma}{\omega}\right)qi + \eta n - (d+2\varepsilon)n\right) \\ &\geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2}. \end{aligned}$$

Since $d_{R,i} = d_R(V_i) \geq d_R(V_j)$, we have that (5.2) holds. ∎

We will also apply the following well-known fact.

FACT 5.4. *Let $0 < \varepsilon < \alpha$ and $\varepsilon' := \max\{\varepsilon/\alpha, 2\varepsilon\}$. Let (A, B) be an ε-regular pair of density d. Suppose $A' \subseteq A$ and $B' \subseteq B$, where $|A'| \geq \alpha|A|$ and $|B'| \geq \alpha|B|$. Then (A', B') is an ε'-regular pair with density d', where $|d' - d| < \varepsilon$.*

LEMMA 5.5 (key lemma [10]). *Suppose that $0 < \varepsilon < d$, that $q, t \in \mathbb{N}$, and that R is a graph where $V(R) = \{v_1, \dots, v_k\}$. We construct a graph G as follows: Replace every vertex $v_i \in V(R)$ by a set V_i of q vertices, and replace each edge of R by an ε-regular pair of density at least d. For each $v_i \in V(R)$, let U_i denote the set of t vertices in $R(t)$ corresponding to v_i. Let H be a subgraph of $R(t)$ with maximum degree Δ, and set $h := |H|$. Set $\delta := d - \varepsilon$ and $\varepsilon_0 := \delta^{\Delta}/(2+\Delta)$. If $\varepsilon \leq \varepsilon_0$ and $t - 1 \leq \varepsilon_0 q$, then there are at least $(\varepsilon_0 q)^h$ labeled copies of H in G so that if $x \in V(H)$ lies in U_i, then x is embedded into V_i in G.*

6. Tools for proving Theorem 4.1. In this section we provide further tools that we will need to prove Theorem 4.1. The following lemma is a special case of Lemma 11 in [8] (which in turn is easily implied by the key lemma above).

LEMMA 6.1. *Set $0 < \beta < 1/2$, and let B be the bottle graph with neck σ and width ω. Set $d \in (0, 1)$. Then there exists an $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$, the following holds for all $q \in \mathbb{N}$: Let G be a graph constructed from B by replacing every vertex of B by q vertices and replacing the edges of B with ε-regular pairs of density at least d. Then G has a B-tiling covering all but at most a β-proportion of the vertices in G.*

Given a bottle graph B, the next lemma ensures that various blown-up copies of graphs contain perfect B-tilings.

LEMMA 6.2. *Set $m \in \mathbb{N}$. Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$ and $\sigma < \omega$. Define B' to be the r-partite bottle graph with neck σ and width $\omega - 1$, and let $B^* := B(m)$. Define $t := (\omega - \sigma)b$. Then $B(mt)$, $B^*(mt)$, $B'(mt)$, and $K_r(mt)$ all have perfect B^*-tilings.*

Proof. Clearly, $B(mt)$ and $B^*(mt)$ both have perfect B^*-tilings. It remains to show that $B'(mt)$ and $K_r(mt)$ have perfect B^*-tilings.

For $K_r(mt)$, tile $(\omega - \sigma)r$ copies of B^* into $K_r(mt)$ such that their (σm)-classes are distributed evenly amongst the r vertex classes of $K_r(mt)$. Indeed, we can view this as tiling $(\omega - \sigma)$ collections of r copies of B^* into $K_r(mt)$ such that for each collection C, each vertex class of $K_r(mt)$ contains the (σm)-class of precisely one copy of B^* in C.

For $B'(mt)$, first tile $(\omega - 1 - \sigma)b$ vertex-disjoint copies of B^* into $B'(mt)$ such that each (σm)-class is placed into the (σmt)-class in $B'(mt)$. So our current B^*-tiling covers all but $\sigma mt - \sigma m(\omega - 1 - \sigma)b = \sigma mb$ vertices in the (σmt)-class in $B'(mt)$ and all but $(\omega - 1)mt - \omega m(\omega - 1 - \sigma)b = \sigma mb$ vertices in each (ωmt)-class in $B'(mt)$. Then the remaining vertices to be covered in $B'(mt)$ form a $K_r(\sigma mb)$ which can be tiled with σr copies of B^*. ∎

The next result states that the degree sequence of G in Theorem 4.1 is inherited by any blown-up copy of G.

PROPOSITION 6.3. *Let $n, s \in \mathbb{N}$ and $b, \omega, \sigma > 0$ such that $\omega n > b$ and $\omega + \sigma \leq b$. Set $\eta > 0$. Suppose G is a graph on n vertices with degree sequence $d_{G,1} \leq d_{G,2} \leq \cdots \leq d_{G,n}$ such that*

$$d_{G,i} \geq \frac{b - \omega - \sigma}{b} n + \frac{\sigma}{\omega} i + \eta n \quad \text{for all } 1 \leq i \leq \tfrac{\omega n}{b}.$$

Then $\bar{G} := G(s)$ has degree sequence $d_{\bar{G},1} \leq d_{\bar{G},2} \leq \cdots \leq d_{\bar{G},ns}$ such that

$$d_{\bar{G},i} \geq \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega} i + \left(\eta n - \frac{\sigma}{\omega}\right) s \quad \text{for all } 1 \leq i \leq \tfrac{\omega ns}{b}.$$

Proof. For any $1 \leq j \leq ns$ we see that

$$d_{\bar{G},j} = s \cdot d_{G,\lceil j/s \rceil}.$$

Suppose that $j \leq \frac{\omega ns}{b} - s$. Then $\lceil j/s \rceil \leq \frac{\omega n}{b}$, and we have

$$d_{\bar{G},j} \geq \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega} \lceil j/s \rceil s + \eta ns \geq \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega} j + \eta ns.$$

In particular, if we take any $i \leq \frac{\omega ns}{b}$, we have

$$d_{\bar{G},i} \geq \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega}(i - s) + \eta ns = \frac{b - \omega - \sigma}{b} ns + \frac{\sigma}{\omega} i + \left(\eta n - \frac{\sigma}{\omega}\right). \qquad ∎$$

The following result acts as a springboard from which to begin the proof of Lemma 7.1.

PROPOSITION 6.4. *Set $\eta > 0$ and $m \in \mathbb{N}$, and let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. Define $B^* := B(m)$. Then there exists $n_0 \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, where*

$$d_i \geq \frac{b-\omega-\sigma}{b}n + \frac{\sigma}{\omega}i + \eta n \quad \text{for all } 1 \leq i \leq \tfrac{\omega n}{b}.$$

Then there exists a copy of B^* in G.

Proof. Set $\Delta := \Delta(B^*)$. Let n be sufficiently large, and define constants $\varepsilon, d > 0$ and $M' \in \mathbb{N}$ such that $0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll 1/b, \eta, 1/\Delta$. Let G be an n-vertex graph as in the statement of the proposition. Applying Lemma 5.2 with parameters ε, d, and M' to G, we obtain clusters $V_1, \ldots, V_k$, an exceptional set V_0, and a pure graph G'. Set $q := |V_1| = \cdots = |V_k|$. Let R be the reduced graph of G with parameters ε, d, and M', where $k := |R|$. By Lemma 5.3 we have that R has degree sequence $d_{R,1} \leq d_{R,2} \leq \cdots \leq d_{R,k}$, where

$$d_{R,i} \geq \frac{b-\omega-\sigma}{b}k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \quad \text{for all} \quad 1 \leq i \leq \frac{\omega k}{b}. \tag{6.1}$$

By doing the following steps, we find a set $\{x_1, \ldots, x_r\} \subseteq V(R)$ such that $\{x_1, \ldots, x_r\}$ induces a copy of K_r in R:

Step 1: Choose a vertex $x_1 \in V(R)$ such that

$$d_R(x_1) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}.$$

Such a vertex exists by (6.1).

Step i for each $i \in \{2, \ldots, r-1\}$: We have that $\{x_1, x_2, \ldots, x_{i-1}\}$ induces a copy of K_{i-1} in R and

$$d_R(x_1), d_R(x_2), \ldots, d_R(x_{i-1}) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}.$$

Let $N_R(x_1, x_2, \ldots, x_{i-1}) := N_R(x_1) \cap N_R(x_2) \cap \cdots \cap N_R(x_{i-1})$. Then

$$\begin{aligned} |N_R(x_1, x_2, \ldots, x_{i-1})| &\geq k - \frac{(i-1)\omega}{b}k + \frac{(i-1)\eta k}{3} \\ &\geq \frac{b-(r-2)\omega}{b}k + \frac{(i-1)\eta k}{3} = \frac{\omega+\sigma}{b}k + \frac{(i-1)\eta k}{3}. \end{aligned}$$

Here the last equality follows as $b = \sigma + (r-1)\omega$. Hence, by (6.1) there exists $y \in N_R(x_1, x_2, \ldots, x_{i-1})$ such that $d_R(y) \geq k - \frac{\omega}{b}k + \frac{\eta k}{3}$. Let $x_i := y$.

Step r: We have that $\{x_1, x_2, \ldots, x_{r-1}\}$ induces a copy of K_{r-1} in R. Moreover,

$$|N_R(x_1, x_2, \ldots, x_{r-1})| \geq \frac{\sigma}{b}k + \frac{(r-1)\eta k}{3}.$$

Choose x_r to be any vertex in $N_R(x_1, x_2, \ldots, x_{r-1})$.

Therefore, there exists a copy of K_r in R, which implies that there exists a copy of B^* in $R(\omega m)$. By Lemma 5.5 we have that there exists a copy of B^* in G. ∎

A crucial tool in the proof of Theorem 4.1 is Lemma 7.1 below. Before stating this lemma, we need two more definitions.

DEFINITION 6.5. Set $\ell \in \mathbb{N}$. Let G be a graph on n vertices, and B be a bottle graph with neck σ and width ω. Suppose that there exists a B-tiling T of G, and let $\{z_1, \dots, z_\ell\} \subseteq V(G) \setminus V(T)$. We say that $\{z_1, \dots, z_\ell\}$ is an *expanding set of size ℓ for T in G* if the following is true: There exists an injection $f : \{z_1, \dots, z_\ell\} \to T$ such that z_i has a neighbor in every ω-vertex class of $f(z_i)$ for each $1 \le i \le \ell$.

DEFINITION 6.6. Set $k, \ell, m \in \mathbb{N}$. Let G be a graph on n vertices, and let $(v_1, v_2, \dots, v_n)$ be an ordering of the vertices of G. Let B be a bottle graph with neck σm and width ωm. Suppose that there exists a B-tiling T of G, and let $\{z_1, \dots, z_\ell\} \subseteq V(G) \setminus V(T)$. Denote by Ω_T the set of all vertices in $V(G)$ that belong to ωm-classes of copies of B in T. Let $z \in \{z_1, \dots, z_\ell\}$ and $y \in \Omega_T$, and denote by B_y the copy of B in T that contains y. Then there exist $1 \le i, j \le n$ such that $z := v_i$, $y := v_j$, and $i \ne j$. We say that (z, y) is a *k-swapping pair with respect to* $(v_1, \dots, v_n)$ if the following is true: z is adjacent to at least σ vertices in the σm-class of B_y, z, is adjacent to at least ω vertices in each ωm-class of B_y that does not contain y, and $j \ge i + k$. We say that $\{z_1, \dots, z_\ell\}$ is a *k-swapping set of size ℓ for T in G with respect to* $(v_1, \dots, v_n)$ if there exists a set of ℓ vertices $\{y_1, \dots, y_\ell\} \subseteq \Omega_T$ such that (z_i, y_i) is a k-swapping pair with respect to $(v_1, \dots, v_n)$ for each $1 \le i \le \ell$ and $B_{y_p} \ne B_{y_q}$ for all $p \ne q$.

Suppose $\mathcal{B}$ is a B-tiling in a reduced graph R. Very roughly speaking, the purpose of expanding sets is to extend $\mathcal{B}$ to a larger tiling, while swapping sets allows us to "rotate" which vertices are uncovered by our tiling (which helps for future expansion of $\mathcal{B}$ to a larger tiling).

7. Almost perfect H-tilings in graphs.

LEMMA 7.1. *Let B be an r-partite bottle graph with neck σ and width ω, where $b := |B|$. Set $\eta, \gamma > 0$ and $n, m \in \mathbb{N}$ such that $0 < 1/n \ll \gamma \ll 1/m \ll \eta \ll 1/b$. Set $B^* := B(m)$. Let G be a graph on n vertices with degree sequence $d_1 \le d_2 \le \cdots \le d_n$, where*

$$d_i \ge \frac{b - \omega - \sigma}{b} n + \frac{\sigma}{\omega} i + \eta n \quad \text{for all } 1 \le i \le \tfrac{\omega n}{b}.$$

Let $V(G) = \{v_1, \dots, v_n\}$ such that $d_G(v_i) = d_i$ for all $1 \le i \le n$. Suppose the largest B^*-tiling in G covers precisely $n' \le (1 - \eta)n$ vertices. Then for any B^*-tiling T covering n' vertices in G there exists an expanding set of size γn for T in G or an $\frac{\omega\gamma n}{\sigma}$-swapping set of size γn for T in G with respect to $(v_1, \dots, v_n)$.

Proof. By repeatedly applying Proposition 6.4, we see that $n' \ge \eta n/2$. Define a bijection $I : V(G) \to [n]$, where $I(x) = i$ implies that $d_G(x) = d_i$. Let $V(G) := \{v_1, \dots, v_n\}$ such that $I(v_i) = i$. Set $n'' := n - n'$, and let $G'' := G \setminus V(T)$. Let $V(G'') = \{x_1, \dots, x_{n''}\}$, where $I(x_1) < I(x_2) < \cdots < I(x_{n''})$. For each $1 \le i \le n''$, set $s_i := I(x_i)$. Then $d_G(x_i) = d_{s_i}$. Choose j to be the largest integer such that

$$d_G(x_j) \le \frac{b - \omega}{b} n + (\eta - 2\gamma)n.$$

Notice that $s_j \le \omega n/b$. We will refer to $x_1, \dots, x_j$ as *small* vertices and $x_{j+1}, \dots, x_{n''}$ as *big* vertices.

Case 1: Suppose we have γn big vertices $z_1, \dots, z_{\gamma n} \in V(G'')$ such that

$$d_G(z_i, G'') \le \frac{b - \omega}{b} n'' + \frac{\eta n}{4} \quad \text{for all } \ 1 \le i \le \gamma n. \tag{7.1}$$

Then

$$d_G(z_i, T) \geq \frac{b-\omega}{b} n' + \frac{\eta n}{4} \quad \text{for all} \quad 1 \leq i \leq \gamma n.$$

Set $\omega^* := \omega m$. For each $1 \leq i \leq \gamma n$, we see that z_i can be adjacent to at most a $\frac{b-\omega}{b}$-proportion of the vertices in T without having a neighbor in each ω^*-class of some copy of B^* in T. Since $\gamma \ll 1/m \ll \eta \ll 1/b$, for each $1 \leq i \leq \gamma n$ there are at least

$$\frac{\left(\frac{b-\omega}{b} n' + \frac{\eta n}{4}\right) - \left(\frac{b-\omega}{b} n'\right)}{\omega^*} = \frac{\eta n}{4\omega^*} \geq \gamma n$$

copies of B^* in T that have at least one neighbor of z_i in each of their ω^*-classes. Thus, we can define an injection $f : \{z_1, \ldots, z_{\gamma n}\} \to T$ such that z_i has a neighbor in each ω^*-class of $f(z_i)$ for each $1 \leq i \leq \gamma n$. Hence, $\{z_1, \ldots, z_{\gamma n}\}$ is an expanding set of size γn for T in G.

Case 2: We may assume there does not exist an expanding set of size γn for T in G.

In particular, there are at most $\gamma n - 1$ vertices in $V(G'')$ that have a neighbor in every ω^*-class of γn copies of B^* in T. (Note that these could be small or big vertices.) Remove such vertices from $V(G'')$, and call the remaining graph G_1''. In particular, no big vertex in G_1'' satisfies (7.1). Set $n_1'' := |G_1''|$.

Subcase A: Suppose we have γn small vertices $x_{i_1}, \ldots, x_{i_{\gamma n}} \in V(G_1'')$ such that

$$d_G(x_{i_\ell}, G_1'') \leq \frac{b-\omega-\sigma}{b} n_1'' + \frac{\sigma}{\omega} i_\ell + 2\gamma n \quad \text{for all} \quad 1 \leq \ell \leq \gamma n. \tag{7.2}$$

Then by (7.2) and the degree sequence condition of the lemma, we have

$$d_G(x_{i_\ell}, T) \geq \frac{b-\omega-\sigma}{b} n' + \left(\frac{\sigma}{\omega} s_{i_\ell} - \frac{\sigma}{\omega} i_\ell\right) + \frac{\eta n}{2} \quad \text{for all} \quad 1 \leq \ell \leq \gamma n. \tag{7.3}$$

Let $k \in \{1, \ldots, \gamma n\}$. Denote by Ω_T^* the set of all vertices in G that belong to ω^*-classes of copies of B^* in T. Set $\sigma^* := \sigma m$. We aim to count the number of vertices $y \in \Omega_T^*$ such that (x_{i_k}, y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair (with respect to $(v_1, \ldots, v_n)$). Let T_1 denote the subcollection of copies B_1 of B^* in T such that x_{i_k} is adjacent to a vertex in every ω^*-class of B_1. Then since we removed earlier all vertices that have a neighbor in every ω^*-class of γn copies of B^* in T, we have

$$d_G(x_{i_k}, T_1) \leq (\gamma n - 1) bm.$$

Suppose $y \in \Omega_T^*$, and let B_y^* be the copy of B^* in T containing y. We say y is *swappable* with x_{i_k} if x_{i_k} is adjacent to at least σ vertices in the σ^*-class of B_y^* and at least ω vertices in each ω^*-class of B_y^* that does not contain y. Denote the set of vertices that are swappable with x_{i_k} by $S(x_{i_k})$. Let T_2 denote the subcollection of copies B_2 of B^* in $T \setminus T_1$ such that B_2 does not contain any vertex in $S(x_{i_k})$. Then

$$d_G(x_{i_k}, T_2) \leq (bm - \omega m - \sigma m + \sigma - 1)|T_2|.$$

Note that the $-\omega m$ term is present since x_{i_k} cannot be adjacent to a vertex in every ω^*-class of any copy of B^* in $T \setminus T_1$. Let $T_3 := T \setminus (T_1 \cup T_2)$. Then

$$d_G(x_{i_k}, T_3) \leq (bm - \omega m)|T_3|.$$

Observe that $|T_1| + |T_2| + |T_3| = n'/bm$. Then

$$\begin{aligned} d_G(x_{i_k}, T) &= d_G(x_{i_k}, T_1) + d_G(x_{i_k}, T_2 \cup T_3) \\ &\leq (\gamma n - 1)bm + \left(\frac{b-\omega-\sigma}{b}n' + \frac{(\sigma-1)}{bm}n'\right) + \sigma m|T_3|. \end{aligned} \tag{7.4}$$

Using (7.3) and (7.4) we see that

$$|T_3| \geq \frac{s_{i_k} - i_k}{\omega m} + \frac{\eta n}{2\sigma m} - \frac{(\gamma n - 1)b}{\sigma} - \frac{(\sigma-1)n'}{b\sigma m^2} \geq \frac{s_{i_k} - i_k}{\omega m} + \frac{\eta n}{8\sigma m},$$

where the the last inequality follows as $\gamma \ll 1/m \ll \eta \ll 1/b$.

Note that as $T_3 \cap T_2 = \emptyset$, every copy B_3 of B^* in T_3 must contain a vertex from $S(x_{i_k})$. By definition of swappable, this in fact implies that every copy B_3 of B^* in T_3 must contain ω^* vertices from $S(x_{i_k})$. Hence, there are at least $s_{i_k} - i_k + \frac{\omega\eta n}{8\sigma}$ vertices in $S(x_{i_k})$. Not all vertices in $S(x_{i_k})$ may form an $\omega\gamma n/\sigma$-swapping pair with x_{i_k} (with respect to $(v_1, \ldots, v_n)$). Indeed, there are at most $s_{i_k} - i_k + \frac{\omega\gamma n}{\sigma}$ vertices $y \in S(x_{i_k})$ with $I(y) < s_{i_k} + \frac{\omega\gamma n}{\sigma}$ (and so do not form an $\omega\gamma n/\sigma$-swapping pair with x_{i_k}). Hence, since $\gamma \ll 1/m, \eta, 1/b$, there are at least

$$\frac{\omega\eta n}{16\sigma} \geq bm\gamma n$$

vertices $y \in \Omega_T^*$ such that (x_{i_k}, y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair. Therefore, since $k \in \{1, \ldots, \gamma n\}$ was arbitrary, for each $\ell \in \{1, \ldots, \gamma n\}$ there exist at least $bm\gamma n$ vertices $y \in \Omega_T^*$ such that (x_{i_ℓ}, y) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair. Hence, there exists a set of vertices $\{y_1, \ldots, y_{\gamma n}\} \subseteq \Omega_T^*$ such that (x_{i_ℓ}, y_ℓ) is an $\frac{\omega\gamma n}{\sigma}$-swapping pair for each $1 \leq \ell \leq \gamma n$ and $B^*_{y_i} \neq B^*_{y_j}$ for all $i \neq j$. Thus, $\{x_{i_1}, \ldots, x_{i_{\gamma n}}\}$ is an $\frac{\omega\gamma n}{\sigma}$-swapping set of size γn for T in G.

Subcase B: Assume there does not exist an $\frac{\omega\gamma n}{\sigma}$-swapping set of size γn for T in G.

Then there are at most $\gamma n - 1$ small vertices $x \in V(G_1'')$ that satisfy (7.2). Remove such vertices from $V(G_1'')$, call the remaining graph G_2'', and set $n_2'' := |G_2''|$. Then for every small vertex $x_i \in V(G_2'')$ we have

$$d_{G_2''}(x_i) \geq \frac{b-\omega-\sigma}{b}n_2'' + \frac{\sigma}{\omega}i + \gamma n_2''.$$

For every big vertex $y \in V(G_2'')$, recall that y does not satisfy (7.1). So since $|G'' \setminus G_2''| \leq 2\gamma n$, we have

$$d_{G_2''}(y) \geq \frac{b-\omega}{b}n_2'' + \gamma n_2''.$$

Thus, G_2'' has degree sequence $d_{G_2'',1} \leq d_{G_2'',2} \leq \cdots \leq d_{G_2'',n_2''}$ such that

$$d_{G_2'',i} \geq \frac{b-\omega-\sigma}{b}n_2'' + \frac{\sigma}{\omega}i + \gamma n_2'' \text{ for all } 1 \leq i \leq \frac{\omega n_2''}{b}.$$

Hence, by Proposition 6.4 there exists a copy of B^* in G_2'', contradicting that the largest B^*-tiling in G covers n' vertices.

Thus, Lemma 7.1 holds. ∎

With Lemma 7.1 at hand we now can prove Theorem 4.1.

Proof of Theorem 4.1. If $\sigma = \omega$, then Theorem 4.1 is equivalent to the (nondirected) graph version of [16, Theorem 4.2].

So we may assume that $\sigma < \omega$. Set $\sigma' := (r-1)\sigma$ and $\omega' := (r-1)\omega$. Let B be the r-partite bottle graph with neck σ' and width ω', set $b := |B|$, and observe that B has a perfect H-tiling. Let $t := (\omega' - \sigma')b$. Note that it suffices to prove the theorem under the additional assumption that $\eta \ll 1/b$. Define additional constants $\varepsilon, d, \gamma \in \mathbb{R}$ and $M', m \in \mathbb{N}$ such that

$$0 < 1/n \ll 1/M' \ll \varepsilon \ll d \ll \gamma \ll 1/m \ll \eta \ll 1/b.$$

Let $B^* := B(m)$, and set

$$S := \frac{2\sigma}{\omega\gamma^2},\ Q := \lceil 1/\gamma \rceil \ \text{ and } \ z := Q(S+1).$$

Note that B^* has a perfect H-tiling.

Suppose G is an n-vertex graph as in the statement of the theorem. Apply Lemma 5.2 with parameters ε, d, and M' to G. This gives us clusters $V_1, \dots, V_k$, an exceptional set V_0, and a pure graph G', where $|V_0| \le \varepsilon n$ and $|V_1| = \cdots = |V_k| =: q$. Let R be the reduced graph of G with parameters ε, d and M'; thus, $k = |R|$. By Lemma 5.3, R has degree sequence $d_{R,1} \le d_{R,2} \le \cdots \le d_{R,k}$ such that

$$\begin{aligned} d_{R,i} &\ge \left(1 - \frac{\omega+\sigma}{h}\right)k + \frac{\sigma}{\omega}i + \frac{\eta k}{2} \\ &= \left(1 - \frac{\omega'+\sigma'}{b}\right)k + \frac{\sigma'}{\omega'}i + \frac{\eta k}{2} \ \text{ for all } 1 \le i \le \frac{\omega k}{h} = \frac{\omega' k}{b}. \end{aligned}$$

In what follows, when we consider an s-swapping set in some blow-up $R(w)$ of R, we always implicitly mean an s-swapping set in $R(w)$ with respect to $(v_1, \dots, v_{kw})$, where $V(R(w)) = \{v_1, \dots, v_{kw}\}$ and $d_{R(w)}(v_1) \le d_{R(w)}(v_2) \le \cdots \le d_{R(w)}(v_{kw})$. That is, each blow-up $R(w)$ of R comes equipped with an ordering of its vertices based on the degrees; these orderings are defined by the functions I_j below.

Claim 7.2. *$R' := R((mt)^z)$ contains a B^*-tiling $\mathcal{T}$ covering at least $(1-\eta/2)k(mt)^z = (1-\eta/2)|R'|$ vertices.*

Proof of Claim 7.2. If R contains a B^*-tiling covering at least $(1-\eta/2)k$ vertices, then Lemma 6.2 implies that Claim 7.2 holds. Suppose then that the largest B^*-tiling T in R covers exactly c vertices, where $c < (1-\eta/2)k$. Then by Lemma 7.1, there exists an expanding set of size γk for T in R or an $\frac{\omega\gamma k}{\sigma}$-swapping set of size γk for T in R. Define B' to be the r-partite bottle graph with neck σ' and width $\omega' - 1$. Set $\omega^* := \omega' m$.

Step 1: *Find a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$.*

Case 1: There exists an expanding set $\{z_1, \dots, z_{\gamma k}\}$ for T and hence also an associated injection $f : \{z_1, \dots, z_{\gamma k}\} \to T$.

In this case we do the following: For each $1 \le i \le \gamma k$, separate $R[z_i \cup f(z_i)]$ into a copy of K_r (containing z_i and one vertex from each ω^*-class of $f(z_i)$), a copy of B', and a copy of $B(m-1)$. Then we have a $(B^*, B(m-1), B', K_r)$-tiling in R covering at least $c + \gamma k$ vertices. By Lemma 6.2, $R(mt)$ contains a B^*-tiling covering at least $(c+\gamma k)mt$ vertices. Further applying Lemma 6.2 we obtain a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$, as desired.

Case 2: There does not exist an expanding set of size γk for T in R.

For each $1 \leq j \leq S$, Proposition 6.3 implies that $R((mt)^j)$ has degree sequence $d_{R((mt)^j),1} \leq d_{R((mt)^j),2} \leq \cdots \leq d_{R((mt)^j),k(mt)^j}$ such that

$$d_{R((mt)^j),i} \geq \left(1 - \frac{\omega+\sigma}{h}\right) k(mt)^j + \frac{\sigma}{\omega} i + \left(\frac{\eta k}{2} - \frac{\sigma}{\omega}\right)(mt)^j \quad \text{for all } 1 \leq i \leq \tfrac{\omega k(mt)^j}{h}.$$

Define for $0 \leq j \leq S$ bijections $I_j : V(R((mt)^j)) \to [k(mt)^j]$, where $I_j(x) := i$ implies that $d_{R((mt)^j)}(x) = d_{R((mt)^j),i}$. In particular, suppose that $x \in V(R)$, and let $x_1, \ldots, x_{(mt)^j}$ denote the $(mt)^j$ vertices in $R((mt)^j)$ that correspond to x. Suppose that $I_0(x) = i$. Then we may assume that

$$I_j(x_s) = (i-1)(mt)^j + s > (I_0(x) - 1)(mt)^j \text{ for each } 1 \leq s \leq (mt)^j. \tag{7.5}$$

To put all this another way, one can view I_0 as an ordering of the vertices in R in terms of the vertex degrees; I_j is the ordering of $R((mt)^j)$ "inherited" from the ordering I_0.

Note that for all $0 \leq j \leq S$,

$$\left(\sum_{x \in V(R((mt)^j))} I_j(x)\right) \leq k^2 (mt)^{2j}. \tag{7.6}$$

Denote by Ω^*_T the set of all vertices in $V(R)$ that belong to ω^*-classes of copies of B^* in T. As there does not exist an expanding set of size γk for T in R, then there exists an $\frac{\omega\gamma k}{\sigma}$-swapping set $\{z_1, \ldots, z_{\gamma k}\}$ for T in R. Hence, there also exists a set $\{y_1, \ldots, y_{\gamma k}\} \subseteq \Omega^*_T$ such that (z_i, y_i) is an $\frac{\omega\gamma k}{\sigma}$-swapping pair for each $1 \leq i \leq \gamma k$ such that $B^*_{y_i} \neq B^*_{y_j}$ [1] for all $i \neq j$ and such that $I_0(y_i) \geq I_0(z_i) + \frac{\omega\gamma k}{\sigma}$ for all $1 \leq i \leq \gamma k$.

For each $1 \leq i \leq \gamma k$, note that $R[(z_i \cup V(B^*_{y_i})) \setminus \{y_i\}]$ can be separated into a copy of B containing z_i and a copy of $B(m-1)$. Then we have a $(B^*, B(m-1), B)$-tiling T_1 covering c vertices in R. Further, since each (z_i, y_i) is an $\frac{\omega\gamma k}{\sigma}$-swapping pair, we have that

$$\left(\sum_{x \in V(R) \setminus V(T_1)} I_0(x)\right) \geq \left(\sum_{x \in V(R) \setminus V(T)} I_0(x)\right) + \frac{\omega\gamma^2 k^2}{\sigma}. \tag{7.7}$$

By Lemma 6.2, $T_1(mt)$ contains a perfect B^*-tiling T'; i.e., T' is a B^*-tiling covering $c(mt)$ vertices in $R(mt)$. Observe that T' in $R(mt)$ covers proportionally the same amount of vertices as T in R. Further, (7.5) and (7.7) imply that

$$\begin{aligned} \sum_{x \in V(R(mt)) \setminus V(T')} I_1(x) &\geq \left(\sum_{x \in V(R) \setminus V(T_1)} (I_0(x) - 1)\right)(mt)^2 \\ &\geq \left(\left(\sum_{x \in V(R) \setminus V(T)} I_0(x)\right) + \frac{\omega(\gamma k)^2}{2\sigma}\right)(mt)^2. \end{aligned} \tag{7.8}$$

Denote by $\Omega^*_{T'}$ the set of all vertices in $R(mt)$ that belong to ω^*-classes of copies of B^* in T'. Suppose that there does not exist an expanding set of size γkmt for T' in

[1] As in Definition 6.6.

$R(mt)$. Then by Lemma 7.1 there must exist an $\frac{\omega\gamma kmt}{\sigma}$-swapping set of size γkmt for T' in $R(mt)$. As before we can produce a $(B^*, B(m-1), B)$-tiling T_1' covering $c(mt)$ vertices in $R(mt)$. Then by Lemma 6.2, $T_1'(mt)$ contains a perfect B^*-tiling T''; i.e., T'' is a B^*-tiling covering $c(mt)^2$ vertices in $R((mt)^2)$. Observe, similarly as before, that T'' in $R((mt)^2)$ covers proportionally the same amount of vertices as T in R and

$$\sum_{x\in V(R((mt)^2))\setminus V(T'')} I_2(x) \geq \left(\left(\sum_{x\in V(R(mt))\setminus V(T')} I_1(x)\right) + \frac{\omega(\gamma kmt)^2}{2\sigma}\right)(mt)^2$$
$$\overset{(7.8)}{\geq} \left(\left(\sum_{x\in V(R)\setminus V(T)} I_0(x)\right) + \frac{\omega(\gamma k)^2}{\sigma}\right)(mt)^4.$$

Note that (7.6) implies that one can repeat this argument at most S times; that is, for some $j \leq S$ we must obtain an expanding set of size $\gamma k(mt)^j$ in $R((mt)^j)$. More precisely, we obtain a B^*-tiling $T^{(j)}$ in $R((mt)^j)$ covering $c(mt)^j$ vertices such that there exists an expanding set of size $\gamma k(mt)^j$ for $T^{(j)}$ in $R((mt)^j)$. Then as before, one can use this expanding set and Lemma 6.2 to obtain a B^*-tiling covering at least $(c+\gamma k)(mt)^{S+1}$ vertices in $R((mt)^{S+1})$, as desired.

General step: Repeating the whole argument from Step 1 at most Q times we see that $R((mt)^{Q(S+1)}) = R((mt)^z) = R'$ has a B^*-tiling $\mathcal{T}$ covering at least $(1-\eta/2)|R'|$ vertices. Thus, Claim 7.2 holds. ∎

Now for each $1 \leq i \leq k$, partition V_i into classes $V_i^*, V_{i,1}, \ldots, V_{i,(mt)^z}$, where $q' := |V_{i,j}| = \lfloor q/(mt)^z \rfloor \geq q/(2(mt)^z)$ for all $1 \leq j \leq (mt)^z$. Lemma 5.2 implies that $qk \geq (1-\varepsilon)n$; therefore,

$$q'|R'| = \lfloor q/(mt)^z \rfloor k(mt)^z \geq qk - k(mt)^z \geq (1-2\varepsilon)n. \tag{7.9}$$

Fact 5.4 tells us that for each ε-regular pair $(V_{i_1}, V_{i_2})_{G'}$ with density at least d we have that $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $2\varepsilon(mt)^z$-regular with density at least $d-\varepsilon \geq d/2$ (for all $1 \leq j_1, j_2 \leq (mt)^z$). Note that $2\varepsilon(mt)^z \leq \varepsilon^{1/2}$. So we can label the vertex set of R' so that $V(R') = \{V_{i,j} : 1 \leq i \leq k, 1 \leq j \leq (mt)^z\}$ and see that if $V_{i_1,j_1}V_{i_2,j_2} \in E(R')$, then $(V_{i_1,j_1}, V_{i_2,j_2})_{G'}$ is $\varepsilon^{1/2}$-regular with density at least $d/2$.

We know by Claim 7.2 that R' has a B^*-tiling $\mathcal{T}$ that covers at least $(1-\eta/2)|R'|$ vertices. Let $\hat{B}^*$ be a copy of B^* in $\mathcal{T}$, and label the vertices of $\hat{B}^*$ so that $V(\hat{B}^*) = \{V_{i_1,j_1}, V_{i_2,j_2}, \ldots, V_{i_{bm},j_{bm}}\}$. Set $V' := V_{i_1,j_1} \cup V_{i_2,j_2} \cup \cdots \cup V_{i_{bm},j_{bm}}$. Applying Lemma 6.1 with $\eta^2, q', d/2, \varepsilon^{1/2}$ playing the roles of β, q, d, ε, we have that $G'[V']$ has a B^*-tiling covering at least $(1-\eta^2)q'bm$ vertices. Applying Lemma 6.1 in this way to each copy of B^* in $\mathcal{T}$ we see that $G' \subseteq G$ has a B^*-tiling covering at least

$$\left(\left(1-\eta^2\right)q'bm\right) \times \left(\left(1-\eta/2\right)|R'|\right)/bm \overset{(7.9)}{\geq} \left(1-\eta^2\right)\left(1-\eta/2\right)\left(1-2\varepsilon\right)n \geq (1-\eta)n$$

vertices. Since each copy of B^* has a perfect H-tiling, G contains an H-tiling covering all but at most ηn vertices. ∎

Theorem 1.4 easily implies Theorem 1.5.

Proof of Theorem 1.5. Let H, $x \in (0,1)$, and $\eta > 0$ be as in the statement of the theorem. Suppose n is sufficiently large, and let G be an n-vertex graph as in the statement of the theorem.

Note that it suffices to prove the result in the case when $x \in (0,1) \cap \mathbb{Q}$. Thus, there exist $a, b \in \mathbb{N}$ such that $x = a/b$. Define $\sigma_1 := a(r-1)\sigma$ and $\omega_1 := a(r-1)\omega + (b-a)h = bh - a\sigma$. Let H_1 be the r-partite bottle graph with neck σ_1 and width ω_1, and observe that $\sigma_1 < \omega_1$ and $|H_1| = b(r-1)h$.

Claim. H_1 contains an H-tiling covering $x|H_1|$ vertices.

The claim follows since one can tile H_1 with $a(r-1)$ copies of H, where each σ-class lies in the σ_1-class of H_1. Thus, we have an H-tiling covering $a(r-1)h = x|H_1|$ vertices in H_1, as desired.

Note that

$$d_i \geq \left(g_H(x) - \frac{x\sigma}{h}\right) n + \frac{(r-1)x\sigma}{h - x\sigma} i = \left(1 - \frac{\omega_1 + \sigma_1}{b(r-1)h}\right) n + \frac{\sigma_1}{\omega_1} i$$

for all $i \leq (\frac{h - x\sigma}{(r-1)h})n = \frac{\omega_1 n}{b(r-1)h}$. Thus, applying Theorem 1.4 with H_1 playing the role of H, we produce an H_1-tiling in G covering all but at most ηn vertices. Then the claim implies that we have an H-tiling in G covering at least $x(1-\eta)n > (x-\eta)n$ vertices. ∎

8. Concluding remarks. In this paper we have given a particular degree sequence condition that forces a graph to contain an almost perfect H-tiling (Theorem 1.4). In fact, in general for a fixed graph H, Theorem 1.4 yields a whole class of degree sequences that force an almost perfect H-tiling. Indeed, we have the following consequence of Theorem 1.4.

THEOREM 8.1. *Let $\eta > 0$ and H be a graph with $\chi(H) = r$ and $h := |H|$. Set $\sigma \in \mathbb{R}$ such that $\sigma(H) \leq \sigma \leq h/r$ and $\omega := (h - \sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, \sigma, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \geq n_0$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_i \geq \left(1 - \frac{\omega + \sigma}{h}\right) n + \frac{\sigma}{\omega} i \quad \text{for all} \quad 1 \leq i \leq \tfrac{\omega n}{h}.$$

Then G contains an H-tiling covering all but at most ηn vertices.

Proof. Note that it suffices to prove the theorem under the assumption that $\sigma \in \mathbb{Q}$. To prove Theorem 8.1, we define a certain bottle graph H^* and then apply Theorem 1.4 with input H^* to conclude our result.

Since $\sigma \in \mathbb{Q}$, there exist $a, b \in \mathbb{N}$ such that $\sigma = a/b$. Let $\omega(H) := (h - \sigma(H))/(r-1)$ and $t := b(r-1)(\omega(H) - \sigma(H))$. We define H^* to be the r-partite bottle graph with neck σt and width ωt (note $\sigma t, \omega t \in \mathbb{N}$). Also, notice that $|H^*| = ht$.

Claim. H^* contains a perfect H-tiling.

We tile t copies of H into H^*. First, tile $b(r-1)(\omega(H) - \sigma)$ copies of H into H^* such that the $\sigma(H)$-classes are all placed in the σt-class of H^*. This leaves

$$\sigma b(r-1)(\omega(H) - \sigma(H)) - \sigma(H)b(r-1)(\omega(H) - \sigma) = \omega(H)b(r-1)(\sigma - \sigma(H))$$

vertices in the σt-class of H^* to be covered and

$$\begin{aligned}
&\omega b(r-1)(\omega(H) - \sigma(H)) - \omega(H)b(r-1)(\omega(H) - \sigma) \\
&\quad = b((r-1)\omega(\omega(H) - \sigma(H)) - (r-1)\omega(H)(\omega(H) - \sigma)) \\
&\quad = b((h-\sigma)(\omega(H) - \sigma(H)) - (h - \sigma(H))(\omega(H) - \sigma)) \\
&\quad = b(h - \omega(H))(\sigma - \sigma(H))
\end{aligned}$$

vertices in each ωt-class of H^* to be covered. Let $\overline{H}$ be the r-partite complete graph with one vertex class of size $(r-1)\omega(H)$ and $(r-1)$ vertex classes of size $(r-2)\omega(H)+\sigma(H)$. Observe that $\overline{H}$ has a perfect H-tiling (using $r-1$ copies of H). To cover the remaining vertices of H^*, tile $b(\sigma-\sigma(H))$ copies of $\overline{H}$ into H^* such that every vertex class of size $(r-1)\omega(H)$ is placed in the σt class of H^*. Observe that

$$((r-2)\omega(H)+\sigma(H))b(\sigma-\sigma(H)) = b(h-\omega(H))(\sigma-\sigma(H)).$$

Hence, H^* contains a perfect H-tiling, and the claim holds.

Suppose G is as in the statement of Theorem 8.1. Applying Theorem 1.4 with input G and H^*, we obtain that G contains an H^*-tiling covering all but at most ηn vertices. (Note the degree sequence in Theorem 8.1 is precisely the degree sequence of Theorem 1.4 with input H^*.) Since each copy of H^* has a perfect H-tiling, we conclude that G contains an H-tiling covering all but at most ηn vertices. ∎

In a similar way, Theorem 1.5 yields a class of degree sequences forcing an almost x-proportional H-tiling in G.

THEOREM 8.2. *Let $x \in (0,1)$ and H be a graph with $\chi(H) = r$ and $h := |H|$. Set $\eta > 0$. Let $\sigma \in \mathbb{R}$ such that $\sigma(H) \le \sigma \le h/r$ and $\omega := (h-\sigma)/(r-1)$. Then there exists an $n_0 = n_0(\eta, x, \sigma, H) \in \mathbb{N}$ such that the following holds: Suppose G is a graph on $n \ge n_0$ vertices with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ such that*

$$d_i \ge \left(g_H(x) - \frac{x\sigma}{h}\right)n + \frac{(r-1)x\sigma}{h-x\sigma}i \quad \text{for all} \quad 1 \le i \le \left(\tfrac{h-x\sigma}{(r-1)h}\right)n.$$

Then G contains an H-tiling covering at least $(x-\eta)n$ vertices.

Proof. Define H^* as in the proof of Theorem 8.1. Applying Theorem 1.5 with input H^*, we obtain that G contains an H^*-tiling covering all but at most $(x-\eta)n$ vertices. Since each copy of H^* has a perfect H-tiling, we conclude that G contains an H-tiling covering all but at most $(x-\eta)n$ vertices. ∎

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