# Nearly Work-Efficient Parallel Algorithm for Digraph Reachability 

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#### Abstract

One of the simplest problems on directed graphs is that of identifying the set of vertices reachable from a designated source vertex. This problem can be solved easily sequentially by performing a graph search, but efficient parallel algorithms have eluded researchers for decades. For sparse high-diameter graphs in particular, there is no known work-efficient parallel algorithm with nontrivial parallelism. This amounts to one of the most fundamental open questions in parallel graph algorithms: Is there a parallel algorithm for digraph reachability with nearly linear work? This paper shows that the answer is yes.

This paper presents a randomized parallel algorithm for digraph reachability and related problems with expected work $\tilde{O}(m)$ and span $\tilde{O}\left(n^{2 / 3}\right)$, and hence parallelism $\tilde{\Omega}\left(m / n^{2 / 3}\right)=$ $\tilde{\Omega}\left(n^{1 / 3}\right)$, on any graph with $n$ vertices and $m$ arcs. This is the first parallel algorithm having both nearly linear work and strongly sublinear span, i.e., span $\tilde{O}\left(n^{1-\epsilon}\right)$ for any constant $\epsilon>0$. The algorithm can be extended to produce a directed spanning tree, determine whether the graph is acyclic, topologically sort the strongly connected components of the graph, or produce a directed ear decomposition, all with work $\tilde{O}(m)$ and $\operatorname{span} \tilde{O}\left(n^{2 / 3}\right)$.

The main technical contribution is an efficient Monte Carlo algorithm that, through the addition of $\tilde{O}(n)$ shortcuts, reduces the diameter of the graph to $\tilde{O}\left(n^{2 / 3}\right)$ with high probability. While both sequential and parallel algorithms are known with those combinatorial properties, even the sequential algorithms are not efficient, having sequential runtime $\Omega\left(m n^{\Omega(1)}\right)$. This paper presents a surprisingly simple sequential algorithm that achieves the stated diameter reduction and runs in $\tilde{O}(m)$ time. Parallelizing that algorithm yields the main result, but doing so involves overcoming several other challenges.


## 1 Introduction

There are essentially no good parallel algorithms known for the most basic problems on general directed graphs, especially when the graph is sparse. This paper yields several.

A good parallel algorithm should have polynomial parallelism and be (nearly) work efficient. The work $W(n)$ of a parallel algorithm on a size- $n$ problem is the total number of primitive operations performed. Ideally, the work of the parallel algorithm should be similar to the best sequential running time $T^{*}(n)$ known for the problem. An algorithm is work efficient if $W(n) \in$ $O\left(T^{*}(n)\right)$ and nearly work efficient if $W(n) \in \tilde{O}\left(T^{*}(n)\right)=O\left(T^{*}(n) \cdot \operatorname{poly}(\log n)\right)$, where $\tilde{O}$ hides logarithmic factors 11 (As a slight abuse of notation, $\tilde{O}(1)$ is used to mean $O(\operatorname{poly}(\log n))$, where the $n$ should be clear from context. ${ }^{2}$ The span $S(n)$, also called depth, of a parallel algorithm is the length of the longest chain of sequential dependencies $3^{3}$ By Brent's scheduling principle [1], such an algorithm can generally be scheduled to run in $O(W(n) / p)$ time on $p \leq W(n) / S(n)$ processors; adding more processors beyond that point does not yield asymptotic speedup. The limit $W(n) / S(n)$ is called the parallelism of the algorithm; an algorithm is moderately parallel if the parallelism is $\Omega\left(n^{\epsilon}\right)$, for some constant $\epsilon>0$, and highly parallel if the span is $\tilde{O}(1)$. The goal is to achieve speedup with respect to the best sequential algorithm, which is why work efficiency matters. A nearly work-efficient algorithm runs in $\tilde{O}\left(T^{*}(n) / p\right)$ time on $p \leq W(n) / S(n)$ processors, but inefficient algorithms may require enormous numbers of processors to beat the sequential algorithm.

Remark. Aside from the context provided in this introduction and high-level ideas, most of the paper does not require any specific knowledge of parallel algorithms; the challenge lies in producing an algorithm with properties amenable to parallelization. Most implementation details are straightforward, so the parallel model and implementation details are deferred to Section 5,

Problem and history. Perhaps the most basic problem on directed graphs is the single-source reachability problem: given a directed graph $G=(V, E)$ and source vertex $s \in V$, identify the set of vertices reachable by a directed path originating from $s$. Throughout, let $n=|V|$ be the number of vertices and $m=|E|$ be the number of arcs, and for conciseness of bounds assume that $m \in \Omega(n)$. This problem has simple sequential solutions: both breadth-first search (BFS) and depth-first search (DFS) solve the problem in $O(m)$ time. There are two natural parallel algorithms for the reachability problem, which seem to be folklore. See Table $\mathbb{1}$ for a comparison. Parallel transitive closure [9, which amounts to repeated squaring of the adjacency matrix, is highly parallel but far from work efficient even for dense graphs. Parallel BFS is similar to sequential BFS, except that arcs from each layer (vertices with the same distance) are explored in parallel. Parallel BFS is work efficient (see, e.g., [14]), but the span is proportional to the diameter, which is $\Theta(n)$ in the worst case. Both algorithms fall short of our goals, but they remain the state of the art.

The only other progress on general graphs are work/span tradeoffs. Ullman and Yannakakis 20] raised the question over 25 years ago of whether it is possible to solve digraph reachability with sublinear work without sacrificing work efficiency. Instead, their algorithm [20], henceforth termed

[^0]|  | Work | Span | Nearly <br> work efficient? | Number of processors to <br> achieve $\tilde{O}(n / k)$ runtime, <br> for $m \in \Theta(n)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| parallel BFS | $O(m)$ | $\tilde{O}(n)$ | Yes | Not Possible unless $k=\tilde{O}(1)$ |  |
| parallel Trans. Closure | $\tilde{O}(M(n))$ | $\tilde{O}(1)$ | No | $k M(n) / n \gg n k$ for $k \leq n$ |  |
| Spencer's [18] | $\tilde{O}\left(m+n \rho^{2}\right)$ | $\tilde{O}(n / \rho)$ | if $\rho=\tilde{O}(\sqrt{m / n})$ | $k^{3}$ | for $k \leq n$ |
| UY [20] |  | $\tilde{O}\left(m \rho+\rho^{4} / n\right)$ | $\tilde{O}(n / \rho)$ | if $\rho=\tilde{O}(1)$ | $k^{2}$ |
| This paper* | $\tilde{O}(m)$ | $\tilde{O}\left(n^{2 / 3}\right)$ | Yes | $k$ | for $k \leq n^{2 / 3} \dagger$ |

Table 1: Comparison of parallel algorithms for single-source reachability. Two of the algorithms are parameterized by $\rho, 1 \leq \rho \leq n$, which trades off work and span. $M(n)$ is the work of the best highly parallel $n \times n$ matrix multiplication, which is at least the current best sequential time of $O\left(n^{2.372869}\right)$ [15].
*: the algorithm is randomized. Bounds are with high probability.
${ }^{\dagger}$ : for higher $k$, the dependence on $k$ becomes worse and more complicated to state.

UY, and Spencer's algorithm [18] exhibit tradeoffs between work and span. Though not originally described in the same terms, both algorithms can be parameterized by a value $\rho, 1 \leq \rho \leq n$. Table 1 summarizes the performance bounds 4 For $\rho=1$, both algorithms are a parallel BFS. As $\rho$ increases, the span decreases but the work increases. When $\rho=n$, both algorithms converge to transitive closure via regular $\Theta\left(n^{3}\right)$-work matrix multiplication. They differ for intermediate $\rho$. Spencer's algorithm is deterministic and, for sufficiently dense graphs, can be nearly work efficient with moderate parallelism. In contrast, UY is randomized and never simultaneously work efficient and moderately parallel, but it exhibits a better work/span tradeoff for sparse graphs.

Other work focuses on either restricted graph classes or sequential preprocessing. Kao and Klein [10] give an algorithm for reachability on planar digraphs with $\tilde{O}(n)$ work and $\tilde{O}(1)$ span. Klein [13] gives an algorithm that preprocesses the graph in $O(n p)$ sequential time, where $p \geq 1$ is a parameter; after the preprocessing, reachability can be solved in $O(m / p)$ time on $p$ processors.

### 1.1 Shortcutting Approach and Contributions

The high-level approach is intuitive: (1) reduce the diameter of the graph through the addition of shortcuts, or arcs whose addition does not change the transitive closure of the graph; (2) run parallel BFS on the shortcutted graph. UY [20] fits this general strategy (and parallel BFS and transitive closure are extreme cases), but Spencer's algorithm [18] does not. If the BFS phase is to complete with $\tilde{O}(m)$ work, then the number of shortcuts added must be limited to $\tilde{O}(m)$.

Ignoring the cost of computing the shortcuts, $O(n)$ shortcuts are known to be sufficient to reduce the diameter of any graph to $\tilde{O}(\sqrt{n})$. UY [20] with $\rho=\sqrt{n}$, for example, accomplishes this task. Ignoring logarithmic factors, this is the best diameter reduction known for general graphs using a linear number of shortcuts. As Hesse [8] shows, however, there exist graphs requiring $\Omega\left(\mathrm{mn}^{1 / 17}\right)$ shortcuts to reduce their diameter below $\Theta\left(n^{1 / 17}\right) 5$ Hesse's lower bound implies a lower bound on reachability via shortcutting: any nearly work-efficient algorithm must have $\tilde{\Omega}\left(n^{1 / 17}\right)$ span.

The main technical challenge is to produce the shortcuts efficiently, which is a challenge even ignoring parallelism. There is no $\tilde{O}(m)$-time sequential algorithm known to reduce every graph's diameter to $\tilde{O}\left(n^{1-\epsilon}\right)$, for any constant $\epsilon>0$. For contrast, consider the most natural approach (similar to UY [20]): sample $\sqrt{n}$ vertices, perform a graph search from each, and add shortcuts

[^1]```
Algorithm 1: Sequential algorithm for shortcutting
\(\operatorname{SeqSC1}(G=(V, E))\)
    1 if \(V=\emptyset\) then return \(\emptyset\)
    2 select a pivot \(x \in V\) uniformly at random
    3 let \(R^{+}\)denote the set of vertices reachable from \(x\)
    4 let \(R^{-}\)denote the set of vertices that can reach \(x\)
    \(5 S:=\left\{(x, v) \mid v \in R^{+}\right\} \cup\left\{(u, x) \mid u \in R^{-}\right\} \quad / /\) add shortcuts to/from vertices
    // having paths from/to \(x\), resp.
    \(6 V_{B}:=R^{+} \cap R^{-} ; \quad V_{S}:=R^{+} \backslash V_{B} ; \quad V_{P}:=R^{-} \backslash V_{B} ; \quad V_{R}:=V \backslash\left(V_{B} \cup V_{S} \cup V_{P}\right)\)
    7 return \(S \cup \operatorname{SeqSC1}\left(G\left[V_{S}\right]\right) \cup \operatorname{SeqSC1}\left(G\left[V_{P}\right]\right) \cup \operatorname{SeqSC1}\left(G\left[V_{R}\right]\right)\)
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between all related pairs of samples. It is straightforward to prove that this algorithm yields a graph with $O(\sqrt{n} \log n)$ diameter, but the running time of the $\sqrt{n}$ independent searches is $O(m \sqrt{n})$.

This paper has the following main contributions:

- (Section 3) An $\tilde{O}(m)$-time sequential Monte Carlo algorithm that shortens the diameter of any graph to $\tilde{O}\left(n^{2 / 3}\right)$, with high probability, through the addition of $\tilde{O}(n)$ shortcuts.
- (Sections 4 and 5 ) A Monte Carlo parallel algorithm having $\tilde{O}(m)$ work and $\tilde{O}\left(n^{2 / 3}\right)$ span that shortens the diameter of any graph to $\tilde{O}\left(n^{2 / 3}\right)$, with high probability, through the addition of $\tilde{O}(n)$ shortcuts.
- Applying the diameter reduction then parallel BFS yields a Las Vegas algorithm for singlesource reachability with $\tilde{O}(m)$ work and $\tilde{O}\left(n^{2 / 3}\right)$ span, with high probability.
- (Section 6) An extension that finds a directed spanning tree of $G$ rooted at source $s$, i.e., a tree rooted at $s$ including only arcs in $G$ and containing all vertices reachable from $s$.
Applying existing reductions yields the following Las Vegas randomized parallel algorithms, both with $\tilde{O}(m)$ work and $\tilde{O}\left(n^{2 / 3}\right)$ span with high probability:
- An algorithm that identifies and sorts the strongly connected components of the graph. (Use the new reachability algorithm in Schudy's algorithm [17].)
- An algorithm that finds a directed ear decomposition of any strongly connected graph. (Use the new directed spanning tree algorithm with Kao and Klein's algorithm [10].)


### 1.2 Algorithm and Analysis Overview

The sequential algorithm is simple enough that the main subroutine is given immediately. (See also Algorithm (1).) The algorithm is recursive. First select a random vertex $x$, called the pivot. Perform a graph search forwards and backwards from $x$ to identify subsets $R^{+}$and $R^{-}$, respectively. Add shortcuts from $R^{-}$to $x$ and from $x$ to $R^{+}$. The graph is next partitioned into four subsets of vertices: $V_{B}$ comprises vertices in both $R^{+}$and $R^{-}, V_{S}$ comprises vertices in $R^{+}$but not $R^{-}, V_{P}$ comprises vertices in $R^{-}$but not $R^{+}$, and $V_{R}$ is all remaining vertices. Recurse on the subgraphs induced by the three subsets $V_{P}, V_{S}$, and $V_{R}$.

Ignoring the addition of shortcuts, Algorithm 1 is essentially the divide-and-conquer algorithm for topologically sorting the strongly connected components of a graph described by Coppersmith et al. [3. Their proof thus carries over to prove that this algorithm runs in $O(m \log n)$ sequential time in expectation, but they do not address the diameter problem.

What should be surprising is that Algorithm $\mathbb{1}$ reduces the graph's diameter, captured by the following lemma. The proof is not obvious and leverages new insights and techniques.

Lemma 1.1. Let $G=(V, E)$ be a directed graph, and consider any vertices $u, v \in V$ such that there exists a directed path from $u$ to $v$ in $G$. Let $S$ be the shortcuts produced by an execution of

Algorithm 1. Then with probability at least $1 / 2$ (over random choices in Algorithm 1), there exists a directed path from $u$ to $v$ in $G_{S}=(V, E \cup S)$ consisting of $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$ arcs.

As a corollary, through a simple application of a Chernoff bound and union bound across $\leq n^{2}$ related pairs, the union of shortcuts across $O(\log n)$ independent executions of Algorithm 1 reduces the diameter of the graph to $\tilde{O}\left(n^{2 / 3}\right)$ with high probability.

Unusual aspects and insight. The analysis focuses on shortcutting a particular path. But unlike most divide-and-conquer analyses, the division step here does not seem to effect progress. Partitioning a graph is good for reducing the problem size (which is what Coppersmith et al. [3] leverage), but it is not good for preserving paths - and once vertices fall in different subproblems, there can be no subsequent shortcuts between them. This feature is likely why previous algorithms, such as UY [20], perform independent searches on the original graph.

A key insight in the analysis is that the partitioning step also reduces by a constant factor the number of vertices that could cause the path to split again later. In doing so, the probability of splitting the path goes down, and hence the probability of shortcutting it goes up. The end effect is that the path is likely to be significantly shortcutted before it is divided into too many pieces.

The proof of this filtering insight (Lemma (3.4) leverages antisymmetric relationships between certain vertices. Interestingly, the lack of symmetry in directed graphs is exactly the feature that makes good parallel algorithms for digraphs so elusive, but here asymmetry is crucial to the proof.

Building a parallel algorithm. The main obstacle to parallelizing Algorithm 1 is the graph searches employed to find $R^{+}$and $R^{-}$. In fact, these searches are exactly the single-source reachability problem that we want to solve. The obvious solution to try is to instead limit the searches to a distance of $\tilde{O}\left(n^{2 / 3}\right)$, but unfortunately doing so causes other problems. The parallel algorithm and the analysis are thus more involved. Section 4 provides a sequential algorithm with distance-limited searches. Given that, the parallel implementation (Section 5) is straightforward.

## 2 Preliminaries

This section provides definitions, notations, and the main probabilistic tools used throughout.
The subgraph of $G=(V, E)$ induced by vertices $V^{\prime} \subseteq V$ is denoted by $G\left[V^{\prime}\right]$.
If there is a directed path (possibly empty) from $u$ to $v$ in digraph $G=(V, E)$, then $u$ precedes $v$ and $v$ succeeds $u$, denoted $u \preceq v$. We say also that $u$ can reach $v$ and that $v$ can be reached by $u$. If $u \preceq v$ and/or $v \preceq u$, then $u$ and $v$ are related; otherwise they are unrelated. The successors or forward reach of $x$ is the set of nodes $R^{+}(G, x)=\{v \mid x \preceq v\}$. The predecessors or backwards reach of $x$ is the set $R^{-}(G, x)=\{u \mid u \preceq x\}$.

A shortcut is any arc $(u, v)$ such that $u \preceq v$ in $G$.

Paths and nonstandard notation. The analysis considers paths as well as the relationships between paths and vertices. A path $P=\left\langle v_{0}, v_{1}, \ldots, v_{\ell}\right\rangle$ is denoted by the sequence of its constituent vertices, with the arcs between consecutive pairs implied. The first and last vertex of the path are denoted by head $(P)$ and $\operatorname{tail}(P)$, and the length of the path, denoted length $(P)$, is the number of arcs. For the path $P$ given, $\operatorname{head}(P)=v_{0}, \operatorname{tail}(P)=v_{\ell}$, and length $(P)=\ell$. Two (possibly empty) disjoint paths $P_{1}$ and $P_{2}$ may be concatenated, denoted $P_{1} \mapsto P_{2}$, as long as the arc (tail $\left(P_{1}\right)$, head $\left.\left(P_{2}\right)\right)$ exists. Splitting a path $P$ into $k$ pieces means partitioning it into subpaths $P_{1}, P_{2}, \ldots, P_{k}$ such that $P=P_{1} \mapsto P_{2} \mapsto \cdots \mapsto P_{k}$.

A vertex $x$ and a path $P$ can be compared in the following ways. The vertex $x$ is a bridge of $P$ if $x$ can reach and can be reached by vertices on the path, i.e., if there exists $v_{i}, v_{j} \in P$ such that $v_{i} \preceq x$ and $x \preceq v_{j}$. Note that every vertex on the path is a bridge. A vertex $x$ is an ancestor of $P$ if $x$ can reach some vertex on the path, but $x$ cannot be reached by any vertex on the path. Similarly, $x$ is a descendent of $P$ if $x$ can be reached by some vertex on the path, but $x$ cannot reach any vertex on the path. The set of all bridges, ancestors, and descendents of $P$ are denoted $\operatorname{Bridge}(G, P), \operatorname{Anc}(G, P)$, and $\operatorname{Desc}(G, P)$, respectively. Note that these sets are all disjoint by definition. If a vertex $x$ is a bridge, ancestor, or descendent of the path $P$, then $x$ and $P$ are related. Otherwise, they are unrelated.

Tools. The analysis employs one relatively uncommon probabilistic tool - a special case of Karp's [11 probabilistic recurrence relations, restated next. Roughly speaking, this theorem relates two processes: (1) a random process where in each round the problem "size" ( $\Phi$ in the theorem) reduces by a constant factor in expectation, and (2) a deterministic process where the problem size reduces by exactly that constant factor. The theorem says that if the random process uses a few extra rounds, it is very likely to experience at least the size reduction of the deterministic process.

Theorem 2.1 (Restatement of special case of Theorem 1.36 in [11). Consider a random process of the following form. Let $\mathcal{I}$ denote the set of all problem instances, and let $I_{0} \in \mathcal{I}$ denote the initial problem instance. In the rth round, the process makes random choices and transforms the instance from $I_{r-1}$ to $I_{r}$ (a random variable). Let $\Phi: \mathcal{I} \rightarrow \mathbb{R}$ be any function satisfying $0 \leq \Phi\left(I_{r}\right) \leq \Phi\left(I_{r-1}\right)$ for all relevant $r \geq 1$ and all feasible sequences $I_{0}, I_{1}, I_{2}, \ldots$ of instance outcomes.

Suppose there exists some constant $p<1$ such that $E\left[\Phi\left(I_{r}\right) \mid I_{0}, I_{1}, \ldots, I_{r-1}\right] \leq p \cdot \Phi\left(I_{r-1}\right)$, and consider any integers $k \geq 0$ and $w \geq 0$. Then $\operatorname{Pr}\left\{\Phi\left(I_{k+w+2}\right)>p^{k} \cdot \Phi\left(I_{0}\right)\right\} \leq p^{w}$.

## 3 Sequential Diameter Reduction

This section focuses on proving the following theorem. The unmodified $G$ is used to refer to subgraphs $G=(V, E)$. When the original input graph is intended, $\hat{G}$ is employed instead. Throughout, $x$ denotes the pivot, and the vertex sets $V_{B}, V_{S}, V_{P}$, and $V_{R}$ are used with meaning as setup in Algorithm 1 .

Theorem 3.1. There exists a randomized sequential algorithm that takes as input a directed graph $\hat{G}=(\hat{V}, \hat{E})$ and has the following guarantees, where $n=|\hat{V}|, m=|\hat{E}|$, and without loss of generality $m \geq n / 2$ : (1) the running time is $O\left(m \log ^{2} n\right)$, (2) the algorithm produces a size- $O\left(n \log ^{2} n\right)$ set $S^{*}$ of shortcuts, and (3) with high probability ${ }^{7}$, the diameter of $G_{S^{*}}=\left(V, E \cup S^{*}\right)$ is $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$.

As mentioned in Section 1, the algorithm entails taking the union of shortcuts from $\Theta(\log n)$ runs of Algorithm T To make the running time worst case, there will be one minor modification introduced later: namely, an extra base case to truncate the recursion.

Sections 3.1 and 3.2 set up the main ideas for proof of Lemma 1.1 but instead proves a weaker distance bound of $O\left(n^{1 / \lg (8 / 3)}\right)=O\left(n^{0.7067}\right)$. Section 3.3 tightens the distance bound to

[^2]$O\left(n^{2 / 3} \log ^{4 / 3} n\right)$, thereby proving Lemma 1.1, It is worth emphasizing that Sections 3.2 and 3.3 use exactly the same algorithm - the only difference is the details of the analysis. Finally, Section 3.4 completes the proof of Theorem 3.1 by analyzing the running time and number of shortcuts.

### 3.1 Setup of the Analysis

Fix any simple path $\hat{P}=\left\langle v_{0}, \ldots, v_{\ell}\right\rangle$ in the graph up front. By partitioning the graph, each call to SeqSC1 also splits the path into subpaths. The analysis tracks a collection of calls whose subgraphs contain subpaths of $\hat{P}$.

More precisely, a path-relevant subproblem, denoted by pair $(G, P)$, corresponds to a call SeqSC1 ( $G$ ) and an associated nonempty subpath $P$ of $\hat{P}$ to shortcut. The starting subproblem is $(\hat{G}, \hat{P})$. The path-relevant subproblems are most subproblems for which $G \cap \hat{P} \neq \emptyset$, except with a base case occurring when a subpath $P$ is shortcutted to two hops - all recursive subproblems arising beyond that point are not path relevant. The following lemma characterizes the pathrelevant subproblems that arise when executing the call SeqSC1 ( $G$ ) with associated path $P$.

It is worth emphasizing that the algorithm has no knowledge of the path $P$; associating the subpath with the subproblem is an analysis tool only.

Lemma 3.2. Let $P=\left\langle v_{0}, \ldots, v_{\ell}\right\rangle$ be a nonempty path in $G=(V, E)$, and consider the effect of a single call SeqSC1 ( $G$ ) in Algorithm 1. The following are the outcomes depending on pivot $x$ :

1. (Base case.) If $x$ is a bridge of $P$, then the shortcuts $\left(v_{0}, x\right)$ and $\left(x, v_{\ell}\right)$ are created. There are no path-relevant subproblems.
2. If $x$ and $P$ are unrelated, then $P$ is entirely contained in $G\left[V_{R}\right]$; the one path-relevant subproblem is thus $\left(G\left[V_{R}\right], P\right)$.
3. If $x$ is an ancestor of $P$, then $P=P_{1} \mapsto P_{2}$ for $P_{1}=P \cap G\left[V_{R}\right]$ and $P_{2}=P \cap G\left[V_{s}\right]$. There are thus at most two path relevant subproblems: if $P_{1}$ is nonempty, $\left(G\left[V_{R}\right], P_{1}\right)$ is path relevant; if $P_{2}$ is nonempty, $\left(G\left[V_{S}\right], P_{2}\right)$ is path relevant.
4. If $x$ is a descendent of $P$, then $P=P_{1} \mapsto P_{2}$ for $P_{1}=P \cap G\left[V_{P}\right]$ and $P_{2}=P \cap G\left[V_{R}\right]$. This case gives rise to at most two path-relevant subproblems, as above.

Proof. The proof follows from the definitions. Consider for example the last case, that $x$ is a descendent of $P$. Then there is some latest vertex $v_{k}$ on the path such that $v_{k} \preceq x$. Then consider subpaths $\left\langle v_{0}, \ldots, v_{k}\right\rangle$ and $\left\langle v_{k+1}, \ldots, v_{\ell}\right\rangle$. For all $v_{i}$ with $i \leq k$, we have $v_{i} \preceq v_{k} \preceq x$, and hence $P_{1}=\left\langle v_{0}, \ldots, v_{k}\right\rangle$ is entirely in $V_{P}$. All $v_{j}$ with $j>k$ are unrelated to $x$ and hence in $V_{R}$.

Cases 3 and 4 seem like bad cases because the number of path-relevant subproblems, and hence arcs in the final path, increases. Section 3.2 argues that these cases do make progress.

The path-relevant subproblems that arise during the execution of the algorithm induce a pathrelevant subproblem tree, where each node $s$ corresponds to a call of SeqSC1 on some pathrelevant subproblem $s=(G, P)$. For the analysis, it is convenient to consider the flattened pathrelevant tree, where each node corresponding to case 2 in Lemma 3.2 is merged with its only child. Viewed algorithmically, a node in the flattened path-relevant tree corresponds to sampling multiple pivots $x$ (and discarding some of the graph) until finally getting one that is related to the path $P$.

The analysis considers levels in the flattened path-relevant tree in aggregate, i.e., executing the algorithm in a breadth-first fashion. The point is to later fit the analysis to Theorem 2.1. Specifically, the analysis consists of a sequence of rounds, where the instance $I_{r}$ in round $r$ is the collection of subproblems defined by the nodes at depth $r$ in the flattened path-relevant tree. We have the following lemma immediately. All that remains is bounding the lengths (Section 3.2).

Lemma 3.3. Consider any graph $\hat{G}=(\hat{V}, \hat{E})$ and any path $\hat{P}$ from $u$ to $v$. Consider an execution of Algorithm 1, let $S$ be the shortcuts produced, and let $\left\{\left(G_{1}, P_{1}\right), \ldots,\left(G_{k}, P_{k}\right)\right\}$ denote the set of path-relevant subproblems at level/depth $r$ in the flattened path-relevant tree. Then there is a u-to-v path in $G_{S}=(\hat{V}, \hat{E} \cup S)$ of length at most $2^{r}+2^{r-1}+\sum_{i=1}^{k}$ length $\left(P_{i}\right)$.

Proof. Let $L_{i}$ denote the set of paths associated with leaves in the tree at depth $i$. Then a simple induction over levels proves that: the set of paths $\left\{P_{1}, \ldots, P_{k}\right\} \cup\left(\bigcup_{i=1}^{r-1} L_{i}\right)$ constitute a splitting (partition) of path $\hat{P}$. To perform the inductive step, apply Lemma 3.2 at each internal node.

It remains to bound the path-length in $G_{S}$ by positing a specific path: the concatenation of the shortcutted paths for the leaves and the full unshortcutted paths for the remaining subproblems. Each concatenation adds 1 arc, each leaf's path uses 2 shortcuts, and each remaining non-leaf path $P_{i}$ has length $\left(P_{i}\right)$ arcs. Since the degree of each node is at most 2 (Lemma 3.2), the number of leaves above level $r$ is at most $2^{r-1}$, and the number of internal nodes (concatenations) above level $r$ is also is at most $2^{r-1}$. Adding everything together gives the bound.

### 3.2 Asymmetry Leads to Progress

This section proves that with probability at least $1 / 2$, the distance between $u$ and $v$ is at most $O\left(n^{1 / \log (8 / 3)}\right)$. The main tools are Theorem 2.1 and a proof that the number of path-related vertices decreases by a constant fraction, on average, with each level in the flattened path-relevant tree. More precisely, a vertex $v$ is path active at level $r$ if (1) $v$ is part of some path-relevant subproblem at level $r$ in the flattened tree, and (2) $v$ is related to the path in that subproblem. The goal is to argue that the expected number of path-active vertices decreases with each level.

Recall that the each node in the flattened tree corresponds to a sequence of calls to SeqSC1, where the last call happens to draw a pivot $x$ that is path related. The analysis focuses on that last choice of $x$. But consider instead the equivalent process of choosing $x$ by first tossing a weighted coin to determine whether $x$ is bridge, ancestor, or descendent, then choosing the specific vertex from within that set uniformly at random. The following lemma considers the effect of choosing $x$ from all path ancestors. Choosing from path descendents is symmetric.

Lemma 3.4. Consider any subproblem $(G, P)$. Suppose that $x$ is drawn uniformly at random from $\operatorname{Anc}(G, P)$, let $\alpha=|\operatorname{Anc}(G, P)|$, and let $\alpha^{\prime}$ be denote the number of vertices in Anc $(G, P)$ that are path active at the next level, i.e., after calling SeqSC1 ( $G$ ). Then $E\left[\alpha^{\prime} \mid x \in \operatorname{Anc}(G, P)\right]<\alpha / 2$.

Proof. Define the following binary relation over vertices in $\operatorname{Anc}(G, P)$ : u preserves $v$ means that if $x=u$, then $v$ remains path active. The relation is irreflexive by virtue of the fact that $x \in\left(R^{-}(G, x) \cap R^{+}(G, x)\right)$ and hence not contained in any subproblems. The goal is to prove that it is also antisymmetric. Assuming the asymmetry, the total number of pairs satisfying the preserves relation is at most $\binom{\alpha}{2}$. The number of vertices preserved by $x$ is $\alpha^{\prime}$, and hence $E\left[\alpha^{\prime}\right] \leq$ $\binom{\alpha}{2} / \alpha=(\alpha-1) / 2$. It remains to prove that the preserves relation is antisymmetric.

Consider any pair with $u$ preserves $v$, let $P=\left\langle v_{0}, v_{1}, \ldots, v_{\ell}\right\rangle$, and let $v_{k}$ be the earliest vertex in $P$ with $u \preceq v_{k}$. Then selecting $x=u$ splits the path into $P_{1}^{(u)}=\left\langle v_{0}, \ldots, v_{k-1}\right\rangle$ and $P_{2}^{(u)}=$ $\left\langle v_{k}, \ldots, v_{\ell}\right\rangle$, as stated in Lemma 3.2, where the superscript ${ }^{(u)}$ indicates $x=u$. There are two ways that $v$ could be preserved: either $v \in V_{R}^{(u)}$ and $v \in \operatorname{Anc}\left(G\left[V_{R}^{(u)}\right], P_{1}^{(u)}\right)$, or $v \in V_{S}^{(u)}$ and $v \in \operatorname{Anc}\left(G\left[V_{S}^{(u)}\right], P_{2}^{(u)}\right)$. (No relationships are added in the subproblems, so $v$ cannot, e.g., become a path descendent.) Suppose the latter is true. Then, by definition of $V_{S}^{(u)}$ in Algorithm $1, u \preceq v$ and $v \npreceq u$. It follows that if $x=v, u \in V_{P}^{(v)}$ and hence $v$ does not preserve $u$.

Suppose instead that $v \in V_{R}^{(u)}$, implying $u$ and $v$ are unrelated. Then $v$ can only preserve $u$ if $u \in \operatorname{Anc}\left(G\left[V_{R}^{(v)}\right], P_{1}^{(v)}\right)$. But $v \in \operatorname{Anc}\left(G\left[V_{R}^{(u)}\right], P_{1}^{(u)}\right)$ implies $P_{1}^{(v)}$ is a subpath of $P_{1}^{(u)}$, which is unrelated to $u$. So $u \notin \operatorname{Anc}\left(G, P_{1}^{(v)}\right) \supseteq \operatorname{Anc}\left(G\left[V_{R}^{(v)}\right], P_{1}^{(v)}\right)$, and hence $v$ does not preserve $v$.

Lemma 3.4 states that if an ancestor is selected, the number of path-active ancestors decreases by half. The following lemma extends the analysis to consider the effect on the total number of pathactive vertices. The worst case is that the number of ancestors equals the number of descendents. Then Lemma 3.4 indicates that half of the vertices decrease by half, i.e., a $3 / 4$ reduction in total.

Lemma 3.5. Let $\eta$ denote the number of path-active vertices in some level- $(r-1)$ subproblem $(G, P)$, and let $\eta^{\prime}$ be a random variable denoting the number of those vertices that are path active at level-r. Then $E\left[\eta^{\prime}\right]<(3 / 4) \eta$.

Proof. Let $\alpha, \beta$, and $\delta$ denote the number of ancestors, bridges, and descendents, respectively, of path $P$ in $G$, with $\alpha+\beta+\delta=\eta$. Selecting a pivot $x$ that is unrelated to $P$ can only decreases the number of path-related vertices, decreasing $\eta$ even further, so these choices can be ignored. Consider the first $x$ that is related to $P$. If, for example, that $x$ is a path ancestor, Lemma 3.4 states $E\left[\eta^{\prime} \mid x \in \operatorname{Anc}(G, P)\right]<\alpha / 2+\beta+\delta$. If $x$ is a bridge, then $\eta^{\prime}=0$ because there are no path-relevant subproblems. Adding up all three cases and scaling by their probabilities, we have

$$
\begin{array}{rlr}
E\left[\eta^{\prime}\right] & =\left(\frac{\alpha}{\eta}\right) E\left[\eta^{\prime} \mid x \in \operatorname{Anc}(G, P)\right]+\left(\frac{\delta}{\eta}\right) \cdot E\left[\eta^{\prime} \mid x \in \operatorname{Desc}(G, P)\right] \\
& <\left(\frac{\alpha}{\eta}\right)(\alpha / 2+\beta+\delta)+\left(\frac{\delta}{\eta}\right)(\alpha+\beta+\delta / 2)  \tag{Lemma3.4}\\
& =\frac{(\alpha+\delta)(\alpha / 2+\beta+\delta / 2)}{\eta}+\frac{\alpha \delta}{\eta} \\
& =\frac{(\eta-\beta)(\eta+\beta)}{2 \eta}+\frac{(\sqrt{\alpha \delta})^{2}}{\eta} \\
& \leq \frac{\eta^{2}}{2 \eta}+\frac{((\alpha+\delta) / 2)^{2}}{\eta} \\
& \leq(3 / 4) \eta .
\end{array} \quad(\eta=\alpha+\beta+\delta)
$$

For subproblem $s=(G, P)$, define $\phi(s)$ to be the number of path-active vertices in $s$. Define $\Phi\left(I_{r}\right)=\sum_{s \in I_{r}} \phi(s)$, where $I_{r}$ is the collection of subproblems at level- $r$ in the flattened tree. Then we have the following. Applying Theorem 2.1 then gives the main lemma.

Corollary 3.6. Given any collection $I_{r-1}$ of subproblems, $E\left[\Phi\left(I_{r}\right) \mid I_{r-1}\right] \leq(3 / 4) \Phi\left(I_{r-1}\right)$.
Proof. Lemma 3.5 states that for each $s \in I_{r}$, we have $E\left[\phi\left(s_{1}\right)+\phi\left(s_{2}\right)\right] \leq(3 / 4) \phi(s)$, where $s_{1}$ and $s_{2}$ are random variables for the at most two path-relevant subproblems of $s$. The claim follows by linearity of expectation over all $s$.
Lemma 3.7. Let $\hat{G}=(\hat{V}, \hat{E})$ be a directed graph, and consider any vertices $u, v \in V$ such that there exists a directed path from $u$ to $v$ in $\hat{G}$. Let $S$ be the shortcuts produced by an execution of Algorithm $\mathbb{1}$ and let $n=|\hat{V}|$. Then with probability at least $1 / 2$, there exists a directed path from $u$ to $v$ in $G_{S}=(\hat{V}, \hat{E} \cup S)$ consisting of $O\left(n^{1 / \lg (8 / 3)}\right)$ arcs.

Proof. Choose an arbitrary simple path $\hat{P}$ from $u$ to $v$ in $\hat{G}$. At most every vertex is path active, so $\Phi\left(I_{0}\right) \leq n$. By Theorem [2.1 with Corollary 3.6. $\operatorname{Pr}\left\{\Phi\left(I_{r+5}\right)>(3 / 4)^{r} n\right\}<1 / 2$. Observe that $\Phi\left(I_{r+5}\right)$ is at least the number of bridge nodes that are still active in round $r+5$, and each node on an active subpath is a bridge node. Thus, by Lemma 3.3, running the algorithm to level $r+5$ is enough to yield a shortcutted path length of at most $O\left(2^{r}\right)+\Phi\left(I_{r+5}\right) \leq O\left(2^{r}\right)+(3 / 4)^{r} n$ with probability at least $1 / 2$. Setting both terms equal and solving for $r$ gives $r=\log _{8 / 3} n$. Thus, with probability at least $1 / 2$, the shortcutted path has length $O\left(2^{r}\right)=O\left(2^{\log _{8 / 3} n}\right)=O\left(n^{1 / \lg (8 / 3)}\right)$.

### 3.3 A Tighter Path-Length Bound (Lemma 1.1)

This section tightens the path-length bound to $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$, thereby proving Lemma 1.1 .
The main difference versus Section 3.2 is a better potential function associated with subproblems. The $3 / 4$ bound reduction in the number of path-active vertices, as stated in Lemma 3.5, is indeed tight in the worst case. But the worst case only occurs when the number of ancestors is equal to the number of descendents. When there is imbalance between the two, the reduction is better. Consider, for example, the extreme that there are no descendents - then the number of path active vertices reduces by $1 / 2$ according to Lemma 3.4

This section uses the following potential function for a subproblem $s=(G, P)$ :

$$
\phi(s)=\phi_{1}(s) C_{\phi}+\phi_{2}(s), \quad \phi_{1}(s)=\sqrt{(\alpha+\beta)(\delta+\beta)}, \quad \phi_{2}(s)=\eta=(\alpha+\beta+\delta),
$$

where $C_{\phi}>1$ is a parameter to be set later, $\alpha=|\operatorname{Anc}(G, P)|, \beta=|\operatorname{Bridge}(G, P)|$, and $\delta=$ $|\operatorname{Desc}(G, P)|$. The main idea of $\phi_{1}$ is to capture imbalance by the geometric mean of the number of ancestors and descendents, but bridges are included in both counts because bridges can eventually become ancestors or descendents in induced subgraphs. This imbalance term is more important, and hence weighted by $C_{\phi}$. The second term $\phi_{2}$ is added to ensure the following lemma:

Lemma 3.8. $\eta \leq \phi(s) \leq\left(C_{\phi}+1\right) \eta$, for $\eta=|\operatorname{Anc}(G, P)|+|\operatorname{Bridge}(G, P)|+|\operatorname{Desc}(G, P)|$.
Proof. The first inequality is trivial: $\phi(s) \geq \phi_{2}(s)=\eta$. For the second, $\phi_{1}(s)$ is maximized when $\beta=\eta$, giving a total of $\phi(s) \leq \phi_{1}(s) C_{\phi}+\phi_{2}(s)=\eta C_{\phi}+\eta$.

The next step is to prove a bound analogous to Lemma 3.5 but for $\phi_{1}(s)$.
Lemma 3.9. Consider any path-relevant subproblem $s=(G, P)$. Let $\alpha^{\prime}$, $\beta^{\prime}$, and $\delta^{\prime}$ be random variables denoting the total number of path ancestors, bridges, and descendents, respectively, in any child path-relevant subproblems. Let $\phi_{1}^{\prime}=\sqrt{\left(\alpha^{\prime}+\beta^{\prime}\right)\left(\delta^{\prime}+\beta^{\prime}\right)}$. Then $E\left[\phi_{1}^{\prime}\right] \leq \phi_{1}(s) / \sqrt{2}$.

Proof. First, observe that any pivots $x$ sampled that are not related to the path can only reduce $\phi_{1}$ either by inactivating an ancestor, descendent, or bridge, or by converting a bridge to an ancestor or descendent. Thus, the proof focuses on the final choice of $x$. Let $\alpha, \beta$, and $\delta$ denote the number of ancestors, bridges, and descendents, respectively, of path $P$ just prior to the final choice of $x$ at this point. Thus $\sqrt{(\alpha+\beta)(\delta+\beta)} \leq \phi_{1}(s)$. Let $\eta=\alpha+\beta+\delta$.

Note that $\beta \geq 1$ because a path-relevant subproblem must have a nonempty path and hence at least one bridge. The implication is that all of the divisors below are nonzero.

The proof focuses on the sums $\alpha^{\prime}+\beta^{\prime}$ or $\delta^{\prime}+\beta^{\prime}$. The value $\alpha^{\prime}$ can increase by changing bridges to ancestors, but the sum $\alpha^{\prime}+\beta^{\prime}$ cannot. Applying Lemma 3.4 in the current notation, $E\left[\alpha^{\prime}+\beta^{\prime}\right] \leq \alpha / 2+\beta$.

The remainder of the proof is analogous to proof of Lemma 3.5, with $\phi_{1}^{\prime}=0$ if $x$ is a bridge.

$$
\begin{aligned}
E\left[\phi_{1}^{\prime}\right] & =\left(\frac{\alpha}{\eta}\right) E\left[\phi_{1}^{\prime} \mid x \in \operatorname{Anc}(G, P)\right]+\left(\frac{\delta}{\eta}\right) E\left[\phi_{1}^{\prime} \mid x \in \operatorname{Desc}(G, P)\right] \\
& \leq\left(\frac{\alpha}{\eta}\right) E\left[\sqrt{\left(\alpha^{\prime}+\beta^{\prime}\right)(\delta+\beta)}\right]+\left(\frac{\delta}{\eta}\right) E\left[\sqrt{(\alpha+\beta)\left(\delta^{\prime}+\beta^{\prime}\right)}\right] \\
& \leq\left(\frac{\alpha}{\eta}\right) \sqrt{\left(E\left[\alpha^{\prime}+\beta^{\prime}\right]\right)(\delta+\beta)}+\left(\frac{\delta}{\eta}\right) \sqrt{(\alpha+\beta)\left(E\left[\delta^{\prime}+\beta^{\prime}\right]\right)} \quad \quad \text { (by Jensen's inequality) } \\
& <\left(\frac{\alpha}{\eta}\right) \sqrt{(\alpha / 2+\beta)(\delta+\beta)}+\left(\frac{\delta}{\eta}\right) \sqrt{(\alpha+\beta)(\delta / 2+\beta)} \quad \quad \text { (Lemma 3.4) } \\
& =\left(\frac{\alpha}{\eta}\right) \sqrt{(1 / 2)\left(1+\frac{\beta}{\alpha+\beta}\right)(\alpha+\beta)(\delta+\beta)}+\left(\frac{\delta}{\eta}\right) \sqrt{(1 / 2)\left(1+\frac{\beta}{\delta+\beta}\right)(\alpha+\beta)(\delta+\beta)} \\
& =\left(\frac{\phi_{1}(s)}{\sqrt{2}}\right) \frac{\alpha \sqrt{1+\frac{\beta}{\alpha+\beta}}+\delta \sqrt{1+\frac{\beta}{\delta+\beta}}}{\eta} \\
& \left.\leq\left(\frac{\phi_{1}(s)}{\sqrt{2}}\right) \frac{\alpha\left(1+\frac{\beta}{2(\alpha+\beta)}\right)+\delta\left(1+\frac{\beta}{2(\delta+\beta)}\right)}{\eta} \quad \quad \text { (because } \sqrt{1+y} \leq 1+y / 2 \text { for } y \geq 0\right) \\
& \leq\left(\frac{\phi_{1}(s)}{\sqrt{2}}\right) \frac{(\alpha+\beta / 2)+(\delta+\beta / 2)}{\eta} \\
& =\phi_{1}(s) / \sqrt{2} .
\end{aligned}
$$

Before extending the bound to the full function $\phi$, the following lemma says that partitioning ancestors, bridges, and descendents arbitrarily across subproblems does not increase the potential. This statement is obvious for $\phi_{2}$, so the lemma focuses on $\phi_{1}$.

Lemma 3.10. Consider any integers $\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}, \delta_{1}, \delta_{2} \geq 0$ with such that $\alpha_{i}+\delta_{i}>0 \Longrightarrow \beta_{i}>0$. Let $\alpha=\alpha_{1}+\alpha_{2}, \beta=\beta_{1}+\beta_{2}$, and $\delta=\delta_{1}+\delta_{2}$. Then $\sqrt{\left(\alpha_{1}+\beta_{1}\right)\left(\delta_{1}+\beta_{1}\right)}+\sqrt{\left(\alpha_{2}+\beta_{2}\right)\left(\delta_{2}+\beta_{2}\right)} \leq$ $\sqrt{(\alpha+\beta)(\delta+\beta)}$.

Proof. If $\beta_{1}=0$ or $\beta_{2}=0$, the claim is trivial. (By assumption, $\beta_{1}=0$ for example implies that $\alpha_{1}=0$ and $\delta_{1}=0$.) Suppose instead that neither is zero.

Let $y=\alpha+\beta$ and $y_{i}=\alpha_{i}+\beta_{i}$. Similarly let $z=\delta+\beta$ and $z_{i}=\delta_{i}+\beta_{i}$. Both $y>0$ and $z>0$ by assumption on $\beta>0$. Let $\epsilon_{y}=y_{1} / y$ and $\epsilon_{z}=z_{1} / z$. The proof focuses on a more general split of $y=y_{1}+y_{2}$ and $z=z_{1}+z_{2}$.

It suffices to show that for all $0 \leq \epsilon_{y}, \epsilon_{z} \leq 1$ that $\sqrt{\epsilon_{y} y \cdot \epsilon_{z} z}+\sqrt{\left(1-\epsilon_{y}\right) y \cdot\left(1-\epsilon_{z}\right) z} \leq \sqrt{y z}$, or equivalently (dividing both sides by $\sqrt{y z}$ ) that $\sqrt{\epsilon_{y} \epsilon_{z}}+\sqrt{\left(1-\epsilon_{y}\right)\left(1-\epsilon_{z}\right)} \leq 1$. Fix any $\epsilon_{y}$, treating $\sqrt{\epsilon_{y}}$ and $\sqrt{1-\epsilon_{y}}$ as constants. The expression is maximized at $\epsilon_{z}=\epsilon_{y}$, so $\sqrt{\epsilon_{y} \epsilon_{z}}+$ $\sqrt{\left(1-\epsilon_{y}\right)\left(1-\epsilon_{z}\right)} \leq \epsilon_{y}+\left(1-\epsilon_{y}\right)=1$, which completes the proof.

Finally, the following lemma considers the full potential $\phi$.
Lemma 3.11. Consider any path-relevant subproblem $s=(G, P)$, and let $s_{1}$ and $s_{2}$ be random variables denoting its child subproblems in the flattened path-relevant tree (with $\phi\left(s_{i}\right)=0$ if the child does not exist). Then $E\left[\phi\left(s_{1}\right)+\phi\left(s_{2}\right)\right] \leq \phi(s)\left(1 / \sqrt{2}+2 / \sqrt{C_{\phi}}\right)$.

Proof. Let $\alpha, \beta$, and $\delta$ denote the number of ancestors, bridges, and descendents, respectively, of path $P$ initially. Let $\alpha^{\prime}, \beta^{\prime}$, and $\delta^{\prime}$ be random variables denoting the total number of pathactive ancestors, bridges, and descendents after partitioning around $x$. Let $\eta=\alpha+\beta+\delta$ and $\eta^{\prime}=\alpha^{\prime}+\beta^{\prime}+\delta^{\prime}$. Observe that $\eta=\phi_{2}(s)$ and $\eta^{\prime}=\phi_{2}\left(s_{1}\right)+\phi_{2}\left(s_{2}\right)$.

Lemma 3.9 already bounds the impact of the partitioning on $\phi_{1}$, so the goal here is to consider the contribution of $\phi_{2}$ to the total. There are two cases depending on the degree of imbalance between $\alpha$ and $\delta$. Assume without loss of generality that $\alpha \geq \delta$.

Case 1: $\delta \leq \alpha / C_{\phi}$. In this case, the imbalance causes $\phi_{2}$ to decrease significantly. We have

$$
\begin{align*}
E\left[\eta^{\prime}\right] & =(\alpha / \eta) E\left[\eta^{\prime} \mid x \in A n c(G, P)\right]+(\delta / \eta) E\left[\eta^{\prime} \mid x \in \operatorname{Desc}(G, P)\right] \\
& \leq(\alpha / \eta) \cdot\left(E\left[\alpha^{\prime}+\beta^{\prime}\right]+\delta\right)+(\delta / \eta) \cdot(\eta) \\
& \leq(\alpha / \eta) \cdot(\alpha / 2+\beta+\delta)+\delta  \tag{Lemma3.4}\\
& \leq \frac{\alpha(\alpha+\beta)+\alpha \beta}{2 \eta}+2 \delta \\
& \leq \frac{\eta^{2}}{2 \eta}+\frac{\alpha \beta}{2 \eta}+\frac{2 \alpha}{C_{\phi}} \\
& \leq \eta / 2+\frac{((\alpha+\beta) / 2)^{2}}{2 \eta}+\frac{2 \eta}{C_{\phi}} \\
& \leq \eta / 2+\eta / 8+2 \eta / C_{\phi} \\
& <\phi_{2}(s) / \sqrt{2}+2 \phi(s) / C_{\phi} .
\end{align*}
$$

Adding the contribution from $\phi_{1}$ and $\phi_{2}$ gives

$$
\begin{array}{rlr}
E\left[\phi\left(s_{1}\right)+\phi\left(s_{2}\right)\right] & =E\left[\phi_{1}\left(s_{1}\right)+\phi_{1}\left(s_{2}\right)\right] \cdot C_{\phi}+E\left[\phi_{2}\left(s_{1}\right)+\phi_{2}\left(s_{1}\right)\right] & \\
& \leq E\left[\sqrt{\left.\left(\alpha^{\prime}+\beta^{\prime}\right)\left(\delta^{\prime}+\beta^{\prime}\right)\right]} \cdot C_{\phi}+E\left[\eta^{\prime}\right]\right. & \text { (Lemma 3.10) } \\
& \leq \phi_{1}(s) / \sqrt{2} \cdot C_{\phi}+E\left[\eta^{\prime}\right] & \text { (Lemma 3.9) } \\
& \leq \phi_{1}(s) / \sqrt{2} \cdot C_{\phi}+\phi_{2}(s) / \sqrt{2}+2 \phi(s) / C_{\phi} & \text { (reduction to } \eta^{\prime} \text { above) } \\
& =\phi(s) / \sqrt{2}+2 \phi(s) / C_{\phi} & \\
& =\phi(s)\left(1 / \sqrt{2}+2 / C_{\phi}\right) . &
\end{array}
$$

$$
7
$$

$\qquad$

```
Algorithm 2: Modified sequential algorithm for shortcutting
\(\operatorname{SeqSC2}(G=(V, E))\)
    if the recursion depth is \(\lg n\) then return \(\emptyset\)
    \(S:=\emptyset\)
    while \(V \neq \emptyset\) do
        select a vertex \(x \in V\) uniformly at random
        \(R^{+}:=R^{+}(G, x)\)
        \(R^{-}:=R^{-}(G, x)\)
        \(S:=S \cup\left\{(x, v) \mid v \in R^{+}\right\} \cup\left\{(u, x) \mid u \in R^{-}\right\} \quad / /\) add shortcuts to/from vertices
        // having paths from/to \(x\), resp.
        \(V_{B}:=R^{+} \cap R^{-} ; \quad V_{S}:=R^{+} \backslash V_{B} ; \quad V_{P}:=R^{-} \backslash V_{B} ; \quad V_{R}:=V \backslash\left(V_{B} \cup V_{S} \cup V_{P}\right)\)
        \(S:=S \cup \operatorname{SeqSC2}\left(G\left[V_{S}\right]\right) \cup \operatorname{SeqSC2}\left(G\left[V_{P}\right]\right)\)
        \(G:=G\left[V_{R}\right]\)
    return \(S\)
```

Proof Lemma 1.1. The proof is analogous to Lemma 3.7. There are two key differences: the initial potential is higher, at $\Phi\left(I_{0}\right) \leq\left(C_{\phi}+1\right) n \leq 9 \lg ^{2} n$ according to Lemma 3.8, and the reduction of active vertices with each round is slightly worse, at $(1 / \sqrt{2})(1+1 / \lg n)$ from Corollary 3.12. Assuming $n \geq 16$ so that $(1 / \sqrt{2})^{6}(1+1 / \lg n)^{6}<1 / 2$, Theorem [2.1] implies $\operatorname{Pr}\left\{\Phi\left(I_{r+8}\right)>(1 / \sqrt{2})^{r}(1+1 / \lg n)^{r}\left(9 n \lg ^{2} n\right)\right\}<1 / 2$. As before, Lemma 3.8 states that $\Phi\left(I_{r+8}\right)$ is an upper bound on the number of active vertices and hence also the total length of all remaining subpaths. Thus, by Lemma3.3, running the algorithm to level $r+8$ is enough to yield a shortcutted path of length at most $O\left(2^{r}\right)+\Phi\left(I_{r+8}\right) \leq O\left(2^{r}\right)+O\left((1 / \sqrt{2})^{r}(1+1 / \lg n)^{r} n \lg ^{2} n\right)$ with probability at least $1 / 2$. For $r=O(\lg n)$, this reduces to $O\left(2^{r}\right)+\Phi\left(I_{r+8}\right) \leq O\left(2^{r}\right)+O\left((1 / \sqrt{2})^{r} n \lg ^{2} n\right)$. Choosing $r=(2 / 3)(\lg n+2 \lg \lg n)+\Theta(1)$ balances the terms and yields a path of length $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$.

### 3.4 Runtime and Number of Shortcuts

This section completes the proof of Theorem 3.1 by analyzing the running time and number of shortcuts added. As stated, however, the running time of Algorithm 1 is not worst case, so it does not meet the promise of a Monte Carlo algorithm.

This section instead analyzes Algorithm 2. Algorithm 2 is obtained from Algorithm 1 by replacing one of the recursive calls (specifically $\operatorname{SeqSC1}\left(G\left[V_{R}\right]\right)$ ) with a loop. There is also a new base case after $\lg n$ levels of recursion to make the bounds worst case, where (as always) $n$ here refers to the number of vertices in the original graph $\hat{G}$. Aside from this one change, Algorithm 1 and Algorithm 2 are equivalent.

The following lemma indicates that the main path-length lemmas (Lemmas 3.7 and 1.1) still hold even with the truncated execution. More precisely, proof of those lemmas only relies on the execution reaching a depth much less than $\lg n$ in the flattened path-relevant tree.

Lemma 3.13. Consider an execution of Algorithm 1 and the corresponding flattened path-relevant tree. When mapped to an execution of Algorithm 圆 with the same random choices, the first $\lg n-1$ levels of the flattened tree all have recursion depth $<\lg n$ in Algorithm 园,

Proof. The flattened tree only merges some of the calls corresponding to $G\left[V_{R}\right]$. Algorithm 2 merges all such nodes, which can only reduce the depth of nodes further.

The next lemmas bound the number of shortcuts and running time.

Lemma 3.14. Consider a graph $\hat{G}=(\hat{V}, \hat{E})$, and let $n=|\hat{V}|$. Each execution of Algorithm 圆 creates $O(n \log n)$ shortcuts.

Proof. Consider a call to $\operatorname{SeqSC2}(G)$ on $G=(V, E)$. Each shortcut added removes a vertex: if, e.g., $(x, v)$ is created, then either $x \in V_{B}$ or $x \in V_{S}$, both of which sets are removed from $G$ at the end of the iteration. Thus, there can be at most $|V|$ arcs added.

There are potentially many recursive subproblems, but by the same argument they are all disjoint subgraphs. Thus, the total number of arcs added at each level of recursion is $O(n)$. There are $O(\lg n)$ levels by construction, which completes the proof.

Lemma 3.15. Consider a graph $\hat{G}=(\hat{V}, \hat{E})$, and let $n=|\hat{V}|$ and $m=|\hat{E}|$. Algorithm 圆 can be implemented to run in $O(m \log n)$ time.
Proof. Proof is similar to Lemma 3.14, getting $O(m)$ total time at each level of recursion, assuming that the call SeqSC2 (G) can be made to run in $O(|V|+|E|)$ time. Given a sample $x \in V$, it is straightforward to implement each search, and build the induced subgraphs, to run in time $O(a)$ where $a$ is the number of arcs explored. Each arc is only explored by one search in each direction, so the total number of arcs visited is $O(|E|)$. Finally, sampling vertices can be achieved by randomly permuting the vertices up front, iterating over that list, and checking whether the vertex has already been visited by a search. This takes a total of $O(|V|)$ time

Proof of Theorem 3.1. The full algorithm consists of $\Theta(\log n)$ independent runs of Algorithm 2, For each related pair $u \preceq v$, each run has probability $\geq 1 / 2$ of reducing the distance between those vertices to $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$ by Lemma 1.1. Thus, a Chernoff bound across $\Theta(\log n)$ runs gives a high-probability result, i.e., failure probability at most $1 / n^{c+2}$ for any constant $c$. There are only $n^{2}$ related pairs of vertices, so a union bound across all of them gives a failure probability of at most $1 / n^{c}$. If there are no failures, then the stated diameter is achieved. The running time and number of shortcuts are obtained by multiplying the bounds from Lemmas 3.15 and 3.14 by the $\Theta(\log n)$ runs.

## 4 An Algorithm with Distance-Limited Searches

This section presents a modified algorithm that is more amenable to being parallelized. For now, this algorithm can be viewed as a sequential algorithm - discussion of the parallel implementation is deferred to Section 5. But the main ideas are guided by certain sequential bottlenecks. As in Section 3, $\hat{G}=(\hat{V}, \hat{E})$ and $n=|\hat{V}|$ are used only to refer to the original graph.

There are two main obstacles to parallelizing Algorithm 2, but the first is more serious. Finding the set $R^{-}(G, x)$ or $R^{+}(G, x)$ entails a graph search, which can have linear span in a high-diameter graph. The solution for this problem is to modify the algorithm to use a $D$-limited $\boldsymbol{B F S}$, returning only the vertices within $D$ hops of the source $x$, but doing so introduces some other difficulties. This section thus focuses on modifying the algorithm to work with distance-limited searches for appropriate distance $D$.

The second obstacle is best exhibited by the loop in Algorithm 2. If there are no arcs in the graph, for example, the loop requires $\Omega(n)$ iterations. The solution is to perform multiple pivots in parallel, but in a controlled way that does not sacrifice much performance.

The full algorithm is given in pseudocode as Algorithm 3. Section 4.1 walks through the ideas incrementally, guided by rough intuitions behind the analysis. The key performance lemma, analogous to Lemma 1.1, is the following:

Lemma 4.1. Let $\hat{G}=(\hat{V}, \hat{E})$ be a directed graph, let $n=|\hat{V}|$, let $m=|\hat{E}|$, and assume without loss of generality that $m \geq n / 2$.

Consider any directed path $\hat{P}$ from $u$ to $v$ with length $(\hat{P}) \leq D$, for $D=\Theta\left(n^{2 / 3} \log ^{4 / 3} n\right)$. Let $S$ be the shortcuts produced by an execution of Algorithm $3 \hat{3}$ starting with $h=\lg n$. Then with probability at least 1/2: (1) there exists a path from $u$ to $v$ in $G_{S}=(\hat{V}, \hat{E} \cup S)$ with length at most $D / 2$, (2) the number of shortcuts produced is $|S|=O\left(n \log ^{2} n\right)$, and (3) the total number of vertices and arcs visited by searches is $O\left(m \log ^{2} n\right)$. Moreover, the maximum distance used for any search is $O\left(n^{2 / 3} \log ^{14} n\right)$;

Using multiple runs of Algorithm 3 (see Section 4.3) yields the following:
Theorem 4.2. There exists a randomized algorithm that takes as input a directed graph $\hat{G}=$ $(\hat{V}, \hat{E})$ and uses distance-limited searches with the following guarantees. Let $n=|\hat{V}|, m=|\hat{E}|$, and without loss of generality $m \geq n / 2$. Then (1) the maximum distance used for any search is $O\left(n^{2 / 3} \log ^{14} n\right)$; (2) the algorithm produces a size- $O\left(n \log ^{4} n\right)$ set $S^{*}$ of shortcuts; (3) the total number of vertices and arcs visited by searches is $O\left(m \log ^{4} n+n \log ^{8} n\right)$, and the searches dominate the overall number of primitive operations performed; and (4) with high probability, the diameter of $G_{S^{*}}$ is $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$,

Updated notation. If there exists a path with length at most $d$ from $u$ to $v$, then $u \preceq_{d} v$. If $u \preceq_{d} v$ or $v \preceq_{d} u$, then $u$ and $v$ are $d$-related. All of the notations and definitions in Section 2 that depend on $\preceq$ (i.e., successors, predecesors, ancestors, descendents, bridges) are augmented with the term "d-limited" and a subscript $d$ to indicate that the $\preceq$ in the definition should be replaced by $\preceq_{d}$. For example, $R_{d}^{+}(G, x)=\left\{v \mid x \preceq_{d} v\right\}$ denotes the d-limited successors of $x$.

### 4.1 The Algorithm

The main goal is to replace the searches $R^{+}(G, x)$ in Algorithms $\square$ and 2 with $D$-limited searches, for $D=\tilde{O}\left(n^{2 / 3}\right)$. The good news is that Lemma 3.4 still holds when restricted to pivots drawn from $D$-limited ancestors. The bad news is that Lemma 3.2 does not hold. For concreteness, consider a path $\left\langle v_{0}, v_{1}, \ldots, v_{\ell}\right\rangle$. It is possible, for example, that $x \preceq_{D} v_{2 k}$ but $x \preceq_{D} v_{2 k+1}$ for all $0 \leq k<\ell / 2$. Thus all the even vertices would be in $V_{S}$ and all the odd ones would be in $V_{R}$, splitting the path into $\Theta(\ell)$ pieces with no potential to shortcut them later. In contrast, when the search is not $D$-limited, $x \preceq v_{i}$ implies $x \preceq v_{k}$ for all $k \geq i$, so $V_{S}$ contains a single contiguous subpath.

The solution is to extend the search a little further and duplicate vertices. That is, start with a distance of $d D$, for some $d=\tilde{O}(1)$. Any vertices reached this way are called core vertices, and they are treated similarly to reached vertices in Algorithm [1. Then extend the search a little farther: to a distance of $(d+1) D$. Vertices discovered in the extended search are called fringe vertices, denoted by $F^{+}$and $F^{-}$in the code. Fringe vertices $F^{+}$and incident arcs are duplicated (similarly for $F^{-}$), belonging to both $G\left[V_{R}\right]$ and $G\left[V_{S} \cup F^{+}\right]$.

The addition of fringe vertices fixes the path-splitting problem, giving an analog of Lemma 3.2, at least for paths of length $\ell \leq D$. Consider again the bad example where $x \preceq_{d D} v_{2 k}$ but $x \preceq_{d D}$ $v_{2 k+1}$. All of the even vertices are core vertices, but now all of the odd vertices are fringe vertices. Thus, the entire path is indeed contained in the subgraph $G\left[V_{S} \cup F^{+}\right]$. We still reason about the path being split across subproblems, but fringe vertices on the path can be treated as belonging to whichever subproblem is better.

Unfortunately, duplicating fringe vertices introduces another problem - path-related fringe vertices can be active in multiple subproblems, thereby destroying the progress bound on $\Phi$. In the

```
Algorithm 3: Shortcutting algorithm with distance-limited searches.
\(\operatorname{ParSC}(G=(V, E), h)\)
    /* The value \(h\) indicates how many more levels of recursion to perform. \(\epsilon_{\pi}, N_{k}, N_{L}\),
        and \(D\) are all global parameters (independent of subproblem) set later. */
    if \(h=0\) then return \(\emptyset\)
    \(S:=\emptyset\)
    randomly permute \(V\), giving vertex sequence \(X=x_{1}, x_{2}, \ldots, x_{|V|}\). Mark each \(x_{j}\) live
    split \(X\) into subsequences \(X_{1}, X_{1}, \ldots, X_{2 k}\), with \(\left|X_{i}\right|=\left|X_{k-i+1}\right|=\left\lfloor\left(1+\epsilon_{\pi}\right)^{i}\right\rfloor\) for \(i<k\)
                and \(\left|X_{k}\right|=\left|X_{k+1}\right| \leq\left\lfloor\left(1+\epsilon_{\pi}\right)^{k}\right\rfloor\)
    for \(i:=1\) to \(2 k\) do
        choose random \(d \in\left\{1,2, \ldots N_{L}-1\right\}\)
        \(d:=d+h N_{k} N_{L}-i N_{L} \quad / /\) add the distance offset
        foreach live \(x_{j} \in X_{i}\) do
            \(R_{j}^{-}:=R_{d D}^{-}\left(G, x_{j}\right) ; \quad R_{j}^{+}:=R_{d D}^{+}\left(G, x_{j}\right) \quad / /\) core vertices
            \(F_{j}^{-}:=R_{(d+1) D}^{-}\left(G, x_{j}\right) \backslash R_{j}^{-} ; \quad F_{j}^{+}:=R_{(d+1) D}^{+}\left(G, x_{j}\right) \backslash R_{j}^{+} \quad / /\) fringe vertices
            \(S:=S \cup\left\{\left(x_{j}, v\right) \mid v \in R_{j}^{+} \cup F_{j}^{+}\right\} \cup\left\{\left(u, x_{j}\right) \mid u \in R_{j}^{-} \cup F_{j}^{-}\right\} \quad / /\) add shortcuts
            append a tag of \(j\) to all vertices in \(R_{j}^{+} \cup R_{j}^{-}\)
        foreach live \(x_{j} \in X_{i}\) do
            remove from \(R_{j}^{+}, R_{j}^{-}, F_{j}^{+}\), and \(F_{j}^{-}\)vertices with a tag \(<j \quad / /\) first core search wins
                \(V_{B, j}:=R_{j}^{+} \cap R_{j}^{-} ; \quad V_{S, j}:=R_{j}^{+} \backslash V_{B, j} ; \quad V_{P, j}:=R_{j}^{-} \backslash V_{B, j}\)
                \(S:=S \cup \operatorname{ParSC}\left(G\left[V_{S, j} \cup F_{j}^{+}\right], h-1\right) \cup \operatorname{ParSC}\left(G\left[V_{P, j} \cup F_{j}^{-}\right], h-1\right) \quad / /\) include fringe
        mark all vertices in \(\bigcup_{j}\left(R_{j}^{+} \cup R_{j}^{-}\right)\)as dead in \(X\)
        \(V_{R}:=V \backslash \bigcup_{j}\left(R_{j}^{+} \cup R_{j}^{-}\right)\)
        \(G:=G\left[V_{R}\right]\)
    return \(S\)
```

worst case, almost all of the active vertices could be fringe vertices, and the total number of active vertices could thus increase drastically after partitioning around pivot $x$.

The solution is to select $d$ (for search distance $d D$ ) randomly from the range $d \in\left\{1,2, \ldots, N_{L}-1\right\}$, for some $N_{L}=\tilde{O}(1)$ to be chosen later ${ }^{8}$ Any vertices in the fringe for distance $d D$ are in the core for distances $d^{\prime} D, d^{\prime}>d$. Thus, on average, only an $O\left(1 / N_{L}\right)$ fraction of vertices are on the fringe. For large enough $N_{L}$, the addition of these fringe vertices does not impact $\phi(s)$ much.

It is also important that the distances searched never increases. This is because any progress towards the number of active vertices is with respect to a particular search distance $d D$. The algorithm therefore selects from $N_{L}$ distance options, but offset by some value to reflect future decreases. With each choice of pivot(s), the offset decreases by at least $N_{L}$. More precisely, as in Algorithm 2, the main subroutine consists of a sequence of iterations, where some pivots are chosen in each iteration. The total number of iterations is bounded by some value $N_{k}$, meaning that the full range of distances effectively owned by a single call has size $N_{k} N_{L}$. Each time the recursion depth increases, the offset decreases accordingly by $N_{k} N_{L}$. We thus use a starting offset of $h N_{k} N_{L}$, where $h$ is the number of levels of recursion to perform. As long as $h=\tilde{O}(1), N_{k}=\tilde{O}(1)$, and $N_{L}=\tilde{O}(1)$, the maximum distance searched is $\tilde{O}(D)=\tilde{O}\left(n^{2 / 3}\right)$ as desired.

[^3]```
Algorithm 4: Diameter reduction with distance-limited searches.
\(\operatorname{ParDiam}(\hat{G}=(\hat{V}, \hat{E}))\)
    for \(i:=1\) to \(\Theta(\log n)\) do
        foreach \(j \in\{1,2, \ldots, \Theta(\log n)\}\) do
            \(S_{j}:=\operatorname{ParSC}\left(G^{\prime}, \lg n\right)\), aborting if number of shortcuts or work exceeds Lemma 4.1
        \(\hat{E}:=\hat{E} \cup\left(\bigcup_{j} S_{j}\right) \quad\) // add more arcs to \(\hat{G}\)
    return \(\hat{G}\)
```


## Searches from Multiple Pivots

In addition to being more parallelizable, searching from multiple pivots is also necessary to keep $N_{k}=\Theta\left(\log _{1+\epsilon_{\pi}} n\right)=\Theta\left(\log n / \epsilon_{\pi}\right)$ low, where $0<\epsilon_{\pi} \leq 1$ is chosen later.

A single recursive call ParSC consists of a sequence of iterations, like Algorithm 2 flattening the recursion of $G\left[V_{R}\right]$. Each iteration proceeds as follows. First, sample a set $\left\{x_{j}\right\}$ of pivots and perform independent searches from each of them, determining both the $d D$-limited core ( $R_{j}^{+}$and $R_{j}^{-}$) and the $(d+1) D$-limited fringe ( $F_{j}^{+}$and $F_{j}^{-}$) of each pivot. Add shortcuts to and from all reached vertices. To roughly simulate the effect of selecting one pivot at a time, if a vertex is part of $x_{j}$ 's core, then it is removed from the core and fringe sets for any $x_{j}^{\prime}$ with $j^{\prime}>j$. Next, calculate the sets $V_{S, j}$ and $V_{P, j}$ as in Algorithms 1 and 2 and launch the recursive subproblems $G\left[V_{P, j} \cup F_{j}^{-}\right]$ and $G\left[V_{S, j} \cup F_{j}^{+}\right]$. Finally, remove all core vertices from the graph and start the next iteration.

Algorithm 3 uses the following process to control the pivot sampling. Randomly permute all of the vertices at the start of the call, creating a sequence $x_{1}, x_{2}, \ldots$ of pivots to consider. All pivots are initially live; the live pivots are those still in the graph. In each iteration, select the next group of pivots from the sequence, where the size of the group is discussed below. Perform searches from each live pivot, and ignore the dead ones. When a core vertex is removed from the graph, the vertex is also marked dead in the pivot sequence.

Number of pivots. The number of pivots (live or dead) selected in each iteration is controlled by the parameter $0<\epsilon_{\pi} \leq 1$. For the first $\Theta\left(1 / \epsilon_{\pi}\right)$ iterations, only one pivot is used. In subsequent iterations, the number of pivots increases geometrically by roughly $\left(1+\epsilon_{\pi}\right)$. Were the only goal to keep the number of times a vertex is reached in a search to $O(\log n)$, setting $\epsilon_{\pi}=1$ and following the geometric increase would be sufficient. To bound the number of times a path can split in a single iteration, however, it is important to achieve a tighter bound. There are $2 k$ iterations total, where $k$ is chosen to be large enough to include all vertices according to the following group sizes. The first $k$ iterations follow a geometric increase, and the next $k$ iterations follow a geometric decreases. More precisely, the number of pivots considered in both iteration $i$ and $2 k-i+1$ is $\left\lfloor\left(1+\epsilon_{\pi}\right)^{i}\right\rfloor$, but iterations $2 k$ and $2 k+1$ can be smaller.

### 4.2 Notation and Shorthand

It is often convenient to refer to iterations of the loop in Algorithm 3. During iteration $i$, quite a bit happens: some pivots are processed, some searches are performed, some induced subgraphs are built, etc, and the claims throughout refer to those objects. Defining every term concretely in every lemma statement or proof gets tedious and unwieldy. Instead, this paper adopts some notational conventions consistent with the pseudocode in Algorithm 3, using the variables to implicitly adopt the meaning of the code.

Concretely, for iteration $i$ on graph $G=(V, E)$, the following notations are used with the same meaning as the pseudocode: $h, X_{i}$ meaining the pivot sequence, and $d$ meaning the random distance chosen. Moreover, for each $x_{j} \in X_{i}$, whenever notations $R_{j}^{+}, R_{j}^{-}, F_{j}^{+}, F_{j}^{-}, V_{S, j}$, or $V_{P, j}$ appear, they should also be interpreted to have the meaning laid out in the pseudocode.

Min and max distances. In each iteration $i$, the algorithm chooses a random distances in some size- $\left(N_{L}-1\right)$ range, but at an offset that depends on the iteration. The values $\bar{d}$ and $\hat{d}$ denote the bounds of the range, i.e., drawing random $d \in\{\bar{d}, \bar{d}+1, \ldots, \bar{d}-1\}$. Here $\bar{d}=h N_{k} N_{L}-i N_{L}+1$, and $\widehat{d}=h N_{k} N_{L}-(i-1) N_{L}$. The minimum possible search distance for a core search is $\bar{d} D$. The maximum possible search distance for a fringe search is $\hat{d} D$. Note that $\bar{d}$ and $\hat{d}$ both rely on the current iteration $i$ and recursion height $h$, which is consistent with the general notational shorthand. These min and max distances are useful for classifying vertex relationships as follows:

Definition 4.3. Consider any iteration $i$ of Algorithm 3 .

- Vertices $u$ and $v$ are never related if $u \not \nwarrow_{\widehat{d D}} v$ and $v \not \not_{\widehat{d D}} u$.
- Vertices $u$ and $v$ are partly related if $u \preceq_{\widehat{d} D}$ v or $v \preceq_{\widehat{d D}} u$.
- Vertices $u$ and $v$ are fully related if $u \preceq_{\bar{d} D} v$ or $v \preceq_{\bar{d} D} u$. If $u$ and $v$ are fully related, then they are also partly related.
When comparing a vertex $v$ and a path $P$, the same terms apply in the natural way. For example, if $v$ is fully related with any vertex in $P$, then $v$ and $P$ are fully related.


### 4.3 Full Diameter-Reduction Algorithm and Proof of Theorem 4.2

Like the algorithm in Section 3, to achieve diameter reduction with high probability requires multiple passes of Algorithm 3. But now more passes are necessary. The full algorithm, shown in Algorithm [4 is as follows. Perform $\Theta(\log n)$ iterations. In each iteration, perform $\Theta(\log n)$ independent executions of Algorithm 3 on the current graph. Add to the graph all of the shortcuts produced thus far, and continue to the next iteration on the updated graph.

The main reason for the extra passes of Algorithm 3is that, due to the $\tilde{O}(D)$-limited searches, the analysis only considers paths of length $D$. The distance $D$ is chosen to be large enough so that each iteration is enough to reduce the length of the path to $D / 2$, with high probability, but a longer path needs to be subdivided.

Proof of Theorem 4.2, assuming Lemma 4.1. Consider any two vertices $u \prec v \in V$. Let $\Delta_{i}$ denote the length of the shortest path from $u$ to $v$ in the graph after iteration $i$ of the outer loop of Algorithm 4. The main claim is that with high probability, $\Delta_{i} \leq D \cdot \max n /\left(D 2^{i}\right), 1$. Thus, when Algorithm 4 returns, the diameter bound is met.

The proof is by induction on $i$. For $i=0$, the length of the shortest path is at most $n$, so $\Delta_{0} \leq n=D \cdot n /\left(D 2^{0}\right)$. For the inductive step (going from iteration $i$ to $i+1$ ), consider the shortest path $P$ from $u$ to $v$ in the current graph. If length $(P) \leq D$, then the path is already short enough. Otherwise, subdivide the path into at most $\left(n /\left(D 2^{i}\right)\right)$ subpaths, each of subpaths, each of length at most $D$. Consider each subpath. By Lemma 4.1, a single execution of Algorithm 3 shortens the subpath's length to $D / 2$ with constant probability. Thus, using a Chernoff bound, $\Theta(\log n)$ runs shorten the subpath to $D / 2$ with high probability. Taking a union bound over all subpaths gives high probability that all subpaths are shortened. Concatenating the subpaths yields a path of length $(D / 2) \cdot n /\left(D 2^{i}\right)=n /\left(D 2^{i+1}\right)$.

The search distance follows directly from Lemma 4.1. The number of shortcuts follows from Lemma 4.1 by multiplying by the number of $\Theta\left(\log ^{2} n\right)$ runs. As for the bound on total number of arcs visited, observe that the graph size is at most $\hat{E}+O\left(n \log ^{4} n\right)$ at the end. Thus, by Lemma 4.1, each run of Algorithm 3 visits $O\left(\left(m \log ^{4} n\right) \log ^{2} n\right)=O\left(m \log ^{6} n\right)$ arcs. Multiplying by $\Theta(\lg n)$ runs completes the proof.

### 4.4 Bounds on Number of Vertices Searched

This section bounds the number of times each vertex can be searched in each iteration of the main loop in Algorithm 3. The main lemma, stated next, is used to prove two corollaries. The first corollary says that, with high probability, a vertex is not searched more than $O(\log n)$ times, where $n=|\hat{V}|$. The second corollary gives a tighter bound, but only in expectation.

Lemma 4.4. Consider any iteration $i$ of the loop in the call $\operatorname{ParSC}(G=(V, E), h)$. Let $y=$ $\sum_{i^{\prime}=1}^{i-1}\left|X_{i}\right|$ be the number of pivots processed before iteration $i$ begins. Let $G_{i}=\left(V_{i}, E_{i}\right)$ be the remaining subgraph at the start of the iteration and let $\tau=|V| \ln n / y$ for any $n \geq 29^{9}$ Then for every $v \in V$ and constant $c \geq 2$ :

With probability at least $1-1 / n^{c-1}$ : either $v \notin V_{i}$, or $\left|R_{\widehat{d D}}^{-}\left(G_{i}, v\right)\right| \leq c \tau$ and $\left|R_{d D}^{+}\left(G_{i}, v\right)\right| \leq c \tau$.
Proof. All searches before iteration $i$ have distance larger than $\widehat{d} D$. Moreover, arcs are not added to the graph on each iteration, so the number of $\hat{d} D$-limited predecessors of $v$ can only decrease or stay the same with each iteration. Thus, if $v$ is to end with $>c \tau$ live predecessors, it must have $>c \tau$ live predecessors the entire time. The remainder of the proof bounds the probability of that event occurring. The argument for successors is symmetric.

For $y<c \ln n$, the claim is vacuous, so consider instead that $y \geq c \ln n$. Let $x_{1}, x_{2}, \ldots, x_{y}$ denote the sequence of pivots chosen before iteration $i$ begins. While the number of $\hat{d} D$-limited predecessors is above threshold, there are at most $|V|$ choices of $x_{j}$ so we have $\operatorname{Pr}\left\{x_{j}\right.$ live and $\left.x_{j} \preceq_{\widehat{d D}} v\right\} \geq$ $c \tau /|V|=c \ln n / y$. Thus, for $y \geq c \ln n, \prod_{j=1}^{y} \operatorname{Pr}\left\{x_{j}\right.$ dead or $\left.x_{j} \npreceq \widehat{d D}^{v}\right\} \leq(1-c \ln n / y)^{y} \leq$ $1 / e^{c \ln n}=1 / n^{c}$. Taking the union bound across the two failure events (predecessors and successors) gives failure probability $2 / n^{c}<1 / n^{c-1}$ for $n \geq 2$.

Corollary 4.5. Choose any $\epsilon_{\pi} \leq 1$. Consider any iteration $i$ and let $X_{i}$ be the random set of pivots selected. Then with high probability with respect to $n$, no vertex is $\hat{d} D$-related to more than $O(\log n)$ live pivots in $X_{i}$.

Proof. Use $G=(V, E)$ to refer to the graph at the beginning of the call, before the first iteration of the loop. Let $y$ be the number of pivots considered before the iteration in question, and let $v$ be a vertex to analyze. By choice of $\epsilon_{\pi},\left|X_{i}\right| \leq 3 y$. (It can only be this large due to roundoff.)

Lemma 4.4 has failure probability $1 / n^{c-1}$, where we can choose whatever constant $c$ we want. We can add the failure probabilities at the end by a union bound, so suppose for now that Lemma 4.4 applies. Then $v$ has $O(|V| \log n / y) \hat{d} D$-limited predecessors and successors.

There are two cases. If $y>|V| / 8$, then $O(|V| \log n / y)=O(\log n)$. Searching from every remaining vertex would thus only result in $O(\log n)$ searches reaching $v$. To complete the proof, take a union bound across all $v$ to get at most a $n^{c-2}$ failure probability.

[^4]If $y \leq|V| / 8$, then $y+\left|X_{i}\right| \leq|V| / 2$. For every pivot position $x_{j} \in X_{i}$, there are thus at least $|V| / 2$ options to draw from. So we have $\operatorname{Pr}\left\{x_{j}\right.$ live and $\left.x_{j} \preceq_{d D} v\right\}=O(|V| \log n /(y|V|))=$ $O(\log n / y)=O\left(\log n /\left|X_{i}\right|\right)$, where the last step follows from $\left|X_{i}\right| \leq 3 y$. Applying a Chernoff bound across all $\left|X_{i}\right|$ pivots, the number of searches that reach $v$ is $O(\log n)$ with high probability. To complete the proof, take a union bound across all $v$, adding the failure probabilities from the Chernoff bound and Lemma 4.4.
Corollary 4.6. Consider any iteration $i$ and vertex $v$, and suppose that $\epsilon_{\pi} \geq 1 / n$ for $n=|\hat{V}|$. As long as $\left|X_{i}\right|>1$, then the expected number of times that vertex $v$ is visited by searches is $O\left(\epsilon_{\pi} \log n\right)$.

Proof. Let $G=(V, E)$ denote the graph before the first iteration of the loop. Suppose that Lemma 4.4 applies, so $v$ has $O(|V| \log n / y) d D$-limited predecessors and successors, where $y$ is the number of pivots processed before this iteration. The failure event can only increase the expectation by an additive $\operatorname{Pr}\{$ failure $\} \cdot|V| \leq\left(1 / n^{c}\right) n=1 / n=O\left(\epsilon_{\pi}\right)$.

The main observation is that when $\left|X_{i}\right|>1, i$ is not in one of the first or last $\Theta\left(1 / \epsilon_{\pi}\right)$ iterations, and the number of vertices $\left|X_{i}\right|$ processed in this iteration is at most an $O\left(\epsilon_{\pi}\right)$-fraction of the number of remaining vertices. Similarly, $\left|X_{i}\right|=O\left(\epsilon_{\pi} y\right)$.

If $y>|V| / 8$, then $O(|V| \log n / y)=O(\log n)$, meaning that $v$ has only $O(\log n) d D$-limited predecessors or ancestors remaining. This iteration thus processes only $O\left(\epsilon_{\pi} \log n\right)$ of them in expectation.

If $y \leq|V| / 8$, let $Y_{j}$ be an indicator random variable for the event that $x_{j} \preceq_{d D} v$. Since there are $\Omega(|V|)$ pivots to choose from, $\operatorname{Pr}\left\{Y_{j}\right\}=O(|V| \log n /(y|V|))=O(\log n / y)$. It follows that $E\left[\sum_{x_{j} \in X_{i}} Y_{j}\right]=\left|X_{i}\right| \cdot O(\log n / y)=O\left(\epsilon_{\pi} \log n\right)$.

The next lemma bounds the number of fringe vertices and arcs explored in a single iteration $i$. Note that a particular vertex may be a fringe vertex for multiple searches. The lemma counts the total number of times that each vertex appears on the fringe. Similarly, arcs may be explored by multiple fringe searches instance of the vertex, but just once per search. (An arc $(u, v)$ is explored by $x_{j}$ 's fringe search if either $u$ or $v$ is a fringe vertex.)

Lemma 4.7. Consider an iteration $i$ and $N_{L} \geq 2$. Let $V^{\prime}$ be any subset of the vertices remaining in the graph, and let $E^{\prime}$ be any subset of the arcs remaining in the graph. Then the total number of fringe vertices also in $V^{\prime}$ is $O\left(\left|V^{\prime}\right| \log n / N_{L}\right)$ in expectation over choice of distance. Similarly, the total number of arcs explored by fringe searches is also in $E^{\prime}$ is $O\left(\left|E^{\prime}\right| \log n / N_{L}\right)$ in expectation.

Proof. By Corollary 4.5, with high probability each vertex in $V^{\prime}$ is visited by at most $O(\log n)$ searches (fringe or core). A failure event can only increase the expectation by an additive $\operatorname{Pr}\{$ failure $\}$. $|\hat{V}|^{2} \leq\left(1 / n^{c}\right) n^{2} \ll \log n$ for appropriate choice of constant $c$ in the high-probability bound. Thus the remainder of the proof assumes that each vertex is not visited too many times.

Fix any arbitrary set of pivots that satisfies Corollary 4.5. Consider any $v \in V^{\prime}$ and pivot $x_{j}$. Let $Y_{j}^{v}$ be an indicator random variable for the event that $v$ is on $x_{j}$ 's fringe, where $Y_{j}^{v}=0$ if $x_{j}$ and $v$ are never related. For a partly related $x_{j}, v$ is only on the fringe at one distance, so $\operatorname{Pr}\left\{Y_{j}^{v}\right\} \leq$ $1 /\left(N_{L}-1\right) \leq 2 / N_{L}$. The total number of times $v$ is on the fringe is thus $E\left[\sum_{x_{j} \in X_{i}} Y_{j}^{v}\right]=O(\log n)$. $\left(2 / N_{L}\right)=O\left(\log n / N_{L}\right)$. Summing across all $v$ gives $E\left[\sum_{v \in V^{\prime}} \sum_{x_{j} \in X_{i}} Y_{j}^{v}\right]=O\left(\left|V^{\prime}\right| \log n / N_{L}\right)$.

The same argument applies to arcs, observing that the arc is explored whenever its endpoints are at the right distance.

### 4.5 Setting Up the Path-Relevant Tree

The definition of path-relevant subproblems and the path-relevant tree differ slightly from Section 3.1 to account for the key changes. Algorithm 2 looks closer to Algorithm 3 than Algorithm 1 does, so it is worth contrasting the changes with Algorithm 2.

Nodes in the (unflattened) path-relevant tree are analogous to those in Section 3.1. Each node corresponds to an iteration of the for loop in Algorithm 3, and associated with the node is the path-relevant subproblem $(G, P)$ at the start of the iteration. Analogously, each node in Section 3.1 corresponds to an iteration of while loop in Algorithm 2,

The following lemma, analogous to Lemma 3.2, considers the effect of an iteration on the pathrelevant subproblems. Unlike Lemma 3.2, there may be more than two path-relevant subproblems.

Lemma 4.8. Let $P=\left\langle v_{0}, \ldots, v_{\ell}\right\rangle$ be a nonempty path in $G=(V, E)$ with $\ell \leq D$ and consider the effect of a single iteration of the for loop in Algorithm 3. Let $X_{i}$ be the set of pivots selected for this iteration, and let $d$ be the distance chosen for the core search. Then the following are the outcomes:

1. (Base case.) If $X_{i}$ contains a live $d D$-limited bridge of $P$, then the shortcuts $\left(v_{0}, x\right)$ and $\left(x, v_{\ell}\right)$ are created. There are no path-relevant subproblems.
2. If none of the live pivots in $X_{i}$ are $d D$-related to $P$, then $P$ is entirely contained in $G\left[V_{R}\right]$; the one path-relevant subproblem is thus $\left(G\left[V_{R}\right], P\right)$ - the next iteration of the for loop.
3. Suppose that just one live pivot in $x_{j} \in X_{i}$ is $d D$-related to $P$ but it is not a bridge. Then there exists a 2-way splitting $P=P_{1} \mapsto P_{2}$ splitting of path $P$ such that either (i) $P_{1}$ is fully contained in $G\left[V_{R}\right]$ and $P_{2}$ is fully contained in $G\left[V_{S, j} \cup F_{j}^{+}\right]$, or (ii) $P_{1}$ is fully contained in $G\left[V_{P, j} \cup F_{j}^{-}\right]$and $P_{2}$ is fully contained in $G\left[V_{R}\right]$.
4. Suppose that $r$ of the live pivots in $X_{i}$ are $d D$-related to $P$ but none of them are bridges. Then there exists an $(r+1)$-way splitting $P=P_{1} \mapsto P_{2} \mapsto \cdots \mapsto P_{r+1}$ corresponding to $r$ consecutive applications, in pivot order, of the 2-way split above. (Some of the paths $P_{i}$ may be empty.) It follows that $(G, P)$ gives rise to at most $r+1$ path-relevant subproblems.

Proof. (Case 1.) Suppose that some $d D$-related bridge $x_{j}$ is selected. Then by definition there exist vertices $v_{a}, v_{b} \in P$ such that $v_{a} \preceq_{d D} x_{j} \preceq_{d D} v_{b}$. Since the path has length at most $D$, it follows that $v_{0} \preceq_{(d+1) D} x_{j} \preceq_{(d+1) D} v_{\ell}$. Since $v_{0}$ and $v_{\ell}$ are within the fringe search distance, the claimed shortcuts are added.
(Case 2.) None of the vertices are $d D$-related to $P$. Then none of the vertices in $P$ are removed from $V_{R}$.
(Case 3.) If $x_{j}$ is a $d D$-limited ancestor. (The case for a descendent is symmetric.) Then there exists a vertex $v_{j} \in P$ such that $x_{j} \preceq_{d D} v_{j}$. The $v_{j}$ be the earliest such vertex. Then $x_{j} \preceq_{(d+1) D} v_{j^{\prime}}$ for all $j^{\prime} \geq j$. Since $x_{j}$ is the first (and only) $d D$-related pivot in pivot sequence, none of those vertices are in another pivot's core. So $v_{j}, v_{j+1}, \ldots, v_{\ell} \in R_{j}^{+} \cup F_{j}^{+}$. Moreover, $x_{j}$ is not a bridge, so none of them are in $R_{j}^{-}$, meaning they are in $V_{S, j} \cup F_{j}^{+}$.
(Case 4.) Consider the path-related pivots in permutation order, applying the above case inductively.

As in Section 3, the bulk of the analysis is with respect to the flattened path-relevant tree. The question is when nodes should be flattened. The most natural choice - flatten when case 2 applies - turns out not to work. The problem is that knowing whether the searches reached the path reveals information about the distance chosen, which changes the distributed over fringe searches. Notably, Lemma 4.7 requires the full range of distance choices.

Instead, an iteration is merged with the next iteration whenever the pivot $x_{j}$ is never related with the path $P$. A node in the flattened tree thus consists of a sequence of iterations such that only the last iteration selects a pivot that partly related with the path. A partly related vertex pivot may, depending on choice of distance, result in the path splitting. But the analysis pessimistically charges for the split.

The following lemma, analogous to Lemma 3.3, bounds the length of the path at depth- $r$ in the tree. Since the degree of each node in the tree is now a random variable, the bound here holds with failure probability $\leq 1 / 8$.
Lemma 4.9. Consider any graph $\hat{G}=(\hat{V}, \hat{E})$ and any path $\hat{P}$ from $u$ to $v$ with length $(P) \leq D$. Consider an execution of Algorithm 3 starting with $h=\lg n$, for $n=|\hat{V}|$, with parameter $\epsilon_{\pi}$ satisfying $\epsilon_{\pi} \leq 1 / \lg ^{3} n$. Let $S$ be the shortcuts produced, and let $\left\{\left(G_{1}, P_{1}\right), \ldots,\left(G_{k}, P_{k}\right)\right\}$ denote the set of path-relevant subproblems at level $r \leq \lg n$ in the flattened path-relevant tree. Then with probability $\geq 7 / 8$, there is a u-to-v in $G_{S}=(\hat{V}, \hat{E} \cup S)$ of length at most $O\left(2^{r}\right)+\sum_{i=1}^{k}$ length $\left(P_{i}\right)$.
Proof. This proof focuses on showing that, with probability at least $7 / 8$, the number of nodes in the flattened path-relevant tree is $O\left(2^{r}\right)$. Otherwise the proof is the same as Lemma 3.3.

Number the nodes in a particular row as $1,2, \ldots, q$. Each node selects at least one partly path-related pivot. Let $z_{t}$ be random variables denoting the number of additional partly-pathrelated pivots selected in node $t$ of the tree. By Lemma 4.8, each pivot gives rise to at most one additional child node. So the number of nodes in the next row is $\leq \sum_{j=1}^{q}\left(2+z_{t}\right)=2 q+\sum_{t=1}^{q} z_{t}$. If only one pivot is in the pivot set, then $E\left[z_{t}\right]=0$; otherwise, Corollary 4.6 implies $E\left[z_{t}\right]=$ $O\left(\epsilon_{\pi} \log n\right)=O\left(1 / \log ^{2} n\right)$ in general. By Markov's inequality, $\operatorname{Pr}\left\{\sum_{t=1}^{q} z_{t} \geq 8 \lg n \cdot E\left[\sum_{t=1}^{q} z_{t}\right]\right\} \leq$ $1 /(8 \lg n)$. Thus, with probability at least $1-1 /(8 \lg n)$, the number of nodes in the next row is $\leq 2 q+8 \lg n \cdot E\left[\sum_{t=1}^{q} z_{t}\right]=2 q+8 \lg n \cdot O\left(1 / \log ^{2} n\right)=2 q(1+O(1 / \log n))$. Taking a union bound across all $r \leq \lg n$ rows, the probability that even one row increases by more than $2(1+O(1 / \log n))$ factor is at most $1 / 8$. Assuming no such larger increase occurs, the total number of nodes in the tree is $\leq \sum_{r^{\prime}=0}^{r} 2^{r^{\prime}}(1+O(1 / \log n))^{r^{\prime}}=\sum_{r^{\prime}=0}^{r} 2^{r^{\prime}} e^{O(1)}=O\left(2^{r}\right)$.

### 4.6 Progress on Path-Active Vertices

This section is analogous to Section 3.3. The goal is to argue that after $(2 / 3) \lg n+o(\log n)$ rounds (or levels in the tree), the number of active vertices drops below $D / 2$ with constant probability. (Or rather, to select the right choice of $D$ so that this is true.) Fortunately, the key lemmas from Section 3.3 can still be applied, since they only rely on the asymmetry of Lemma 3.4. And although multiple pivots may be selected in any iteration, the analysis only leverages the progress caused by the first pivot in sequence order.

There are some differences to the analysis as well. Most notably, the additional of fringe nodes increases the number of path-active vertices. That increase has an effect on the expected reduction to the potential, which must be bounded here. It is possible for the potential to increase when advancing to the next round. Theorem 2.1, however, requires that $\Phi$ never increase. This section updates the definition of $\Phi$ and adds a little extra machinery.

Due to the distance-limited searches, the definition of path active is updated with respect to being partly related. That is, a vertex $v$ is path active at some level $r$ in the flattened tree if (1) $v$ is part of some path-relevant subproblem at level $r$, and (2) $v$ is partly related to the path in that subproblem.

The potential function of a subproblem is the same as in Section 3.3, but with the updated
definition of being path active. Concretely, for subproblem $(G, P)$,

$$
\phi(s)=\phi_{1}(s) C_{\phi}+\phi_{2}(s), \quad \phi_{1}(s)=\sqrt{(\alpha+\beta)(\delta+\beta)}, \quad \phi_{2}(s)=\eta=(\alpha+\beta+\delta),
$$

where $\alpha=\left|A n c_{\widehat{d D}}(G, P)\right|, \beta=\left|\operatorname{Bridge}_{\widehat{d D}}(G, P)\right|$, and $\delta=\left|\operatorname{Desc}_{\widehat{d D}}(G, P)\right|$.
Fringe nodes. It is useful to track the provenance of vertices, treating fringe vertices as new vertices. Specifically, a path-active vertex $v$ is said to be preserved by the execution of a specific node in the flattened tree if (1) $v$ is still path active in a child node $s^{\prime}$, and (2) $v$ was not added to $s^{\prime}$ due to a fringe search. That is, if for example $s^{\prime}$ operates on the graph $G\left[V_{S, j} \cup F_{j}^{+}\right]$, and $v \in F_{j}^{+}$, then $v$ is not preserved here. (But the vertex $v$ could be preserved with respect to a different subproblem.) Given this view of fringe nodes, the analysis can adopt the tools from Section 3 to consider the effects of the core searches.

How to reason about the randomness. There is one key difference in the reasoning. Think of the randomness in the following way: first, reveal enough of the randomness in the pivots just to reveal which pivots are partly path related and fully path related, but not any more specific than that. This first step is enough to determine whether the iteration is the final iteration of the flattened node. Then determine the distance searched. Finally, resolve the specific pivot choices. This process has the same probabilities of any outcome as Algorithm 3, but reasoning about the randomness in this way helps.

For concreteness, here is a restatement of Lemma 3.4 in the new context. This version only states a progress argument if the pivot selected is fully related to the path - the reason is that a different argument will be applied for partly related pivots. Specifically, active vertices that are not fully related to the path are, by definition, not related at any distance less than $\bar{d} D$. All such vertices are always inactive in child subproblems, regardless of random choices. The only unknown is thus what happens to the fully-related active vertices.

Lemma 4.10. Consider any subproblem $(G, P)$. Let $A=A n c_{\bar{d} D}(G, P)$ be the fully-related ancestors of path $P$. Consider any search distance $d D \geq \bar{d} D$. Let $x_{j}$ be the first fully-related pivot selected, and suppose that $x_{j}$ is drawn uniformly at random from $A$. Let $\bar{\alpha}=|A|$. Let $\bar{\alpha}^{\prime} \leq \bar{\alpha}$ denote the number of vertices in A that are preserved. (Recall that preserved means with respect to core searches only.) Then $E\left[\bar{\alpha}^{\prime} \mid x \in A\right]<\bar{\alpha} / 2$.

Proof. The proof is similar to Lemma [3.4, except that a subset of vertices is considered, and all relationships are with respect to $\preceq_{d D}$ instead of $\preceq$. As before, the goal is to show that the preserves relation is antisymmetric for all $u, v \in A$.

It is possible $u \in A$ be a $d D$-limited bridge. (It is not a bridge at distance $\bar{d} D$, but it could be a bridge at greater distance.) Bridges do not have any path-relevant subproblems, so they do not preserve any other vertices.

Consider any pair of vertices $u, v \in A$ such that $u$ preserves $v$. The logic follows proof of Lemma 3.4 with the same two cases, summarized briefly here. If $u \preceq_{d D} v$, then $v$ cannot preserve $u$. If $u$ and $v$ are $d D$-unrelated, then consider the earliest vertices $v_{a}$ with $u \preceq_{d D} v_{a}$ and $v_{b}$ with $v \preceq_{d D} v_{b}$ on the path. For $u$ to preserve $v$, it must be that $b<a$, so $v$ cannot also preserve $u$.

The other lemmas in Section [3.3, e.g., Lemma 3.11 essentially just build algebraically off Lemma 3.4, so the analogs with respect to Lemma 4.10 hold with the same proof, but for two issues that require care when applying the lemmas - Lemma 4.10 neither copes with fringe nodes
nor with active vertices that are partly related but not fully related. Since the algebra remains the same, these lemmas are not reproved.

Much of the complexity that follows arises from the specific choice of potential function. Using just the linear function $\phi_{2}$, as in Section [3.2, would simplify many of the details. But the bound would be worse.

The following lemma implies that fringe nodes can be cleanly factored-out of the $\phi$. The number $f$ of fringe nodes is included twice in the $\phi_{1}$ term, which overcharges the contribution of fringe nodes.

Lemma 4.11. Consider any $\alpha \geq 0, \beta \geq 0, \delta \geq 0$, and let $\eta=\alpha+\beta+\delta$. Let $\phi_{1}=\sqrt{(\alpha+\beta)(\delta+\beta)}$ and $\phi=\phi_{1} \cdot C_{\phi}+\eta$. For any $f \geq 0$ and $C_{\phi} \geq 2$,

$$
\sqrt{(\alpha+\beta+f)(\delta+\beta+f)} \cdot C_{\phi}+\eta+f \leq \phi\left(1+1 / \sqrt{C_{\phi}}\right)+3 f C_{\phi}^{2} .
$$

Proof. Let $y=\alpha+\beta$ and $z=\delta+\beta$, and without loss of generality suppose that $y \leq z$. Consider the $\phi_{1}$ term first, and for ease of reference, use superscript ${ }^{f}$ to refer to the version of the terms including $f$, e.g., $\phi_{1}^{f}=\sqrt{(y+f)(z+f)}$. We have $\phi_{1}^{f} \leq \sqrt{y z}+\sqrt{y f+z f}+f \leq \phi_{1}+\sqrt{2 f z}+f$. There are two cases.

Case 1: $2 f \leq z / C_{\phi}^{3 / 2}$. Then $\sqrt{2 z f} \leq \sqrt{z^{2} / C_{\phi}^{3}}=z / C_{\phi}^{3 / 2} \leq \eta / C_{\phi}^{3 / 2}$. Putting everything together gives $\phi^{f}=\phi_{1}^{f} C_{\phi}+\eta+f \leq\left(\phi_{1}+\eta / C_{\phi}^{3 / 2}+f\right) C_{\phi}+\eta+f=\phi+\eta / \sqrt{C_{\phi}}+f\left(C_{\phi}+1\right) \leq$ $\phi\left(1+1 / \sqrt{C_{\phi}}\right)+2 f C_{\phi}$.

Case 2: $2 f>z / C_{\phi}^{3 / 2}$. Then $\sqrt{2 f z} \leq \sqrt{(2 f)\left(2 f C_{\phi}^{3 / 2}\right)}=2 f C_{\phi}^{3 / 4}<2 f C_{\phi}$. Putting everything together gives $\phi^{f}=\phi_{1}^{f} C_{\phi}+\eta+f \leq\left(\phi_{1}+2 f C_{\phi}+f\right) C_{\phi}+\eta+f=\phi+2 f C_{\phi}^{2}+f C_{\phi}+f \leq \phi+3 f C_{\phi}^{2}$ for $C_{\phi} \geq 2$.

Taking the larger of the two cases for each term proves the claim.
The next lemmas provide tools to address the active vertices that are not fully related to the path. The lemmas themselves are purely algebraic and do not directly consider the random choices. In the lemma, $\phi_{1}$ is meant to capture the potential of the subproblem, whereas $\bar{\phi}_{1}$ captures the potential just with respect to those vertices fully related to the path. The motivation for these lemmas is that the potential always decreases down to $\bar{\phi}$ regardless of what pivot is selected. If a fully related pivot gets selected, the potential reduces even further. The goal is to quantify the balance between the cases, showing that the situation is at least as good as the situation in which all vertices are fully path related. When Lemma 4.12 is applied, $q$ shall be used to mean the result of Lemma 3.11, i.e., $q=\left(1 / \sqrt{2}+O\left(1 / C_{\phi}\right)\right)$.

Lemma 4.12. Let $g(\alpha, \beta, \delta)$ be any function over the number of ancestors, bridges, and descendents. Let $\bar{\alpha}, \bar{\beta}$, and $\bar{\delta}$ be counts that could arise by removing relationships between vertices, i.e., reducing any counts or moving bridges to ancestors/descendents. Let $p=(\bar{\alpha}+\bar{\beta}+\bar{\delta}) /(\alpha+\beta+\delta \leq 1$.

Suppose $g$ satisfies $g(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq \sqrt{p} \cdot g(\alpha, \beta, \delta)$ for all valid input values. Then for any $q \geq 2 / 3$,

$$
p \cdot q \cdot g(\bar{\alpha}, \bar{\beta}, \bar{\delta})+(1-p) \cdot g(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq q \cdot g(\alpha, \beta, \delta) .
$$

Proof. Substituting in $g(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq \sqrt{p} \cdot g(\alpha, \beta, \delta)$ gives $p \cdot q \cdot g(\bar{\alpha}, \bar{\beta}, \bar{\delta})+(1-p) \cdot g(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq$ $p q \sqrt{p} \cdot g(\alpha, \beta, \delta)+(1-p) \sqrt{p} \cdot g(\alpha, \beta, \delta)=\sqrt{p}(p q-p+1) g(\alpha, \beta, \delta)$. The claim follows as long as $\sqrt{p}(p q-p+1) \leq q$. Treat $q$ as a constant and observe how the function of $p$ changes. For $q=2 / 3$, the expression on the left is maximized at $p=1$, solving to exactly $q$. As $q$ increases, the maximum of the function shifts even further to the right, meaning that the expression is still maximized for $p \in[0,1]$ at $p=1$.

Lemma 4.13. The function $\phi$ satisfies the conditions of Lemma 4.12. Notably use $\phi$ to mean the potential applied to $\alpha, \beta$, and $\delta$, and $\bar{\phi}$ to mean the potential for $\bar{\alpha}, \bar{\beta}$, and $\bar{\delta}$, where the remainder of the notation is the same as Lemma 4.12. Then $\bar{\phi} \leq \sqrt{p} \cdot \phi$.

Proof. Also define $\eta=\alpha+\beta+\delta$ and $\bar{\eta}=\bar{\alpha}+\bar{\beta}+\bar{\delta}$. Use $\phi_{1}$ and $\phi_{2}$, respectively, to mean the potential applied to values $\alpha, \beta$, and $\delta$. Similarly use $\bar{\phi}_{1}$ and $\bar{\phi}_{2}$ for $\bar{\alpha}, \bar{\beta}$, and $\bar{\delta}$.

Bound $\phi_{1}$ and $\phi_{2}$ separately. The latter is trivial - $\phi_{2}=\eta$, so $\bar{\eta}=p \eta$ implies $\bar{\phi}_{2}=p \bar{\phi}_{2} \leq \sqrt{p} \bar{\phi}_{2}$ for $p \leq 1$. The $\phi_{1}$ bound is a little harder because $\beta$ is double-counted in $\phi_{1}$.

To bound $\phi_{1}$, consider the following. The fraction $p$ dictates how much $\alpha, \beta$, and/or $\delta$ must be reduced by in total. The worst case for the desired inequality is to maximize $\bar{\phi}_{1}$. Without loss of generality, suppose $\alpha \leq \delta$, The $\bar{\phi}_{1}$ term is maximized when $\bar{\alpha}$ and $\bar{\delta}$ are kept as large and as balanced as possible, so $\delta$ should be reduced first. If $p(\delta+\beta)>(\alpha+\beta)$, then only $\delta$ should be reduced, giving $\bar{\delta}+\bar{\beta} \leq p(\delta+\beta)$. The potential thus becomes $\bar{\phi}_{1}=\sqrt{(\bar{\alpha}+\bar{\beta})(\bar{\delta}+\bar{\beta})} \leq \sqrt{(\alpha+\beta)(p(\delta+\beta))}=\sqrt{p}$. $\phi_{1}$. If $p$ is smaller, then consider two phases $p_{1}$ and $p_{2}$ with $p=p_{1} p_{2}$ and $p_{1}=(\alpha+\beta) /(\delta+\beta)$. During the first, the potential is maximized as above, leaving $\bar{\phi}_{1}=\sqrt{p} \cdot \phi_{1}$ as above. For the second phase, $\alpha=\delta$, so the best choice to keep the expression maximized is to balance the reductions from both counts simultaneously. For this regime, $\bar{\phi}_{1}=\sqrt{(\bar{\alpha}+\bar{\beta})(\bar{\delta}+\bar{\beta})} \leq \sqrt{p_{2}(\alpha+\beta) p_{2}(\delta+\beta)}=p_{2} \phi_{1}$. Multiplying the two together gives a maximum value of $p_{2} \sqrt{p_{1}} \cdot \phi_{1} \leq \sqrt{p_{1} p_{2}} \cdot \phi_{1}$.

The next lemma pulls together all the pieces to argue that the expected reduction on $\phi(s)$ is still almost as good as previously.

Lemma 4.14. Consider any path-relevant subproblem $s=(G, P)$. Let $s_{1}, s_{2}, \ldots$ be random variables denoting its child subproblems in the flattened path-relevant tree, and suppose $N_{L} \geq$ $C_{\phi}^{2.5} \log n=\Omega\left(\log ^{6} n\right)$. Then $E\left[\phi\left(s_{1}\right)+\phi\left(s_{2}\right)+\cdots\right] \leq(\phi(s) / \sqrt{2})\left(1+O\left(1 / \sqrt{C_{\phi}}\right)\right)$.

Proof. Let $i$ denote the iteration during which at least one pivot $x_{j}$ that is partly related to the path $P$ is selected. Before that, the potential can only decrease. Let $\alpha=\left|A n c_{\widehat{d} D}(G, P)\right|$ denote the number of partly path-related vertices in the graph $G$ at the start of the iteration. Define $\beta$ and $\delta$ similarly for bridges and descendents, respectively. Let $\bar{\alpha}=\left|A n c_{\bar{d} D}(G, P)\right|$ denote the number of fully path-related ancestors at the start. Similarly for $\bar{\beta}$ and $\bar{\delta}$.

If any of the pivots is a bridge, there are no path-relevant subproblems (Lemma 4.8) and the potential is 0 .

Suppose instead that no pivot beyond the first is a bridge, which can only increase the potential. Then consider the pivots and corresponding partition steps in order, as in Lemma 4.8. Let $s_{1}$ denote the recursive subproblem generated by the first pivot, and let $s_{1, R}$ denote the nonrecursive subproblem corresponding to vertices not found by a the core searches. The second pivot partitions $s_{1, R}$ into recursive problem $s_{2}$ and remainder $s_{2, R}$. The third pivot partitions $s_{2, R}$, and so on. For $r$ pivots, the subproblems are $s_{1}, s_{2}, \ldots, s_{r}, s_{r+1}$, where $s_{r+1}=s_{r, R}$.

The goal is to bound $E\left[\sum_{i=1}^{r+1} \phi\left(s_{i}\right)\right]$ given that there is at least one partly path-related pivot. For each of the subproblems, let $\bar{\alpha}_{i}, \bar{\beta}_{i}$, and $\bar{\delta}_{i}$ denote the number of fully path-related vertices that are preserved, i.e., part of core searches. Let $\bar{\eta}_{i}=\bar{\alpha}_{1}+\bar{\beta}_{i}+\bar{\delta}_{i}$. Only the fully related vertices need be considered as these are the only ones that can be active at the next level. Also consider the result $\bar{\alpha}_{1, R}, \bar{\beta}_{1, R}$ and $\bar{\delta}_{1, R}$ of the first search. Let $f_{i}$ denote the number of path-active fringe nodes added, and let $f=\sum_{i=1}^{r+1} f_{i}$. For simplicity, double-count the fringe vertices, giving us the
following:

$$
\begin{align*}
& \sum_{i=1}^{r+1} \phi\left(s_{i}\right) \leq \phi\left(s_{1}\right)+\sum_{i=2}^{r+1}\left(\phi_{1}\left(s_{i}\right)+\phi_{2}\left(s_{i}\right)\right) \\
& \leq \phi\left(s_{i}\right)+\left(\sqrt{\left(\bar{\alpha}_{i}+\bar{\beta}_{i}+f_{i}\right)\left(\bar{\delta}_{i}+\bar{\beta}_{i}+f_{i}\right)}+\left(\bar{\eta}_{i}+f_{i}\right)\right) \\
& \leq \sqrt{\left(\bar{\alpha}_{1}+\bar{\beta}_{1}+f_{1}\right)\left(\bar{\delta}_{1}+\bar{\beta}_{1}+f_{1}\right)}+\left(\bar{\eta}_{1}+f_{1}\right) \\
&+\sqrt{\left(\sum_{i=2}^{r+1}\left(\bar{\alpha}_{i}+\bar{\beta}_{i}+f_{i}\right)\right)\left(\sum_{i=2}^{r+1}\left(\bar{\delta}_{i}+\bar{\beta}_{i}+f_{i}\right)\right)}+\sum_{i=2}^{r+1}\left(\bar{\eta}_{i}+f_{i}\right)  \tag{Lemma3.10}\\
& \leq\left(1+\frac{1}{\left.\sqrt{C_{\phi}}\right)\left(\sqrt{\left(\bar{\alpha}_{1}+\bar{\beta}_{1}\right)\left(\bar{\delta}_{1}+\bar{\beta}_{1}\right)}+\left(\bar{\eta}_{1}\right)\right.}\right. \\
&\left.\quad+\sqrt{\left(\sum_{i=2}^{r+1}\left(\bar{\alpha}_{i}+\bar{\beta}_{i}\right)\right)\left(\sum_{i=2}^{r+1}\left(\bar{\delta}_{i}+\bar{\beta}_{i}\right)\right)}+\sum_{i=2}^{r+1} \bar{\eta}_{i}\right)+3 f C_{\phi}^{2} \\
&=\left(1+\frac{1}{\left.\sqrt{C_{\phi}}\right)\left(\sqrt{\left(\bar{\alpha}_{1}+\bar{\beta}_{1}\right)\left(\bar{\delta}_{1}+\bar{\beta}_{1}\right)}+\bar{\eta}_{1}\right.}\right. \\
&\left.+\sqrt{\left(\bar{\alpha}_{1, R}+\bar{\beta}_{1, R}\right)\left(\bar{\delta}_{1, R}+\bar{\beta}_{1, R}\right)}+\bar{\eta}_{1, R}\right)+3 f C_{\phi}^{2} \tag{1}
\end{align*}
$$

(Lemma 4.11)

At this point, the multiple pivots and the fringe nodes have all been extracted from the main expression. Let $\bar{\phi}^{\prime}=\sqrt{\left(\bar{\alpha}_{1}+\bar{\beta}_{1}\right)\left(\bar{\delta}_{1}+\bar{\beta}_{1}\right)}+\bar{\eta}_{1}$ and $\bar{\phi}^{\prime \prime}=\sqrt{\left(\bar{\alpha}_{1, R}+\bar{\beta}_{1, R}\right)\left(\bar{\delta}_{1, R}+\bar{\beta}_{1, R}\right)}+\bar{\eta}_{1, R}$. The sum $\bar{\phi}^{\prime}+\bar{\phi}^{\prime \prime}$ looks exactly like the random variables generated from a single partition without fringe nodes. Note that to this point, no expectation has yet been applied; all manipulations thus far are just algebra on the random variables. Lemma 3.11 can thus be applied as long as the expectation is performed in a way consistent with Lemma 4.10 .

Lemma 4.10 applies if the pivot is fully path related. It does not if the pivot is only partly path related. There are thus cases depending on how the first pivot $x$ is classified. Let $p=$ $(\bar{\alpha}+\bar{\beta}+\bar{\delta}) /(\alpha+\beta+\delta)$ be the fraction of partly path-related vertices that are fully related. Let $F_{x}$ be the event that $x$ is fully related with the path. Let $\bar{\phi}=\sqrt{(\bar{\alpha}+\bar{\beta})(\bar{\delta}+\bar{\beta})}+\bar{\alpha}+\bar{\beta}+\bar{\delta}$. Then

$$
\begin{align*}
E\left[\bar{\phi}^{\prime}+\bar{\phi}^{\prime \prime}\right] & =\operatorname{Pr}\left\{F_{x}\right\} \cdot E\left[\bar{\phi}^{\prime}+\bar{\phi}^{\prime \prime} \mid F_{x}\right]+\left(1-\operatorname{Pr}\left\{F_{x}\right\}\right) \cdot E\left[\bar{\phi}^{\prime}+\bar{\phi}^{\prime \prime} \mid \neg F_{x}\right] \\
& \leq p \cdot E\left[\bar{\phi}^{\prime}+\bar{\phi}^{\prime \prime} \mid F_{x}\right]+(1-p) \cdot \bar{\phi}  \tag{Lemma3.10}\\
& \leq p \cdot\left(\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{C_{\phi}}}\right) \bar{\phi}+(1-p) \cdot \bar{\phi}  \tag{Lemma3.11}\\
& \leq\left(\frac{1}{\sqrt{2}}+\frac{2}{\sqrt{C_{\phi}}}\right) \phi(s)
\end{align*}
$$

(Lemmas 4.12 and 4.13)

Substituting back into Equation 1 gives

$$
\begin{array}{rlr}
E\left[\sum_{i=1}^{r+1} \phi\left(s_{i}\right)\right] & \leq \frac{1}{\sqrt{2}}\left(1+1 / \sqrt{C_{\phi}}\right)\left(1+2 \sqrt{2} / \sqrt{C_{\phi}}\right) \phi(s)+E\left[3 f C_{\phi}^{2}\right] \\
& \leq \frac{1}{\sqrt{2}}\left(1+6 / \sqrt{C_{\phi}}\right) \phi(s)+3 C_{\phi}^{2} \cdot E[f] & \quad \text { for } C_{\phi} \geq 4 \\
& \leq \frac{1}{\sqrt{2}}\left(1+6 / \sqrt{C_{\phi}}\right) \phi(s)+3 C_{\phi}^{2} \cdot O\left((\bar{\alpha}+\bar{\beta}+\bar{\delta}) \log n / N_{L}\right) \quad \text { (Lemma 4.7) } \\
& \leq \frac{1}{\sqrt{2}}\left(1+6 / \sqrt{C_{\phi}}\right) \phi(s)+3 C_{\phi}^{2} \cdot O\left(\phi(s) \log n / N_{L}\right) \\
& \leq \frac{1}{\sqrt{2}}\left(1+6 / \sqrt{C_{\phi}}\right) \phi(s)+\phi(s) \cdot O\left(1 / \sqrt{C_{\phi}}\right) \quad \text { for } N_{L} \geq C_{\phi}^{2.5} \log n \\
& =\frac{1}{\sqrt{2}}\left(1+O\left(\frac{1}{\sqrt{C_{\phi}}}\right)\right) \phi(s) &
\end{array}
$$

### 4.7 Analyzing the Layers in the Tree

Define the total potential $\Phi$ of a level as follows:

$$
\Phi\left(I_{r}\right)=\left(1+c_{\Phi} / \lg n\right)^{\lg n-r} \sum_{s \in I_{r}} \phi(s),
$$

where $I_{r}$ is the collection of subproblems corresponding to level $r$ in the flattened tree and $c_{\Phi}$ is a constant to be set later.

Corollary 4.15. Suppose $C_{\phi}=\Theta\left(\lg ^{2} n\right)$ and $N_{L}=\Omega\left(\lg ^{6} n\right)$ Then there exists a large-enough constant $c_{\Phi}$ such that $E\left[\Phi\left(I_{r}\right) \mid I_{r-1}\right] \leq \Phi\left(I_{r-1}\right) / \sqrt{2}$.

Proof. Choose $c_{\Phi}$ large enough so that $1+c_{\Phi} / \lg n$ is greater than the $\left(1+O\left(1 / \sqrt{C_{\phi}}\right)\right)=1+$ $O(1 / \log n))$ term in Lemma 4.14. The claim then follows from linearity of expectation over subproblems.

The real purpose of the extra $\left(1+c_{\Phi} / \lg n\right)^{\lg n-r}$ factor is to offset any potential increases to the subproblem potentials $\phi$. With an unlucky number of active fringe vertices, it is possible that $\Phi$ increase when going from one row to the next. Such an increase, called a fringe failure, would preclude the application of Theorem 2.1. The next lemma shows that fringe failures are unlikely.

Lemma 4.16. There exist constants $c_{\Phi}$ and $c_{L}$ such that: for $C_{\phi}=\Theta\left(\lg ^{2} n\right)$ and $N_{L} \geq c_{L} C_{\phi}^{3} \lg n=$ $\Omega\left(\log ^{7} n\right), \operatorname{Pr}\left\{\Phi\left(I_{r+1}\right)>\Phi\left(I_{r}\right)\right\} \leq 1 /(8 \lg n)$.

Proof. Even if all active vertices are preserved, Lemma 3.10 states that the subproblem potentials $\sum_{s} \phi(s)$ cannot increase without the addition of fringe nodes. The active fringe nodes themselves have two contributions (see Lemma 4.11): a multiplicative $\left(1+O\left(1 / \sqrt{C_{\phi}}\right)\right)$ overhead, and an additive $3 C_{\phi}^{2} f$. The former does not depend on the number of fringe nodes, so choose $c_{\Phi} / \lg n$ to be say twice as large as the $O\left(1 / \sqrt{C_{\phi}}\right)$ term. Thus for $\Phi$ to increase would require that the total contribution from $f$ fringe nodes exceed $f \geq c_{\Phi} /(2 \lg n) \sum_{s \in I_{r}} \phi(s)$. For large enough $N_{L}$, the expected number of fringe nodes is $E[f]=O\left(\sum_{s \in I_{r}} \phi(s) \log n / N_{L}\right) \leq \sum_{s \in I_{r}} \phi(s) /\left(3 C_{\phi}^{3}\right)$ giving
an expected potential contribution of $\sum_{s \in I_{r}} \phi(s) / C_{\phi}$. For large enough $c_{\Phi}$ and $C_{\phi}=\Theta\left(\lg ^{2} n\right)$, this expectation is at $\operatorname{most} c_{\Phi} /\left(16 \lg ^{2} n\right) \sum_{s \in I_{r}} \phi(s)$. Reaching the target threshold would require being $8 \lg n$ times the expectation, which occurs with probability at most $1 /(8 \lg n)$ by Markov's inequality.

To prevent $\Phi$ from increasing at all, instead define $\Phi^{\prime}$ to be equal to $\Phi$, except that it drops to 0 when a fringe failure occurs. It follows that (1) $\Phi^{\prime}$ never increases, and (2) $E\left[\Phi^{\prime}\left(I_{r+1} \mid I_{r}\right] \leq\right.$ $E\left[\Phi\left(I_{r+1} \mid I_{r}\right]\right.$. Theorem 2.1 can now be applied.

Lemma 4.17. Let $\hat{G}=(\hat{V}, \hat{E})$ be a directed graph, and let $n=|\hat{V}|$. There exists a setting of $D=\Theta\left(n^{2 / 3} \log ^{4 / 3} n\right), C_{\phi}=\Theta\left(\log ^{2} n\right), N_{L}=\Theta\left(\log ^{7} n\right)$, and $\epsilon_{\pi}=O\left(1 / \log ^{3} n\right)$ such that the following holds.

Consider any directed path $\hat{P}$ with length $(\hat{P}) \leq D$ from $u$ to $v$. Let $S$ be the shortcuts produced by an execution of Algorithm 图 on $\hat{G}$ with starting $h=\lg n$. Then with probability at least $5 / 8$ : there exists a path from $u$ to $v$ in $G_{S}=(\hat{V}, \hat{E} \cup S)$ consisting of at most $D / 2$ arcs.

Proof. The starting value of $\Phi^{\prime}\left(I_{0}\right) \leq\left(1+c_{\Phi} / \lg n\right)^{\lg n}\left(C_{\phi}+1\right) n=O\left(n \log ^{2} n\right)$. For large enough constant $w$, Theorem 2.1 states that $\operatorname{Pr}\left\{\Phi^{\prime}\left(I_{r+w}\right)>(1 / \sqrt{2})^{r} O(n \lg n)\right\}<1 / 8$. Then for $r=$ $(2 / 3) \lg n+(4 / 3) \lg \lg n+\Theta(1)$, this expression reduces to $\operatorname{Pr}\left\{\Phi^{\prime}\left(I_{r+w}\right)>c n^{2 / 3} \lg ^{4 / 3} n\right\}<1 / 8$, for some constant $c$.

If a fringe failure occurs, the bound on $\Phi^{\prime}$ is meaningless. The probability of a fringe failure is at most the union bound over $r<\lg n$ levels of the failure probability $1 /(8 \lg n)$ from Lemma 4.16, which reduces to $1 / 8$. If neither of these failures occurs, the bound on $\Phi^{\prime}$ implies a bound on active unshortcutted subpaths, as all path vertices counted as bridges towards $\Phi^{\prime}$. Thus, with failure probability $1 / 4$, the total length of all subpaths in path-relevant subproblems in level- $(r+2)$ is at most $O\left(n^{2 / 3} \lg ^{4 / 3} n\right)$.

Finally, consider the concatenations and shortcutted leaves via Lemma 4.9. With failure probability $1 / 8$, the total shortcutted length is thus $O\left(n^{2 / 3} \lg ^{4 / 3} n\right)$. Choose $D$ to be a constant factor larger than the constant hidden inside the big-O.

### 4.8 Completing the Proof of Lemma 4.1

This section proves bounds on the number of shortcuts and overall work performed. The main goal is to show that, with probability at least $7 / 8$, the number of vertices (and hence shortcuts) and arcs visited by searches is consistent with Lemma 4.17. Thus, with probability at least $1 / 2$, both Lemma 4.17 and this bound hold. Combining with Lemma 4.17, this yields a proof of Lemma 4.1 and hence also Theorem 4.2,

The settings used are $D=\Theta\left(n^{2 / 3} \log ^{4 / 3} n\right), C_{\phi}=\Theta\left(\log ^{2} n\right), N_{L}=\Theta\left(\log ^{7} n\right), \epsilon_{\pi}=\Theta\left(1 / \log ^{3} n\right)$, and $N_{k}=\Theta\left(\log ^{4} n\right)$, as dictated by constraints offered in previous lemmas. The maximum search distance is immediate: it is at most $h D N_{k} N_{L}=O\left(\log n \cdot n^{2 / 3} \log ^{4 / 3} n \cdot \log ^{4} n \cdot \log ^{7} n\right)=$ $O\left(n^{2 / 3} \log ^{14} n\right)$.

Consider each level of recursion in Algorithm 3. Corollary 4.5 holds with high probability, so assume that it holds for every node at every iteration. Consider any vertex in the iteration in which it is visited by a core search. By assumption of Corollary 4.5, the vertex, and hence its incident arcs, is visited by at most $O(\log n)$ searches.

The number of vertices (and hence arcs) may increases with each level in the tree due to fringe searches. The final step of the proof is to argue that the total size of all subproblems in the final
level of recursion is $O(n)$ vertices and $O(m)$ arcs, and hence the total cost of all levels is $O\left(n \log ^{2} n\right)$ vertices and $O\left(m \log ^{2} n\right)$ arcs.

By Lemma 4.7 with $N_{L}=\log ^{7} n$, the number of vertices and arcs increases with each level of recursion by an additive $O\left(n^{\prime} / \log ^{2} n\right)$ and $O\left(m^{\prime} / \log ^{2} n\right)$ in expectation, where $n^{\prime}$ and $m^{\prime}$ are the current numbers at that level. Thus, by Markov's inequality, with probability at most $1 /(8 \lg n)$, the increases is not more than a multiplicative $1+O(1 / \log n)$. Since there are $\lg n$ levels of recursion, this results in probability at most $1 / 8$ of exceeding a total of $n(1+O(1 / \log n))^{\lg n}=O(n)$ vertices and $m(1+O(1 / \log n))^{\lg n}=O(m)$ arcs.

## 5 Parallel Version

This section analyzes a parallel version of Algorithm 3 and Algorithm 4. This section assumes the reader is comfortable enough with parallel algorithms to infer the details, instead focusing only on the interesting issues.

The main results are as follows.
Theorem 5.1. There exists a randomized parallel algorithm taking as input a directed graph $\hat{G}=$ $(\hat{V}, \hat{E})$ with the following guarantees. Let $n=|\hat{V}|, m=|\hat{E}|$, and without loss of generality assume $m \geq n / 2$. Then (1) the algorithm produces a size- $O\left(n \log ^{4} n\right)$ set $S^{*}$ of shortcuts; (2) the algorithm has $O\left(m \log ^{6} n+n \log ^{10} n\right)$ work; (3) the algorithm has $O\left(n^{2 / 3} \log ^{21} n\right)$ span; and (4) with high probability, the diameter of $G_{S^{*}}=\left(\hat{V}, \hat{E} \cup S^{*}\right)$ is $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$.

Corollary 5.2. There exists a randomized parallel CREW algorithm for digraph reachability that has work $O\left(m \log ^{6} n+n \log ^{10} n\right)$ work and $O\left(n^{2 / 3} \log ^{21} n\right)$ span, both with high probability.

Proof. Perform the diameter reduction algorithm, then run a standard parallel BFS but limited to $O\left(n^{2 / 3} \log ^{4 / 3}\right)$ hops. The work and span of the diameter reduction dominates. If the BFS completes in the prescribed number of rounds, the algorithm terminates. Otherwise, keep repeating the diameter reduction and BFS until successful.

Model. This paper adopts the de facto standard work-span model [4, also called work-time [9] or work-depth model, which abstracts low-level details of the machine such as the number of processors or how parallel tasks are scheduled. The work-span model allows algorithms to be expressed through the inclusion of parallel loops, i.e., a parallel foreach. A parallel foreach indicates that each task corresponding to a loop iteration may execute in parallel, and that all parallel tasks must complete before continuing to the next step after the loop. It is generally straightforward to map algorithms from the work-span model to a PRAM model; see, e.g., [9, 12]. Like the asynchronous PRAM model [6, the work-span model requires that algorithmic correctness not be tied to any assumptions about how tasks are scheduled beyond the explicit ordering imposed by the loops. That is to say, it should not be assumed that the instructions across iterations execute in lock step.

The work of an algorithm is the same as the sequential running time in a RAM model if all parallel loops are replaced by sequential loops. When multiple tasks are combined through a parallel loop, the span of the composition is the maximum of the span of the individual subproblems, plus the span of the loop itself. There are several variants to the work-span model. In a binary-forking model such as [4], the span of a $k$-way loop is $\Theta(\lg k)$. Much of the literature on parallel algorithms, however, adopts an unlimited-forking model, where the span of launching $k$ parallel tasks adds $O(1)$ to the span. Since many of the subroutines employed are analyzed in this model, this paper
adopts the unlimited-forking model. PRAM algorithms, for example, correspond to an unlimited forking model. Both models only differ by logarithmic factors in the span.

The algorithm is a concurrent-read exclusive-write ( $\boldsymbol{C R E W}$ ) algorithm. CREW means that multiple parallel tasks may read the same data, but they may not write to the same location 10

## Performing Concurrent Searches

The key subroutine in Algorithm 3 are the $d D$-limited searches to find, e.g., $R_{j}^{+}$. One might simply replace the foreach loops by parallel loops, but the question is how the bookkeeping should be performed. Ordinarily, a BFS keeps track of already-visited vertices by either annotating vertices in the graph directly, or equivalently by keeping an extra array indexed by vertex. A natural way to perform multiple searches in parallel using a CREW algorithm would thus be to duplicate the bookkeeping efforts for each parallel search, but doing so would increase the work dramatically just to copy the graph or initialize the arrays.

The key property that allows an efficient implementation is Corollary 4.5 - with high probability, no vertex is visited by more than $O(\log n)$ parallel searches. The implementation may assume that this is the case, and just abort by returning immediately if a vertex gets visited too many times.

The main goal is to support the following for each call to ParSC.
Lemma 5.3. Consider an iteration $i$ in call to ParSC on graph $G=(V, E)$. Let $n_{e}$ be the total number of arcs traversed by searches, counting each arc once per search that reaches it. Then an iteration can be implemented with $O\left(n_{e} \log ^{2} n+\left|X_{i}\right| \log n\right)$ work and $O\left(n^{2 / 3} \log ^{15}\right)$ span.

The remainder of the section is devoted to exhibiting an algorithm that proves Lemma 5.3.
The set of searches from $X_{i}$ (in one direction) are grouped together as a single modified BFS. Rather than marking a vertex with a single bit indicating whether it has been discovered, a vertex is tagged with a list of IDs of the pivots that have reached it. Every time this list of IDs changes, the vertex may be re-added to the frontier and all of its outgoing arcs explored again. Since a vertex is not visited too many times, the overhead is not too high.

In more detail, the algorithm is as follows. At the start of the call to ParSC, initialize $\Theta(\log n)$ space for each vertex to record the ID tags, initally all null. Use an array to store the frontier vertices along with the ID of the pivot from which this search originated; a vertex may appear in the frontier multiple times from different pivots. Save all frontiers so as to identify all vertices reached by the searches at the end and also to record all new shortcuts.

To start a set of searches from $\left|X_{i}\right|$, copy all live pivots $x_{j}$ to the frontier array and associate with each pivot its own ID as the originator of the search. Also update each pivot's tag list to include itself.

Each round of the BFS operates as follows. Foreach vertex in the frontier in parallel, identify the number of outgoing arcs. Next, perform parallel prefix sums so that each arc has a distinct index in the next frontier array. Foreach arc $(u, v)$ in parallel, let $x_{j}$ be the associated pivot ID. Check whether $v$ 's ID set includes $x_{j}$; this check can be performed in $O(\log n)$ sequential time (both work and span) by scanning through $v$ 's tag list. If $x_{j}$ is not present, record $v$ and $x_{j}$ in $(u, v$ )'s slot in the next frontier; otherwise record null.

At this point, a vertex may appear many times in the frontier list, even from a single search. Sort the frontier list by vertex (high priority) and pivot ID (lower priority). Remove duplicate

[^5]entries with a compaction pass. Now each vertex appears at most once for each search, so $O(\log n)$ times in total. For each slot $j$ in the next frontier in parallel, let $v$ be the vertex stored there. Check whether this is the first slot for vertex $v$, i.e., if $j-1$ stores a different vertex. If so, scan through the $O(\log n)$ next slots (sequentially), and for each entry of $v$ append the pivot tag to $v$ 's tag list.

Repeat this process for the number of rounds dictated by the distance $d D$ for the core searches. Extend the search another $D$ rounds for the fringe search, but use a different tag list and frontier array going forward.

When the searches complete, sort the arrays of all vertices reached by core searches and by fringe searches. Foreach vertex $v$ in core searches, in parallel, identify the lowest ID pivot reaching $v$. Again use parallel prefix sums and then copy the lowest-ID occurrence of $v$ to a new array for the recursive searches. Finally, sort the new array by pivot ID so that all vertices in the same induced subgraph are adjacent. Building the induced subgraphs for recursive calls can again be accomplished with arc counting, prefix sums, and sorting.

Updating $G\left[V_{R}\right]$. One could build $G\left[V_{R}\right]$ explicitly, but doing so would require processing the full graph. The goal expressed by Lemma 5.3 is to have work proportional to the number of arcs reached, but $G\left[V_{R}\right]$ could be much larger. Instead, simply mark vertices in $V$ as dead when they have been reached by a core search. Augment the search to ignore dead vertices.

Completing the proof of Lemma 5.3. The basic subroutines used in each round such as prefix sums, compaction, etc, can all be performed in linear work and $O(\log n)$ span. (See e.g., [9.) Scanning the list of tags also requires $O(\log n)$ work per arc on the frontier and $O(\log n)$ span as it is performed sequentially. Using Cole's merge sort [2], the cost of a sort is $O(\log n)$ work per element sorted and $O(\log n)$ span. Multiplying the search distance by $O(\log n)$ thus gives the overall span bound. Since each arc may be reached by $O(\log n)$ searches, the bound is $O\left(\log ^{2} n\right)$ work per arc visited.

## Aborting Algorithm 3

To make the work (and shortcut) bound deterministic, Algorithm 4 needs the ability to abort any runs of Algorithm 3 that exceed the target work bound. (Exceeding the shortcut bound can be handled simply discarding the result - a true abort is not necessary there.)

Unfortunately, the proof of Lemma 4.1 examines the work in aggregate across levels in the recursion tree. It is likely that individual recursive subproblems will do more work, so abort decisions are not local.

One simple option is to augment the algorithm to check the elapsed time, and to return immediately if some threshold has been reached. Technically, however, this solution violates the work-span model as the target time bound would depend on both on how efficiently the program is scheduled and on the number of processors employed.

There is a solution in the work-span model - logically implement the recursive steps of the algorithm as a BFS. That is, maintain an array of subproblems, initially just $\operatorname{ParSC}(\hat{G}, \lg n)$. To implement a level of recursive, perform prefix sums to add up the total number of vertices across all subproblem, and give each vertex (pivot) a specific slot to put its recursive subproblem. Instead of launching the recursive subproblems immediately, simply record them in the appropriate slot. When all subproblems at this level of recursion complete, launch all problems at the next level (in parallel).

The work bound can only be exceeded if the total number of arcs in the next set of subproblems grows too large. This number can be counted with a parallel reduce after each level of recursion
completes. None of these steps increase the work or span asymptotically.
Proof of Theorem [5.1. The diameter and shortcut bounds are directly from Theorem 4.2, Multiplying the cost per arc of Lemma 5.3 with the number of arcs searched in Theorem 4.2 gives the work bound. Shortcuts can be gathered and larger graphs built for each iteration of Algorithm 4 by sorting, but that is dominated by the other work performed.

The span bound is obtained by multiplying the maximum search distance of $O\left(n^{2 / 3} \log ^{14} n\right)$ by the $O(\log n)$ span per BFS round, the $N_{k}=O\left(\log ^{4} n\right)$ iterations in a call, the $O(\log n)$ levels of recursion in a run of Algorithm 3, and the $O(\log n)$ iterations of the outer loop of Algorithm 4, Note that the inner loop of Algorithm 4 can be implemented in parallel. All together, that gives $O\left(n^{2 / 3} \log ^{21} n\right)$ span.

## 6 Building a Directed Spanning Tree

This section discusses how to augmented the algorithm to produce a directed spanning tree. It is not immediately obvious how to do so even for the sequential algorithm of Section 3. To illustrate the issues, consider the following graph: $s \rightarrow u \rightarrow v \rightleftarrows w$. If $w$ is selected as a pivot first, then a shortcut $(s, w)$ is added and a BFS from $s$ in the shortcutted graph may discover the following path: $s \rightsquigarrow w \rightarrow v$. Simply splicing in the path corresponding to the shortcut would result in $s \rightarrow u \rightarrow v \rightarrow w \rightarrow v$, which is no longer a simple path. The goal is to do this splicing, but in a way that avoids repeated vertices. The situation is slightly more challenge in the case of Algorithm 4 because the arcs shortcutted could themselves be shortcuts, but the result is just that several iterations are needed.

The algorithm for building the directed spanning tree is a postprocessing step performed after the full execution of Algorithm 4. The algorithm references the BFS trees used to build shortcuts, however, so all BFS trees need to be saved as Algorithm 4 executes. Each shortcut must also be augmented with a reference to the BFS tree that produced it. The forward-search BFS trees are directed out from the root, whereas the backward-search BFS trees are directed towards the root. In this way, the BFS trees correspond to arcs in some graph.

Let $G_{0}, G_{1}, \ldots, G_{k=\Theta(\log n)}$, where $G_{0}=\hat{G}$, denote the sequence of graphs built after each iteration of the outer loop of Algorithm 4. Running BFS on the resulting graph $G_{k}$ yields a directed spanning tree $T_{k}$ in $G_{k}$. This section describes how to transform a directed spanning tree $T_{i}$ in graph $G_{i}$ to a directed spanning tree $T_{i-1}$ in $G_{i-1}$. Iterating $\Theta(\log n)$ times gives a spanning tree in the original graph.

Start each iteration by labeling every vertex $v$ in the tree with label $\operatorname{low}(v)=0$ and $\operatorname{high}(v)$, where $\operatorname{high}(v)$ is $v$ 's distance from the root in $T_{i}$. This can be accomplished in linear work and logarithmic span using the Euler-tour method [19.

For the next step, the shortcuts in both directions are treated differently. The goal is to essentially splice in the paths, which results in vertices appearing multiple times. This multiplicity will be resolved afterwards.

For each vertex $u \in T_{i}$ in parallel, traverse all forward-search BFS trees created in iteration $i$ and rooted at $u$. Label those vertices $v$ with $\operatorname{high}(v)=\operatorname{high}(u)$, and $\operatorname{low}(v)$ is $v$ 's depth (or distance from $u$ ) in the tree. Note that these labelings should be performed on the BFS trees themselves, not on $T_{i}$ or $G_{i}$, as each vertex may belong to multiple trees and may otherwise be labeled multiple times concurrently 11

[^6]For the backward direction, consider all $\operatorname{arcs}(u, v)$ in $T_{i}$ in parallel. If $(u, v)$ is a shortcut on a backward-search BFS tree rooted at $v$, traverse the path from $u$ to $v$ in the BFS tree and label each vertex $w$ on the path by $\operatorname{high}(w)=\operatorname{high}(u)$. Also label low $(w)$ with $w$ 's distance from $u$ on the path.

Finally, sort all arcs $(u, v)$ in the collection of BFS trees traversed in the above process, as well as the arcs in $T_{i}$ that also exist in $G_{i-1}$, by three values: $v$ 's ID (most significant), high (u), and low $(u)$ (least significant). Foreach $\operatorname{arc}(u, v)$ in the sorted list in parallel, if it is the first arc directed toward $v$ in sorted order then include the arc $(u, v)$ in $T_{i-1}$.

Lemma 6.1. Suppose that $T_{i}$ is a directed spanning tree in $G_{i}$ rooted at vertex $s$. Then the $T_{i-1}$ produced is a directed spanning tree rooted at vertex $s$ consisting of only arcs present in $G_{i-1}$.

Proof. Since only arcs present in $G_{i-1}$ are considered in the last step of the algorithm, $T_{i-1}$ is a subgraph of $G_{i-1}$. It is not obvious, however, that it is a tree, nor is it obvious that it spans.

The first step is to show that every vertex, except $s$, has an incoming arc in $T_{i-1}$. Consider a vertex $v$ and its incoming arc $(u, v)$ in $T_{i}$. If $(u, v)$ is present in $G_{i-1}$ as well, then it is in consideration the last step, so $v$ must select an arc. If $(u, v)$ is a shortcut, then it corresponds to some path in a BFS tree. All arcs in that path, and specifically the arc directed toward $v$, are also in consideration. Thus, $v$ has an incoming arc.

For each vertex, let the final label be the lowest label associated with any of its copies. If all arcs go from lower label to higher label, then there are no cycles. To prove this is the case, the claim is that every copy of each vertex other than the source (and in particular the lowest-label copy) has an incoming arc from a vertex with a lower label. Since the minimum incoming arc is the one used, that would imply that all arcs are from lower to higher label.

To prove the claim, consider a copy of vertex $v$. There are three cases. If $v$ is in a forward-search BFS tree and not the root, then $v$ has depth (and hence low $(v)$ label) one higher than its parent in the tree. If $v$ is in a backward-search path and not the source, the same argument holds.

Otherwise, $v$ 's label is the same as in $T_{i}$. In $T_{i}, v$ 's incoming $\operatorname{arc}(u, v)$ satisfies $\operatorname{high}(u)<\operatorname{high}(v)$ by construction. If $(u, v)$ is in $G_{i-1}$, then this arc satisfies the claim. Otherwise, $v$ the non-root of a BFS tree with a strictly lower high value, and hence one of the first two cases applies.

## 7 Conclusions

This work makes the first major progress toward work-efficient parallel algorithms for directed graphs, but it also exposes several new questions. First, can the performance be improved? Shaving logarithmic factors would be nice, but doing so seems premature - it is quite likely that $\tilde{O}\left(n^{2 / 3}\right)$ is not the final answer. I would conjecture that an $n^{1 / 2+o(1)}$-diameter reduction is possible using a more sophisticated algorithm based on the one presented herein.

Is true work efficiency, i.e., $O(m)$ work, possible for the diameter-reduction problem? Achieving that would require first producing an $O(m)$-time sequential algorithm for the problem.

Hesse's lower bound provides a lower bound on work-efficient diameter reduction, but that is not a general lower bound on digraph reachability. Can digraph reachability be improved by relaxing the shortcutting requirements, perhaps by adopting some ideas from Spencer's algorithm? Are there good general lower bounds for work/span tradeoffs of these algorithms?

Finally, can the algorithm be extended to solve unweighted shortest paths?
tree in the first place.

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[^0]:    ${ }^{1}$ In addition to uncluttering the bounds, ignoring logarithmic factors is particularly convenient when comparing parallel algorithms - the precise bounds depend on the specifics of the parallel model, but the bounds typically only vary by logarithmic factors (see [9] for discussion) - allowing us to focus on the high-level discussion.
    ${ }^{2}$ The standard definition for soft- $O$ is that $f(n) \in \tilde{O}(g(n))$ if $f(n) \in O(g(n)$ poly $(\log g(n)))$. This paper uses $f(n) \in \tilde{O}(g(n))$ to mean $f(n) \in O(g(n)$ poly $(\log n))$, with the only relevant difference being the meaning of $\tilde{O}(1)$.
    ${ }^{3}$ Older PRAM literature often characterizes algorithms by a number of processors and parallel running time. Span here is generally equivalent to parallel time, and work corresponds to the product of processors and time.

[^1]:    ${ }^{4}$ The work bound stated by Ullman and Yannakakis [20] is worse, for small $\rho$, than the bound displayed in Table 1 . The table shows the improved bound observed by Schudy 17 .
    ${ }^{5}$ Closing the gap between $n^{1 / 17}$ and $\sqrt{n}$ is an interesting open question but not addressed by this paper.

[^2]:    ${ }^{6}$ Karp states the theorem very differently. The process described here corresponds to his recurrence $T(I)=$ $a(\Phi(I))+T(h(I))$, where $a(x)=0, x<d$ and $a(x)=1, x \geq d$, for $d=p^{k} \cdot \Phi\left(I_{0}\right)$. This recurrence counts the number of steps to reach the target size. (Note that $d$ depends only on the initial instance and is constant in the recurrence.) The deterministic counterpart is $\tau(x)=a(x)+\tau(p x)$, which has solution $u\left(\Phi\left(I_{0}\right)\right)=\left\lceil\log _{1 / p}\left(\Phi\left(I_{0}\right) / d\right)\right\rceil \leq k+1$.
    ${ }^{7}$ With high probability means the failure probability can be driven down to $1 / n^{c}$ for any constant $c$ by increasing the constants hidden inside the big- $O$ notation (specifically the running time and number of shortcuts here).

[^3]:    ${ }^{8} \operatorname{Read} N_{L}$ as "number of layers".

[^4]:    ${ }^{9}$ The $n$ here can be anything, but the intent is $n=|\hat{V}|$, the number of vertices in the full graph.

[^5]:    ${ }^{10}$ CREW is usually a restriction applied to the PRAM 5 7 76 machine model, e.g., a CREW PRAM. In contrast, the work-span model is an algorithmic cost model, not a machine model. This paper proposes lifting the CREW qualifier to the work-span level rather than the PRAM level.

[^6]:    ${ }^{11}$ The Euler-tour technique could be applied to each tree, but a parallel BFS is sufficient here as the trees have depth $O\left(n^{2 / 3} \log ^{4 / 3} n\right)$ by construction; the work and span would be at most the work and span of constructing the

