# $\left(2 P_{2}, K_{4}\right)$-Free Graphs are 4-Colorable 

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#### Abstract

In this paper, we show that every $\left(2 P_{2}, K_{4}\right)$-free graph is 4 -colorable. The bound is attained by the five-wheel and the complement of the seven-cycle. This answers an open question by Wagon [19] in the 1980s. Our result can also be viewed as a result in the study of the Vizing bound for graph classes. A major open problem in the study of computational complexity of graph coloring is whether coloring can be solved in polynomial time for ( $4 P_{1}, C_{4}$ )-free graphs. Lozin and Malyshev [15] conjecture that the answer is yes. As an application of our main result, we provide the first positive evidence to the conjecture by giving a 2 -approximation algorithm for coloring $\left(4 P_{1}, C_{4}\right)$-free graphs.


Keywords: graph coloring; $\chi$-bound; forbidden induced subgraphs; approximation algorithm.

AMS subject classifications: $68 \mathrm{R} 10,05 \mathrm{C} 15,05 \mathrm{C} 75,05 \mathrm{C} 85$.

## 1 Introduction

All graphs in this paper are finite and simple. We say that a graph $G$ contains a graph $H$ if $H$ is isomorphic to an induced subgraph of $G$. A graph $G$ is $H$-free if it does not contain $H$. For a family of graphs $\mathcal{H}, G$ is $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. In case $\mathcal{H}$ consists of two graphs, we write $\left(H_{1}, H_{2}\right)$-free instead of $\left\{H_{1}, H_{2}\right\}$-free. As usual, let $P_{n}$ and $C_{n}$ denote the path and the cycle on $n$ vertices, respectively. The complete graph on $n$ vertices is denoted by $K_{n}$. The $n$-wheel $W_{n}$ is the graph obtained from $C_{n}$ by adding a new vertex and making it adjacent to every vertex in $C_{n}$. For two graphs $G$ and $H$, we use $G+H$ to denote the disjoint union of $G$ and $H$. For a positive integer $r$, we use $r G$ to denote the disjoint union of $r$ copies of $G$. The complement of $G$ is denoted by $\bar{G}$. A hole in a graph is an induced cycle of length at least 4. A hole is odd if it is of odd length.

A $q$-coloring of a graph $G$ is a function $\phi: V(G) \longrightarrow\{1, \ldots, q\}$ such that $\phi(u) \neq \phi(v)$ whenever $u$ and $v$ are adjacent in $G$. We say that $G$ is $q$-colorable if $G$ admits a $q$-coloring. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number $q$ such that $G$ is $q$-colorable. The clique number of $G$, denoted by $\omega(G)$, is the size of a largest clique in $G$. Obviously, $\chi(G) \geq \omega(G)$ for any graph $G$. The maximum degree of a graph $G$ is denoted by $\Delta(G)$.

A family $\mathcal{G}$ of graphs is said to be $\chi$-bounded if there exists a function $f$ such that for every graph $G \in \mathcal{G}$ and every induced subgraph $H$ of $G$ it holds that $\chi(H) \leq f(\omega(H))$. The function $f$ is called a $\chi$-binding function for $\mathcal{G}$. The class of perfect graphs (a graph $G$ is perfect if for every induced subgraph $H$ of $G$ it holds that $\chi(H)=\omega(H)$ ), for instance, is a $\chi$-bounded family with $\chi$-binding function $f(x)=x$. Therefore, $\chi$-boundedness is a generalization of perfection. The notion of $\chi$-bounded families was introduced by Gyárfás [10] who make the following conjecture.

[^0]Conjecture 1 (Gyárfás [9]). For every forest $T$, the class of $T$-free graphs is $\chi$-bounded.
Gyárfás [10] proved the conjecture for $T=P_{t}$ : every $P_{t}$-free graph $G$ has $\chi(G) \leq(t-$ $1)^{\omega(G)-1}$. The result was slightly improved by Gravier, Hoàng and Maffray in [8] that every $P_{t}$-free graph $G$ has $\chi(G) \leq(t-2)^{\omega(G)-1}$. This implies that every $P_{5}$-free graph $G$ has $\chi(G) \leq 3^{\omega(G)-1}$. Note that this $\chi$-binding function is exponential in $\omega(G)$. For $\omega(G)=3$, Esperet, Lemoine, Maffray and Morel [3] obtained the optimal bound on the chromatic number: every $\left(P_{5}, K_{4}\right)$-free graph is 5-colorable. They also demonstrated a $\left(P_{5}, K_{4}\right)$-free graph whose chromatic number is 5 . On the other hand, a polynomial $\chi$-binding function for the class of $2 P_{2}$ free graphs was shown by Wagon [19] who proved that every such graph has $\chi(G) \leq\binom{\omega(G)+1}{2}$. This implies that every $\left(2 P_{2}, K_{4}\right)$-free graph is 6 -colorable. In [19] it was asked if there exists a $\left(2 P_{2}, K_{4}\right)$-free graph whose chromatic number is 5 or 6 . We observe that the $\left(P_{5}, K_{4}\right)$-free graph with chromatic number 5 given in [3] contains an induced $2 P_{2}$.

In this paper we settle Wagon's question [19] by proving the following theorem.
Theorem 1. Every $\left(2 P_{2}, K_{4}\right)$-free graph $G$ has $\chi(G) \leq 4$.
The bound in Theorem 1 is attained by the five-wheel $W_{5}$ and the complement of a sevencycle $\overline{C_{7}}$. Hence, we obtain the optimal $\chi$-bound for the class of $2 P_{2}$-free graphs when the clique number is 3 . A family $\mathcal{G}$ of graph is said to satisfy the Vizing bound if $f(x)=x+1$ is a $\chi$-binding function for $\mathcal{G}$. The definition was motivated by the classical Vizing's Theorem [18] on the chromatic index $\chi^{\prime}(G)$ of graphs which states that $\chi^{\prime}(G) \leq \Delta(G)+1$ for any graph $G$. This is equivalent to say that the class of line graphs satisfies the Vizing bound. Our result (Theorem 1) shows that the class of $\left(2 P_{2}, K_{4}\right)$-free graphs also satisfies the Vizing bound. We refer to Randerath and Schiermeyer [17] and Fan, Xu, Ye and Yu [4] for more results on the Vizing bound for various $\mathcal{H}$-free graphs.

We also note that our proofs of Theorem 1 below are algorithmic: one can easily follow the steps of the proof and give a 4-coloring of the input graph in polynomial time.
An application. Let Coloring denoted the computational problem of determining the chromatic number of a graph. In the past two decades, there has been an overwhelming attention on the complexity of Coloring $\mathcal{H}$-free graphs. The starting point is a result due to Král', Kratochvíl, Tuza, and Woeginger [14] who gave a complete classification of the complexity of Coloring for the case where $\mathcal{H}$ consists of a single graph $H$ : if $H$ is an induced subgraph of $P_{4}$ or of $P_{1}+P_{3}$, then Coloring restricted to $H$-free graphs is polynomial-time solvable, otherwise it is NP-complete. Afterwards, researchers started to study Coloring restricted to $\left(H_{1}, H_{2}\right)$-free graphs. Despite much efforts of top researchers in the area the complexity of Coloring are known only for some pairs of $H_{1}$ and $H_{2}$, see [6] for a summary of the known partial results. Even solving the problem for particular pairs of $H_{1}$ and $H_{2}$ requires substantial work, see $[2,16,11,12,15,13]$ for instance. Lozin and Malyshev [15] demonstrated that the classification is already problematic even if both $H_{1}$ and $H_{2}$ are 4 -vertex graphs: they determined the complexity of Coloring for all such pairs with three exceptions. One of the three unknown pairs is $\left(4 P_{1}, C_{4}\right)$. Lozin and Malyshev [15] conjecture that Coloring can be solved in polynomial time for $\left(4 P_{1}, C_{4}\right)$-free graphs. The problem was listed as an important open problem in the survey on the computational complexity of coloring graphs with forbidden subgraphs by Golovach, Johnson, Paulusma and Song [6].

Here we use Theorem 1 to give a 2 -approximation algorithm for coloring ( $4 P_{1}, C_{4}$ )-free graphs. This is the first general result towards a polynomial-time algorithm for the problem, although Fraser, Hamel, Hoàng, Holmes, and LaMantia showed that the problem is polynomial time solvable for a subclass of $\left(4 P_{1}, C_{4}\right)$-free graphs [5]. For a graph $G$ and a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$. A graph is chordal if it is $C_{t}$-free for each $t \geq 4$.
Theorem 2. There exists a polynomial-time 2-approximation algorithm for coloring $\left(4 P_{1}, C_{4}\right)$ free graphs.


Figure 1: Two special graphs $H_{1}$ and $H_{2}$.

Proof. Let $G$ be a $\left(4 P_{1}, C_{4}\right)$-free graph. Then $\bar{G}$ is $\left(2 P_{2}, K_{4}\right)$-free. By Theorem 1, we have that $\bar{G}$ can be partitioned into 4 stable sets. So, $G$ can be partitioned into 4 cliques $K_{i}$ for $1 \leq i \leq 4$, and this partition can be found in polynomial time. Since $G$ is $C_{4}$-free, both $G\left[K_{1} \cup K_{2}\right]$ and $G\left[K_{3} \cup K_{4}\right]$ are chordal. It is well-known that the chromatic number of a chordal graph can be determined in linear time, see [7] for example. Therefore, the value $\chi\left(G\left[K_{1} \cup K_{2}\right]\right)+\chi\left(G\left[K_{3} \cup K_{4}\right]\right)$ provides a 2-approximation for $\chi(G)$.

We now turn to the proof of Theorem 1. The neighborhood of a vertex $v$ in a graph $G$, denoted by $N_{G}(v)$, is the set of neighbors of $v$. We simply write $N(v)$ if the graph $G$ is clear from the context. Two nonadjacent vertices $u$ and $v$ in $G$ are comparable if either $N(v) \subseteq N(u)$ or $N(u) \subseteq N(v)$. Observe that if $N(u) \subseteq N(v)$, then $\chi(G-u)=\chi(G)$. Therefore, it suffices to prove Theorem 1 for every connected $\left(2 P_{2}, K_{4}\right)$-free graph with no pair of comparable vertices. We do so by proving a number of lemmas below. The idea is that we assume the occurrence of some induced subgraph $H$ in $G$ and then argue that the theorem holds in this case. Afterwards, we can assume that $G$ is $H$-free in addition to being $\left(2 P_{2}, K_{4}\right)$-free. We then pick a different induced subgraph as $H$ and repeat. In the end, we are able to show that the theorem holds if $G$ contains a $C_{5}$ (see Lemma 2-Lemma 5 below). Therefore, the remaining case is that $G$ is (odd hole, $K_{4}$ )-free. In this case, the theorem follows from a known result by Chudnovsky, Robertson, Seymour and Thomas [1] that every (odd hole, $K_{4}$ )-free graph is 4-colorable. This proves Theorem 1.

Th proof idea is based on a paper by Esperet et al. [3] who proved that every ( $P_{5}, K_{4}$ )-free graph is 5 -colorable. In particular, the graph $H_{1}$ (see Figure 1) that plays an important role in our proof was also used in [3]. However, to prove 4 -colorability we need to use the argument of comparable vertices and extensively extend the structural analysis in [3]. The remainder of the paper is organized as follows. In section 2 we present some preliminary results. In section 3 and section 4 we prove Lemma 2 and Lemma 3, respectively. We then prove Lemma 4 and Lemma 5 in section 5 .

## 2 Preliminaries

We present the structure around a five-cycle in $\left(2 P_{2}, K_{4}\right)$-free graphs that will be used in section 4 and section 5 . Let $G$ be a $\left(2 P_{2}, K_{4}\right)$-free graph and $C=12345$ be an induced $C_{5}$ of $G$. All indices below are modulo 5 . We partition $V \backslash C$ into the following subsets:

$$
\begin{aligned}
Z & =\left\{v \in V \backslash C: N_{C}(v)=\emptyset\right\}, \\
R_{i} & =\left\{v \in V \backslash C: N_{C}(v)=\{i-1, i+1\}\right\}, \\
Y_{i} & =\left\{v \in V \backslash C: N_{C}(v)=\{i-2, i, i+2\}\right\}, \\
F_{i} & =\left\{v \in V \backslash C: N_{C}(v)=C \backslash\{i\}\right\}, \\
U & =\left\{v \in V \backslash C: N_{C}(v)=C\right\} .
\end{aligned}
$$

Lemma 1. Let $G$ be a $\left(2 P_{2}, K_{4}\right)$-free graph and $C=12345$ be an induced $C_{5}$ of $G$. Then $V(G)=C \cup Z \cup\left(\bigcup_{i=1}^{5} R_{i}\right) \cup\left(\bigcup_{i=1}^{5} Y_{i}\right) \cup\left(\bigcup_{i=1}^{5} F_{i}\right) \cup U$.
Proof. Suppose that there is a vertex $v \in V(G) \backslash C$ that does not belong to any of $Z, R_{i}, Y_{i}$, $F_{i}$ and $U$. Note that $v$ has at least one and at most three neighbors on $C$. Moreover, these neighbors must be consecutive on $C$. Without loss of generality, we may assume that $v$ is adjacent to 1 and not adjacent to 3 and 4 . Now 34 and $1 v$ induce a $2 P_{2}$.

We now prove some structural properties of these sets.
(2.1) $Z \cup R_{i}$ is an independent set.

If $Z \cup R_{i}$ contains an edge $x y$, then $x y$ and $(i-2)(i+2)$ induce a $2 P_{2}$, a contradiction.
(2.2) $U \cup Y_{i}$ and $U \cup F_{i}$ are independent sets.

If either $U \cup Y_{i}$ or $U \cup F_{i}$ contains an edge $x y$, then $\{x, y, i-2, i+2\}$ induces a $K_{4}$.
(2.3) $R_{i}$ and $R_{i+1}$ are complete.

It suffices to prove for $i=1$. If $r_{1} \in R_{1}$ and $r_{2} \in R_{2}$ are not adjacent, then $5 r_{1}$ and $3 r_{2}$ induce a $2 P_{2}$.
(2.4) $Y_{i}$ and $Y_{i+1}$ are complete.

It suffices to prove for $i=1$. If $y_{1} \in Y_{1}$ and $y_{2} \in Y_{2}$ are not adjacent, then $5 y_{2}$ and $3 y_{1}$ induce a $2 P_{2}$.
(2.5) $R_{i}$ and $Y_{i}$ are complete.

It suffices to prove for $i=1$. If $r_{1} \in R_{1}$ and $y_{1} \in Y_{1}$ are not adjacent, then $5 r_{1}$ and $3 y_{1}$ induce a $2 P_{2}$.
(2.6) Either $R_{i}$ and $Y_{i+1}$ are anti-complete or $R_{i+1}$ and $Y_{i}$ are anti-complete.

Suppose, by contradiction, that there exist vertices $r_{i} \in R_{i}, r_{i+1} \in R_{i+1}, y_{i} \in Y_{i}, y_{i+1} \in$ $Y_{i+1}$ such that $r_{i}$ and $r_{i+1}$ are adjacent to $y_{i+1}$ and $y_{i}$, respectively. Then it follows from (2.3), (2.4) and (2.5) that $\left\{r_{i}, r_{i+1}, y_{i}, y_{i+1}\right\}$ induces a $K_{4}$.
(2.7) Each vertex in $Y_{i}$ is anti-complete to either $Y_{i-2}$ or $Y_{i+2}$.

It suffices to prove for $i=1$. If $y_{1} \in Y_{1}$ is adjacent to a vertex $y_{i} \in Y_{i}$ for $i=3,4$, then $\left\{1, y_{1}, y_{3}, y_{4}\right\}$ induces a $K_{4}$ by (2.4).
(2.8) $F_{i}$ is complete to $Y_{i-2} \cup Y_{i+2}$ and anti-complete to $Y_{i-1} \cup Y_{i} \cup Y_{i+1}$.

It suffices to prove for $i=5$. Let $f \in F_{5}$. Recall that $f$ is adjacent to $1,2,3,4$ but not adjacent to 5 by the definition of $F_{5}$. Suppose first that $f$ is not adjacent to a vertex $y \in Y_{2} \cup Y_{3}$. Note that $y$ is adjacent to 5 by the definition of $Y_{2}$ and $Y_{3}$. Now either $3 f$ or $2 f$ forms a $2 P_{2}$ with $5 y$ depending on whether $y \in Y_{2}$ or $y \in Y_{3}$. This proves the first part of (2.8). Suppose now that $f$ is adjacent to a vertex $y \in Y_{i}$ for some $i \in\{1,4,5\}$. Since $i \notin\{2,3\}$, it follows that $5 \notin\{i-2, i+2\}$. Therefore, $f$ is adjacent to $i-2$ and $i+2$. This implies that $\{f, y, i-2, i+2\}$ induces a $K_{4}$. This proves the second part of (2.8).
(2.9) $F_{i}$ is complete to $R_{i-1} \cup R_{i+1}$.

It suffices to prove $i=5$. If $f \in F_{5}$ is not adjacent to $r \in R_{1} \cup R_{4}$, then either $f 3$ or $f 2$ forms a $2 P_{2}$ with $5 r$ depending on whether $r \in R_{1}$ or $r \in R_{4}$.
(2.10) If $U \neq \emptyset$, then $Y_{i}$ and $Y_{i+2}$ are anti-complete.

Let $u \in U$. If $y_{i} \in Y_{i}$ and $y_{i+2} \in Y_{i+2}$ are adjacent, then $y_{i} y_{i+2}$ and $u(i+1)$ induce a $2 P_{2}$ since $u$ is adjacent to neither $y_{i}$ nor $y_{i+2}$ by (2.2), a contradiction.
(2.11) Either $F_{i}$ or $F_{i+2}$ is empty.

It suffices to prove for $i=3$. Suppose that $F_{i}$ contains a vertex $f_{i} \in F_{i}$ for $i=3,5$. Then either $3 f_{5}$ and $5 f_{3}$ induce a $2 P_{2}$ or $\left\{1,2, f_{3}, f_{5}\right\}$ induces a $K_{4}$ depending on whether $f_{3}$ and $f_{5}$ are nonadjacent or not.
(2.12) If $G$ is $H_{1}$-free, then the following holds: if $F_{i} \neq \emptyset$, then $R_{i+1}$ is anti-complete to $Y_{i+2} \cup Y_{i}$ and $R_{i-1}$ is anti-complete to $Y_{i-2} \cup Y_{i}$.
It suffices to prove for $i=5$. Let $f \in F_{5}$. Suppose, by contradiction, that there exists vertices $r \in R_{1}$ and $y \in Y_{2} \cup Y_{5}$ such that $r$ and $y$ are adjacent. Note that $f$ is adjacent to $r$ by (2.9). If $y \in Y_{2}$, then $f$ is adjacent to $y$ by (2.8) and this implies that $\{f, y, r, 2\}$ induces a $K_{4}$. If $y \in Y_{5}$, then $f$ is not adjacent to $y$ by (2.8) and this implies that $C \cup \backslash\{1\} \cup\{f, y, r\}$ induces an $H_{1}$ (see Figure 1). This proves that $R_{1}$ is anti-complete to $Y_{2} \cup Y_{5}$. The proof for the second part is symmetric.
(2.13) Each vertex in $R_{i}$ is anti-complete to either $Y_{i+1}$ or $Y_{i+2}$. By symmetry, each vertex in $R_{i}$ is anti-complete to either $Y_{i-1}$ or $Y_{i-2}$
Suppose, by contradiction, that there exists a vertex $r_{i} \in R_{i}$ such that $r_{i}$ is adjacent to a vertex $y_{i+1} \in Y_{i+1}$ and a vertex $y_{i+2} \in Y_{i+2}$. By (2.4), $y_{i+1}$ and $y_{i+2}$ are adjacent. This implies that $\left\{r_{i}, y_{i+1}, y_{i+2}, i-1\right\}$ induces a $K_{4}$.

## 3 Eliminate $H_{1}$

In this section we show that our main theorem, Theorem 1, holds when $G$ is connected, has no pair of comparable vertices, and contains $H_{1}$ as an induced subgraph.

Lemma 2. Let $G$ be a connected $\left(2 P_{2}, K_{4}\right)$-free graph with no pair of comparable vertices. If $G$ contains an induced $H_{1}$, then $\chi(G) \leq 4$.
Proof. Let $H=C \cup\{w\}$ be an induced $H_{1}$ in $G$ where $C=\{1,2,3,4,5,6\}$ induces a $\overline{C_{6}}$ such that $i j$ is an edge if and only if $|i-j| \neq 1$, and $w$ is adjacent to $1,2,4$ and 5 (See Figure 1). All the indices below are modulo 6 . We partition $V(G)$ into following subsets:

$$
\begin{aligned}
Z & =\left\{v \in V \backslash C: N_{C}(v)=\emptyset\right\}, \\
D_{i, i+1} & =\left\{v \in V \backslash C: N_{C}(v)=\{i, i+1\}\right\}, \\
T_{i} & =\left\{v \in V \backslash C: N_{C}(v)=\{i-1, i, i+1\}\right\}, \\
F_{i, i+1} & =\left\{v \in V \backslash C: N_{C}(v)=\{i-1, i, i+1, i+2\}\right\}, \\
W & =\left\{v \in V \backslash C: N_{C}(v)=N_{C}(w)=\{1,2,4,5\}\right\} .
\end{aligned}
$$

Let $D=\bigcup_{i=1}^{6} D_{i, i+1}, T=\bigcup_{i=1}^{6} T_{i}$ and $F=\bigcup_{i=1}^{6} F_{i, i+1}$. Without loss of generality, we assume $H$ has been chosen such that $|T|+|F|$ is maximized. We first show that $V(G)=C \cup Z \cup D \cup$ $T \cup F \cup W$.
(3.1) There is no vertex $v \in V \backslash C$ such that $v$ is adjacent to $i$ but adjacent to neither $i-1$ nor $i+1$ for any $1 \leq i \leq 6$.
Suppose that such a vertex $v$ exists. Then it follows that $v i$ and $(i-1)(i+1)$ induce a $2 P_{2}$.
(3.2) If a vertex in $V \backslash C$ has at most two neighbors on $C$, then $v \in Z \cup D$.

Suppose not. Let $v \in V \backslash C$ that has at most two neighbors on $C$ and $v \notin Z \cup D$. Then either $v$ has exactly one neighbor on $C$ or has two neighbors on $C$ that are not consecutive. By symmetry, we may assume that $v$ is adjacent to 1 but not adjacent to 2 and 6. This contradicts (3.1).
(3.3) If a vertex $v \in V \backslash C$ that has exactly three neighbors on $C$, then $v \in T$.

Suppose not. Let $v \in V \backslash C$ that has exactly at three neighbors on $C$. By symmetry, we may assume that $v$ is adjacent to 1 . It follows from (3.1) that $v$ is adjacent to either 2 or 6 , say 2 . If $v$ is not adjacent to 3 or 6 , then it contradicts (3.1) for $i=4$ or $i=5$. Therefore, $v \in T_{1}$ or $v \in T_{2}$.
(3.4) If a vertex $v \in V \backslash C$ that has exactly four neighbors on $C$, then $v \in F \cup W$.

By (3.1), $v$ must have two consecutive neighbors on $C$. If $v$ has three consecutive neighbors on $C$, then all four neighbors must be consecutive by (3.1) and so $v \in F$. Now $N_{C}(v)=$ $\{i, i+1, i+3, i+4\}$ for some $i$. If $i=1$, then $v \in W$. Suppose that $i=2$ (and the case $i=3$ is symmetric). Then either $w 1$ and $v 6$ induce a $2 P_{2}$ or $\{w, v, 2,5\}$ induces a $K_{4}$, depending on whether $w$ and $v$ are nonadjacent or not.
(3.5) There is no vertex in $V \backslash C$ that has more than four neighbors.

Suppose not. Let $v \in V \backslash C$ have at least five neighbors on $C$. By symmetry, we may assume that $v$ is adjacent to $i$ for each $1 \leq i \leq 5$. Then $\{1,3,5, v\}$ induces a $K_{4}$.

It follows from (3.2)-(3.5) that $V(G)=C \cup Z \cup D \cup T \cup F \cup W$. Note that each of the subsets defined is an independent set since $G$ is $\left(2 P_{2}, K_{4}\right)$-free. We further investigate the adjacency among those subsets.
(3.6) The set $W$ is anti-complete to $Z$.

If $w \in W$ and $z \in Z$ are adjacent, then $w z$ and 36 induce a $2 P_{2}$, a contradiction.
(3.7) The set $W$ is complete to $D_{i, i+1}$ for $i \in\{2,3,5,6\}$ and anti-complete to $D_{i, i+1}$ for $i \in\{1,4\}$. Suppose that $w \in W$ is not adjacent some vertex $d \in D_{i, i+1}$ for some $i \in\{2,3,5,6\}$. By symmetry, we may assume that $i=2$. Then $d 3$ and $w 4$ induce a $2 P_{2}$, a contradiction. Suppose that $w \in W$ is adjacent some vertex $d \in D_{1,2} \cup D_{4,5}$. Then $d w$ and 36 induce a $2 P_{2}$, a contradiction.
(3.8) The set $W$ is complete to $T_{1} \cup T_{2} \cup T_{4} \cup T_{5}$ and anti-complete to $T_{3} \cup T_{6}$.

Suppose that $w \in W$ is not adjacent some vertex $t \in T_{i}$ for some $i \in\{1,2,4,5\}$. By symmetry, we may assume that $i=1$. Then $t 6$ and $w 5$ induce a $2 P_{2}$. Suppose that $w \in W$ is adjacent some vertex $t \in T_{i}$ for some $i \in\{3,6\}$. By symmetry, we may assume that $i=3$. Then $\{w, t, 2,4\}$ induces a $K_{4}$.
(3.9) The set $W$ is anti-complete to $F_{i, i+1}$ for $i \in\{2,3,5,6\}$ and complete to $F_{i, i+1}$ for $i \in\{1,4\}$. Suppose that $w \in W$ is adjacent some vertex $f \in F_{i, i+1}$ for some $i \in\{2,3,5,6\}$. By symmetry, we may assume that $i=2$. Then $\{f, w, 1,4\}$ induces a $K_{4}$. Suppose that $w \in W$ is not adjacent some vertex $f \in F_{i, i+1}$ for some $i \in\{1,4\}$. By symmetry, we may assume that $i=1$. Then $6 f$ and $5 w$ induce a $2 P_{2}$.
(3.10) The set $Z$ is anti-complete to $D \cup T \cup\left(F \backslash\left(F_{1,2} \cup F_{4,5}\right)\right)$.

Suppose that $z \in Z$ is adjacent to some vertex $x \in D \cup T \cup\left(F \backslash\left(F_{1,2} \cup F_{4,5}\right)\right)$. If $x \in D \cup T$, then there exists a vertex $i \in C$ such that $x$ is not adjacent to $i-1$ and $i+1$. Then $z x$ and $(i-1)(i+1)$ induce a $2 P_{2}$. If $x \in F_{i, i+1}$ for some $i=2,3,5,6$, then $x w \notin E$ by (3.9). Moreover, there exists a vertex $j \in N_{C}(w)$ such that $x j \notin E$. Then $w j$ and $z x$ induce a $2 P_{2}$.

It follows from and (3.6) and (3.10) that any vertex in $Z$ has neighbors only in $F_{1,2} \cup F_{4,5}$. On the other hand, $w$ is complete to $F_{1,2} \cup F_{4,5}$ by (3.9). Since $G$ contains no pair of comparable vertices, it follows that $Z=\emptyset$.
(3.11) For each $i, D_{i, i+1}$ is anti-complete to $D_{i+1, i+2}$, complete to $D_{i+2, i+3}$ and anti-complete to $D_{i+3, i+4}$.
By symmetry, it suffices to prove the claim for $i=1$. Let $d \in D_{1,2}$. If $d$ is adjacent to $d^{\prime} \in D_{2,3}$, then 46 and $d d^{\prime}$ induce a $2 P_{2}$. If $d$ is not adjacent to $d^{\prime} \in D_{3,4}$, then $2 d$ and $3 d^{\prime}$ induce a $2 P_{2}$. If $d$ is adjacent to $d^{\prime} \in D_{4,5}$, then 36 and $d d^{\prime}$ induce a $2 P_{2}$.
(3.12) For each $i, F_{i, i+1}$ is anti-complete to $F_{i+1, i+2} \cup F_{i+3, i+4}$ and complete to $F_{i+2, i+3}$.

By symmetry, it suffices to prove the claim for $i=1$. Let $f \in F_{1,2}$. If $f$ is adjacent to a vertex $f^{\prime} \in F_{2,3}$, then $\left\{1,3, f, f^{\prime}\right\}$ induces a $K_{4}$. If $f$ is not adjacent to a vertex $f^{\prime} \in F_{3,4}$, then $5 f^{\prime}$ and $6 f$ induce a $2 P_{2}$. If $f$ is adjacent to a vertex $f^{\prime} \in F_{4,5}$, then $\left\{3,6, f, f^{\prime}\right\}$ induces a $K_{4}$.
(3.13) The sets $T_{i}$ and $T_{i+1}$ are anti-complete for $i \in\{1,4\}$.

By symmetry, it suffices to prove this for $i=1$. If $t_{1} \in T_{1}$ and $t_{2} \in T_{2}$ are adjacent, then $w$ is adjacent to both $t_{1}$ and $t_{2}$ by (3.8). But now $\left\{t_{1}, t_{2}, w, 1\right\}$ induces a $K_{4}$.
(3.14) The sets $T_{3}$ and $T_{1} \cup T_{5}$ are complete. By symmetry, $T_{6}$ and $T_{2} \cup T_{4}$ are complete.

Suppose that $t_{3} \in T_{3}$ is not adjacent to some vertex $t \in T_{1} \cup T_{5}$. By (3.8), $w$ is adjacent to $t$ but not to $t_{3}$. Then $3 t_{3}$ and $w t$ induce a $2 P_{2}$, a contradiction.
(3.15) The sets $T_{i}$ and $T_{i+3}$ are complete for each $1 \leq i \leq 6$.

By symmetry, it suffices to prove this for $i=1$. If $t_{1} \in T_{1}$ and $t_{4} \in T_{4}$ are not adjacent, then $2 t_{1}$ and $3 t_{4}$ induce a $2 P_{2}$.
(3.16) For each $i, D_{i, i+1}$ is anti-complete to $T_{i-1} \cup T_{i} \cup T_{i+1} \cup T_{i+2}$ and complete to $T_{i+3} \cup T_{i+4}$.

We note that $D_{1,2}$ and $D_{4,5}$ are symmetric, and $D_{2,3}, D_{3,4}, D_{5,6}$ and $D_{6,1}$ are symmetric. So, it suffices to prove the claim for $D_{1,2}$ and $D_{2,3}$.
Let $d \in D_{1,2}$. Suppose that $d$ is adjacent to some vertex $t \in T_{6} \cup T_{1} \cup T_{2} \cup T_{3}$. By symmetry, we may assume that $i \in\{1,3\}$. If $i=1$, then $t d$ and 35 induce a $2 P_{2}$. If $i=3$, then $w$ is not adjacent to $d$ and $t$ by (3.7) and (3.8). Then $d t$ and $w 5$ induce a $2 P_{2}$. Now suppose that $d$ is not adjacent to some vertex $t \in T_{4} \cup T_{5}$. By symmetry, we may assume that $t \in T_{4}$. Then $d 2$ and $t 3$ induce a $2 P_{2}$. This proves the claim for $D_{1,2}$.

Let $d \in D_{2,3}$. Suppose that $d$ is adjacent to some vertex $t \in T_{2} \cup T_{3}$. By symmetry, we may assume that $t \in T_{2}$. Then $d t$ and 46 induce a $2 P_{2}$. Suppose that $d$ is not adjacent to some vertex $t \in T_{5} \cup T_{6}$. By symmetry, we may assume that $t \in T_{5}$. Then $d 3$ and $t 4$ induce a $2 P_{2}$.
By (3.7) and (3.8), $\{2, w\}$ is complete to $D_{2,3} \cup T_{1}$. It follows from $K_{4}$-freeness of $G$ that $D_{2,3}$ is anti-complete to $T_{1}$. It remains to show that $D_{2,3}$ is anti-complete to $T_{4}$. Suppose that $d$ is adjacent to some vertex $t_{4} \in T_{4}$. Note that $C^{\prime}=C \backslash\{1\} \cup\left\{t_{4}\right\}$ induces a $\overline{C_{6}}$ and $H^{\prime}=C^{\prime} \cup\{w\}$ induces a subgraph isomorphic to $H_{1}$. By (3.13) and (3.14), all vertices in $T_{1} \cup T_{4} \cup T_{5} \cup T_{6}$ remain to be $T$-vertices with respect to $C^{\prime}$. Moreover, all vertices in $T_{3} \cup F$ remain to be $F$-vertices or $T$-vertices. By the choice of $C$, there exists a vertex $t \in T_{2}$ that is not adjacent to $t_{4}$. Then $d t_{4}$ and $1 t_{2}$ induce a $2 P_{2}$, a contradiction. This proves the claim for $D_{2,3}$.
(3.17) For each $i, F_{i, i+1}$ is anti-complete to $T_{i} \cup T_{i+1}$ and complete to $T_{i+3} \cup T_{i+4}$

By symmetry of $C$, it suffices to prove this for $i=1$. Let $f \in F_{1,2}$. If $f$ is adjacent to some vertex $t \in T_{1} \cup T_{2}$, then either $\{6,2, f, t\}$ or $\{1,3, f, t\}$ induces a $K_{4}$ depending on whether $t \in T_{1}$ or $t \in T_{2}$. Suppose that $f$ is not adjacent to some vertex $t \in T_{4} \cup T_{5}$. By symmetry, we may assume that $t \in T_{4}$. Then $6 f$ and $5 t$ induce a $2 P_{2}$, a contradiction.
(3.18) The sets $F_{i, i+1}$ and $T_{i-1}$ are complete for $i \in\{2,5\}$, and $F_{i, i+1}$ and $T_{i+2}$ are complete for $i \in\{3,6\}$.
Let $f \in F_{i, i+1}$ and $t \in T_{i}$ be nonadjacent. By (3.9) and (3.8), $w$ is adjacent to $t$ but not $f$. It can be readily checked that in each of the cases $w t$ and $f 3$ or $w t$ and $f 6$ induce a $2 P_{2}$.
(3.19) The set $D_{1,2}$ is anti-complete to $F_{6,1} \cup F_{2,3}$ and complete to $F_{45}$.

The set $D_{4,5}$ is anti-complete to $F_{3,4} \cup F_{5,6}$ and complete to $F_{12}$.
The set $D_{2,3}$ is anti-complete to $F_{1,2}$ and complete to $F_{5,6} \cup F_{6,1}$.
The set $D_{3,4}$ is anti-complete to $F_{4,5}$ and complete to $F_{5,6} \cup F_{6,1}$.
The set $D_{6,1}$ is anti-complete to $F_{1,2}$ and complete to $F_{2,3} \cup F_{3,4}$.
The set $D_{5,6}$ is anti-complete to $F_{4,5}$ and complete to $F_{2,3} \cup F_{3,4}$.
Note that $D_{1,2}$ and $D_{4,5}$ are symmetric, and $D_{2,3}, D_{3,4}, D_{5,6}$ and $D_{6,1}$ are symmetric. So, it suffices to prove the claim for $D_{1,2}$ and $D_{2,3}$. Let $d \in D_{1,2}$. If $d$ is adjacent to some vertex $f \in F_{6,1} \cup F_{2,3}$, then $w$ is not adjacent to $d$ and $f$ by (3.7) and (3.9). Now $d f$ and $w 4$ or $d f$ and $w 5$ induce a $2 P_{2}$ depending on whether $f \in F_{6,1}$ or $f \in F_{2,3}$. If $d$ is not adjacent to some vertex $f \in F_{4,5}$, then $d 2$ and $f 3$ induce a $2 P_{2}$. This proves the claim for $D_{1,2}$.
Now let $d \in D_{2,3}$. By (3.7), it follows that $w d \in E$. If $d$ is adjacent to a vertex $f \in F_{1,2}$, then $\{d, f, 2, w\}$ induces a $K_{4}$ by (3.9). If $d$ is not adjacent to a vertex $f \in F_{5,6} \cup F_{6,1}$, then $6 f$ and $w d$ induce a $2 P_{2}$ by (3.9). This proves the claim for $D_{2,3}$.

We proceed with a few claims that help to show that certain sets are empty.
Claim 1. Either $D_{1,2}$ or $D_{4,5}$ is empty.
Proof of Claim 1. Suppose not. Let $d_{12} \in D_{1,2}$ and $d_{45} \in D_{4,5}$. By (3.7)-(3.19), $N\left(d_{12}\right) \subseteq$ $N(w)$ unless $d_{12}$ has a neighbor $f \in F_{3,4} \cup F_{5,6}$. Similarly, $N\left(d_{45}\right) \subseteq N(w)$ unless $d_{45}$ has a neighbor $f^{\prime} \in F_{3,4} \cup F_{5,6}$. By (3.11) and (3.19), $d_{12} f$ and $d_{45} f^{\prime}$ induce a $2 P_{2}$, a contradiction.

Claim 2. Each vertex in $T_{1}$ has a non-neighbor in $T_{5}$ and each vertex in $T_{5}$ has a non-neighbor in $T_{1}$. By symmetry, each vertex in $T_{2}$ has a non-neighbor in $T_{4}$ and each vertex in $T_{4}$ has a non-neighbor in $T_{2}$.

Proof of Claim 2. Let $t_{1} \in T_{1}$. Let

$$
X=\{6,1,2\} \cup W \cup D_{3,4} \cup D_{4,5} \cup T_{3} \cup T_{4} \cup F_{2,3} \cup F_{3,4} \cup F_{4,5} .
$$

Note that $N(4)=X \cup T_{5} \cup F_{5,6}$ and $N\left(t_{1}\right) \subseteq X \cup T_{5} \cup F_{5,6} \cup T_{6}$ by the properties we have proved. Since $G$ contains no pair of comparable vertices, $t_{1}$ has a neighbor $t_{6} \in T_{6}$ and there exists a vertex $t \in N(4) \backslash N\left(t_{1}\right)$. Clearly, $t \in F_{5,6} \cup T_{5}$. If $t \in F_{5,6}$, then $4 t$ and $t_{1} t_{6}$ induce a $2 P_{2}$ since $F_{56}$ and $T_{6}$ are anti-complete by (3.17). This shows that $t_{1}$ has a non-neighbor $t \in T_{5}$. By symmetry, each vertex in $T_{5}$ has a non-neighbor in $T_{1}$.

Claim 3. Each vertex in $T_{6}$ has a neighbor in $T_{1} \cup T_{5}$. By symmetry, each vertex in $T_{3}$ has a neighbor in $T_{2} \cup T_{4}$.

Proof of Claim 3. Let $t_{6} \in T_{6}$. Let

$$
X=\{5,6,1\} \cup D_{2,3} \cup D_{3,4} \cup T_{2} \cup T_{3} \cup T_{4} \cup F_{2,3} \cup F_{3,4} .
$$

Note that $N(3)=X \cup F_{1,2} \cup F_{4,5}$ and $N\left(t_{6}\right) \subseteq X \cup T_{1} \cup T_{5} \cup F_{12} \cup F_{45}$. Since $G$ contains no pair of comparable vertices, $t_{6}$ has a neighbor in $T_{1} \cup T_{5}$.


Figure 2: The adjacency among $T_{i}$ and $D_{i, i+1}$. A thick line between two sets means that the two sets are complete, a thin line means the edges between the two sets can be arbitrary, and no line means that the two sets are anti-complete. For clarity, edges between two $D_{i, i+1}$ are not shown.

Claim 4. If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $T_{2}$ and $T_{4}$ are complete. By symmetry, if $D_{2,3} \cup D_{3,4} \neq \emptyset$, then $T_{1}$ and $T_{5}$ are complete.

Proof of Claim 4. Let $d \in D_{5,6} \cup D_{6,1}$. Suppose that $t_{2} \in T_{2}$ and $t_{4} \in T_{4}$ are not adjacent. If $d \in D_{5,6}$, then $d t_{2} \in E$ and $d t_{4} \notin E$ by (3.16). Thus, $d t_{2}$ and $4 t_{4}$ induce a $2 P_{2}$. If $d \in D_{6,1}$, then $d t_{4} \in E$ and $d t_{2} \notin E$ by (3.16). Thus, $d t_{4}$ and $2 t_{2}$ induce a $2 P_{2}$.

Claim 5. One of $F_{6,1}, F_{1,2}$ and $F_{2,3}$ is empty. By symmetry, one of $F_{3,4}, F_{4,5}$ and $F_{5,6}$ is empty.

Proof of Claim 5. Suppose that $f_{61} \in F_{6,1}, f_{12} \in F_{1,2}$, and $f_{23} \in F_{2,3}$. Then $f_{61} f_{23}$ and $f_{12} w$ induce a $2 P_{2}$ by (3.9) and (3.12).

By Claim 1, we may assume that $D_{4,5}=\emptyset$. It follows from (3.13), (3.14) and (3.15) that either $T_{1}$ and $T_{5}$ are complete or $T_{2}$ and $T_{4}$ are complete for otherwise $G$ would contain a $2 P_{2}$ (see Figure 2). By symmetry, we may assume that $T_{1}$ and $T_{5}$ are complete. It then follows from Claim 2 and Claim 3 that $T_{1} \cup T_{5} \cup T_{6}=\emptyset$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $T_{2} \cup T_{3} \cup T_{4}=\emptyset$ due to Claim 2-Claim 4. In the following we shall use this fact without explicitly mentioning it. We divide our proof into four cases depending on whether $F_{1,2}$ and $F_{4,5}$ are empty or not. One can verify that each of the partitions of $V(G)$ into 4 subsets in the following is a 4 -coloring of $G$ using the properties we have proved. For convenience, we draw Figure 3 to visulize the adjacency among $D_{i, i+1}$ and $F_{i, i+1}$. From Figure 3 it can be seen that if $T_{2} \cup T_{3} \cup T_{4}=\emptyset$, then we can use the symmetry of $H$ under its automorphism $f: V(H) \rightarrow V(H)$ with $f(1)=2, f(2)=1, f(3)=6, f(4)=5, f(5)=4$, $f(6)=3$ and $f(w)=w$.

Case 1. Both $F_{1,2}$ and $F_{4,5}$ are not empty. Let $f_{12} \in D_{1,2}$ and $f_{45} \in D_{4,5}$. We first show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3} \cup D_{3,4} \cup D_{5,6} \cup D_{6,1}$. By symmetry, it suffices to show that $F_{1,2} \cup F_{4,5}$ is anti-complete to $D_{2,3}$. Suppose that $d \in D_{2,3}$ and $f \in F_{1,2} \cup F_{4,5}$ are adjacent. By (3.19), $f \in F_{4,5}$. Then $d f$ and $1 f_{12}$ induce a $2 P_{2}$. On the other hand, it follows from Claim 5 and (3.12) that at most one of $F_{2,3}, F_{3,4}, F_{5,6}$ and $F_{6,1}$ is not empty.


Figure 3: The adjacency among $F_{i, i+1}$ and $D_{i, i+1}$. A thick line between two sets means that the two sets are complete, a thin line means the edges between the two sets can be arbitrary, and no line means that the two sets are anti-complete. For clarity, edges between two $D_{i, i+1}$ are not shown.

- If $F_{2,3} \neq \emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{4,5} \cup D_{2,3} \cup D_{3,4} \cup\{1\} \cup T_{4}, \\
& F_{2,3} \cup D_{1,2} \cup W \cup\{6\} \cup T_{3}, \\
& F_{1,2} \cup\{4,5\} \cup T_{2}, \\
& D_{5,6} \cup D_{6,1} \cup\{2,3\} .
\end{aligned}
$$

- Suppose that $F_{6,1} \neq \emptyset$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{4,5} \cup D_{5,6} \cup D_{6,1} \cup\{2\}, \\
& F_{6,1} \cup D_{1,2} \cup W \cup\{3\}, \\
& F_{1,2} \cup\{4,5\}, \\
& D_{2,3} \cup D_{3,4} \cup\{1,6\} .
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{4,5} \cup\{1,2\} \cup T_{4}, \\
& F_{6,1} \cup D_{1,2} \cup W \cup\{3\}, \\
& F_{1,2} \cup\{4,5\} \cup T_{2}, \\
& D_{2,3} \cup D_{3,4} \cup\{6\} \cup T_{3} .
\end{aligned}
$$

- Suppose that $F_{3,4} \neq \emptyset$. Note first that no vertex $d \in D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$ for otherwise a neighbor of $d$ in $F_{1,2}$, a neighbor of $d$ in $F_{3,4}, d$ and 2 induce a $K_{4}$. Let
$D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{3,4}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$. Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{4,5} \cup D_{2,3} \cup D_{3,4} \cup\{1\} \cup T_{4}, \\
& F_{3,4} \cup D_{1,2}^{\prime} \cup W \cup\{6\} \cup T_{3}, \\
& F_{1,2} \cup D_{1,2}^{\prime \prime} \cup\{4,5\} \cup T_{2}, \\
& D_{5,6} \cup D_{6,1} \cup\{2,3\} .
\end{aligned}
$$

- Suppose that $F_{5,6} \neq \emptyset$. Note first that no vertex $d \in D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$ for otherwise a neighbor of $d$ in $F_{1,2}$, a neighbor of $d$ in $F_{5,6}, d$ and 1 induce a $K_{4}$. Let $D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{5,6}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$. By (3.17) and (3.18), $F_{5,6}$ and $T_{3} \cup T_{4}$ are complete. Since $G$ is $K_{4}$-free, $T_{3}$ and $T_{4}$ are anti-complete. Then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{4,5} \cup D_{5,6} \cup D_{6,1} \cup\{2\}, \\
& F_{5,6} \cup D_{1,2}^{\prime} \cup W \cup\{3\}, \\
& F_{1,2} \cup D_{1,2}^{\prime \prime} \cup\{4,5\} \cup T_{2}, \\
& D_{2,3} \cup D_{3,4} \cup\{1,6\} \cup T_{3} \cup T_{4} .
\end{aligned}
$$

Case 2. Both $F_{1,2}$ and $F_{4,5}$ are empty. By (3.12) and the fact that $G$ is $2 P_{2}$-free, one of $F_{2,3}$, $F_{3,4}, F_{5,6}$ and $F_{6,1}$ is empty. By (3.11), (3.19), (3.12) and $K_{4}$-freeness of $G$, either $D_{5,6}$ and $F_{5,6}$ are anti-complete or $D_{3,4}$ and $F_{3,4}$ are anti-complete.

- Suppose that $F_{6,1}=\emptyset$.

If $D_{5,6}$ and $F_{5,6}$ are anti-complete, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{2,3} \cup F_{3,4} \cup W \cup\{6\} \cup T_{3}, \\
& F_{5,6} \cup D_{5,6} \cup\{2,3\}, \\
& D_{1,2} \cup D_{6,1} \cup\{4,5\} \cup T_{2}, \\
& D_{2,3} \cup D_{3,4} \cup\{1\} \cup T_{4} .
\end{aligned}
$$

Now assume that $D_{3,4}$ and $F_{3,4}$ are anti-complete.
If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{2,3} \cup F_{5,6} \cup W, \\
& F_{3,4} \cup D_{3,4} \cup\{6,1\}, \\
& D_{1,2} \cup D_{2,3} \cup\{4,5\}, \\
& D_{5,6} \cup D_{6,1} \cup\{2,3\} .
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{2,3} \cup D_{1,2} \cup W \cup\{6\} \cup T_{3}, \\
& F_{3,4} \cup D_{3,4} \cup\{1\} \cup T_{4}, \\
& F_{5,6} \cup\{2,3\}, \\
& D_{2,3} \cup\{4,5\} \cup T_{2} .
\end{aligned}
$$

- Suppose that $F_{2,3}=\emptyset$.

Suppose first that $D_{3,4}$ and $F_{3,4}$ are anti-complete.
If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{6,1} \cup F_{5,6} \cup W \cup\{3\}, \\
& F_{3,4} \cup D_{3,4} \cup\{6,1\}, \\
& D_{1,2} \cup D_{2,3} \cup\{4,5\}, \\
& D_{6,1} \cup D_{5,6} \cup\{2\} .
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{6,1} \cup F_{5,6} \cup W \cup\{3\}, \\
& F_{3,4} \cup D_{3,4} \cup\{6\} \cup T_{3}, \\
& D_{1,2} \cup D_{2,3} \cup\{4,5\} \cup T_{2}, \\
& \{1,2\} \cup T_{4} .
\end{aligned}
$$

Suppose now that $D_{3,4}$ and $F_{3,4}$ are not anti-complete and that $D_{5,6}$ and $F_{5,6}$ are anticomplete. By (3.16) and (3.17), $D_{3,4} \cup F_{3,4}$ are anti-complete to $T_{3} \cup T_{4}$. Since $G$ is $2 P_{2}$-free, it follows that $T_{3}$ and $T_{4}$ are anti-complete. Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{6,1} \cup F_{3,4} \cup W, \\
& F_{5,6} \cup D_{5,6} \cup\{2,3\}, \\
& D_{1,2} \cup D_{6,1} \cup\{4,5\} \cup T_{2}, \\
& D_{2,3} \cup D_{3,4} \cup\{6,1\} \cup T_{3} \cup T_{4} .
\end{aligned}
$$

- Suppose that $F_{5,6}=\emptyset$. If $F_{6,1}=\emptyset$, then $G$ has a 4 -coloring as above. So, we can assume that $F_{6,1} \neq \emptyset$. Let $f_{61} \in F_{6,1}$. If $d \in D_{2,3}$ and $f \in F_{2,3}$ are adjacent, then $\left\{2, f_{61}, d, f\right\}$ induces a $K_{4}$ by (3.12) and (3.19). So, $D_{2,3}$ and $F_{2,3}$ are anti-complete. By (3.17) and (3.18), $F_{6,1}$ and $T_{2} \cup T_{3}$ are complete. Since $G$ is $K_{4}$-free, $T_{2}$ and $T_{3}$ are anti-complete. Then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{3,4} \cup F_{6,1} \cup W, \\
& F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup\{5,6\} \cup T_{2} \cup T_{3}, \\
& D_{3,4} \cup\{1,2\} \cup T_{4}, \\
& D_{5,6} \cup D_{6,1} \cup\{3,4\} .
\end{aligned}
$$

- Suppose that $F_{3,4}=\emptyset$. If $F_{2,3}=\emptyset$, then $G$ has a 4 -coloring as above. So, we can assume that $F_{2,3} \neq \emptyset$. Let $f_{23} \in F_{2,3}$. If $d \in D_{6,1}$ and $f \in F_{6,1}$ are adjacent, then $\left\{1, f_{23}, d, f\right\}$ induces a $K_{4}$ by (3.12) and (3.19). So, $D_{6,1}$ and $F_{6,1}$ are anti-complete.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{5,6} \cup F_{2,3} \cup W, \\
& F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup\{3,4\}, \\
& D_{5,6} \cup\{1,2\}, \\
& D_{3,4} \cup D_{2,3} \cup\{5,6\} .
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{5,6} \cup F_{6,1} \cup W \cup\{3\}, \\
& F_{2,3} \cup D_{1,2} \cup\{6\} \cup T_{3}, \\
& D_{2,3} \cup\{4,5\} \cup T_{2}, \\
& D_{3,4} \cup\{1,2\} \cup T_{4} .
\end{aligned}
$$

Case 3. The set $F_{1,2}=\emptyset$ but the set $F_{4,5} \neq \emptyset$. By Claim 5, either $F_{3,4}=\emptyset$ or $F_{5,6}=\emptyset$. By (3.11), (3.19), (3.12) and $K_{4}$-freeness of $G$, either $D_{2,3}$ and $F_{2,3}$ are anti-complete or $D_{6,1}$ and $F_{6,1}$ are anti-complete.

- Suppose that $F_{5,6}=\emptyset$.

If $D_{6,1}$ and $F_{6,1}$ are anti-complete, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{2,3} \cup F_{3,4} \cup W \cup\{6\} \cup T_{3}, \\
& F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup\{3,4\}, \\
& F_{4,5} \cup D_{5,6} \cup\{1,2\} \cup T_{4}, \\
& D_{2,3} \cup D_{3,4} \cup\{5\} \cup T_{2} .
\end{aligned}
$$

Now assume that $D_{2,3}$ and $F_{2,3}$ are anti-complete.
If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{3,4} \cup F_{6,1} \cup W \\
& F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup\{5,6\}, \\
& F_{4,5} \cup D_{3,4} \cup\{1,2\} \\
& D_{5,6} \cup D_{6,1} \cup\{3,4\}
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{3,4} \cup W \cup\{6\} \cup T_{3} \\
& F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup\{5\} \cup T_{2}, \\
& F_{4,5} \cup D_{3,4} \cup\{1,2\} \cup T_{4}, \\
& F_{6,1} \cup\{3,4\}
\end{aligned}
$$

- Suppose that $F_{3,4}=\emptyset$. Suppose first that $D_{2,3}$ and $F_{2,3}$ are anti-complete.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{5,6} \cup F_{6,1} \cup W \cup\{3\} \\
& F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup\{5,6\} \\
& F_{4,5} \cup D_{3,4} \cup\{1,2\} \\
& D_{5,6} \cup D_{6,1} \cup\{4\}
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{5,6} \cup F_{6,1} \cup W \cup\{3\}, \\
& F_{2,3} \cup D_{1,2} \cup D_{2,3} \cup\{6\} \cup T_{3}, \\
& F_{4,5} \cup D_{3,4} \cup\{1,2\} \cup T_{4}, \\
& \{4,5\} \cup T_{2} .
\end{aligned}
$$

Now suppose that $D_{2,3}$ and $F_{2,3}$ are not anti-complete and that $D_{6,1}$ and $F_{6,1}$ are anticomplete. Then $T_{2}$ and $T_{3}$ are anti-complete for otherwise an edge between $T_{2}$ and $T_{3}$ and an edge between $D_{2,3}$ and $F_{2,3}$ induce a $2 P_{2}$ by (3.16) and (3.17). Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{5,6} \cup F_{2,3} \cup W, \\
& F_{6,1} \cup D_{1,2} \cup D_{6,1} \cup\{3,4\}, \\
& F_{4,5} \cup D_{5,6} \cup\{1,2\} \cup T_{4}, \\
& D_{2,3} \cup D_{3,4} \cup\{5,6\} \cup T_{2} \cup T_{3} .
\end{aligned}
$$

Case 4. The set $F_{4,5}=\emptyset$ but the set $F_{1,2} \neq \emptyset$. By Claim 5, either $F_{2,3}=\emptyset$ or $F_{6,1}=\emptyset$. By (3.19) and (3.12), $F_{3,4}$ is complete to $D_{5,6} \cup F_{5,6}$. So, if $F_{3,4} \neq \emptyset$, then $D_{5,6}$ and $F_{5,6}$ are anti-complete for otherwise $G$ would contain a $K_{4}$. By symmetry, if $F_{5,6} \neq \emptyset$, then $D_{3,4}$ and $F_{3,4}$ are anti-complete. Moreover, either $D_{3,4}$ and $F_{3,4}$ are anti-complete or $D_{5,6}$ and $F_{5,6}$ are anticomplete. Similarly, either $D_{2,3}$ and $F_{3,4}$ are anti-complete or $D_{6,1}$ and $F_{5,6}$ are anti-complete.

- Suppose that $F_{6,1}=\emptyset$. If both $F_{3,4}$ and $F_{5,6}$ are not empty, then consider the following 4-coloring of $G-\left(D_{2,3} \cup D_{6,1}\right)$ :

$$
\begin{aligned}
I_{1} & =F_{2,3} \cup D_{1,2} \cup W \cup\{6\} \cup T_{3} \\
I_{2} & =F_{3,4} \cup D_{3,4} \cup\{1\} \cup T_{4} \\
I_{3} & =F_{5,6} \cup D_{5,6} \cup\{2,3\} \\
I_{4} & =F_{1,2} \cup\{4,5\} \cup T_{2}
\end{aligned}
$$

If $D_{2,3}$ and $F_{3,4}$ are anti-complete, then $G$ has a 4-coloring: $I_{1}, I_{2} \cup D_{2,3}, I_{3}$ and $I_{4} \cup D_{6,1}$. If $D_{6,1}$ and $F_{5,6}$ are anti-complete, then $G$ has a 4-coloring: $I_{1}, I_{2}, I_{3} \cup D_{6,1}$ and $I_{4} \cup D_{2,3}$. It reamains to consider the case where at least one of $F_{3,4}$ and $F_{5,6}$ is empty.

Suppose that $F_{5,6}=\emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$. Let $D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$. Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{1,2} \cup D_{1,2}^{\prime} \cup\{4,5\} \cup T_{2} \\
& F_{2,3} \cup F_{3,4} \cup D_{1,2}^{\prime \prime} \cup W \cup\{6\} \cup T_{3}, \\
& D_{2,3} \cup D_{3,4} \cup\{1\} \cup T_{4}, \\
& D_{5,6} \cup D_{6,1} \cup\{2,3\}
\end{aligned}
$$

Suppose now that $F_{5,6} \neq \emptyset$ and $F_{3,4}=\emptyset$. Note that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$. Let $D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$. Moreover, recall that since $F_{5,6} \neq \emptyset, T_{3}$ and $T_{4}$ are anti-complete. Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{1,2} \cup D_{2,3} \cup D_{1,2}^{\prime} \cup\{4,5\} \cup T_{2} \\
& F_{2,3} \cup F_{5,6} \cup D_{1,2}^{\prime \prime} \cup W \\
& D_{3,4} \cup\{6,1\} \cup T_{3} \cup T_{4} \\
& D_{5,6} \cup D_{6,1} \cup\{2,3\}
\end{aligned}
$$

- Suppose that $F_{2,3}=\emptyset$. If both $F_{3,4}$ and $F_{5,6}$ are not empty, then consider the following 4-coloring of $G-\left(D_{2,3} \cup D_{6,1}\right)$ :

$$
\begin{aligned}
& I_{1}=F_{6,1} \cup D_{1,2} \cup W \cup\{3\}, \\
& I_{2}=F_{5,6} \cup D_{5,6} \cup\{2\}, \\
& I_{3}=F_{3,4} \cup D_{3,4} \cup\{6,1\} \cup T_{3} \cup T_{4}, \\
& I_{4}=F_{1,2} \cup\{4,5\} \cup T_{2} .
\end{aligned}
$$

If $D_{2,3}$ and $F_{3,4}$ are anti-complete, then $G$ has a 4 -coloring: $I_{1}, I_{2}, I_{3} \cup D_{2,3}$ and $I_{4} \cup D_{6,1}$. If $D_{6,1}$ and $F_{5,6}$ are anti-complete, then $G$ has a 4-coloring: $I_{1}, I_{2} \cup D_{6,1}, I_{3}$ and $I_{4} \cup D_{2,3}$. So, one of $F_{3,4}$ and $F_{5,6}$ is empty.

Suppose that $F_{5,6} \neq \emptyset$. So, $F_{3,4}=\emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{5,6}$. Let $D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$. Moreover, $T_{3}$ and $T_{4}$ are anti-complete. Then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{1,2} \cup D_{1,2}^{\prime} \cup\{4,5\} \cup T_{2}, \\
& F_{6,1} \cup F_{5,6} \cup D_{1,2}^{\prime \prime} \cup W \cup\{3\}, \\
& D_{6,1} \cup D_{5,6} \cup\{2\}, \\
& D_{2,3} \cup D_{3,4} \cup\{6,1\} \cup T_{3} \cup T_{4} .
\end{aligned}
$$

Suppose now that $F_{5,6}=\emptyset$. Recall that no vertex in $D_{1,2}$ can have a neighbor in both $F_{1,2}$ and $F_{3,4}$. Let $D_{1,2}^{\prime}$ be the set of vertices in $D_{1,2}$ that are anti-complete to $F_{1,2}$ and $D_{1,2}^{\prime \prime}=D_{1,2} \backslash D_{1,2}^{\prime}$.

If $D_{5,6} \cup D_{6,1} \neq \emptyset$, then $G$ has a 4 -coloring:

$$
\begin{aligned}
& F_{1,2} \cup D_{6,1} \cup D_{1,2}^{\prime} \cup\{4,5\}, \\
& F_{6,1} \cup F_{3,4} \cup D_{1,2}^{\prime \prime} \cup W, \\
& D_{5,6} \cup\{2,3\}, \\
& D_{2,3} \cup D_{3,4} \cup\{6,1\} .
\end{aligned}
$$

If $D_{5,6} \cup D_{6,1}=\emptyset$, then $G$ has a 4-coloring:

$$
\begin{aligned}
& F_{1,2} \cup D_{2,3} \cup D_{1,2}^{\prime} \cup\{4,5\} \cup T_{2}, \\
& F_{3,4} \cup D_{1,2}^{\prime \prime} \cup W \cup\{6\} \cup T_{3}, \\
& F_{5,6} \cup\{3\}, \\
& D_{3,4} \cup\{1,2\} \cup T_{4} .
\end{aligned}
$$

In each case we have found a 4 -coloring of $G$. This completes our proof.

## 4 Eliminate $\mathrm{H}_{2}$

In this section we show that our main theorem, Theorem 1, holds when $G$ is connected, has no pair of comparable vertices, does not contain $H_{1}$ as an induced subgraph, but contains $H_{2}$ as an induced subgraph.

Lemma 3. Let $G$ be a connected $\left(2 P_{2}, K_{4}, H_{1}\right)$-free graph with no pair of comparable vertices. If $G$ contains an induced $H_{2}$, then $\chi(G) \leq 4$.

Proof. Let $H=C \cup\{f\}$ be an induced $H_{2}$ where $C=12345$ induces a $C_{5}$ and $f$ is adjacent to $1,2,3$ and 4. We partition $V \backslash C$ into subsets of $Z, R_{i}, Y_{i}, F_{i}$ and $U$ as in section 2. By the fact that $G$ is $H_{1}$-free and (2.11), it follows that $F_{i}=\emptyset$ for $i \neq 5$. Note that $f \in F_{5}$. We choose $H$ such that

- $|U|$ is minimum, and
- $\left|F_{5}\right|$ is minimum subject to the previous condition.
(4.1) $U$ is complete to $R_{i}$ for $1 \leq i \leq 5$.

Suppose not. Let $u \in U$ be nonadjacent to $r_{i} \in R_{i}$ for some $i$. Suppose first that $1 \leq i \leq 4$. Note that $C^{\prime}=C \backslash\{i\} \cup\left\{r_{i}\right\}$ induces a $C_{5}$ and $H^{\prime}=C^{\prime} \cup\{u\}$ induces an $H_{2}$. Since $5 \in C^{\prime}$, it follows that $F_{5} \cap U^{\prime}=\emptyset$ and $U^{\prime} \subseteq U$. Moreover, $u \in U$ is not in $U^{\prime}$ since $u$ is not adjacent to $r_{i}$. This implies that $\left|U^{\prime}\right|<|U|$, contradicting the choice of $H$.

Now suppose that $i=5$. Note that $C^{\prime}=C \backslash\{5\} \cup\left\{r_{5}\right\}$ induces a $C_{5}$ and $H^{\prime}=C^{\prime} \cup\{u\}$ induces an $H_{2}$. Note that $U^{\prime} \subseteq F_{5} \cup U$ and $u \notin U^{\prime}$ since $u$ is not adjacent to $r_{i}$. By the chocie of $H$, there exists a vertex $f^{\prime} \in F_{5}$ such that $f^{\prime}$ is adjacent to $r_{5}$. By (2.2), $u$ and $f$ are not adjacent. But then $f r_{5}$ and $5 u$ indcue a $2 P_{2}$.
(4.2) If $U \neq \emptyset$, then $R_{i}$ and $R_{i+2}$ are anti-complete.

Let $u \in U$. If $r_{i} \in R_{i}$ and $r_{i+2} \in R_{i+2}$ are not adjacent, then $\left\{r_{i}, r_{i+2}, i+1, u\right\}$ induces a $K_{4}$, since $u$ is adjacent to $r_{i}$ and $r_{i+2}$ by (4.1).

Suppose first that $U \neq \emptyset$. By (4.2), $R_{i}$ and $R_{i+2}$ are anti-complete. Recall that $Y_{i}$ and $Y_{i+2}$ are anti-complete by (2.10). By (2.12), $R_{1}$ is anti-complete to $Y_{5} \cup Y_{2}$ and $R_{4}$ is anti-complete to $Y_{5} \cup Y_{3}$. By (2.8), $F_{5}$ is anti-complete to $Y_{1} \cup Y_{4}$. By (2.6), either $Y_{3}$ and $R_{2}$ are anti-complete or $Y_{2}$ and $R_{3}$ are anti-complete.

If $Y_{3}$ and $R_{2}$ are anti-complete, then $G$ admits the following 4-coloring:

$$
\begin{align*}
& Y_{1} \cup Y_{4} \cup U \cup F_{5}  \tag{2.10}\\
& Y_{2} \cup Y_{5} \cup R_{1} \cup\{1\}  \tag{2.10}\\
& Y_{3} \cup R_{2} \cup R_{4} \cup\{2,4\}  \tag{4.2}\\
& R_{3} \cup R_{5} \cup Z \cup\{3,5\} \tag{4.2}
\end{align*}
$$

If $Y_{2}$ and $R_{3}$ are anti-complete, then $G$ admits the following 4-coloring:

$$
\begin{align*}
& Y_{1} \cup Y_{4} \cup U \cup F_{5}  \tag{2.10}\\
& Y_{3} \cup Y_{5} \cup R_{4} \cup\{4\}  \tag{2.10}\\
& Y_{2} \cup R_{1} \cup R_{3} \cup\{1,3\}  \tag{4.2}\\
& R_{2} \cup R_{5} \cup Z \cup\{2,5\} \tag{4.2}
\end{align*}
$$

This shows that if $U \neq \emptyset$, then $G$ has a 4-coloring. Therefore, we can assume in the following that $U=\emptyset$.
(4.3) Each vertex in $R_{2} \cup R_{3}$ is either complete or anti-complete to $F_{5}$.

Suppose not. Let $r \in R_{2} \cup R_{3}$ be adjacent to $f \in F_{5}$ and not adjacent to $f^{\prime} \in F_{5}$. By symmetry, we may assume that $r \in R_{2}$. Note that $C^{\prime}=C \backslash\{2\} \cup\{r\}$ induces a $C_{5}$ and $H^{\prime}=C^{\prime} \cup\{f\}$ induces an $H_{2}$. Clearly, $f^{\prime} \notin F_{5}^{\prime}$. By the choice of $H$, there exists a vertex $y \in Y$ such that $y \in F_{5}^{\prime}$. This means that $y$ is not adjacent to 5 but adjacent to $1,3,4$ and $r_{2}$. This implies that $y \in Y_{1}$. By (2.8), $f^{\prime}$ and $y$ are not adjacent. But now $f^{\prime} 2$ and $y r_{2}$ induce a $2 P_{2}$.

By (2.8), (2.9) and (4.3), only vertices in $R_{5} \cup Z$ can distinguish two vertices in $F_{5}$. By (2.1), $R_{5} \cup Z$ is an independent set and so ( $\left.F_{5}, R_{5} \cup Z\right)$ is a $2 P_{2}$-free bipartite graph. This implies that $F_{5}=\{f\}$ since any two vertices in $F$ are comparable. Let $R_{i}^{\prime}=N(f) \cap R_{i}$ and $R_{i}^{\prime \prime}=R_{i} \backslash R_{i}^{\prime}$ for $i=2,3,5$. We now prove properties of $R_{i}^{\prime}$ and $R_{i}^{\prime \prime}$.
(4.4) $R_{5}^{\prime}$ is anti-complete to $R_{2}^{\prime} \cup R_{3}^{\prime}$.

Suppose that $r_{5}^{\prime} \in R_{5}^{\prime}$ and $r_{2}^{\prime} \in R_{2}^{\prime}$ are adjacent. Then $\left\{r_{5}^{\prime}, r_{2}^{\prime}, 1, f\right\}$ induces a $K_{4}$.
(4.5) $R_{5}^{\prime}$ is anti-complete to $Y_{2} \cup Y_{3}$.

Suppose that $r_{5}^{\prime} \in R_{5}^{\prime}$ and $y_{2} \in Y_{2}$ are adjacent. By (2.8), $f$ and $y_{2}$ are adjacent. Then $\left\{r_{5}^{\prime}, 4, y_{2}, f\right\}$ induces a $K_{4}$.
(4.6) $R_{2}^{\prime}$ is anti-complete to $R_{4}$. By symmetry, $R_{3}^{\prime}$ is anti-complete to $R_{1}$.

Suppose that $r_{2}^{\prime} \in R_{2}^{\prime}$ and $r_{4} \in R_{4}$ are adjacent. By (2.9), $f$ and $r_{4}$ are adjacent. Then $\left\{r_{2}^{\prime}, r_{4}, 3, f\right\}$ induces a $K_{4}$.
(4.7) $R_{5}^{\prime \prime}$ is anti-complete to $R_{2}^{\prime \prime} \cup R_{3}^{\prime \prime}$.

Suppose that $r_{5}^{\prime \prime} \in R_{5}^{\prime \prime}$ and $r_{2}^{\prime \prime} \in R_{2}^{\prime \prime}$ are adjacent. Then $r_{5}^{\prime \prime} r_{2}^{\prime \prime}$ and $f 2$ induce a $2 P_{2}$.
(4.8) $Y_{5}$ is anti-complete to $R_{2}^{\prime \prime} \cup R_{3}^{\prime \prime}$.

Suppose that $y_{5} \in Y_{5}$ and $r_{2}^{\prime \prime} \in R_{2}^{\prime \prime}$ are adjacent. By (2.8), $f$ and $y$ are not adjacent. Then $y_{5} r_{2}^{\prime \prime}$ and $f 4$ induce a $2 P_{2}$.
(4.9) $R_{5}^{\prime \prime}$ is anti-complete to $Y_{1} \cup Y_{4}$.

Suppose that $r_{5}^{\prime \prime} \in R_{5}^{\prime \prime}$ and $y_{4} \in Y_{4}$ are adjacent. By (2.8), $f$ and $y_{4}$ are not adjacent. Then $r_{5}^{\prime \prime} y_{4}$ and $f 2$ induce a $2 P_{2}$.
(4.10) $R_{2}^{\prime \prime}$ is anti-complete to $Y_{1}$. By symmetry, $R_{3}^{\prime \prime}$ is anti-complete to $Y_{4}$.

Suppose that $r_{2}^{\prime \prime} \in R_{2}^{\prime \prime}$ and $y_{1} \in Y_{1}$ are adjacent. By (2.8), $f$ and $y_{1}$ are not adjacent. Then $r_{2}^{\prime \prime} y_{1}$ and $f 2$ induce a $2 P_{2}$.
(4.11) $R_{2}^{\prime}$ is anti-complete to $Y_{3}$. By symmetry, $R_{3}^{\prime}$ is anti-complete to $Y_{2}$.

Suppose that $r_{2}^{\prime} \in R_{2}^{\prime}$ and $y_{3} \in Y_{3}$ are adjacent. By (2.9), $f$ and $y_{3}$ are adjacent. Then $\left\{r_{2}^{\prime}, y_{3}, 3, f\right\}$ induces a $K_{4}$.
(4.12) $Y_{5}$ is complete to $R_{2}^{\prime} \cup R_{3}^{\prime}$.

Suppose that $y_{5} \in Y_{5}$ and $r_{2}^{\prime} \in R_{2}^{\prime}$ are not adjacent. By (2.8), $f$ and $y_{5}$ are not adjacent. Then $f r_{2}^{\prime}$ and $5 y_{5}$ induce a $2 P_{2}$.

We now prove properties of $Z$.
(4.13) Any vertex in $Z$ is anti-complete to either $Y_{2}$ or $Y_{3}$.

Suppose not. Then there exists a vertex $z \in Z$ that is adjacent to a vertex $y_{i} \in Y_{i}$ for $i=2,3$. By (2.8), $f$ is adjacent to $y_{2}$ and $y_{3}$. Moreover, $y_{2}$ and $y_{3}$ are adjacent by (2.4). This implies that $f$ and $z$ are not adjacent for otherwise $\left\{f, z, y_{i}, y_{i+1}\right\}$ would induce a $K_{4}$.
We now show that $z$ is anti-complete to $Y_{1} \cup Y_{4} \cup Y_{5}$. Suppose not. Let $z$ be adjacent to a vertex $y \in Y_{1} \cup Y_{4} \cup Y_{5}$. Note that there exists a vertex $i \in N_{C}(f)$ such that $i$ is not adjacent to $y$. Moreover, $f$ and $y$ are not adjacent by (2.8). Then $z y$ and if induce a $2 P_{2}$. This shows that $z$ is anti-complete to $Y_{1} \cup Y_{4} \cup Y_{5}$. Recall that $Z$ is anti-complete to $R_{i}$ for each $i$ by (2.1). Therefore, $N(z) \subseteq Y_{2} \cup Y_{3} \subseteq N(f)$, contradicting the assumption that $G$ has no pair of comparable vertices.
(4.14) If $z \in Z$ is not adjacent to $y_{i} \in Y_{i}$, then $y_{i}$ is complete to $N(z) \backslash Y_{i}$.

It suffices to prove for $i=1$ by symmetry. Note that $N(z) \backslash Y_{1}=\left(N(z) \cap\left(Y_{2} \cup Y_{5}\right)\right) \cup$ $\left(N(z) \cap\left(Y_{3} \cup Y_{4}\right)\right)$. By (2.4), $y_{1}$ is complete to $N(z) \cap\left(Y_{2} \cup Y_{5}\right)$. It remains to show that $y_{1}$ is complete to $N(z) \cap\left(Y_{3} \cup Y_{4}\right)$. Suppose not. Let $y \in N(z) \cap\left(Y_{3} \cup Y_{4}\right)$ be nonadjacent to $y_{1}$. By symmetry, we may assume that $y \in Y_{3}$. Then $z y$ and $y_{1} 4$ induce a $2 P_{2}$.
(4.15) If $z$ is anti-complete to $Y_{i}$ for some $i \in\{2,3\}$, then $Y_{i}=\emptyset$.

Suppose that $z$ is anti-complete to $Y_{2}$ and $Y_{2}$ contains a vertex $y_{2}$. It follows from (4.14) that $N(z) \subseteq N\left(y_{2}\right)$, contradicting the assumption that $G$ contains no pair of comparable vertices.

If $Y_{5}=\emptyset$, then $N(5)=\{1,4\} \cup R_{1} \cup R_{4} \cup Y_{2} \cup Y_{3} \subseteq N(f)$ by (2.8) and (2.9). This contradicts the assumption that $G$ contains no pair of comparable vertices. So, we assume in the following that $Y_{5}$ contains a vertex $y_{5}$. We claim now that either $R_{2}^{\prime \prime}$ or $R_{3}^{\prime \prime}$ is empty. Suppose not. Let $r_{i}^{\prime \prime} \in R_{i}^{\prime \prime}$ for $i=2,3$. By (2.3), $r_{2}^{\prime \prime}$ and $r_{3}^{\prime \prime}$ are adjacent. Moreover, $y_{5}$ is not adjacent to $r_{2}^{\prime \prime}$ and $r_{3}^{\prime \prime}$ by (4.8). Then $r_{2}^{\prime \prime} r_{3}^{\prime \prime}$ and $5 y_{5}$ induce a $2 P_{2}$. This proves that either $R_{2}^{\prime \prime}$ or $R_{3}^{\prime \prime}$ is empty. We consider two cases depending on whether $f$ has a neighbor in $R_{5}$.
Case 1. $R_{5}^{\prime}=\emptyset$, i.e., $f$ has no neighbor in $R_{5}$. Therefore, $R_{5}=R_{5}^{\prime \prime}$. Recall that either $R_{2}^{\prime \prime}$ or $R_{3}^{\prime \prime}$ is empty. By symmetry, we may assyme that $R_{2}^{\prime \prime}=\emptyset$. Then $R_{2}=R_{2}^{\prime}$ and so $R_{2}$ and $R_{4}$ are anti-complete by (4.6). Let $Y_{2}^{\prime}=\left\{y \in Y_{2}: y\right.$ is anti-complete to $\left.Y_{5}\right\}$ and $Y_{2}^{\prime \prime}=Y_{2} \backslash Y_{2}^{\prime}$. Note that each vertex in $Y_{2}^{\prime \prime}$ has a neighbor in $Y_{5}$ by the definition and so is anti-complete to $Y_{4}$ by (2.7). Then the following is a 4 -coloring $\phi$ of $G-\left(R_{3} \cup Z\right)$ :

$$
\begin{align*}
& I_{1}=Y_{2}^{\prime} \cup Y_{5} \cup R_{1} \cup\{1\}  \tag{2.12}\\
& I_{2}=Y_{2}^{\prime \prime} \cup Y_{4} \cup R_{3} \cup\{3\} \\
& I_{3}=R_{2}\left(=R_{2}^{\prime}\right) \cup R_{4} \cup Y_{3} \cup\{2,4\}  \tag{2.12}\\
& I_{4}=Y_{1} \cup R_{5}\left(=R_{5}^{\prime \prime}\right) \cup\{f, 5\} \tag{2.8}
\end{align*}
$$

Definition of $Y_{2}^{\prime \prime}$

We now extend $\phi$ to $R_{3}$ as follows. Since $R_{3}$ is an independent set by (2.1), it suffices to explain how to extend $\phi$ to each vertex in $R_{3}$ independently. Let $r_{3} \in R_{3}$ be an arbitrary vertex. Suppose first that $r_{3} \in R_{3}^{\prime}$. By (4.6) and (4.11), $r_{3}$ is anti-complete to $R_{1} \cup Y_{2}$. By (2.13), $r_{3}$ is anti-complete to either $Y_{4}$ or $Y_{5}$. Therefore, we can add $r_{3}$ to either $I_{1}$ or $I_{2}$. Now suppose that $r_{3} \in R_{3}^{\prime \prime}$. By (4.7) and (4.10), $r_{3}$ is anti-complete to $Y_{4} \cup R_{5}$. By (2.13), $r_{3}$ is anti-complete to either $Y_{1}$ or $Y_{2}$. Therefore, we can add $r_{3}$ to either $I_{2}$ or $I_{4}$. This shows that $G-Z$ admits a 4-coloring $\phi^{\prime}=\left(I_{1}^{\prime}, I_{2}^{\prime}, I_{3}^{\prime}, I_{4}^{\prime}\right)$ with $I_{i} \subseteq I_{i}^{\prime}$ for each $1 \leq i \leq 4$.

We now obtain a 4 -coloring of $G$ by either extending $\phi^{\prime}$ to $Z$ or by finding another 4-coloring of $G$. If $Z$ is anti-complete to $Y_{3}$, then we can extend $\phi^{\prime}$ by adding $Z$ to $I_{3}^{\prime}$. So, we assume that there is a vertex $z \in Z$ that is adjacent to a vertex in $Y_{3}$. It then follows from (4.13) and (4.15) that $Y_{2}=\emptyset$. If each vertex in $Z$ is anti-complete to one of $Y_{3}, Y_{4}$ and $Y_{5}$, then we can extend $\phi^{\prime}$ to $Z$ by adding each vertex in $Z$ to $I_{1}^{\prime}, I_{2}^{\prime}$ or $I_{3}^{\prime}\left(\right.$ since $\left.Y_{2}=\emptyset\right)$. Therefore, let $z \in Z$ be adjacent to $y_{i} \in Y_{i}$ for $i \in\{3,4,5\}$. We prove some additional properties using the existence of $y_{3}, y_{4}$ and $y_{5}$. First of all, $R_{1}$ and $R_{4}$ are anti-complete. Suppose not. Let $r_{1} \in R_{1}$ and $r_{4} \in R_{4}$ be adjacent. By (2.12), $y_{5}$ is not adjacent to $r_{1}$ and $r_{4}$. Then $r_{1} r_{4}$ and $z y_{5}$ induce a $2 P_{2}$. Secondly, $y_{3}$ and $y_{5}$ are not adjacent for otherwise $\left\{y_{3}, y_{4}, y_{5}, z\right\}$ induces a $K_{4}$. Thirdly, $Y_{1}$ and $Y_{4}$ are anti-complete to each other. Suppose not. Then $Y_{1}$ contains a vertex $y_{1}$ that is not anit-complete to $Y_{4}$. By (2.7), $y_{1}$ is anti-complete to $Y_{3}$. Then $f y_{3}$ and $y_{1} y_{5}$ induce a $2 P_{2}$. Now $G$ admits the following 4-coloring:

$$
\begin{align*}
& Y_{1} \cup R_{5}^{\prime \prime}\left(=R_{5}\right) \cup Y_{4} \cup\{f, 5\}  \tag{4.9}\\
& Y_{3} \cup R_{2}^{\prime}\left(=R_{2}\right) \cup\{2\}  \tag{4.11}\\
& R_{1} \cup R_{4} \cup Y_{5} \cup\{1,4\}  \tag{2.12}\\
& R_{3} \cup Z \cup\{3\} \tag{2.1}
\end{align*}
$$

Case 2. $R_{5}^{\prime} \neq \emptyset$. Let $r_{5}^{\prime} \in R_{5}^{\prime}$. If $r_{1} \in R_{1}$ and $r_{4} \in R_{4}$ are adjacent, then $\left\{r_{1}, r_{4}, r_{5}^{\prime}, f\right\}$ induces a $K_{4}$ by (2.3) and (2.9). So, $R_{1}$ and $R_{4}$ are anti-complete. We now consider two subcases.

Case 2.1. $R_{2}^{\prime \prime}$ and $Y_{3}$ are not anti-complete. Let $r_{2}^{\prime \prime} \in R_{2}^{\prime \prime}$ and $y_{3} \in Y_{3}$ be adjacent. We claim first that $Y_{1}$ and $Y_{4}$ are anti-complete. Suppose not. Let $y_{1} \in Y_{1}$ and $y_{4} \in Y_{4}$ be adjacent. Then $y_{3}$ and $y_{4}$ are adjacent by (2.4). By (2.7), $y_{1}$ is not adjacent to $y_{3}$. Moreover, $y_{1}$ is not adjacent to $r_{2}^{\prime \prime}$ by (4.10). But now $4 y_{1}$ and $y_{3} r_{2}^{\prime \prime}$ induce a $2 P_{2}$. This shows that $Y_{1}$ and $Y_{4}$ are anti-complete. Moreover, $Y_{2}$ and $R_{3}$ are anti-complete by (2.6). Therefore, the following is a 4-coloring $\phi$ of $G-\left(R_{2}^{\prime \prime} \cup Z\right)$.

$$
\begin{align*}
& I_{1}=R_{4} \cup Y_{5} \cup R_{1} \cup\{1,4\}  \tag{2.12}\\
& I_{2}=Y_{1} \cup R_{5}^{\prime \prime} \cup Y_{4} \cup\{f, 5\}  \tag{4.9}\\
& I_{3}=R_{3} \cup Y_{2} \cup\{3\}  \tag{2.6}\\
& I_{4}=Y_{3} \cup R_{2}^{\prime} \cup R_{5}^{\prime} \cup\{2\} \tag{4.4}
\end{align*}
$$

We now explain how to extend $\phi$ to each vertex in $R_{2}^{\prime \prime} \cup Z$. Since $R_{2}^{\prime \prime} \cup Z$ is an independent set by (2.1), this will give a 4 -coloring of $G$. By (4.13), we can add each vertex in $Z$ to either $I_{3}$ or $I_{4}$. Let $s^{\prime \prime} \in R_{2}^{\prime \prime}$ be an arbitrary vertex. Then $s^{\prime \prime}$ is anti-complete to $R_{5}^{\prime \prime} \cup Y_{1}$ by (4.7) and (4.10). If $s^{\prime \prime}$ is not anti-complete to $Y_{3}$, then $s^{\prime \prime}$ is anti-complete to $Y_{4}$ by (2.13) and thus we can add $s^{\prime \prime}$ to $I_{2}$. Now $s^{\prime \prime}$ is anti-complete to $Y_{3}$. We claim that $s^{\prime \prime}$ is anti-complete to $R_{5}^{\prime}$. Suppose not. Then $s^{\prime \prime}$ is adjacent to some vertex $r^{\prime} \in R_{5}^{\prime}$. Note that $y_{3}$ is not adjacent to $s^{\prime \prime}$ by our assumption. Moreover, $y_{3}$ is not adjacent to $r^{\prime}$ by (4.5). Then $s^{\prime \prime} r^{\prime}$ and $5 y_{3}$ induce a $2 P_{2}$. This shows that $s^{\prime \prime}$ is anti-complete to $R_{5}^{\prime}$ and thus we can add $s^{\prime \prime}$ to $I_{4}$.
Case 2.2. $R_{2}^{\prime \prime}$ and $Y_{3}$ are anti-complete. By symmetry, $R_{3}^{\prime \prime}$ and $Y_{2}$ are anti-complete. It follows from (4.11) that $R_{2}$ and $Y_{3}$ are anti-complete and $R_{3}$ and $Y_{2}$ are anti-complete. Recall that either $R_{2}^{\prime \prime}$ or $R_{3}^{\prime \prime}$ is empty. By symmetry, we may assume that $R_{2}^{\prime \prime}=\emptyset$. Then $R_{2}=R_{2}^{\prime}$. We now claim that $R_{3}^{\prime}$ is anti-complete to $Y_{4}$. Suppose not. Let $r_{3}^{\prime} \in R_{3}^{\prime}$ be adjacent to $y_{4} \in Y_{4}$. By (4.12), $r_{3}^{\prime}$ is adjacent to $y_{5}$. But this contradicts (2.13). So, $R_{3}^{\prime}$ is anti-complete to $Y_{4}$. By symmetry, $R_{2}^{\prime}$ is anti-complete to $Y_{1}$. This together with (4.10) implies that $R_{3}$ and $R_{2}$ are anti-complete to $Y_{4}$ and $Y_{1}$, respectively. Let $Y_{4}^{\prime}=\left\{y \in Y_{4}: y\right.$ is anti-complete to $\left.Y_{1}\right\}$ and $Y_{4}^{\prime \prime}=Y_{4} \backslash Y_{4}^{\prime}$. Note that each vertex in $Y_{4}^{\prime \prime}$ has a neighbor in $Y_{1}$ and so is anti-complete to $Y_{2}$ by (2.7). Now $G-Z$ admits a 4 -coloring $\phi$ :

$$
\begin{align*}
& I_{1}=R_{4} \cup Y_{5} \cup R_{1} \cup\{1,4\}  \tag{2.12}\\
& I_{2}=Y_{1} \cup R_{5}^{\prime \prime} \cup Y_{4}^{\prime} \cup\{f, 5\}  \tag{4.9}\\
& I_{3}=R_{3} \cup Y_{2} \cup Y_{4}^{\prime \prime} \cup\{3\}  \tag{2.6}\\
& I_{4}=Y_{3} \cup R_{2}^{\prime} \cup R_{5}^{\prime} \cup\{2\} \tag{4.4}
\end{align*}
$$

We now explain how to obtain a 4-coloring of $G$ based on $\phi$. If $Z$ is anti-complete to $Y_{3}$, then we can add $Z$ to $I_{4}$. So, assume that there exists a vertex in $Z$ that is adjacent to some vertex in $Y_{3}$. It then follows from (4.13) and (4.15) that $Y_{2}=\emptyset$. If each vertex in $Z$ is anti-complete to one of $Y_{3}, Y_{4}^{\prime \prime}$ and $Y_{5}$, then we can extend $\phi^{\prime}$ to $Z$ by adding each vertex in $Z$ to $I_{1}, I_{3}$ or $I_{4}$ (since $Y_{2}=\emptyset$ ). Therefore, let $z \in Z$ be adjacent to $y_{i} \in Y_{i}$ for $i \in\{3,5\}$ and be adjacent to $y_{4} \in Y_{4}^{\prime \prime}$. Note that $y_{3}$ and $y_{5}$ are not adjacent for otherwise $\left\{y_{3}, y_{4}, y_{5}, z\right\}$ induces a $K_{4}$. We claim that $Y_{1}$ and $Y_{4}$ are anti-complete to each other. Suppose not. Then $Y_{1}$ contains a vertex $y_{1}$ that is not anit-complete to $Y_{4}$. By (2.7), $y_{1}$ is anti-complete to $Y_{3}$. Then $f y_{3}$ and $y_{1} y_{5}$ induce a $2 P_{2}$. Now $G$ admits the following 4-coloring:

$$
\begin{align*}
& R_{4} \cup Y_{5} \cup R_{1} \cup\{1,4\}  \tag{2.12}\\
& Y_{1} \cup R_{5}^{\prime \prime} \cup Y_{4} \cup\{f, 5\}  \tag{4.9}\\
& R_{3} \cup Z \cup\{3\}  \tag{2.1}\\
& Y_{3} \cup R_{2}^{\prime} \cup R_{5}^{\prime} \cup\{2\} \tag{4.4}
\end{align*}
$$

This completes the proof.

## 5 Eliminate $W_{5}$ and $C_{5}$

In this section we prove two lemmas. The fist one states that our main theorem, Theorem 1, holds when $G$ is connected, has no pair of comparable vertices, does not contain $H_{1}$ or $H_{2}$ as an induced subgraph, but contains the 5 -wheel as an induced subgraph. The second lemma then assumes that $G$ is $W_{5}$-free as well, but contains an induced $C_{5}$.

Lemma 4. Let $G$ be a $\left(2 P_{2}, K_{4}, H_{1}, H_{2}\right)$-free graph with no pair of comparable vertices. If $G$ contains an induced $W_{5}$, then $\chi(G) \leq 4$.

Proof. Let $W=C \cup\{u\}$ be an induced $W_{5}$ such that $C=12345$ induces a $C_{5}$ in this order and $u$ is complete to $C$. We partition $V \backslash C$ into subsets of $Z, R_{i}, Y_{i}, F_{i}$ and $U$ as in section 2. Note that $u \in U$. Since $G$ is $H_{2}$-free, it follows that $F_{i}=\emptyset$ for each $i$. We prove the following properties.
(5.1) $U$ is complete to $R$.

If $u^{\prime} \in U$ is not adjacent to $r_{i} \in R_{i}$, then $C \backslash\{i\} \cup\left\{r_{i}, u\right\}$ induces an $H_{2}$. This contradicts our assumption that $G$ is $H_{2}$-free.
(5.2) $R_{i}$ and $R_{i+2}$ are anti-complete.

Suppose that $r_{i} \in R_{i}$ and $r_{i+2} \in R_{i+2}$ are not adjacent. By (5.1), $u$ is adjacent to both $r_{i}$ and $r_{i+2}$. This implies that $\left\{r_{i}, r_{i+2}, i+1, u\right\}$ induces a $K_{4}$.
(5.3) $R_{i}$ and $Y_{i+1}$ are anti-complere.

It suffices to prove for $i=1$. If $r_{1} \in R_{1}$ and $y_{2} \in Y_{2}$ are adjacent, then $C \backslash\{1\} \cup\left\{r_{1}, y_{2}\right\}$ induces an $H_{2}$, a contradiction.
(5.4) $Y_{i}$ and $Y_{i+2}$ are anti-complete.

Since $U \neq \emptyset$, (5.4) follows directly from (2.10).
It follows from (5.2)-(5.4) and (2.1)-(2.2) that $G$ admits the following 4-coloring:

$$
\begin{align*}
& R_{1} \cup R_{3} \cup Z \cup\{1,3\}  \tag{5.2}\\
& R_{2} \cup Y_{3} \cup R_{4} \cup\{2,4\}  \tag{5.2}\\
& Y_{1} \cup R_{5} \cup Y_{4} \cup\{5\}  \tag{5.3}\\
& Y_{2} \cup Y_{5} \cup U \tag{5.4}
\end{align*}
$$

This completes our proof.
Lemma 5. Let $G$ be a connected $\left(2 P_{2}, K_{4}, H_{1}, H_{2}, W_{5}\right)$-free graph with no pair of comparable vertices. If $G$ contains an induced $C_{5}$, then $\chi(G) \leq 4$.

Proof. Let $C=12345$ be an induced $C_{5}$ in this order. We partition $V \backslash C$ into subsets of $Z$, $R_{i}, Y_{i}, F_{i}$ and $U$ as in section 2. Since $G$ is $\left(H_{2}, W_{5}\right)$-free, both $U$ and $F_{i}$ are empty. It then follows from Lemma 1 that $V(G)=C \cup Z \cup\left(\bigcup_{i=1}^{5} R_{i}\right) \cup\left(\bigcup_{i=1}^{5} Y_{i}\right)$. We first prove the following properties of $R_{i}$ and $Z$.
(5.1) Each vertex in $R_{i}$ is anti-complete to either $R_{i-2}$ or $R_{i+2}$.

It suffices to prove for $i=4$. Suppose that $r_{4} \in R_{4}$ is adjacent to a vertex $r_{i} \in R_{i}$ for $i=1,2$. By (2.3), $r_{1}$ and $r_{2}$ are adjacent. This implies that $\left\{r_{1}, r_{2}, 3,4,5, r_{4}\right\}$ induces a subgraph isomorphic to $H_{2}$. This contradicts the assumption that $G$ is $H_{2}$-free.
(5.2) $R_{i}$ and $Y_{i+1}$ are anti-complete.

It suffices to prove for $i=1$. If $r_{1} \in R_{1}$ and $y_{2} \in Y_{2}$ are adjacent, then $C \backslash\{1\} \cup\left\{r_{1}, y_{2}\right\}$ induces an $\mathrm{H}_{2}$.
(5.3) Each vertex in $Z$ cannot have a neighbor in each of $Y_{i}$ for $1 \leq i \leq 5$.

Suppose that $z \in Z$ has a neighbor $y_{i} \in Y_{i}$ for each $1 \leq i \leq 5$. By (2.4), $y_{i}$ and $y_{i+1}$ are adjacent. This implies that $y_{i}$ and $y_{i+2}$ are not adjacent, for otherwise $\left\{y_{i}, y_{i+1}, y_{i+2}, z\right\}$ induces a $K_{4}$. But now $\left\{y_{1}, y_{2}, y_{3}, y_{4}, y_{5}, z\right\}$ induces a $W_{5}$.
(5.4) If $z \in Z$ has a neighbor in each of $Y_{i}, Y_{i+1}, Y_{i+2}$ and $Y_{i+3}$, then $Y_{i+4}$ is anti-complete to $N(z)$.
It suffices to prove for $i=1$. Let $y_{i} \in Y_{i}$ be a neighbor of $z$ for $1 \leq i \leq 4$. By (5.3), $z$ is anti-complete to $Y_{5}$ and so $N(z) \subseteq Y_{1} \cup Y_{2} \cup Y_{3} \cup Y_{4}$ by (2.1). Let $y_{5}$ be an arbitrary vertex in $Y_{5}$. By (2.4), $y_{5}$ is complete to $Y_{1} \cup Y_{4}$. Therefore it remains to show that $y_{5}$ is complete to $N(z) \cap\left(Y_{2} \cup Y_{3}\right)$. If $y_{5}$ is not adjacent to a vertex $y \in N(z) \cap\left(Y_{2} \cup Y_{3}\right)$, then either $3 y_{5}$ or $2 y_{5}$ forms a $2 P_{2}$ with $z y$ depending on whether $y \in Y_{2}$ or $y \in Y_{3}$.
(5.5) If $Z$ contains a vertex that has a neighbor in $Y_{i}, Y_{i+1}, Y_{i+2}$ and $Y_{i+3}$, then $Y_{i+4}=\emptyset$.

Let $z \in Z$ have neighbor in $Y_{i}$ for $1 \leq i \leq 4$. By (5.3), $z$ is anti-complete to $Y_{5}$. If $Y_{5}$ contains a vertex $y$, then $N(z) \subseteq N(y)$ by (5.4). This contradicts the assumption that $G$ contains no pair of comparable vertices.

Let $Y_{4}^{\prime}=\left\{y \in Y_{4}: y\right.$ is anti-complete to $\left.Y_{1}\right\}$ and $Y_{4}^{\prime \prime}=Y_{4} \backslash Y_{4}^{\prime}$. Note that each vertex in $Y_{4}^{\prime \prime}$ has a neighbor in $Y_{1}$ by the definition and so is anti-complete to $Y_{2}$ by (2.7). Similarly, let $R_{4}^{\prime}=\left\{r \in R_{4}: r\right.$ is anti-complete to $\left.R_{1}\right\}$ and $R_{4}^{\prime \prime}=R_{4} \backslash R_{4}^{\prime}$. By (5.1), $R_{4}^{\prime \prime}$ is anti-complete to $R_{2}$. We now consider the following two cases.
Case 1. $Z$ contains a vertex that has a neighbor in four $Y_{i}$. It then follows from (5.5) that $Y_{j}=\emptyset$ for some $j$. We may assume by symmetry that $j=5$. These facts and (5.2) imply that $G$ admits the following 4-coloring:

$$
\begin{aligned}
& Y_{1} \cup R_{5} \cup Y_{4}^{\prime} \cup\{5\} \\
& Y_{2} \cup R_{3} \cup Y_{4}^{\prime \prime} \cup\{3\} \\
& R_{1} \cup Z \cup R_{4}^{\prime} \cup\{1\} \\
& R_{2} \cup Y_{3} \cup R_{4}^{\prime \prime} \cup\{2,4\}
\end{aligned}
$$

Case 2. Each vertex in $Z$ has a neighbor in at most three $Y_{i}$. Note that $G-Z$ admits the following 4-coloring $\phi$ by (5.2):

$$
\begin{aligned}
& I_{1}=Y_{1} \cup R_{5} \cup Y_{4}^{\prime} \cup\{5\} \\
& I_{2}=Y_{2} \cup R_{3} \cup Y_{4}^{\prime \prime} \cup\{3\} \\
& I_{3}=R_{1} \cup Y_{5} \cup R_{4}^{\prime} \cup\{1\} \\
& I_{4}=R_{2} \cup Y_{3} \cup R_{4}^{\prime \prime} \cup\{2,4\} .
\end{aligned}
$$

We now explain how to extend $\phi$ to $Z$. For this purpose we partition $Z$ into the following two subsets:

$$
\begin{aligned}
& Z_{1}=\left\{z \in Z: z \text { is anti-complete to either } Y_{3} \text { or } Y_{5}\right\}, \\
& Z_{2}=Z \backslash Z_{1}
\end{aligned}
$$

We first claim that each vertex in $Z_{2}$ has a neighbor in $Y_{4}$. Suppose not. Let $z \in Z_{2}$ be a vertex such that $z$ is anti-complete to $Y_{4}$. Since $z$ has a neighbor in both $Y_{3}$ and $Y_{5}, z$ is anti-complete to either $Y_{1}$ or $Y_{2}$ by the assumption that each vertex in $z$ has a neighbor in at most three $Y_{i}$. If $z$ is anti-complete to $Y_{1}$, then $N(z) \subseteq Y_{2} \cup Y_{3} \cup Y_{5} \subseteq N(5)$. If $z$ is anti-complete to $Y_{2}$, then $N(z) \subseteq Y_{1} \cup Y_{3} \cup Y_{5} \subseteq N(3)$. In either case, it contradicts the assumption that $G$ contains no pair of comparable veritces. This proves the claim. Consequently, $Z_{2}$ is anticomplete to $Y_{1} \cup Y_{2}$. We now claim that each vertex in $Z_{2}$ is anti-complete to either $Y_{4}^{\prime}$ or $Y_{4}^{\prime \prime}$. Suppose not. Let $z \in Z_{2}$ have a neighbor $y_{4}^{\prime} \in Y_{4}$ and a neighbor $y_{4}^{\prime \prime} \in Y_{4}^{\prime \prime}$. By the definition of $Y_{4}^{\prime \prime}$, it follows that $y_{4}^{\prime \prime}$ has a neighbor $y_{1} \in Y_{1}$. Then $3 y_{1}$ and $y_{4}^{\prime} z$ induce a $2 P_{2}$ since $y_{4}^{\prime}$ is not adjacent to $y_{1}$. Now we can extend $\phi$ to $Z$ by adding each vertex in $Z_{1}$ to $I_{3}$ or $I_{4}$ and by adding each vertex in $Z_{2}$ to $I_{1}$ or $I_{2}$.

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