

# Asymptotics of the gradient of solutions to the perfect conductivity problem

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## Abstract

In the perfect conductivity problem of composite material, the gradient of solutions can be arbitrarily large when two inclusions are located very close. To characterize the singular behavior of the gradient in the narrow region between two inclusions, we capture the leading term of the gradient and give a fairly sharp description of such asymptotics.

## 1 Introduction and main results

It is important from an engineering point of view to study gradient estimates for solutions to a class of elliptic equations of divergence form with piecewise constant coefficients, which models the conductivity problem of a composite material, frequently consisting of inclusions and background media. When the conductivity of inclusions degenerates to be infinity, we call it a perfect conductivity problem. It is known that the electric field, expressed as the gradient, in the narrow region between inclusions may become arbitrarily large when the distance between two inclusions tends to zero. In this paper we characterize such blow-up rates of the gradient with respect to the distance and establish its asymptotic formula in dimensions two and three, two physically relevant dimensions, for two adjacent general convex inclusions.

Before stating our results, we first describe the nature of our domains. Let  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ , be a bounded open set with  $C^2$  boundary, and let  $D_1^*$  and  $D_2^*$  be two open sets whose closure belonging to  $\Omega$ , touching at the origin with the inner normal of  $\partial D_1^*$  being the positive  $x_n$ -axis. We write variable  $x$  as  $(x', x_n)$ . Translating  $D_1^*$  and  $D_2^*$  by  $\frac{\varepsilon}{2}$  along  $x_n$ -axis, we obtain

$$D_1^\varepsilon := D_1^* + (0', \frac{\varepsilon}{2}), \quad \text{and} \quad D_2^\varepsilon := D_2^* - (0', \frac{\varepsilon}{2}).$$

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When there is no possibility of confusion, we drop the superscripts  $\varepsilon$  and denote  $D_1 := D_1^\varepsilon$  and  $D_2 := D_2^\varepsilon$ .

The conductivity problem can be modeled by the following boundary value problem of the scalar equation with piecewise constant coefficients

$$\begin{cases} \operatorname{div}(a_k(x)\nabla u_k) = 0 & \text{in } \Omega, \\ u_k = \varphi(x) & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\varphi \in C^2(\partial\Omega)$  is given, and

$$a_k(x) = \begin{cases} k \in [0, 1) \cup (1, \infty] & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \widetilde{\Omega} := \Omega \setminus \overline{D_1 \cup D_2}. \end{cases}$$

When  $k$  is away from 0 and  $\infty$ , the gradient of the solution of (1.1),  $\nabla u_k$ , is bounded by a constant, independent of the distance  $\varepsilon$ . Babuška, Andersson, Smith, and Levin [6] computationally analyzed the damage and fracture in fiber composite materials where the Lamé system is used. They observed numerically that  $|\nabla u_k|$  remains bounded when the distance  $\varepsilon$  tends to zero. Bonnetier and Vogelius [14] proved that  $|\nabla u_k|$  remains bounded for touching disks  $D_1$  and  $D_2$  in dimension  $n = 2$ . The bound depends on the value of  $k$ . Li and Vogelius [27] extended the result to general divergence form second order elliptic equations with piecewise Hölder continuous coefficients in all dimensions, and they proved that  $|\nabla u_k|$  remains bounded as  $\varepsilon \rightarrow 0$ . They also established stronger,  $\varepsilon$ -independent,  $C^{1,\alpha}$  estimates for solutions in the closure of each of the regions  $D_1$ ,  $D_2$  and  $\widetilde{\Omega}$ . This extension covers domains  $D_1$  and  $D_2$  of arbitrary smooth shapes. Li and Nirenberg [26] extended the results in [27] to general divergence form second order elliptic systems including systems of elasticity.

In this paper, we consider the perfect conductivity problem when  $k = +\infty$ . It was proved by Ammari, Kang and Lim [1] and Ammari, Kang, H. Lee, J. Lee and Lim [4] that, when  $D_1$  and  $D_2$  are disks of comparable radii embedded in  $\Omega = \mathbb{R}^2$ , the blow-up rate of the gradient of the solution to the perfect conductivity problem is  $\varepsilon^{-1/2}$  as  $\varepsilon$  goes to zero; with the lower bound given in [1] and the upper bound given in [4]. Yun in [31, 32] generalized the above mentioned result by establishing the same lower bound,  $\varepsilon^{-1/2}$ , for two strictly convex subdomains in  $\mathbb{R}^2$ . More finer results in this line, see [5, 28]. Bao, Li and Yin [7] introduced a linear functional  $Q_\varepsilon[\varphi]$  and obtained the optimal bounds

$$\frac{\rho_n(\varepsilon)|Q_\varepsilon[\varphi]|}{C\varepsilon} \leq \|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq \frac{C\rho_n(\varepsilon)|Q_\varepsilon[\varphi]|}{\varepsilon} + C\|\varphi\|_{C^2(\partial\Omega)},$$

where  $C$  is independent of  $\varepsilon$  or  $\varphi$ , and

$$\rho_n(\varepsilon) = \begin{cases} \sqrt{\varepsilon}, & \text{for } n = 2; \\ |\log \varepsilon|^{-1}, & \text{for } n = 3; \\ 1, & \text{for } n \geq 4. \end{cases}$$

It may happen that for some  $\varphi$ ,  $|Q_\varepsilon[\varphi]|$  has positive lower and upper bounds independent of  $\varepsilon$ . It may also happen that for some  $\varphi \neq 0$  (independent of  $\varepsilon$ ),  $Q_\varepsilon[\varphi] = 0$ . A similar

result for  $p$ -Laplace equation was investigated by Gorb and Novikov [18]. In particular, for  $p = 2$ , they proved that

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon \|\nabla u\|_{L^\infty(\widetilde{\Omega})}}{\rho_n(\varepsilon)} = \frac{\mathcal{R}_o}{C_o}, \quad \text{for } n = 2, 3,$$

where  $\mathcal{R}_o$  is a constant multiple of  $Q_\varepsilon[\varphi]$ ,  $C_o$  is an explicitly computable constant. The rate at which the  $L^\infty$  norm of the gradient of a special solution for two identical circular inclusions in  $\mathbb{R}^2$  has been shown in [22] to be  $\varepsilon^{-1/2}$ , see also [15, 30].

After knowing the blow-up rate of  $|\nabla u|$  with respect to  $\varepsilon$ , it is desirable and important from the viewpoint of practical applications in engineering to capture such blow-up. Recently, Kang, Lim and Yun [20] characterize asymptotically the singular part of the solution for two adjacent circular inclusions  $B_1$  and  $B_2$  in  $\mathbb{R}^2$  of radius  $r_1$  and  $r_2$  with  $\varepsilon$  apart,

$$u(x) = \frac{2r_1r_2}{r_1+r_2}(\vec{n} \cdot \nabla H)(p) \left( \ln|x-p_1| - \ln|x-p_2| \right) + g(x),$$

for  $x \in \mathbb{R}^2 \setminus (B_1 \cup B_2)$ , where  $H$  is a given entire harmonic function in  $\mathbb{R}^2$ ,  $p_1 \in B_1$  and  $p_2 \in B_2$  are the fixed point of  $R_1R_2$  and  $R_2R_1$  respectively,  $R_j$  is the reflection with respect to  $\partial B_j$ ,  $j = 1, 2$ ,  $\vec{n}$  is the unit vector in the direction of  $p_2 - p_1$ , and  $p$  is the middle point of the shortest line segment connecting  $\partial B_1$  and  $\partial B_2$ , and  $|\nabla g(x)|$  is bounded independent of  $\varepsilon$  on any bounded subset of  $\mathbb{R}^2 \setminus (B_1 \cup B_2)$ . Then

$$\nabla u(x) = \frac{2r_1r_2}{r_1+r_2}(\vec{n} \cdot \nabla H)(p) \left( \frac{1}{|x-p_1|} - \frac{1}{|x-p_2|} \right) + \nabla g(x).$$

In  $\mathbb{R}^3$ , an analogous estimate is obtained by Kang, Lim, and Yun in [21] in the narrow region between two balls with the same radius  $r$  and when  $\sqrt{x_1^2 + x_2^2} \leq r|\log \varepsilon|^{-2}$ . Ammari, Ciraolo, Kang, Lee, Yun [2] extended the result in [20] to the case that inclusions  $D_1$  and  $D_2$  are strictly convex domains in  $\mathbb{R}^2$ . For two adjacent spherical inclusions in  $\mathbb{R}^3$ , it was studied by Kang, Lim and Yun [21]. Bonnetier and Triki [13] derived the asymptotics of the eigenvalues of the Poincaré variational problem as the distance between the inclusions tends to zero. The gradient estimates for Lamé system with partially infinite coefficients were recently obtained in [9, 10, 11]. For more related works, see [3, 8, 12, 16, 17, 24, 25, 29] and the references therein.

In this paper, we obtain estimates for perfect conductivity problems in bounded domains in  $\mathbb{R}^n$ ,  $n = 2, 3$ , analogous to [20, 21] in the whole space. Our estimates in bounded domains in  $\mathbb{R}^3$  improve those in [21] with a higher order asymptotic expansion. One of the main ingredients in achieving these is an asymptotic expansion of the Dirichlet energy of the harmonic function  $v_i$  in  $\widetilde{\Omega}$  satisfying  $v_i = 1$  on  $\partial D_i$ , and  $v_i = 0$  on  $\partial \widetilde{\Omega} \setminus \partial D_i$ , defined by the following

$$\begin{cases} \Delta v_i = 0 & \text{in } \widetilde{\Omega}, \\ v_i = \delta_{ij} & \text{on } \partial D_j, \quad i, j = 1, 2, \\ v_i = 0 & \text{on } \partial \widetilde{\Omega}. \end{cases} \quad (1.2)$$

Our method in deriving the asymptotics of the gradients are very different from that in [2, 20, 21].

We assume that near the origin,  $\partial D_i^*$  are respectively the graph of two  $C^2$  functions  $h_1$  and  $h_2$ , and for some  $R_0, \kappa > 0$ ,

$$h_1(x') > h_2(x'), \quad \text{for } 0 < |x'| < R_0,$$

$$h_1(0') = h_2(0') = 0, \quad \nabla_{x'} h_1(0') = \nabla_{x'} h_2(0') = 0, \quad (1.3)$$

$$\nabla_{x'}^2 (h_1(0') - h_2(0')) \geq \kappa I, \quad (1.4)$$

where  $I$  denotes the  $(n-1) \times (n-1)$  identity matrix.

Here is the above mentioned ingredient, which has its independent interest.

**Theorem 1.1.** *Assume the above with  $n = 2, 3$ ,  $\partial D_i^*$  and  $\partial\Omega$  are of  $C^{k,1}$ ,  $k \geq 3$ . Let  $v_i \in H^1(\widetilde{\Omega})$  be the solution of (1.2),  $i = 1, 2$ . There exist  $\varepsilon$ -independent constants  $M_i$ ,  $i = 1, 2$ , and  $C$ , such that*

$$\left| \int_{\widetilde{\Omega}} |\nabla v_i|^2 - \left( \frac{\kappa_n}{\rho_n(\varepsilon)} + M_i \right) \right| \leq \begin{cases} C\varepsilon^{\frac{1}{4} - \frac{1}{2k}}, & \text{if } n = 2, \\ C\varepsilon^{\frac{1}{2} - \frac{1}{2k}} |\log \varepsilon|, & \text{if } n = 3, \end{cases} \quad i = 1, 2,$$

where  $\kappa_2 = \frac{\sqrt{2}\pi}{\sqrt{\lambda_1}}$ ,  $\kappa_3 = \frac{2\pi}{\sqrt{\lambda_1\lambda_2}}$ , and  $\lambda_1$  and  $\lambda_2$  are the eigenvalues of  $\nabla_{x'}^2 (h_1 - h_2)(0')$ .

Consider the perfect conductivity problem in the bounded domain  $\Omega$ :

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ \nabla u = 0 & \text{on } \overline{D}_i, \quad i = 1, 2, \\ u|_+ = u|_- & \text{on } \partial D_i, \quad i = 1, 2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu} = 0 & i = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where  $\varphi \in C^0(\partial\Omega)$ , and for  $x \in \partial D_i$

$$\frac{\partial u}{\partial \nu^-}(x) := \lim_{t \rightarrow 0^+} \frac{u(x) - u(x + t\nu)}{t}.$$

Here and throughout this paper  $\nu$  is the outward unit normal to the domain and the subscript  $\pm$  indicates the limit from outside and inside the domain, respectively.  $u$  is the weak limit of  $u_k \in H^1(\Omega)$ , the solution of (1.1), as  $k \rightarrow +\infty$ . The existence, uniqueness and regularity of solutions to (1.5) can be found in the appendix of [7].

We rewrite (1.5) as

$$\begin{cases} \Delta u = 0 & \text{in } \widetilde{\Omega}, \\ u = C_i & \text{on } \partial D_i, \quad i = 1, 2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu} = 0 & i = 1, 2, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases} \quad (1.6)$$

where  $C_1$  and  $C_2$  are constants uniquely determined by the third line. As in [7], we decompose the solution  $u$  of (1.6) as follows

$$u(x) = C_1 v_1(x) + C_2 v_2(x) + v_0(x), \quad \text{in } \widetilde{\Omega}, \quad (1.7)$$

where  $v_1, v_2$  are defined by (1.2) and  $v_0$  is the solution of

$$\begin{cases} \Delta v_0 = 0 & \text{in } \widetilde{\Omega}, \\ v_0 = 0 & \text{on } \partial D_1 \cup \partial D_2, \\ v_0 = \varphi(x) & \text{on } \partial \Omega. \end{cases}$$

For  $0 \leq r \leq R_0$ , let

$$\Omega_r := \left\{ (x', x_n) \in \mathbb{R}^n \mid -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x'| < r \right\}.$$

In order to obtain the asymptotic expansion of  $v_1$ , we introduce an auxiliary function  $\bar{u} \in C^{k,1}(\widetilde{\Omega})$ , such that  $\bar{u} = 1$  on  $\partial D_1$ ,  $\bar{u} = 0$  on  $\partial D_2 \cup \partial \Omega$ ,

$$\bar{u}(x) = \frac{x_n - h_2(x') + \frac{\varepsilon}{2}}{\varepsilon + h_1(x') - h_2(x')}, \quad \text{in } \Omega_{R_0}, \quad (1.8)$$

and

$$\|\bar{u}\|_{C^{k,1}(\mathbb{R}^n \setminus \Omega_{R_0})} \leq C. \quad (1.9)$$

Similarly, we can study the asymptotic expansion of  $v_2$  through an auxiliary function  $\tilde{u} := 1 - \bar{u}$  in  $\Omega_{R_0}$ , with  $\|\tilde{u}\|_{C^{k,1}(\mathbb{R}^n \setminus \Omega_{R_0})} \leq C$ . Recalling the assumption (1.3) and (1.4), a direct computation yields

$$|\partial_{x'} \bar{u}(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad \partial_{x_n} \bar{u}(x) = \frac{1}{\varepsilon + h_1(x') - h_2(x')}, \quad x \in \Omega_{R_0}. \quad (1.10)$$

Now define a linear functional  $Q$  and a constant  $\Theta$  as follows:

$$Q[\varphi] := \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu^-} \int_{\partial \Omega} \frac{\partial v_2^*}{\partial \nu} - \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial \nu^-} \int_{\partial \Omega} \frac{\partial v_1^*}{\partial \nu}, \quad (1.11)$$

and

$$\Theta := -\kappa_n \int_{\partial \Omega} \frac{\partial(v_1^* + v_2^*)}{\partial \nu} = \kappa_n \int_{\widetilde{\Omega}^*} |\nabla(v_1^* + v_2^*)|^2, \quad (1.12)$$

where  $\kappa_2 = \frac{\sqrt{2}\pi}{\sqrt{\lambda_1}}$ ,  $\kappa_3 = \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}}$ , and for  $x \in \partial \Omega$

$$\frac{\partial u}{\partial \nu}(x) := \lim_{t \rightarrow 0^+} \frac{u(x) - u(x - t\nu)}{t}.$$

Note that  $\Theta/\kappa_n$  is the condenser capacity of  $\partial D_1^* \cup \partial D_2^*$  relative to  $\partial \Omega$ . Here  $v_1^*, v_2^*$  and  $v_0^*$  are defined, respectively, by

$$\begin{cases} \Delta v_i^* = 0 & \text{in } \widetilde{\Omega}^* := \Omega \setminus \overline{D_1^* \cup D_2^*}, \\ v_i^* = \delta_{ij} & \text{on } \partial D_j^* \setminus \{0\}, \\ v_i^* = 0 & \text{on } \partial \Omega, \end{cases} \quad i = 1, 2, \quad \text{and} \quad \begin{cases} \Delta v_0^* = 0 & \text{in } \widetilde{\Omega}^*, \\ v_0^* = 0 & \text{on } \partial D_1^* \cup \partial D_2^*, \\ v_0^* = \varphi(x) & \text{on } \partial \Omega. \end{cases} \quad (1.13)$$

The well-definedness and the boundedness that  $0 \leq v_1^*, v_2^* \leq 1$  can be seen from lemma 3.1 of [7] and above that.

We have the asymptotic expression of  $\nabla u$  in the narrow region between  $D_1$  and  $D_2$  as follows:

**Theorem 1.2.** *Let  $\Omega, D_1^*, D_2^*$  be defined as the above. For  $\varphi \in C^0(\partial\Omega)$ , let  $u \in H^1(\Omega) \cap C^1(\bar{\Omega} \setminus (D_1 \cup D_2))$  be the solution to (1.5). Then*

(i) *if  $n = 2$ ,  $\partial D_i^*$  and  $\partial\Omega$  are of  $C^{3,1}$ ,*

$$\nabla u = \frac{Q[\varphi] \sqrt{\varepsilon}}{\Theta} \nabla \bar{u} + O(1) \|\varphi\|_{C^0(\partial\Omega)}, \quad \text{in } \bar{\Omega}; \quad (1.14)$$

(ii) *if  $n = 3$ ,  $\partial D_i^*$  and  $\partial\Omega$  are of  $C^{k,1}$ ,  $k \geq 3$ , there exists a positive  $\varepsilon$ -independent constant  $\tilde{M}$ , such that*

$$\nabla u = \frac{Q[\varphi]}{\Theta} \left( \frac{1}{|\log \varepsilon| - \tilde{M}} + O(1) \varepsilon^{\frac{1}{2} - \frac{1}{2k}} |\log \varepsilon|^{-1} \right) \nabla \bar{u} + O(1) \|\varphi\|_{C^0(\partial\Omega)}, \quad \text{in } \bar{\Omega}, \quad (1.15)$$

where  $O(1)$  denotes some quantity satisfying  $|O(1)| \leq C$  for some  $\varepsilon$ -independent constant  $C$ .

The rest of this paper is organized as follows. In section 2, we first reduce the proof of Theorem 1.2 to Proposition 2.1 and Proposition 2.2 below, one for the estimates of  $\|\nabla(v_1 - \bar{u})\|_{L^\infty(\bar{\Omega})}$ , the other for the estimate of  $C_1 - C_2$ , then prove them in Subsection 2.2 and Subsection 2.3, respectively. Finally, we give the proof of Theorem 1.1 in Section 3.

**Acknowledgements.** The research of H.G. Li is partially supported by NSFC (11571042), (11631002) and Fok Ying Tung Education Foundation (151003). H.G. Li would like to thank Professor Jiguang Bao for his suggestions and constant encouragement to this work. The research of Y.Y. Li is partially supported by NSF grant DMS-1501004. We thank the referees for helpful suggestions which improve the exposition.

## 2 The proof of Theorem 1.2

### 2.1 The strategy to prove Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We make use of the energy method to single out the singular term of  $\nabla u$ . We only need to prove (1.14) and (1.15) with  $\|\varphi\|_{C^0(\partial\Omega)}$  replaced by  $\|\varphi\|_{C^2(\partial\Omega)}$ . Indeed, since  $u_k$  in (1.1) satisfies  $\|u_k\|_{L^\infty(\Omega)} \leq \|\varphi\|_{C^0(\partial\Omega)}$ , we have by the convergence of  $u_k$  to  $u$  (see Appendix in [7]),  $\|u\|_{L^\infty(\Omega)} \leq \|\varphi\|_{C^0(\partial\Omega)}$ . Taking a slightly smaller domain  $\Omega_1 \subset\subset \Omega$ , then  $\varphi_1 := u|_{\partial\Omega_1}$  satisfies  $\|\varphi_1\|_{C^2(\partial\Omega_1)} \leq C\|u\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{C^0(\partial\Omega)}$  in view of interior derivative estimates for harmonic functions. The desired identity (1.14) follows by working with  $u$ ,  $\Omega_1$  and  $\varphi_1$ . Without loss of generality, we assume that  $\|\varphi\|_{C^2(\partial\Omega)} = 1$ , by considering  $u/\|\varphi\|_{C^2(\partial\Omega)}$  if  $\|\varphi\|_{C^2(\partial\Omega)} > 0$ . If  $\varphi|_{\partial\Omega} = 0$  then  $u \equiv 0$ .

From (1.7), we have

$$\nabla u = (C_1 - C_2) \nabla v_1 + C_2 \nabla(v_1 + v_2) + \nabla v_0.$$

Noting that  $u = C_i$  on  $\partial D_i$  and  $\|u\|_{H^1(\tilde{\Omega})} \leq C$  (independent of  $\varepsilon$ ), using the trace embedding theorem, we have

$$|C_1| + |C_2| \leq C. \quad (2.1)$$

Since  $\Delta v_0 = 0$  in  $\tilde{\Omega}$  with  $v_0 = 0$  on  $\partial D_1 \cup \partial D_2$ , and  $\Delta(v_1 + v_2 - 1) = 0$  in  $\tilde{\Omega}$  with  $v_1 + v_2 - 1 = 0$  on  $\partial D_1 \cup \partial D_2$ , it follows from lemma 2.3 in [7] (or theorem 1.1 in [23]) and the standard elliptic theory that

$$\|\nabla v_0\|_{L^\infty(\tilde{\Omega})} \leq C, \quad \text{and} \quad \|\nabla(v_1 + v_2)\|_{L^\infty(\tilde{\Omega})} \leq C. \quad (2.2)$$

Recalling the definition of  $\bar{u}$  in  $\Omega_{R_0}$ , (1.8), we first prove that the  $L^\infty$  norm of  $\nabla(v_1 - \bar{u})$  is bounded.

**Proposition 2.1.** *Under the assumptions of Theorem 1.1, let  $v_1 \in H^1(\tilde{\Omega})$  be the weak solution of (1.2). Then*

$$\|\nabla(v_1 - \bar{u})\|_{L^\infty(\tilde{\Omega})} \leq C. \quad (2.3)$$

Consequently,

$$\frac{1}{C(\varepsilon + |x'|^2)} \leq |\nabla v_1(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0}, \quad (2.4)$$

and

$$\|\nabla v_1\|_{L^\infty(\tilde{\Omega} \setminus \Omega_{R_0})} \leq C. \quad (2.5)$$

The proof will be given in Section 2.2.

On the other hand, from the third line of (1.6) and (1.7), the constants  $C_1$  and  $C_2$  are determined by the following linear system

$$\begin{cases} C_1 \int_{\partial D_1} \frac{\partial v_1}{\partial \nu^-} + C_2 \int_{\partial D_1} \frac{\partial v_2}{\partial \nu^-} + \int_{\partial D_1} \frac{\partial v_0}{\partial \nu^-} = 0, \\ C_1 \int_{\partial D_2} \frac{\partial v_1}{\partial \nu^-} + C_2 \int_{\partial D_2} \frac{\partial v_2}{\partial \nu^-} + \int_{\partial D_2} \frac{\partial v_0}{\partial \nu^-} = 0. \end{cases} \quad (2.6)$$

Similarly, as in [7], we denote

$$a_{ij} := \int_{\partial D_i} \frac{\partial v_j}{\partial \nu^-}, \quad b_i := - \int_{\partial D_i} \frac{\partial v_0}{\partial \nu^-}, \quad i, j = 1, 2.$$

Then (2.6) can be written as

$$\begin{cases} a_{11}C_1 + a_{12}C_2 = b_1, \\ a_{21}C_1 + a_{22}C_2 = b_2. \end{cases}$$

By using Cramer's rule, we have

$$C_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad C_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad \text{and} \quad C_1 - C_2 = \frac{\begin{vmatrix} b_1 & a_{11} + a_{12} \\ b_2 & a_{21} + a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

By the Green's formula, it is easy to see that  $a_{12} = a_{21}$ , and

$$a_{11} + a_{12} = a_{11} + a_{21} = - \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \quad a_{21} + a_{22} = a_{12} + a_{22} = - \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu}.$$

In view of

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{11} + a_{12} \\ a_{21} & a_{21} + a_{22} \end{vmatrix},$$

and denoting

$$Q_\varepsilon[\varphi] := \int_{\partial D_1} \frac{\partial v_0}{\partial \nu^-} \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} - \int_{\partial D_2} \frac{\partial v_0}{\partial \nu^-} \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \quad (2.7)$$

and

$$\Theta_\varepsilon := - \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_1}{\partial \nu^-} \right) \int_{\partial\Omega} \frac{\partial v_2}{\partial \nu} + \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_2}{\partial \nu^-} \right) \int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}, \quad (2.8)$$

we have

$$C_1 - C_2 = \rho_n(\varepsilon) \frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon}.$$

The following asymptotic expansion of  $\frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon}$  in term of  $\rho_n(\varepsilon)$  is an essential part in the proof of Theorem 1.2.

**Proposition 2.2.** *Under the same assumptions of Theorem 1.2, let  $Q_\varepsilon[\varphi]$  and  $\Theta_\varepsilon$  be defined by (2.7) and (2.8),  $Q[\varphi]$  and  $\Theta$  be defined by (1.11) and (1.12), respectively. Then*

$$\frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon} - \frac{Q[\varphi]}{\Theta} = \frac{Q[\varphi]}{\Theta} \frac{\widetilde{M}\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon), \quad n = 2, 3,$$

where  $\widetilde{M}$  is an  $\varepsilon$ -independent constant, and

$$E_n(\varepsilon) = \begin{cases} O\left(\varepsilon^{\frac{1}{12}}\right), & \text{if } n = 2, \\ O\left(\varepsilon^{\frac{1}{2} - \frac{1}{2k}} |\log \varepsilon|\right), & \text{if } n = 3. \end{cases}$$

The proof of Proposition 2.2 will be given in Section 2.3. We are now in a position to prove Theorem 1.2 by using Proposition 2.1 and Proposition 2.2.

*Proof of Theorem 1.2.* By using (2.1), (2.2) and (2.3),

$$\nabla u = (C_1 - C_2)\nabla \bar{u} + O(1).$$

It follows from Proposition 2.2 that

$$\begin{aligned} \frac{C_1 - C_2}{\rho_n(\varepsilon)} - \frac{Q[\varphi]}{\Theta} &= \frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon} - \frac{Q[\varphi]}{\Theta} \\ &= \frac{Q[\varphi]}{\Theta} \frac{\widetilde{M}\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon). \end{aligned}$$

So that

$$\begin{aligned}
C_1 - C_2 &= \rho_n(\varepsilon) \left( \frac{Q[\varphi]}{\Theta} + \frac{Q[\varphi]}{\Theta} \frac{\widetilde{M}\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon) \right) \\
&= \frac{Q[\varphi]}{\Theta} \rho_n(\varepsilon) \left( 1 + \frac{\widetilde{M}\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon) \right) \\
&= \frac{Q[\varphi]}{\Theta} \rho_n(\varepsilon) \left( \frac{1}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon) \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\nabla u(x) &= (C_1 - C_2) \nabla \bar{u}(x) + O(1) \\
&= \frac{Q[\varphi]}{\Theta} \rho_n(\varepsilon) \left( \frac{1}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon) \right) \nabla \bar{u}(x) + O(1).
\end{aligned}$$

Theorem 1.2 follows easily from the above and Proposition 2.1.  $\square$

## 2.2 Proof of Proposition 2.1

*Proof of Proposition 2.1.* We denote

$$w := v_1 - \bar{u}.$$

By the definition of  $v_1$  in (1.2), and the fact that  $v_1 = \bar{u}$  on  $\partial D_1 \cup \partial D_2 \cup \partial \Omega$ , we have

$$\begin{cases} -\Delta w = \Delta \bar{u} & \text{in } \widetilde{\Omega}, \\ w = 0 & \text{on } \partial \widetilde{\Omega}. \end{cases} \quad (2.9)$$

Recalling the definition of  $\bar{u}$ , (1.8) and (1.9),

$$\|\bar{u}\|_{C^{k,1}(\widetilde{\Omega} \setminus \Omega_{R_0/2})} \leq C. \quad (2.10)$$

By standard elliptic theories, we know that

$$|w| + |\nabla w| + |\nabla^2 w| \leq C, \quad \text{in } \widetilde{\Omega} \setminus \Omega_{R_0}.$$

Therefore, to show (2.3), we only need to prove

$$\|\nabla w\|_{L^\infty(\Omega_{R_0})} \leq C.$$

For  $(z', z_n) \in \Omega_{2R_0}$ , denote

$$\delta(z') := \varepsilon + h_1(z') - h_2(z'). \quad (2.11)$$

The rest of the proof is divided into three steps.

**STEP 1.** Boundedness of the energy of  $w$  in  $\widetilde{\Omega}$ :

$$\int_{\widetilde{\Omega}} |\nabla w|^2 \leq C. \quad (2.12)$$

By the maximum principle,  $0 < v_1 < 1$ . Recalling the definition of  $\bar{u}$ ,  $\bar{u}$  is also bounded. Hence

$$\|w\|_{L^\infty(\bar{\Omega})} \leq C. \quad (2.13)$$

A direct computation yields,

$$|\partial_{x_i x_j} \bar{u}(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad |\partial_{x_i x_n} \bar{u}(x)| \leq \frac{C|x'|}{(\varepsilon + |x'|^2)^2}, \quad \partial_{x_n x_n} \bar{u}(x) = 0, \quad \text{for } (x', x_n) \in \Omega_{R_0}.$$

So that

$$|\Delta \bar{u}| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_{R_0}. \quad (2.14)$$

Now multiply the equation in (2.9) by  $w$ , integrate by parts, and make use of (2.10), (2.13) and (2.14),

$$\int_{\bar{\Omega}} |\nabla w|^2 = \int_{\bar{\Omega}} w (\Delta \bar{u}) \leq \|w\|_{L^\infty(\bar{\Omega})} \left( \int_{\Omega_{R_0}} |\Delta \bar{u}| + C \right) \leq C.$$

Thus, (2.12) is proved.

**STEP 2.** Proof of

$$\frac{1}{|\Omega_\delta(z')|} \int_{\Omega_\delta(z')} |\nabla w|^2 dx \leq C, \quad \text{for } n \geq 2, \quad (2.15)$$

where

$$\Omega_\delta(z') = \left\{ x \in \mathbb{R}^n \mid -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), |x' - z'| < \delta \right\},$$

and  $\delta := \delta(z')$  is defined in (2.11).

The proof is in spirit similar to that in [23] and [10, 11], see in particular, the proof of proposition 3.2 in [10]. For reader's convenience, we outline the proof here. For  $0 < t < s < R_0$ , let  $\eta$  be a smooth cutoff function satisfying  $\eta(x') = 1$  if  $|x' - z'| < t$ ,  $\eta(x') = 0$  if  $|x' - z'| > s$ ,  $0 \leq \eta(x') \leq 1$  if  $t \leq |x' - z'| \leq s$ , and  $|\nabla_{x'} \eta(x')| \leq \frac{2}{s-t}$ . Multiplying the equation in (2.9) by  $w\eta^2$  and integrating by parts leads to

$$\int_{\Omega_t(z')} |\nabla w|^2 \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |w|^2 + (s-t)^2 \int_{\Omega_s(z')} |\Delta \bar{u}|^2. \quad (2.16)$$

**Case 1.** For  $\sqrt{\varepsilon} \leq |z'| \leq R_0$ . For  $0 < s < \frac{2|z'|}{3}$ , note that

$$\int_{\Omega_s(z')} |w|^2 \leq C|z'|^4 \int_{\Omega_s(z')} |\nabla w|^2, \quad \text{if } 0 < s < \frac{2|z'|}{3}. \quad (2.17)$$

Substituting it into (2.16) and denoting

$$\widehat{F}(t) := \int_{\Omega_t(z')} |\nabla w|^2,$$

we have

$$\widehat{F}(t) \leq \left( \frac{C_0 |z'|^2}{s-t} \right)^2 \widehat{F}(s) + C(s-t)^2 \int_{\Omega_s(z')} |\Delta \bar{u}|^2, \quad \forall 0 < t < s < \frac{2|z'|}{3}, \quad (2.18)$$

where  $C_0$  is a positive universal constant.

Let  $k = \left\lceil \frac{1}{4C_0|z'|} \right\rceil$  and  $t_i = \delta + 2C_0i|z'|^2$ ,  $i = 0, 1, 2, \dots, k$ . Taking  $s = t_{i+1}$  and  $t = t_i$  in (2.18), and in view of (2.14),

$$\int_{\Omega_{t_{i+1}}(z')} |\Delta \bar{u}|^2 \leq \int_{|x'-z'| < t_{i+1}} \frac{C}{\varepsilon + |x'|^2} dx' \leq \frac{C t_{i+1}^{n-1}}{|z'|^2} \leq C(i+1)^{n-1} |z'|^{2(n-2)}. \quad (2.19)$$

we obtain the iteration formula

$$\widehat{F}(t_i) \leq \frac{1}{4} \widehat{F}(t_{i+1}) + C(i+1)^{n-1} |z'|^{2n}.$$

After  $k$  iterations, using (2.12),

$$\widehat{F}(t_0) \leq (1/4)^k \widehat{F}(t_k) + C|z'|^{2n} \sum_{l=1}^k (1/4)^{l-1} l^{n-1} \leq C|z'|^{2n}.$$

This implies that

$$\int_{\Omega_\delta(z')} |\nabla w|^2 \leq C|z'|^{2n}.$$

**Case 2.** For  $0 \leq |z'| \leq \sqrt{\varepsilon}$ . Estimate (2.17) becomes

$$\int_{\Omega_s(z')} |w|^2 \leq C\varepsilon^2 \int_{\Omega_s(z')} |\nabla w|^2, \quad \text{if } 0 < s < \sqrt{\varepsilon}.$$

Estimate (2.18) becomes, in view of (2.16),

$$\widehat{F}(t) \leq \left( \frac{C_1 \varepsilon}{s-t} \right)^2 \widehat{F}(s) + C(s-t)^2 \int_{\Omega_s(z')} |\Delta \bar{u}|^2, \quad \forall 0 < t < s < \sqrt{\varepsilon}, \quad (2.20)$$

where  $C_1$  is another positive universal constant. Let  $k = \left\lceil \frac{1}{4C_1\sqrt{\varepsilon}} \right\rceil$  and  $t_i = \delta + 2C_1i\varepsilon$ ,  $i = 0, 1, 2, \dots, k$ . Then by (2.20) with  $s = t_{i+1}$  and  $t = t_i$ , and using, instead of estimate (2.19),

$$\int_{\Omega_{t_{i+1}}(z')} |\Delta \bar{u}|^2 \leq \int_{|x'-z'| < t_{i+1}} \frac{C}{\varepsilon + |x_1|^2} dx_1 \leq \frac{C t_{i+1}^{n-1}}{\varepsilon} \leq C(i+1)^{n-1} \varepsilon^{n-2}, \quad \text{if } 0 < s < \sqrt{\varepsilon}. \quad (2.21)$$

we have

$$\widehat{F}(t_i) \leq \frac{1}{4} \widehat{F}(t_{i+1}) + C(i+1)^{n-1} \varepsilon^n.$$

After  $k$  iterations, using (2.12),

$$\widehat{F}(t_0) \leq (1/4)^k \widehat{F}(t_k) + C \sum_{l=1}^k (1/4)^{l-1} l^{n-1} \varepsilon^n \leq C(1/4)^{\frac{1}{C\sqrt{\varepsilon}}} + C\varepsilon^n \leq C\varepsilon^n.$$

This implies that

$$\int_{\Omega_\delta(z')} |\nabla w|^2 \leq C\varepsilon^n.$$

In view of the definition of  $\delta(z')$ , (2.15) is proved.

**STEP 3.** Proof of (2.3).

By using the following scaling and translating of variables

$$\begin{cases} x' - z' = \delta y', \\ x_n = \delta y_n, \end{cases}$$

then  $\Omega_\delta(z')$  becomes  $Q_1$ , where

$$Q_r = \left\{ y \in \mathbb{R}^n \mid -\frac{\varepsilon}{2\delta} + \frac{1}{\delta}h_2(\delta y' + z') < y_n < \frac{\varepsilon}{2\delta} + \frac{1}{\delta}h_1(\delta y' + z'), |y'| < r \right\}, \quad \text{for } r \leq 1,$$

and the top and bottom boundaries respectively become

$$y_n = \hat{h}_1(y') := \frac{1}{\delta} \left( \frac{\varepsilon}{2} + h_1(\delta y' + z') \right), \quad |y'| < 1,$$

and

$$y_n = \hat{h}_2(y') := \frac{1}{\delta} \left( -\frac{\varepsilon}{2} + h_2(\delta y' + z') \right), \quad |y'| < 1.$$

Then

$$\hat{h}_1(0') - \hat{h}_2(0') := \frac{1}{\delta} (\varepsilon + h_1(z') - h_2(z')) = 1,$$

and by (1.3),

$$|\nabla_{x'} \hat{h}_1(0')| + |\nabla_{x'} \hat{h}_2(0')| \leq C|z'|, \quad |\nabla_{x'}^2 \hat{h}_1(0')| + |\nabla_{x'}^2 \hat{h}_2(0')| \leq C.$$

Since  $R_0$  is small,  $\|\hat{h}_1\|_{C^{1,1}((-1,1)^{n-1})}$  and  $\|\hat{h}_2\|_{C^{1,1}((-1,1)^{n-1})}$  are small and  $Q_1$  is essentially a unit square (or a unit cylinder for  $n = 3$ ) as far as applications of the Sobolev embedding theorem and classical  $L^p$  estimates for elliptic equations are concerned. Let

$$U(y', y_n) := \bar{u}(z' + \delta y', \delta y_n), \quad W(y', y_n) := w(z' + \delta y', \delta y_n), \quad y \in Q'_1,$$

then by (2.9),

$$-\Delta W = \Delta U, \quad y \in Q_1,$$

where

$$|\Delta U| = \delta^2 |\Delta \bar{u}|.$$

Since  $W = 0$  on the top and bottom boundaries of  $Q_1$ , using the Poincaré inequality,

$$\|W\|_{H^1(Q_1)} \leq C \|\nabla W\|_{L^2(Q_1)}.$$

By  $W^{2,p}$  estimates for elliptic equations (see e.g. [19]) and the Sobolev embedding theorems, with  $p > n$ ,

$$\|\nabla W\|_{L^\infty(Q_{1/2})} \leq C \|W\|_{W^{2,p}(Q_{1/2})} \leq C (\|\nabla W\|_{L^2(Q_1)} + \|\Delta U\|_{L^\infty(Q_1)}).$$

It follows from  $\nabla W = \delta \nabla w$  that

$$\|\nabla w\|_{L^\infty(\Omega_{\delta/2}(z'))} \leq C (\delta^{-n/2} \|\nabla W\|_{L^2(\Omega_\delta(z'))} + \delta \|\Delta \bar{u}\|_{L^\infty(\Omega_\delta(z'))}).$$

Using (2.14), (2.15), and the definition of  $\Omega_\delta(z')$ , Proposition 2.1 is established.  $\square$

*Remark 2.1.* We point out that the estimate involving  $\Delta \bar{u}$  is very crucial in the above proof, such as (2.19), (2.21) for  $\int_{\Omega_{r+1}(z')} |\Delta \bar{u}|^2$  and  $\delta \|\Delta \bar{u}\|_{L^\infty(\Omega_\delta(z'))}$ , so that it is essentially important to select an auxiliary function  $\bar{u}$  to obtain appropriate estimates (2.14).

### 2.3 The proof of Proposition 2.2

Since

$$\frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon} - \frac{Q[\varphi]}{\Theta} = \frac{Q_\varepsilon[\varphi] - Q[\varphi]}{\Theta_\varepsilon} + \frac{Q[\varphi]}{\Theta} \frac{\Theta - \Theta_\varepsilon}{\Theta_\varepsilon},$$

it follows that the proof of Proposition 2.2 can be reduced to the establishment of three Lemmas in the following.

**Lemma 2.3.** *Let  $\Theta$  and  $\Theta_\varepsilon$  be defined as (1.12) and (2.8), respectively. There exists some universal constant  $\delta_0 > 0$  such that*

$$\Theta \geq \delta_0,$$

and  $\lim_{\varepsilon \rightarrow 0} \Theta_\varepsilon = \Theta$ . Consequently, for sufficiently small  $\varepsilon$ ,

$$\Theta_\varepsilon \geq \delta_0/2.$$

**Lemma 2.4.** *Let  $\Theta$  and  $\Theta_\varepsilon$  be defined as (1.12) and (2.8), respectively. Then*

$$\Theta - \Theta_\varepsilon = \rho_n(\varepsilon) \left( M_1 \int_{\partial\Omega} \frac{\partial(v_1^* + v_2^*)}{\partial\nu} + \left( \int_{\partial\Omega} \frac{\partial v_1^*}{\partial\nu} \right)^2 \right) + E_n(\varepsilon)\rho_n(\varepsilon), \quad (2.22)$$

where  $M_1$  is the constant determined in Theorem 1.1. Consequently,

$$\frac{\Theta - \Theta_\varepsilon}{\Theta} = \widetilde{M}\rho_n(\varepsilon) + E_n(\varepsilon)\rho_n(\varepsilon),$$

where

$$\widetilde{M} := -\frac{M_1}{\kappa_n} + \frac{(\alpha_1^*)^2}{\Theta}, \quad \alpha_i^* = \int_{\partial\Omega} \frac{\partial v_i^*}{\partial\nu},$$

which depend only on  $D_1^*$ ,  $D_2^*$  and  $\Omega$ .

**Lemma 2.5.** *Let  $Q[\varphi]$  and  $Q_\varepsilon[\varphi]$  be defined as (1.11) and (2.7), respectively. Then*

$$Q_\varepsilon[\varphi] - Q[\varphi] = \begin{cases} O(\varepsilon^{3/4}), & \text{if } n = 2; \\ O(\varepsilon|\log \varepsilon|), & \text{if } n = 3. \end{cases} \quad (2.23)$$

We first prove Proposition 2.2 by using Lemma 2.3–2.5, whose proofs will be given later.

*Proof of Proposition 2.2.* By Lemma 2.3–2.5, for  $n = 2$ ,

$$\begin{aligned} \frac{Q_\varepsilon[\varphi]}{\Theta_\varepsilon} - \frac{Q[\varphi]}{\Theta} &= \frac{Q[\varphi]}{\Theta} \frac{\frac{\Theta - \Theta_\varepsilon}{\Theta}}{1 - \frac{\Theta - \Theta_\varepsilon}{\Theta}} + \frac{Q_\varepsilon[\varphi] - Q[\varphi]}{\Theta_\varepsilon} \\ &= \frac{Q[\varphi]}{\Theta} \frac{\widetilde{M}\rho_n(\varepsilon) + E_n(\varepsilon)\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon) - E_n(\varepsilon)\rho_n(\varepsilon)} + O(\varepsilon^{3/4}) \\ &= \frac{Q[\varphi]}{\Theta} \frac{\widetilde{M}\rho_n(\varepsilon)}{1 - \widetilde{M}\rho_n(\varepsilon)} + E_n(\varepsilon)\rho_n(\varepsilon). \end{aligned}$$

For  $n = 3$ , we only need to replace  $O(\varepsilon^{3/4})$  by  $O(\varepsilon|\log \varepsilon|)$  in the second line of the above equalities. The proof of Proposition 2.2 is completed.  $\square$

## 2.4 Proof of Lemma 2.3

*Proof of Lemma 2.3.* By the definition of  $v_1^*$  and  $v_2^*$ , (1.13), we have

$$\begin{cases} \Delta(v_1^* + v_2^*) = 0 & \text{in } \widetilde{\Omega}^*, \\ v_1^* + v_2^* = 1 & \text{on } \partial D_1^* \cup \partial D_2^*, \\ v_1^* + v_2^* = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the Hopf Lemma, we have

$$\frac{\partial(v_1^* + v_2^*)}{\partial\nu} < 0, \quad \text{on } \partial\Omega.$$

Since  $0 < v_1^* + v_2^* < 1$  in  $\widetilde{\Omega}^*$  and  $v_1^* + v_2^* = 1$  on  $\partial D_1^* \cup \partial D_2^*$ , the boundary gradient estimates of a harmonic function implies that there exists a ball  $B(\bar{x}, 2\bar{r}) \subset \widetilde{\Omega}^*$ , such that  $v_1^* + v_2^* > 1/2$  in  $B(\bar{x}, 2\bar{r})$ , where  $\bar{r}$  is independent of  $\varepsilon$ . Let  $\rho \in C^2(\widetilde{\Omega}^* \setminus \overline{B(\bar{x}, \bar{r})})$  be the solution to

$$\begin{cases} \Delta\rho = 0 & \text{in } \Omega \setminus \overline{B(\bar{x}, \bar{r})}, \\ \rho = 1/2 \text{ on } \partial B(\bar{x}, \bar{r}), \quad \rho = 0 \text{ on } \partial\Omega. \end{cases}$$

By the maximum principle,  $0 < \rho < 1/2$  in  $\widetilde{\Omega}^* \setminus \overline{B(\bar{x}, \bar{r})}$ . Using the Hopf Lemma again,

$$\frac{\partial\rho}{\partial\nu} \leq -\frac{1}{C}, \quad \text{on } \partial\Omega.$$

On the other hand, since  $\rho \leq v_1^* + v_2^*$  on the boundary of  $\widetilde{\Omega}^* \setminus \overline{B(\bar{x}, 2\bar{r})}$ , it follows from the maximum principle that  $0 < \rho \leq v_1^* + v_2^*$  in  $\widetilde{\Omega}^* \setminus \overline{B(\bar{x}, 2\bar{r})}$ . In view of  $\rho = v_1^* + v_2^* = 0$  on  $\partial\Omega$ ,

$$\frac{\partial\rho}{\partial\nu} \geq \frac{\partial(v_1^* + v_2^*)}{\partial\nu}, \quad \text{on } \partial\Omega.$$

Thus,

$$\int_{\partial\Omega} \frac{\partial(v_1^* + v_2^*)}{\partial\nu} \leq -\frac{1}{C}|\partial\Omega|.$$

This implies that

$$\Theta \geq \frac{1}{C}.$$

Therefore, using  $\int_{\partial\Omega} \frac{\partial v_i}{\partial\nu} \rightarrow \int_{\partial\Omega} \frac{\partial v_i^*}{\partial\nu}$ ,  $i = 1, 2$ , as  $\varepsilon \rightarrow 0$ , see [7], there exists some positive constant  $\delta_0$  such that  $\Theta \geq \delta_0$ , and  $\Theta_\varepsilon \geq \delta_0/2$  for sufficiently small  $\varepsilon$ .  $\square$

## 2.5 Proof of Lemma 2.4

In the following Lemmas for  $v_i$  and  $v_i^*$ ,  $i = 1, 2$ , we only give the proofs for case  $i = 1$ , since the case  $i = 2$  is the same.

**Lemma 2.6.** *Let  $v_i$  and  $v_i^*$  be defined as (1.2) and (1.13), respectively. Then*

$$\|v_i - v_i^*\|_{L^\infty(\Omega \setminus ((D_1 \cup D_1^*) \cup (D_2 \cup D_2^*) \cup \Omega_{\varepsilon^{1/4}}))} \leq C\varepsilon^{1/2}, \quad i = 1, 2. \quad (2.24)$$

*Proof.* We will first consider the difference  $v_1 - v_1^*$  on the boundary of  $\Omega \setminus (D_1 \cup D_2 \cup D_1^* \cup D_2^* \cup \Omega_{\varepsilon^{1/2-\beta}})$ , where  $0 < \beta < 1/2$  (small, to be determined later), then use the maximum principle and boundary estimates for elliptic equations to obtain (2.24).

**STEP 1.** First consider the parts on the boundary  $\partial(D_1 \cup D_1^*)$ . It can be divided into two parts: (a)  $\partial D_1^* \setminus D_1$  and (b)  $\partial D_1 \setminus D_1^*$ .

(a) For  $x \in \partial D_1^* \setminus D_1$ , we introduce a cylinder

$$C_r := \left\{ x \in \mathbb{R}^n \mid |x'| < r, \quad -\frac{\varepsilon}{2} + 2 \min_{|x'|=r} h_2(x') \leq x_n \leq \frac{\varepsilon}{2} + 2 \max_{|x'|=r} h_1(x') \right\},$$

for  $r \leq R_0$ .

(a1) For  $x \in \partial D_1^* \cap (C_{R_0} \setminus C_{\varepsilon^{1/2-\beta}})$ , by mean value theorem and estimate (2.4), we have, for some  $\theta_\varepsilon \in (0, 1)$

$$\begin{aligned} |v_1(x) - v_1^*(x)| &= |v_1(x) - 1| = \left| v_1(x', h_1(x')) - v_1(x', \frac{\varepsilon}{2} + h_1(x')) \right| \\ &= \left| \partial_{x_n} v_1(x', \frac{\theta_\varepsilon \varepsilon}{2} + h_1(x')) \right| \cdot \frac{\varepsilon}{2} \\ &\leq \frac{C}{\varepsilon + |x'|^2} \cdot \frac{\varepsilon}{2} \leq \frac{C\varepsilon}{\varepsilon^{1-2\beta}} = C\varepsilon^{2\beta}. \end{aligned}$$

(a2) For  $x \in \partial D_1^* \cap (\widetilde{\Omega} \setminus \Omega_{R_0})$ , there exists  $y_\varepsilon \in \partial D_1 \cap \overline{\widetilde{\Omega} \setminus \Omega_{R_0/2}}$  such that  $|x - y_\varepsilon| < C\varepsilon$  (note that  $v_1(y_\varepsilon) = 1$ ). By (2.5), then for some  $\theta_\varepsilon \in (0, 1)$

$$|v_1(x) - v_1^*(x)| = |v_1(x) - 1| = |v_1(x) - v_1(y_\varepsilon)| \leq |\nabla v_1((1 - \theta_\varepsilon)x + \theta_\varepsilon y_\varepsilon)| |x - y_\varepsilon| \leq C\varepsilon.$$

(b) For  $x \in \partial D_1 \setminus D_1^*$ , since  $0 < v_1 < 1$  in  $\widetilde{\Omega}$  and  $\Delta v_1 = 0$  in  $\widetilde{\Omega}$ , it follows from the boundary estimates of harmonic function that there exists  $y_x \in \widetilde{\Omega}$ ,  $|y_x - x| \leq C\varepsilon$  such that  $v_1(y_x) = v_1^*(x)$ . Using (2.5) again,

$$|v_1(x) - v_1^*(x)| = |v_1(x) - v_1(y_x)| \leq \|\nabla v_1\|_{L^\infty(\widetilde{\Omega} \setminus \Omega_{R_0})} |x - y_x| \leq C\varepsilon.$$

Therefore,

$$|v_1(x) - v_1^*(x)| \leq \frac{C\varepsilon}{\varepsilon^{1-2\beta}} = C\varepsilon^{2\beta}, \quad \text{for } x \in \partial(D_1 \cup D_1^*) \setminus C_{\varepsilon^{1/2-\beta}}. \quad (2.25)$$

Similarly, we have

$$|v_1(x) - v_1^*(x)| \leq C\varepsilon^{2\beta}, \quad \text{for } x \in \partial(D_2 \cup D_2^*) \setminus C_{\varepsilon^{1/2-\beta}}. \quad (2.26)$$

**STEP 2.** Now consider the line segments (or the cylindrical shaped surface in dimension  $n = 3$ ) between  $\partial D_1^*$  and  $\partial D_2^*$ ,  $S_{1/2-\beta} := \{(x', x_n) \mid |x'| = \varepsilon^{1/2-\beta}, h_2(x') \leq x_n \leq h_1(x')\}$ . By using Propostion 2.1 and the fact that  $(v_1 - \bar{u}) = 0$  on  $\partial D_2$ , we have, for  $x \in S_{1/2-\beta}$ ,

$$|(v_1 - \bar{u})(x)| \leq \|\nabla(v_1 - \bar{u})\|_{L^\infty(S_\beta)} |\varepsilon + h_1(x') - h_2(x')| \leq C(\varepsilon + |x'|^2) \leq C\varepsilon^{1-2\beta}. \quad (2.27)$$

Similarly, we define  $\bar{u}^*$ , such that  $\bar{u}^* = 1$  on  $\partial D_1^* \setminus \{0\}$ ,  $\bar{u}^* = 0$  on  $\partial D_2^* \cup \partial \Omega$ ,

$$\bar{u}^* = \frac{x_n - h_2(x')}{h_1(x') - h_2(x')} \quad \text{in } \Omega_{R_0}^* := \{(x', x_n) \mid h_2(x') \leq x_n \leq h_1(x'), |x'| \leq R_0\},$$

and  $\|\bar{u}^*\|_{C^{k,1}(\bar{\Omega}^* \setminus \Omega_{R_0}^*)} \leq C$ . It is easy to see that

$$\bar{u}^* = \lim_{\varepsilon \rightarrow 0} \bar{u}, \quad \text{in } C^k(\Omega_{R_0}^* \setminus \{0\}),$$

and

$$|\partial_{x'} \bar{u}^*(x)| \leq \frac{C}{|x'|}, \quad \partial_{x_n} \bar{u}^*(x) = \frac{1}{h_1(x') - h_2(x')}, \quad x \in \Omega_{R_0}^* \setminus \{0\}. \quad (2.28)$$

By the proof of Proposition 2.1, we also have

$$\|\nabla(v_1^* - \bar{u}^*)\|_{L^\infty(\bar{\Omega}^*)} \leq C. \quad (2.29)$$

Therefore, using  $(v_1^* - \bar{u}^*) = 0$  on  $\partial D_2^*$ , we have, for  $x \in S_{1/2-\beta}$ ,

$$|(v_1^* - \bar{u}^*)(x)| \leq \|\nabla(v_1^* - \bar{u}^*)\|_{L^\infty(S_{1/2-\beta})} |h_1(x') - h_2(x')| \leq C|x'|^2 \leq C\varepsilon^{1-2\beta}. \quad (2.30)$$

Finally, by the definitions of  $\bar{u}$  and  $\bar{u}^*$ , for  $x \in S_{1/2-\beta}$ ,

$$\begin{aligned} & |(\bar{u} - \bar{u}^*)(x)| \\ & \leq \left| \bar{u}(x', h_2(x')) - \bar{u}(x', -\frac{\varepsilon}{2} + h_2(x')) \right| + \|\partial_{x_n}(\bar{u} - \bar{u}^*)\|_{L^\infty(S_{1/2-\beta})} |h_1(x') - h_2(x')| \\ & \leq \left| \partial_{x_n} \bar{u}(x', -\frac{\theta_\varepsilon \varepsilon}{2} + h_2(x')) \right| \cdot \frac{\varepsilon}{2} + C \left( \frac{1}{h_1(x') - h_2(x')} - \frac{1}{\varepsilon + h_1(x') - h_2(x')} \right) |x'|^2 \\ & \leq \frac{C\varepsilon}{\varepsilon + |x'|^2} + \frac{C\varepsilon}{|x'|^2(\varepsilon + |x'|^2)} |x'|^2 \leq C\varepsilon^{2\beta}. \end{aligned} \quad (2.31)$$

Taking  $\beta = 1/4$ , by (2.27), (2.30) and (2.31), we have, for  $x \in S_{1/4}$ ,

$$|(v_1 - v_1^*)(x)| \leq |(v_1 - \bar{u})(x)| + |(\bar{u} - \bar{u}^*)(x)| + |(\bar{u}^* - v_1^*)(x)| \leq C\varepsilon^{1/2}.$$

Combining with (2.25), (2.26) for  $\beta = 1/4$ , recalling  $v_1 - v_1^* \equiv 0$  on  $\partial \Omega$ , and using maximum principle, we obtain (2.24).  $\square$

Outside of  $\Omega_{R_0}$ , we have the following improvement of Lemma 2.6.

**Lemma 2.7.** *Let  $v_i$  and  $v_i^*$  be defined as (1.2) and (1.13), respectively. Then*

$$\|v_i - v_i^*\|_{L^\infty(\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}))} \leq \begin{cases} C\varepsilon^{3/4}, & \text{if } n = 2; \\ C\varepsilon |\log \varepsilon|, & \text{if } n = 3, \end{cases} \quad i = 1, 2. \quad (2.32)$$

*Proof.* Let  $k_1$  be  $2^{-k_1-1} \leq \varepsilon^{1/4} \leq 2^{-k_1}$ , and  $k_0$  be  $2^{-k_0-1} \leq R_0/2 \leq 2^{-k_0}$ , since  $R_0 < 1$ . Since for sufficiently small  $\varepsilon$ ,  $\partial(D_i \cup D_i^*) \cap C_{R_0/2} = \partial D_i^* \cap C_{R_0/2}$ , we denote

$$E_i^k := (C_{2^{-k}} \setminus C_{2^{-k-1}}) \cap \partial D_i^*, \quad \text{for } k_0 \leq k \leq k_1, \quad i = 1, 2.$$

Then

$$\bigcup_{k=k_0}^{k_1} E_i^k = (C_{R_0/2} \setminus C_{\varepsilon^{1/4}}) \cap \partial D_i^*, \quad i = 1, 2.$$

It follows from (2.25) that

$$|v_1(x) - v_1^*(x)| \leq C\varepsilon \cdot 2^{2k}, \quad x \in E_i^k, \quad \text{for } k_0 \leq k \leq k_1, \quad i = 1, 2. \quad (2.33)$$

For each  $E_i^k$ , we will construct a positive harmonic function  $\xi_i^k$  as below. We will use

$$\xi_i := \sum_{k=k_0}^{k_1} \xi_i^k$$

as one of the few harmonic functions to bound  $\pm(v_1 - v_1^*)$  on  $\partial(D_i \cup D_i^*)$  from above in the following. Let  $\tilde{\xi}_i^k$  be the solution of

$$\begin{cases} \Delta \tilde{\xi}_i^k = 0, & \text{in } \mathbb{R}^n \setminus D_i^*, \\ \tilde{\xi}_i^k = 1, \text{ on } E_i^k, \quad \tilde{\xi}_i^k = 0, \text{ on } \partial D_i^* \setminus E_i^k, \\ \tilde{\xi}_i^k \in L^\infty(\mathbb{R}^n \setminus D_i^*), & \text{if } n = 2, \\ \tilde{\xi}_i^k \rightarrow 0 \text{ as } |x| \rightarrow \infty, & \text{if } n = 3. \end{cases}$$

By the representation formula for the solution of the above boundary value problem using Green's function, we have

$$\tilde{\xi}_i^k(x) = \int_{E_i^k} \frac{\partial G_i}{\partial \nu^-}(x, y) dS(y),$$

where  $G_i(x, y)$  is the Green's function for the domain  $\mathbb{R}^n \setminus D_i$  which satisfies

$$|\nabla_y G_i(x, y)| \leq C \quad \text{if } y \in E_i^k \quad \text{and } x \in \tilde{\Omega}^* \cap \partial C_{R_0}, \quad \forall k_0 \leq k \leq k_1. \quad (2.34)$$

In view of (2.33), we take

$$\xi_i^k := C\varepsilon \cdot 2^{2k} \tilde{\xi}_i^k(x)$$

and use

$$\xi_i(x) = \sum_{k=k_0}^{k_1} \xi_i^k, \quad i = 1, 2,$$

to bound  $\pm(v_1 - v_1^*)$  on  $(C_{R_0/2} \setminus C_{\varepsilon^{1/4}}) \cap \partial D_i^*$  from above.

Let  $B_r$  denote the ball of radius  $r$  centered at the origin in  $\mathbb{R}^n$ . Now due to (2.24),  $|(v_1 - v_1^*)(x)| \leq C\varepsilon^{1/2}$  at  $\tilde{\Omega}^* \cap \partial B_{2\varepsilon^{1/4}}$ , we will construct another auxiliary function to bound  $\pm(v_1 - v_1^*)$  on  $\tilde{\Omega}^* \cap \partial B_{2\varepsilon^{1/4}}$  from above. Define

$$\Sigma_\delta := \{(x_1, x_2, \dots, x_n) \in \partial B_1 \mid |x_n| < \delta\}.$$

Let  $u_\delta$  be the solution of

$$\begin{cases} \Delta u_\delta = 0, & \text{in } B_1 \subset \mathbb{R}^n, \\ u_\delta(x) = 1, & \text{on } \Sigma_{C_0\delta}, \\ u_\delta(x) = 0, & \text{on } \partial B_1 \setminus \Sigma_{C_0\delta}, \end{cases}$$

where  $C_0$  is a constant such that  $\sum_{i=1}^n \lambda_i \leq C_0$ . From the Green's representation, we have

$$u_\delta(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\Sigma_{C_0\delta}} \frac{1}{|x - y|^{n-2}} dS_y,$$

where  $\alpha(n) = |B_1|$ . Then for  $|x| \leq \frac{3}{4}$ ,

$$0 < u_\delta(x) \leq C \int_{\partial B_1 \cap \{|x_n| < C_0\delta\}} dS_y \leq C\delta.$$

By the Kelvin transformation, let

$$\tilde{u}_\delta(x) := \frac{1}{|x|^{n-2}} u_\delta\left(\frac{x}{|x|^2}\right), \quad \text{for } |x| > 1,$$

then  $\tilde{u}_\delta(x) = 1$  on  $\Sigma_{C_0\delta}$ ,

$$\Delta \tilde{u}_\delta(x) = 0, \quad \tilde{u}_\delta(x) > 0, \quad \text{for } |x| > 1,$$

and as  $|x| \rightarrow \infty$ ,  $\tilde{u}_\delta(x) \rightarrow u_\delta(0) = \frac{|\Sigma_{C_0\delta}|}{|\partial B_1|}$  if  $n = 2$ ;  $\tilde{u}_\delta(x) \rightarrow 0$  for  $n \geq 3$ . Furthermore, for  $|x| \geq \frac{4}{3}$ , we have

$$\tilde{u}_\delta(x) \leq \frac{C\delta}{|x|^{n-2}}. \quad (2.35)$$

Take

$$\xi_0 := \bar{C}\varepsilon^{1/2} \tilde{u}_\delta(x) \left(\frac{x}{\delta}\right)$$

with  $\delta = 2\varepsilon^{1/4}$ , where  $\bar{C}$  is the same constant  $C$  in (2.24). Because of the choice of  $C_0$  and (2.24), we can see

$$\xi_0 = \bar{C}\varepsilon^{1/2} \geq \pm(v_1 - v_1^*) \quad \text{on } \tilde{\Omega}^* \cap \partial B_{2\varepsilon^{1/4}}.$$

And according to (2.35)

$$\xi_0 \leq C\varepsilon^{\frac{n+1}{4}} \quad \text{on } \tilde{\Omega}^* \cap \partial C_{R_0}.$$

Due to  $\|\nabla v_1\|_{L^\infty(\tilde{\Omega} \setminus \Omega_{R/2})} \leq C$  and  $\|\nabla v_1^*\|_{L^\infty(\tilde{\Omega}^* \setminus \Omega_{R/2}^*)} \leq C$ , we have

$$\pm(v_1 - v_1^*) \leq C\varepsilon, \quad \text{on } \partial(D_1 \cup D_1^* \cup D_2 \cup D_2^*) \setminus C_{R_0/2}.$$

In view of  $v_1 - v_1^* = 0$  on  $\partial\Omega$  and the positivity of  $\xi_i$ ,  $i = 0, 1, 2$ , we have

$$\pm(v_1 - v_1^*) \leq \xi_0 + \xi_1 + \xi_2 + C\varepsilon, \quad \text{on } \partial(\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup B_{2\varepsilon^{1/4}})).$$

By using the maximum principle in  $\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup B_{2\varepsilon^{1/4}})$ , we have

$$\pm(v_1 - v_1^*) \leq \xi_0 + \xi_1 + \xi_2 + C\varepsilon, \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup B_{2\varepsilon^{1/4}}). \quad (2.36)$$

Next, in order to prove (2.32), we need to further estimate  $\xi_i$  on  $\widetilde{\Omega}^* \cap \partial C_{R_0}$ ,  $i = 1, 2$ . Making use of (2.34),

$$\tilde{\xi}_i^k(x) \leq C|E_i^k| \leq \frac{C}{2^{(n-1)k}}, \quad x \in \widetilde{\Omega}^* \cap \partial C_{R_0}.$$

Thus

$$\xi_i(x) \leq C \sum_{k=k_0}^{k_1} \varepsilon \cdot 2^{2k} \frac{C}{2^{(n-1)k}} = C \sum_{k=k_0}^{k_1} \frac{\varepsilon}{2^{(n-3)k}}, \quad x \in \widetilde{\Omega}^* \cap \partial C_{R_0}.$$

Hence, if  $n = 2$ , recalling  $k_1 \sim \frac{1}{4 \log 2} |\log \varepsilon|$ ,

$$\xi_i(x) \leq C \sum_{k=k_0}^{k_1} \varepsilon 2^k \leq C \varepsilon 2^{k_1} \leq C \varepsilon^{3/4}, \quad x \in \widetilde{\Omega}^* \cap \partial C_{R_0};$$

if  $n = 3$ ,

$$\xi_i(x) \leq C \varepsilon k_1 \leq C \varepsilon |\log \varepsilon|, \quad x \in \widetilde{\Omega}^* \cap \partial C_{R_0}.$$

Combining these estimates above with (2.36), we have, on  $\partial(\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}))$ ,

$$\pm(v_1 - v_1^*) \leq \xi_0 + \xi_1 + \xi_2 + C\varepsilon \leq \begin{cases} C\varepsilon^{3/4} + C\varepsilon^{3/4} + C\varepsilon, & \text{if } n = 2; \\ C\varepsilon + C\varepsilon |\log \varepsilon| + C\varepsilon, & \text{if } n = 3. \end{cases}$$

By using the maximum principle again,

$$|v_1 - v_1^*| \leq \begin{cases} C\varepsilon^{3/4}, & \text{if } n = 2; \\ C\varepsilon |\log \varepsilon|, & \text{if } n = 3, \end{cases} \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}).$$

The proof is completed.  $\square$

An immediate consequence of Lemma 2.7 and the boundary estimates for elliptic equations is as follows:

**Lemma 2.8.** *Let  $v_i$  and  $v_i^*$  be defined as (1.2) and (1.13), respectively. Then*

$$\left| \int_{\partial\Omega} \frac{\partial v_i}{\partial \nu} - \int_{\partial\Omega} \frac{\partial v_i^*}{\partial \nu} \right| \leq \begin{cases} C\varepsilon^{3/4}, & \text{if } n = 2; \\ C\varepsilon |\log \varepsilon|, & \text{if } n = 3, \end{cases} \quad i = 1, 2. \quad (2.37)$$

Now we prove Lemma 2.4.

*Proof of Lemma 2.4.* Since

$$\int_{\partial D_1} \frac{\partial v_1}{\partial \nu^-} = \int_{\widetilde{\Omega}} |\nabla v_1|^2,$$

it follows from Theorem 1.1 that

$$\rho_n(\varepsilon) \int_{\widetilde{\Omega}} |\nabla v_i|^2 - \kappa_n = M_i \rho_n(\varepsilon) + E_n(\varepsilon) \rho_n(\varepsilon), \quad i = 1, 2.$$

In view of the definitions of  $v_1$  and  $v_2$  and the Green's formula, we obtain the following identity

$$\int_{\partial D_1} \frac{\partial v_2}{\partial v^-} = \int_{\partial D_2} \frac{\partial v_1}{\partial v^-} = - \int_{\partial D_1} \frac{\partial v_1}{\partial v^-} - \int_{\partial \Omega} \frac{\partial v_1}{\partial v}.$$

Thus, recalling the definition of  $\Theta_\varepsilon$ , (2.8),

$$\begin{aligned} \Theta_\varepsilon &= - \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_1}{\partial v^-} \right) \int_{\partial \Omega} \frac{\partial v_2}{\partial v} + \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_2}{\partial v^-} \right) \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \\ &= - \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_1}{\partial v^-} \right) \int_{\partial \Omega} \frac{\partial(v_1 + v_2)}{\partial v} - \rho_n(\varepsilon) \left( \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \right)^2. \end{aligned}$$

Recalling the definition of  $\Theta$ , (1.12),

$$\Theta = -\kappa_n \int_{\partial \Omega} \frac{\partial(v_1^* + v_2^*)}{\partial v}$$

and using Lemma 2.8 and Theorem 1.1, we have,

$$\begin{aligned} \Theta - \Theta_\varepsilon &= \left( \rho_n(\varepsilon) \int_{\partial D_1} \frac{\partial v_1}{\partial v^-} - \kappa_n \right) \int_{\partial \Omega} \frac{\partial(v_1^* + v_2^*)}{\partial v} + \rho_n(\varepsilon) \left( \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v} \right)^2 \\ &\quad + O(\varepsilon^{\frac{3}{4}}) \text{ (or } O(\varepsilon |\log \varepsilon|)) \\ &= (M_1 \rho_n(\varepsilon) + E_n(\varepsilon) \rho_n(\varepsilon)) \int_{\partial \Omega} \frac{\partial(v_1^* + v_2^*)}{\partial v} + \rho_n(\varepsilon) \left( \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v} \right)^2 \\ &\quad + O(\varepsilon^{\frac{3}{4}}) \text{ (or } O(\varepsilon |\log \varepsilon|)) \\ &= \rho_n(\varepsilon) \left( M_1 \int_{\partial \Omega} \frac{\partial(v_1^* + v_2^*)}{\partial v} + \left( \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v} \right)^2 \right) + E_n(\varepsilon) \rho_n(\varepsilon). \end{aligned}$$

(2.22) is proved.  $\square$

## 2.6 Proof of Lemma 2.5

To prove (2.23), besides (2.37), we need

**Lemma 2.9.** *Let  $v_0$  and  $v_0^*$  be defined in (1.2) and (1.13), respectively. Then*

$$\left| \int_{\partial D_i} \frac{\partial v_0}{\partial v^-} - \int_{\partial D_i^*} \frac{\partial v_0^*}{\partial v^-} \right| \leq C \|\varphi\|_{L^\infty(\partial \Omega)} \begin{cases} \varepsilon^{3/4}, & \text{if } n = 2, \\ \varepsilon |\log \varepsilon|, & \text{if } n = 3, \end{cases} \quad i = 1, 2. \quad (2.38)$$

*Proof.* Using the Green's formula,

$$\int_{\partial D_1} \frac{\partial v_0}{\partial v^-} = \int_{\partial D_1} \frac{\partial v_0}{\partial v^-} v_1 = \int_{\partial \bar{\Omega}} \frac{\partial v_0}{\partial v} v_1 = \int_{\partial \bar{\Omega}} \frac{\partial v_1}{\partial v} v_0 = \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \varphi,$$

and

$$\int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} = \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} v_1^* = \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v} \varphi.$$

So that, by Lemma 2.8,

$$\left| \int_{\partial D_1} \frac{\partial v_0}{\partial v^-} - \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} \right| = \left| \int_{\partial \Omega} \frac{\partial(v_1 - v_1^*)}{\partial v} \varphi \right| \leq \begin{cases} C\varepsilon^{3/4}, & \text{if } n = 2, \\ C\varepsilon |\log \varepsilon|, & \text{if } n = 3. \end{cases}$$

This completes the proof, with the assumption  $\|\varphi\|_{C^2(\partial \Omega)} = 1$ .  $\square$

*Proof of Lemma 2.5.* Recall that

$$Q_\varepsilon[\varphi] = \int_{\partial D_1} \frac{\partial v_0}{\partial v^-} \int_{\partial \Omega} \frac{\partial v_2}{\partial v} - \int_{\partial D_2} \frac{\partial v_0}{\partial v^-} \int_{\partial \Omega} \frac{\partial v_1}{\partial v},$$

and

$$Q[\varphi] = \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} \int_{\partial \Omega} \frac{\partial v_2^*}{\partial v} - \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial v^-} \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v}.$$

Using (2.37) and (2.38), we have

$$\begin{aligned} & |Q_\varepsilon[\varphi] - Q[\varphi]| \\ & \leq \left| \left( \int_{\partial D_1} \frac{\partial v_0}{\partial v^-} - \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} \right) \int_{\partial \Omega} \frac{\partial v_2}{\partial v} \right| + \left| \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial v^-} \left( \int_{\partial \Omega} \frac{\partial v_2}{\partial v} - \int_{\partial \Omega} \frac{\partial v_2^*}{\partial v} \right) \right| \\ & \quad + \left| \left( \int_{\partial D_2} \frac{\partial v_0}{\partial v^-} - \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial v^-} \right) \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \right| + \left| \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial v^-} \left( \int_{\partial \Omega} \frac{\partial v_1}{\partial v} - \int_{\partial \Omega} \frac{\partial v_1^*}{\partial v} \right) \right| \\ & \leq C \|\varphi\|_{L^\infty(\partial \Omega)} \begin{cases} \varepsilon^{3/4}, & \text{if } n = 2, \\ \varepsilon |\log \varepsilon|, & \text{if } n = 3. \end{cases} \end{aligned}$$

So (2.23) is proved.  $\square$

### 3 Proof of Theorem 1.1

Using Lemma 2.6, we have

**Lemma 3.1.** *Assume that  $v_1$  and  $v_1^*$  are solutions of (1.2) and (1.13), respectively. If  $\partial D_1^*$  and  $\partial D_2^*$  are of  $C^{k,1}$ ,  $k \geq 3$ , then for  $\varepsilon^{1/4} \leq |x'| \leq R_0$ , we have*

$$|\nabla v_1(x)| \leq C|x'|^{-2}, \quad x \in \Omega_{R_0} \setminus \Omega_{\varepsilon^{1/4}}, \quad |\nabla v_1^*(x)| \leq C|x'|^{-2}, \quad x \in \Omega_{R_0}^* \setminus \Omega_{\varepsilon^{1/4}}^*; \quad (3.1)$$

and

$$|\nabla(v_1 - v_1^*)(x)| \leq C\varepsilon^{1/2(1-\frac{1}{k})}|x'|^{-2}, \quad \text{in } \Omega_{R_0}^* \setminus \Omega_{\varepsilon^{1/4}}^*. \quad (3.2)$$

*Proof.* For  $\varepsilon^{1/4} \leq |z'| \leq R_0$ , use the change of variable as before

$$\begin{cases} x' - z' = |z'|^2 y', \\ x_n = |z'|^2 y_n, \end{cases}$$

to rescale  $\Omega_{|z'|+|z'|^2} \setminus \Omega_{|z'|}$  into a nearly unit-size square (or cylinder)  $Q_1$ , and  $\Omega_{|z'|+|z'|^2}^* \setminus \Omega_{|z'|}^*$  into  $Q_1^*$ . Let

$$V_1(y) = v_1(z' + |z'|^2 y', |z'|^2 y_n), \quad \text{in } Q_1,$$

and

$$V_1^*(y) = v_1^*(z' + |z'|^2 y', |z'|^2 y_n), \quad \text{in } Q_1^*.$$

Since  $0 < V_1, V_1^* < 1$ , using the standard elliptic estimate, we have

$$|\nabla^k V_1| \leq C(k), \quad \text{in } Q_1, \quad \text{and } |\nabla^k V_1^*| \leq C(k), \quad \text{in } Q_1^*.$$

By using an interpolation with (2.24), we have

$$|\nabla(V_1 - V_1^*)| \leq C(k)\varepsilon^{1/2(1-\frac{1}{k})}, \quad \text{in } Q_1^*.$$

Thus, back to  $v_1 - v_1^*$ , we have

$$|\nabla(v_1 - v_1^*)(x)| \leq C\varepsilon^{1/2(1-\frac{1}{k})}|z'|^{-2}, \quad x \in \Omega_{|z'|+|z'|^2}^* \setminus \Omega_{|z'|}^*.$$

(3.2) follows. By the way,

$$|\nabla v_1(x)| \leq C|z'|^{-2}, \quad x \in \Omega_{|z'|+|z'|^2} \setminus \Omega_{|z'|}, \quad |\nabla v_1^*(x)| \leq C|z'|^{-2}, \quad x \in \Omega_{|z'|+|z'|^2}^* \setminus \Omega_{|z'|}^*.$$

So (3.1) follows.  $\square$

*Proof of Theorem 1.1.* We only prove the case for  $i = 1$  that

$$\rho_n(\varepsilon) \int_{\bar{\Omega}} |\nabla v_1|^2 - \kappa_n = M_1 \rho_n(\varepsilon) + \begin{cases} O\left(\varepsilon^{\frac{1}{4}-\frac{1}{2k}}\right) \rho_n(\varepsilon), & \text{if } n = 2, \\ O\left(\varepsilon^{\frac{1}{2}-\frac{1}{2k}} |\log \varepsilon|\right) \rho_n(\varepsilon), & \text{if } n = 3. \end{cases} \quad (3.3)$$

The case for  $i = 2$  is the same.

**STEP 1.** For  $0 < \gamma \leq 1/4$ , we divide the integral into three parts:

$$\int_{\bar{\Omega}} |\nabla v_1|^2 = \int_{\Omega_{\varepsilon^\gamma}} |\nabla v_1|^2 + \int_{\Omega_{R_0} \setminus \Omega_{\varepsilon^\gamma}} |\nabla v_1|^2 + \int_{\bar{\Omega} \setminus \Omega_{R_0}} |\nabla v_1|^2 =: \text{I} + \text{II} + \text{III}.$$

(i) For the first term I,

$$\int_{\Omega_{\varepsilon^\gamma}} |\nabla v_1|^2 = \int_{\Omega_{\varepsilon^\gamma}} |\nabla \bar{u}|^2 + 2 \int_{\Omega_{\varepsilon^\gamma}} \nabla \bar{u} \cdot \nabla (v_1 - \bar{u}) + \int_{\Omega_{\varepsilon^\gamma}} |\nabla (v_1 - \bar{u})|^2. \quad (3.4)$$

Recalling (1.10), we have

$$\int_{\Omega_{\varepsilon^\gamma}} |\partial_{x'} \bar{u}|^2 \leq C \int_{|x'| < \varepsilon^\gamma} \frac{|x'|^2}{\varepsilon + |x'|^2} dx' \leq C \int_{|x'| < \varepsilon^\gamma} dx' = O\left(\varepsilon^{(n-1)\gamma}\right).$$

By combining Proposition 2.1, we have

$$2 \int_{\Omega_{\varepsilon^\gamma}} \nabla \bar{u} \cdot \nabla (v_1 - \bar{u}) + \int_{\Omega_{\varepsilon^\gamma}} |\nabla (v_1 - \bar{u})|^2 = O\left(\varepsilon^{(n-1)\gamma}\right).$$

Hence, it follows from (3.4) that

$$\text{I} = \int_{\Omega_{\varepsilon^\gamma}} |\nabla v_1|^2 = \int_{\Omega_{\varepsilon^\gamma}} |\partial_{x_n} \bar{u}|^2 + O\left(\varepsilon^{(n-1)\gamma}\right) = \int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} + O\left(\varepsilon^{(n-1)\gamma}\right).$$

(ii) For the second term, we divide it further as follows:

$$\begin{aligned}\Pi &= \int_{\Omega_{R_0} \setminus \Omega_{\varepsilon^\gamma}} |\nabla v_1|^2 = \int_{(\Omega_{R_0} \setminus \Omega_{\varepsilon^\gamma}) \setminus (\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*)} |\nabla v_1|^2 + \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla(v_1 - v_1^*)|^2 \\ &\quad + 2 \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} \nabla v_1^* \cdot \nabla(v_1 - v_1^*) + \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla v_1^*|^2 \\ &=: \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4.\end{aligned}$$

Noting that the thickness of  $(\Omega_{R_0} \setminus \Omega_{\varepsilon^\gamma}) \setminus (\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*)$  is  $\varepsilon$ , and using Lemma 3.1,

$$\Pi_1 = \int_{(\Omega_{R_0} \setminus \Omega_{\varepsilon^\gamma}) \setminus (\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*)} |\nabla v_1|^2 \leq C\varepsilon \int_{\varepsilon^\gamma < |x'| < R_0} \frac{dx'}{|x'|^4} \leq C\varepsilon^{1+(n-5)\gamma}.$$

For any  $\varepsilon^\gamma \leq |z'| \leq R_0$ ,  $0 < \gamma \leq 1/4$ , by Lemma 3.1, if  $\partial D_1^*$  and  $\partial D_2^*$  are of  $C^{k,1}$ ,  $k \geq 3$ , then we have

$$\begin{aligned}\Pi_2 &= \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla(v_1 - v_1^*)|^2 \leq C\varepsilon^{1-\frac{1}{k}} \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |x'|^{-4} dx' dx_n \\ &\leq C\varepsilon^{1-\frac{1}{k}} \int_{\varepsilon^\gamma < |x'| < R_0} \frac{dx'}{|x'|^2} \\ &\leq \begin{cases} C\varepsilon^{1-\frac{1}{k}-\gamma}, & \text{if } n = 2, \\ C\gamma\varepsilon^{1-\frac{1}{k}} |\log \varepsilon|, & \text{if } n = 3, \end{cases}\end{aligned}$$

and

$$|\Pi_3| \leq \left| 2 \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} \nabla v_1^* \cdot \nabla(v_1 - v_1^*) \right| \leq \begin{cases} C\varepsilon^{1/2(1-\frac{1}{k})-\gamma}, & \text{if } n = 2, \\ C\gamma\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|, & \text{if } n = 3. \end{cases}$$

Now, we use the explicit function  $\bar{u}^*$  to approximate  $\nabla v_1^*$ . Using (2.28) and (2.29), a similar argument as in  $I$  yields

$$\begin{aligned}\Pi_4 &= \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla v_1^*|^2 = \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla \bar{u}^*|^2 + 2 \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} \nabla \bar{u}^* \cdot \nabla(v_1^* - \bar{u}^*) + \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\nabla(v_1^* - \bar{u}^*)|^2 \\ &= \int_{\Omega_{R_0}^* \setminus \Omega_{\varepsilon^\gamma}^*} |\partial_{x_n} \bar{u}^*|^2 + A_1 + O(\varepsilon^{(n-1)\gamma}) \\ &= \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{h_1(x') - h_2(x')} + A_1 + O(\varepsilon^{(n-1)\gamma}),\end{aligned}$$

where

$$A_1 := 2 \int_{\Omega_{R_0}^*} \nabla \bar{u}^* \cdot \nabla(v_1^* - \bar{u}^*) + \int_{\Omega_{R_0}^*} (|\nabla(v_1^* - \bar{u}^*)|^2 + |\partial_{x'} \bar{u}^*|^2)$$

is independent of  $\varepsilon$ .

For  $0 < \gamma \leq 1/4$ ,  $\varepsilon^{1+(n-5)\gamma} \leq \varepsilon^{(n-1)\gamma}$ , it follows from these estimates above that

$$\Pi = \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{h_1(x') - h_2(x')} + A_1 + O(\varepsilon^{(n-1)\gamma}) + \begin{cases} O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}), & \text{if } n = 2, \\ O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|), & \text{if } n = 3. \end{cases}$$

(iii) For term III, since

$$\Delta(v_1 - v_1^*) = 0, \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}),$$

and

$$0 < v_1, v_1^* < 1, \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}),$$

it follows that provided  $\partial D_1^*$ ,  $\partial D_2^*$  and  $\partial \Omega$  are of  $C^{k,1}$ ,  $k \geq 3$ ,

$$|\nabla^k(v_1 - v_1^*)| \leq C(k), \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}),$$

where  $C(k)$  is independent of  $\varepsilon$ . By an interpolation inequality with (2.24), we have

$$|\nabla(v_1 - v_1^*)| \leq C\varepsilon^{1/2(1-\frac{1}{k})}, \quad \text{in } \Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0}). \quad (3.5)$$

In view of the boundedness of  $|\nabla v_1|$  in  $(D_1^* \cup D_2^*) \setminus (D_1 \cup D_2 \cup \Omega_{R_0})$  and  $(D_1 \cup D_2) \setminus (D_1^* \cup D_2^*)$ , and the fact that the volume of  $(D_1^* \cup D_2^*) \setminus (D_1 \cup D_2 \cup \Omega_{R_0})$  and  $(D_1 \cup D_2) \setminus (D_1^* \cup D_2^*)$  is less than  $C\varepsilon$ , and by using (3.5), we have

$$\begin{aligned} \text{III} &= \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0})} |\nabla v_1|^2 + O(\varepsilon) \\ &= \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0})} |\nabla v_1^*|^2 + 2 \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0})} \nabla v_1^* \nabla(v_1 - v_1^*) \\ &\quad + \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup D_2^* \cup \Omega_{R_0})} |\nabla(v_1 - v_1^*)|^2 + O(\varepsilon) \\ &= \int_{\tilde{\Omega}^* \setminus \Omega_{R_0}^*} |\nabla v_1^*|^2 + O(\varepsilon^{1/2(1-\frac{1}{k})}). \end{aligned}$$

Now combining (i) (ii) and (iii) and using  $0 < \gamma \leq 1/4$ , we obtain

$$\begin{aligned} \int_{\tilde{\Omega}} |\nabla v_1|^2 &= \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{h_1(x') - h_2(x')} + \int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} \\ &\quad + A_2 + O(\varepsilon^{(n-1)\gamma}) + \begin{cases} O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}), & \text{if } n = 2; \\ O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|), & \text{if } n = 3, \end{cases} \end{aligned} \quad (3.6)$$

where

$$A_2 := \int_{\tilde{\Omega}^* \setminus \Omega_{R_0}^*} |\nabla v_1^*|^2 + A_1.$$

**STEP 2.** After a rotation of the coordinates if necessary, we assume that

$$h_1(x') - h_2(x') = \sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2 + \sum_{|\alpha|=3} C_\alpha x'^\alpha + O(|x'|^4), \quad |x'| \leq R_0, \quad (3.7)$$

where  $\text{diag}(\lambda_1, \dots, \lambda_{n-1}) = \nabla_{x'}^2(h_1 - h_2)(0')$ ,  $C_\alpha$  are some constants,  $\alpha$  is an  $(n-1)$ -dimensional multi-index. We call  $\lambda_1, \dots, \lambda_{n-1}$  the relative principal curvatures of  $\partial D_1$  and  $\partial D_2$ .

To evaluate the first two terms in (3.6), we would like to replace  $h_1(x') - h_2(x')$  by the quadratic polynomial  $\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2$ . First, under the assumption (1.3)–(1.4) and (3.7), we have

$$\begin{aligned}
& \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{h_1(x') - h_2(x')} - \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} \\
&= \int_{R_0 > |x'| > \varepsilon^\gamma} \left[ \frac{1}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2 + \sum_{|\alpha|=3} C_\alpha x'^\alpha + O(|x'|^4)} - \frac{1}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} \right] dx' \\
&= \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{1}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} \left[ \left( 1 + \frac{\sum_{|\alpha|=3} C_\alpha x'^\alpha}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} + O(|x'|^2) \right)^{-1} - 1 \right] dx' \\
&= \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{1}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} \left[ \left( 1 - \frac{\sum_{|\alpha|=3} C_\alpha x'^\alpha}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} + O(|x'|^2) \right) - 1 \right] dx',
\end{aligned}$$

where in last line we use Taylor expansion due to the smallness of  $R_0$ . Note that  $\frac{\sum_\alpha C_\alpha x'^\alpha}{\sum_j (\lambda_j/2) x_j^2}$  is odd and the integrating domain is symmetric, we have

$$\int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{h_1(x') - h_2(x')} - \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} = \int_{R_0 > |x'| > \varepsilon^\gamma} O(1) dx' = \tilde{C} + O(\varepsilon^{(n-1)\gamma}),$$

where  $\tilde{C}$  is some constant depending on  $n, R_0, \lambda_j$  but not  $\varepsilon$ . Similarly, we have

$$\int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} - \int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + \sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} = \int_{|x'| < \varepsilon^\gamma} O(1) dx' = O(\varepsilon^{(n-1)\gamma}).$$

Therefore, (3.6) becomes

$$\begin{aligned}
\int_{\Omega} |\nabla v_1|^2 &= \int_{R_0 > |x'| > \varepsilon^\gamma} \frac{dx'}{\sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} + \int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + \sum_{j=1}^{n-1} \frac{\lambda_j}{2} x_j^2} \\
&\quad + A_3 + O(\varepsilon^{(n-1)\gamma}) + \begin{cases} O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}), & \text{if } n = 2; \\ O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|), & \text{if } n = 3, \end{cases} \tag{3.8}
\end{aligned}$$

where

$$A_3 := A_2 + \tilde{C}.$$

**STEP 3.** Now we deal with the first two explicit terms in (3.8).

(i) For  $n = 2$ , we have

$$\begin{aligned}
& 2 \left( \int_{\varepsilon^\gamma}^{R_0} \frac{dx_1}{\frac{\lambda_1}{2} x_1^2} + \int_0^{\varepsilon^\gamma} \frac{dx_1}{\varepsilon + \frac{\lambda_1}{2} x_1^2} \right) \\
&= 2 \left( \int_{\varepsilon^\gamma}^{R_0} \frac{dx_1}{\frac{\lambda_1}{2} x_1^2} - \int_{\varepsilon^\gamma}^{\infty} \frac{dx_1}{\frac{\lambda_1}{2} x_1^2} \right) + 2 \int_0^{\infty} \frac{dx_1}{\varepsilon + \frac{\lambda_1}{2} x_1^2} + O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}) \\
&= -\frac{4}{\lambda_1 R_0} + \frac{1}{\rho_n(\varepsilon)} \frac{\sqrt{2}\pi}{\sqrt{\lambda_1}} + O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}),
\end{aligned}$$

where we use in second line,

$$\left| \int_{\varepsilon^\gamma}^{\infty} \frac{1}{\varepsilon + \frac{\lambda_1}{2} x_1^2} - \frac{1}{\frac{\lambda_1}{2} x_1^2} dx_1 \right| \leq C\varepsilon \int_{\varepsilon^\gamma}^{\infty} \frac{dx_1}{x_1^4} = O(\varepsilon^{1-3\gamma}) \leq O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}).$$

Therefore,

$$\int_{\bar{\Omega}} |\nabla v_1|^2 = \frac{1}{\rho_n(\varepsilon)} \frac{\sqrt{2}\pi}{\sqrt{\lambda_1}} + (A_3 - \frac{4}{\lambda_1 R_0}) + O(\varepsilon^{(n-1)\gamma}) + O(\varepsilon^{1/2(1-\frac{1}{k})-\gamma}).$$

(ii) For  $n = 3$ ,

$$\begin{aligned} & \int_{\varepsilon^\gamma < |x'| < R_0} \frac{dx'}{\frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} + \int_{|x'| < \varepsilon^\gamma} \frac{dx'}{\varepsilon + \frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} \\ &= \int_{|x'| < R_0} \frac{dx'}{\varepsilon + \frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} + O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|), \end{aligned} \quad (3.9)$$

where we used that

$$\begin{aligned} \left| \int_{\varepsilon^\gamma < |x'| < R_0} \frac{1}{\varepsilon + \frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} - \frac{1}{\frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} dx' \right| &\leq C\varepsilon \int_{\varepsilon^\gamma < |x'| < R_0} \frac{dx'}{|x'|^4} = O(\varepsilon^{1-2\gamma}) \\ &\leq O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|). \end{aligned}$$

Denote  $R(\theta) := R_0(\frac{2}{\lambda_1} \cos^2 \theta + \frac{2}{\lambda_2} \sin^2 \theta)^{-1/2}$ . After a change of variables, the first term of (3.9) becomes

$$\begin{aligned} \int_{|x'| < R_0} \frac{dx'}{\varepsilon + \frac{\lambda_1}{2} x_1^2 + \frac{\lambda_2}{2} x_2^2} &= \frac{2}{\sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \int_0^{R(\theta)} \frac{r}{\varepsilon + r^2} dr d\theta \\ &= \frac{1}{\sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \ln(\varepsilon + r^2) \Big|_{r=0}^{R(\theta)} d\theta \\ &= \frac{1}{\rho_n(\varepsilon)} \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}} + \frac{1}{\sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \ln(R(\theta)^2) + \ln(1 + \frac{\varepsilon}{R(\theta)^2}) d\theta \\ &= \frac{1}{\rho_n(\varepsilon)} \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}} + \frac{2}{\sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \ln R(\theta) d\theta + O(\varepsilon), \end{aligned} \quad (3.10)$$

where we use the fact that  $R(\theta)^2$  has a positive lower bound that is greater than  $\varepsilon$ , and the Taylor expansion of  $\ln(1+x)$ , for  $|x| < 1$ . Combining (3.9) and (3.10), we conclude that for  $n = 3$ ,

$$\begin{aligned} \int_{\bar{\Omega}} |\nabla v_1|^2 &= \frac{1}{\rho_n(\varepsilon)} \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}} + \left( A_3 + \frac{2}{\sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \ln R(\theta) d\theta \right) \\ &\quad + O(\varepsilon^{(n-1)\gamma}) + O(\varepsilon^{1/2(1-\frac{1}{k})} |\log \varepsilon|). \end{aligned}$$

We now define

$$\kappa_n := \begin{cases} \frac{\sqrt{2\pi}}{\sqrt{\lambda_1}}, & n = 2, \\ \frac{2\pi}{\sqrt{\lambda_1\lambda_2}}, & n = 3, \end{cases} \quad \text{and } M_1 := \begin{cases} A_3 - \frac{4}{\lambda_1 R_0}, & n = 2, \\ A_3 + \frac{2}{\sqrt{\lambda_1\lambda_2}} \int_0^{2\pi} \ln R(\theta) d\theta, & n = 3. \end{cases}$$

Taking  $\gamma = 1/4$ ,  $\varepsilon^{(n-1)\gamma}$ ,  $\varepsilon^{1/2(1-\frac{1}{k})-\gamma}$  (or  $\varepsilon^{1/2(1-\frac{1}{k})|\log \varepsilon|}$ ) are smaller than  $\varepsilon^{\frac{1}{4}-\frac{1}{2k}}$  (or  $\varepsilon^{\frac{1}{2}-\frac{1}{2k}|\log \varepsilon|}$ ), and (3.3) is proved. It is not difficult to prove that  $M_1$  is independent of  $R_0$ . If not, suppose that there exist  $M_1(R_0)$  and  $M_1(\tilde{R}_0)$ , both independent of  $\varepsilon$ , such that (3.3) holds, then

$$M_1(R_0) - M_1(\tilde{R}_0) = \begin{cases} O\left(\varepsilon^{\frac{1}{4}-\frac{1}{2k}}\right), & \text{if } n = 2, \\ O\left(\varepsilon^{\frac{1}{2}-\frac{1}{2k}|\log \varepsilon|}\right), & \text{if } n = 3, \end{cases}$$

which implies that  $M_1(R_0) = M_1(\tilde{R}_0)$ .  $\square$

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