$(2P_2, K_4)$ -Free Graphs are 4-Colorable

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Abstract

In this paper, we show that every $(2P_2, K_4)$ -free graph is 4-colorable. The bound is attained by the five-wheel and the complement of the seven-cycle. This answers an open question by Wagon [19] in the 1980s. Our result can also be viewed as a result in the study of the Vizing bound for graph classes. A major open problem in the study of computational complexity of graph coloring is whether coloring can be solved in polynomial time for $(4P_1, C_4)$ -free graphs. Lozin and Malyshev [15] conjecture that the answer is yes. As an application of our main result, we provide the first positive evidence to the conjecture by giving a 2-approximation algorithm for coloring $(4P_1, C_4)$ -free graphs.

Keywords: graph coloring; χ -bound; forbidden induced subgraphs; approximation algorithm.

AMS subject classifications: 68R10, 05C15, 05C75, 05C85.

1 Introduction

All graphs in this paper are finite and simple. We say that a graph G contains a graph H if H is isomorphic to an induced subgraph of G. A graph G is H-free if it does not contain H. For a family of graphs \mathcal{H} , G is \mathcal{H} -free if G is H-free for every $H \in \mathcal{H}$. In case \mathcal{H} consists of two graphs, we write (H_1, H_2) -free instead of $\{H_1, H_2\}$ -free. As usual, let P_n and C_n denote the path and the cycle on n vertices, respectively. The complete graph on n vertices is denoted by K_n . The n-wheel W_n is the graph obtained from C_n by adding a new vertex and making it adjacent to every vertex in C_n . For two graphs G and H, we use G + H to denote the disjoint union of G and H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. The complement of G is denoted by \overline{G} . A hole in a graph is an induced cycle of length at least 4. A hole is odd if it is of odd length.

A q-coloring of a graph G is a function $\phi : V(G) \longrightarrow \{1, \ldots, q\}$ such that $\phi(u) \neq \phi(v)$ whenever u and v are adjacent in G. We say that G is q-colorable if G admits a q-coloring. The chromatic number of G, denoted by $\chi(G)$, is the minimum number q such that G is q-colorable. The clique number of G, denoted by $\omega(G)$, is the size of a largest clique in G. Obviously, $\chi(G) \geq \omega(G)$ for any graph G. The maximum degree of a graph G is denoted by $\Delta(G)$.

A family \mathcal{G} of graphs is said to be χ -bounded if there exists a function f such that for every graph $G \in \mathcal{G}$ and every induced subgraph H of G it holds that $\chi(H) \leq f(\omega(H))$. The function fis called a χ -binding function for \mathcal{G} . The class of perfect graphs (a graph G is *perfect* if for every induced subgraph H of G it holds that $\chi(H) = \omega(H)$), for instance, is a χ -bounded family with χ -binding function f(x) = x. Therefore, χ -boundedness is a generalization of perfection. The notion of χ -bounded families was introduced by Gyárfás [10] who make the following conjecture.

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Conjecture 1 (Gyárfás [9]). For every forest T, the class of T-free graphs is χ -bounded.

Gyárfás [10] proved the conjecture for $T = P_t$: every P_t -free graph G has $\chi(G) \leq (t - 1)^{\omega(G)-1}$. The result was slightly improved by Gravier, Hoàng and Maffray in [8] that every P_t -free graph G has $\chi(G) \leq (t-2)^{\omega(G)-1}$. This implies that every P_5 -free graph G has $\chi(G) \leq 3^{\omega(G)-1}$. Note that this χ -binding function is exponential in $\omega(G)$. For $\omega(G) = 3$, Esperet, Lemoine, Maffray and Morel [3] obtained the optimal bound on the chromatic number: every (P_5, K_4) -free graph is 5-colorable. They also demonstrated a (P_5, K_4) -free graph whose chromatic number is 5. On the other hand, a polynomial χ -binding function for the class of $2P_2$ -free graphs was shown by Wagon [19] who proved that every such graph has $\chi(G) \leq {\omega(G)+1 \choose 2}$. This implies that every $(2P_2, K_4)$ -free graph is 6-colorable. In [19] it was asked if there exists a $(2P_2, K_4)$ -free graph whose chromatic number is 5 or 6. We observe that the (P_5, K_4) -free graph with chromatic number 5 given in [3] contains an induced $2P_2$.

In this paper we settle Wagon's question [19] by proving the following theorem.

Theorem 1. Every $(2P_2, K_4)$ -free graph G has $\chi(G) \leq 4$.

The bound in Theorem 1 is attained by the five-wheel W_5 and the complement of a sevencycle $\overline{C_7}$. Hence, we obtain the optimal χ -bound for the class of $2P_2$ -free graphs when the clique number is 3. A family \mathcal{G} of graph is said to satisfy the Vizing bound if f(x) = x + 1 is a χ -binding function for \mathcal{G} . The definition was motivated by the classical Vizing's Theorem [18] on the chromatic index $\chi'(G)$ of graphs which states that $\chi'(G) \leq \Delta(G) + 1$ for any graph G. This is equivalent to say that the class of line graphs satisfies the Vizing bound. Our result (Theorem 1) shows that the class of $(2P_2, K_4)$ -free graphs also satisfies the Vizing bound. We refer to Randerath and Schiermeyer [17] and Fan, Xu, Ye and Yu [4] for more results on the Vizing bound for various \mathcal{H} -free graphs.

We also note that our proofs of Theorem 1 below are algorithmic: one can easily follow the steps of the proof and give a 4-coloring of the input graph in polynomial time.

An application. Let COLORING denoted the computational problem of determining the chromatic number of a graph. In the past two decades, there has been an overwhelming attention on the complexity of COLORING H-free graphs. The starting point is a result due to Král', Kratochvíl, Tuza, and Woeginger [14] who gave a complete classification of the complexity of COLORING for the case where \mathcal{H} consists of a single graph H: if H is an induced subgraph of P_4 or of $P_1 + P_3$, then COLORING restricted to *H*-free graphs is polynomial-time solvable, otherwise it is NP-complete. Afterwards, researchers started to study COLORING restricted to (H_1, H_2) -free graphs. Despite much efforts of top researchers in the area the complexity of COLORING are known only for some pairs of H_1 and H_2 , see [6] for a summary of the known partial results. Even solving the problem for particular pairs of H_1 and H_2 requires substantial work, see [2, 16, 11, 12, 15, 13] for instance. Lozin and Malyshev [15] demonstrated that the classification is already problematic even if both H_1 and H_2 are 4-vertex graphs: they determined the complexity of COLORING for all such pairs with three exceptions. One of the three unknown pairs is $(4P_1, C_4)$. Lozin and Malyshev [15] conjecture that COLORING can be solved in polynomial time for $(4P_1, C_4)$ -free graphs. The problem was listed as an important open problem in the survey on the computational complexity of coloring graphs with forbidden subgraphs by Golovach, Johnson, Paulusma and Song [6].

Here we use Theorem 1 to give a 2-approximation algorithm for coloring $(4P_1, C_4)$ -free graphs. This is the first general result towards a polynomial-time algorithm for the problem, although Fraser, Hamel, Hoàng, Holmes, and LaMantia showed that the problem is polynomial time solvable for a subclass of $(4P_1, C_4)$ -free graphs [5]. For a graph G and a subset $S \subseteq V(G)$, we denote by G[S] the subgraph of G induced by S. A graph is *chordal* if it is C_t -free for each $t \ge 4$.

Theorem 2. There exists a polynomial-time 2-approximation algorithm for coloring $(4P_1, C_4)$ -free graphs.

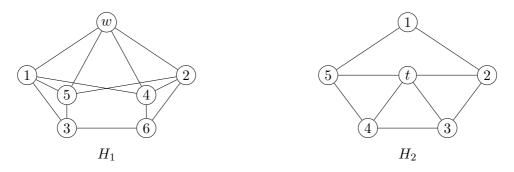


Figure 1: Two special graphs H_1 and H_2 .

Proof. Let G be a $(4P_1, C_4)$ -free graph. Then \overline{G} is $(2P_2, K_4)$ -free. By Theorem 1, we have that \overline{G} can be partitioned into 4 stable sets. So, G can be partitioned into 4 cliques K_i for $1 \le i \le 4$, and this partition can be found in polynomial time. Since G is C_4 -free, both $G[K_1 \cup K_2]$ and $G[K_3 \cup K_4]$ are chordal. It is well-known that the chromatic number of a chordal graph can be determined in linear time, see [7] for example. Therefore, the value $\chi(G[K_1 \cup K_2]) + \chi(G[K_3 \cup K_4])$ provides a 2-approximation for $\chi(G)$.

We now turn to the proof of Theorem 1. The neighborhood of a vertex v in a graph G, denoted by $N_G(v)$, is the set of neighbors of v. We simply write N(v) if the graph G is clear from the context. Two nonadjacent vertices u and v in G are comparable if either $N(v) \subseteq N(u)$ or $N(u) \subseteq N(v)$. Observe that if $N(u) \subseteq N(v)$, then $\chi(G-u) = \chi(G)$. Therefore, it suffices to prove Theorem 1 for every connected $(2P_2, K_4)$ -free graph with no pair of comparable vertices. We do so by proving a number of lemmas below. The idea is that we assume the occurrence of some induced subgraph H in G and then argue that the theorem holds in this case. Afterwards, we can assume that G is H-free in addition to being $(2P_2, K_4)$ -free. We then pick a different induced subgraph as H and repeat. In the end, we are able to show that the theorem holds if G contains a C_5 (see Lemma 2-Lemma 5 below). Therefore, the remaining case is that Gis (odd hole, K_4)-free. In this case, the theorem follows from a known result by Chudnovsky, Robertson, Seymour and Thomas [1] that every (odd hole, K_4)-free graph is 4-colorable. This proves Theorem 1.

The proof idea is based on a paper by Esperet et al. [3] who proved that every (P_5, K_4) -free graph is 5-colorable. In particular, the graph H_1 (see Figure 1) that plays an important role in our proof was also used in [3]. However, to prove 4-colorability we need to use the argument of comparable vertices and extensively extend the structural analysis in [3]. The remainder of the paper is organized as follows. In section 2 we present some preliminary results. In section 3 and section 4 we prove Lemma 2 and Lemma 3, respectively. We then prove Lemma 4 and Lemma 5 in section 5.

2 Preliminaries

We present the structure around a five-cycle in $(2P_2, K_4)$ -free graphs that will be used in section 4 and section 5. Let G be a $(2P_2, K_4)$ -free graph and C = 12345 be an induced C_5 of G. All indices below are modulo 5. We partition $V \setminus C$ into the following subsets:

$$Z = \{v \in V \setminus C : N_C(v) = \emptyset\},\$$

$$R_i = \{v \in V \setminus C : N_C(v) = \{i - 1, i + 1\}\},\$$

$$Y_i = \{v \in V \setminus C : N_C(v) = \{i - 2, i, i + 2\}\},\$$

$$F_i = \{v \in V \setminus C : N_C(v) = C \setminus \{i\}\},\$$

$$U = \{v \in V \setminus C : N_C(v) = C\}.$$

Lemma 1. Let G be a $(2P_2, K_4)$ -free graph and C = 12345 be an induced C_5 of G. Then $V(G) = C \cup Z \cup (\bigcup_{i=1}^5 R_i) \cup (\bigcup_{i=1}^5 Y_i) \cup (\bigcup_{i=1}^5 F_i) \cup U.$

Proof. Suppose that there is a vertex $v \in V(G) \setminus C$ that does not belong to any of Z, R_i , Y_i , F_i and U. Note that v has at least one and at most three neighbors on C. Moreover, these neighbors must be consecutive on C. Without loss of generality, we may assume that v is adjacent to 1 and not adjacent to 3 and 4. Now 34 and 1v induce a $2P_2$.

We now prove some structural properties of these sets.

- (2.1) $Z \cup R_i$ is an independent set. If $Z \cup R_i$ contains an edge xy, then xy and (i-2)(i+2) induce a $2P_2$, a contradiction.
- (2.2) $U \cup Y_i$ and $U \cup F_i$ are independent sets.

If either $U \cup Y_i$ or $U \cup F_i$ contains an edge xy, then $\{x, y, i-2, i+2\}$ induces a K_4 .

(2.3) R_i and R_{i+1} are complete.

It suffices to prove for i = 1. If $r_1 \in R_1$ and $r_2 \in R_2$ are not adjacent, then $5r_1$ and $3r_2$ induce a $2P_2$.

(2.4) Y_i and Y_{i+1} are complete.

It suffices to prove for i = 1. If $y_1 \in Y_1$ and $y_2 \in Y_2$ are not adjacent, then $5y_2$ and $3y_1$ induce a $2P_2$.

(2.5) R_i and Y_i are complete.

It suffices to prove for i = 1. If $r_1 \in R_1$ and $y_1 \in Y_1$ are not adjacent, then $5r_1$ and $3y_1$ induce a $2P_2$.

- (2.6) Either R_i and Y_{i+1} are anti-complete or R_{i+1} and Y_i are anti-complete.
 Suppose, by contradiction, that there exist vertices r_i ∈ R_i, r_{i+1} ∈ R_{i+1}, y_i ∈ Y_i, y_{i+1} ∈ Y_{i+1} such that r_i and r_{i+1} are adjacent to y_{i+1} and y_i, respectively. Then it follows from (2.3), (2.4) and (2.5) that {r_i, r_{i+1}, y_i, y_{i+1}} induces a K₄.
- (2.7) Each vertex in Y_i is anti-complete to either Y_{i-2} or Y_{i+2} . It suffices to prove for i = 1. If $y_1 \in Y_1$ is adjacent to a vertex $y_i \in Y_i$ for i = 3, 4, then $\{1, y_1, y_3, y_4\}$ induces a K_4 by (2.4).
- (2.8) F_i is complete to $Y_{i-2} \cup Y_{i+2}$ and anti-complete to $Y_{i-1} \cup Y_i \cup Y_{i+1}$.

It suffices to prove for i = 5. Let $f \in F_5$. Recall that f is adjacent to 1, 2, 3, 4 but not adjacent to 5 by the definition of F_5 . Suppose first that f is not adjacent to a vertex $y \in Y_2 \cup Y_3$. Note that y is adjacent to 5 by the definition of Y_2 and Y_3 . Now either 3f or 2f forms a $2P_2$ with 5y depending on whether $y \in Y_2$ or $y \in Y_3$. This proves the first part of (2.8). Suppose now that f is adjacent to a vertex $y \in Y_i$ for some $i \in \{1, 4, 5\}$. Since $i \notin \{2, 3\}$, it follows that $5 \notin \{i - 2, i + 2\}$. Therefore, f is adjacent to i - 2 and i + 2. This implies that $\{f, y, i - 2, i + 2\}$ induces a K_4 . This proves the second part of (2.8).

(2.9) F_i is complete to $R_{i-1} \cup R_{i+1}$.

It suffices to prove i = 5. If $f \in F_5$ is not adjacent to $r \in R_1 \cup R_4$, then either f3 or f2 forms a $2P_2$ with 5r depending on whether $r \in R_1$ or $r \in R_4$.

(2.10) If $U \neq \emptyset$, then Y_i and Y_{i+2} are anti-complete.

Let $u \in U$. If $y_i \in Y_i$ and $y_{i+2} \in Y_{i+2}$ are adjacent, then $y_i y_{i+2}$ and u(i+1) induce a $2P_2$ since u is adjacent to neither y_i nor y_{i+2} by (2.2), a contradiction.

(2.11) Either F_i or F_{i+2} is empty.

It suffices to prove for i = 3. Suppose that F_i contains a vertex $f_i \in F_i$ for i = 3, 5. Then either $3f_5$ and $5f_3$ induce a $2P_2$ or $\{1, 2, f_3, f_5\}$ induces a K_4 depending on whether f_3 and f_5 are nonadjacent or not.

(2.12) If G is H_1 -free, then the following holds: if $F_i \neq \emptyset$, then R_{i+1} is anti-complete to $Y_{i+2} \cup Y_i$ and R_{i-1} is anti-complete to $Y_{i-2} \cup Y_i$.

It suffices to prove for i = 5. Let $f \in F_5$. Suppose, by contradiction, that there exists vertices $r \in R_1$ and $y \in Y_2 \cup Y_5$ such that r and y are adjacent. Note that f is adjacent to r by (2.9). If $y \in Y_2$, then f is adjacent to y by (2.8) and this implies that $\{f, y, r, 2\}$ induces a K_4 . If $y \in Y_5$, then f is not adjacent to y by (2.8) and this implies that $C \cup \setminus \{1\} \cup \{f, y, r\}$ induces an H_1 (see Figure 1). This proves that R_1 is anti-complete to $Y_2 \cup Y_5$. The proof for the second part is symmetric.

(2.13) Each vertex in R_i is anti-complete to either Y_{i+1} or Y_{i+2} . By symmetry, each vertex in R_i is anti-complete to either Y_{i-1} or Y_{i-2}

Suppose, by contradiction, that there exists a vertex $r_i \in R_i$ such that r_i is adjacent to a vertex $y_{i+1} \in Y_{i+1}$ and a vertex $y_{i+2} \in Y_{i+2}$. By (2.4), y_{i+1} and y_{i+2} are adjacent. This implies that $\{r_i, y_{i+1}, y_{i+2}, i-1\}$ induces a K_4 .

3 Eliminate H_1

In this section we show that our main theorem, Theorem 1, holds when G is connected, has no pair of comparable vertices, and contains H_1 as an induced subgraph.

Lemma 2. Let G be a connected $(2P_2, K_4)$ -free graph with no pair of comparable vertices. If G contains an induced H_1 , then $\chi(G) \leq 4$.

Proof. Let $H = C \cup \{w\}$ be an induced H_1 in G where $C = \{1, 2, 3, 4, 5, 6\}$ induces a $\overline{C_6}$ such that ij is an edge if and only if $|i - j| \neq 1$, and w is adjacent to 1, 2, 4 and 5 (See Figure 1). All the indices below are modulo 6. We partition V(G) into following subsets:

$$Z = \{ v \in V \setminus C : N_C(v) = \emptyset \},$$

$$D_{i,i+1} = \{ v \in V \setminus C : N_C(v) = \{i, i+1\} \},$$

$$T_i = \{ v \in V \setminus C : N_C(v) = \{i-1, i, i+1\} \},$$

$$F_{i,i+1} = \{ v \in V \setminus C : N_C(v) = \{i-1, i, i+1, i+2\} \},$$

$$W = \{ v \in V \setminus C : N_C(v) = N_C(w) = \{1, 2, 4, 5\} \}.$$

Let $D = \bigcup_{i=1}^{6} D_{i,i+1}$, $T = \bigcup_{i=1}^{6} T_i$ and $F = \bigcup_{i=1}^{6} F_{i,i+1}$. Without loss of generality, we assume H has been chosen such that |T| + |F| is maximized. We first show that $V(G) = C \cup Z \cup D \cup T \cup F \cup W$.

(3.1) There is no vertex $v \in V \setminus C$ such that v is adjacent to i but adjacent to neither i-1 nor i+1 for any $1 \le i \le 6$.

Suppose that such a vertex v exists. Then it follows that vi and (i-1)(i+1) induce a $2P_2$.

(3.2) If a vertex in $V \setminus C$ has at most two neighbors on C, then $v \in Z \cup D$.

Suppose not. Let $v \in V \setminus C$ that has at most two neighbors on C and $v \notin Z \cup D$. Then either v has exactly one neighbor on C or has two neighbors on C that are not consecutive. By symmetry, we may assume that v is adjacent to 1 but not adjacent to 2 and 6. This contradicts (3.1).

(3.3) If a vertex $v \in V \setminus C$ that has exactly three neighbors on C, then $v \in T$.

Suppose not. Let $v \in V \setminus C$ that has exactly at three neighbors on C. By symmetry, we may assume that v is adjacent to 1. It follows from (3.1) that v is adjacent to either 2 or 6, say 2. If v is not adjacent to 3 or 6, then it contradicts (3.1) for i = 4 or i = 5. Therefore, $v \in T_1$ or $v \in T_2$.

(3.4) If a vertex $v \in V \setminus C$ that has exactly four neighbors on C, then $v \in F \cup W$.

By (3.1), v must have two consecutive neighbors on C. If v has three consecutive neighbors on C, then all four neighbors must be consecutive by (3.1) and so $v \in F$. Now $N_C(v) =$ $\{i, i + 1, i + 3, i + 4\}$ for some i. If i = 1, then $v \in W$. Suppose that i = 2 (and the case i = 3 is symmetric). Then either w1 and v6 induce a $2P_2$ or $\{w, v, 2, 5\}$ induces a K_4 , depending on whether w and v are nonadjacent or not.

(3.5) There is no vertex in $V \setminus C$ that has more than four neighbors.

Suppose not. Let $v \in V \setminus C$ have at least five neighbors on C. By symmetry, we may assume that v is adjacent to i for each $1 \leq i \leq 5$. Then $\{1, 3, 5, v\}$ induces a K_4 .

It follows from (3.2)-(3.5) that $V(G) = C \cup Z \cup D \cup T \cup F \cup W$. Note that each of the subsets defined is an independent set since G is $(2P_2, K_4)$ -free. We further investigate the adjacency among those subsets.

(3.6) The set W is anti-complete to Z.

If $w \in W$ and $z \in Z$ are adjacent, then wz and 36 induce a $2P_2$, a contradiction.

- (3.7) The set W is complete to $D_{i,i+1}$ for $i \in \{2,3,5,6\}$ and anti-complete to $D_{i,i+1}$ for $i \in \{1,4\}$. Suppose that $w \in W$ is not adjacent some vertex $d \in D_{i,i+1}$ for some $i \in \{2,3,5,6\}$. By symmetry, we may assume that i = 2. Then d3 and w4 induce a $2P_2$, a contradiction. Suppose that $w \in W$ is adjacent some vertex $d \in D_{1,2} \cup D_{4,5}$. Then dw and 36 induce a $2P_2$, a contradiction.
- (3.8) The set W is complete to $T_1 \cup T_2 \cup T_4 \cup T_5$ and anti-complete to $T_3 \cup T_6$.

Suppose that $w \in W$ is not adjacent some vertex $t \in T_i$ for some $i \in \{1, 2, 4, 5\}$. By symmetry, we may assume that i = 1. Then t6 and w5 induce a $2P_2$. Suppose that $w \in W$ is adjacent some vertex $t \in T_i$ for some $i \in \{3, 6\}$. By symmetry, we may assume that i = 3. Then $\{w, t, 2, 4\}$ induces a K_4 .

- (3.9) The set W is anti-complete to $F_{i,i+1}$ for $i \in \{2, 3, 5, 6\}$ and complete to $F_{i,i+1}$ for $i \in \{1, 4\}$. Suppose that $w \in W$ is adjacent some vertex $f \in F_{i,i+1}$ for some $i \in \{2, 3, 5, 6\}$. By symmetry, we may assume that i = 2. Then $\{f, w, 1, 4\}$ induces a K_4 . Suppose that $w \in W$ is not adjacent some vertex $f \in F_{i,i+1}$ for some $i \in \{1, 4\}$. By symmetry, we may assume that i = 1. Then 6f and 5w induce a $2P_2$.
- (3.10) The set Z is anti-complete to $D \cup T \cup (F \setminus (F_{1,2} \cup F_{4,5}))$.

Suppose that $z \in Z$ is adjacent to some vertex $x \in D \cup T \cup (F \setminus (F_{1,2} \cup F_{4,5}))$. If $x \in D \cup T$, then there exists a vertex $i \in C$ such that x is not adjacent to i - 1 and i + 1. Then zx and (i - 1)(i + 1) induce a $2P_2$. If $x \in F_{i,i+1}$ for some i = 2, 3, 5, 6, then $xw \notin E$ by (3.9). Moreover, there exists a vertex $j \in N_C(w)$ such that $xj \notin E$. Then wj and zx induce a $2P_2$.

It follows from and (3.6) and (3.10) that any vertex in Z has neighbors only in $F_{1,2} \cup F_{4,5}$. On the other hand, w is complete to $F_{1,2} \cup F_{4,5}$ by (3.9). Since G contains no pair of comparable vertices, it follows that $Z = \emptyset$.

- (3.11) For each i, D_{i,i+1} is anti-complete to D_{i+1,i+2}, complete to D_{i+2,i+3} and anti-complete to D_{i+3,i+4}.
 By symmetry, it suffices to prove the claim for i = 1. Let d ∈ D_{1,2}. If d is adjacent to d' ∈ D_{2,3}, then 46 and dd' induce a 2P₂. If d is not adjacent to d' ∈ D_{3,4}, then 2d and 3d' induce a 2P₂. If d is adjacent to d' ∈ D_{4,5}, then 36 and dd' induce a 2P₂.
- (3.12) For each i, $F_{i,i+1}$ is anti-complete to $F_{i+1,i+2} \cup F_{i+3,i+4}$ and complete to $F_{i+2,i+3}$.

By symmetry, it suffices to prove the claim for i = 1. Let $f \in F_{1,2}$. If f is adjacent to a vertex $f' \in F_{2,3}$, then $\{1, 3, f, f'\}$ induces a K_4 . If f is not adjacent to a vertex $f' \in F_{3,4}$, then 5f' and 6f induce a $2P_2$. If f is adjacent to a vertex $f' \in F_{4,5}$, then $\{3, 6, f, f'\}$ induces a K_4 .

- (3.13) The sets T_i and T_{i+1} are anti-complete for $i \in \{1, 4\}$. By symmetry, it suffices to prove this for i = 1. If $t_1 \in T_1$ and $t_2 \in T_2$ are adjacent, then w is adjacent to both t_1 and t_2 by (3.8). But now $\{t_1, t_2, w, 1\}$ induces a K_4 .
- (3.14) The sets T_3 and $T_1 \cup T_5$ are complete. By symmetry, T_6 and $T_2 \cup T_4$ are complete. Suppose that $t_3 \in T_3$ is not adjacent to some vertex $t \in T_1 \cup T_5$. By (3.8), w is adjacent to t but not to t_3 . Then $3t_3$ and wt induce a $2P_2$, a contradiction.
- (3.15) The sets T_i and T_{i+3} are complete for each $1 \le i \le 6$. By symmetry, it suffices to prove this for i = 1. If $t_1 \in T_1$ and $t_4 \in T_4$ are not adjacent, then $2t_1$ and $3t_4$ induce a $2P_2$.
- (3.16) For each i, $D_{i,i+1}$ is anti-complete to $T_{i-1} \cup T_i \cup T_{i+1} \cup T_{i+2}$ and complete to $T_{i+3} \cup T_{i+4}$. We note that $D_{1,2}$ and $D_{4,5}$ are symmetric, and $D_{2,3}$, $D_{3,4}$, $D_{5,6}$ and $D_{6,1}$ are symmetric. So, it suffices to prove the claim for $D_{1,2}$ and $D_{2,3}$.

Let $d \in D_{1,2}$. Suppose that d is adjacent to some vertex $t \in T_6 \cup T_1 \cup T_2 \cup T_3$. By symmetry, we may assume that $i \in \{1,3\}$. If i = 1, then td and 35 induce a $2P_2$. If i = 3, then w is not adjacent to d and t by (3.7) and (3.8). Then dt and w5 induce a $2P_2$. Now suppose that d is not adjacent to some vertex $t \in T_4 \cup T_5$. By symmetry, we may assume that $t \in T_4$. Then d2 and t3 induce a $2P_2$. This proves the claim for $D_{1,2}$.

Let $d \in D_{2,3}$. Suppose that d is adjacent to some vertex $t \in T_2 \cup T_3$. By symmetry, we may assume that $t \in T_2$. Then dt and 46 induce a $2P_2$. Suppose that d is not adjacent to some vertex $t \in T_5 \cup T_6$. By symmetry, we may assume that $t \in T_5$. Then d3 and t4 induce a $2P_2$.

By (3.7) and (3.8), $\{2, w\}$ is complete to $D_{2,3} \cup T_1$. It follows from K_4 -freeness of G that $D_{2,3}$ is anti-complete to T_1 . It remains to show that $D_{2,3}$ is anti-complete to T_4 . Suppose that d is adjacent to some vertex $t_4 \in T_4$. Note that $C' = C \setminus \{1\} \cup \{t_4\}$ induces a $\overline{C_6}$ and $H' = C' \cup \{w\}$ induces a subgraph isomorphic to H_1 . By (3.13) and (3.14), all vertices in $T_1 \cup T_4 \cup T_5 \cup T_6$ remain to be T-vertices with respect to C'. Moreover, all vertices in $T_3 \cup F$ remain to be F-vertices or T-vertices. By the choice of C, there exists a vertex $t \in T_2$ that is not adjacent to t_4 . Then dt_4 and $1t_2$ induce a $2P_2$, a contradiction. This proves the claim for $D_{2,3}$.

(3.17) For each i, $F_{i,i+1}$ is anti-complete to $T_i \cup T_{i+1}$ and complete to $T_{i+3} \cup T_{i+4}$

By symmetry of C, it suffices to prove this for i = 1. Let $f \in F_{1,2}$. If f is adjacent to some vertex $t \in T_1 \cup T_2$, then either $\{6, 2, f, t\}$ or $\{1, 3, f, t\}$ induces a K_4 depending on whether $t \in T_1$ or $t \in T_2$. Suppose that f is not adjacent to some vertex $t \in T_4 \cup T_5$. By symmetry, we may assume that $t \in T_4$. Then 6f and 5t induce a $2P_2$, a contradiction.

- (3.18) The sets F_{i,i+1} and T_{i-1} are complete for i ∈ {2,5}, and F_{i,i+1} and T_{i+2} are complete for i ∈ {3,6}.
 Let f ∈ F_{i,i+1} and t ∈ T_i be nonadjacent. By (3.9) and (3.8), w is adjacent to t but not f. It can be readily checked that in each of the cases wt and f3 or wt and f6 induce a 2P₂.
- (3.19) The set $D_{1,2}$ is anti-complete to $F_{6,1} \cup F_{2,3}$ and complete to F_{45} .

The set $D_{4,5}$ is anti-complete to $F_{3,4} \cup F_{5,6}$ and complete to F_{12} .

The set $D_{2,3}$ is anti-complete to $F_{1,2}$ and complete to $F_{5,6} \cup F_{6,1}$.

The set $D_{3,4}$ is anti-complete to $F_{4,5}$ and complete to $F_{5,6} \cup F_{6,1}$.

The set $D_{6,1}$ is anti-complete to $F_{1,2}$ and complete to $F_{2,3} \cup F_{3,4}$.

The set $D_{5,6}$ is anti-complete to $F_{4,5}$ and complete to $F_{2,3} \cup F_{3,4}$.

Note that $D_{1,2}$ and $D_{4,5}$ are symmetric, and $D_{2,3}$, $D_{3,4}$, $D_{5,6}$ and $D_{6,1}$ are symmetric. So, it suffices to prove the claim for $D_{1,2}$ and $D_{2,3}$. Let $d \in D_{1,2}$. If d is adjacent to some vertex $f \in F_{6,1} \cup F_{2,3}$, then w is not adjacent to d and f by (3.7) and (3.9). Now df and w4 or df and w5 induce a $2P_2$ depending on whether $f \in F_{6,1}$ or $f \in F_{2,3}$. If d is not adjacent to some vertex $f \in F_{4,5}$, then d2 and f3 induce a $2P_2$. This proves the claim for $D_{1,2}$.

Now let $d \in D_{2,3}$. By (3.7), it follows that $wd \in E$. If d is adjacent to a vertex $f \in F_{1,2}$, then $\{d, f, 2, w\}$ induces a K_4 by (3.9). If d is not adjacent to a vertex $f \in F_{5,6} \cup F_{6,1}$, then 6f and wd induce a $2P_2$ by (3.9). This proves the claim for $D_{2,3}$.

We proceed with a few claims that help to show that certain sets are empty.

Claim 1. Either $D_{1,2}$ or $D_{4,5}$ is empty.

Proof of Claim 1. Suppose not. Let $d_{12} \in D_{1,2}$ and $d_{45} \in D_{4,5}$. By (3.7)-(3.19), $N(d_{12}) \subseteq N(w)$ unless d_{12} has a neighbor $f \in F_{3,4} \cup F_{5,6}$. Similarly, $N(d_{45}) \subseteq N(w)$ unless d_{45} has a neighbor $f' \in F_{3,4} \cup F_{5,6}$. By (3.11) and (3.19), $d_{12}f$ and $d_{45}f'$ induce a $2P_2$, a contradiction.

Claim 2. Each vertex in T_1 has a non-neighbor in T_5 and each vertex in T_5 has a non-neighbor in T_1 . By symmetry, each vertex in T_2 has a non-neighbor in T_4 and each vertex in T_4 has a non-neighbor in T_2 .

Proof of Claim 2. Let $t_1 \in T_1$. Let

$$X = \{6, 1, 2\} \cup W \cup D_{3,4} \cup D_{4,5} \cup T_3 \cup T_4 \cup F_{2,3} \cup F_{3,4} \cup F_{4,5}.$$

Note that $N(4) = X \cup T_5 \cup F_{5,6}$ and $N(t_1) \subseteq X \cup T_5 \cup F_{5,6} \cup T_6$ by the properties we have proved. Since G contains no pair of comparable vertices, t_1 has a neighbor $t_6 \in T_6$ and there exists a vertex $t \in N(4) \setminus N(t_1)$. Clearly, $t \in F_{5,6} \cup T_5$. If $t \in F_{5,6}$, then 4t and t_1t_6 induce a $2P_2$ since F_{56} and T_6 are anti-complete by (3.17). This shows that t_1 has a non-neighbor $t \in T_5$. By symmetry, each vertex in T_5 has a non-neighbor in T_1 .

Claim 3. Each vertex in T_6 has a neighbor in $T_1 \cup T_5$. By symmetry, each vertex in T_3 has a neighbor in $T_2 \cup T_4$.

Proof of Claim 3. Let $t_6 \in T_6$. Let

$$X = \{5, 6, 1\} \cup D_{2,3} \cup D_{3,4} \cup T_2 \cup T_3 \cup T_4 \cup F_{2,3} \cup F_{3,4}.$$

Note that $N(3) = X \cup F_{1,2} \cup F_{4,5}$ and $N(t_6) \subseteq X \cup T_1 \cup T_5 \cup F_{12} \cup F_{45}$. Since G contains no pair of comparable vertices, t_6 has a neighbor in $T_1 \cup T_5$.

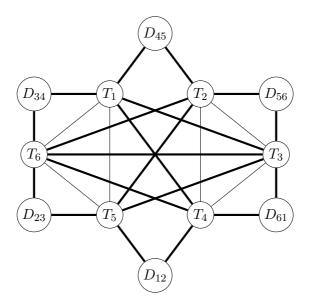


Figure 2: The adjacency among T_i and $D_{i,i+1}$. A thick line between two sets means that the two sets are complete, a thin line means the edges between the two sets can be arbitrary, and no line means that the two sets are anti-complete. For clarity, edges between two $D_{i,i+1}$ are not shown.

Claim 4. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then T_2 and T_4 are complete. By symmetry, if $D_{2,3} \cup D_{3,4} \neq \emptyset$, then T_1 and T_5 are complete.

Proof of Claim 4. Let $d \in D_{5,6} \cup D_{6,1}$. Suppose that $t_2 \in T_2$ and $t_4 \in T_4$ are not adjacent. If $d \in D_{5,6}$, then $dt_2 \in E$ and $dt_4 \notin E$ by (3.16). Thus, dt_2 and $4t_4$ induce a $2P_2$. If $d \in D_{6,1}$, then $dt_4 \in E$ and $dt_2 \notin E$ by (3.16). Thus, dt_4 and $2t_2$ induce a $2P_2$.

Claim 5. One of $F_{6,1}$, $F_{1,2}$ and $F_{2,3}$ is empty. By symmetry, one of $F_{3,4}$, $F_{4,5}$ and $F_{5,6}$ is empty.

Proof of Claim 5. Suppose that $f_{61} \in F_{6,1}$, $f_{12} \in F_{1,2}$, and $f_{23} \in F_{2,3}$. Then $f_{61}f_{23}$ and $f_{12}w$ induce a $2P_2$ by (3.9) and (3.12).

By Claim 1, we may assume that $D_{4,5} = \emptyset$. It follows from (3.13), (3.14) and (3.15) that either T_1 and T_5 are complete or T_2 and T_4 are complete for otherwise G would contain a $2P_2$ (see Figure 2). By symmetry, we may assume that T_1 and T_5 are complete. It then follows from Claim 2 and Claim 3 that $T_1 \cup T_5 \cup T_6 = \emptyset$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $T_2 \cup T_3 \cup T_4 = \emptyset$ due to Claim 2-Claim 4. In the following we shall use this fact without explicitly mentioning it. We divide our proof into four cases depending on whether $F_{1,2}$ and $F_{4,5}$ are empty or not. One can verify that each of the partitions of V(G) into 4 subsets in the following is a 4-coloring of G using the properties we have proved. For convenience, we draw Figure 3 to visulize the adjacency among $D_{i,i+1}$ and $F_{i,i+1}$. From Figure 3 it can be seen that if $T_2 \cup T_3 \cup T_4 = \emptyset$, then we can use the symmetry of H under its automorphism $f: V(H) \to V(H)$ with f(1) = 2, f(2) = 1, f(3) = 6, f(4) = 5, f(5) = 4, f(6) = 3 and f(w) = w.

Case 1. Both $F_{1,2}$ and $F_{4,5}$ are not empty. Let $f_{12} \in D_{1,2}$ and $f_{45} \in D_{4,5}$. We first show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3} \cup D_{3,4} \cup D_{5,6} \cup D_{6,1}$. By symmetry, it suffices to show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3}$. Suppose that $d \in D_{2,3}$ and $f \in F_{1,2} \cup F_{4,5}$ are adjacent. By (3.19), $f \in F_{4,5}$. Then df and $1f_{12}$ induce a $2P_2$. On the other hand, it follows from Claim 5 and (3.12) that at most one of $F_{2,3}$, $F_{3,4}$, $F_{5,6}$ and $F_{6,1}$ is not empty.

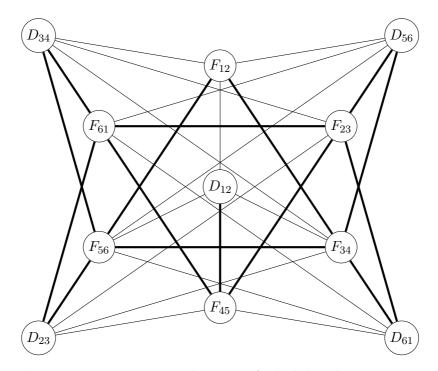


Figure 3: The adjacency among $F_{i,i+1}$ and $D_{i,i+1}$. A thick line between two sets means that the two sets are complete, a thin line means the edges between the two sets can be arbitrary, and no line means that the two sets are anti-complete. For clarity, edges between two $D_{i,i+1}$ are not shown.

• If $F_{2,3} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{4,5} \cup D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4, \\ F_{2,3} \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\ F_{1,2} \cup \{4,5\} \cup T_2, \\ D_{5,6} \cup D_{6,1} \cup \{2,3\}. \end{split}$$

• Suppose that $F_{6,1} \neq \emptyset$. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{4,5} \cup D_{5,6} \cup D_{6,1} \cup \{2\}, \\ F_{6,1} \cup D_{1,2} \cup W \cup \{3\}, \\ F_{1,2} \cup \{4,5\}, \\ D_{2,3} \cup D_{3,4} \cup \{1,6\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{4,5} \cup \{1,2\} \cup T_4, \\ F_{6,1} \cup D_{1,2} \cup W \cup \{3\}, \\ F_{1,2} \cup \{4,5\} \cup T_2, \\ D_{2,3} \cup D_{3,4} \cup \{6\} \cup T_3. \end{split}$$

• Suppose that $F_{3,4} \neq \emptyset$. Note first that no vertex $d \in D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$ for otherwise a neighbor of d in $F_{1,2}$, a neighbor of d in $F_{3,4}$, d and 2 induce a K_4 . Let

 $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{3,4}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Then G has a 4-coloring:

$$F_{4,5} \cup D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4,$$

$$F_{3,4} \cup D'_{1,2} \cup W \cup \{6\} \cup T_3,$$

$$F_{1,2} \cup D''_{1,2} \cup \{4,5\} \cup T_2,$$

$$D_{5,6} \cup D_{6,1} \cup \{2,3\}.$$

• Suppose that $F_{5,6} \neq \emptyset$. Note first that no vertex $d \in D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$ for otherwise a neighbor of d in $F_{1,2}$, a neighbor of d in $F_{5,6}$, d and 1 induce a K_4 . Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{5,6}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. By (3.17) and (3.18), $F_{5,6}$ and $T_3 \cup T_4$ are complete. Since G is K_4 -free, T_3 and T_4 are anti-complete. Then G has a 4-coloring:

$$\begin{split} F_{4,5} &\cup D_{5,6} \cup D_{6,1} \cup \{2\}, \\ F_{5,6} &\cup D'_{1,2} \cup W \cup \{3\}, \\ F_{1,2} &\cup D''_{1,2} \cup \{4,5\} \cup T_2, \\ D_{2,3} &\cup D_{3,4} \cup \{1,6\} \cup T_3 \cup T_4 \end{split}$$

Case 2. Both $F_{1,2}$ and $F_{4,5}$ are empty. By (3.12) and the fact that G is $2P_2$ -free, one of $F_{2,3}$, $F_{3,4}$, $F_{5,6}$ and $F_{6,1}$ is empty. By (3.11), (3.19), (3.12) and K_4 -freeness of G, either $D_{5,6}$ and $F_{5,6}$ are anti-complete or $D_{3,4}$ and $F_{3,4}$ are anti-complete.

• Suppose that $F_{6,1} = \emptyset$.

If $D_{5,6}$ and $F_{5,6}$ are anti-complete, then G has a 4-coloring:

$$\begin{split} F_{2,3} \cup F_{3,4} \cup W \cup \{6\} \cup T_3, \\ F_{5,6} \cup D_{5,6} \cup \{2,3\}, \\ D_{1,2} \cup D_{6,1} \cup \{4,5\} \cup T_2, \\ D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4. \end{split}$$

Now assume that $D_{3,4}$ and $F_{3,4}$ are anti-complete. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{2,3} \cup F_{5,6} \cup W, \\ F_{3,4} \cup D_{3,4} \cup \{6,1\}, \\ D_{1,2} \cup D_{2,3} \cup \{4,5\}, \\ D_{5,6} \cup D_{6,1} \cup \{2,3\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{2,3} \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\ F_{3,4} \cup D_{3,4} \cup \{1\} \cup T_4, \\ F_{5,6} \cup \{2,3\}, \\ D_{2,3} \cup \{4,5\} \cup T_2. \end{split}$$

• Suppose that $F_{2,3} = \emptyset$. Suppose first that $D_{3,4}$ and $F_{3,4}$ are anti-complete. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{6,1} \cup F_{5,6} \cup W \cup \{3\}, \\ F_{3,4} \cup D_{3,4} \cup \{6,1\}, \\ D_{1,2} \cup D_{2,3} \cup \{4,5\}, \\ D_{6,1} \cup D_{5,6} \cup \{2\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{6,1} \cup F_{5,6} \cup W \cup \{3\}, \\ F_{3,4} \cup D_{3,4} \cup \{6\} \cup T_3, \\ D_{1,2} \cup D_{2,3} \cup \{4,5\} \cup T_2, \\ \{1,2\} \cup T_4. \end{split}$$

Suppose now that $D_{3,4}$ and $F_{3,4}$ are not anti-complete and that $D_{5,6}$ and $F_{5,6}$ are anticomplete. By (3.16) and (3.17), $D_{3,4} \cup F_{3,4}$ are anti-complete to $T_3 \cup T_4$. Since G is $2P_2$ -free, it follows that T_3 and T_4 are anti-complete. Then G has a 4-coloring:

$$\begin{split} F_{6,1} \cup F_{3,4} \cup W, \\ F_{5,6} \cup D_{5,6} \cup \{2,3\}, \\ D_{1,2} \cup D_{6,1} \cup \{4,5\} \cup T_2, \\ D_{2,3} \cup D_{3,4} \cup \{6,1\} \cup T_3 \cup T_4. \end{split}$$

• Suppose that $F_{5,6} = \emptyset$. If $F_{6,1} = \emptyset$, then G has a 4-coloring as above. So, we can assume that $F_{6,1} \neq \emptyset$. Let $f_{61} \in F_{6,1}$. If $d \in D_{2,3}$ and $f \in F_{2,3}$ are adjacent, then $\{2, f_{61}, d, f\}$ induces a K_4 by (3.12) and (3.19). So, $D_{2,3}$ and $F_{2,3}$ are anti-complete. By (3.17) and (3.18), $F_{6,1}$ and $T_2 \cup T_3$ are complete. Since G is K_4 -free, T_2 and T_3 are anti-complete. Then G has a 4-coloring:

$$\begin{split} F_{3,4} \cup F_{6,1} \cup W, \\ F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5,6\} \cup T_2 \cup T_3, \\ D_{3,4} \cup \{1,2\} \cup T_4, \\ D_{5,6} \cup D_{6,1} \cup \{3,4\}. \end{split}$$

• Suppose that $F_{3,4} = \emptyset$. If $F_{2,3} = \emptyset$, then G has a 4-coloring as above. So, we can assume that $F_{2,3} \neq \emptyset$. Let $f_{23} \in F_{2,3}$. If $d \in D_{6,1}$ and $f \in F_{6,1}$ are adjacent, then $\{1, f_{23}, d, f\}$ induces a K_4 by (3.12) and (3.19). So, $D_{6,1}$ and $F_{6,1}$ are anti-complete.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{5,6} \cup F_{2,3} \cup W, \\ F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3,4\}, \\ D_{5,6} \cup \{1,2\}, \\ D_{3,4} \cup D_{2,3} \cup \{5,6\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{5,6} \cup F_{6,1} \cup W \cup \{3\}, \\ F_{2,3} \cup D_{1,2} \cup \{6\} \cup T_3, \\ D_{2,3} \cup \{4,5\} \cup T_2, \\ D_{3,4} \cup \{1,2\} \cup T_4. \end{split}$$

Case 3. The set $F_{1,2} = \emptyset$ but the set $F_{4,5} \neq \emptyset$. By Claim 5, either $F_{3,4} = \emptyset$ or $F_{5,6} = \emptyset$. By (3.11), (3.19), (3.12) and K_4 -freeness of G, either $D_{2,3}$ and $F_{2,3}$ are anti-complete or $D_{6,1}$ and $F_{6,1}$ are anti-complete.

• Suppose that $F_{5,6} = \emptyset$.

If $D_{6,1}$ and $F_{6,1}$ are anti-complete, then G has a 4-coloring:

$$\begin{split} F_{2,3} \cup F_{3,4} \cup W \cup \{6\} \cup T_3, \\ F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3,4\}, \\ F_{4,5} \cup D_{5,6} \cup \{1,2\} \cup T_4, \\ D_{2,3} \cup D_{3,4} \cup \{5\} \cup T_2. \end{split}$$

Now assume that $D_{2,3}$ and $F_{2,3}$ are anti-complete. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{3,4} \cup F_{6,1} \cup W, \\ F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5,6\}, \\ F_{4,5} \cup D_{3,4} \cup \{1,2\}, \\ D_{5,6} \cup D_{6,1} \cup \{3,4\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{3,4} \cup W \cup \{6\} \cup T_3, \\ F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5\} \cup T_2, \\ F_{4,5} \cup D_{3,4} \cup \{1,2\} \cup T_4, \\ F_{6,1} \cup \{3,4\}. \end{split}$$

• Suppose that $F_{3,4} = \emptyset$. Suppose first that $D_{2,3}$ and $F_{2,3}$ are anti-complete. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{5,6} \cup F_{6,1} \cup W \cup \{3\}, \\ F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{5,6\}, \\ F_{4,5} \cup D_{3,4} \cup \{1,2\}, \\ D_{5,6} \cup D_{6,1} \cup \{4\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{5,6} \cup F_{6,1} \cup W \cup \{3\}, \\ F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup \{6\} \cup T_3, \\ F_{4,5} \cup D_{3,4} \cup \{1,2\} \cup T_4, \\ \{4,5\} \cup T_2. \end{split}$$

Now suppose that $D_{2,3}$ and $F_{2,3}$ are not anti-complete and that $D_{6,1}$ and $F_{6,1}$ are anticomplete. Then T_2 and T_3 are anti-complete for otherwise an edge between T_2 and T_3 and an edge between $D_{2,3}$ and $F_{2,3}$ induce a $2P_2$ by (3.16) and (3.17). Then G has a 4-coloring:

$$\begin{split} F_{5,6} \cup F_{2,3} \cup W, \\ F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup \{3,4\}, \\ F_{4,5} \cup D_{5,6} \cup \{1,2\} \cup T_4, \\ D_{2,3} \cup D_{3,4} \cup \{5,6\} \cup T_2 \cup T_3 \end{split}$$

Case 4. The set $F_{4,5} = \emptyset$ but the set $F_{1,2} \neq \emptyset$. By Claim 5, either $F_{2,3} = \emptyset$ or $F_{6,1} = \emptyset$. By (3.19) and (3.12), $F_{3,4}$ is complete to $D_{5,6} \cup F_{5,6}$. So, if $F_{3,4} \neq \emptyset$, then $D_{5,6}$ and $F_{5,6}$ are anti-complete for otherwise G would contain a K_4 . By symmetry, if $F_{5,6} \neq \emptyset$, then $D_{3,4}$ and $F_{3,4}$ are anti-complete. Moreover, either $D_{3,4}$ and $F_{3,4}$ are anti-complete or $D_{5,6}$ and $F_{5,6}$ are anti-complete. Similarly, either $D_{2,3}$ and $F_{3,4}$ are anti-complete or $D_{6,1}$ and $F_{5,6}$ are anti-complete.

• Suppose that $F_{6,1} = \emptyset$. If both $F_{3,4}$ and $F_{5,6}$ are not empty, then consider the following 4-coloring of $G - (D_{2,3} \cup D_{6,1})$:

$$\begin{split} I_1 &= F_{2,3} \cup D_{1,2} \cup W \cup \{6\} \cup T_3, \\ I_2 &= F_{3,4} \cup D_{3,4} \cup \{1\} \cup T_4, \\ I_3 &= F_{5,6} \cup D_{5,6} \cup \{2,3\}, \\ I_4 &= F_{1,2} \cup \{4,5\} \cup T_2. \end{split}$$

If $D_{2,3}$ and $F_{3,4}$ are anti-complete, then G has a 4-coloring: $I_1, I_2 \cup D_{2,3}, I_3$ and $I_4 \cup D_{6,1}$. If $D_{6,1}$ and $F_{5,6}$ are anti-complete, then G has a 4-coloring: $I_1, I_2, I_3 \cup D_{6,1}$ and $I_4 \cup D_{2,3}$. It reamains to consider the case where at least one of $F_{3,4}$ and $F_{5,6}$ is empty.

Suppose that $F_{5,6} = \emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Then G has a 4-coloring:

$$\begin{split} F_{1,2} \cup D_{1,2}' \cup \{4,5\} \cup T_2, \\ F_{2,3} \cup F_{3,4} \cup D_{1,2}'' \cup W \cup \{6\} \cup T_3, \\ D_{2,3} \cup D_{3,4} \cup \{1\} \cup T_4, \\ D_{5,6} \cup D_{6,1} \cup \{2,3\}. \end{split}$$

Suppose now that $F_{5,6} \neq \emptyset$ and $F_{3,4} = \emptyset$. Note that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Moreover, recall that since $F_{5,6} \neq \emptyset$, T_3 and T_4 are anti-complete. Then Ghas a 4-coloring:

$$F_{1,2} \cup D_{2,3} \cup D'_{1,2} \cup \{4,5\} \cup T_2,$$

$$F_{2,3} \cup F_{5,6} \cup D''_{1,2} \cup W,$$

$$D_{3,4} \cup \{6,1\} \cup T_3 \cup T_4,$$

$$D_{5,6} \cup D_{6,1} \cup \{2,3\}.$$

• Suppose that $F_{2,3} = \emptyset$. If both $F_{3,4}$ and $F_{5,6}$ are not empty, then consider the following 4-coloring of $G - (D_{2,3} \cup D_{6,1})$:

$$I_1 = F_{6,1} \cup D_{1,2} \cup W \cup \{3\},$$

$$I_2 = F_{5,6} \cup D_{5,6} \cup \{2\},$$

$$I_3 = F_{3,4} \cup D_{3,4} \cup \{6,1\} \cup T_3 \cup T_4,$$

$$I_4 = F_{1,2} \cup \{4,5\} \cup T_2.$$

If $D_{2,3}$ and $F_{3,4}$ are anti-complete, then G has a 4-coloring: $I_1, I_2, I_3 \cup D_{2,3}$ and $I_4 \cup D_{6,1}$. If $D_{6,1}$ and $F_{5,6}$ are anti-complete, then G has a 4-coloring: $I_1, I_2 \cup D_{6,1}, I_3$ and $I_4 \cup D_{2,3}$. So, one of $F_{3,4}$ and $F_{5,6}$ is empty.

Suppose that $F_{5,6} \neq \emptyset$. So, $F_{3,4} = \emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. Moreover, T_3 and T_4 are anti-complete. Then G has a 4-coloring:

$$\begin{split} F_{1,2} \cup D_{1,2}' \cup \{4,5\} \cup T_2, \\ F_{6,1} \cup F_{5,6} \cup D_{1,2}'' \cup W \cup \{3\}, \\ D_{6,1} \cup D_{5,6} \cup \{2\}, \\ D_{2,3} \cup D_{3,4} \cup \{6,1\} \cup T_3 \cup T_4. \end{split}$$

Suppose now that $F_{5,6} = \emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$. Let $D'_{1,2}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D''_{1,2} = D_{1,2} \setminus D'_{1,2}$. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{1,2} \cup D_{6,1} \cup D'_{1,2} \cup \{4,5\}, \\ F_{6,1} \cup F_{3,4} \cup D''_{1,2} \cup W, \\ D_{5,6} \cup \{2,3\}, \\ D_{2,3} \cup D_{3,4} \cup \{6,1\}. \end{split}$$

If $D_{5,6} \cup D_{6,1} = \emptyset$, then G has a 4-coloring:

$$\begin{split} F_{1,2} \cup D_{2,3} \cup D'_{1,2} \cup \{4,5\} \cup T_{2,2} \\ F_{3,4} \cup D''_{1,2} \cup W \cup \{6\} \cup T_{3}, \\ F_{5,6} \cup \{3\}, \\ D_{3,4} \cup \{1,2\} \cup T_{4}. \end{split}$$

In each case we have found a 4-coloring of G. This completes our proof.

4 Eliminate H_2

In this section we show that our main theorem, Theorem 1, holds when G is connected, has no pair of comparable vertices, does not contain H_1 as an induced subgraph, but contains H_2 as an induced subgraph.

Lemma 3. Let G be a connected $(2P_2, K_4, H_1)$ -free graph with no pair of comparable vertices. If G contains an induced H_2 , then $\chi(G) \leq 4$. *Proof.* Let $H = C \cup \{f\}$ be an induced H_2 where C = 12345 induces a C_5 and f is adjacent to 1, 2, 3 and 4. We partition $V \setminus C$ into subsets of Z, R_i , Y_i , F_i and U as in section 2. By the fact that G is H_1 -free and (2.11), it follows that $F_i = \emptyset$ for $i \neq 5$. Note that $f \in F_5$. We choose H such that

- |U| is minimum, and
- $|F_5|$ is minimum subject to the previous condition.
- (4.1) U is complete to R_i for $1 \le i \le 5$.

Suppose not. Let $u \in U$ be nonadjacent to $r_i \in R_i$ for some *i*. Suppose first that $1 \leq i \leq 4$. Note that $C' = C \setminus \{i\} \cup \{r_i\}$ induces a C_5 and $H' = C' \cup \{u\}$ induces an H_2 . Since $5 \in C'$, it follows that $F_5 \cap U' = \emptyset$ and $U' \subseteq U$. Moreover, $u \in U$ is not in U' since *u* is not adjacent to r_i . This implies that |U'| < |U|, contradicting the choice of H.

Now suppose that i = 5. Note that $C' = C \setminus \{5\} \cup \{r_5\}$ induces a C_5 and $H' = C' \cup \{u\}$ induces an H_2 . Note that $U' \subseteq F_5 \cup U$ and $u \notin U'$ since u is not adjacent to r_i . By the chocie of H, there exists a vertex $f' \in F_5$ such that f' is adjacent to r_5 . By (2.2), u and f are not adjacent. But then fr_5 and 5u indcue a $2P_2$.

(4.2) If $U \neq \emptyset$, then R_i and R_{i+2} are anti-complete.

Let $u \in U$. If $r_i \in R_i$ and $r_{i+2} \in R_{i+2}$ are not adjacent, then $\{r_i, r_{i+2}, i+1, u\}$ induces a K_4 , since u is adjacent to r_i and r_{i+2} by (4.1).

Suppose first that $U \neq \emptyset$. By (4.2), R_i and R_{i+2} are anti-complete. Recall that Y_i and Y_{i+2} are anti-complete by (2.10). By (2.12), R_1 is anti-complete to $Y_5 \cup Y_2$ and R_4 is anti-complete to $Y_5 \cup Y_3$. By (2.8), F_5 is anti-complete to $Y_1 \cup Y_4$. By (2.6), either Y_3 and R_2 are anti-complete or Y_2 and R_3 are anti-complete.

If Y_3 and R_2 are anti-complete, then G admits the following 4-coloring:

$Y_1 \cup Y_4 \cup U \cup F_5$	(2.10)(2.2)(2.8)
$Y_2 \cup Y_5 \cup R_1 \cup \{1\}$	(2.10)(2.12)
$Y_3 \cup R_2 \cup R_4 \cup \{2,4\}$	(4.2)(2.12)
$R_3 \cup R_5 \cup Z \cup \{3,5\}$	(4.2)(2.2)

If Y_2 and R_3 are anti-complete, then G admits the following 4-coloring:

$Y_1 \cup Y_4 \cup U \cup F_5$	(2.10)(2.2)(2.8)
$Y_3 \cup Y_5 \cup R_4 \cup \{4\}$	(2.10)(2.12)
$Y_2 \cup R_1 \cup R_3 \cup \{1,3\}$	(4.2)(2.12)
$R_2 \cup R_5 \cup Z \cup \{2,5\}$	(4.2)(2.2)

This shows that if $U \neq \emptyset$, then G has a 4-coloring. Therefore, we can assume in the following that $U = \emptyset$.

(4.3) Each vertex in $R_2 \cup R_3$ is either complete or anti-complete to F_5 .

Suppose not. Let $r \in R_2 \cup R_3$ be adjacent to $f \in F_5$ and not adjacent to $f' \in F_5$. By symmetry, we may assume that $r \in R_2$. Note that $C' = C \setminus \{2\} \cup \{r\}$ induces a C_5 and $H' = C' \cup \{f\}$ induces an H_2 . Clearly, $f' \notin F'_5$. By the choice of H, there exists a vertex $y \in Y$ such that $y \in F'_5$. This means that y is not adjacent to 5 but adjacent to 1, 3, 4 and r_2 . This implies that $y \in Y_1$. By (2.8), f' and y are not adjacent. But now f'_2 and yr_2 induce a $2P_2$. By (2.8), (2.9) and (4.3), only vertices in $R_5 \cup Z$ can distinguish two vertices in F_5 . By (2.1), $R_5 \cup Z$ is an independent set and so $(F_5, R_5 \cup Z)$ is a $2P_2$ -free bipartite graph. This implies that $F_5 = \{f\}$ since any two vertices in F are comparable. Let $R'_i = N(f) \cap R_i$ and $R''_i = R_i \setminus R'_i$ for i = 2, 3, 5. We now prove properties of R'_i and R''_i .

- (4.4) R'_5 is anti-complete to $R'_2 \cup R'_3$. Suppose that $r'_5 \in R'_5$ and $r'_2 \in R'_2$ are adjacent. Then $\{r'_5, r'_2, 1, f\}$ induces a K_4 .
- (4.5) R'_5 is anti-complete to $Y_2 \cup Y_3$. Suppose that $r'_5 \in R'_5$ and $y_2 \in Y_2$ are adjacent. By (2.8), f and y_2 are adjacent. Then $\{r'_5, 4, y_2, f\}$ induces a K_4 .
- (4.6) R'_2 is anti-complete to R_4 . By symmetry, R'_3 is anti-complete to R_1 . Suppose that $r'_2 \in R'_2$ and $r_4 \in R_4$ are adjacent. By (2.9), f and r_4 are adjacent. Then $\{r'_2, r_4, 3, f\}$ induces a K_4 .
- (4.7) R_5'' is anti-complete to $R_2'' \cup R_3''$. Suppose that $r_5'' \in R_5''$ and $r_2'' \in R_2''$ are adjacent. Then $r_5''r_2''$ and f2 induce a $2P_2$.
- (4.8) Y_5 is anti-complete to $R_2'' \cup R_3''$. Suppose that $y_5 \in Y_5$ and $r_2'' \in R_2''$ are adjacent. By (2.8), f and y are not adjacent. Then y_5r_2'' and f4 induce a $2P_2$.
- (4.9) R_5'' is anti-complete to $Y_1 \cup Y_4$. Suppose that $r_5'' \in R_5''$ and $y_4 \in Y_4$ are adjacent. By (2.8), f and y_4 are not adjacent. Then $r_5''y_4$ and f_2 induce a $2P_2$.
- (4.10) R_2'' is anti-complete to Y_1 . By symmetry, R_3'' is anti-complete to Y_4 . Suppose that $r_2'' \in R_2''$ and $y_1 \in Y_1$ are adjacent. By (2.8), f and y_1 are not adjacent. Then $r_2''y_1$ and f_2 induce a $2P_2$.
- (4.11) R'_2 is anti-complete to Y_3 . By symmetry, R'_3 is anti-complete to Y_2 . Suppose that $r'_2 \in R'_2$ and $y_3 \in Y_3$ are adjacent. By (2.9), f and y_3 are adjacent. Then $\{r'_2, y_3, 3, f\}$ induces a K_4 .
- (4.12) Y_5 is complete to $R'_2 \cup R'_3$. Suppose that $y_5 \in Y_5$ and $r'_2 \in R'_2$ are not adjacent. By (2.8), f and y_5 are not adjacent. Then fr'_2 and $5y_5$ induce a $2P_2$.

We now prove properties of Z.

(4.13) Any vertex in Z is anti-complete to either Y_2 or Y_3 .

Suppose not. Then there exists a vertex $z \in Z$ that is adjacent to a vertex $y_i \in Y_i$ for i = 2, 3. By (2.8), f is adjacent to y_2 and y_3 . Moreover, y_2 and y_3 are adjacent by (2.4). This implies that f and z are not adjacent for otherwise $\{f, z, y_i, y_{i+1}\}$ would induce a K_4 .

We now show that z is anti-complete to $Y_1 \cup Y_4 \cup Y_5$. Suppose not. Let z be adjacent to a vertex $y \in Y_1 \cup Y_4 \cup Y_5$. Note that there exists a vertex $i \in N_C(f)$ such that i is not adjacent to y. Moreover, f and y are not adjacent by (2.8). Then zy and if induce a $2P_2$. This shows that z is anti-complete to $Y_1 \cup Y_4 \cup Y_5$. Recall that Z is anti-complete to R_i for each i by (2.1). Therefore, $N(z) \subseteq Y_2 \cup Y_3 \subseteq N(f)$, contradicting the assumption that G has no pair of comparable vertices. (4.14) If $z \in Z$ is not adjacent to $y_i \in Y_i$, then y_i is complete to $N(z) \setminus Y_i$.

It suffices to prove for i = 1 by symmetry. Note that $N(z) \setminus Y_1 = (N(z) \cap (Y_2 \cup Y_5)) \cup (N(z) \cap (Y_3 \cup Y_4))$. By (2.4), y_1 is complete to $N(z) \cap (Y_2 \cup Y_5)$. It remains to show that y_1 is complete to $N(z) \cap (Y_3 \cup Y_4)$. Suppose not. Let $y \in N(z) \cap (Y_3 \cup Y_4)$ be nonadjacent to y_1 . By symmetry, we may assume that $y \in Y_3$. Then zy and y_14 induce a $2P_2$.

(4.15) If z is anti-complete to Y_i for some $i \in \{2, 3\}$, then $Y_i = \emptyset$.

Suppose that z is anti-complete to Y_2 and Y_2 contains a vertex y_2 . It follows from (4.14) that $N(z) \subseteq N(y_2)$, contradicting the assumption that G contains no pair of comparable vertices.

If $Y_5 = \emptyset$, then $N(5) = \{1, 4\} \cup R_1 \cup R_4 \cup Y_2 \cup Y_3 \subseteq N(f)$ by (2.8) and (2.9). This contradicts the assumption that G contains no pair of comparable vertices. So, we assume in the following that Y_5 contains a vertex y_5 . We claim now that either R''_2 or R''_3 is empty. Suppose not. Let $r''_i \in R''_i$ for i = 2, 3. By (2.3), r''_2 and r''_3 are adjacent. Moreover, y_5 is not adjacent to r''_2 and r''_3 by (4.8). Then $r''_2r''_3$ and $5y_5$ induce a $2P_2$. This proves that either R''_2 or R''_3 is empty. We consider two cases depending on whether f has a neighbor in R_5 .

Case 1. $R'_5 = \emptyset$, i.e., f has no neighbor in R_5 . Therefore, $R_5 = R''_5$. Recall that either R''_2 or R''_3 is empty. By symmetry, we may assyme that $R''_2 = \emptyset$. Then $R_2 = R'_2$ and so R_2 and R_4 are anti-complete by (4.6). Let $Y'_2 = \{y \in Y_2 : y \text{ is anti-complete to } Y_5\}$ and $Y''_2 = Y_2 \setminus Y'_2$. Note that each vertex in Y''_2 has a neighbor in Y_5 by the definition and so is anti-complete to Y_4 by (2.7). Then the following is a 4-coloring ϕ of $G - (R_3 \cup Z)$:

$$I_{1} = Y_{2}' \cup Y_{5} \cup R_{1} \cup \{1\}$$

$$I_{2} = Y_{2}'' \cup Y_{4} \cup R_{3} \cup \{3\}$$

$$I_{3} = R_{2}(= R_{2}') \cup R_{4} \cup Y_{3} \cup \{2, 4\}$$

$$I_{4} = Y_{1} \cup R_{5}(= R_{5}'') \cup \{f, 5\}$$

$$(2.12)$$

$$(2.12)$$

$$(4.11)$$

$$(2.8)$$

$$(4.9)$$

We now extend ϕ to R_3 as follows. Since R_3 is an independent set by (2.1), it suffices to explain how to extend ϕ to each vertex in R_3 independently. Let $r_3 \in R_3$ be an arbitrary vertex. Suppose first that $r_3 \in R'_3$. By (4.6) and (4.11), r_3 is anti-complete to $R_1 \cup Y_2$. By (2.13), r_3 is anti-complete to either Y_4 or Y_5 . Therefore, we can add r_3 to either I_1 or I_2 . Now suppose that $r_3 \in R''_3$. By (4.7) and (4.10), r_3 is anti-complete to $Y_4 \cup R_5$. By (2.13), r_3 is anti-complete to either Y_1 or Y_2 . Therefore, we can add r_3 to either I_2 or I_4 . This shows that G - Z admits a 4-coloring $\phi' = (I'_1, I'_2, I'_3, I'_4)$ with $I_i \subseteq I'_i$ for each $1 \leq i \leq 4$.

We now obtain a 4-coloring of G by either extending ϕ' to Z or by finding another 4-coloring of G. If Z is anti-complete to Y_3 , then we can extend ϕ' by adding Z to I'_3 . So, we assume that there is a vertex $z \in Z$ that is adjacent to a vertex in Y_3 . It then follows from (4.13) and (4.15) that $Y_2 = \emptyset$. If each vertex in Z is anti-complete to one of Y_3 , Y_4 and Y_5 , then we can extend ϕ' to Z by adding each vertex in Z to I'_1 , I'_2 or I'_3 (since $Y_2 = \emptyset$). Therefore, let $z \in Z$ be adjacent to $y_i \in Y_i$ for $i \in \{3, 4, 5\}$. We prove some additional properties using the existence of y_3 , y_4 and y_5 . First of all, R_1 and R_4 are anti-complete. Suppose not. Let $r_1 \in R_1$ and $r_4 \in R_4$ be adjacent. By (2.12), y_5 is not adjacent to r_1 and r_4 . Then r_1r_4 and zy_5 induce a $2P_2$. Secondly, y_3 and y_5 are not adjacent for otherwise $\{y_3, y_4, y_5, z\}$ induces a K_4 . Thirdly, Y_1 and Y_4 are anti-complete to each other. Suppose not. Then Y_1 contains a vertex y_1 that is not anit-complete to Y_4 . By (2.7), y_1 is anti-complete to Y_3 . Then fy_3 and y_1y_5 induce a $2P_2$. Now G admits the following 4-coloring:

$Y_1 \cup R_5''(=R_5) \cup Y_4 \cup \{f,5\}$	(4.9)
$Y_3 \cup R'_2(=R_2) \cup \{2\}$	(4.11)
$R_1 \cup R_4 \cup Y_5 \cup \{1,4\}$	(2.12)
$R_3 \cup Z \cup \{3\}$	(2.1)

Case 2. $R'_5 \neq \emptyset$. Let $r'_5 \in R'_5$. If $r_1 \in R_1$ and $r_4 \in R_4$ are adjacent, then $\{r_1, r_4, r'_5, f\}$ induces a K_4 by (2.3) and (2.9). So, R_1 and R_4 are anti-complete. We now consider two subcases.

Case 2.1. R''_2 and Y_3 are not anti-complete. Let $r''_2 \in R''_2$ and $y_3 \in Y_3$ be adjacent. We claim first that Y_1 and Y_4 are anti-complete. Suppose not. Let $y_1 \in Y_1$ and $y_4 \in Y_4$ be adjacent. Then y_3 and y_4 are adjacent by (2.4). By (2.7), y_1 is not adjacent to y_3 . Moreover, y_1 is not adjacent to r''_2 by (4.10). But now $4y_1$ and $y_3r''_2$ induce a $2P_2$. This shows that Y_1 and Y_4 are anti-complete. Moreover, Y_2 and R_3 are anti-complete by (2.6). Therefore, the following is a 4-coloring ϕ of $G - (R''_2 \cup Z)$.

$$I_{1} = R_{4} \cup Y_{5} \cup R_{1} \cup \{1, 4\}$$

$$I_{2} = Y_{1} \cup R_{5}'' \cup Y_{4} \cup \{f, 5\}$$

$$I_{3} = R_{3} \cup Y_{2} \cup \{3\}$$

$$I_{4} = Y_{3} \cup R_{2}' \cup R_{5}' \cup \{2\}$$

$$(2.12)$$

$$(4.9)$$

$$(4.9)$$

$$(4.9)$$

$$(4.4)$$

$$(4.1)$$

We now explain how to extend ϕ to each vertex in $R''_2 \cup Z$. Since $R''_2 \cup Z$ is an independent set by (2.1), this will give a 4-coloring of G. By (4.13), we can add each vertex in Z to either I_3 or I_4 . Let $s'' \in R''_2$ be an arbitrary vertex. Then s'' is anti-complete to $R''_5 \cup Y_1$ by (4.7) and (4.10). If s'' is not anti-complete to Y_3 , then s'' is anti-complete to Y_4 by (2.13) and thus we can add s'' to I_2 . Now s'' is anti-complete to Y_3 . We claim that s'' is anti-complete to R'_5 . Suppose not. Then s'' is adjacent to some vertex $r' \in R'_5$. Note that y_3 is not adjacent to s'' by our assumption. Moreover, y_3 is not adjacent to r' by (4.5). Then s''r' and $5y_3$ induce a $2P_2$. This shows that s'' is anti-complete to R'_5 and thus we can add s'' to I_4 .

Case 2.2. R''_2 and Y_3 are anti-complete. By symmetry, R''_3 and Y_2 are anti-complete. It follows from (4.11) that R_2 and Y_3 are anti-complete and R_3 and Y_2 are anti-complete. Recall that either R''_2 or R''_3 is empty. By symmetry, we may assume that $R''_2 = \emptyset$. Then $R_2 = R'_2$. We now claim that R'_3 is anti-complete to Y_4 . Suppose not. Let $r'_3 \in R'_3$ be adjacent to $y_4 \in Y_4$. By (4.12), r'_3 is adjacent to y_5 . But this contradicts (2.13). So, R'_3 is anti-complete to Y_4 . By symmetry, R'_2 is anti-complete to Y_1 . This together with (4.10) implies that R_3 and R_2 are anti-complete to Y_4 and Y_1 , respectively. Let $Y'_4 = \{y \in Y_4 : y \text{ is anti-complete to } Y_1\}$ and $Y''_4 = Y_4 \setminus Y'_4$. Note that each vertex in Y''_4 has a neighbor in Y_1 and so is anti-complete to Y_2 by (2.7). Now G - Z admits a 4-coloring ϕ :

$$I_{1} = R_{4} \cup Y_{5} \cup R_{1} \cup \{1, 4\}$$

$$I_{2} = Y_{1} \cup R_{5}'' \cup Y_{4}' \cup \{f, 5\}$$

$$I_{3} = R_{3} \cup Y_{2} \cup Y_{4}'' \cup \{3\}$$

$$I_{4} = Y_{3} \cup R_{2}' \cup R_{5}' \cup \{2\}$$

$$(2.12)$$

$$(4.9)$$

$$(2.12)$$

$$(4.9)$$

$$(4.9)$$

$$(2.6)$$

$$(4.4)$$

$$(4.5)$$

$$(4.11)$$

We now explain how to obtain a 4-coloring of G based on ϕ . If Z is anti-complete to Y_3 , then we can add Z to I_4 . So, assume that there exists a vertex in Z that is adjacent to some vertex in Y_3 . It then follows from (4.13) and (4.15) that $Y_2 = \emptyset$. If each vertex in Z is anti-complete to one of Y_3 , Y''_4 and Y_5 , then we can extend ϕ' to Z by adding each vertex in Z to I_1 , I_3 or I_4 (since $Y_2 = \emptyset$). Therefore, let $z \in Z$ be adjacent to $y_i \in Y_i$ for $i \in \{3, 5\}$ and be adjacent to $y_4 \in Y''_4$. Note that y_3 and y_5 are not adjacent for otherwise $\{y_3, y_4, y_5, z\}$ induces a K_4 . We claim that Y_1 and Y_4 are anti-complete to each other. Suppose not. Then Y_1 contains a vertex y_1 that is not anit-complete to Y_4 . By (2.7), y_1 is anti-complete to Y_3 . Then fy_3 and y_1y_5 induce a $2P_2$. Now G admits the following 4-coloring:

$R_4 \cup Y_5 \cup R_1 \cup \{1,4\}$	(2.12)
$Y_1 \cup R_5'' \cup Y_4 \cup \{f, 5\}$	(4.9)
$R_3 \cup Z \cup \{3\}$	(2.1)
$Y_3 \cup R'_2 \cup R'_5 \cup \{2\}$	(4.4)(4.5)(4.11)

This completes the proof.

5 Eliminate W_5 and C_5

In this section we prove two lemmas. The fist one states that our main theorem, Theorem 1, holds when G is connected, has no pair of comparable vertices, does not contain H_1 or H_2 as an induced subgraph, but contains the 5-wheel as an induced subgraph. The second lemma then assumes that G is W_5 -free as well, but contains an induced C_5 .

Lemma 4. Let G be a $(2P_2, K_4, H_1, H_2)$ -free graph with no pair of comparable vertices. If G contains an induced W_5 , then $\chi(G) \leq 4$.

Proof. Let $W = C \cup \{u\}$ be an induced W_5 such that C = 12345 induces a C_5 in this order and u is complete to C. We partition $V \setminus C$ into subsets of Z, R_i , Y_i , F_i and U as in section 2. Note that $u \in U$. Since G is H_2 -free, it follows that $F_i = \emptyset$ for each i. We prove the following properties.

(5.1) U is complete to R.

If $u' \in U$ is not adjacent to $r_i \in R_i$, then $C \setminus \{i\} \cup \{r_i, u\}$ induces an H_2 . This contradicts our assumption that G is H_2 -free.

(5.2) R_i and R_{i+2} are anti-complete.

Suppose that $r_i \in R_i$ and $r_{i+2} \in R_{i+2}$ are not adjacent. By (5.1), u is adjacent to both r_i and r_{i+2} . This implies that $\{r_i, r_{i+2}, i+1, u\}$ induces a K_4 .

- (5.3) R_i and Y_{i+1} are anti-complete. It suffices to prove for i = 1. If $r_1 \in R_1$ and $y_2 \in Y_2$ are adjacent, then $C \setminus \{1\} \cup \{r_1, y_2\}$ induces an H_2 , a contradiction.
- (5.4) Y_i and Y_{i+2} are anti-complete.

Since $U \neq \emptyset$, (5.4) follows directly from (2.10).

It follows from (5.2)–(5.4) and (2.1)–(2.2) that G admits the following 4-coloring:

$R_1\cup R_3\cup Z\cup\{1,3\}$	(5.2)(2.1)
$R_2 \cup Y_3 \cup R_4 \cup \{2,4\}$	(5.2)(5.3)
$Y_1 \cup R_5 \cup Y_4 \cup \{5\}$	(5.3)(5.4)
$Y_2 \cup Y_5 \cup U$	(5.4)(2.2)

This completes our proof.

Lemma 5. Let G be a connected $(2P_2, K_4, H_1, H_2, W_5)$ -free graph with no pair of comparable vertices. If G contains an induced C_5 , then $\chi(G) \leq 4$.

Proof. Let C = 12345 be an induced C_5 in this order. We partition $V \setminus C$ into subsets of Z, R_i, Y_i, F_i and U as in section 2. Since G is (H_2, W_5) -free, both U and F_i are empty. It then follows from Lemma 1 that $V(G) = C \cup Z \cup (\bigcup_{i=1}^5 R_i) \cup (\bigcup_{i=1}^5 Y_i)$. We first prove the following properties of R_i and Z.

(5.1) Each vertex in R_i is anti-complete to either R_{i-2} or R_{i+2} .

It suffices to prove for i = 4. Suppose that $r_4 \in R_4$ is adjacent to a vertex $r_i \in R_i$ for i = 1, 2. By (2.3), r_1 and r_2 are adjacent. This implies that $\{r_1, r_2, 3, 4, 5, r_4\}$ induces a subgraph isomorphic to H_2 . This contradicts the assumption that G is H_2 -free.

(5.2) R_i and Y_{i+1} are anti-complete.

It suffices to prove for i = 1. If $r_1 \in R_1$ and $y_2 \in Y_2$ are adjacent, then $C \setminus \{1\} \cup \{r_1, y_2\}$ induces an H_2 .

(5.3) Each vertex in Z cannot have a neighbor in each of Y_i for $1 \le i \le 5$.

Suppose that $z \in Z$ has a neighbor $y_i \in Y_i$ for each $1 \le i \le 5$. By (2.4), y_i and y_{i+1} are adjacent. This implies that y_i and y_{i+2} are not adjacent, for otherwise $\{y_i, y_{i+1}, y_{i+2}, z\}$ induces a K_4 . But now $\{y_1, y_2, y_3, y_4, y_5, z\}$ induces a W_5 .

(5.4) If $z \in Z$ has a neighbor in each of Y_i , Y_{i+1} , Y_{i+2} and Y_{i+3} , then Y_{i+4} is anti-complete to N(z).

It suffices to prove for i = 1. Let $y_i \in Y_i$ be a neighbor of z for $1 \le i \le 4$. By (5.3), z is anti-complete to Y_5 and so $N(z) \subseteq Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ by (2.1). Let y_5 be an arbitrary vertex in Y_5 . By (2.4), y_5 is complete to $Y_1 \cup Y_4$. Therefore it remains to show that y_5 is complete to $N(z) \cap (Y_2 \cup Y_3)$. If y_5 is not adjacent to a vertex $y \in N(z) \cap (Y_2 \cup Y_3)$, then either $3y_5$ or $2y_5$ forms a $2P_2$ with zy depending on whether $y \in Y_2$ or $y \in Y_3$.

(5.5) If Z contains a vertex that has a neighbor in Y_i , Y_{i+1} , Y_{i+2} and Y_{i+3} , then $Y_{i+4} = \emptyset$.

Let $z \in Z$ have neighbor in Y_i for $1 \le i \le 4$. By (5.3), z is anti-complete to Y_5 . If Y_5 contains a vertex y, then $N(z) \subseteq N(y)$ by (5.4). This contradicts the assumption that G contains no pair of comparable vertices.

Let $Y'_4 = \{y \in Y_4 : y \text{ is anti-complete to } Y_1\}$ and $Y''_4 = Y_4 \setminus Y'_4$. Note that each vertex in Y''_4 has a neighbor in Y_1 by the definition and so is anti-complete to Y_2 by (2.7). Similarly, let $R'_4 = \{r \in R_4 : r \text{ is anti-complete to } R_1\}$ and $R''_4 = R_4 \setminus R'_4$. By (5.1), R''_4 is anti-complete to R_2 . We now consider the following two cases.

Case 1. Z contains a vertex that has a neighbor in four Y_i . It then follows from (5.5) that $Y_j = \emptyset$ for some j. We may assume by symmetry that j = 5. These facts and (5.2) imply that G admits the following 4-coloring:

$$\begin{split} &Y_1 \cup R_5 \cup Y'_4 \cup \{5\}, \\ &Y_2 \cup R_3 \cup Y''_4 \cup \{3\}, \\ &R_1 \cup Z \cup R'_4 \cup \{1\}, \\ &R_2 \cup Y_3 \cup R''_4 \cup \{2,4\}. \end{split}$$

Case 2. Each vertex in Z has a neighbor in at most three Y_i . Note that G - Z admits the following 4-coloring ϕ by (5.2):

$$I_1 = Y_1 \cup R_5 \cup Y'_4 \cup \{5\},$$

$$I_2 = Y_2 \cup R_3 \cup Y''_4 \cup \{3\},$$

$$I_3 = R_1 \cup Y_5 \cup R'_4 \cup \{1\},$$

$$I_4 = R_2 \cup Y_3 \cup R''_4 \cup \{2,4\}$$

We now explain how to extend ϕ to Z. For this purpose we partition Z into the following two subsets:

$$Z_1 = \{ z \in Z : z \text{ is anti-complete to either } Y_3 \text{ or } Y_5 \}$$
$$Z_2 = Z \setminus Z_1.$$

We first claim that each vertex in Z_2 has a neighbor in Y_4 . Suppose not. Let $z \in Z_2$ be a vertex such that z is anti-complete to Y_4 . Since z has a neighbor in both Y_3 and Y_5 , z is anti-complete to either Y_1 or Y_2 by the assumption that each vertex in z has a neighbor in at most three Y_i . If z is anti-complete to Y_1 , then $N(z) \subseteq Y_2 \cup Y_3 \cup Y_5 \subseteq N(5)$. If z is anti-complete to Y_2 , then $N(z) \subseteq Y_1 \cup Y_3 \cup Y_5 \subseteq N(3)$. In either case, it contradicts the assumption that G contains no pair of comparable veritces. This proves the claim. Consequently, Z_2 is anticomplete to $Y_1 \cup Y_2$. We now claim that each vertex in Z_2 is anti-complete to either Y'_4 or Y''_4 . Suppose not. Let $z \in Z_2$ have a neighbor $y'_4 \in Y_4$ and a neighbor $y''_4 \in Y''_4$. By the definition of Y''_4 , it follows that y''_4 has a neighbor $y_1 \in Y_1$. Then $3y_1$ and y'_4z induce a $2P_2$ since y'_4 is not adjacent to y_1 . Now we can extend ϕ to Z by adding each vertex in Z_1 to I_3 or I_4 and by adding each vertex in Z_2 to I_1 or I_2 .

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