RANDOM SAMPLING AND EFFICIENT ALGORITHMS FOR MULTISCALE PDES*

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Abstract. We describe a numerical framework that uses random sampling to efficiently capture low-rank local solution spaces of multiscale PDE problems arising in domain decomposition. In contrast to existing techniques, our method does not rely on detailed analytical understanding of specific multiscale PDEs, in particular, their asymptotic limits. We present the application of the framework on two examples — a linear kinetic equation and an elliptic equation with rough media. On these two examples, this framework achieves the asymptotic preserving property for the kinetic equations and numerical homogenization for the elliptic equations.

Key word. Randomized sampling, multiscale PDE, finite element method, domain decomposition

AMS subject classifications. 65N30, 65N55

1. Introduction. Partial differential equations (PDEs) that involve multiple temporal and spatial scales are numerically challenging to solve. The current generation of efficient solvers exploits the analytic solution structures that are intrinsic to each specific multiscale problem. In this work, we exploit instead the "low-rank" property of the solution spaces that is common to many multiscale problems that are "homogenizable", and design a general framework in which analytic structures of solutions are discovered automatically by the algorithms without the need for any problem-specific analysis.

We consider the following boundary value problem:

(1.1)
$$\mathcal{L}^{\varepsilon} u^{\varepsilon} = 0$$
, with $\mathcal{B} u^{\varepsilon} = f$,

where $\mathcal{L}^{\varepsilon}$ is a linear PDE operator with multiscale structure, with ε representing the small scale. \mathcal{B} is the boundary operator and f is the boundary condition. The solution u^{ε} contains information at both coarse scale x and fine scale x/ε . A naive numerical scheme for (1.1) would require a fine discretization: The mesh size h must resolve ε (that is, $h \ll \varepsilon$), and thus the number of grid points (the degrees of freedom) N_{ε} is of the order of ε^{-d} , with d being the dimension of the problem. For small ε , the computational cost is prohibitive. These observations have motivated research into algorithms for multiscale PDE problems that are much more efficient than such naive schemes.

One strategy commonly used by efficient algorithms is to exploit the asymptotic behavior of the multiscale problems as $\varepsilon \to 0$. In particular, the "effective equations" that capture the behavior of the solution as ε approaches zero have been derived for several specific multiscale problems. More specifically, we seek a homogenized operator \mathcal{L}^* , with no dependence on ε , such that the solution u^* of the "effective equation"

(1.2)
$$\mathcal{L}^* u^* = 0, \quad \text{with} \quad \mathcal{B} u^* = f$$

satisfies

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(1.3)
$$||u^{\varepsilon} - u^*|| \to 0 \quad \text{as } \varepsilon \to 0$$

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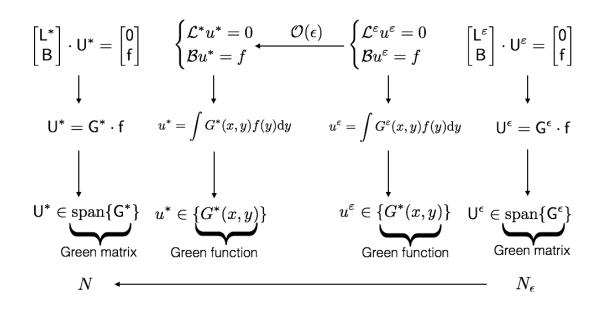


FIG. 1. Green's function superposition in discrete and continuous setting, for both $\mathcal{L}^{\varepsilon}$ and the asymptotic limit \mathcal{L}^* . The column space of G^{ε} , which typically has high dimension, can be well approximated well by the (lower-dimensional) column space of G^* .

in a proper norm. Since u^* is asymptotically equivalent to u^{ε} (1.3) and no small-scale oscillation is present, solving (1.2) can typically be done in a much more efficient manner than directly solving (1.1) with small ε .

Identifying the effective operator \mathcal{L}^* , however, is mostly nontrivial. Different techniques are needed for different equations. The hydrodynamic limit of kinetic equations is based on moment expansions and entropic closures; the homogenization of elliptic equations with oscillatory media is based on corrector equations and two-scale convergence analysis; and the semiclassical limit of Schrödinger equations is based on WKB expansion and Wigner transformations. Each of these analytical tools leads to a different algorithmic approach, so there is a wide variation in algorithms for different multiscale problems.

We describe in this paper a general approach to designing efficient algorithms for multiscale PDE problems that does not rely on detailed analytical knowledge of the PDE and applies to a wide variety of problems. Our approach not only has the advantage of a unified treatment, but also applies to cases in which the asymptotic limit is not known, or is too complicated to derive. (See, for example, an application in [18].) While the proposed approach might not be the most effective approach for every multiscale problem (for example, many numerical approaches have been developed over the years for elliptic PDEs with rough coefficients), its numerical performance compares favorably with known approaches for particular problems. We believe that the broad applicability of our generic approach is a significant advantage.

Our framework is based on domain decomposition together with random sampling to characterize the local solution space on each patch in the decomposition. We make use of the fact that most multiscale PDEs that have asymptotic limits independent of small scales also have local solution spaces of low dimension.

We illustrate the relationships between the multiscale PDE, its discretization, and its asymptotic

limits in Figure 1. The key points of this diagram are as follows.

- 1. Both u^* and u^{ε} , the solutions to \mathcal{L}^* and $\mathcal{L}^{\varepsilon}$, respectively, are convolutions of Green's functions $G^{\varepsilon/*}$ with the boundary conditions.¹
- 2. In the discrete setting, with L^* and L^{ε} denoting the discrete operators and G^* and G^{ε} the corresponding Green's matrices, the numerical solutions U^* and U^{ε} are in the column space spanned by the respective Green's matrices.
- 3. As discussed above, accurate discretization of $\mathcal{L}^{\varepsilon}$ requires $N_{\varepsilon} \sim \varepsilon^{-d}$ degrees of freedom, while discretization of \mathcal{L}^* usually requires a modest number N of degrees of freedom, independent of ε , with $N \ll N_{\varepsilon}$ for interesting values of ε .

Figure 1 suggests that if U^* and U^{ε} are good numerical approximations to u^* and u^{ε} , respectively, and since u^* and u^{ε} are close when ε is small, then U^* and U^{ε} should also be close to each other. Since U^{ε} and U^* lie in the column spaces of G^{ε} and G^* respectively, the two matrices should therefore have similar column spaces. Without knowing the effective equations, it may not be possible to identify G^* explicitly, but we can still obtain essential information contained in G^* from G^{ε} . For this task, we need to determine, first, how much column-space information is contained in G^{ε} and, second, how to extract this information.

Regarding the first question, we define "numerical rank" to be the minimum number of degrees of freedom required to capture the solution space of a PDE to within a preset error tolerance. The concept is closely connected to Kolmogorov N-width. To address the second question, we employ random sampling: The range of a matrix with low numerical rank can be captured by multiplying the matrix by a set of random vectors. We adapt this strategy to sketch the local solution space of the PDE via random sampling.

Random sampling for numerical PDEs has been explored in previous works, mainly for multiscale elliptic equations. In particular, it has been used to construct local basis functions for the generalized finite element method; see [15, 16, 60] and our previous work [17], in which we report on numerical experiments to determine optimal sampling strategies. In [66, 67], numerical homogenization is reformulated as a Bayesian inference problem through observation of random samplings, where orthogonal basis functions in $H_0^1(\Omega)$ could be obtained by nested measurements of solutions or source terms. This approach is consistent with randomized linear algebra approaches that use random projections of a matrix to provide good approximations to the left/right singular-vector space corresponding to the largest singular values of that matrix. Similar connections to randomized linear algebra have been made in [15, 17] for numerical homogenization of elliptic equations. From another perspective [59, 63], randomized linear algebra algorithms are used to compress the Green's matrix of elliptic equations based on the framework of hierarchical matrices [34]. Most of these works that exploit randomized sampling consider only elliptic equations with oscillatory media, while the method we propose in this paper applies to more general situations. (The authors learnt about [15] while drafting the current paper. That work also seeks low-rank representations based on domain decomposition for elliptic equations, but it does not utilize homogenization theory, nor does it extend to general multiscale PDEs. As mentioned previously, the main value of the proposed approach is that it brings a unified framework for various PDE problems exhibiting multiscale features.)

The remainder of the paper is organized as follows. In section 2, we review two representative case studies of multiscale PDEs: the linear kinetic equation with small Knudsen number and an elliptic PDE with oscillatory media. Motivated by the essential similarity of these multiscale problems, we define in section 3 the notion of numerical rank and design a general framework for efficient algorithms based on domain decomposition and random sampling of local solution space. section 4 and section 5 describe details of the application of our framework to the two problems introduced in section 2. Numerical results

¹With slight danger of confusion, we adopt a generalized notion of "Green's function" in this work, which might vary from conventional terminology for specific PDEs. For example, for elliptic PDEs with Dirichlet boundary condition, the "Green's function" would be given by the Poisson kernel, that is, the derivative of the usual Green's function (Newtonian kernel).

demonstrate that the general methodology yields competitive algorithms, without the need for detailed analytical knowledge of the specific structure of the multiscale problems at hand.

2. Asymptotic preserving scheme and numerical homogenization. In this section we briefly summarize the asymptotic preserving scheme and numerical homogenization. These approaches were developed for two rather different multiscale problems, but they share the similar philosophy of finding a set of "effective equations" that are numerically simpler than the original PDE in some sense, and utilizing these equations in efficient numerical solvers. These approaches are closely related to our randomized methodology and will serve to motivate our approach.

2.1. Asymptotic preserving scheme for kinetic equations. The asymptotic preserving (AP) scheme was developed originally in the context of numerical methods for kinetic theory. We will explain the idea using the radiative transfer equation, a particular linear Boltzmann equation that is a model problem in kinetic theory.

In radiative transfer, we seek a function $u^{\varepsilon}(x, v)$, defined on the phase space $(x, v) \in \mathcal{K} \times \mathbb{V}$, that represents the density of photons at location x with speed v. The equation is

(2.1)
$$-v \cdot \nabla_x u^{\varepsilon} + \frac{1}{\varepsilon} \mathsf{S}[u^{\varepsilon}] = g(x), \quad (x,v) \in \mathcal{K} \times \mathbb{V}$$

where the linear collision operator ${\sf S}$ is defined as follows:

(2.2)
$$\mathsf{S}u(x,v) = \int_{\mathbb{V}} k(x,v,v')u(x,v')\mathrm{d}v' - \int_{\mathbb{V}} k(x,v',v)\mathrm{d}v'u(x,v) \, .$$

In Eq. (2.1), the evolution of photon density is governed by the transport term $v \cdot \nabla_x u^{\varepsilon}$, that describes the photons free streaming with speed v in direction x, and the collision term S that characterizes the interaction of the particles with the background media. The first term in S represents particles with velocity v' that are scattered off to obtain v, while the second term indicates the particles whose velocity changes from v to v'. The specific form of k(x, v, v') depends on the media. When the scattering is homogeneous in velocity, we can write $k(x, v, v') = \sigma(x)$ for some σ , so that (2.2) becomes

(2.3)
$$\mathsf{S}u(x,v) = \sigma(x) \int_{\mathbb{V}} \left(u(x,v') - u(x,v) \right) \mathrm{d}v' \, .$$

In the radiative transfer equation (2.1), the quantity ε , which captures the strength of the collision term, is called the *Knudsen number*. When ε is small, the collision term dominates the transport term, and we have $S[u^{\varepsilon}] = 0$, to leading order. In this case, the solution is close to lying in the null space of S, that is, the solution profile nearly achieves local equilibrium for every x. Via asymptotic expansion, we have that $u^{\varepsilon}(x,v) \to M(v)u^{*}(x)$, where $u^{*}(x)$ solves the heat equation and M(v) (called the *local equilibrium* or the *Maxwellian*) spans NullS. More specifically, we have the following result [11, 50, 69] for homogeneous collision (2.3).

THEOREM 2.1. Suppose that u^{ε} solves (2.1) with collision term (2.3) in \mathcal{K} , which is a bounded domain in \mathbb{R}^3 with C^1 boundary, with $\mathbb{V} = \mathbb{S}^2$, and with boundary condition

(2.4)
$$u^{\varepsilon}(x,v) = \phi(x,v) \quad on \quad x \in \partial \mathcal{K}, \quad v \cdot n_x < 0.$$

Then M(v) = 1 and

(2.5)
$$\|u^{\varepsilon}(x,v) - u^{*}(x)\|_{L_{2}(\mathrm{d}x\mathrm{d}v)} \to 0,$$

where $u^*(x)$ solves

(2.6)
$$\frac{1}{3}\nabla_x \cdot \left(\frac{1}{\sigma}\nabla_x u^*\right) = g(x), \quad x \in \mathcal{K},$$

with the boundary condition

$$u^*(x) = \xi_{\phi}(x), \quad on \quad x \in \partial \mathcal{K},$$

where $\xi_{\phi}(x)$ is obtained by solving the boundary layer equation [69].

This result indicates that the limiting operator as $\varepsilon \to 0$ is $\mathcal{L}^* := (1/3)\nabla_x \cdot ((1/\sigma)\nabla_x)$, which is independent of the velocity variable. The constant changes with the dimension of v; 1/3 is the appropriate value for $\mathcal{K} \subset \mathbb{R}^3$.

Remark 2.2. Here we only present the least complicated case, in which the collision is homogeneous (2.3), and we do not specify the convergence rate in (2.5). If the collision operator S is not homogeneous in v, the Maxwellian M(v) could have a complicated form, and the theorem must be modified accordingly. It was long believed that with the correct boundary-layer equation introduced in [69] to translate the boundary conditions from that of u^{ε} to that of u^* , the convergence rate is first order (that is, $u^{\varepsilon} - u^* = \mathcal{O}(\varepsilon)$). Recently, however, this was shown not to be the case; see [53, 55, 76], which show that the boundary layer corrector can reduce the convergence order to less than 1. The sharpest bound is still unknown.

AP, both as a term and a concept, was coined in [42], although the development of AP in the context of the radiative transfer equation dates back to earlier works [43, 51]. The fundamental idea is that a good numerical method, besides being consistent and stable, should also (for fixed discretization) preserve the asymptotic limit of the original equation. As shown in Figure 2, one designs a method $\mathcal{F}_{\varepsilon}^{h}$ for a system $\mathcal{F}_{\varepsilon}$, and asks (1) whether the discrete system, with fixed h, converges when ε shrinks; and (2) if it does converge, whether the limit as $h \to 0$ correctly discretizes \mathcal{F}_{*} , the limiting system on the continuous level. If $\mathcal{F}_{\varepsilon}^{h}$ satisfies both properties, it is said to be asymptotic preserving.

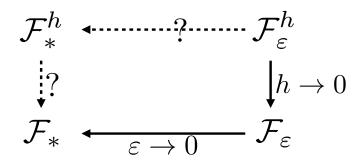


FIG. 2. Commuting diagram of asymptotic preserving (AP) schemes. An AP solver $\mathcal{F}^{h}_{\varepsilon}$ should, in the zero limit of ε for fixed h, capture the solution to \mathcal{F}_{*} .

This AP property is not easy to satisfy in general. For conventional schemes, we need $h \ll \varepsilon$ for accuracy, so we cannot in practice fix h as $\varepsilon \to 0$. AP schemes have to be designed carefully by using analytic knowledge about the limiting operator \mathcal{F}_* . Much progress has been made in the past decade. For linear equations, an even-odd decomposition approach has been designed, with the even part capturing the limit and the odd part capturing the second order expansion [33, 44, 47, 52]. Another approach uses a preconditioned conjugate gradient that exploits the structure of the discrete matrix [6, 57]. In the nonlinear setting, the BGK penalization method was developed in [13, 28] and methods based on the Wild sum [74] were described in [20, 56] (see also [19, 39]). Most of these methods are designed for time-dependent problems. Because of the limited analytic knowledge about kinetic boundary layers, there are very few AP solvers for time-independent problems (see [30, 53, 54]). **2.2. Numerical homogenization.** Consider elliptic equations in divergence form with highly oscillatory media:

(2.7)
$$\begin{cases} \nabla_x \cdot \left(a\left(\frac{x}{\varepsilon}\right) \nabla_x u^{\varepsilon}\right) = g & \text{in } \mathcal{K}, \\ u^{\varepsilon} = f & \text{on } \partial \mathcal{K}. \end{cases}$$

where \mathcal{K} is a bounded Lipchitz domain and $0 < \varepsilon \ll 1$ characterizes the small scale in the problem. We assume $a(\cdot)$ is bounded below and above by positive constants. We also assume $a(\cdot)$ is Hölder continuous and 1-periodic, so problem (2.7) is elliptic with highly-oscillatory media. As ε goes to zero, the solution u^{ε} converges to that of a homogenized equation

(2.8)
$$\begin{cases} \nabla_x \cdot (a^*(x)\nabla_x u^*) = g & \text{in } \mathcal{K}, \\ u^* = f, & \text{on } \partial \mathcal{K} \end{cases}$$

in the sense that

$$||u^{\varepsilon} - u^*||_2 = \mathcal{O}(\varepsilon).$$

Here a^* is termed the *effective media* [5, 14, 64].

THEOREM 2.3 (Theorem 1.1 in [46]). Denote u^{ε} is the solution to Equation (2.7) and u^* the solution to the effective equation (2.8). With same assumptions on \mathcal{K} and $a(\cdot)$ as above, then for any $g \in L^2(\mathcal{K})$ and $f \in H^1(\partial \mathcal{K})$, if $u^* \in H^2(\mathcal{K})$, then we have the strong convergence in L_2 :

(2.9)
$$\|u^{\varepsilon} - u^*\|_{L^2(\mathcal{K})} \le C_{\sigma}\varepsilon |\ln(\varepsilon)|^{\frac{1}{2}+\sigma} \left[\|g\|_{L^2(\mathcal{K})} + \|f\|_{H^1(\partial\mathcal{K})} \right]$$

for any $\sigma > 0$.

The aim of numerical homogenization, or numerical treatment for elliptic equations with rough media to a larger extent, which has a long history, is to develop efficient solvers with two key properties:

1. the discretization is independent of ε ;

2. the numerical solutions capture the correct limiting solutions on the discrete level.

Many methods have been developed for elliptic equations, including the multiscale Finite Element Method (MsFEM) [25, 37, 38], the heterogeneous multiscale method (HMM) [4, 21, 22], the reduced basis type method [2, 3], local orthogonal decomposition [62], subspace decomposition methods [48, 49], local basis construction methods [7, 9, 66, 68], and the global-local approach [8, 40, 65], to name just a few. Many of them have been extended to treat a large class of other equations as well [1, 27, 70]. The focus for these methods are slightly different. For example, MsFEM intends to capture the fine scale oscillation while HMM mainly targets at finding the solution to the effective equation. The comparison of these methods is tangential to the goal of the current work. Interested readers are referred to review papers and books [21, 24, 71].

3. General solution framework based on domain decomposition and random sampling. The asymptotic preserving and numerical homogenization schemes reviewed in the previous section are two efficient schemes for solving multiscale problems with highly oscillatory solutions. Although these schemes tackle different problems in different ways, both schemes achieve efficiency by exploiting the fact that the solutions are close to their asymptotic limits, which lie in a low-dimensional subspace. The design of these schemes relies heavily on a sophisticated understanding of the equation and its asymptotic limit. For many PDEs, this level of understanding is not available [18]. Our goal of this work is to propose a general numerical framework that can be applied to various multiscale problems, capturing the efficient representation of the solutions without an explicit reliance on the analytical understanding.

A first step in developing our framework is to relate the AP and numerical homogenization schemes to the numerical linear algebra concept of *low rank*. When the matrix operator in a linear algebra problem has low rank, the solution lies in a subspace of low dimension; there are efficient numerical schemes, based on random sampling, that exploit this property. Drawing on these ideas from linear algebra, we propose a method under the domain decomposition framework, that utilizes random sampling to search representative modes in the solution space.

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3.1. Numerical rank. In this section, we tackle the questions of low-rankness of a PDE operator and low dimensionality of the solution space in a general setting, and estimate the rank and dimension for several problems of interest. In this way, we aim to unify the AP and numerical homogenization schemes, and develop numerical schemes for more general multiscale problems.

We consider a bounded linear operator \mathcal{A} :

$$\begin{array}{cccc} (3.1) & \mathcal{A}: & \mathcal{X} & \to & \mathcal{Y} \\ & & f & \mapsto & u \end{array}$$

that maps $f \in \mathcal{X}$ to a Hilbert space \mathcal{Y} . In the PDE setting, \mathcal{A} maps the boundary conditions and/or source term to the solution of the problem. We define the following neighborhood of \mathcal{A} that is parametrized by a positive scalar τ :

$$S_{\tau} := \{ \tilde{\mathcal{A}} \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \| \mathcal{A} - \tilde{\mathcal{A}} \|_{\mathcal{X} \to \mathcal{Y}} \le \tau \}.$$

The set S_{τ} is the collection of all operators whose operator norm is within distance τ of \mathcal{A} . When the context is clear, we suppress the subscript in the operator norm $\|\cdot\|_{\mathcal{X}\to\mathcal{Y}}$.

DEFINITION 3.1 (Numerical rank). The numerical τ -rank of \mathcal{A} is the rank of the lowest-rank operator in S_{τ} , that is,

$$k_{\tau}(\mathcal{A}) := \dim \operatorname{ran} \mathcal{A}_{\tau}; \quad \mathcal{A}_{\tau} := \arg \min \{\dim \operatorname{ran} \mathcal{A} : \mathcal{A} \in S_{\tau} \}.$$

That is, \mathcal{A}_{τ} is the operator within distance τ of \mathcal{A} whose range space has the smallest dimension, and $k_{\tau}(\mathcal{A})$ is this dimension. We set $k_{\tau}(\mathcal{A})$ to ∞ if all $\tilde{\mathcal{A}} \in S_{\tau}$ have range spaces of infinite dimension.

The definition of numerical rank is closely related to Kolmogorov N-width, which we define here.

DEFINITION 3.2 (Kolmogorov N-width). Given the linear operator in (3.1), the Kolmogorov N-width $d_N(\mathcal{A})$ is the shortest distance to an N-dimensional space, that is,

$$d_N(\mathcal{A}) := \min_{S:\dim S=N} d(\mathcal{A}, S) = \min_{S:\dim S=N} \sup_{f} \min_{v \in S} \frac{\|\mathcal{A}f - v\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}}.$$

Definition 3.1 and Definition 3.2 are connected through the following proposition:

PROPOSITION 3.3. For the operator \mathcal{A} specified in (3.1), the following are true.

- (a) If the numerical τ -rank is N, then $d_N(\mathcal{A}) \leq \tau$.
- (b) If $d_N(\mathcal{A}) \leq \tau < d_{N-1}(\mathcal{A})$, then the numerical τ -rank is N.

Proof. We use \mathbf{P}_S to denote the projection operator onto a finite dimensional subspace S. Note that the Kolmogorov N-width is a non-increasing function of N.

For (a), let $\mathcal{A}_{\tau} \in S_{\tau}$ be the operator that achieves the numerical τ -rank of N, and denote by S the range of \mathcal{A}_{τ} . We then have

$$\tau \ge \|\mathcal{A} - \mathcal{A}_{\tau}\| = \sup_{f} \frac{\|\mathcal{A}f - \mathcal{A}_{\tau}f\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} \ge \sup_{f} \min_{v \in S} \frac{\|\mathcal{A}f - v\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} \ge d_{N}(\mathcal{A}),$$

where the last inequality is from Definition 3.2.

For (b), suppose that $d_N \leq \tau < d_{N-1}(\mathcal{A})$. First, for an arbitrary (N-1)-dimensional subspace S, we have

$$\tau < d_{N-1}(\mathcal{A}) \le \sup_{f} \min_{v \in S} \frac{\|\mathcal{A}f - v\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} \le \sup_{f} \frac{\|\mathcal{A}f - \mathbf{P}_{S}\mathcal{A}f\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} = \|\mathcal{A} - \mathbf{P}_{S}\mathcal{A}\|,$$

then according to Definition 3.1, there is no (N-1)-dimensional operator that achieves τ accuracy, so we must have $k_{\tau}(\mathcal{A}) \geq N$. Second, since $d_N(\mathcal{A}) \leq \tau$, then there exists a N-dimensional subspace S such that

$$d_N(\mathcal{A}) = \sup_f \min_{v \in S} \frac{\|\mathcal{A}f - v\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} = \sup_f \frac{\|\mathcal{A}f - \mathbf{P}_S \mathcal{A}f\|_{\mathcal{Y}}}{\|f\|_{\mathcal{X}}} = \|\mathcal{A} - \mathbf{P}_S \mathcal{A}\| \le \tau.$$

Defining $\mathcal{A}_{\tau} = \mathbf{P}_{S}\mathcal{A}$, we see that the numerical τ -rank is N.

The numerical rank and the Kolmogorov N-width both depend on *optimal approximations*, which typically require basis set construction that is adaptive to the given problem. The pre-defined basis sets conventionally used in numerical discretization, such as local polynomials and global Fourier functions (as used in finite difference/element methods and spectral methods), are not optimal, except in very special cases (heat equation, for example). In fact, there are counterexamples that show them to be arbitrarily bad; see [73] for the spectral method and [10] for finite elements.

It is important to distinguish between numerical rank and degrees of freedom (DOF). The DOF is the number of variables needed to represent the solutions (to a certain specified error tolerance), when the basis functions are given. Each numerical method utilizes a certain set of pre-specified basis functions, and the DOF changes according to the method used. The numerical rank, however, depends on the *optimal* representation, so is the minimum DOF across all possible methods. We study two examples and give rough computation of DOF using standard finite element methods, thus yielding upper bound of the respective numerical ranks. Numerical rank, as a concept, was explicitly explored in several papers on numerical homogenization, including [7, 16, 32, 62]. In [7], it was proved that the singular values of a confinement map decay almost exponentially. This concept, however, was not as developed in other sub-areas of multiscale computation. We compare numerical rank and DOF explicitly below.

3.1.1. Numerical rank of the radiative transfer equation. To estimate the numerical rank of the solution operator \mathcal{A} for the radiative transfer equation (2.1), (2.2) and its diffusion limit, we consider the following cases. We assume in this section that the boundary condition ϕ in (2.4) satisfies $\phi \in W^{2,\infty}$, so that the solution u and u^{ε} attain the same regularity [26]. The boundary-to-solution map \mathcal{A} thus maps $W^{2,\infty}$ to $W^{2,\infty}$. For simplicity, we study the numerical rank of \mathcal{A} associated with L^2 norm.

a) Let $\varepsilon = 1$ in (2.1). If we use the upwind method for ∂_x and the trapezoidal rule for S, the method is first-order in x and second-order in v. By equating the numerical error estimate to the accuracy required, we have

$$\mathcal{O}(N_x^{-1} + N_v^{-2}) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}(1/\tau), \quad N_v = \mathcal{O}(1/\sqrt{\tau}).$$

For τ -accuracy, we thus obtain the following DOF:

$$N_{\varepsilon=1} = N_x N_v = \mathcal{O}(1/\tau^{3/2}).$$

b) Suppose that ε is extremely small in (2.1) and we use the same method as shown above. Then, defining $C_{\varepsilon} = \|\partial_x^2 u^{\varepsilon}\|_{\infty} = \mathcal{O}(\frac{1}{\varepsilon^2})$, we have that

$$\mathcal{O}(C_{\varepsilon}N_x^{-1} + N_v^{-2}) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}(C_{\varepsilon}/\tau) = \mathcal{O}\left(\frac{1}{\tau\varepsilon^2}\right), N_v = \mathcal{O}(1/\sqrt{\tau}).$$

Note that C_{ε} blows up for small ε , since u^{ε} has sharp transitions. For τ -accuracy, the DOF is

(3.2)
$$N_{\varepsilon} = N_x N_v = \mathcal{O}\left(\frac{1}{\varepsilon^2 \tau^{3/2}}\right) \,.$$

c) If hat functions are used to construct the finite element basis for the limiting Poisson equation (2.6), the method is second-order convergent in x, and we obtain

$$\mathcal{O}(N_x^{-2}) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}\left(1/\sqrt{\tau}\right) \,.$$

The DOF in this case is thus:

$$N_* = N_x = \mathcal{O}(1/\sqrt{\tau}).$$

d) If we make use of the diffusion limit, the triangle inequality yields

$$\|u^{\varepsilon} - U\| \le \|u^{\varepsilon} - u^*\| + \|u^* - U\| \le \mathcal{O}(\varepsilon) + \mathcal{O}(N_x^{-2}),$$

with U being the numerical solution to u^* . By comparing with the tolerance τ and taking the zero limit of ε , we obtain for the DOF that

(3.3)
$$N_{\varepsilon}^{\mathrm{ap}} = \mathcal{O}\left(\frac{1}{\sqrt{|\tau - \varepsilon|}}\right) = \mathcal{O}(1/\sqrt{\tau}) \quad \mathrm{as} \quad \varepsilon \to 0 \,.$$

This is the approximation used by the AP method, hence our notation $N_{\epsilon}^{\rm ap}$.

We see by comparing (3.3) and (3.2) that different schemes produce vastly different DOF. Numerical rank of \mathcal{A} , bounded by the smallest DOF, is thus controlled by N_{ε}^{ap} . The homogenization scheme gives a much sharper bound on numerical rank than the brute-force finite difference method.

3.1.2. Numerical rank of elliptic equation with oscillatory coefficients. A similar analysis to the previous subsection can be conducted for the diffusion equation (2.7) with rough media. Again, we assume $H^{3/2}$ regularity for the boundary condition g, so that the solution u and u^{ε} gain H^2 regularity. We thus consider the solution operator \mathcal{A} to be a mapping from $H^{3/2}$ to H^2 , and study the numerical rank of \mathcal{A} associated with L^2 norm.

a) Let $\varepsilon = 1$ in (2.7). If one uses the classical finite element method with piecewise hat functions as basis functions for $\nabla_x \cdot (a(x, x/\varepsilon)\nabla_x)$, the method is second-order convergent. By equating the numerical error to the required accuracy τ , we obtain

$$\mathcal{O}(N_x^{-2}) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}(1/\sqrt{\tau}),$$

so that the DOF within τ -accuracy is $N_{\varepsilon=1} = N_x = \mathcal{O}(1/\tau^{1/2})$.

b) Suppose that $0 < \varepsilon \ll 1$ in (2.7). If we use the classical finite element method with hat functions, as above, the discretization needs to resolve the oscillations, leading to the estimate

$$\mathcal{O}\left(\frac{1}{\varepsilon^2}N_x^{-2}\right) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}\left(\frac{1}{\varepsilon\sqrt{\tau}}\right) \,,$$

where the factor $1/\varepsilon^2$ arises from Theorem 4.4 in [36]. We thus have

(3.4)
$$N_{\varepsilon} = N_x = \mathcal{O}\left(\frac{1}{\varepsilon\sqrt{\tau}}\right)$$

c) If the finite element method with hat-function basis is applied to the limiting effective equation with smooth media (2.8), the solution is smooth and the derivative is order one. Since the method is second-order, we obtain

$$\mathcal{O}(N_x^{-2}) = \tau \quad \Rightarrow \quad N_x = \mathcal{O}(1/\sqrt{\tau}),$$

which leads to a DOF of $N_* = N_x = \mathcal{O}(1/\sqrt{\tau})$.

d) The homogenization route and the triangle inequality leads to

$$\|u^{\varepsilon} - U\| \le \|u^{\varepsilon} - u^*\| + \|u^* - U\| \le \mathcal{O}(\varepsilon) + \mathcal{O}(N_x^{-2}) \le \tau \,,$$

so that

(3.5)
$$N_{\varepsilon}^{\text{hom}} = N_x \ge \mathcal{O}\left(\frac{1}{\sqrt{|\tau - \varepsilon|}}\right) \to \mathcal{O}(1/\sqrt{\tau}) \text{ as } \varepsilon \to 0.$$

By comparing (3.4) and (3.5), we see that the DOF obtained from homogenization gives a much sharper bound on the numerical rank. Moreover, the numerical rank is finite, even in the zero limit of ε .

The discussions above show that the DOF depends on both the approximate solution space and the choice of basis functions, while numerical rank, by contrast, reflects the size of the basis required to

approximate the solution up to a certain given accuracy. Heuristically, it also implies that the singular values of stiffness matrix decay rapidly, while the size of this matrix explodes as $\varepsilon \to 0$. When DOF is significantly higher than the numerical rank, fast matrix-vector multiplication methods, which may exploit the sparsity of the stiffness matrix, may accelerate the computation. However, this topic is beyond the focus of this paper. We take the alternative route here of identifying lower-dimensional spaces that approximate the solution space well and economically, using techniques that are motivated by randomized algorithms in numerical linear algebra.

Remark 3.4. The discussion above has been justified rigorously in [7] for elliptic equation with rough media. This paper shows the optimal local basis functions are indeed the singular vectors of a restriction operator P, and that the Kolmogorov N-width of P is exponentially decaying, that is,

$$d_N(P) \lesssim e^{-n^{1/(d+1)}}$$

where d is dimension of physical space. Therefore, the numerical τ -rank of P is small and the optimal representation of solution of elliptic equation has small DOF. The work [7] constructed optimal basis via an eigenvalue problem, whereas our work proposes to use a randomized algorithm.

3.2. Random sampling in numerical linear algebra. Random sampling algorithms have a long history in numerical linear algebra [29, 31, 35, 41, 45, 61, 72]; we will focus here on those related to low-rank approximations of a matrix. Given a matrix $A \in \mathbb{R}^{m \times n}$ that is known to be approximately low rank, a standard way to obtain the most important modes in its range is via the singular value decomposition (SVD). Without loss of generality, we assume $m \geq n$ and write the singular value decomposition as

(3.6)
$$\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^{\top} = \sum_{i=1}^{n} \sigma_{i} u_{i} v_{i}^{\top},$$

where $U = [u_1, u_2, \ldots, u_n] \in \mathbb{R}^{m \times n}$ contains the left singular vectors, $V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n \times n}$ contains the right singular vectors and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ contains the singular values in descending order: $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$. U and V are orthogonal matrices. It is well known that the best k-rank approximation to A (in spectral norm) is given by thresholding the singular value decomposition at k-th order, termed A_k here:

$$\mathsf{A}_k = \mathsf{U}_k \Sigma_k V_k = \sum_{i=1}^k \sigma_i u_i v_i^\top,$$

where U_k and V_k contain the first k columns in U and V. We say the matrix is approximately rank-k if $||A - A_k|| = \sigma_{k+1} \ll \sigma_1$. In this case, we have

$$\|A - A_k\| = \|A - U_k U_k^\top A\| = \sigma_{k+1} \ll \sigma_1 = \|A\|.$$

In terms of the discussion in the previous subsection, the range space of A is approximately the same as the range space of A_k , which equals the span of the columns of U_k , which is the subspace we seek. Computation of the SVD (3.6) is a classical problem in numerical linear algebra, requiring $\mathcal{O}(mn^2)$ operations.

Randomized SVD efficiently computes the low-rank approximation of a given matrix by means of random sampling of its column space. The particular version of the algorithm we describe here was developed in [58, 75]; see [35] for a review.

The idea behind the algorithm is simple: if an $m \times n$ matrix A is of approximate low rank k, the matrix maps an n-dimensional sphere to an m-dimensional ellipsoid that is "skinny:" k of its axes are significantly larger than the rest. With high probability, vectors that are randomly sampled vector on the n-dimensional sphere are mapped by A to vectors that lie mostly in a k-dimensional subspace of \mathbb{R}^m , which is the range of A. An approximation to A_k can be obtained by projecting onto this subspace.

The precise statement of the randomized SVD algorithm and its error estimates are recalled in the following theorem.

THEOREM 3.5 (Theorems 10.6 and 10.8 of [35]). Let A be defined as in (3.6) and let the target rank k be at least 2. Define

$$(3.7) Y = A\Omega,$$

where $\Omega = [\omega_1, \ldots, \omega_{k+p}]$ is a matrix of size $n \times (k+p)$ with its entries randomly drawn from i.i.d. normal distribution, where p is an oversampling parameter. If A is approximately k-rank, then with large probability, $P_Y(A)$, the projection of A onto the space spanned by Y, defined by

$$\mathsf{P}_{\mathsf{Y}}(\mathsf{A}) = \mathsf{Y}(\mathsf{Y}\mathsf{Y}^{\top})^{-1}\mathsf{Y}^{\top}\mathsf{A}\,,$$

yields the following error bounds. a) Average spectral error:

$$\mathbb{E} \|\mathsf{A} - \mathsf{P}_{\mathsf{Y}}(\mathsf{A})\| \le \left(1 + \frac{k}{p-1}\right)\sigma_{k+1} + \frac{e\sqrt{k+p}}{p} \left(\sum_{j>k} \sigma_j^2\right)^{1/2} \ll \sigma_1 \,.$$

b) Deviation bound:

$$\|\mathsf{A} - \mathsf{P}_{\mathsf{Y}}(\mathsf{A})\| \le \left[\left(1 + t\sqrt{\frac{3k}{p+1}} \right) \sigma_{k+1} + t \frac{e\sqrt{k+p}}{p+1} \left(\sum_{j>k} \sigma_j^2 \right)^{1/2} \right] + ut \frac{e\sqrt{k+p}}{p+1} \sigma_{k+1} \ll \sigma_1,$$

with failure probability at most $2t^{-p} + e^{-u^2/2}$, for all u, t > 1.

We emphasize two advantages of the algorithm: It captures the approximate range within n(k + p) operations (p is fixed and small), and it does not require full knowledge of A, only the ability to evaluate the matrix-vector product A Ω .

3.3. General solution framework for multiscale problems. Finding a low-rank representation of solution space is the key to reducing complexity. In this section, we adapt the low-rank approximation scheme from numerical linear algebra into a general methodology for solving multiscale PDEs. The method requires limited knowledge on the specific structure of the solution spaces, so the solvers are expected to be applicable to a large class of multiscale problems. Our framework uses domain decomposition to sketch the local solution space via randomized sampling, in an offline step. This is followed by an online step, in which the solution is patched together by imposing continuity conditions across the domains.

We wish to solve the problem (1.1), that is,

(3.8)
$$\begin{cases} (\mathcal{L}^{\varepsilon}u^{\varepsilon})(x) = 0, & x \in \mathcal{K}, \\ \mathcal{B}u(x) = \phi(x), & x \in \Gamma, \end{cases}$$

where \mathcal{B} is the boundary condition operator, Γ the boundary associated with domain \mathcal{K} and f the boundary data. We adopt the domain decomposition approach, partitioning \mathcal{K} into M non-overlapping subdomains, as follows:

$$\mathcal{K} = \bigcup_{m=1}^{M} \mathcal{K}_m$$
, with $\mathcal{K}_m^{\circ} \cap \mathcal{K}_n^{\circ} = \emptyset \quad (m \neq n)$,

where \mathcal{K}_m denotes the *m*-th local patch. Accordingly, we denote by Γ_m the boundary associated with \mathcal{K}_m . Different types of equations require various kinds of boundary conditions, as we will make explicit in section 4 and section 5. Each subdomain is further discretized with a conformal mesh. We denote h

as the largest meshsize and assume that it is fine enough such that $h \ll \varepsilon$. The number of subdomains M does not depend on ε .

- Domain decomposition approach consists of two stages, as follows.
- (1) Offline stage: Prepare local solution space. Denote by G_m the collection of local solutions in each local patch \mathcal{K}_m , $m = 1, 2, \ldots, M$, that is,

$$\mathsf{G}_m = [b_{m,1}, b_{m,2} \dots] ,$$

where each local function $b_{m,n}$ is one solution to the equation on the subdomain \mathcal{K}_m , that is,

$$\mathcal{L}^{\varepsilon}b_{m,n} = 0, \quad x \in \mathcal{K}_m,$$

with boundary condition on Γ_m . These solutions are computed on fine grids with discretization h. (2) **Online stage**: The global solution is written as

$$u = \sum_{m=1}^{M} u_m = \sum_{m=1}^{M} \mathsf{G}_m c_m,$$

with u_m being u confined on \mathcal{K}_m . c_m is a vector of coefficients determined by the boundary conditions ϕ and conditions that enforce continuity across patches.

The online stage is a standard step in domain decomposition. Its cost is governed by the number of basis functions chosen in the offline step. In the offline stage, there are many ways to construct the local solution space G_m . Since this space contains all possible local solutions, it can be regarded as a full library of all Green's functions. One possible way to define G_m is to define the boundary conditions on the *m*th patch to be delta functions defined over a grid on the boundary Γ_m , that is,

$$\begin{cases} \mathcal{L}^{\varepsilon} b_{m,n} = 0, & x \in \mathcal{K}_m \\ b_{m,n} = \delta_{m,n}, & x \in \Gamma_m \end{cases}$$

where $\delta_{m,n}$ is the Kronecker delta function that takes the value 1 at the *n*-th grid point on Γ_m and zero on the other grid points on $\partial \mathcal{K}_m$. Since $h \ll \varepsilon$, the number of functions n_m in G_m grows as ε shrinks. This strategy, summarized in Algorithm DETLOCALSOLU, is referred to as the *full-basis approach*.

An alternative way to construct basis functions for each patch also makes use of a grid defined on the boundary Γ_m , but takes the boundary conditions for each function $b_{m,n}$ to be a set of random values on the grid points, rather than a δ function. Specifically, we have

$$\begin{cases} \mathcal{L}^{\varepsilon} r_{m,n} = 0, & x \in \mathcal{K}_m, \\ r_{m,n} = \omega_{m,n}, & x \in \Gamma_m, \end{cases}$$

where $\omega_{m,n}$ is defined to have a random value drawn i.i.d. from a normal distribution at each grid point in Γ_m . Since the local solution space is homogenizable and low rank, we expect that the number of basis functions k_m required to represent it adequately will be much smaller than n_m defined above, and independent of ε . This strategy, which we refer to as the *randomized reduced-basis approach*, is summarized in Algorithm RANDLOCALSOLU. In practice, one could add a QR-decomposition at the end of algorithm RANDLOCALSOLU to return basis functions that are orthonormal. This would improve the condition number of the global online problems (for example, (4.13) and (5.14)).

Denote by G_m^b the collection of full basis $\{b_{m,n}\}$ and G_m^r the collection of random reduced basis $\{r_{m,n}\}$, we have the following relationship:

$$\mathsf{G}_m^r = \mathsf{G}_m^b \Omega \,,$$

where Ω is a random i.i.d. matrix with entries $\omega_{m,n}$.

Algorithm 3.1 Multiscale solver for $\mathcal{L}^{\varepsilon} u^{\varepsilon} = 0$ over \mathcal{K} with $\mathcal{B} u = f$ on Γ

1: Domain Decomposition

- Partition domain into non-overlapping patches $\mathcal{K} = \bigcup_{m=1}^{M} \mathcal{K}_m$. Form the ansatz $u = \sum_{m=1}^{M} u_m = \sum_{m=1}^{M} \mathsf{G}_m \vec{c}_m$. 2:
- 3:
- 4:**Offline Stage:**
- Call function $G_m = DETLOCALSOLU(\mathcal{K}_m)$ or $G_m = RANLOCALSOLU(\mathcal{K}_m)$. 5:
- 6: **Online Stage:**
- Use continuity condition and global boundary data f to determine $[\vec{c}_1, \ldots, \vec{c}_M]$. 7:
- 8: **Return:** approximated global solution $\hat{u} = \sum_{i=1}^{M} \mathsf{G}_m \vec{c}_m$.

1: function DETLOCALSOLU(\mathcal{K}_m)

- Prepare full list of numerical delta functions $\delta_{m,i}$, $i = 1, \ldots, n_m$ on Γ_m . 2:
- 3: Call function $u_{m,i}$ =LOCALPDESOLVER $(\mathcal{K}_m, \delta_{m,i})$ for $i = 1, 2, \ldots, n_m$.
- **Return:** Local solution space span $G_m = [u_{m,1}, \ldots, u_{m,n_m}].$ 4:
- 5: end function

1: function RANLOCALSOLU(\mathcal{K}_m)

- Prepare k_m random i.i.d. Gaussian vector $\omega_{m,i}, i = 1, \ldots, k_m$ on Γ_m . 2:
- Call function $u_{m,i}$ =LOCALPDESOLVER $(\mathcal{K}_m, \omega_{m,i})$ for $i = 1, 2, \ldots, k_m$. 3:
- **Return:** Approximated local solution space span $G_m = [u_{m,1}, \ldots, u_{m,k_m}]$. 4:
- 5: end function

1: function LOCALPDESOLVER (Local domain \mathcal{K}_m , Boundary condition ϕ)

- Use standard Finite Element/Difference Methods to solve PDE $\mathcal{L}^{\varepsilon}u_{m}^{\varepsilon}=0$ over \mathcal{K}_{m} with $u_{m}^{\varepsilon}=\phi$ 2: over Γ_m , for solution u_m^{ε} .
- **Return:** Local solution u_m^{ε} . 3:
- 4: end function

The complete scheme, which includes the two alternative implementations of the offline stage described above, is specified as Algorithm 3.1.

In practice, for RANDLOCALSOLU, we often use a slightly larger patch $\mathcal{K}_m \supset \mathcal{K}_m$ that augments \mathcal{K}_m by a buffer zone. The local solution is obtained on \mathcal{K}_m , with random boundary conditions on its associated boundary Γ_m , and then restricted on \mathcal{K}_m , as follows:

$$\begin{cases} \mathcal{L}^{\varepsilon} \tilde{b}_{m,n} = 0, & x \in \widetilde{\mathcal{K}}_m \\ \tilde{b}_{m,n} = \omega_{m,n}, & x \in \widetilde{\Gamma}_m. \end{cases}$$

Use of the buffer zone helps to remove boundary layer effects and the effect of the singularity at the boundary. This technique will be discussed further for the particular PDEs considered in the next two sections.

Remark 3.6. We emphasize that such connection between PDE and linear algebra has been observed by several previous works, including [15, 66, 68]. Our proposed method especially coincides with that of [68], in which the author explicitly connects the random sampling in H^{-1} (seen in the source) to the representative basis functions in H^1 (seen in the solution space). In our case the random sampling is done on the boundary condition, but the method shares the same spirit as reported in [68].

4. Example 1: Radiative transfer equation. We now describe the application of our framework to the radiative transfer equation with zero source, which is

(4.1)
$$\mathcal{L}^{\varepsilon}u^{\varepsilon} = v\partial_{x}u^{\varepsilon}(x,v) - \frac{1}{\varepsilon}\mathsf{S}[u^{\varepsilon}] = 0, \quad (x,v) \in \mathcal{K} = \Omega \times \mathbb{V} = [0,1] \times [-1,1],$$

where the collision term **S** is given by

$$\mathsf{S}u(x,v) = \int_{-1}^{1} k(x,v,v')u(x,v')\mathrm{d}v' - \int_{-1}^{1} k(x,v',v)\mathrm{d}v'u(x,v)$$

We use the Henyey-Greenstein model, in which the scattering coefficient is defined by

(4.2)
$$k(x,v,v') = \frac{1}{2} \frac{1-g^2}{1+g^2+2g(vv')}$$

where $g \in (-1, 1)$ is a specified constant. To impose boundary conditions properly for radiative transfer equations, we denote by Γ_{\pm} the outgoing / incoming part of the boundary:

$$\Gamma_{\pm} = \{ (x, v) : x \in \partial\Omega, \ \pm v \cdot n_x > 0 \},\$$

where n_x is the exterior normal direction at $x \in \partial \Omega$. In particular, for the problem (4.1) on the spatial domain $\Omega = [0, 1]$, we have

$$\Gamma_{-} = \left\{ (x=0,v>0) \right\} \cup \left\{ (x=1,v<0) \right\}, \quad \Gamma_{+} = \left\{ (x=0,v<0) \right\} \cup \left\{ (x=1,v>0) \right\}.$$

The equation (4.1) is well-posed if a Dirichlet boundary condition is imposed on the incoming boundary, also known as the incoming boundary condition: $u^{\varepsilon}|_{\Gamma_{-}} = \phi$.

To implement domain decomposition, we partition the domain as follows:

(4.3)
$$\mathcal{K} = [0,1] \times [-1,1] = \bigcup_{m=1}^{M} \mathcal{K}_m, \text{ with } \mathcal{K}_m = [x_{m-1}, x_m] \times [-1,1],$$

where $x_m = m/M$ forms a set of (M + 1) equi-spaced grid points on [0, 1] and \mathcal{K}_m is the *m*-th patch of the domain. The incoming / outgoing parts of the boundary of each patch are

$$\Gamma_{m,-} = \{(x_{m-1}, v > 0)\} \cup \{(x_m, v < 0)\}, \text{ and } \Gamma_{m,+} = \{(x_{m-1}, v < 0)\} \cup \{(x_m, v > 0)\}.$$

We denote by $L_{m,m+1} = \mathcal{K}_m \cap \mathcal{K}_{m+1} = \{(x_m, v) : v \in [-1, 1]\}$ the line segment that separates \mathcal{K}_m and \mathcal{K}_{m+1} . The geometry of the domain and the patches is plotted in Figure 3.

As described in section 3, the domain decomposition approach prepares the local solution space in the offline step and patches together solutions via continuity and boundary conditions in the online step. We describe the two options for constructing the basis functions — the full-basis approach and the randomized reduced-basis approach — in the following two subsections.

4.1. Full basis approach.

Offline step. We prepare a full basis of the local solution space by enumerating all possible boundary conditions, up to a discretization. Since the problem (4.1) is linear, we can obtain each basis function by solving a problem over a patch with a Dirichlet boundary condition that is nonzero at only one grid point. Specifically, for the patch \mathcal{K}_m , each basis function $b_{m,i}$ is obtained by solving

(4.4)
$$\begin{cases} v\partial_x b_{m,i} - \frac{1}{\varepsilon} \mathsf{S}[b_{m,i}] = 0, \quad (x,v) \in \mathcal{K}_m, \\ b_{m,i}|_{\Gamma_{m,-}} = \delta_{m,i}, \end{cases}$$

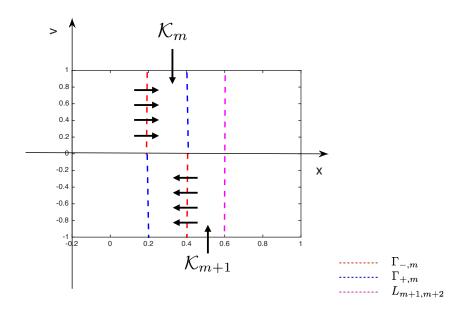


FIG. 3. Domain decomposition for RTE and the boundaries of the local patch.

where $\delta_{m,i}$ is a numerical delta function supported on a grid point on $\Gamma_{m,-}$ and the index *i* enumerates all grid points on the incoming boundary. The full basis for the local solution space is then given by

(4.5)
$$\mathsf{G}_m^{\varepsilon} = [b_{m,1}, \dots, b_{m,n_m}] ,$$

where $\mathsf{G}_m^{\varepsilon}$ is a Green's matrix whose columns are the basis functions $b_{m,i}$. Here, n_m is the total number of grid points on the incoming boundary $\Gamma_{m,-}$ of \mathcal{K}_m . In other words, the Green's matrix $\mathsf{G}_m^{\varepsilon}$ is the analog of the operator $\mathcal{A}_m^{\varepsilon}: f \mapsto b$ defined by

$$\begin{cases} v\partial_x b - \frac{1}{\varepsilon}\mathsf{S}[b] = 0, \quad (x,v) \in \mathcal{K}_m, \\ b|_{\Gamma_{m,-}} = f. \end{cases}$$

Online step. The online step obtains the global solution as a linear combination of all local basis functions, as follows:

(4.6)
$$u^{\varepsilon} = \sum_{m} u_{m}^{\varepsilon} = \sum_{m} \sum_{i} c_{m,i} b_{m,i}$$

where the coefficients $c_{m,i}$ are chosen to satisfy the following conditions:

* Continuity: $u_m(L_{m,m+1}) = u_{m+1}(L_{m,m+1})$, which can be stated in more detail as

(4.7)
$$\begin{cases} u_m(\Gamma_{m,+} \cap L_{m,m+1}) = u_{m+1}(\Gamma_{m,+} \cap L_{m,m+1}) = u_{m+1}(\Gamma_{m+1,-} \cap L_{m,m+1}), \\ u_m(\Gamma_{m,-} \cap L_{m,m+1}) = u_{m+1}(\Gamma_{m,-} \cap L_{m,m+1}) = u_{m+1}(\Gamma_{m+1,+} \cap L_{m,m+1}). \end{cases}$$

In both equations, the first equality comes from the continuity condition and the second equality follows from

$$\Gamma_{m,\pm} \cap L_{m,m+1} = \Gamma_{m+1,\mp} \cap L_{m,m+1},$$

as illustrated in Figure 3.

* Boundary condition:

$$(4.8) u|_{\Gamma_{-}} = \phi.$$

Algebraically, we denote by M_m the matrix that maps inflow boundary condition $c_m = u_m(\Gamma_{m,-})$ to outflow data $u_m(\Gamma_{m,+})$, and denote by I_m^l (resp. I_m^r) the restriction operator on the left edge $L_{m-1,m}$ (resp. the right edge $L_{m,m+1}$) of patch \mathcal{K}_m . Using this notation, (4.7) and (4.8) can be written as follows:

$$\begin{bmatrix} \mathbf{I}_m^r \mathbf{M}_m & -\mathbf{I}_{m+1}^l \\ -\mathbf{I}_m^r & \mathbf{I}_{m+1}^l \mathbf{M}_{m+1} \end{bmatrix} \begin{bmatrix} c_m \\ c_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{I}_1^l & 0 \\ 0 & \mathbf{I}_M^r \end{bmatrix} \begin{bmatrix} c_1 \\ c_M \end{bmatrix} = \begin{bmatrix} \mathbf{I}_1^l \phi \\ \mathbf{I}_M^r \phi \end{bmatrix}.$$

Assembling these conditions over all patches, we obtain

$$(4.9) Pc = d$$

where

$$\mathsf{P} = \begin{bmatrix} \mathsf{I}_1^l & 0 & 0 & \dots & 0\\ \mathsf{I}_1^r \mathsf{M}_1 & -\mathsf{I}_2^l & 0 & \dots & 0\\ -\mathsf{I}_1^r & \mathsf{I}_2^l \mathsf{M}_2 & 0 & \dots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & \mathsf{I}_{M-1}^r \mathsf{M}_{M-1} & -\mathsf{I}_M^l\\ 0 & \dots & 0 & 0 & \mathsf{I}_M^r \end{bmatrix}, \quad c = \begin{bmatrix} c_1\\ c_2\\ \vdots\\ c_M \end{bmatrix}, \quad d = \begin{bmatrix} \mathsf{I}_1^l \phi\\ 0\\ \vdots\\ 0\\ \mathsf{I}_M^r \phi \end{bmatrix}.$$

We obtain the solution by substituting the coefficients $\{c_{m,i} : i = 1, ..., n_m, m = 1, ..., M\}$ from (4.9) into (4.6).

4.2. Reduced basis approach. An approximation to the local solution space for a patch \mathcal{K}_m starts by defining the larger "buffered" patch $\widetilde{\mathcal{K}}_m \supset \supset \mathcal{K}_m$. The buffered patch has boundaries $\widetilde{\Gamma}_{\pm,m}$, as illustrated in Figure 4. We denote by $\widetilde{\mathsf{G}}_m^{\varepsilon}$ the Green's matrix obtained by solving the local equation on the buffered patch $\widetilde{\mathcal{K}}_m$ with all possible boundary conditions, as in the construction of (4.5), but restricted to the domain \mathcal{K}_m . More precisely, we can obtain $\widetilde{b}_{m,i}$ by solving

$$\begin{cases} v\partial_x \widetilde{b}_{m,i} - \frac{1}{\varepsilon} \mathsf{S}[\widetilde{b}_{m,i}] = 0, \quad (x,v) \in \widetilde{\mathcal{K}}_m, \\ \widetilde{b}_{m,i}|_{\widetilde{\Gamma}_{m,-}} = \delta_i, \end{cases}$$

where $\widetilde{\Gamma}_{m,-}$ is the incoming portion of the boundary of $\partial \widetilde{\mathcal{K}}_m$, and then define

$$\widetilde{\mathsf{G}}_{m}^{\varepsilon} = \left[\widetilde{b}_{m,1}|_{\mathcal{K}_{m}}, \dots, \widetilde{b}_{m,\widetilde{n}_{m}}|_{\mathcal{K}_{m}}\right],$$

where \tilde{n}_m is the number of incoming boundary grid points. It is clear that each column of $\tilde{\mathsf{G}}_m^{\varepsilon}$ solves (4.4) inside \mathcal{K}_m , and thus is in span $\mathsf{G}_m^{\varepsilon}$ (since the latter consists of all possible local solutions). Moreover, the solution to the global equation restricted to \mathcal{K}_m also lies in span $\tilde{\mathsf{G}}_m^{\varepsilon}$.

Due to the diffusion limit, as discussed in subsection 2.1, the Green's matrix $\widetilde{G}_m^{\varepsilon}$ is approximately low-rank and can be compressed through random sampling.² As in subsection 3.3, we solve the following system with randomized boundary conditions to obtain each basis function $\tilde{r}_{m,i}$:

(4.10)
$$\begin{cases} v\partial_x \widetilde{r}_{m,i} - \frac{1}{\varepsilon} \mathsf{S}[\widetilde{r}_{m,i}] = 0, \quad (x,v) \in \widetilde{\mathcal{K}}_m, \\ \widetilde{r}_{m,i}|_{\widetilde{\Gamma}_{m,-}} = \omega_{m,i}, \end{cases}$$

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²We do not directly approximate G_m^{ε} , which is not low-rank due to the singularity near $\partial \mathcal{K}_m$ caused by the incoming Dirichlet boundary condition at $\Gamma_{m,-}$. For $\widetilde{G}_m^{\varepsilon}$, because of the presence of the buffer, this singularity does not appear in \mathcal{K}_m , causing $\widetilde{G}_m^{\varepsilon}$ to be approximately low-rank. The use of a buffer is similar to the oversampling approach in the multiscale finite element method [37].

where $\omega_{m,i}$ takes i.i.d. standard Gaussian at all grid points on the boundary $\Gamma_{m,-}$ and *i* is the index of random samples corresponds to different realizations of the boundary data. We then take restrictions $r_{m,i} = \tilde{r}_{m,i}|_{\mathcal{K}_m}$ and assemble them into local reduced Green's matrix:

$$\mathsf{G}_m^{\varepsilon,r} = [r_{m,1}, \ldots, r_{m,k_m}] = \widetilde{\mathsf{G}}_m^{\varepsilon} [\omega_{m,1}, \ldots, \omega_{m,k_m}] \; .$$

According to Theorem 3.5, we have with high probability that

$$\frac{\|\mathsf{G}_m^\varepsilon-\mathsf{P}_{\widetilde{\mathsf{G}}_m^{\varepsilon,r}}(\mathsf{G}_m^\varepsilon)\|}{\|\widetilde{\mathsf{G}}_m^\varepsilon\|}\ll 1.$$

Because of the approximate low-rank property, we can take $k_m \ll n_m$, thus reducing significantly the dimension of the local solution space (and also the dimension of the global linear system in the online step). For m = 1 and m = M (for which the patch \mathcal{K}_m is at the boundary of full domain), we use the full basis matrix $\mathbf{G}_m^{\varepsilon,r} = \mathbf{G}_m^{\varepsilon}$, so that we can capture the boundary conditions that are imposed on the full domain.

In the online step, we write the solution as

(4.11)
$$u^{\varepsilon} = \sum_{m} u_{m}^{\varepsilon} \approx \sum_{m} \sum_{i} \tilde{c}_{m,i} r_{m,i}$$

with $\{\tilde{c}_{m,i}, m = 1, 2, \ldots, M, i = 1, 2, \ldots, k_m\}$ being the coefficients for the reduced basis. We denote by \widetilde{M}_m and \widetilde{W}_m the matrix that maps \tilde{c}_m to outflow data $\sum_i \tilde{c}_{m,i} r_{m,i}(\Gamma_{m,+})$ and inflow data $\sum_i \tilde{c}_{m,i} r_{m,i}(\Gamma_{m,-})$ respectively. Note that the analogous W would become identity in the full basis approach. By imposing the continuity condition and exterior boundary condition, we obtain

$$\begin{bmatrix} \mathsf{I}_m^r \widetilde{\mathsf{M}}_m & -\mathsf{I}_{m+1}^l \widetilde{\mathsf{W}}_{m+1} \\ -\mathsf{I}_m^r \widetilde{\mathsf{W}}_m & \mathsf{I}_{m+1}^l \widetilde{\mathsf{M}}_{m+1} \end{bmatrix} \begin{bmatrix} \widetilde{c}_m \\ \widetilde{c}_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathsf{I}_1^l \widetilde{\mathsf{W}}_1 & 0 \\ 0 & \mathsf{I}_M^r \widetilde{\mathsf{W}}_M \end{bmatrix} \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_M \end{bmatrix} = \begin{bmatrix} \mathsf{I}_1^l \phi \\ \mathsf{I}_M^r \phi \end{bmatrix}$$

Assembling these equations, we obtain

$$(4.12) \qquad \qquad \widetilde{\mathsf{P}}\widetilde{c} = d$$

where

$$\widetilde{\mathsf{P}} = \begin{bmatrix} \mathsf{I}_{1}^{l} \widetilde{\mathsf{W}}_{1} & 0 & 0 & \dots & 0\\ \mathsf{I}_{1}^{r} \widetilde{\mathsf{M}}_{1} & -\mathsf{I}_{2}^{l} \widetilde{\mathsf{W}}_{2} & 0 & \dots & 0\\ -\mathsf{I}_{1}^{r} \widetilde{\mathsf{W}}_{1} & \mathsf{I}_{2}^{l} \widetilde{\mathsf{M}}_{2} & 0 & \dots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & \mathsf{I}_{M-1}^{r} \widetilde{\mathsf{M}}_{M-1} & -\mathsf{I}_{M}^{l} \widetilde{\mathsf{W}}_{M}\\ 0 & \dots & 0 & 0 & \mathsf{I}_{M-1}^{r} \widetilde{\mathsf{W}}_{M-1} & \mathsf{I}_{M}^{l} \widetilde{\mathsf{M}}_{M}\\ 0 & \dots & 0 & 0 & \mathsf{I}_{M}^{r} \widetilde{\mathsf{W}}_{M} \end{bmatrix}, \quad \widetilde{c} = \begin{bmatrix} \widetilde{c}_{1} \\ \widetilde{c}_{2} \\ \vdots \\ \widetilde{c}_{M} \end{bmatrix}, \quad d = \begin{bmatrix} \mathsf{I}_{1}^{l} \phi \\ 0 \\ \vdots \\ 0 \\ \mathsf{I}_{M}^{r} \phi \end{bmatrix}$$

Since we are working in an approximate local solution space due to the random sampling, this global linear system constraint is overdetermined and cannot be solved exactly in general. Instead, we use the least-squares solution defined by

(4.13)
$$\tilde{c} = \arg\min_{e} \|\widetilde{\mathsf{P}}e - d\|_2 \quad \Rightarrow \quad \tilde{c} = (\widetilde{\mathsf{P}}^{\top}\widetilde{\mathsf{P}})^{-1}\widetilde{\mathsf{P}}^{\top}d.$$

Remark 4.1. The matrix $\tilde{\mathsf{P}}$ is of size $M_p \times N_p$ where $M_p = \sum_{m=1}^M n_m$ and $N_p = \sum_{m=1}^M k_m$. The typical time complexity for this linear regression problem is of order $\mathcal{O}\left(N_p^2(M_P + N_p)\right)$ whereas for the full basis approach (4.9), the matrix P is of size M_p by M_p and time complexity is $\mathcal{O}(M_p^3)$. Because $N_p \ll M_p$, the computation cost of our approach is considerably lower.

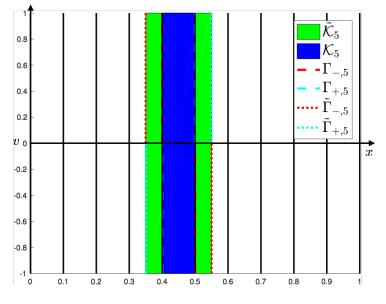


FIG. 4. Buffered domain decomposition

4.3. Numerical test. We set g = 1/2 in (4.2), and decompose the domain as in (4.3) with M = 10. In the velocity domain, we use the grid points $v_j = -1 + \frac{j}{N_v}$ with $N_v = 120$ so that the mesh size in the velocity domain is $\Delta v = \frac{1}{60}$. We define the buffered patches $\tilde{\mathcal{K}}_m$ to be twice as large as the original patches \mathcal{K}_m , with equal margins on each side. When solving the local problems, we use spatial discretization with fine mesh size $\Delta x = 0.01$. The setup is shown in Figure 4.

4.3.1. Local test. In Figure 5, we show the *normalized* singular values (that is the ratio σ_j/σ_1 for j = 1, 2, ...) of Green's matrix G_2^{ε} and $\tilde{G}_2^{\varepsilon}$ for the second local patch and the buffered patch, with Knudsen number $\varepsilon = 2^{-6}$. Note that singular values enjoy fast decay when ε is small and that the use of a buffer induces faster decay. In Figure 6, we plot a measure of relative error for different values of k_m and ε . The quantity plotted is defined by

$$\operatorname{error} = \frac{\|\widetilde{\mathsf{G}}_2^\varepsilon - \mathsf{Q}\mathsf{Q}^\top \widetilde{\mathsf{G}}_2^\varepsilon\|_2}{\|\widetilde{\mathsf{G}}_2^\varepsilon\|_2}\,,\quad \text{with}\quad \mathsf{G}_2^{\varepsilon,r} = \mathsf{Q}\mathsf{R}\,,$$

that is, Q is obtained from a QR decomposition of $\mathsf{G}_2^{\varepsilon,r}$, for which the number of columns increases as k_m increases. As k_m increases, the range of $\mathsf{G}_2^{\varepsilon,r}$ captures the range of $\widetilde{\mathsf{G}}_2^{\varepsilon}$ more and more accurately, and that the approximation is satisfactory only for small values of ε .

In Figure 7, we construct random local solution space span{ $\mathsf{G}_2^{\varepsilon,r}$ } with $k_2 = 50$ and show how well this random solution space can capture the first 3 left singular modes of $\mathsf{G}_2^{\varepsilon}$ with $\varepsilon = 2^{-6}$.

4.3.2. Global test. In the global test, we consider solving (4.1) with boundary data

$$\phi(v) = \begin{cases} 3 + \sin(2\pi v), & (x = 0, v > 0) \\ 2 + \sin(2\pi v), & (x = 1, v < 0), \end{cases}$$

and compare the numerical solutions of the full-basis and randomized reduced-basis approaches. Figure 8 shows three solutions: reference solution, the solution obtained from the reduced basis with $k_m = 10$, and the solution obtained from the reduced basis with $k_m = 50$. Results are given for $\varepsilon = 2^0$ and $\varepsilon = 2^{-6}$. We see that the information contained in $n_m = 120$ bases is largely captured by the random bases with $k_m = 10$ (for all m) when $\varepsilon = 2^{-6}$, at considerably lower computational cost. The quantitative error-decay as a function of k_m is plotted in Figure 9.

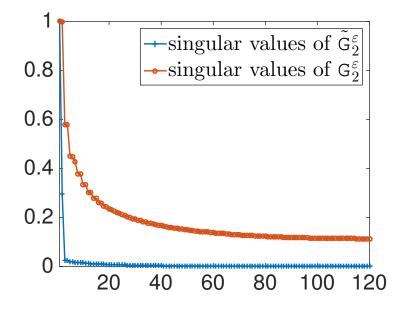


FIG. 5. A comparison of normalized singular values of G_2^{ε} and $\widetilde{G}_2^{\varepsilon}$ when $\varepsilon = 2^{-6}$. Use of a buffer zone ensures that Green's matrix enjoys faster decay in its singular values.

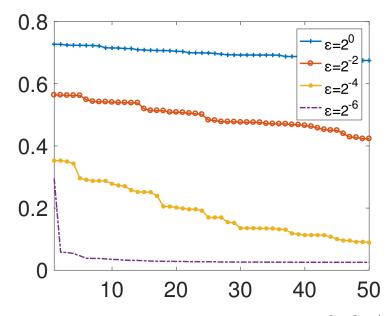


FIG. 6. Effectiveness of the random sampling for different Knudsen number: $\varepsilon = 2^0, 2^{-2}, 2^{-4}, 2^{-6}$ on buffered domain $\tilde{\mathcal{K}}_2$. For each ε , the approximate range captured by $G_2^{\varepsilon,r}$ improves as the number of random modes k_m increases. Much better approximations are obtained for smaller ε than for larger values.

5. Example 2: elliptic equation with highly oscillatory media. We now consider elliptic equations with oscillatory media on the domain $\mathcal{K} = [0,1]^2$ with Dirichlet boundary conditions. The problem is

(5.1)
$$\nabla_x \cdot \left(a\left(x, \frac{x}{\varepsilon}\right) \nabla_x u^{\varepsilon} \right) = 0, \quad \text{in } \mathcal{K} = [0, 1]^2,$$

(5.2)
$$u^{\varepsilon} = \phi(x), \quad \text{on } \Gamma = \partial \mathcal{K},$$

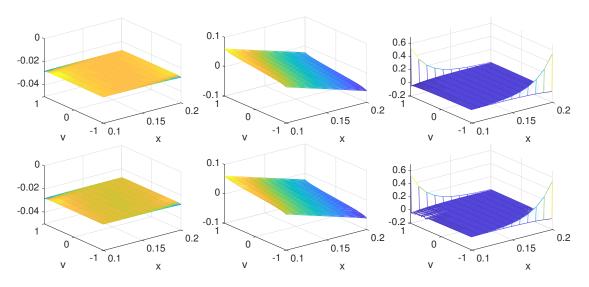


FIG. 7. For small Knudsen number $\varepsilon = 2^{-6}$, the first row shows the first three singular vectors of $\mathsf{G}_2^{\varepsilon}$ and the second row shows their projection into $\operatorname{span}\{\mathsf{G}_2^{\varepsilon,r}\}$ with $k_2 = 50$.

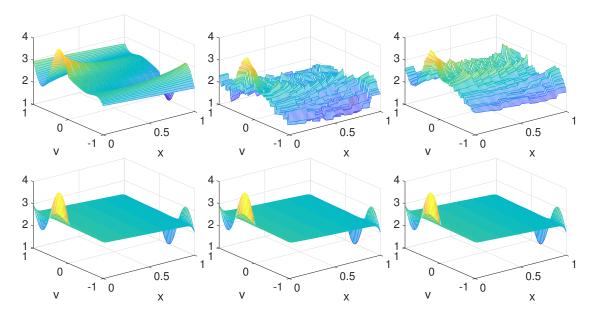


FIG. 8. The two rows of plots are for $\varepsilon = 1$ and $\varepsilon = 2^{-6}$ respectively. The leftmost column show a reference solution obtained with fine grids. The middle column and the rightmost column are solutions obtained from randomized reduced bases with $k_m = 10$ and $k_m = 50$ (for all m), respectively.

where the coefficient field $a = a(x, x/\varepsilon)$ is oscillatory because of its explicit dependence on the fast variable x/ε . (ε indicates the scale of oscillation in the coefficient field.)

We solve (5.1) on a coarse mesh $\{(x_{m_1}, y_{m_2}) \mid x_{m_1} = m_1 H, y_{m_2} = m_2 H\}$ with H = 1/M. The coarse mesh size H is chosen independent of the small parameter ε . The domain \mathcal{K} is decomposed into patches defined by

(5.3)
$$\mathcal{K} = \bigcup_{m} \mathcal{K}_{m}, \quad \text{with} \quad \mathcal{K}_{m} = [x_{m_{1}-1}, x_{m_{1}}] \times [y_{m_{2}-1}, y_{m_{2}}],$$

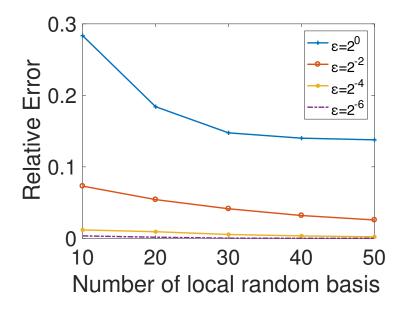


FIG. 9. The global error as a function of k_m . As k_m , the number of random modes per patch increases, the relative error decreases. For fixed number of random modes, the relative error is better for small ε .

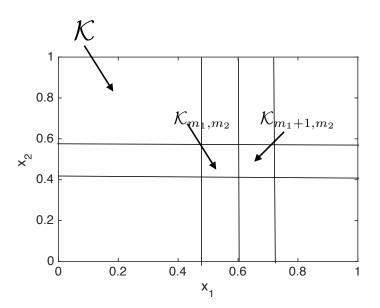


FIG. 10. Domain \mathcal{K} is decomposed into patches, each defined by a multi-index $m = (m_1, m_2)$.

where $m = (m_1, m_2)$ is a multi-index. Two patches \mathcal{K}_m and \mathcal{K}_n share boundaries if they are adjacent, and we define the shared edge as follows:

$$L_{mn} = \mathcal{K}_m \cap \mathcal{K}_n.$$

Thus L_{mn} is nontrivial only if $(m_1, m_2) = (n_1 \pm 1, n_2)$ or $(m_1, m_2) = (n_1, n_2 \pm 1)$; see Figure 10. Note too that $L_{mn} = L_{nm}$.

5.1. Full basis approach.

Offline. In the full-basis scheme, we prepare the local solution space functions $b_{m,i}$ in the offline step by solving (5.1) in every patch \mathcal{K}_m with boundary conditions that are non-vanishing at one just grid point *i* on the boundary of the patch:

(5.4)
$$\begin{cases} \nabla_x \cdot \left(a\left(x, \frac{x}{\varepsilon}\right) \nabla_x b_{m,i}\right) = 0, & x \in \mathcal{K}_m, \\ b_{m,i} = \delta_i, & x \in \Gamma_m = \partial \mathcal{K}_m, \end{cases}$$

where $\delta_i = 1$ at the *i*-th boundary grid point of Γ_m and is zero at all other grid points in Γ_m . The solutions $\{b_{m,i}, i = 1, \ldots, n_m\}$ span the space of local solutions with all possible boundary conditions, and we assemble them into the local Green's matrix for \mathcal{K}_m :

(5.5)
$$\mathbf{G}_m^{\varepsilon} = [b_{m,1}, \dots, b_{m,n_m}] \; .$$

Note that to compute the basis functions, we use fine discretization with meshsize $h \ll \varepsilon$, which leads to $n_m = \mathcal{O}(1/\varepsilon)$. (Details of the fine mesh are discussed in subsection 5.3.)

Online. We write the global solution as a linear combination of all local basis functions, with coefficients $c_{m,i}$:

(5.6)
$$u^{\varepsilon} = \sum_{m} u_{m}^{\varepsilon} = \sum_{m} \sum_{i=1}^{n_{m}} c_{m,i} b_{m,i}$$

The coefficients are determined by enforcing the following constraints:

- * Continuity across edges L_{mn} : $u_m(L_{mn}) = u_n(L_{mn})$ and $a\partial_n u_m(L_{mn}) = -a\partial_n u_n(L_{mn})$ if $L_{mn} \neq 0$
- \emptyset , where ∂_n denotes the outer normal derivative on the boundary;
- * Boundary condition on $\partial \mathcal{K}$: $u|_{\partial \mathcal{K}} = \phi$.

Denote by $M_{m,n}$ the matrix that maps c_m to $u_m(L_{mn})$, and by $W_{m,n}$ the matrix that maps c_m to $a\partial_n u_n(L_{mn})$, that is,

$$\mathsf{M}_{m,n}c_m = u_m(L_{mn}), \quad \mathsf{W}_{m,n}c_m = a\partial_n u_m(L_{mn})$$

(Note that $M_{m,n}$ is a submatrix of G_m^{ε} .) From the continuity condition, we have

(5.7)
$$\begin{cases} \mathsf{M}_{m,n}c_m - \mathsf{M}_{n,m}c_n = 0, & x \in L_{mn} \\ \mathsf{W}_{m,n}c_m + \mathsf{W}_{n,m}c_n = 0, & x \in L_{mn}. \end{cases}$$

Similarly, we define by $M_{m,ext}$ the matrix that maps c_m to the intersection of $\partial \mathcal{K}_m$ with $\partial \mathcal{K}$. From the boundary condition, we have

(5.8)
$$\mathsf{M}_{m,\mathrm{ext}}c_m = \phi, \quad x \in \partial \mathcal{K}_m \cap \partial \mathcal{K}.$$

By assembling the conditions (5.7) and (5.8) for all m and n, and solving for the coefficients $\{c_{m,i}, m = 1, \ldots, M, i = 1, \ldots, n_m\}$, we obtain u^{ε} from (5.6). The linear system has the form

$$(5.9) Pc = d,$$

with $d = [\phi, 0]$, $c = [c_{m,i}]$ and P is formed by the collection of $M_{m,n}$, $W_{m,n}$, and $M_{m,ext}$.

5.2. Reduced basis approach. As in subsection 4.2, we define buffered patches \mathcal{K}_m such that $\mathcal{K}_m \subset \subset \widetilde{\mathcal{K}}_m$, and solve a local problem on each buffered patch. When we restrict the local solutions to \mathcal{K}_m , we find that (as before) these solutions lie approximately in a lower-dimensional space. Similarly to Equation (5.4), we define the local problems as follows:

(5.10)
$$\begin{cases} \nabla_x \cdot \left(a(x, \frac{x}{\varepsilon}) \nabla_x \widetilde{b}_{m,i} \right) = 0, \quad x \in \widetilde{\mathcal{K}}_m, \\ \widetilde{b}_{m,i}|_{\partial \widetilde{\mathcal{K}}_m} = \delta_i, \end{cases}$$

then define the local solution space via the following Green's matrix:

(5.11)
$$\widetilde{\mathsf{G}}_{m}^{\varepsilon} = \left[\widetilde{b}_{m,1}|_{\mathcal{K}_{m}}, \dots, \widetilde{b}_{m,\widetilde{n}_{m}}|_{\mathcal{K}_{m}}\right]$$

Since span $\widetilde{G}_m^{\varepsilon}$ contains all local solutions, we seek a good approximation to span $\widetilde{G}_m^{\varepsilon}$ for the interior cells during the offline stage. As shown in [12], and similarly to subsection 4.2, the matrix $\widetilde{G}_m^{\varepsilon}$ is low rank and can be compressed through random sampling. We solve (5.10) with the boundary condition δ_i replaced by a function w_i which takes on random values (specifically, i.i.d. normal random variables) at the grid points of the boundary $\partial \widetilde{\mathcal{K}}_m$, that is,

(5.12)
$$\begin{cases} \nabla_x \cdot \left(a(x, \frac{x}{\varepsilon}) \nabla_x \widetilde{r}_{m,i} \right) = 0, \quad x \in \widetilde{\mathcal{K}}_m, \\ \widetilde{r}_{m,i}|_{\partial \widetilde{\mathcal{K}}_m} = \omega_i. \end{cases}$$

We do this for k_m choices of random boundary function w_i and assemble the local reduced Green's matrix from the restricted solutions $r_{m,i} = \tilde{r}_{m,i}|_{\mathcal{K}_m}$, $i = 1, 2, \ldots, k_m$:

$$\mathbf{G}_{m}^{\varepsilon,r} = [r_{m,1}, \ldots, r_{m,k_{m}}] = \widetilde{\mathbf{G}}_{m}^{\varepsilon} [\omega_{m,1}, \ldots, \omega_{m,k_{m}}] \Big|_{\mathcal{K}_{m}}.$$

As done in the full basis approach, the coefficients are determined in the online step, namely, we express the solution as

$$u^{\varepsilon} = \sum_{m} u_{m}^{\varepsilon} \approx \sum_{m} \sum_{i=1}^{k_{m}} \tilde{c}_{m,i} r_{m,i}$$

and determine the coefficients $\tilde{c}_{m,i}$ by imposing the continuity conditions in the interior boundaries and and boundary conditions on the exterior boundary.

Similar to the full basis approach, denote $\widetilde{\mathsf{M}}_{m,n}$ and $\widetilde{\mathsf{W}}_{m,n}$ the matrices that map \tilde{c}_m to $u_m(L_{mn})$ and $a\partial_n u_n(L_{mn})$ respectively, that is,

$$\widetilde{\mathsf{M}}_{m,n}\widetilde{c}_m = u_m(L_{mn}), \quad \widetilde{\mathsf{W}}_{m,n}\widetilde{c}_m = a\partial_n u_m(L_{mn}).$$

By imposing the continuity condition and the exterior boundary condition, we obtain

(5.13)
$$\begin{cases} \widetilde{\mathsf{M}}_{m,n}\widetilde{c}_m - \widetilde{\mathsf{M}}_{n,m}\widetilde{c}_n = 0, & x \in L_{mn} \\ \widetilde{\mathsf{W}}_{m,n}\widetilde{c}_m + \widetilde{\mathsf{W}}_{n,m}\widetilde{c}_n = 0, & x \in L_{mn} \end{cases}, \text{ and } \widetilde{\mathsf{M}}_{m,\text{ext}}c_m = \phi, & x \in \partial\mathcal{K}_m \cap \partial\mathcal{K}. \end{cases}$$

Assembling the equations, we obtain:

$$\mathsf{P}\tilde{c}=d$$

However, since the number of coefficients in the reduced basis approach is significantly smaller than that in the full basis approach $(k_m \ll n_m \text{ in every patch } \mathcal{K}_m)$, while the number of continuity condition and the boundary condition is not changed, the system is overdetermined. We thus consider the leastsquares solution, that is:

(5.14)
$$\tilde{c} = \operatorname{argmin}_{e} \| \widetilde{\mathsf{P}}e - d \|_{2} \quad \Rightarrow \quad \tilde{c} = (\widetilde{\mathsf{P}}^{\top} \widetilde{\mathsf{P}})^{-1} \widetilde{\mathsf{P}}^{\top} d \,.$$

Alternatively, we could enforce the boundary conditions exactly and relax only the continuity condition, as in the following constrained least-squares formulation: such that:

$$\min_{c} \sum_{m,n} \|\widetilde{\mathsf{M}}_{m,n}c_m - \widetilde{\mathsf{M}}_{n,m}c_n\|_{2,mn}^2 + \|\widetilde{\mathsf{W}}_{m,n}c_m + \widetilde{\mathsf{W}}_{n,m}c_n\|_{2,mn}^2, \quad \text{such that} \quad \widetilde{\mathsf{M}}_{m,\text{ext}}c_m = \phi.$$

Here $\|\cdot\|_{2,mn}$ denotes L_2 norm confined on $x \in L_{mn}$. If we assume a uniform mesh with n_m constant for all patches, then matrix $\tilde{\mathsf{P}}$ is of size $M_p \times N_p$ where $M_p = \frac{1}{2}\sqrt{M}(\sqrt{M}+1)n_m$ and $N_p = \sum_{m=1}^M k_m$. Similar to the case of RTE, the typical time complexity for linear regression problem (5.14) is of order $\mathcal{O}(N_p^2(M_p + N_p))$. Numerically, we obtain satisfactory results from (5.14), which we present in the next subsection.

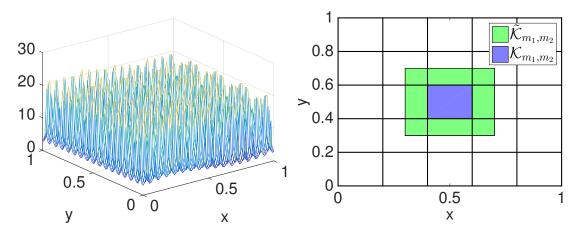


FIG. 11. Left: Media used in elliptic equation (5.1). Right: illustration of buffered domain decomposition

5.3. Numerical test. We set the domain to be $\mathcal{K} = [0,1]^2$ and define the media as follows, for $x = (x_1, x_2) \in \mathcal{K}$:

$$a\left(x,\frac{x}{\varepsilon}\right) = 2 + \sin(2\pi x_1)\cos(2\pi x_2) + \frac{2 + 1.8\sin(\frac{2\pi x_1}{\varepsilon})}{2 + 1.8\cos(\frac{2\pi x_2}{\varepsilon})} + \frac{2 + \sin(\frac{2\pi x_2}{\varepsilon})}{2 + 1.8\cos(\frac{2\pi x_1}{\varepsilon})}$$

For the domain decomposition we set M = 5 (for a total of 25 patches), and each local patch is further divided into a 20 by 20 fine mesh so that the mesh parameter h = 0.01 can resolve the smallest scales $\varepsilon = 2^{-4}$. A complete basis on each patch is formed from $n_m = 80$ basis functions. These functions are computed from a standard finite element P_1 method with bilinear nodal basis. The buffered patch $\tilde{\mathcal{K}}_m$ is set to be a square concentric with \mathcal{K}_m but with all sides twice as long. Figure 11 illustrates the setup, for $\varepsilon = 2^{-4}$.

5.3.1. Local test. In Figure 12 we show the rank of the Green's matrices $\mathsf{G}_{2,2}^{\varepsilon}$ and $\mathsf{G}_{2,2}^{\varepsilon}$ (defined by (5.5) and (5.11), respectively) for the (2,2) patch, with $\varepsilon = 2^{-4}$. Use of buffers yields rapid decays in the singular values of $\widetilde{\mathsf{G}}_{2,2}^{\varepsilon}$. We then define the relative error between $\mathsf{G}_{2,2}^{\varepsilon,r}$ and $\mathsf{G}_{2,2}^{\varepsilon}$ as follows:

$$\operatorname{error} = \frac{\|\widetilde{\mathsf{G}}_{2,2}^{\varepsilon} - \mathsf{Q}\mathsf{Q}^{\top}\widetilde{\mathsf{G}}_{2,2}^{\varepsilon}\|_{2}}{\|\widetilde{\mathsf{G}}_{2,2}^{\varepsilon}\|_{2}}, \quad \text{with} \quad \mathsf{G}_{2,2}^{\varepsilon,r} = \mathsf{Q}\mathsf{R},$$

where Q is obtained from QR decomposition of $\mathsf{G}_{2,2}^{\varepsilon,r}$. We see in Figure 12 that the relative error decays exponentially fast as k_m increases. In Figure 13, we plot the first three left singular vectors of $\widetilde{\mathsf{G}}_{2,2}^{\varepsilon,r}$ and their projections onto $\operatorname{span}\{\mathsf{G}_{2,2}^{\varepsilon,r}\}$ with $k_{2,2} = 6$. This plot shows that, visually, $\operatorname{span}\{\mathsf{G}_{2,2}^{\varepsilon,r}\}$ captures well the leading singular vectors of the full-basis Green's matrix.

5.3.2. Global test. In the global test, the boundary condition is the sine function over the boundary $\partial \mathcal{K}$. Equation (5.9) is computed with G_m^{ε} for the reference solution u_{ref} , and (5.14) is computed for the approximate solution u_{approx} . Figure 14 shows the reference solution u_{ref} along with the approximated solutions u_{approx} obtained using $k_m = 10$ and $k_m = 50$, respectively. The decay in relative error

relative error =
$$\frac{\|u_{\text{ref}} - u_{\text{approx}}\|_2}{\|u_{\text{ref}}\|_2}$$

as a function of k_m is plotted in Figure 15.

5.4. Comparison with MsFEM and GMsFEM. A number of successful existing numerical homogenization methods share with our proposed method the property that that optimal basis functions

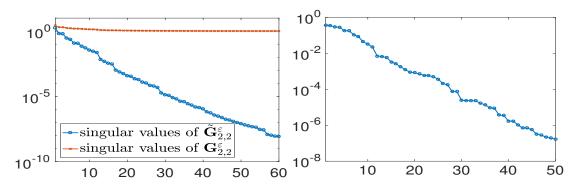


FIG. 12. The plot on the left shows the singular values of $G_{2,2}^{\varepsilon}$ (from (5.5)) and $\tilde{G}_{2,2}^{\varepsilon}$ (from (5.11)), with $\varepsilon = 2^{-4}$. Use of the buffer zone in the calculation of $\tilde{G}_{2,2}^{\varepsilon}$ causes fast decay of singular values, making this matrix approximately low-rank. The plot on the right panel shows relative error between $G_{2,2}^{\varepsilon,r}$ and $G_{2,2}^{\varepsilon}$ as we increase the number of random modes k_m from 1 to 50.

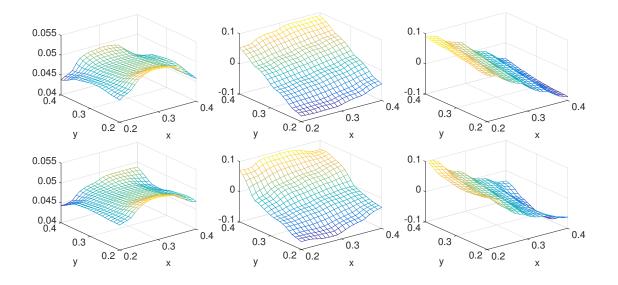


FIG. 13. The first row shows the first three singular vectors of $\tilde{G}_{2,2}^{\varepsilon,r}$ and the second row shows projection of them onto span $\{G_{2,2}^{\varepsilon,r}\}$ with $k_{2,2} = 6$. Visually, six random sampled basis are enough to capture the leading modes from the full-basis Green's matrix.

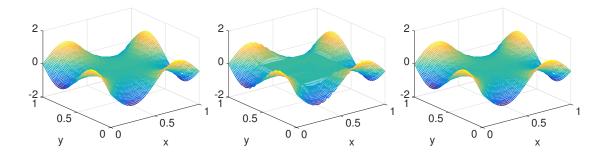


FIG. 14. Computed solutions. Left panel shows the reference solution obtained with fine grids. Middle panel and right panel show solutions obtained from (4.10), (4.11), (4.12) with $k_m = 10$ and $k_m = 50$ (for all m), respectively.

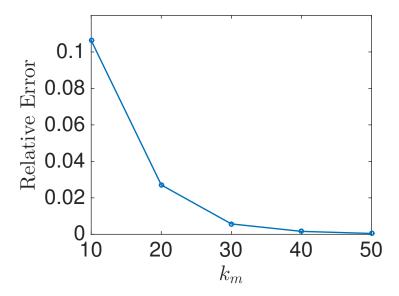


FIG. 15. The global error as a function of number of random nodes per patch k_m . Note the rapid decay of error as k_m increases.

are constructed offline. MsFEM (Multiscale Finite Element Method) [37] and GMsFEM (Generalized MsFEM) [23] have been used with success in many examples and with excellent numerical performance. MsFEM builds four basis functions by solving the local equation for *a*-harmonic functions that set 1 at the four nodal points, while GMsFEM, prepares a full list of Green's functions over the subdomain and select the optimal ones according to a carefully designed spectral criterion (that translates into a generalized eigenvalue problem). On the theoretical level, MsFEM has been shown to have good convergence (see [38] for periodic media), and the theory for GMsFEM can be found in [23]. In this subsection we compare our methods with these two approaches, for a challenging example in which the media contains both multiscale structures and high contrasts:

$$a = 1 + 1000 \mathbf{1}_{S}(x, y), \quad S = \{(x, y) \in [0, 1]^{2} : (x \cos(100\sqrt{(x - 0.5)^{2} + (y - 0.5)^{2}})) \le y - 0.5\}.$$

We plot the media in Figure 16, noting that our comparison is imperfect because the analytical result for MsFEM assumes periodicity. Upon dividing the domain into fine mesh with $h = \frac{1}{100}$ and coarse mesh with $H = \frac{1}{5}$, we investigate the behavior of three different methods on the subdomain $\mathcal{K}_{2,2}$. We compute the reference optimal basis function by first looping over the boundary to build the entire Green's function list, then performing SVD. Figure 17 shows that the random sampling method (using merely 6 samples) can quickly capture the three leading basis functions and gives a higher accuracy, in comparison with MsFEM. In Table 1, we report the CPU time needed for the three methods (MsFEM, GMsFEM and random sampling) vs the reference solution computed directly from performing SVD, and report the relative error in capturing the first three basis functions. Here the relative error is defined by:

(5.15)
$$\operatorname{Error} = \frac{\|(\mathbf{I} - \mathbf{Q}_k \mathbf{Q}_k^{\top})\mathbf{U}_3\|_2}{\|\mathbf{U}_3\|_2}, \quad e_i = \frac{\|(\mathbf{I} - \mathbf{Q}_k \mathbf{Q}_k^{\top})\mathbf{u}_i\|_2}{\|\mathbf{u}_i\|_2}$$

where Q_k collects the orthonormal first k-basis constructed via different methods and $U_n = [u_1, u_2, \ldots, u_n]$ collects the first *n* optimal basis functions u_i . It is clear that GMsFEM is rather expensive while MsFEM is the cheapest of the three approaches. In terms of the error, random sampling strategies performs much better than MsFEM and similar to GMsFEM. We note that GMsFEM selects basis functions according to a spectral method reflected via a generalized eigenvalue problem. Since it has a different definition for "optimality", the comparison is not truly fair.

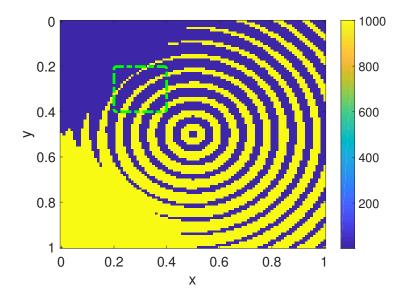


FIG. 16. High contrast media with yellow part indicating $a\left(x, \frac{x}{\varepsilon}\right) = 1000$ and blue part indicating $a\left(x, \frac{x}{\varepsilon}\right) = 1$. The green box shows local patch $\mathcal{K}_{2,2}$.

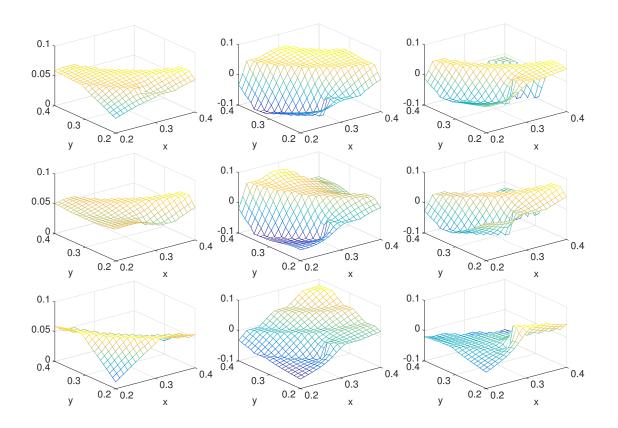


FIG. 17. Optimal basis functions and their projections onto the approximate spaces. First row plots the first three singular vectors of $\tilde{G}_{2,2}$. Second row plots their projection onto the space spanned by the random sampled basis with $k_{2,2} = 6$. Third row shows projection onto MsFEM space. Random sampled basis provide much better accuracy.

	SVD (ref.)	MsFEM	GMsFEM			Random sampling
	5 V D (161.)		snapshots	ensemble	spectral	Random sampling
CPU Time (s)	6.6569	0.1663	7.1168	0.2068	0.0051	0.3164
e_1		0.2043	0.0867			0.1108
e_2		0.5930	0.1236			0.1101
e_3		0.7581	0.0451			0.0567
Error		0.8206		0.1557		0.1289

TABLE 1

CPU time and error comparison of the methods MsFEM, GMsFEM and random sampling (proposed method). Error is defined in (5.15).

References.

- J. AARNES AND Y. EFENDIEV, An adaptive multiscale method for simulation of fluid flow in heterogeneous porous media, Multiscale Modeling & Simulation, 5 (2006), pp. 918–939.
- [2] A. ABDULLE AND Y. BAI, Reduced basis finite element heterogeneous multiscale method for highorder discretizations of elliptic homogenization problems, Journal of Computational Physics, 231 (2012), pp. 7014 – 7036.
- [3] A. ABDULLE, Y. BAI, AND G. VILMART, Reduced basis finite element heterogeneous multiscale method for quasilinear elliptic homogenization problems, Discrete and Continuous Dynamical Systems - Series S, 8 (2015), pp. 91–118.
- [4] A. ABDULLE AND C. SCHWAB, Heterogeneous multiscale fem for diffusion problems on rough surfaces, Multiscale Modeling & Simulation, 3 (2005), pp. 195–220.
- [5] G. ALLAIRE, Homogenization and two-scale convergence, SIAM Journal on Mathematical Analysis, 23 (1992), pp. 1482–1518.
- [6] Y. AZMY, Unconditionally stable and robust adjacent-cell diffusive preconditioning of weighteddifference particle transport methods is impossible, J. Comput. Phys., 182 (2002), p. 213.
- [7] I. BABUŠKA AND R. LIPTON, Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, Multiscale Model. Simul., 9 (2011), pp. 373–406.
- [8] I. BABUŠKA, R. LIPTON, AND M. STUEBNER, The penetraion function and its application to microscale problems, BIT Numer. Math., 48 (2008), pp. 167–187.
- [9] I. BABUŠKA AND J. MELENK, The partition of unity method, International Journal for Numerical Methods in Engineering, 40 (1997), pp. 727–758.
- [10] I. BABUŠKA AND J. E. OSBORN, Can a finite element method perform arbitrarily badly?, Mathematics of Computation, 69 (2000), pp. 443–462.
- [11] C. BARDOS, S. SANTOS, AND R. SENTIS, Diffusion approximation and computation of the critical size, Trans. Amer. Math. Soc., 284 (1984), pp. 617–649.
- [12] M. BEBENDORF AND W. HACKBUSCH, Existence of H-matrix approximants to the inverse FE-matrix of elliptic operators with l[∞]-coefficients, Numerische Mathematik, 95 (2003), pp. 1–28.
- [13] M. BENNOUNE, M. LEMOU, AND L. MIEUSSENS, Uniformly stable numerical schemes for the Boltzmann equation preserving the compressible Navier-Stokes asymptotics, J. Comput. Phys., 227 (2008), pp. 3781–3803.
- [14] A. BENSOUSSAN, J. LIONS, AND G. PAPANICOLAOU, Asymptotic Analysis for Periodic Structures, AMS Chelsea Publishing Series, American Mathematical Society, 2011.
- [15] A. BUHR AND K. SMETANA, Randomized local model order reduction, arXiv prepring arXiv: 1706.09179, (2017).
- [16] V. M. CALO, Y. EFENDIEV, J. GALVIS, AND G. LI, Randomized oversampling for generalized multiscale finite element methods, Multiscale Modeling & Simulation, 14 (2016), pp. 482–501.
- [17] K. CHEN, Q. LI, J. LU, AND S. WRIGHT, Randomized sampling for basis functions construction in generalized finite element methods, 2018. preprint, arXiv:1801.06938.

- [18] K. CHEN, Q. LI, J. LU, AND S. J. WRIGHT, A low-rank schwarz method for radiative transport equation with heterogeneous scattering coefficient, arXiv preprint arXiv:1906.02176, (2019).
- [19] P. DEGOND, Asymptotic-preserving schemes for fluid models of plasmas, in Numerical models for fusion, vol. 39/40 of Panor. Synthèses, Soc. Math. France, Paris, 2013, pp. 1–90.
- [20] G. DIMARCO AND L. PARESCHI, Exponential Runge-Kutta methods for stiff kinetic equations, SIAM J. Numer. Anal., 49 (2011), pp. 2057–2077.
- [21] W. E AND B. ENGQUIST, The heterogeneous multi-scale methods, Commun. Math. Sci., 1 (2003), pp. 87–133.
- [22] W. E, P. MING, AND P. ZHANG, Analysis of the heterogeneous multiscale method for elliptic homogenization problems, J. Amer. Math. Soc., 18 (2005), pp. 121–156.
- [23] Y. EFENDIEV, J. GALVIS, AND T. Y. HOU, Generalized multiscale finite element methods (gmsfem), Journal of Computational Physics, 251 (2013), pp. 116 – 135.
- [24] Y. EFENDIEV AND T. Y. HOU, Multiscale finite element methods: theory and applications, vol. 4, Springer Science & Business Media, 2009.
- [25] Y. EFENDIEV, T. Y. HOU, AND X.-H. WU, Convergence of a nonconforming multiscale finite element method, SIAM J. Numer. Anal., 37 (2000), pp. 888–910.
- [26] H. EGGER AND M. SCHLOTTBOM, An lp theory for stationary radiative transfer, Applicable Analysis, 93 (2014), pp. 1283–1296.
- [27] B. ENGQUIST AND Y.-H. TSAI, Heterogeneous multiscale methods for stiff ordinary differential equations, Mathematics of computation, 74 (2005), pp. 1707–1742.
- [28] F. FILBET AND S. JIN, A class of asymptotic-preserving schemes for kinetic equations and related problems with stiff sources, J. Comput. Phys., 229 (2010), pp. 7625–7648.
- [29] A. FRIEZE, R. KANNAN, AND S. VEMPALA, Fast Monte-Carlo algorithms for finding low-rank approximations, in FOCS 1998: 39th Annual Symposium on Foundations of Computer Science, 1998, pp. 370–378.
- [30] F. GOLSE AND A. KLAR, A numerical method for computing asymptotic states and outgoing distributions for kinetic linear half-space problems, J. Stat. Phys., 80 (1995), pp. 1033–1061.
- [31] S. GOREINOV, E. TYRTYSHNIKOV, AND N. ZAMARASHKIN, A theory of pseudoskeleton approximations, Linear Algebra Appl., 261 (1997), pp. 1–21.
- [32] L. GRASEDYCK, I. GREFF, AND S. SAUTER, The al basis for the solution of elliptic problems in heterogeneous media, Multiscale Modeling & Simulation, 10 (2012), pp. 245–258.
- [33] J.-L. GUERMOND AND G. KANSCHAT, Asymptotic analysis of upwind discontinuous galerkin approximation of the radiative transport equation in the diffusive limit, SIAM Journal on Numerical Analysis, 48 (2010), pp. 53–78.
- [34] W. HACKBUSCH, *Hierarchical Matrices: Algorithms and Analysis*, Springer Series in Computational Mathematics, Springer, 2015.
- [35] N. HALKO, P. G. MARTINSSON, AND J. A. TROPP, Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions, SIAM Review, 53 (2011), pp. 217–288.
- [36] T. HOU, X.-H. WU, AND Z. CAI, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, Mathematics of Computation of the American Mathematical Society, 68 (1999), pp. 913–943.
- [37] T. Y. HOU AND X.-H. WU, A multiscale finite element method for elliptic problems in composite materials and porous media, J. Comput. Phys., 134 (1997), pp. 169 – 189.
- [38] T. Y. HOU, X.-H. WU, AND Z. CAI, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, Math. Comp., 68 (1999), pp. 913–943.
- [39] J. HU, S. JIN, AND Q. LI, Asymptotic-Preserving schemes for multiscale hyperbolic and kinetic equations, in Handbook of Numerical Methods for Hyperbolic Problems, R. Abgrall and C.-W. Shu, eds., North Holland/Elsevier, to appear.
- [40] Y. HUANG, J. LU, AND P. MING, A concurrent global-local numerical method for multiscale PDEs,

J. Sci. Comput., 76 (2018), pp. 1188–1215.

- [41] M. F. HUTCHINSON, A stochastic estimator of the trace of the influence matrix for Laplacian smoothing splines, J. Comm. Stat. Simul., 19 (1990), pp. 433–450.
- [42] S. JIN, Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations, SIAM J. Sci. Comput., 21 (1999), pp. 441–454.
- [43] S. JIN AND D. LEVERMORE, Fully discrete numerical transfer in diffusive regimes, Transp. Theory Stat. Phys., 22 (1993), pp. 739–791.
- [44] S. JIN, L. PARESCHI, AND G. TOSCANI, Uniformly accurate diffusive relaxation schemes for multiscale transport equations, SIAM J. Num. Anal., 38 (2000), pp. 913–936.
- [45] R. KANNAN AND S. VEMPALA, Randomized algorithms in numerical linear algebra, Acta Numer., (2017), pp. 95–135.
- [46] C. E. KENIG, F. LIN, AND Z. SHEN, Convergence rates in l2 for elliptic homogenization problems, Archive for Rational Mechanics and Analysis, 203 (2012), pp. 1009–1036.
- [47] A. KLAR, An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit, SIAM J. Numer. Anal., 35 (1998), pp. 1097–1094.
- [48] R. KORNHUBER, D. PETERSEIM, AND H. YSERENTANT, An analysis of a class of variational multiscale methods based on subspace decomposition, Mathematics of Computation, 87 (2018), pp. 2765– 2774.
- [49] R. KORNHUBER AND H. YSERENTANT, Numerical homogenization of elliptic multiscale problems by subspace decomposition, Multiscale Modeling & Simulation, 14 (2016), pp. 1017–1036.
- [50] E. LARSEN AND J. KELLER, Asymptotic solution of neutron transport problems for small mean free paths, J. Math. Phys., 15 (1974), pp. 75–81.
- [51] E. LARSEN AND J. MOREL, Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. ii, J. Comput. Phys., 83 (1989), pp. 212–236.
- [52] M. LEMOU AND L. MIEUSSENS, New asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit, SIAM J. Sci. Comput., 31 (2008), pp. 334–368.
- [53] Q. LI, J. LU, AND W. SUN, Diffusion approximations and domain decomposition method of linear transport equations: Asymptotics and numerics, Journal of Computational Physics, 292 (2015), pp. 141 – 167.
- [54] Q. LI, J. LU, AND W. SUN, A convergent method for linear half-space kinetic equations, ESAIM: M2AN, 51 (2017), pp. 1583–1615.
- [55] Q. LI, J. LU, AND W. SUN, Validity and regularization of classical half-space equations, Journal of Statistical Physics, 166 (2017), pp. 398–433.
- [56] Q. LI AND L. PARESCHI, Exponential Runge-Kutta for the inhomogeneous Boltzmann equations with high order of accuracy, J. Comput. Phys., 259 (2014), pp. 402–420.
- [57] Q. LI AND L. WANG, Implicit asymptotic preserving method for linear transport equations, Communications in Computational Physics, 22 (2017), pp. 157–181.
- [58] E. LIBERTY, F. WOOLFE, P.-G. MARTINSSON, V. ROKHLIN, AND M. TYGERT, Randomized algorithms for the low-rank approximation of matrices, Proc. Natl. Acad. Sci. USA, 104 (2007), pp. 20167–20172.
- [59] L. LIN, J. LU, AND L. YING, Fast construction of hierarchical matrix representation from matrixvector multiplication, J. Comput. Phys., 230 (2011), pp. 4071–4087.
- [60] R. LIPTON, P. SINZ, AND M. STUEBNER, Uncertain loading and quantifying maximum energy concentration within composite structures, Journal of Computational Physics, 325 (2016), pp. 38–52.
- [61] M. MAHONEY, Randomized algorithms for matrices and data, Found. Trends Mach. Learning, 3 (2011), pp. 123–224.
- [62] A. MÅLQVIST AND D. PETERSEIM, Localization of elliptic multiscale problems, Math. Comp., 83 (2014), pp. 2583–2603.
- [63] P. G. MARTINSSON, A fast randomized algorithm for computing a hierarchically semiseparable representation of a matrix, SIAM Journal on Matrix Analysis and Applications, 32 (2011), pp. 1251–1274.

- [64] S. MOSKOW AND M. VOGELIUS, First-order corrections to the homogenised eigenvalues of a periodic composite medium. a convergence proof, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 127 (1997), pp. 1263 – 1299.
- [65] J. ODEN AND K. VEMAGANTI, Estimation of local modeling error and goal-oriented adaptive modeling of heterogeneous materials, I. error estimates and adaptive algorithms, J. Comput. Phys., 164 (2000), pp. 22–47.
- [66] H. OWHADI, Bayesian numerical homogenization, Multiscale Modeling & Simulation, 13 (2015), pp. 812–828.
- [67] H. OWHADI, Multigrid with rough coefficients and multiresolution operator decomposition from hierarchical information games, SIAM Review, 59 (2017), pp. 99–149.
- [68] H. OWHADI, L. ZHANG, AND L. BERLYAND, Polyharmonic homogenization, rough polyharmonic splines and sparse super-localization, M2AN Math. Model. Numer. Anal., 48 (2014), pp. 517–552.
- [69] G. PAPANICOLAOU, Asymptotic analysis of transport process, Bull. American Math. Soc., 81 (1975), pp. 330–392.
- [70] D. PETERSEIM, Eliminating the pollution effect in helmholtz problems by local subscale correction, Mathematics of Computation, 86 (2017), pp. 1005–1036.
- [71] D. PETERSEIM, Numerical homogenization beyond scale separation and periodicity. AMSI Winter School on Computational Modeling of Hetergogeneous Media, July 2019.
- [72] G. STEWART, Four algorithms for the efficient computation of truncated QR approximations to a sparse matrix, Numer. Math., 83 (1999), pp. 313–323.
- [73] E. TADMOR, Convergence of spectral methods for nonlinear conservation laws, SIAM Journal on Numerical Analysis, 26 (1989), pp. 30–44.
- [74] E. WILD, On Boltzmann's equation in the kinetic theory of gases, Mathematical Proceedings of the Cambridge Philosophical Society, 47 (1951), pp. 602 – 609.
- [75] F. WOOLFE, E. LIBERTY, V. ROKHLIN, AND M. TYGERT, A fast randomized algorithm for the approximation of matrices, Appl. Comput. Harmon. Anal., 25 (2008), pp. 335 366.
- [76] L. WU AND Y. GUO, Geometric correction for diffusive expansion of steady neutron transport equation, Communications in Mathematical Physics, 336 (2015), pp. 1473–1553.