On the second largest component of random hyperbolic graphs

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Abstract

We show that in the random hyperbolic graph model as formalized by [GPP12] in the most interesting range of $\frac{1}{2} < \alpha < 1$ the size of the second largest component is $\Theta((\log n)^{1/(1-\alpha)})$. Our research is motivated by the question raised in [BFM13] regarding the uniqueness of linear size components in random hyperbolic graphs which naturally leads to the question regarding the size of the second largest component. We also show that for $\alpha = \frac{1}{2}$ with constant probability the corresponding size is $\Theta(\log n)$, whereas for $\alpha = 1$ it is $\Omega(n^b)$ for some $b > 0$.

1 Introduction

The model of random hyperbolic graphs introduced by Krioukov et al. [KPK+10] has attracted quite a bit of interest due to its key properties also observed in large real-world networks. One convincing demonstration of this fact was given by Boguñá et al. in [BnPK10] where a compelling (heuristic) maximum likelihood fit of autonomous systems of the internet graph in hyperbolic space was computed. A second reason for why the model initially caught attention is due to the experimental results reported by Krioukov et al. [KPK+10, § X] confirming that the model exhibits the algorithmic small-world phenomenon established by the groundbreaking letter forwarding experiment of Milgram from the 60's [TM67].

Another important aspect of the random graph model introduced in [KPK+10] is its mathematically elegant specification and the fact that it is amenable to mathematical analysis. This partly explains why the model has been studied not only empirically by the

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networking community but also analytically by theoreticians. For the latter, it is natural to first consider those issues that played a crucial role in the development of the theory of other random graph models. Among these, the Erdős-Rényi random graph model is undisputedly the most relevant. One of the most, if not the most, studied aspect of the Erdős-Rényi model is the evolution (as a function of the graph density) of the size and number of its connected components [ER60], specially the size of the largest one, but also the size of the second largest. These studies have played a crucial role in the development of mathematical techniques and significantly contributed to the understanding of the Erdős-Rényi random graph model. For the random hyperbolic graph model, the study of the largest component's size was started by Bode, Fountoulakis and Müller [BFM13] and recently refined by Fountoulakis and Müller [FM18]. A logarithmic lower bound and polylogarithmic upper bound for the size of the second largest component of random hyperbolic graphs (when $\frac{1}{2} < \alpha < 1$) were first established in [KM18]. In this paper we improve on these bounds and determine the precise order of the size of the second largest component of random hyperbolic graphs.

Model specification: In the original model of Krioukov et al. [KPK$^+$10] an n-vertex size graph G was obtained by first randomly choosing n points in $B_O(R)$ (the disk of radius $R = R(n)$ centered at the origin O of the hyperbolic plane). From a probabilistic point of view it is arguably more natural to consider the Poissonized version of this model. Formally, the Poissonized model is the following (see also [GPP12] for the same description in the uniform model): for each $n \in \mathbb{N}$, consider a Poisson point process on the hyperbolic disk of radius $R := 2\log(n/\nu)$ for some positive constant $\nu \in \mathbb{R}^+$ (log denotes here and throughout the paper the natural logarithm) and denote its point set by V (the choice of V is due to the fact that we will identify points of the Poisson process with vertices of the graph). The intensity function at polar coordinates (r, θ) for $0 \leq r < R$ and $0 \leq \theta < 2\pi$ is equal to

$$g(r,\theta) := \nu e^{\frac{R}{2}} f(r,\theta),$$

where $f(r, \theta)$ is the joint density function with θ chosen uniformly at random in the interval $[0, 2\pi)$ and independently of r, which is chosen according to the density function

$$f(r) := \begin{cases} \dfrac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}, & \text{if } 0 \leq r < R, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this choice of $f(r)$ corresponds to the uniform distribution inside a disk of radius R around the origin in a hyperbolic plane of curvature $-\alpha^2$. Identify then the points of the Poisson process with vertices (that is, identify a point with polar coordinates (r_v, θ_v) with vertex $v \in V$) and make the following graph $G = (V, E)$: for $u, u' \in V$, $u \neq u'$, there is an edge with endpoints u and u' provided the distance (in the hyperbolic plane) between u and u' is at most R, i.e., the hyperbolic distance between u and u', denoted by $\mathrm{d_h} := \mathrm{d_h}(u, u')$, is such that $\mathrm{d_h} \leq R$ where $\mathrm{d_h}$ is obtained by solving

$$\cosh \mathrm{d_h} := \cosh r_u \cosh r_{u'} - \sinh r_u \sinh r_{u'} \cos(\theta_u - \theta_{u'}). \tag{1}$$

For a given $n \in \mathbb{N}$, we denote this model by $\mathrm{Poi}_{\alpha,\nu}(n)$. Note in particular that

$$\iint g(r,\theta)d\theta dr = \nu e^{\frac{R}{2}} = n,$$

and thus $\mathbb{E}|V| = n$. The main advantage of defining V as a Poisson point process is motivated by the following two properties: the number of points of V that lie in any region $A \subseteq B_O(R)$ follows a Poisson distribution with mean given by $\int_A g(r,\theta)drd\theta = n\mu(A)$, and the numbers of points of V in disjoint regions of the hyperbolic plane are independently distributed.

The restriction $\alpha > \frac{1}{2}$ and the role of R, informally speaking, guarantee that the resulting graph has bounded average degree (depending on α and ν only): if $\alpha < \frac{1}{2}$, then the degree sequence is so heavy tailed that this is impossible (the graph is with high probability connected in this case, as shown in [BFM16]), and if $\alpha > 1$, then as the number of vertices grows, the largest component of a random hyperbolic graph has sublinear order [BFM15, Theorem 1.4]. In fact, although some of our results hold for a wider range of α, we will always assume $\frac{1}{2} < \alpha < 1$; only in the concluding remarks we discuss the cases $\alpha = \frac{1}{2}$ and $\alpha = 1$.

It is known that for $\frac{1}{2} < \alpha < 1$, with high probability the graph G has a linear size component [BFM15, Theorem 1.4] and all other components are of polylogarithmic order [KM15, Corollary 13], which justifies referring to the linear size component as *the giant component*. Implicit in the proof of [BFM15, Theorem 1.4] is that the giant component of a random hyperbolic graph G is the one that contains all vertices whose radial coordinates are at most $\frac{R}{2}$. More precise results including a law of large numbers for the largest component in these networks were established recently in [FM18].

Main result and proof overview: in this paper we determine the exact order of the size of the second largest component, which we denote by $L_2(G)$.

We say that an event holds *asymptotically almost surely (a.a.s.)*, if it holds with probability tending to 1 as $n \to \infty$. We use the standard Bachmann-Landau notation for the asymptotic behaviour of sequences: For two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, we write $a_n = O(b_n)$ to denote the existence of a constant $C > 0$ and a non-negative integer n_0 such that $|a_n| \leq C|b_n|$ for all $n \geq n_0$. Moreover, we write $a_n = \Omega(b_n)$ if $b_n = O(a_n)$, and $a_n = \Theta(b_n)$ if both $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. Finally, $a_n = o(b_n)$ if for every constant $C > 0$ there is a non-negative integer n_0 such that $|a_n| \leq C|b_n|$ for all $n \geq n_0$, and moreover $a_n = \omega(b_n)$ if $b_n = o(a_n)$. The main result of this paper is the following:

Theorem 1. *Let $\frac{1}{2} < \alpha < 1$. If $G = (V,E)$ is chosen according to $\mathrm{Poi}_{\alpha,\nu}(n)$, then a.a.s.,*

$$L_2(G) = \Theta(\log^{\frac{1}{1-\alpha}} n).$$

Moreover, for some sufficiently small constant $b > 0$, there are $\Omega(n^b)$ components in G, each one of size $\Theta(\log^{\frac{1}{1-\alpha}} n)$.

To establish the lower bound, we partition the disk into sectors, so that close to the central axis of each sector, one can find a chain (a path) of vertices within a certain distance from the boundary so that the expected number of vertices with larger radius and in the same sector is of the desired order. While it is relatively easy to show that a constant fraction of these large radii vertices indeed connects to the chain (hence, belong to the same connected component), it is more work to show that none of these vertices in fact is connected to the giant component. Technically, this is tedious since vertices at all radii might potentially be connected to the giant component; vertices with smaller radii might be more dangerous to have neighbors with smaller radii, whilst vertices with larger radii (close to the boundary of $B_O(R)$) might be more dangerous to being reachable from vertices of larger radii that connect to the giant component.

An original aspect of our lower bound analysis consists in identifying "walls", that is, regions $\mathcal{W}$, inside $B_O(R)$ and close to its boundary (specifically, a collection of connected points at distance at least $\ell := R - O(\log R)$ from the origin) which satisfy the following conflicting properties: (i) they do not contain vertices, and (ii) for a sector Φ of $B_O(R)$ strictly containing $\mathcal{W}$, the region $\Phi \setminus B_O(\ell)$ is partitioned into connected regions $\mathcal{W}', \mathcal{W}, \mathcal{W}''$ in such a way that the hyperbolic distance between a point in $\mathcal{W}'$ and a point in $\mathcal{W}''$ is greater than R. The abundance of walls coupled with the fact that the subgraph of G induced by the vertices in $B_O(R) \setminus B_O(\ell)$ contains many vertices (belonging to connected components which we refer to as *pre-components*) reduces the problem of bounding $L_2(G)$ from below to one of showing that there are sectors of $B_O(R)$, say Φ, for which $\Phi \cap B_O(R) \setminus B_O(\ell)$ contains a relatively large connected component while $\Phi \cap B_O(\ell)$ is unlikely to contain vertices of G (these latter regions are the ones where neighbors of pre-components can potentially lie).

Interestingly, the mentioned abundance of walls also partly explains the hierarchical structure close to the boundary of $B_O(R)$ that random hyperbolic graphs exhibit (see Figure 1).

The upper bound of Theorem 1, easier than the lower bound, makes use of the fact that all vertices that are not too close to the boundary of $B_O(R)$ belong to the giant component. We can thus find in every sector of not too big angle a vertex belonging to the giant component, and by simple known geometric properties of random hyperbolic graphs any other component must be squeezed between two such sectors. Since the number of vertices in such a sector is concentrated, we get an upper bound on the size of the second component.

To conclude our study of the size of the second largest component of random hyperbolic graphs we consider the relevant remaining cases where $\alpha = \frac{1}{2}$ or $\alpha = 1$. In the former case, we show that a.a.s. every vertex of the second largest component must be within $C = \Theta(1)$ of the boundary of $B_O(R)$. Moreover, by some geometric considerations, such a component must be contained in a sector Φ of $B_O(R)$ for which $\Phi \cap B_O(R-C)$ does not contain vertices of G. An analysis of the likely maximum angle such a sector Φ can have and of the number of vertices that can be found in $\Phi \setminus B_O(R-C)$ yields the following:

Proposition 2. *For $\alpha = \frac{1}{2}$ and ν small enough, with constant probability, $L_2(G) = \Theta(\log n)$.*

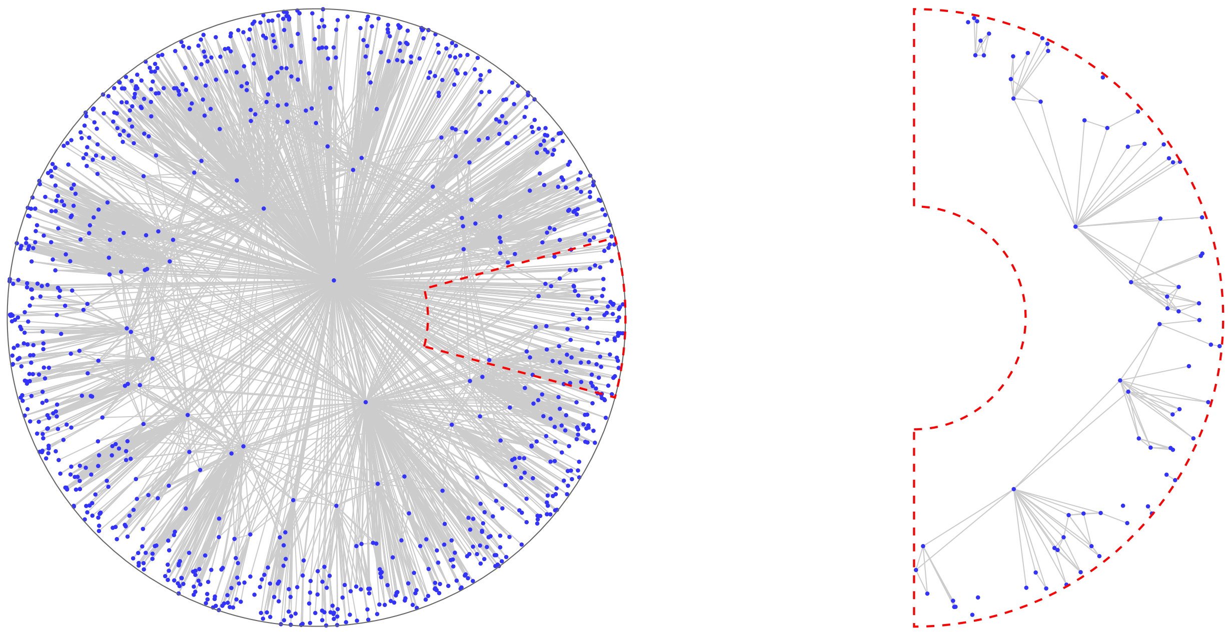

Figure 1: (Left) An instance G of Krioukov et. al.'s random hyperbolic graph model with parameters $n = 1000$, $\alpha = 0.7$, and $\nu = 1.1$. (Right) The subgraph of G induced by the vertices inside the dashed region shown on the left side, where angular coordinates have been scaled by a factor of 6 in order to better elicit the hierarchical structure of the induced graph.

Observe that this result is rather surprising, as the size of the second largest component is discontinuous as α tends to $\frac{1}{2}$ from above: the formula $\Theta((\log n)^{\frac{1}{1-\alpha}})$ would suggest a size of $\Theta(\log^2 n)$, contradicting Proposition 2. For the $\alpha = 1$ case, we show that there is a $\frac{1}{2} < \lambda < 1$ for which a.a.s. there is a vertex of degree $\Theta(n^{1-\lambda})$ that belongs to a component separated from the giant (if the latter exists), so we obtain the following:

Proposition 3. *For $\alpha = 1$ there exists γ, $0 < \gamma < 1$ such that a.a.s., $L_2(G) = \Omega(n^\gamma)$. Moreover, there exists some $0 < \delta < \gamma$ so that for some sufficiently small constant $b > 0$, a.a.s. there are $\Omega(n^b)$ components in G, each one of size $\Omega(n^\delta)$.*

Related work: Although the random hyperbolic graph model was relatively recently introduced [KPK$^+$10], several of its key properties have already been established. As already mentioned, in [GPP12], the degree distribution, the expected value of the maximum degree and global clustering coefficient were determined, and in [BFM15], the existence of a giant component as a function of α.

The threshold in terms of α for the connectivity of random hyperbolic graphs was given in [BFM16]. Concerning diameter and graph distances, except for the aforementioned papers of [KM15] and [FK15], the average distance of two points belonging to the giant component was investigated in [ABF17]. Results on the global clustering coefficient of the so called binomial model of random hyperbolic graphs were obtained in [CF16], and on the evolution of graphs on more general spaces with negative curvature in [Fou15]. Finally, the spectral gap of the Laplacian of this model was studied in [KM18].

The model of random hyperbolic graphs for $\frac{1}{2} < \alpha < 1$ is very similar to two different models studied in the literature: the model of inhomogeneous long-range percolation in $\mathbb{Z}^d$ as defined in [DvdHH13], and the model of geometric inhomogeneous random graphs, as introduced in [BKL19]. In both cases, each vertex is given a weight, and conditionally on the weights, the edges are independent (the presence of edges depending on one or more parameters). In [DvdHH13] the degree distribution, the existence of an infinite component and the graph distance between remote pairs of vertices in the model of inhomogeneous long-range percolation are analyzed. On the other hand, results on typical distances, diameter, clustering coefficient, separators, and existence of a giant component in the model of geometric inhomogeneous graphs were given in [BKL18, BKL19], bootstrap percolation in the same model was studied in [KL16] and greedy routing in [BKL+17]. Both models are very similar to each other, and similar results were obtained in both cases. The latter model generalizes random hyperbolic graphs.

Notation: All asymptotic notation in this paper is with respect to n. Expressions given in terms of other variables such as $O(R)$ are still asymptotics with respect to n, since these variables still depend on n. We say that an event holds *with extremely high probability (w.e.h.p.)*, if for every $c > 0$, there exists an $n_0 := n_0(c)$ such that for every $n \geq n_0$ the event holds with probability at least $1 - O(n^{-c})$. Note that the union of polynomially (in n) many events (where the degree of the polynomial is not allowed to depend on c) that hold w.e.h.p. is also an event that holds w.e.h.p. In what follows, any union bound is over at most $O(n^2)$ many events.

2 Preliminaries

In this section we collect some of the known properties concerning random hyperbolic graphs.

By the hyperbolic law of cosines (1), the hyperbolic triangle formed by the geodesics between points p', p'', and p, with opposing side segments of length d'_{h}, $\mathrm{d}''_{\mathrm{h}}$, and d_{h} respectively, is such that the angle formed at p is:

$$\theta_{\mathrm{d}_{\mathrm{h}}}(\mathrm{d}'_{\mathrm{h}}, \mathrm{d}''_{\mathrm{h}}) = \arccos\Big(\frac{\cosh \mathrm{d}'_{\mathrm{h}} \cosh \mathrm{d}''_{\mathrm{h}} - \cosh \mathrm{d}_{\mathrm{h}}}{\sinh \mathrm{d}'_{\mathrm{h}} \sinh \mathrm{d}''_{\mathrm{h}}}\Big).$$

Clearly, $\theta_{\mathrm{d}_{\mathrm{h}}}(\mathrm{d}'_{\mathrm{h}}, \mathrm{d}''_{\mathrm{h}}) = \theta_{\mathrm{d}_{\mathrm{h}}}(\mathrm{d}''_{\mathrm{h}}, \mathrm{d}'_{\mathrm{h}})$.

Remark 4. *Recall that* $\cosh(\cdot)$ *is at least* 1 *and strictly increasing in* $\mathbb{R}^+$*. Moreover,* $\cosh^2 x - \sinh^2 x = 1$*. Hence, if* $0 < x, y \leq R$*, then*

$$\frac{\partial}{\partial x}\Big(\frac{\cosh x \cosh y - \cosh R}{\sinh x \sinh y}\Big) = \frac{-\cosh y + \cosh R \cosh x}{\sinh^2 x \sinh y} > \frac{\cosh R - \cosh y}{\sinh^2 x \sinh y} \geq 0.$$

Since $\arccos(\cdot)$ *is strictly decreasing, it follows that* $\theta_R(\cdot, y)$ *is strictly decreasing for fixed* $0 < y \le R$. *By symmetry, a similar claim holds for* $\theta_R(x, \cdot)$.

Next, we state a very handy approximation for $\theta_R(\cdot, \cdot)$.

Lemma 5 ([GPP12, Lemma 3.1])**.** *If* $0 \le \min\{\mathrm{d}'_\mathrm{h}, \mathrm{d}''_\mathrm{h}\} \le R \le \mathrm{d}'_\mathrm{h} + \mathrm{d}''_\mathrm{h}$, *then*

$$\theta_R(\mathrm{d}'_\mathrm{h}, \mathrm{d}''_\mathrm{h}) = 2e^{\frac{1}{2}(R-\mathrm{d}'_\mathrm{h}-\mathrm{d}''_\mathrm{h})}\big(1 + \Theta(e^{R-\mathrm{d}'_\mathrm{h}-\mathrm{d}''_\mathrm{h}})\big).$$

Remark 6. *We will use the previous lemma also in this form: let* p' *and* p'' *be two points at distance* R *from each other such that* $r_{p'}, r_{p''} > \frac{R}{2}$ *and* $\min\{r_{p'}, r_{p''}\} \le R$. *Then, taking* $\mathrm{d}'_\mathrm{h} = r_{p'}$ *and* $\mathrm{d}''_\mathrm{h} = r_{p''}$ *in Lemma 5, we get*

$$\theta_R(r_{p'}, r_{p''}) := 2e^{\frac{1}{2}(R-r_{p'}-r_{p''})}\big(1 + \Theta(e^{R-r_{p'}-r_{p''}})\big).$$

Throughout, we will need estimates for measures of regions of the hyperbolic plane, and more specifically, for regions obtained by performing some set algebra involving a few balls. For a point p of the hyperbolic plane $\mathbb{H}^2$, the ball of radius ρ centered at p will be denoted by $B_p(\rho)$, i.e., $B_p(\rho) := \{q \in \mathbb{H}^2 : \mathrm{d}_\mathrm{h}(p, q) \le \rho\}$.

Also, we denote by $\mu(S)$ the measure of a set $S \subseteq \mathbb{H}^2$, i.e., $\mu(S) := \displaystyle\int_S f(r, \theta) dr d\theta$.

Next, we collect a few results for such measures.

Lemma 7 ([GPP12, Lemma 3.2])**.** *If* $0 \le \rho < R$, *then* $\mu(B_O(\rho)) = e^{-\alpha(R-\rho)}(1 + o(1))$.

A direct consequence of Lemma 7 is

Corollary 8. *If* $0 \le \rho'_O < \rho_O < R$, *then*

$$\mu(B_O(\rho_O) \setminus B_O(\rho'_O)) = e^{-\alpha(R-\rho_O)}(1 - e^{-\alpha(\rho_O-\rho'_O)} + o(1)).$$

By standard estimates for Poisson random variables, we have the following lemma:

Lemma 9 ([KM18, Lemma 12])**.** *Let* V *be the vertex set of a graph chosen according to* $\mathrm{Poi}_{\alpha,\nu}(n)$. *For every* $c > 0$, *there is a sufficiently large constant* $c' = c'(c)$ *such that if* $S \subseteq B_O(R)$ *with* $\mu(S) \ge c' \log n/n$, *then with probability at least* $1 - n^{-c}$, $|S \cap V| = \Theta(n\mu(S))$. *If moreover* $S \subseteq B_O(R)$ *is such that* $\mu(S) = \omega(\log n/n)$, *then w.e.h.p.* $|S \cap V| = \Theta(n\mu(S))$.

We need one more lemma.

Lemma 10 ([FK15, Lemma 9])**.** *Let* $p, p', p'' \in B_O(R)$ *be such that* $\theta_p \le \theta_{p'} \le \theta_{p''}$ *and let* $\mathrm{d}_\mathrm{h}(p, p'') \le R$. *Then the following holds:*

(i).- if $r_{p'} \le \min\{r_p, r_{p''}\}$, *then* $\mathrm{d}_\mathrm{h}(p, p'), \mathrm{d}_\mathrm{h}(p', p'') \le R$.

(ii).- if $r_{p'} \le r_{p''}$, *then* $\mathrm{d}_\mathrm{h}(p, p') \le R$.

3 Intermediate regime of α

In this section we prove the main result of this article which concerns the regime where α takes values strictly between $\frac{1}{2}$ and 1. Since our results are asymptotic, we may and will ignore floors in the following calculations, and assume that certain expressions such as $R - \frac{\log R}{1-\alpha}$, $R - \frac{\log R}{1-\alpha} - L$ for some constant L or the like are integers, if needed. When working with a Poisson point process V, for a positive integer ℓ, we refer to the vertices of G that belong to $B_O(\ell) \setminus B_O(\ell-1)$ as the ℓ-th *band* or *layer* and denote it by $V_\ell := V_\ell(G)$, i.e., $V_\ell := V \cap B_O(\ell) \setminus B_O(\ell-1)$. Throughout this section we always assume that $\frac{1}{2} < \alpha < 1$.

3.1 Upper bound

We start with some observations that simplify arguing about the giant component of random hyperbolic graphs. Henceforth, we call *center component* the connected component containing all vertices of a random hyperbolic graph that are within distance $\frac{R}{2}$ of the origin (since all the latter vertices are within distance R of each other, they belong to the same connected component). Concerning the relation between the giant component and the center component, the following is known:

Proposition 11 (Bode et al. [BFM15]). *W.e.h.p. the giant and center component coincide.*

Next we establish that it is likely that vertices bounded away from the boundary of $B_O(R)$ belong to the center component. A similar but slightly weaker result was already proven in [KM18].

Lemma 12. *Let $\ell(L) := R - \frac{\log R}{1-\alpha} - L$ and $G = (V, E)$ be chosen according to $\mathrm{Poi}_{\alpha,\nu}(n)$. For every $c > 0$, there is a sufficiently large constant $L := L(c) > 0$ such that with probability at least $1 - O(n^{-c})$, all vertices in $V \cap B_O(\ell)$, $\ell = \ell(L)$, belong to the center component.*

Proof. It suffices to show that for a sufficiently large L and every vertex $v \in V_i$ with $\frac{R}{2} \leq i \leq \ell$ with probability at least $1 - O(n^{-c})$, there exists a path connecting v to a vertex in $V \cap B_O(\frac{R}{2})$. Taking a union bound, and iterating the argument with $i-1$ instead of i until $i = \frac{R}{2}$, it is enough to show (as proved next) that for a fixed vertex $v \in V_i$ with i as before, with probability at least $1 - O(n^{-(c+1)})$, vertex v has a neighbor in V_{i-1}.

By Remark 6, v is connected to vertex $u \in V_{i-1}$ if the angle at the origin between u and v is $O(\theta_R(i,i))$. By Corollary 8, we have

$$\mu(B_v(R) \cap B_O(i-1) \setminus B_O(i-2)) = \Theta(e^{-\alpha(R-i)} e^{\frac{1}{2}(R-2i)}) = \Theta(e^{(1-\alpha)(R-i)}/n).$$

Since $\alpha < 1$, this expression is clearly decreasing in i, and plugging in our upper bound on i, we obtain

$$\mu(B_v(R) \cap B_O(i-1) \setminus B_O(i-2)) = \Omega(e^{(1-\alpha)(R-\ell)}/n) = \Omega(\log n/n),$$

where the constant hidden in the asymptotic expression can be made arbitrarily large by choosing L large enough so that applying Lemma 9 guarantees that with probability at least

$1-O(n^{-(c+1)})$, vertex v has $\Omega(\log n)$ neighbors in V_{i-1}. By definition, v is connected by an edge to any such vertex, and hence in particular with probability at least $1-O(n^{-(c+1)})$, vertex v has a neighbor in V_{i-1}. □

Define next a ϕ-sector Φ to be a sector of $B_O(R)$, that contains all points in $B_O(R)$ making an angle of at most ϕ at the origin with an arbitrary but fixed reference point.

We deduce from the previous lemma that in any not too small angle there will be at least one vertex belonging to the giant component:

Lemma 13. *For every $c>0$, if $L := L(c)$ and $L' := L'(c)$ are sufficiently large constants, then, for $\ell = \ell(L) := R - \frac{\log R}{1-\alpha} - L$ and $\phi = \phi(L') := \frac{L'}{n}(\log n)^{1/(1-\alpha)}$, with probability at least $1-O(n^{-c})$, every 2ϕ-sector Φ contains at least one vertex $v \in V_\ell$.*

Proof. Partition $B_O(R)$ into ϕ-sectors $\Phi_1, \dots, \Phi_{2\pi/\phi}$. By Corollary 8, we get

$$\mu(\Phi_i \cap B_O(\ell) \setminus B_O(\ell-1)) = \Theta(\phi e^{-\alpha(R-\ell)}) = \Theta(\log n/n).$$

For L' sufficiently large, the constant hidden in the asymptotic notation can be made as large as required by Lemma 9 to get that, with probability at least $1-O(n^{-(c+1)})$, the number of vertices in $V_\ell \cap \Phi_i$ is $\Theta(\log n)$. By taking a union bound over all ϕ-sectors Φ_i (there are $2\pi/\phi = O(n)$ of them), this holds with probability at least $1-O(n^{-c})$ in all of them simultaneously. The statement then follows since every 2ϕ-sector Φ has to contain entirely a ϕ-sector Φ_i, and by a union bound over all events. □

We are now ready for the upper bound on the second largest component.

Proposition 14. *Let $G=(V,E)$ be chosen according to* $\mathrm{Poi}_{\alpha,\nu}(n)$. *W.e.h.p.,*

$$L_2(G) = O(\log^{\frac{1}{1-\alpha}} n).$$

Proof. Let $c>0$, $L := L(c+1)$, $\ell := \ell(L)$, $L' := L'(c+1)$, and $\phi := \phi(L')$ be as in the statement of Lemma 13. By a union bound and appropriate choices of L and L', Lemma 12 and Lemma 13 imply that, with probability at least $1-O(n^{-(c+1)})$, all vertices in $B_O(\ell)$ belong to the center component and every 2ϕ-sector contains at least one vertex $v \in B_O(\ell)$. Then, every vertex x outside the center component belongs to $B_O(R) \setminus B_O(\ell)$. Now, consider a component C distinct from the center component and let u, u' be vertices in C such that $|\theta_{u'} - \theta_u| = \max_{x,x'} |\theta_x - \theta_{x'}|$, where the maximum is taken over all pairs of vertices x, x' belonging to C. If we had $|\theta_{u'} - \theta_u| \geq 2\phi$, then by our conditioning there would be a vertex $v \in B_O(\ell)$ (thus in the center component) such that $\theta_u \leq \theta_v \leq \theta_{u'}$. Since there exists a path in C between u and u' containing only vertices u_j with $r_{u_j} > \ell$, in such a path there must be a pair of vertices, say u_i, u_j, with $r_v \leq r_{u_i}, r_{u_j}$, $u_i u_j \in E$, and $\theta_{u_i} \leq \theta_v \leq \theta_{u_j}$. By Lemma 10, also $u_i v \in E$ and $u_j v \in E$, and hence u and u' are connected to the center component. Therefore, by our conditioning we may assume that $|\theta_{u'} - \theta_u| < 2\phi$. Note that conditioning on the distribution of vertices inside $B_O(\ell)$ does not change the distribution of vertices in $B_O(R) \setminus B_O(\ell)$. Hence, since $\phi = \omega(\log n/n)$, by Lemma 9, w.e.h.p. we get $|C| = O(\phi n) = O((\log n)^{\frac{1}{1-\alpha}})$. By a union bound over all events, with probability at least $1-O(n^{-c})$, it holds that connected components distinct from the center component are of size $O((\log n)^{\frac{1}{1-\alpha}})$, and the statement follows from Proposition 11. □

3.2 Lower bound

We next turn to prove a lower bound matching the bound of Proposition 14. Let $M =: M(\alpha,\nu)$ throughout this subsection be a sufficiently large constant. Partition $B_O(R)$ into ψ-sectors with $\psi := (\nu/n)^{1-\beta}$ for a sufficiently small constant $\beta := \beta(\alpha, M, \nu)$ (first, M has to be chosen sufficiently large as a function of the model parameters α and ν, independent of β, and then, β has to be chosen small enough). Fix throughout this subsection $\ell := R - \frac{\log R}{1-\alpha} + \frac{M}{1-\alpha}$ (recall that we suppose that ℓ is an integer). Let $\phi := 9\theta_R(\ell,\ell)$. By Lemma 5 and Remark 6 thereafter, and since $R = 2\log\frac{n}{\nu}$,

$$\theta_R(\ell,\ell) = (2+o(1))\frac{\nu}{n}e^{R-\ell} = (2+o(1))\frac{\nu}{n}R^{\frac{1}{1-\alpha}}e^{-\frac{M}{1-\alpha}}. \tag{2}$$

For each ψ-sector Ψ, consider the region $\Upsilon_\ell := \Upsilon_\ell(\Psi)$ consisting of those points of $B_O(\ell) \setminus B_O(\ell-1)$ that belong to the ϕ-sector having the same bisector as Ψ. Formally, defining Φ_Ψ as the ϕ-sector having the same bisector as Ψ, we have $\Upsilon_\ell = \Phi_\Psi \cap B_O(\ell) \setminus B_O(\ell-1)$. Next, we establish a lower bound on the probability that $V \cap \Upsilon_\ell$ induces a connected component of G. Actually, we establish a stronger fact. In the ensuing discussion, unless we say otherwise, the ψ-sector Ψ is assumed to be given and all regions as well as subgraphs mentioned depend on Ψ.

Lemma 15. *Let $\Upsilon'_1, \ldots, \Upsilon'_{18}$ be a partition of Υ_ℓ into 18 parts, each Υ'_i obtained as the intersection of Υ_ℓ and a $\frac{\phi}{18}$-sector. The following hold:*

(i).- Let $\mathcal{B}$ be the event that $V \cap \Upsilon'_i$ is non-empty for every $i = 1, ..., 18$. Then, $\mathcal{B}$ occurs a.a.s.

(ii).- For sufficiently large n, all vertices in $V \cap \Upsilon_\ell$ belong to the same connected component.

Proof. To prove (i), observe that by our choice of ϕ, Corollary 8, and (2), for each i,

$$\mu(\Upsilon'_i) = \frac{1}{2}\theta_R(\ell,\ell)\mu(B_O(\ell)\setminus B_O(\ell-1)) = (1+o(1))(1-e^{-\alpha})\frac{\nu}{n}e^{(1-\alpha)(R-\ell)}.$$

Clearly, the events $V \cap \Upsilon'_1 \neq \emptyset, ..., V \cap \Upsilon'_{18} \neq \emptyset$ are independent. Hence, by our choice of ℓ,

$$\mathbf{P}(\mathcal{B}) = (1 - e^{-\nu(1+o(1))(1-e^{-\alpha})e^{(1-\alpha)(R-\ell)}})^{18} = 1 + o(1).$$

To prove (ii), note that (by Remark 4) two vertices in $B_O(\ell)\setminus B_O(\ell-1)$ (and thus the same holds for vertices in $\Upsilon_\ell \cap B_O(\ell)\setminus B_O(\ell-1)$) are neighbors if they form an angle at the origin of at most $\theta_R(\ell,\ell)$. Thus, every vertex in Υ'_i is connected by an edge to every vertex in $\Upsilon'_{i-1} \cup \Upsilon'_i \cup \Upsilon'_{i+1}$, since the maximal angle such pairs of vertices form is, by our choice of ϕ, at most $2\frac{\phi}{18} = \theta_R(\ell,\ell)$. Because of (i), we get that all vertices in $V \cap \Upsilon_\ell$ must be connected. $\square$

Henceforth, for two points $p, p' \in B_O(R)$ let $\Delta\phi_{p,p'}$ denote the (smaller) angle in $[0,\pi)$ between p and p' formed at the origin, i.e., $\Delta\phi_{p,p'} := \min\{|\theta_p - \theta_{p'}|, |2\pi - \theta_p + \theta_{p'}|\}$. By definition of $\theta_R(\cdot,\cdot)$, we know that $\mathrm{d_h}(p,p') \leq R$ if and only if $\Delta\phi_{p,p'} \leq \theta_R(r_p, r_{p'})$. Now, for

$i \in \{0, \ldots, R-\ell\}$, let $\Upsilon_{\ell+i}$ be the collection of points in $B_O(\ell+i) \setminus B_O(\ell+i-1)$ that belong to the $(2\upsilon_{\ell+i})$-sector with the same bisector as Ψ where

$$\upsilon_{\ell+i} := \frac{\phi}{2} + \sum_{j=0}^{i-1} \theta_R(\ell-1+j, \ell+j).$$

(Note that the preceding definition of Υ_ℓ is consistent with the one given before Lemma 15.)

Similarly, for $i \in \{0, \ldots, R-\ell\}$, let $\Xi_{\ell+i}$ be the collection of points in $B_O(\ell+i) \setminus B_O(\ell+i-1)$ that belong to the $(2\xi_{\ell+i})$-sector with the same bisector as Ψ where

$$\xi_{\ell+i} := \theta_R(\ell-1+i, \ell-1+i) + \frac{\phi}{2} + \xi,$$

and $\xi := \displaystyle\sum_{j=0}^{R-\ell-1} \theta_R(\ell-1+j, \ell+j)$.

Finally, let $\Xi := \displaystyle\bigcup_{i=0}^{R-\ell} \Xi_{\ell+i}$ and $\Upsilon := \displaystyle\bigcup_{i=0}^{R-\ell} \Upsilon_{\ell+i}$ (see Figure 2). Clearly, $\Upsilon \subseteq \Xi$.

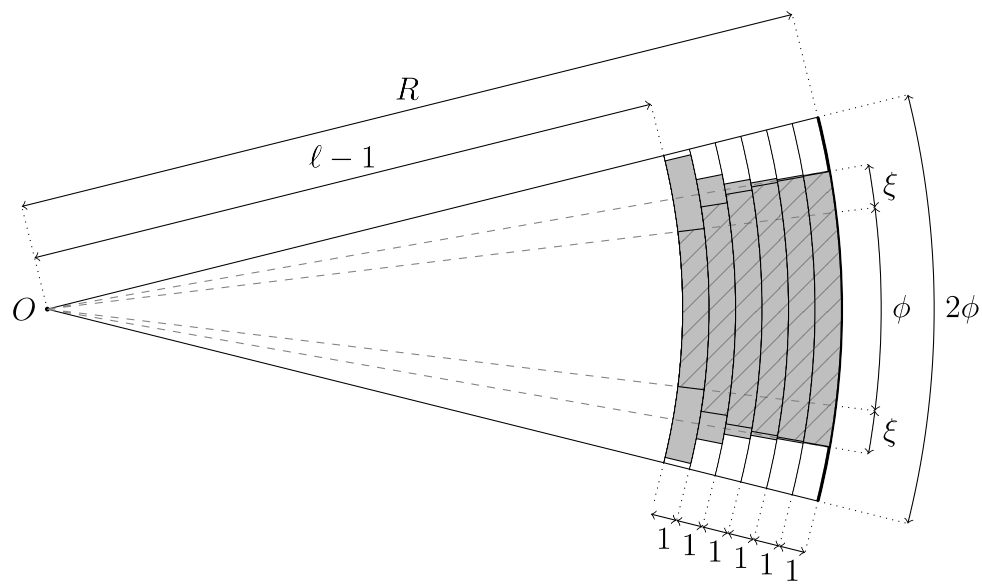

Figure 2: Region Ξ is shown shaded in gray and region Υ diagonally hatched (not to scale). The two shaded non-hatched regions correspond to walls provided they do not contain vertices.

Note that $\Xi \setminus \Upsilon$ is comprised of two similar connected geometric regions of $B_O(R)$. Denote by $\mathcal{W}$ either one of them. We say that a region $\mathcal{W}' \subseteq B_O(R)$ that is obtained by a rotation around the origin of $\mathcal{W}$ is a *wall* if it does not contain any element of V, i.e., $V \cap \mathcal{W}' = \emptyset$.

Next, we establish several facts concerning regions Ξ and Υ, but first we bound ξ just defined. By Lemma 5, the formula for the sum of a geometric series, since $R = 2\log\frac{n}{\nu}$, and by our choice of ℓ

$$\xi = (2+o(1))\frac{\nu}{n}e^{R-\ell+\frac{1}{2}}\sum_{j=0}^{R-\ell-1} e^{-j} = (2+o(1))\frac{e^{3/2}}{e-1}\frac{\nu}{n}e^{R-\ell}. \tag{3}$$

Since $e^{3/2}/(e-1) < 3$, by (2), and our choice of ϕ, for sufficiently large n,

$$\xi < 3\theta_R(\ell,\ell) = \frac{1}{3}\phi. \tag{4}$$

Let $\mathcal{C}$ be the event that there is no vertex in $\Xi \setminus \Upsilon$. The next lemma says that $\Xi \setminus \Upsilon$ contains, with a not too small probability, two disjoint walls (one in each side of the bisector of Υ).

Lemma 16. *For sufficiently large n, the following hold:*

(i).- For some constant $C_0 = C_0(\alpha)$ depending only on α, the probability that $\mathcal{C}$ occurs is at least $e^{-C_0\nu Re^{-M}}$.

(ii).- If $p \in (B_O(R) \setminus B_O(\ell-1)) \setminus \Xi$ and $p' \in \Upsilon$, then $\mathrm{d_h}(p,p') > R$.

Proof. To prove (i), observe that by Lemma 5, Corollary 8, by definition of $\Xi_{\ell+i}$, the formula for the sum of a geometric series and since $\sqrt{e}/(e-1) < 1$,

$$\begin{aligned}
\mu(\Xi_{\ell+i} \setminus \Upsilon_{\ell+i}) &= 2(\xi_{\ell+i} - \upsilon_{\ell+i})\mu(B_O(\ell+i) \setminus B_O(\ell-1+i)) \\
&= 2\Big((2e+o(1))\frac{\nu}{n}e^{R-\ell-i} + (2\sqrt{e}+o(1))\frac{\nu}{n}e^{R-\ell}\sum_{j=i}^{R-\ell-1} e^{-j}\Big)(1-e^{-\alpha})e^{-\alpha(R-\ell-i)} \\
&\leq (8e+o(1))(1-e^{-\alpha})\frac{\nu}{n}e^{(1-\alpha)(R-\ell-i)}.
\end{aligned}$$

Hence, again by the formula for the sum of a geometric series and our choice of ℓ,

$$\mu(\Xi \setminus \Upsilon) \leq \frac{(8e+o(1))(1-e^{-\alpha})}{1-e^{-(1-\alpha)}}\frac{\nu}{n}e^{(1-\alpha)(R-\ell)}(1-e^{-(1-\alpha)(R-\ell+1)}) < C_0\frac{\nu}{n}Re^{-M},$$

where C_0 is a constant depending only on α. The sought after lower bound on the probability that $V \cap \Xi \setminus \Upsilon$ is empty follows immediately.

Next, consider (ii). To prove that $\mathrm{d_h}(p,p') > R$ it suffices to show that $\Delta\phi_{p,p'} > \theta_R(r_p, r_{p'})$. Assume $p \in (B_O(\ell+i) \setminus B_O(\ell+i-1)) \setminus \Xi$ and $p' \in \Upsilon_{\ell+i'}$. Hence,

$$\begin{aligned}
\Delta\phi_{p,p'} > \xi_{\ell+i} - \upsilon_{\ell+i'} &= \theta_R(\ell-1+i, \ell-1+i) + \sum_{j=i'}^{R-\ell-1} \theta_R(\ell-1+j, \ell+j) \\
&= (2e+o(1))\frac{\nu}{n}e^{R-\ell}\big(e^{-i} + e^{-i'}\frac{\sqrt{e}}{e-1}(1-e^{-(R-\ell-i')})\big),
\end{aligned}$$

where the last equality follows from Lemma 5, since $R = 2\log\frac{n}{\nu}$, and the formula for the sum of a geometric series. If $i' < R-\ell$, then $e^{-(R-\ell-i')} \le e^{-1}$, and since $\sqrt{e}(1-e^{-1})/(e-1) \approx 0.61$, applying Jensen's inequality we obtain that for sufficiently large n,

$$\Delta\phi_{p,p'} > (2e + o(1))\frac{\nu}{n}e^{R-\ell}e^{-\frac{1}{2}(i+i')} = \theta_R(\ell-1+i, \ell-1+i').$$

If $i' = R-\ell$, then $e^{-i} \ge e^{-\frac{1}{2}(i+i')}$ and $e^{-(R-\ell-i')} = 1$, so by Remark 4,

$$\Delta\phi_{p,p'} > \theta_R(\ell-1+i, \ell-1+i) \ge \theta_R(\ell-1+i, \ell-1+i').$$

Now, by Remark 4, Lemma 5, and again since $R = 2\log\frac{n}{\nu}$,

$$\theta_R(r_p, r_{p'}) \le \theta_R(\ell-1+i, \ell-1+i').$$

The last three displayed bounds imply that, for a sufficiently large n (independent of i and i'), we have $\Delta\phi_{p,p'} > \theta_R(r_p, r_{p'})$ as claimed. □

We stress that Lemma 16 part (ii) corresponds exactly to the second property satisfied by walls as described in Section 1.

For a given Ψ, let H be the subgraph of G induced by $V \cap \Upsilon$, where $\Upsilon = \Upsilon(\Psi)$, and denote by $C(\Upsilon)$ the collection of vertices of the connected components of H that contain at least one vertex in $V \cap \Upsilon_\ell$.

Let $\mathcal{G}$ be the event that $|C(\Upsilon)| = \Omega((\log n)^{\frac{1}{1-\alpha}})$. Thus, by definition, $\mathcal{G}$ depends only on what happens inside Υ.

Lemma 17. *The event $\mathcal{G}$ occurs a.a.s.*

Proof. Let $\eta = \eta(\alpha,\nu)$ be a sufficiently large constant, let Φ be the $\frac{\phi}{3}$-sector with the same bisector as Ψ, and let $\ell' := R - \frac{c\log R}{1-\alpha}$ for some small constant $0 < c < 1$. For each vertex $z \in V_{R-\eta} \cap \Phi$, let X_z be the indicator random variable for the event that there is a path $z = z_{R-\eta}, \ldots, z_{\ell'}$ in G so that $z_i \in V_i := V \cap B_O(i) \setminus B_O(i-1)$ for every i.

We claim that for a sufficiently large n, there is a $\delta > 0$ such that if $z \in V_{R-\eta} \cap \Phi$, then the expected value of X_z is at least δ. Indeed, suppose that for some i we found a path until z_{i+1}. By Lemma 5, Remark 4, and Corollary 8, the region $\mathcal{R} \subseteq B_O(i) \setminus B_O(i-1)$ in which the next vertex z_i with the desired properties can be found satisfies

$$\mu(\mathcal{R}) \ge \theta_R(i+1, i)\mu(B_O(i) \setminus B_O(i-1)) = (2+o(1))(1-e^{-\alpha})\frac{\nu}{n}e^{(1-\alpha)(R-i)-\frac{1}{2}}, \tag{5}$$

and hence, with probability at most $e^{-(2+o(1))\nu(1-e^{-\alpha})e^{(1-\alpha)(R-i)-\frac{1}{2}}}$ no such vertex is found. Thus, for some positive constant $\delta > 0$, assuming η was chosen sufficiently large (and also n sufficiently large),

$$\mathbf{E}X_z \ge 1 - \sum_{i=\ell'}^{R-\eta-1} e^{-(2+o(1))\nu(1-e^{-\alpha})e^{(1-\alpha)(R-i)-\frac{1}{2}}} \ge \delta. \tag{6}$$

Now, let $X := \sum_z X_z$ where the summation is over the z's in $V_{R-\eta} \cap \Phi$. We claim that $X = (1+o(1))\mathbf{E}X$ a.a.s. Indeed, by Lemma 5, Corollary 8, and (2), we have $\mu(\Phi \cap B_O(R-\eta) \setminus B_O(R-\eta-1)) = \Theta(\frac{1}{n}R^{\frac{1}{1-\alpha}})$. Thus, by Lemma 9, for η large enough, w.e.h.p., $|V_{R-\eta} \cap \Phi| = \Theta((\log n)^{\frac{1}{1-\alpha}})$, and hence by (6), $\mathbf{E}X = \Theta((\log n)^{\frac{1}{1-\alpha}})$. Moreover, in case there is a path $z = z_{R-\eta}, \ldots, z_{\ell'}$ in G so that $z_i \in V_i$ for every i, the total angle between z and $z_{\ell'}$ is

$$\Delta\phi_{z,z_{\ell'}} \leq \sum_{i=\ell'}^{R-\eta-1} \Delta\phi_{z_i,z_{i+1}} \leq \sum_{i=\ell'}^{R-\eta-1} \theta_R(i-1,i) = O\Big(\frac{\nu}{n}e^{R-\ell'}\Big) = O\Big(\frac{\nu}{n}R^{\frac{c}{1-\alpha}}\Big) = o(\phi).$$

Hence, if two such vertices $z, z' \in V_{R-\eta} \cap \Phi$ are at an angle $\omega\big(\frac{1}{n}(\log n)^{\frac{c}{1-\alpha}}\big)$, then X_z and $X_{z'}$ are independent. Since $c < 1$, most pairs of vertices are at angular distance $\omega\big(\frac{1}{n}(\log n)^{\frac{c}{1-\alpha}}\big)$, and thus $\mathbf{E}(X^2) = (1+o(1))(\mathbf{E}X)^2$, so by Chebyshev's inequality, a.a.s. $X = (1+o(1))\mathbf{E}X$ as claimed.

By the preceding discussion, in order to conclude that a.a.s. $|C(\Upsilon)| = (1+o(1))\mathbf{E}X = \Omega((\log n)^{\frac{1}{1-\alpha}})$ it is enough to show that a.a.s. the following event occurs: for every vertex z in $V_{\ell'} \cap \Phi$ there exists a path $z = z_{\ell'} \ldots z_\ell$ in G with $z_i \in V_i$. This fact follows observing that similar calculations as the ones performed above to estimate $|V_{R-\eta} \cap \Phi|$ yield that w.e.h.p. $|V_{\ell'} \cap \Phi| = O((\log n)^{\frac{1}{1-\alpha}})$. By calculations as in (5) together with a union bound, the desired event does not occur with probability

$$\begin{aligned} O((\log n)^{\frac{1}{1-\alpha}} e^{-\log^c n}) + \mathbf{P}\big(|V_{\ell'} \cap \Phi| = \omega((\log n)^{\frac{1}{1-\alpha}})\big) &= e^{\Theta(\log\log n) - \log^c n} + o(n^{-1}) \\ &= e^{-\Theta(\log^c n)}. \end{aligned}$$

Finally, let z be a vertex in $V_{R-\eta} \cap \Phi$ for which there exists a path $z = z_{R-\eta}, \ldots, z_\ell$ in G with $z_i \in V_i$ for all i. Note that the angle $\Delta\phi_{z,z_\ell}$ between the endvertices z and z_ℓ of the path satisfies, by Remark 4,

$$\Delta\phi_{z,z_\ell} \leq \sum_{i=\ell}^{R-\eta-1} \Delta\phi_{z_i,z_{i+1}} \leq \sum_{i=\ell}^{R-\eta-1} \theta_R(i-1,i) \leq \sum_{i=0}^{R-\ell-1} \theta_R(\ell-1+i, \ell+i) = \xi.$$

Thus, by (4), the total angle between z and z_ℓ is at most $\frac{1}{3}\phi$. Since z is a vertex in Φ, it lies within an angle of at most $\frac{\phi}{6}$ of the bisector of Ψ. Thus, all vertices of the $z, ..., z_\ell$ path are within a ϕ-sector with the same bisector as Υ so by construction are also within Υ, and hence in establishing that $\mathcal{G}$ occurs a.a.s. only $\Upsilon \cap \Phi$ needs to be exposed. $\square$

Now, in order to have a component disconnected from the giant component it is enough that all vertices in Υ have no neighbors in $B_O(R) \setminus \Upsilon$. For vertices in Υ not to have neighbors in $(B_O(R) \setminus B_O(\ell-1)) \setminus \Upsilon$, by Lemma 16 Part (ii), it is enough that $V \cap \Xi \setminus \Upsilon$ is empty, as no vertex in Υ can have a neighbor in $(B_O(R) \setminus B_O(\ell-1)) \setminus \Xi$. However, vertices in Υ could have neighbors in $B_O(\ell-1)$. We next deal with this situation. First, we show that it is unlikely for such neighbors to fall within $B_O(\ell-1) \setminus B_O((1-\frac{\beta}{2})R)$ and then we deal with the possibility of having neighbors in $B_O((1-\frac{\beta}{2})R)$ (recall that $\beta = \beta(M)$ is a sufficiently small constant).

Let $\mathcal{H}$ be the event that no vertex in $B_O(\ell-1) \setminus B_O((1-\frac{\beta}{2})R)$ is within distance R of Υ.

Lemma 18. *There is a constant $C_1 = C_1(\alpha)$ depending only on α so that for sufficiently large n the event $\mathcal{H}$ occurs with probability at least $e^{-C_1\nu Re^{-M}}$. Moreover, all area exposed in $\mathcal{H}$ is inside $\Psi \cap B_O(\ell-1) \setminus B_O((1-\frac{\beta}{2})R)$.*

Proof. Since by definition $\upsilon_{\ell+i}$ increases with i, all points in Υ are within an angle $2\upsilon_R = 2(\frac{\phi}{2}+\xi)$, so recalling (4) also within an angle 2ϕ. Moreover, by Remark 4, between two points within distance at most R one of which is in $B_O(j+1) \setminus B_O(j)$, $(1-\frac{\beta}{2})R \le j \le \ell-2$, and the other one in Υ there is an angle at the origin of at most $\theta_R(j, \ell-1)$. Hence, by Lemma 5 and Lemma 7, and again by our choices for ϕ and ℓ, the expected number of neighbors of the vertices in Υ that are inside $B_O(\ell-1) \setminus B_O((1-\frac{\beta}{2})R)$ is at most

$$\begin{aligned}
& n \sum_{j=(1-\frac{\beta}{2})R}^{\ell-2} 2(\upsilon_R + \theta_R(j,\ell-1))\mu(B_O(j+1) \setminus B_O(j)) \\
&\quad \le 2\phi n\mu(B_O(\ell)) + 2\sum_{j=(1-\frac{\beta}{2})R}^{\ell-2} \theta_R(j,\ell-1) n\mu(B_O(j+1)) \\
&\quad \le 18(2+o(1))\nu e^{(1-\alpha)(R-\ell)} + 2(2e^{3/2-\alpha}+o(1))\nu e^{\frac{1}{2}(R-\ell)} \sum_{k\ge R-\ell} e^{-(\alpha-\frac{1}{2})k} \\
&\quad \le C_1 \nu R e^{-M},
\end{aligned}$$

where C_1 is a constant depending on α, but independent of M. The lower bound on $\mathbb{P}(\mathcal{H})$ immediately follows.

To conclude, observe that all area exposed in $\mathcal{H}$ is inside the ψ-sector Ψ, as all area exposed lies within an angle of at most $2(\upsilon_R + \theta_R(\ell-1,(1-\frac{\beta}{2})R))$, which by the preceding discussion, Lemma 5, and our choices of ψ, ϕ, and ℓ, is at most

$$2\phi + 2(2\sqrt{e}+o(1))e^{\frac{1}{2}(R-\ell-(1-\frac{\beta}{2})R)} = \left(\frac{\nu}{n}\right)^{1-\frac{\beta}{2}+o(1)} = o(\psi).$$

$\square$

If for a sector Ψ the events $\mathcal{B}, \mathcal{C}, \mathcal{G}, \mathcal{H}$ hold, then we have found a *precomponent* of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$: by $\mathcal{B}$ and $\mathcal{G}$, there is a collection of vertices in Υ connected to each other (but perhaps not separated from the giant component) of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$. All events are independent or positively correlated: $\mathcal{B}$ and $\mathcal{G}$ only depend on what happens inside Υ, and $\mathcal{C}$ and $\mathcal{H}$ depend on what happens in disjoint regions outside Υ, so $\mathcal{B} \cap \mathcal{G}$, $\mathcal{C}$, and $\mathcal{H}$ are independent. Moreover, events $\mathcal{B}$ and $\mathcal{G}$ are positively correlated. Hence, by combining Lemmata 15, 16, 17 and 18 we get

$$\mathbf{P}(\mathcal{B} \cap \mathcal{C} \cap \mathcal{G} \cap \mathcal{H}) \ge (1+o(1))e^{-C_0\nu Re^{-M}} e^{-C_1\nu Re^{-M}} = e^{-c_M R} \tag{7}$$

for some constant $c_M = c_M(\alpha, \nu) > 0$ that can be made as small as desired by choosing M sufficiently large. Hence, for a given sector Ψ, the probability to have a precomponent of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$ is at least $e^{-c_M R}$, independent of β. Observe also that all events $\mathcal{B}$, $\mathcal{C}$, $\mathcal{G}$, $\mathcal{H}$ expose only areas inside $\Psi \setminus B_O((1-\frac{\beta}{2})R)$, and thus the events corresponding to the existence of a precomponent in disjoint ψ-sectors are independent.

Now, consider the partition of $B_O(R)$ into ψ-sectors $\Psi_1, \ldots, \Psi_{2\pi/\psi}$. By (7), the probability that there is no ψ_i with a precomponent is therefore at most

$$(1-e^{-c_M R})^{n^{1-\beta+o(1)}} \leq e^{-n^{1-\beta-2c_M+o(1)}}, \tag{8}$$

which tends to 0 faster than the inverse of any fixed polynomial in n, if c_M is chosen small enough so that $1-\beta-2c_M > 0$ (such a choice exists, since c_M is independent of β). Hence, w.e.h.p. there exists a ψ-sector Ψ that contains a precomponent of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$.

Let $\mathcal{S}$ be the event that a randomly chosen ψ-sector Ψ is such that there is no vertex in $B_O((1-\frac{\beta}{2})R)$ at distance R from $\Upsilon(\Psi)$.

Lemma 19. *The event $\mathcal{S}$ holds a.a.s.*

Proof. By Remark 4, points in $B_O(j) \setminus B_O(j-1)$, $j \leq (1-\frac{\beta}{2})R$, at distance at most R from some point in $\Upsilon = \Upsilon(\Psi)$ lie in a sector of angle at most $2\theta_R(\ell-1, j-1) = 2(2e+o(1))e^{\frac{1}{2}(R-\ell-j)}$. Also, as observed at the beginning of the proof of Lemma 18, points inside Υ are within an angle of 2ϕ. Hence, the region $\mathcal{R} \subseteq B_O((1-\frac{\beta}{2})R)$ that needs to be empty in order for $\mathcal{S}$ to hold satisfies

$$\begin{aligned}\mu(\mathcal{R}) &= 2(2e+o(1)) \sum_{j=0}^{(1-\frac{\beta}{2})R} \left(e^{\frac{1}{2}(R-\ell-j)} + 2\phi\right) e^{-\alpha(R-j)} \\ &= O(R^{\frac{1}{1-\alpha}} e^{-\alpha R + (\alpha-\frac{1}{2})(1-\frac{\beta}{2})R}) = n^{-1-\beta(\alpha-\frac{1}{2})+o(1)}.\end{aligned}$$

Thus, the expected number of vertices inside $\mathcal{R}$ is $o(1)$, and by Markov's inequality, the event $\mathcal{S}$ holds a.a.s. □

To prove Theorem 1, observe now that if in addition to the existence of a precomponent the event $\mathcal{S}$ holds, then the precomponent inside the randomly chosen ψ-sector Ψ forms a connected component separated from the giant component. Since by (8) w.e.h.p. there is a precomponent, by Lemma 19, by a union bound, a.a.s. there exists a component of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$. Summarizing, we have established the following:

Proposition 20. *For $\frac{1}{2} < \alpha < 1$, a.a.s. $L_2(G) = \Theta((\log n)^{\frac{1}{1-\alpha}})$.*

In fact, we have established that for some sufficiently small $\beta' > 0$ a.a.s. there are $\Omega(n^{\beta'})$ components of size $\Theta((\log n)^{\frac{1}{1-\alpha}})$: indeed, the partition of $B_O(R)$ into ψ-sectors can be grouped into groups of sectors making for a total angle of $n^{-\beta''}$, where $\beta'' > 0$ is chosen small enough so that (8) holds in each group, and also small enough, so that a union bound of all events over all groups still holds as well.

Proposition 14, Proposition 20, and the argument of the previous paragraph yield Theorem 1.

4 Boundary cases of α

As noted in the introduction, for the hyperbolic random graph model, the interesting range of the parameter is when $\frac{1}{2} \le \alpha \le 1$. In this section we investigate the size of the second largest component when α takes the values $\frac{1}{2}$ or 1.

4.1 Case $\alpha = \frac{1}{2}$

By [BFM16], for $\alpha = \frac{1}{2}$, it is known that for $\nu \ge \pi$, with probability tending to 1, the random graph G is connected, whereas for smaller values of ν, the probability of being connected is a continuous function of ν tending to 0 as $\nu \to 0$.

On the one hand, for any constant ν, there exists a constant C (with C being large as ν being small) so that a.a.s. each vertex $v \in B_O(R-C)$ belongs to the giant component: indeed, for a vertex $v \in B_O(i) \setminus B_O(i-1)$ with $\frac{R}{2} < i \le R-C$, the expected number of neighbors of v that belong to $B_O(j) \setminus B_O(j-1)$ with say $j > \frac{R}{2}$ is $\Theta(e^{\frac{1}{2}(R-i-j)} n e^{-\frac{1}{2}(R-j)}) = \Omega(1)$, where the constant can be made large by making C large. Hence, the probability that v does not find a neighbor in $B_O(\frac{5R}{6}) \setminus B_O(\frac{4R}{5})$ is $e^{-\Omega(R)}$, where the constant in the exponent can be made large by choosing C large. By a similar argument, a.a.s. every vertex in $B_O(\frac{5R}{6}) \setminus B_O(\frac{4R}{5})$ also has a neighbor in $B_O(\frac{R}{2}) \setminus B_O(\frac{R}{4})$. Since all vertices within $B_O(\frac{R}{2})$ form a clique, all vertices in $B_O(R-C)$ thus form a component of linear size. Now, by choosing a sector Φ of angle $C' \log n/n$ with $C' = C'(C)$ sufficiently large, by standard estimates for Poisson random variables, each such sector will a.a.s. contain a vertex in $B_O(R-C)$. Hence, a.a.s. the second component has to be contained in at most two consecutive sectors, as otherwise, by Lemma 10, any path whose vertices are all in $B_O(R) \setminus B_O(R-C)$ spanning two sectors, as well as the component to which such path belongs, would necessarily also have to be connected to a vertex of the giant component. Since the number of vertices in each sector of angle $C' \log n/n$ is a.a.s. $O(\log n)$, this upper bound holds also for the size of the second component.

On the other hand, for ν sufficiently small, we now show that with constant probability there exists a sector Φ of $B_O(R)$ of angle $\varepsilon \log n/n$ with $\varepsilon = \varepsilon(C)$ sufficiently small so that the following three events hold:

(i).- inside Φ there is no vertex v in $B_O(R-C)$,

(ii).- there exists a path of length $\varepsilon' \log n$ (ε' sufficiently small) with all vertices being in $\varepsilon' \log n$ consecutive subsectors of Φ of angle ε''/n (with $\varepsilon'' = \varepsilon''(C)$ small enough), with all but the first and last vertex belonging to $B_O(R-C_1+1) \setminus B_O(R-C_1)$ while the first and last belong to $B_O(R-C_1) \setminus B_O(R-C_1-1)$ (for C_1 a small constant in comparison to C, but not too small so that any two vertices in consecutive subsectors are adjacent; clearly, if a smaller value of C_1 is needed below, then this can be achieved by making ε'' smaller), except for this first and last vertex in all these $\varepsilon' \log n$ subsectors there is no vertex in $B_O(R-C_1) \setminus B_O(R-C)$, and there is no other vertex inside $B_O(R) \setminus B_O(R-C_1)$ in the subsector of the first and the last vertex, (iii).- no vertex of the path is connected to the giant component.

Note that for a fixed sector Φ condition (i) is satisfied with probability $e^{-\Theta(R)}$ with the constant in the exponent small for ε small. Condition (ii) also holds with probability $e^{-\Theta(R)}$ with the constant small for ε' small. The last condition is satisfied if the leftmost and rightmost vertex of the path do not connect to the giant component: indeed, if a vertex outside the $\varepsilon' \log n$ subsectors containing the path is connected by an edge to a vertex that is neither the first nor the last vertex of the path, then by Lemma 10 with p' being the first (last) vertex on the path, p'' being another vertex on the path, and p being the vertex outside, it must hold that p is also connected by an edge to the first (last, respectively) vertex of the path. If a vertex inside Φ is connected by an edge to the giant component, then the vertex must be inside $B_O(R) \setminus B_O(R-C_1)$, since first there are no vertices in $\Phi \setminus B_O(R-C)$, and second, in all subsectors between the first and last vertex of the path there are no vertices in $B_O(R-C_1)$. Since any set of vertices connected to the giant component has to leave Φ, once more by Lemma 10, at least one vertex of this set also has to be connected by an edge to the first (last, respectively) vertex of the path. Hence, condition (iii) again happens with probability $e^{-\Theta(R)}$ (again with a constant in the exponent that can be made small for ν sufficiently small and C_1 still relatively small). The events described in the first two conditions are independent, and their intersection is positively correlated with the event described by the third condition. Thus the expected number of sectors Φ for which all conditions hold is $ne^{-\Theta(R)}/\log n = \omega(1)$ for $\varepsilon, \varepsilon', \varepsilon''$ sufficiently small. A second moment method analogous to the one in Lemma 17 shows that different sectors are "almost" independent (special care is taken of vertices close to the center, that is, at a large constant distance, as in Lemma 18 and Lemma 19). Thus, with constant probability such a sector exists (since there is constant probability that there is no vertex close to the center), and the second largest component is of size $\Omega(\log n)$, and thus we obtain Proposition 2.

4.2 Case $\alpha = 1$

Again by [BFM15], for $\alpha = 1$, for ν sufficiently large, a.a.s. there exists a giant component, whereas for ν small enough, a.a.s. the largest component is sublinear. Choose $\ell := \lambda R$ for some $0.51 < \lambda < 1$ (λ depends on ν and has to be chosen closer to 1 for ν larger) and consider a vertex v_1 in $(B_O(\ell) \setminus B_O(\ell-1)) \cap \Upsilon_1$, where Υ_1 is a sector containing all vertices u with $\theta_u \in [0, \pm C n^{-(\lambda-0.51)})$ for some large constant $C > 0$ (there are w.e.h.p. $\Theta(ne^{-(R-\ell)} n^{-(\lambda-0.51)}) = \Theta(n^{\lambda-0.49})$ such vertices, and hence w.e.h.p. we find such a vertex v_1). Clearly, the component of v_1 is at least the degree of v_1, which is w.e.h.p. $\Theta(n^{1-\lambda})$. We will show that a.a.s. there are polynomially many sectors like Υ_1 containing a vertex of degree $\Theta(n^{1-\lambda})$ having all vertices of its component inside a sector whose angle is three times the angle of Υ_1.

First, by standard estimates for Poisson random variables, a.a.s. there is no vertex in $B_O(0.49R)$. Now, we try to construct a *staircase* around the component of v_1 (a curve essentially like the boundary of the hatched region of Figure 2 but with $\phi = 0$). The *left border* of the staircase has the following anchor points: $(\theta^1_{\ell-1}, r_{\ell-1})$, $(\theta^2_{\ell-1}, r_{\ell-1})$, where $\theta^1_{\ell-1} = \theta_{v_1}$, $r_{\ell-1} = \ell - 1$ and $\theta^2_{\ell-1}$ is chosen so that the point $(\theta^2_{\ell-1}, r_\ell)$ with $r_\ell = \ell$ is exactly at hyperbolic distance R from $(\theta^1_{\ell-1}, r_{\ell-1})$ (and to the left of v_1, that is, its angular coordinate precedes in counterclockwise order θ_{v_1}). Then, iteratively having found the two anchor points $(\theta^1_{\ell'}, r_{\ell'})$, $(\theta^2_{\ell'}, r_{\ell'})$ for some $\ell - 1 \le \ell' \le R-2$, define the new anchor points corresponding to layer $\ell'+1$ as $(\theta^1_{\ell'+1}, r_{\ell'+1})$ and $(\theta^2_{\ell'+1}, r_{\ell'+1})$ with $\theta^1_{\ell'+1} = \theta^2_{\ell'}$, $r_{\ell'+1} = \ell'+1$ and $\theta^2_{\ell'+1}$ chosen so that the point $(\theta^2_{\ell'+1}, r_{\ell'+1})$ is exactly at hyperbolic distance R from $(\theta^2_{\ell'}, r_{\ell'})$ (and to the left of it). For each anchor point $p = (\theta^j_{\ell'}, r_{\ell'})$ with $j \in \{1,2\}$, the expected number of vertices in $B_p(R) \cap B_O(\ell'+1) \setminus B_O(0.49R)$ is again $\Theta(\sum_{i=0.49R}^{\ell'+1} ne^{-(R-i)}e^{\frac{1}{2}(R-\ell'-i)}) = \Theta(1)$, with the constant hidden in the $\Theta(\cdot)$ notation proportional to ν. The events of having no vertex in the mentioned neighborhoods of all anchor points are not independent, but they are positively correlated (conditional under having some regions empty, this only helps to have other regions empty). Hence, given that there are $\Theta(R)$ anchor points, the probability to have all desired regions empty (including the one of v_1) is at least $e^{-\Theta(R)} = n^{-\gamma}$, where $\gamma > 0$ can be made small by choosing λ close to 1. Define the *right border* of the staircase of v_1 in the same way, and the probability that all anchor points on both borders have their corresponding regions empty is at least $n^{-2\gamma}$. Moreover, the angle exposed by all these regions (outside $B_O(0.49R)$) is $\Theta\big(\sum_{\ell'=\ell-1}^{R-1}(e^{\frac{1}{2}(R-\ell'-0.49R)} + e^{\frac{1}{2}(R-2\ell')})\big) = \Theta(e^{\frac{1}{2}(R-\ell-0.49R)}) = \Theta(n^{0.51-\lambda})$, and for C sufficiently large, all exposed area is inside a sector whose bisector is θ_{v_1} and whose angle is twice the angle of Υ_1.

Next, partition $B_O(R)$ into $N = \Theta(n^{\lambda-0.51})$ sectors $\Xi_1, ..., \Xi_N$ each of angle $3Cn^{0.51-\lambda}$ for some C sufficiently large (the same C as before, note that the angle of Ξ_i is three times the angle of Υ_1). Applying the above argument to the middle subsector Υ_i of the three subsectors of angle $Cn^{0.51-\lambda}$ of each sector Ξ_i, and noting that for λ sufficiently close to 1, we have $2\gamma < \lambda - 0.51$, w.e.h.p. we find some $1 \le i \le N$ such that the corresponding middle subsector Υ_i is such that all desired regions corresponding to anchor points of the staircase around the starting vertex v_i of Υ_i are empty. In that case, we claim that there is no edge crossing the (vertical or horizontal lines of the) staircase, and hence the component of v_i is restricted to Ξ_i: indeed, suppose that there is a vertex u in $B_O(\ell') \setminus B_O(\ell'-1)$ "below" the staircase ("below" refers to the following area: connect all anchor points starting from $(\theta^1_{\ell-1}, r_{\ell-1})$ both to the left and to the right by artificial lines in staircase manner, and the last one at radial distance $R-1$ via a straight line to the boundary; this divides $B_O(R)$ into two connected pieces, and "below" refers to the piece not containing the origin, and "above" to the piece containing the origin) that is connected by an edge to a vertex w "above" the staircase. Suppose first that w is such that θ_w is within the smaller angle formed by the leftmost and rightmost anchor point of the staircase, and suppose w.l.o.g. that θ_w is between the leftmost anchor point and θ_v. If θ_w is between $\theta^2_{\ell''}$ and $\theta^1_{\ell''}$, then $p = (\theta^2_{\ell''}, r_{\ell''})$ is, by Lemma 10 applied with $p' = w$ and $p'' = (\theta^1_{\ell''}, r_{\ell''})$ also at distance at most R from w (note that for any ℓ'', the points p and p'' are at distance less than R by monotonicity of cosh), which contradicts having the desired region empty. Otherwise, if w is such that θ_w is not within the smaller angle formed by the leftmost and rightmost anchor point of the staircase, suppose w.l.o.g. that θ_w is to the left of the leftmost anchor point of the staircase. Then, for $r_w \in (r_{\ell''}, r_{\ell''+1})$ with $\ell'' < \ell'$, the anchor point $(\theta^2_{\ell'-1}, r_{\ell'-1})$ is once again by Lemma 10 also at distance at most R from w, contradicting our assumption of having the desired region empty. If $\ell'' \geq \ell'$, then we arrive at a contradiction: on the one hand, by Lemma 10 with $p' = (\theta^2_{\ell'-1}, r_{\ell'-1})$, $p'' = u$ and $p = w$, the distance between p' and w is at most R. On the other hand, $(\theta^2_{\ell'}, r_{\ell'})$ is at hyperbolic distance exactly R from $(\theta^2_{\ell'-1}, r_{\ell'-1})$, and w is in angular distance further away from $(\theta^2_{\ell'-1}, r_{\ell'-1})$ than $(\theta^2_{\ell'}, r_{\ell'})$ and it has also a strictly bigger radial coordinate than $r_{\ell'}$. Thus, by strict monotonicity of cosh (see Remark 4) its hyperbolic distance is bigger than R, hence contradiction. It follows that the component of v_i is inside a sector whose bisector is θ_{v_i} and whose angle is twice the angle of Υ_i, and hence the component is inside Ξ_i.

Since in fact not only one, but w.e.h.p. polynomially many such sectors Υ_i can be found, the argument shows that w.e.h.p. polynomially many polynomial-size components exist (of size $\Omega(n^\delta)$ for some $\delta > 0$), thus establishing Proposition 3. Determining the exponent of the size of the second largest component remains open.

5 Final remarks

For $\frac{1}{2} < \alpha < 1$, the proof argument put forth in this article does not seem strong enough to be able to pinpoint the constant accompanying the $(\log n)^{\frac{1}{1-\alpha}}$ term in the asymptotic expression derived for $L_2(G)$ in Theorem 1. We believe that developing techniques that would allow to do so is a worthwhile and interesting endeavor.

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