

MOMENT IDEALS OF LOCAL DIRAC MIXTURES

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ABSTRACT. In this paper we study ideals arising from moments of local Dirac measures and their mixtures. We provide generators for the case of first order local Diracs and explain how to obtain the moment ideal of the Pareto distribution from them. We then use elimination theory and Prony's method for parameter estimation of finite mixtures. Our results are showcased with applications in signal processing and statistics. We highlight the natural connections to algebraic statistics, combinatorics and applications in analysis throughout the paper.

1. INTRODUCTION

Moments of statistical and stochastic objects have recently gained attention from an algebraic and combinatorial point of view; see [AFS16; IS17; KSS18]. In this paper, we extend those methods to the study of moment ideals of mixture models coming from Dirac measures.

Finite mixture models appear in a wide range of applications in statistics and possess a nice underlying geometric structure as in [Lin83]. They are of use when a population consists of a finite number of homogeneous subpopulations each having its own distribution with density function $\phi_j(x)$. Then the whole population follows a distribution with p.d.f. given by

$$\phi(x) = \sum_{j=1}^r \lambda_j \phi_j(x),$$

where $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^r \lambda_j = 1$. A central problem associated with mixture models is identifying the parameters involved in the distributions of the components as well as the mixing parameters λ_j from a sample. A common approach to this problem is computing the moments of the observed sample and finding the mixture model that best fits the observations.

The moments m_i of a distribution with probability density function ϕ are given by the integrals

$$m_i = \int x^i \phi(x) dx.$$

Moments of mixture distributions are therefore convex combinations of the moments of the components. Despite the integral in the definition, it turns out that for many of the commonly used distributions (Gaussian, Poisson, binomial, ...), the moments are polynomials in the parameters. This allows for a number of algebraic techniques to be used, such as studying determinants of moment matrices in [Lin89], or using polynomial algebra for the Gaussian distribution in work that started with Pearson

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[Pea94], continued with [Mon03] and [Laz04] and was given a systematic algebraic and computational treatment in [AFS16].

In the case when the moments are polynomials or, more generally, rational functions in the parameters, one can study the (projective) variety containing all the points $[m_0 : m_1 : \dots : m_d] \in \mathbb{P}^d$. Using Gröbner basis methods to compute the ideals quickly becomes a computationally intractable problem because these methods are very sensitive to the increase of the number of variables as more mixture components are added. On the geometric side, taking mixtures of a distribution corresponds to obtaining the secants of the moment variety of this distribution, see for example [DSS09, Chapter 4]. Aside from the statistical context, secant varieties play an important role in many other areas, such as in tensor rank and tensor decomposition problems; see [Lan12]. A particular example of this is the symmetric tensor decomposition problem, which can be formulated in terms of homogeneous polynomials and is classically known as Waring's problem: Given a homogeneous polynomial $f \in \mathbb{k}[x_0, \dots, x_n]$ of degree d , find a decomposition $f = L_{\xi_1}^d + \dots + L_{\xi_r}^d$ into powers of linear forms $L_{\xi_j} := \xi_{j0}x_0 + \dots + \xi_{jn}x_n$, $j = 1, \dots, r$, for the smallest possible natural number r . From a measure-theoretic point of view, this involves a mixture of Dirac measures δ_{ξ_j} supported at points $\xi_j \in \mathbb{P}^n$.

Local mixtures are similar to mixture models and they have a fascinating underlying geometric theory; see [Mar02; AM07b]. A finite local mixture of a distribution depending on a parameter ξ with density function $\phi_\xi(x)$ involves adding some variation to the distribution through its derivatives

$$\phi_\xi(x) + \sum_{i=1}^l \alpha_i \phi_\xi^{(i)}(x)$$

for local mixing coefficients α_i . We define l to be the *order* of a local mixture if α_l is non-zero. Adding to the statistics-to-geometry dictionary, the moments of local mixtures of order 1 correspond to taking the tangent variety of the moment variety. Moments of higher-order local mixtures correspond to varieties known as osculating varieties [BCGI07].

In this paper we consider the local mixtures of univariate Dirac measures. We study the varieties associated to their moments and provide a generating set for the first order case in Theorem 3.1 similar to the one in [Eis92]. We use techniques from commutative algebra and combinatorics to prove this result. After reparametrizing the moments of the Pareto distribution, one observes that they are inverses of the first order local Dirac moments, as shown in Section 4. Exploiting this fact and the generators we found, we obtain generators for the moment ideal of the Pareto as well.

In Section 5, we study the problem of identifying the parameters of mixtures of first-order local Dirac distributions. We use the equivalent cumulant coordinates that often simplify the varieties under consideration. In the case of a mixture of two local Diracs the moments are given by

$$m_i = \lambda(\xi_1^i + i\alpha_1\xi_1^{i-1}) + (1 - \lambda)(\xi_2^i + i\alpha_2\xi_2^{i-1}),$$

for a pair of parameters (ξ_i, α_i) from each mixture and a mixing parameter λ . We are able to show that the moment (respectively cumulant) map

$$(\xi_1, \xi_2, \alpha_1, \alpha_2, \lambda) \longmapsto [m_0 : m_1 : \dots : m_d]$$

is finite-to-one for $d = 5$ and one-to-one for $d = 6$. In addition to providing specific polynomials, we also formulate two algorithmic strategies for recovering the parameters of local mixtures. The second algorithm is an extension of Prony's method, which is a tool used in signal processing for recovering mixtures of Dirac distributions; see [PT14] for a survey.

In Section 6, we illustrate the content of the paper numerically by providing an application to the reconstruction of piecewise-polynomial functions from Fourier samples. For smooth functions, the truncated Fourier series provides a good approximation of a signal due to rapid uniform convergence, but in the presence of discontinuities, uniform convergence is lost owing to the occurrence of oscillations known as Gibbs' phenomenon near the singularities. Possible approaches to circumvent this include auto-adaptive spectral approximation [MP09] and Gegenbauer approximation [AG02]. In contrast, we adopt a piecewise-polynomial model, which accounts for discontinuities. Such models are frequently employed in schemes for detection of discontinuities from sampled data [Lee91; Wri10]. Applying our results on parameter recovery of Section 5 allows for the reconstruction of piecewise-polynomial functions from the minimal number of Fourier samples.

The second application comes from local mixture models in statistics [AM07a; AM07b]. These are used for samples whose variation has mostly – but not entirely – been explained by the model. The local components added to the model account for the remaining variation. We express local mixture models as convolutions of a basic model with a local Dirac. We then proceed to present a numerical example where we estimate the mixing parameters using moments coming from a sample of local Gaussians.

2. PRELIMINARIES

2.1. Moments. The k -th moment of the l -th order local mixture of a univariate Dirac measure δ_ξ is given by

$$m_{l,k} = \xi^k + \sum_{i=1}^{\min\{l,k\}} \alpha_i \frac{k!}{(k-i)!} \xi^{k-i}$$

for some given parameters ξ and $\alpha_1, \dots, \alpha_l$; see [Sch73, Chapter 2]. For example, for the first order mixture with $\alpha := \alpha_1$, we obtain $m_0 = 1$, $m_1 = \xi + \alpha$, $m_2 = (\xi + 2\alpha)\xi$, $m_3 = (\xi + 3\alpha)\xi^2$, and so on. Since models focus on some predetermined order l , we omit the l in the subscript whenever the order is clear.

In statistics one has $m_0 = 1$, which leads to the restriction $\sum_{j=0}^l \lambda_j = 1$. For a local mixture, this translates to the condition that the coefficient of the main component, in this case the coefficient of ξ^k , is one; see also [AM07b]. Note that since the components of a local mixture are not necessarily probability distributions one does not require $\alpha_i \geq 0$, but other semialgebraic restrictions are necessary for statistics, see for instance Section 6.2 and [Mar02].

Definition 2.1. Let $f_0, \dots, f_d \in \mathbb{k}(x_1, \dots, x_s)$ be rational functions. The Zariski-closure of the set parametrized by

$$\{[m_0 : \dots : m_d] \in \mathbb{P}^d \mid m_i = f_i(x) \text{ for } x \in \mathbb{k}^s, 0 \leq i \leq d\}$$

is a projective variety of dimension at most s . If the f_i are expressions for the moments of a family of distributions with s parameters, every point in the parametrically given set is a *moment vector* of an element of the family and the variety is called *moment*

variety with respect to d of the family of distributions, which we commonly denote by \mathcal{X}_d .

Throughout this paper, we will denote the homogeneous ideal of \mathcal{X}_d by I_d . Since in statistics the zeroth moment m_0 is equal to 1, we will often work with the corresponding dehomogenized version of the ideal which we denote by \tilde{I}_d .

Even though the distributions we consider are usually defined over the real numbers \mathbb{R} , since the moments are polynomials or rational functions in the parameters, it is often convenient to work with the complexification of the moment varieties, or more generally, to work over any algebraically closed field. Therefore, unless specified otherwise, we will assume \mathbb{k} to be an algebraically closed field of characteristic 0.

2.2. Difference functions. For describing the generators of the moment ideal, the following notion is useful. We define Δ_r as the Vandermonde determinant

$$\Delta_r = \det (X_j^k)_{0 \leq k, j \leq r} = \prod_{0 \leq i < j \leq r} (X_j - X_i),$$

considered as an element of the polynomial ring $\mathbb{k}[X_0, \dots, X_r]$. Thus, for $r = 1$, the powers of Δ_1 are just the higher order differences

$$\begin{aligned} \Delta_1 &= X_1 - X_0, \\ \Delta_1^2 &= X_1^2 - 2X_0X_1 + X_0^2, \\ \Delta_1^3 &= X_1^3 - 3X_0X_1^2 + 3X_0^2X_1 - X_0^3. \end{aligned}$$

We define \mathbb{E} to be the \mathbb{k} -linear map between polynomial rings

$$(2.1) \quad \begin{aligned} \mathbb{E}: \mathbb{k}[X_0, \dots, X_r] &\longrightarrow \mathbb{k}[M_0, M_1, \dots], \\ X_0^{a_0} \dots X_r^{a_r} &\longmapsto M_{a_0} \dots M_{a_r}, \end{aligned}$$

in which we interpret X_0, \dots, X_r as abstract random variables which are viewed as independent replicates of a distribution, that is,

$$\mathbb{E}(X_j^a X_{j'}^{a'}) = \mathbb{E}(X_j^a) \mathbb{E}(X_{j'}^{a'}) = M_a M_{a'}.$$

The map \mathbb{E} can then be understood as the expectation of random variables, mapping any random variable that is a polynomial expression in the variables X_0, \dots, X_r to the abstract moments, which are polynomial expressions in M_a , $a \in \mathbb{N}$. More formally, this interpretation is captured by the concept of Umbral Calculus; see, e.g., [RS00] for a brief overview.

With this notation,

$$\mathbb{E}(X_0^{a_0} \dots X_r^{a_r} \Delta_r^n)$$

is a homogeneous polynomial of degree $r+1$ in the moments M_a , $a \in \mathbb{N}$. For example, for $r = 1$, we get

$$\mathbb{E}(X_0^{a_0} X_1^{a_1} \Delta_1^n) = \sum_{k=0}^n (-1)^k \binom{n}{k} M_{a_0+k} M_{a_1+n-k}.$$

Note that, in the notation of [Eis92], we have $\mathbb{E}(X_0^{a_0} X_1^{a_1} \Delta_1) = \Delta_{a_0, a_1}$ as well as $\mathbb{E}(X_0^{a_0} X_1^{a_1} \Delta_1^3) = \Gamma_{a_0, a_1}$.

When working with finitely many variables M_0, \dots, M_d , we will assume that the map \mathbb{E} is restricted to a suitable subspace.

3. IDEALS OF LOCAL MIXTURES

3.1. Generators of the first order moment ideal. In this section, we focus on the case of local mixtures of order $l = 1$. Our main goal is to find a generating set for the ideal $I_d := \mathcal{I}(\mathcal{X}_d)$, the homogeneous (with respect to the standard grading) ideal of the moment variety. Note that this ideal is also homogeneous with respect to the grading induced by the weight vector $(0, 1, \dots, d)$. The main result we prove here is the following:

Theorem 3.1. *For $d \geq 6$, let J_d be the ideal generated by the $\binom{d-2}{2}$ relations*

$$f_{i,j} := (j - i + 3)M_i M_j - 2(j - i + 2)M_{i-1} M_{j+1} + (j - i + 1)M_{i-2} M_{j+2},$$

for $2 \leq i \leq j \leq d - 2$. Then J_d is equal to I_d , the homogeneous ideal of the moment variety.

We remark here that an alternative equivalent set of generators was given in [Eis92, Section 3]:

Theorem 3.2. *For $d \geq 6$,*

$$I_d = \langle \mathbb{E}(X_0^{a_0} X_1^{a_1} \Delta_1^3) \mid 0 \leq a_0 < a_1 \leq d - 3 \rangle$$

is the ideal generated by $\binom{d-2}{2}$ relations coming from the third powers of Vandermonde determinants.

More explicitly, Theorem 3.2 means that the moment ideal is generated by the $\binom{d-2}{2}$ quadratic relations

$$(3.1) \quad M_i M_{j+3} - 3M_{i+1} M_{j+2} + 3M_{i+2} M_{j+1} - M_{i+3} M_j \text{ for } 0 \leq i < j \leq d - 3.$$

The proof in Eisenbud's paper employs multilinear algebra and representation theory to find the generators of the ideal of the variety. Our proof relies heavily on combinatorics in order to compute the Hilbert functions of the ideals involved, as explained below.

In order to prove Theorem 3.1, we employ the following strategy. First, we work with the dehomogenized version \tilde{J}_d and \tilde{I}_d of the ideals, where we set $M_0 = 1$. The ideal \tilde{J}_d can be seen to be contained in \tilde{I}_d , which is shown in Equation (3.4) below. Then, we use the grading of the polynomial ring $\mathbb{k}[M_1, M_2, \dots, M_d]$ given by the vector $(1, 2, \dots, d)$ as well as combinatorics to show that the ideals in question have the same Hilbert series.

Lemma 3.3. *Let \prec be the monomial order on $\mathbb{k}[M_1, M_2, \dots, M_d]$ that compares monomials with the reverse lexicographical order with $M_1 \prec M_2 \prec \dots \prec M_d$. Then the monomial ideal S_d generated by*

$$\langle M_i M_j \mid 2 \leq i \leq j \leq d - 2 \rangle + \langle M_1 M_i M_{d-1} \mid 2 \leq i \leq d - 2 \rangle + \langle M_1^2 M_{d-1}^2 \rangle$$

is contained in $\text{in}_{\prec} J_d$.

Proof. The degree 2 monomials of S_d are precisely the leading terms of the generators of J_d with respect to \prec . For the degree 3 monomials in the generating set of S_d , one obtains $M_1 M_i M_{d-1}$ for $i = 2, 3, \dots, d - 3$ as the leading term of the S-pair $S(f_{2,i}, f_{2,d-2})$ reduced by the elements in the generating set of J_d . Similarly, the monomial $M_1 M_{d-2} M_{d-1}$ arises as the leading term of the S-pair $S(f_{2,d-2}, f_{d-2,d-2})$ after reducing. Finally, $M_1^2 M_{d-1}^2$ is the leading term of a polynomial in the ideal $\langle f_{2,d-2}, f_{2,d-3}, f_{d-2,d-2} \rangle$. One obtains this polynomial by taking the S-pair of $f_{2,d-2}$

and $f_{d-2,d-2}$, reducing by $f_{2,d-3}$, and then taking the S-pair of the resulting polynomial with $f_{2,d-2}$. \square

Our goal now is to show that the ideal S_d has the same Hilbert series as \tilde{I}_d from which we conclude that it is indeed its initial ideal.

Lemma 3.4. *The ideal S_d defined in Lemma 3.3 has a primary decomposition*

$$(3.2) \quad \langle M_i M_j \mid 1 \leq i \leq j \leq d-2 \rangle \cap \langle M_i M_j \mid 2 \leq i \leq j \leq d-1 \rangle.$$

Proof. Set $C = \langle M_i M_j \mid 2 \leq i \leq j \leq d-2 \rangle$. Then we want to show that

$$S_d = (\langle M_i M_j \mid 1 \leq i \leq d-2 \rangle + C) \cap (C + \langle M_i M_{d-1} \mid 2 \leq i \leq d-1 \rangle).$$

Inclusion of S_d in the other ideal follows by checking that each generator of S_d is divided by some monomial in each part of the decomposition.

For the other inclusion, take a monomial m in the intersection. If the monomial is divided by some monomial in C , it is clearly in S_d . If not, there are two possibilities. The first one is that m is divided by both M_1^2 and M_{d-1}^2 , in which case it is divided by the single generator $M_1^2 M_{d-1}^2$ of S_d . The other one is that m is divided by two monomials $M_1 M_i$ and $M_j M_{d-1}$, with $j \geq 2$ and $i \leq d-2$. In this case, $M_1 M_i M_{d-1}$ and $M_1 M_j M_{d-1}$ both divide m and at least one of them must be in S_d . \square

Consider the polynomial moment map

$$(3.3) \quad \begin{aligned} \text{mom}_d: \mathbb{k}[M_1, M_2, \dots, M_d] &\longrightarrow \mathbb{k}[A, X] \\ M_i &\longmapsto (X + iA)X^{i-1}. \end{aligned}$$

From now on, we consider the grading given by $\deg M_i = i$ on $\mathbb{k}[M_1, M_2, \dots, M_d]$ and the standard grading $\deg A = \deg X = 1$ on $\mathbb{k}[A, X]$. The moment map becomes graded this way. The ideal \tilde{I}_d is the kernel of this map. Using the lemmata above, we are able to compare the Hilbert series of the ideals.

Proof of Theorem 3.1. We show first the equality of the dehomogenized ideals \tilde{I}_d and \tilde{J}_d . The inclusion $\tilde{J}_d \subseteq \tilde{I}_d$ follows from the fact that, for each of the generators of \tilde{J}_d , substituting M_i with its image given by the moment map evaluates to zero. Indeed, we have

$$(3.4) \quad \begin{aligned} &\text{mom}_d((j-i+3)M_i M_j - 2(j-i+2)M_{i-1}M_{j+1} + (j-i+1)M_{i-2}M_{j+2}) \\ &= ((j-i+3) - 2(j-i+2) + (j-i+1))((X^{i+j} + (i+j)AX^{i+j-1} + ijA^2X^{i+j-2}) \\ &= 0. \end{aligned}$$

We now show equality. Let mom_d be the moment map (3.3) and consider the short exact sequences

$$0 \longrightarrow \tilde{I}_d \longrightarrow \mathbb{k}[M_1, M_2, \dots, M_d] \longrightarrow \text{im } \text{mom}_d \longrightarrow 0$$

and

$$0 \longrightarrow S_d \longrightarrow \mathbb{k}[M_1, M_2, \dots, M_d] \longrightarrow \mathbb{k}[M_1, M_2, \dots, M_d]/S_d \longrightarrow 0.$$

Claim 3.5. Let $d \geq 3$. For $n \geq 1$, the vector space in degree n in the image of the moment map has dimension n . Thus, the Hilbert series of the image of the moment map is

$$\text{HS}(\text{im } \text{mom}_d) = \frac{1 - t + t^2}{(1 - t)^2}.$$

Claim 3.6. Let $d \geq 6$. The Hilbert series of the graded algebra $\mathbb{k}[M_1, M_2, \dots, M_d]/S_d$ is

$$\text{HS}(\mathbb{k}[M_1, M_2, \dots, M_d]/S_d) = \frac{1 - t + t^2}{(1 - t)^2}.$$

Since $S_d \subseteq \text{in}_{\prec} \tilde{J}_d \subseteq \text{in}_{\prec} \tilde{I}_d$, for all n we have the inequalities

$$(3.5) \quad \text{HF}(S_d)(n) \leq \text{HF}(\text{in}_{\prec} \tilde{J}_d)(n) \leq \text{HF}(\text{in}_{\prec} \tilde{I}_d)(n) = \text{HF}(\tilde{I}_d)(n).$$

Since the two Hilbert functions of im mom_d and $\mathbb{k}[M_1, M_2, \dots, M_d]/S_d$ coincide, it follows from the two exact sequences above that $\text{HF}(S_d)(n)$ and $\text{HF}(\tilde{I}_d)(n)$ are also equal. Thus, all inequalities in (3.5) are in fact equalities, implying that $\tilde{J}_d = \tilde{I}_d$.

By [CLO15, Section 8.4, Theorem 4], it follows that the Gröbner basis of I_d is the homogenized version of the Gröbner basis for \tilde{I}_d . This can be obtained in both cases by using the Buchberger algorithm on the corresponding generating set given by Theorem 3.1 to obtain polynomials whose initial terms are the ones given by Lemma 3.3. The homogeneous version follows, i.e., I_d is equal to J_d . \square

We finally show the two claims we used in the proof of the Theorem.

Proof of Claim 3.5. We use induction to show that, for each degree n (recall that we are using the standard grading here), the corresponding vector space has n basis elements $X^n + nAX^{n-1}$ and A^iX^{n-i} for $i = 2, 3, \dots, n$ and $n \geq 2$.

For $n = 1$, the corresponding vector space is generated by $X + A$.

For $n = 2$, the only generators are $(X + A)^2$ and the image of M_2 , that is $X^2 + 2AX$. It follows that the vector space in degree 2 is generated by $X^2 + 2AX$ and A^2 .

For $n = 3$, there are three generators $X^3 + 3AX^2$, $(X + A)A^2$ and $(X + A)^3$. One checks that $(X + A)^3 - (X^3 + 3AX^2) = 3A^2X + A^3$ and $(X + A)A^2 = XA^2 + A^3$, which implies the vector space has a basis consisting of the elements $X^3 + 3AX^2$, A^2X and A^3 .

Now assume that the inductive hypothesis is true in degrees n and $n + 1$. Then the basis elements A^iX^{n+2-i} for $i = 4, 5, \dots, n$ in degree $n + 2$ arise by multiplying the corresponding terms in degree n with A^2 . Further, multiplying the terms A^3X^{n-2} , A^2X^{n-1} and $X^{n+1} + (n + 1)AX^n$ of degree $n + 1$ with $X + A$, we obtain $A^3X^{n-1} + A^4X^{n-2}$, $A^2X^n + A^3X^{n-1}$ and $X^{n+2} + (n + 2)AX^{n+1} + (n + 1)A^2X^n$, which give rise to the remaining three required terms.

It remains to show that there exists no combination other than $X^{n+2} + (n + 2)AX^{n+1}$ that involves the terms X^{n+2} , AX^{n+1} . Indeed, any such combination can only arise in the following way: we choose k and l between 0 and $n + 2$ such that $k + l = n + 2$ and we form the product $(X^k + kAX^{k-1})(X + A)^l$. But this is equal to $X^{n+2} + (n + 2)AX^{n+1} + \text{sum of lower terms}$. Thus the given set of polynomials is indeed a vector space basis. \square

Proof of Claim 3.6. We use the primary decomposition of the monomial ideal S_d given by Lemma 3.4 to compute the Hilbert series. By inclusion-exclusion on the primary decomposition, we obtain that $\text{HS}(\mathbb{k}[M_1, M_2, \dots, M_d]/S_d)$ is equal to

$$\begin{aligned} & \text{HS}(\mathbb{k}[M_1, M_2, \dots, M_d]/\langle M_i M_j \mid 1 \leq i \leq j \leq d - 2 \rangle) \\ & + \text{HS}(\mathbb{k}[M_1, M_2, \dots, M_d]/\langle M_i M_j \mid 2 \leq i \leq j \leq d - 1 \rangle) \\ & - \text{HS}(\mathbb{k}[M_1, M_2, \dots, M_d]/\langle M_i M_j \mid 1 \leq i \leq j \leq d - 1 \rangle). \end{aligned}$$

We compute the three Hilbert series separately.

For the first one, note that the vector space in degree n is spanned by monomials in degree n that use the indeterminates M_{d-1} , M_d and maybe one of the M_i for $1 \leq i \leq d-2$ at most once (indeed, any pairwise product of them vanishes in the quotient.) Thus, the cardinality of the vector space is equal to the number of ways of partitioning the number n as a sum using only the numbers $d-1$, d and at most one of the numbers $1, 2, \dots, d-2$ at most once. Hence, the first Hilbert series is

$$\frac{1}{1-t^{d-1}} \frac{1}{1-t^d} (1+t+\dots+t^{d-2}).$$

A similar argument for the second part yields the series

$$\frac{1}{1-t} \frac{1}{1-t^d} (1+t^2+t^3+\dots+t^{d-1}),$$

while for the third one we obtain

$$\frac{1}{1-t^d} (1+t+\dots+t^d).$$

Summing up gives :

$$\begin{aligned} & \frac{1}{1-t^d} \left(\frac{1}{1-t^{d-1}} (1+\dots+t^{d-2}) + \frac{1}{1-t} (1+t^2+t^3+\dots+t^{d-1}) - (1+\dots+t^d) \right) \\ &= \frac{1}{1-t^d} \frac{1}{(1-t)^2} ((1-t) + (1-t) + t^2 - t^d - (1-t)(1-t^{d+1})) \\ &= \frac{1-t+t^2}{(1-t)^2}. \end{aligned}$$

This finishes the proof of the claimed result. \square

Remark 3.7. We have computed above the Hilbert series of our ideals for a special grading. Eisenbud in [Eis92] proved the equivalent result for the standard grading. For $d \geq 6$, the Hilbert series of the ideal I_d is

$$\frac{1 + (d-2)t + (d-2)t^2 + t^3}{(1-t)^3}.$$

Remark 3.8. We apply the methods of [Cil+16] on Cremona linearizations to simplify the description of the moment ideal. For this, let $y_0 := M_0 = 1$ as before and define a polynomial transformation as

$$y_i := \begin{cases} M_i & \text{if } i \leq 3, \\ M_i - z_i & \text{otherwise,} \end{cases}$$

where

$$z_i := \begin{cases} M_i & \text{if } i \leq 3, \\ \frac{1}{2}k(k+1)z_{k-1}z_{k+2} - \frac{1}{2}(k-1)(k+2)z_kz_{k+1} & \text{if } i > 3 \text{ and } i = 2k+1, \\ k^2z_{k-1}z_{k+1} - (k-1)(k+1)z_k^2 & \text{if } i > 3 \text{ and } i = 2k. \end{cases}$$

This map is triangular and thus invertible. Further, one checks that in these variables, if $d \geq 3$, the moment variety is defined by the quartic polynomial equation

$$(3.6) \quad 3y_1^2y_2^2 - 4y_1^3y_3 - 4y_2^3 + 6y_1y_2y_3 - y_3^2 = 0$$

and the equations $0 = y_4 = y_5 = \dots = y_d$. This means that the variety is mapped isomorphically to a surface in a three-dimensional linear subspace.

Note that the polynomial in Equation (3.6) is, up to a constant factor, the discriminant of the polynomial

$$1 + 3y_1X + 3y_2X^2 + y_3X^3.$$

Since the discriminant of a polynomial vanishes whenever it has at least one double root, the moment variety \mathcal{X}_3 is thus parametrized by cubics with a double root. Recall that this is the tangent variety of the Veronese curve. It is depicted in Figure 1.



FIGURE 1. The surface defined by Equation (3.6)

More generally, up to a projective linear transformation, the tangent variety of the Veronese curve has the parametrization

$$\{ [u^{d-1}v] \in \mathbb{P}^d : u, v \text{ linear forms} \}.$$

We refer to [OR14, Corollary, p. 305] for the general description of the generators for the tangential variety of the Veronese variety that also covers the multivariate case, as well as a generalization to tangential varieties of Segre-Veronese varieties.

3.2. Conjectures for higher orders. Note that by replacing i with $i + 2$, the generators of the polynomial in Theorem 3.1 can be written in the more symmetric form $(j - i + 1)M_{i+2}M_j - 2(j - i)M_{i+1}M_{j+1} + (j - i - 1)M_iM_{j+2}$. Computer experiments with Macaulay2 [GS18] seem to suggest the following extrapolation for a 2-local mixture:

Conjecture 3.9. Let $I_{2,d}$ be the ideal of the moments of the second-order local mixture. Then for $d \geq 12$ this ideal is generated by

$$c_0M_{i+3}M_j + c_1M_{i+2}M_{j+1} + c_2M_{i+1}M_{j+2} + c_3M_iM_{j+3}$$

for $i \geq 0, j \geq 0$ and $i \geq j - 3$, where

$$\begin{aligned} c_0 &= (j - i + 1)(j - i + 2) & c_1 &= -3(j - i - 1)(j - i + 2) \\ c_2 &= 3(j - i + 1)(j - i - 2) & c_3 &= -(j - i - 1)(j - i - 2). \end{aligned}$$

The equivalent generators in the notation of Theorem 3.2 are

$$\begin{aligned} \mathbb{E}(X_0^i X_1^j \Delta_1^5) = \\ M_i M_{j+5} - 5M_{i+1} M_{j+4} + 10M_{i+2} M_{j+3} - 10M_{i+3} M_{j+2} + 5M_{i+4} M_{j+1} - M_{i+5} M_j. \end{aligned}$$

Although the second set of generators seems more natural than the first one above, it appears that the methods in [Eis92] do not immediately extend to the higher order case. On the other hand, one could retrace the steps of the proof of Theorem 3.1 to

prove this Conjecture. The main difficulty here is finding the dimension of the vector spaces in the image of the moment map $M_i \mapsto X^i + AiX^{i-1} + B^2i(i-1)X^{i-2}$.

One can generalize this set of generators for higher orders of mixtures:

Conjecture 3.10. Let $I_{l,d}$ be the ideal of the moments of the l -th order local mixture. Then for d sufficiently large, this ideal is generated by the polynomials

$$\mathbb{E}(X_0^{a_0} X_1^{a_1} \Delta_1^{2l+1}) = \sum_{k=0}^{2l+1} (-1)^k \binom{2l+1}{k} M_{a_0+k} M_{a_1+2l+1-k}.$$

4. PARETO DISTRIBUTION

The Pareto distribution was introduced by Vilfredo Pareto as a model for income distribution, see [Arn83]. It is a heavy-tailed continuous probability distribution that finds a wide range of applications, especially in econometrics. In the univariate case, its probability density function is

$$\varphi(x) := \frac{\alpha \xi^\alpha}{x^{\alpha+1}} \mathbb{1}_{\{x \geq \xi\}},$$

where $\alpha, \xi \in \mathbb{R}_{>0}$. The moments of this distribution are given by

$$m_i = \begin{cases} \frac{\alpha}{\alpha-i} \xi^i, & i < \alpha, \\ \infty, & i \geq \alpha; \end{cases}$$

see [Arn83]. We show below how to reparametrize them so that the moments of the Pareto are the inverses of the first order local mixture of Diracs and exploit this fact to obtain generators of the Pareto moment ideal.

4.1. Ideal generators. Algebraically, we are interested in the cases for which the moments are finite, which are described by the image of the map

$$\begin{aligned} \mathbb{R}_{>d} \times \mathbb{R}_{>0} &\longrightarrow \mathbb{P}^d, \\ (\alpha, \xi) &\longmapsto [m_0 : \dots : m_d] = \left[\frac{\alpha}{\alpha-i} \xi^i \right]_{0 \leq i \leq d}, \end{aligned}$$

for a given $d \in \mathbb{N}$, where \mathbb{P}^d denotes the projective space over \mathbb{C} . We define the moment variety of the Pareto distribution as the Zariski-closure over \mathbb{C} of the image of the above map. Since $\mathbb{R}_{>d}$ is Zariski-dense in \mathbb{C} , we may extend the domain of the parametrization to $(\mathbb{C} \setminus \{0, \dots, d\}) \times \mathbb{C}^\times$ without changing the Zariski-closure of the image. Let ρ be the corresponding map $\rho: (\mathbb{C} \setminus \{0, \dots, d\}) \times \mathbb{C}^\times \rightarrow \mathbb{P}^d$.

Proposition 4.1. *Let $\mathcal{Y} := \overline{\text{im}(\rho)} \subseteq \mathbb{P}^d$ be the Pareto moment variety and $\mathcal{X} \subseteq \mathbb{P}^d$ the moment variety of 1-local mixtures, that is, the tangent variety of the Veronese curve. Then \mathcal{X} and \mathcal{Y} are birationally equivalent via the rational map*

$$\psi: \mathbb{P}^d \dashrightarrow \mathbb{P}^d, \quad [m_0 : \dots : m_d] \longmapsto [m_0^{-1} : \dots : m_d^{-1}].$$

Proof. We change the parametrization of the Pareto moment variety via the bijective map

$$\begin{aligned} \eta: \{(\alpha, \xi) \in \mathbb{C}^\times \times \mathbb{C}^\times \mid -\xi\alpha^{-1} \neq 1, \dots, d\} &\longrightarrow (\mathbb{C} \setminus \{0, \dots, d\}) \times \mathbb{C}^\times, \\ (\alpha, \xi) &\longmapsto (-\xi\alpha^{-1}, \xi^{-1}), \end{aligned}$$

which leaves \mathcal{Y} as the closure of the image of $\rho \circ \eta$ unchanged. With this parametrization, the moments are of the form

$$[m_0 : \cdots : m_d] = \rho(\eta(\alpha, \xi)) = \left[\frac{-\xi\alpha^{-1}}{-\xi\alpha^{-1} - i} \xi^{-i} \right]_{0 \leq i \leq d} = \left[\frac{1}{\xi^i + i\alpha\xi^{i-1}} \right]_{0 \leq i \leq d},$$

so ψ maps points from the image of $\rho \circ \eta$ to moment vectors of 1-local mixtures, that is, to points on \mathcal{X} . Then $\psi|_{\mathcal{Y}}: \mathcal{Y} \dashrightarrow \mathcal{X}$ is a rational map that is an isomorphism on the dense subset $\text{im}(\rho \circ \eta)$ of \mathcal{Y} , as $\text{im}(\rho \circ \eta) \subseteq \{m_i \neq 0\}$. Being the tangent variety of an irreducible variety, \mathcal{X} is irreducible by [Lan12, Section 8.1]. Thus, the image of $\psi|_{\mathcal{Y}}$ is dense in \mathcal{X} which proves the claim. \square

Theorem 4.2. *For $d \geq 6$, let $\tilde{I}_d^{\text{inv}} \subseteq R := \mathbb{C}[M_1, \dots, M_d]$ be the ideal generated by the $\binom{d-2}{2}$ polynomials*

$$(j-i+3)M_{i-2}M_{i-1}M_{j+1}M_{j+2} - 2(j-i+2)M_{i-2}M_iM_jM_{j+2} + (j-i+1)M_{i-1}M_iM_jM_{j+1}$$

for $2 \leq i \leq j \leq d-2$, where we assume $M_0 = 1$. Then the affine Pareto moment ideal is equal to the saturation

$$\tilde{I}_d^{\text{inv}} : (M_1 \cdots M_d)^\infty.$$

Proof. Let $\tilde{I}_d \subseteq \mathbb{C}[M_1, \dots, M_d]$ be the dehomogenization of the moment ideal of local mixtures of Diracs which was studied in Section 3.1. In order to restrict to the algebraic torus where the rational map ψ given in Proposition 4.1 is defined, consider $J := R[y]\tilde{I}_d + \langle M_1 \cdots M_d y - 1 \rangle \subseteq R[y]$. The restriction of the map ψ to the torus agrees with the torus automorphism induced by the homomorphism

$$\begin{aligned} \bar{\psi}: R[y] &\longrightarrow R[y], \\ y &\longmapsto M_1 \cdots M_d, \\ M_i &\longmapsto M_1 \cdots M_{i-1} M_{i+1} \cdots M_d y. \end{aligned}$$

Note that we can choose an ideal $I' \subseteq R$ such that $\bar{\psi}(J) = R[y]I' + \langle M_1 \cdots M_d y - 1 \rangle$ by observing that, for any $f \in \tilde{I}_d$, we can choose a suitable $k \in \mathbb{N}$ such that $(M_1 \cdots M_d)^k \bar{\psi}(f) \equiv f' \pmod{\langle M_1 \cdots M_d y - 1 \rangle}$ for some $f' \in R$. In particular, this construction establishes a bijection between the generating set of \tilde{I}_d given in Theorem 3.1 and the generating set of \tilde{I}_d^{inv} . Therefore, we choose $I' := \tilde{I}_d^{\text{inv}}$. In order to describe the affine closure of the image of ψ , we intersect $\bar{\psi}(J)$ with R , which is equal to

$$\bar{\psi}(J) \cap R = \tilde{I}_d^{\text{inv}} : (M_1 \cdots M_d)^\infty$$

by [CLO15, Theorem 4.4.14] from which we conclude. \square

The statement of the Theorem also holds for the ideal we get if, in the construction, we replace the generators of Theorem 3.1 by those in (3.1). Taking the saturation in the construction is necessary in order to prevent the variety from containing additional irreducible components that are supported on the boundary of the algebraic torus, which cannot be part of the Pareto moment variety. In computations we carried out, the ideal \tilde{I}_d^{inv} has a smaller primary decomposition and performs faster with computer algebra systems when compared with its counterpart implied by (3.1).

5. RECOVERY OF PARAMETERS

In this section we use the method of moments [BS06] to estimate the parameters of mixtures of Dirac local mixture distributions. In statistics, one often starts with a measurement from a population or a sample following a particular distribution. From this, one can compute the empirical moments (or equivalently the cumulants) and try to deduce the original parameters of the underlying distribution. By contrast, in signal processing, one usually obtains the moments of a measure directly using Fourier methods.

5.1. Cumulants. So far in this paper, we focused on moment coordinates. Cumulants are an alternative set of associated quantities that are well-known to statisticians and have recently become an object of study for algebraists. The cumulants k_i of a distribution can be given as an invertible polynomial transformation of the moments and they carry the same information. However, for many interesting distributions studied in the literature, they have a simpler form than the moments and doing computations with them empirically turns out to be faster, which becomes very useful when Gröbner basis computations are involved. In the univariate case, let

$$M(t) = \sum_{i \geq 0} m_i \frac{t^i}{i!} \quad \text{and} \quad K(t) = \sum_{i \geq 0} k_i \frac{t^i}{i!}$$

be the moment and cumulant generating functions respectively. One can symbolically compute the relationship between moments and cumulants using the relations

$$M(t) = \exp K(t) \quad \text{and} \quad K(t) = \log M(t).$$

For example, up to degree 5, we obtain

$$\begin{aligned} (5.1) \quad & k_0 = 0, \\ & k_1 = m_1, \\ & k_2 = m_2 - m_1^2, \\ & k_3 = m_3 - 3m_1m_2 + 2m_1^3, \\ & k_4 = m_4 - 4m_1m_3 - 3m_2^2 + 12m_1^2m_2 - 6m_1^4, \\ & k_5 = m_5 - 5m_1m_4 - 10m_2m_3 + 20m_1^2m_3 + 30m_1m_2^2 - 60m_1^3m_2 + 24m_1^5, \end{aligned}$$

and more generally $k_d = m_d + p_d(m_1, m_2, \dots, m_{d-1})$, where p_d is some polynomial. In particular, the first cumulant is the mean and the second cumulant the variance of the distribution.

Notice that the cumulants are given in a triangular form and can therefore be inverted to give the moments as functions thereof. Precise (and in our opinion beautiful) combinatorial formulas giving the cumulants as functions of the moments can be found in Chapter 4 of [Zwi15].

Another useful property of cumulants is translation invariance: Adding a quantity q to each element in a sample only increases the first cumulant by q , while all higher cumulants remain the same. One often exploits this property by assuming that the mean $m_1 = k_1$ is zero.

5.2. Identifiability of finite mixtures by elimination theory. In this subsection we apply elimination theory [CLO15] to recover the parameters of a mixture distribution. We transform polynomials into their cumulant versions instead of moments because computations are significantly faster. We present an algorithm to do this

and explicitly perform computations in the case of a mixture with two components using the computer algebra system Macaulay2 [GS18].

In this case, the i -th moment is given by

$$m_i = \lambda(\xi_1^i + i\alpha_1\xi_1^{i-1}) + (1 - \lambda)(\xi_2^i + i\alpha_2\xi_2^{i-1}).$$

Transforming into cumulants by using the equations (5.1), we obtain polynomials

$$(5.2) \quad \begin{aligned} f_1 &= -k_1 + \lambda(\xi_1 + \alpha_1) + (1 - \lambda)(\xi_2 + \alpha_2), \\ f_2 &= -k_2 + \lambda(\xi_1^2 + 2\alpha_1\xi_1) + (1 - \lambda)(\xi_2^2 + 2\alpha_2\xi_2) \\ &\quad - (\lambda(\xi_1 + \alpha_1) + (1 - \lambda)(\xi_2 + \alpha_2))^2, \\ f_3 &= -k_3 + \lambda(\xi_1^3 + 3\alpha_1\xi_1^2) + (1 - \lambda)(\xi_2^3 + 3\alpha_2\xi_2^2) \\ &\quad - 3(\lambda(\xi_1 + \alpha_1) + (1 - \lambda)(\xi_2 + \alpha_2))(\lambda(\xi_1^2 + 2\alpha_1\xi_1) + (1 - \lambda)(\xi_2^2 + 2\alpha_2\xi_2)) \\ &\quad + 2(\lambda(\xi_1 + \alpha_1) + (1 - \lambda)(\xi_2 + \alpha_2))^3, \end{aligned}$$

and so on. Instead of using elimination theory directly on the variables ξ_i , we rather look at their symmetric polynomials $\xi_1\xi_2$ and $\xi_1 + \xi_2$, as was done in [AFS16].

Define a polynomial $f_s = s - (\xi_1 + \xi_2)$ and consider the ideal generated by $\langle f_1, \dots, f_5, f_s \rangle$. By eliminating the variables $\alpha_1, \alpha_2, \xi_1, \xi_2, \lambda$, we obtain an ideal generated by the single polynomial

$$\begin{aligned} g_s &= (4k_2^3 + k_3^2)s^4 \\ &\quad - (32k_1k_2^3 + 24k_2^2k_3 + 8k_1k_3^2 + 4k_3k_4)s^3 \\ &\quad + (96k_1^2k_2^3 + 24k_2^4 + 144k_1k_2^2k_3 + 24k_1^2k_3^2 + 36k_2k_3^2 + 20k_2^2k_4 + 24k_1k_3k_4 + 4k_4^2 \\ &\quad + 2k_3k_5)s^2 \\ &\quad - (128k_1^3k_2^3 + 96k_1k_2^4 + 288k_1^2k_2^2k_3 + 32k_1^3k_3^2 + 80k_2^3k_3 + 144k_1k_2k_3^2 + 80k_1k_2^2k_4 \\ &\quad + 48k_1^2k_3k_4 + 8k_3^3 + 40k_2k_3k_4 + 16k_1k_4^2 + 8k_2^2k_5 + 8k_1k_3k_5 + 4k_4k_5)s \\ &\quad + (64k_1^4k_2^3 + 96k_1^2k_2^4 + 192k_1^3k_2^2k_3 + 16k_1^4k_3^2 + 160k_1k_2^3k_3 + 144k_1^2k_2k_3^2 + 80k_1^2k_2^2k_4 \\ &\quad + 32k_1^3k_3k_4 + 72k_2^2k_3^2 + 16k_1k_3^3 + 80k_1k_2k_3k_4 + 16k_1^2k_4^2 + 16k_1k_2^2k_5 + 8k_1^2k_3k_5 \\ &\quad + 4k_3^2k_4 + 16k_2k_3k_5 + 8k_1k_4k_5 + k_5^2) \end{aligned}$$

that has degree 4 as a polynomial in the variable s over $\mathbb{k}[k_1, \dots, k_5]$.

If we substitute numerical values of the first five cumulants, we can algebraically identify the parameters of the distribution. Indeed, for every solution for s of the polynomial g_s above, we uniquely recover the original parameters as follows. First, we eliminate the variables $(\alpha_1, \alpha_2, \xi_1, \xi_2, \lambda)$ from the ideal $\langle f_1, \dots, f_5, f_s, f_p \rangle$, where $f_p = p - \xi_1\xi_2$. Inside the generators, we find the polynomial

$$(2sk_2 - 2k_3)p + 6sk_2^2 - s^2k_3 - 10k_2k_3 + 2sk_4 - k_5$$

which is linear in p and thus we can substitute all values to identify p , which allows us to obtain a pair of values for ξ_1 and ξ_2 .

In order to determine the remaining parameters $\lambda, \alpha_1, \alpha_2$, define

$$\lambda'_1 := \lambda\alpha_1, \quad \lambda'_2 := (1 - \lambda)\alpha_2$$

and observe that, in these new parameters, the moments depend only linearly on $\lambda, \lambda'_1, \lambda'_2$, as we have

$$m_i = \lambda\xi_1^i + \lambda'_1 i\xi_1^{i-1} + (1 - \lambda)\xi_2^i + \lambda'_2 i\xi_2^{i-1}.$$

Thus, from a computational point of view, the main difficulty lies in finding the points ξ_1, ξ_2 .

This procedure also implies that the cumulant (respectively moment) map sending $(\alpha_1, \alpha_2, \xi_1, \xi_2, \lambda)$ to (k_1, \dots, k_5) is generically four-to-one.

What is further, using the first six cumulants allows us to rationally identify the parameters. Indeed, let f_6 be the polynomial as in Equation (5.2) that contains the information about the sixth cumulant. By eliminating the five variables $\alpha_1, \alpha_2, \xi_1, \xi_2, \lambda$ from $\langle f_1, \dots, f_6, f_s \rangle$ we obtained an ideal that contains a polynomial of degree 1 in s . This polynomial has 412 terms and is too long to report here. In this case the cumulant map is one-to-one.

Using the polynomials in this section, one can substitute the values of the cumulants coming from any sample of a two component mixture and recover the parameters. The process we describe above can potentially be adjusted for mixtures of Dirac local mixtures with any number of components and local mixture depth.

We phrase the above process for a mixture of two first order local Diracs as an algorithmic strategy for parameter estimation. We remark here that that we write down the following primarily as an experimental process and we do not provide a proof that it works in all cases.

Algorithm 1 Parameter recovery for local Dirac mixtures with r components of order l

Input: The order $l \geq 0$ of the mixture components, the number of mixture components $r \geq 1$ and the moments $m_0 = 1, m_1, \dots, m_{(l+2)r}$.

Output: The parameters ξ_j, λ_j and α_{jk} for $1 \leq j \leq r, 1 \leq k \leq l$.

- 1: Let $f_1, \dots, f_{(l+2)r-1}$ be polynomials whose zeros give the first $(l+2)r-1$ moments (or cumulants) as functions of the parameters, as for example in Equation (5.2).
- 2: Let h_{p_i} be the i -th elementary symmetric polynomial on ξ_j for $i = 1, 2, \dots, r$, so for example $h_{p_1} = \xi_1 + \dots + \xi_r$ and $h_{p_r} = \xi_1 \dots \xi_r$, and set $f_{p_i} = p_i - h_{p_i}$.
- 3: Eliminate the parameter variables ξ_j, λ_j and α_{jk} from the ideal

$$\langle f_1, \dots, f_{(l+2)r-1}, f_{p_1} \rangle$$

to obtain a polynomial g_1 in $\mathbb{K}[m_1, \dots, m_{(l+2)r-1}][p_1]$.

- 4: Using separately each of $f_{p_2}, f_{p_3}, \dots, f_{p_r}$, obtain polynomials g_2, g_3, \dots, g_r in $\mathbb{K}[m_1, \dots, m_{(l+2)r-1}][p_1, p_i]$ for $i = 2, 3, \dots, r$.
 - 5: Substitute the values for the moments (or cumulants) from a sample and solve the equations to get a list of potential values for the p_i .
 - 6: Use the values of the p_i to obtain the ξ_j . From those, the rest of the parameters can be obtained.
 - 7: Use some method to choose the parameters that best fit the sample, such as the additional moment $m_{(l+2)r}$.
-

In case of a probability distribution, a practical thing to do in Step 7 would be to check, for which $((l+2)r-1)$ -tuples of ξ_j, λ_j and α_{jk} , the λ_j are real numbers between 0 and 1 and discard the rest of the solutions. Then one can check which parameter set gives an $(l+2)r$ -th moment that is closest to the empirical moment $m_{(l+2)r}$.

5.3. Prony's method. In the following, we describe an algorithm for parameter recovery that is motivated by Prony's method. Prony's method is a widely used tool

in signal processing that is useful for recovering sparse signals from Fourier samples and dates back to [Pro95]. Here, we closely follow the discussion of Prony's method in [Mou18] because it covers the case of multiplicities. The variant that we use is summed up in Theorem 5.1 below.

We first fix some notation. Denote by $\mathbb{k}[X]_{\leq a}$ the vector subspace of polynomials of degree at most $a \in \mathbb{N}$ and let $\mathbb{k}[X]^* := \text{Hom}(\mathbb{k}[X], \mathbb{k})$ be the dual \mathbb{k} -vector space of the polynomial ring $\mathbb{k}[X]$. Given any sequence $(m_i)_{i \in \mathbb{N}}$, $m_i \in \mathbb{k}$, define $\sigma \in \mathbb{k}[X]^*$ to be the \mathbb{k} -linear functional

$$\sigma: \mathbb{k}[X] \longrightarrow \mathbb{k}, \quad X^i \longmapsto m_i.$$

Hence, σ is the composition of the map \mathbb{E} of (2.1) and the evaluation map $M_i \mapsto m_i$. Further, let M_σ be the \mathbb{k} -linear operator

$$M_\sigma: \mathbb{k}[X] \longrightarrow \mathbb{k}[X]^*, \quad p \longmapsto (q \mapsto \sigma(pq)).$$

In the \mathbb{k} -vector space basis X^i and the dual basis $(X^j)^*$, this map is described by an infinite Hankel matrix with entries m_{i+j} ; see [Mou18, Remark 2.1]. Denote by $M_{a,b}$ the truncation of M_σ to degrees a, b

$$(5.3) \quad M_{a,b}: \mathbb{k}[X]_{\leq b} \longrightarrow \mathbb{k}[X]_{\leq a}^*.$$

The matrix $M_{a,b} = (m_{i+j})_{0 \leq i \leq a, 0 \leq j \leq b}$ is of size $(a+1) \times (b+1)$. If $(m_i)_{i \in \mathbb{N}}$ is a sequence of moments of some distribution, $M_{a,b}$ is called *moment matrix*.

Assume now we are given an r -mixture of local Dirac mixture distributions of the form $\sum_{j=1}^r \lambda_{j0} \delta_{\xi_j} + \lambda_{j1} \delta'_{\xi_j} + \dots + \lambda_{jl_j} \delta_{\xi_j}^{(l_j)}$ for some $\xi_j, \lambda_{j,k_j} \in \mathbb{k}$, $0 \leq k_j \leq l_j$, $1 \leq j \leq r$. Then its moments are of the form

$$(5.4) \quad m_i = \sum_{j=1}^r \sum_{k_j=0}^{l_j} \lambda_{j,k_j} \frac{i!}{(i-k_j)!} \xi_j^{i-k_j} = \sum_{j=1}^r (\Lambda_j(\partial)(X^i))(\xi_j) \in \mathbb{k}, \quad i \in \mathbb{N},$$

where $\Lambda_j(\partial) := \sum_{k=0}^{l_j} \lambda_{j,k} \partial^k \in \mathbb{k}[\partial]$ is a polynomial of degree l_j in ∂ that is applied to the monomial X^i as a differential operator. We cite the following theorem in order to rephrase it in our language. In particular, we specialize it to the univariate case. We remark that the theorem can also be understood in terms of the canonical form of a binary form. For a detailed treatment of this viewpoint, we refer to [IK99] and the references therein.

Theorem 5.1 ([IK99, Theorem 1.43], [Mou18, Theorem 4.1]). *Let \mathbb{k} be an algebraically closed field of characteristic 0 and let $m_0, m_1, \dots, m_{2s} \in \mathbb{k}$ for some $s \in \mathbb{N}$. Let $M_{s-1,s-1}, M_{s,s}$ be the corresponding Hankel matrices as in (5.3). Assume $\text{rk } M_{s-1,s-1} = \text{rk } M_{s,s} = r'$. Then there exists a unique mixture of local mixtures of Diracs*

$$\mu := \sum_{j=1}^r \Lambda_j(\partial) \delta_{\xi_j}$$

for some $r \in \mathbb{N}$, $\xi_j \in \mathbb{k}$, $0 \neq \Lambda_j \in \mathbb{k}[\partial]$, such that $\sum_{j=1}^r 1 + \deg(\Lambda_j) = r'$ and its moments up to degree $2s$ coincide with m_0, \dots, m_{2s} . Further, as ideals of $\mathbb{k}[X]$, it holds that

$$\mathbb{k}[X] \cdot \ker M_{s-1,s} = \bigcap_{j=1}^r \langle X - \xi_j \rangle^{1+\deg \Lambda_j}.$$

Note that the condition $\sum_{j=1}^r 1 + \deg(\Lambda_j) = r'$ is due to the fact that $\deg(\Lambda_j) + 1$ is the dimension of the vector space spanned by Λ_j and all its derivatives, as we specialized to the univariate case.

This leads to a two-step algorithm for recovering the parameters of μ from finitely many moments: first, compute the points ξ_1, \dots, ξ_r from $\ker M_{s-1,s}$ for s sufficiently large; next, determine the weights Λ_j from (5.4). If $\deg \Lambda_j = 0$ for all $j = 1, \dots, r$, this algorithm is classically known as *Prony's method*, and is also referred to as *Sylvester's algorithm* in the classical algebraic geometry literature. See for instance [IK99; Lan12] and the references therein.

In the following, we refine this algorithm for the case of mixtures of local mixtures of Diracs of fixed order $l := l_1 = \dots = l_r$ where $l_j = \deg \Lambda_j$ for $j = 1, \dots, r$. In this case, it is usually possible to recover the parameters from significantly fewer moments.

Proposition 5.2. *Let \mathbb{k} be a field of characteristic 0 and let $\mu := \sum_{j=1}^r \Lambda_j(\partial) \delta_{\xi_j}$ be an r -mixture of l -th order local mixtures of Diracs, i. e., $\xi_j \in \mathbb{k}$ and $\Lambda_j \in \mathbb{k}[\partial]$ with $\deg(\Lambda_j) = l$, $1 \leq j \leq r$. Then, the parameters Λ_j, ξ_j of μ can be recovered from the moments $m_0, m_1, \dots, m_{2(l+1)r-1}$ of μ using Algorithm 2.*

Proof. Let σ and M_σ be defined as above. Then, by [Mou18, Theorem 3.1], we have $\text{rk } M_\sigma = (l+1)r$. It follows from [Mou18, Proposition 3.9] that $\text{rk } M_{a,b} = (l+1)r$ for all $a, b \geq (l+1)r - 1$. In particular, for $s := (l+1)r$, we have

$$\text{rk } M_{s-1,s-1} = \text{rk } M_{s,s} = (l+1)r.$$

The algorithm is based on the following observation. Let p be the polynomial $p := \prod_{j=1}^r (X - \xi_j) = X^r + \sum_{i=0}^{r-1} p_i X^i$, noting that knowledge of p is enough to recover the points ξ_j . By the addendum of Theorem 5.1, we have $p^{l+1} \in (\mathbb{k}[x] \cdot \ker M_{s-1,s}) \otimes_{\mathbb{k}} \bar{\mathbb{k}}$ where $\bar{\mathbb{k}}$ is the algebraic closure of \mathbb{k} . Since also $p^{l+1} \in \mathbb{k}[x]$, it follows in particular that p is mapped to 0 under the composition of the maps

$$\begin{aligned} \mathbb{k}[X]_{\leq r} &\longrightarrow \mathbb{k}[X]_{\leq (l+1)r} \longrightarrow \mathbb{k}[X]_{\leq r-1}^*, \\ q &\longmapsto q^{l+1} \longmapsto M_{r-1,(l+1)r} q^{l+1}, \end{aligned}$$

where the second map is the \mathbb{k} -linear map given by the moment matrix $M_{r-1,(l+1)r}$, which is a submatrix of $M_{s-1,s}$. The first map however is non-linear, defined by taking the $(l+1)$ -th power of q viewed as a polynomial.

For the polynomial p , this yields the following polynomial system of r equations of degree $l+1$ in r variables p_0, \dots, p_{r-1} which are the monomial coefficients of p :

$$(5.5) \quad M_{r-1,(l+1)r} p^{l+1} = 0.$$

By Bézout's theorem, this system of equations either has infinitely many or at most $(l+1)^r$ solutions. If the solution set is infinite, we need to add more algebraic constraints to the system in order to determine p , which is done by adding more rows to the moment matrix.

By hypothesis, we have $\xi_1, \dots, \xi_r \in \mathbb{k}$. Therefore, termination of this algorithm and correct recovery of the points ξ_1, \dots, ξ_r follow from Theorem 5.1.

As for computation of the weights λ_{jk} in Step 5, note that, once the roots ξ_j have been computed, the moments are a linear combination of the monomials ξ_j^i and their

derivatives given by (5.4), so to compute the weights λ_{jk} , we solve the linear system

$$(V_1, \dots, V_r) \begin{pmatrix} \lambda_{10} \\ \vdots \\ \lambda_{1l} \\ \vdots \\ \lambda_{rl} \end{pmatrix} = \begin{pmatrix} m_0 \\ \vdots \\ m_d \end{pmatrix}$$

for $d \geq s$, where (V_1, \dots, V_r) is a *confluent Vandermonde matrix*, for which each block is given by

$$V_j = ((\partial^k X^i)(\xi_j))_{\substack{0 \leq i \leq d, \\ 0 \leq k \leq l}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \xi_j & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \xi_j^d & d\xi_j^{d-1} & \cdots & \frac{d!}{(d-l)!} \xi_j^{d-l} \end{pmatrix}.$$

Since the system is linear, uniqueness of the solution follows from the claim that the confluent Vandermonde matrix is of full rank s . Without loss of generality, we can assume the confluent Vandermonde matrix to be of size $s \times s$ by choosing a suitable submatrix. Then the claim follows from the fact that the Hermite interpolation problem has a unique solution if the points ξ_1, \dots, ξ_r are distinct or, equivalently, from the product formula for the determinant of a square confluent Vandermonde matrix; see [HJ94, Problem 6.1.12]. \square

Algorithm 2 Parameter recovery for mixtures with r components of order l

Input: The (maximum) order $l \geq 0$ of the mixture components, the number of mixture components $r \geq 1$ and the moments $m_0, \dots, m_{(l+2)r}, \dots$

Output: The parameters ξ_j and λ_{jk} for $1 \leq j \leq r$, $0 \leq k \leq l$, satisfying Equation (5.4).

- 1: Solve the polynomial system $M_{r-1, (l+1)r} p^{l+1} = 0$ for p .
 - 2: If the solution set is infinite, increase the number of rows of the moment matrix and repeat.
 - 3: If there is more than one solution, use further information, such as the additional moment $m_{(l+2)r}$, to restrict to a single solution p .
 - 4: Compute the roots ξ_1, \dots, ξ_r of p .
 - 5: Compute the weights λ_{jk} by solving a confluent Vandermonde system.
-

Note that the algorithm is designed to use as few moments as possible. See also Remark 5.4 for a discussion of the number of moments used by this algorithm.

Example 5.3. For $r = 2$, $l = 1$, let m_0, \dots, m_5 be the moments of a corresponding distribution and write $p = p_0 + p_1 X + X^2$. Then the system of equations (5.5) is given by the quadratic equations

$$(5.6) \quad \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & m_4 \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{pmatrix} \begin{pmatrix} p_0^2 \\ 2p_0 p_1 \\ 2p_0 + p_1^2 \\ 2p_1 \\ 1 \end{pmatrix} = 0.$$

If ξ_1, ξ_2 are the points of the underlying distribution, one solution of this system is given by $p = (X - \xi_1)(X - \xi_2)$, that is, $p_1 = -(\xi_1 + \xi_2)$ and $p_0 = \xi_1 \xi_2$. Hence, computing p_1 by eliminating p_0 from System (5.6), and vice versa, is equivalent to the process of recovering the parameters from elementary symmetric polynomials outlined in Section 5.2. However, with the presented new approach, we need to eliminate only a single variable instead of 5, which makes this much more viable computationally.

Remark 5.4. We discuss some of the steps involved in Algorithm 2. Solving the system in Step 1 can be done using symbolic methods, as outlined in the previous sections, or using numerical tools. In Example 6.2 for instance, we use a numerical solver for this which is based on homotopy continuation methods.

Restricting from finitely many solutions to a single one using the additional moment $m_{(l+2)r}$ in Step 3 works by observing that $M_{r,(l+1)r} p^{l+1} = 0$. If a numerical solver is used, the computed solution will only be approximately zero, and one should assert that the selected solution is significantly closer to zero than all other possible choices. Another common approach to check uniqueness of the solution is to perform monodromy loop computations using a homotopy solver.

In the worst case, this algorithm makes use of moments $m_0, \dots, m_{2(l+1)r-1}$ as stated in Proposition 5.2. Then solving the polynomial system in Algorithm 2 simplifies, since the solution is in the kernel of $M_{(l+1)r-1,(l+1)r}$ which is a linear problem. In this case, the algorithm performs the same computation as [Mou18, Algorithm 3.2], so this guarantees termination.

However, as Algorithm 2 solves a more specific problem, it can usually recover the parameters using a smaller number of moments. The polynomial system in Step 1 consists of r equations of degree $l+1$ in r unknowns, so, generically, we expect finitely many solutions in Step 2 already in the first iteration of the algorithm. This means we expect to algebraically identify the parameters from the moments $m_0, \dots, m_{(l+2)r-1}$ and to rationally identify them using one additional moment, so usually we do not need all the moments up to $m_{2(l+1)r-1}$. This is also what we observe in practice, so we do not get infinitely many solutions for generic input if we use the moments up to $m_{(l+2)r-1}$. By a parameter count, we cannot expect to be able to recover the parameters from fewer moments, so the number of moments we use in practice is the minimal number possible.

We use the term *algebraic identifiability* in the same way as in [ARS17], that is, meaning that the map from the parameters to the moments is generically finite-to-one. In this case, the *identifiability degree* is the cardinality of the preimage of a generic point in the image of the moment map (up to permutation). Similarly, *rational identifiability* means that the moment map is generically one-to-one.

Proposition 5.5. *Let $d \geq (l+2)r - 1$. Then algebraic identifiability holds for the moment map sending the parameters ξ_j, Λ_j to the moments m_0, \dots, m_d , where $\deg \Lambda_j = l$, $1 \leq j \leq r$.*

Proof. By [CGG02, Proposition 3.1], the secant varieties of the tangent variety of the Veronese curve are non-defective, that is, for $l = 1$, the dimension of the moment variety in \mathbb{P}^d for mixtures with r components of order l is the expected one: $\min(3r - 1, d)$. In particular this means that the moment variety fills the ambient space sharply if $d = 3r - 1$ and does not fill the ambient space if $d > 3r - 1$. Thus, the moment map is generically finite-to-one if $d \geq 3r - 1$. Note that for $d < 3r - 1$ the cardinality of the preimage of a generic point is infinite for dimension reasons.

Similarly, for $l \geq 2$, the moment variety is a secant variety of the l -th osculating variety to the Veronese curve which is non-defective by [BCGI07, Section 4], so the moment map is generically finite-to-one for $d \geq (l+2)r - 1$. \square

We do not currently know whether rational identifiability holds as soon as $d \geq (l+2)r$, although we expect this to be true as discussed in Remark 5.4.

Open Problem 5.6. The computations we have done in this section suggest that the algebraic identifiability degree for a mixture with r components of order l is $(l+1)^r$, which is the expected number of solutions of Equation (5.5).

Remark 5.7. It would be interesting to generalize Algorithm 2 to the multivariate setting. Note that in this case [Mou18, Algorithm 3.2] can be used to find the decomposition. However, since this does not take into account the special structure of our input, namely that all the mixture distributions have the same order l , this approach might use more moments than necessary. This is similar to the univariate case as explained in Remark 5.4.

Further, the algorithm in [BT18a, Section 6] also computes a generalized decomposition from a given set of moments. This algorithm differs from our Algorithm 2 in that it computes parameters of *any* generalized decomposition explaining the given moments, rather than the unique decomposition in which each term corresponds to the same order l . In the one-dimensional case, when using as few moments as possible, this usually leads to a non-generalized decomposition, which does not recover the parameters we are interested in. See also the related discussion in [BT18a, Section 7.1].

Remark 5.8. We briefly discuss how the problem of parameter recovery of a mixture of 1-local mixtures simplifies, if the mixture components $\delta_{\xi_j} + \alpha_j \delta'_{\xi_j}$, $1 \leq j \leq r$, are known to differ only in the parameters ξ_j , but have a constant parameter $\alpha := \alpha_1 = \dots = \alpha_r$. For this, let X be a distribution with moments $\mathbb{E}(X^i) = \sum_{j=1}^r \lambda_j (\xi_j^i + \alpha i \xi_j^{i-1})$ with a fixed parameter α . Further, let Y be the distribution with moments $\mathbb{E}(Y^i) = \sum_{j=1}^r \lambda_j \xi_j^i$. Then we have

$$\mathbb{E}(X^i) = \mathbb{E}(Y^i) + \alpha i \mathbb{E}(Y^{i-1})$$

and conversely

$$\mathbb{E}(Y^i) = \sum_{k=0}^i \frac{i!}{k!} (-\alpha)^{i-k} \mathbb{E}(X^k).$$

Hence, if α is known, this allows to recover the mixing distribution Y from the moments of X . The parameters of Y can then be recovered, e.g., using Prony's method.

In case α is fixed, but unknown, treating α as a variable in the moment matrix $M_r(Y) = (\mathbb{E}(Y^{i+j}))_{0 \leq i, j \leq r}$, it can be determined as one of the roots of $\det M_r(Y)$, which is a polynomial of degree $r(r+1)$ in α .

6. APPLICATIONS

6.1. Moments and Fourier coefficients. In this section, we show how the tools developed in this paper can be applied to the problem of recovering a piecewise-polynomial function supported on the interval $[-\pi, \pi[$ from Fourier samples; see [PT14]. For this, we describe how moments of a mixture of local mixtures arise as the

Fourier coefficients of a piecewise-polynomial function and illustrate this numerically. For simplicity, we focus on the case $l = 1$ of piecewise-linear functions.

Let $t_j \in [-\pi, \pi]$, $1 \leq j \leq r$, be real points and let $f: [-\pi, \pi[\rightarrow \mathbb{C}$ be the piecewise-linear function given by

$$(6.1) \quad f(x) := \sum_{j=1}^{r-1} (f_j + (x - t_j)f'_j) \mathbb{1}_{[t_j, t_{j+1}[}(x),$$

where $f_j, f'_j \in \mathbb{C}$. In particular, splines of degree 1 are of this form, but we do not require continuity here. The Fourier coefficients of f are defined to be

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

from which we obtain

$$c_k = \frac{1}{2\pi(ik)^2} \sum_{j=1}^r (ik(f_j - f_{j-1} + (t_{j-1} - t_j)f'_{j-1}) + (f'_j - f'_{j-1})) e^{-ikt_j},$$

for $k \in \mathbb{Z} \setminus \{0\}$, where $f_0, f'_0, f_r, f'_r := 0$. Further, let

$$(6.2) \quad \begin{aligned} \xi_j &:= e^{-it_j}, \\ \lambda_j &:= \xi_j^{-s} (f'_j - f'_{j-1} - is(f_j - f_{j-1} + (t_{j-1} - t_j)f'_{j-1})), \\ \lambda'_j &:= \xi_j^{1-s} i(f_j - f_{j-1} + (t_{j-1} - t_j)f'_{j-1}). \end{aligned}$$

Assume now, we are given finitely many Fourier coefficients c_{-s}, \dots, c_s for some $s \in \mathbb{N}$. Then, for $0 \leq k \leq 2s$, $k \neq s$, we define

$$(6.3) \quad m_k := 2\pi(i(k-s))^2 c_{k-s} = \sum_{j=1}^r \lambda_j \xi_j^k + \lambda'_j k \xi_j^{k-1}.$$

Thus, from the knowledge of Fourier coefficients c_{-s}, \dots, c_s of f , we can compute m_k , $k \neq s$, which we interpret as the moments of a mixture of 1-local mixtures with support points ξ_j on the unit circle. Extending the definition to m_s , by construction we have

$$m_s := \sum_{j=1}^r \lambda_j \xi_j^s + \lambda'_j s \xi_j^{s-1} = \sum_{j=1}^r f'_j - f'_{j-1} = 0.$$

All in all, we know the moments m_0, \dots, m_{2s} of this 1-local mixture. Recovering the parameters $\xi_j, \lambda_j, \lambda'_j$ via Algorithm 2 generically requires the moments m_0, \dots, m_{3r} , so we need to choose $2s \geq 3r$. Subsequently retrieving the original parameters t_j, f_j, f'_j from (6.2) is straightforward.

Remark 6.1. The piecewise-linear function f is viewed as a periodic function on the interval $[-\pi, \pi[$, in the discussion above. For simplicity in presentation, we assumed that f is constantly zero outside of $[t_1, t_r[$, representing a constant line segment. More generally, one can adapt the computation to account for an additional (non-zero) line segment there, without changing the number of jumping points or required samples. Thus, the function f consists of r line segments and has r discontinuities.

Example 6.2. We apply the process described above to a piecewise-linear function with $r = 10$ line segments on the interval $[-\pi, \pi[$. The parameters t_j, f_j, f'_j defining the function as in (6.1) are listed in Table 1. The random jump points t_j are chosen uniformly on the interval. The jump heights f_j and slopes f'_j are chosen with respect

to a Gaussian distribution. The function as well as the sampling data are visualized in Figure 2. By Fourier transform, the Fourier coefficients carry the same information as the evaluations of the Fourier partial sum at equidistantly-spaced sampling points. The number of sampling points equals the number of Fourier coefficients needed for reconstruction, namely $3r+1 = 31$. From $2s \geq 3r$, we determine $s = 15$. We compute the Fourier coefficients c_{-s}, \dots, c_s from the given data and add some noise to each of these coefficients, sampled from a Gaussian distribution with standard deviation 10^{-12} .

j	t_j	f_j	f'_j
1	-2.814030328751694	-0.20121264876344414	-0.775069863870378
2	-2.457537611167516	-0.35221920435611676	-0.9795392068942285
3	-1.4536804635810938	-0.9254256123988903	0.26040229778962753
4	-1.1734228328971805	0.4482105605664995	-0.46848914917290574
5	-0.6568874684874002	1.11978779941218	-0.8808972481620518
6	0.54049294753688	0.3012272070859375	0.2777255506414151
7	1.0213620344785337	-0.8357295816882367	1.5239161501048377
8	1.0930147137662223	-0.2071744440917742	-1.7777276640658903
9	1.6867064885416054	0.8681006042361324	-2.9330595087256466
10	2.7678373800858678		

TABLE 1. The parameters of the piecewise-linear function of Example 6.2.

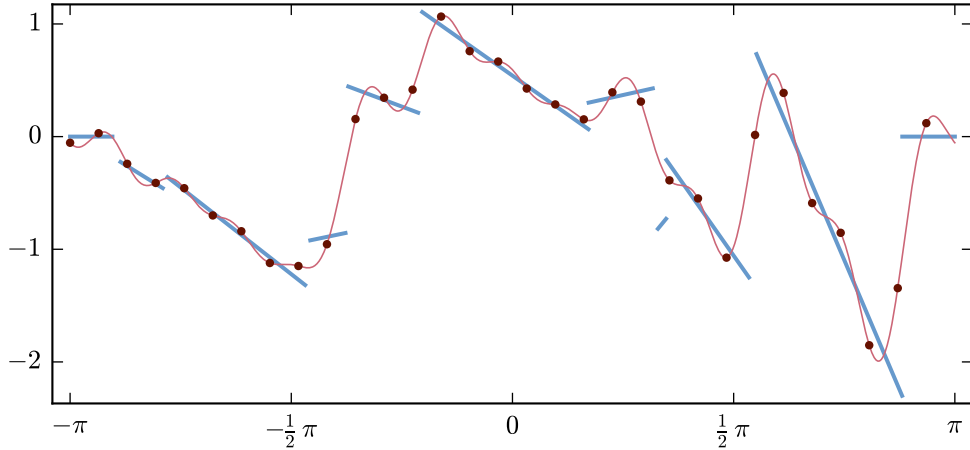


FIGURE 2. The piecewise-linear function of Example 6.2 with $r = 10$ line segments; the Fourier partial sum approximation of order $s = 15$ and $2s + 1 = 31$ equidistantly-spaced sampling points.

In order to reconstruct the piecewise-linear function from the Fourier coefficients, we compute the moments m_0, \dots, m_{3r} via Equation (6.3) and apply Algorithm 2 using numerical tools. From the moments m_0, \dots, m_{3r-1} , we get a system of r quadratic equations in r unknowns, which we solve using the Julia package *HomotopyContinuation.jl* [BT18b], version 0.3.2, from which we obtain up to 2^r finite solutions. From these, we choose the one that best solves the equation system $M_{r,2r}p^2 = 0$ induced by the additional moment m_{3r} . In this example, we obtain 1024 solutions, the best of which has error $1.54 \cdot 10^{-10}$ in the ℓ^2 -norm; the second best solution has error $3.70 \cdot 10^{-4}$, which is significantly larger, so we accept the solution.

Next, we compute the points ξ_j using the Julia package *PolynomialRoots.jl* [SG12], version 0.2.0, and solve an overdetermined confluent Vandermonde system for the

weights λ_j, λ'_j , for which we use a built-in least-squares solver. Lastly, we use (6.2) to compute the parameters t_j, f_j, f'_j . Julia code for these computations can be found in the ancillary files of the ArXiv-version of this article. The numerical computations were carried out using the Julia language [BEKS17], version 1.0.0.

In this example, the total error we get for the reconstructed points t_1, \dots, t_{10} is $3.89 \cdot 10^{-10}$ in the ℓ^2 -norm, whereas for f_j and f'_j , $1 \leq j \leq 9$, we get $2.15 \cdot 10^{-7}$ and $2.35 \cdot 10^{-7}$, respectively. Even though, in this example, one of the line segments is quite far off of the Fourier partial sum, as shown in Figure 2, the sampling data still contains enough information to reconstruct it.

We observe that we cannot always reconstruct the randomly chosen points correctly using homotopy continuation, but many times reconstruction is successful. We expect that the separation distance among the points plays a major part in numerical reconstruction. If the randomly chosen points are badly separated, it will be difficult to distinguish them numerically by just using the moments, as is the case if $l = 0$; see [Moi15].

Further, we observe that, after having obtained the points, solving the confluent Vandermonde system often induces additional errors of about three orders of magnitude, resulting from the possibly bad condition of the confluent Vandermonde matrix. A detailed discussion of this condition number exceeds the scope of this paper, so we leave it for further study.

Remark 6.3. If $l > 1$, one can adapt Algorithm 2 in a similar fashion to the reconstruction of functions that are piecewisely defined by polynomials of degree l . As this requires solving a system of polynomial equations of degree l , the involved computations are more challenging. Note however that, under additional assumptions on the smoothness of the function, computations can be reduced to a polynomial system of smaller degree. For example, if we let $l = 3$ and additionally impose C^1 -continuity, the second derivative is piecewise-linear, so reconstruction can be accomplished by applying the method outlined above.

6.2. Local mixture distributions. Local mixture models have been proposed as a means of dealing with small variation that is unaccounted for when fitting data to a model, see [AM07b] and the references therein. In this case the model is augmented by truncating a Taylor-like expansion of the probability density function.

Definition 6.4. The local mixture model of a regular exponential family $\phi_\xi(x)$ is

$$\psi_\xi(x) := \phi_\xi(x) + \sum_{i=1}^l \alpha_i \phi_\xi^{(i)}(x),$$

for parameters $\alpha_1, \dots, \alpha_l$ such that $\psi_\xi(x) \geq 0$ for all x .

The local mixture model defined this way is a convolution of a local Dirac mixture with the member of the exponential family centered at 0, i. e.,

$$\psi_\xi = \phi_0 * \left(\delta_\xi + \sum_{i=1}^l \alpha_i \delta_\xi^{(i)} \right).$$

Remark 6.5. Let $M_1(t), M_2(t)$ be the moment generating functions of the two distributions in the convolution, respectively, i. e., $M_2(t) := \sum_{k \geq 0} m_k \frac{t^k}{k!}$ where

$$m_k = \xi^k + \sum_{i=1}^{\min\{l, k\}} (-1)^i \alpha_i \frac{k!}{k-i!} \xi^{k-i}.$$

Note that the sign changes are due to the property of the derivative of the Dirac distribution that

$$\int \phi(x) d\delta_\xi^{(i)}(x) = (-1)^i \phi^{(i)}(\xi).$$

Then the product $M_1(t)M_2(t)$ is the moment generating function corresponding to ψ_ξ , so the moments m_k of the underlying local Dirac mixture can be computed if the moments of a random variable with density ψ_ξ as well as the moments corresponding to ϕ_0 are known. Thus, we can reduce the problem of parameter inference to the problem of parameter recovery of a local Dirac mixture.

Example 6.6 (A mixture of two local Gaussians with known common variance). We numerically apply the process outlined above to a Gaussian distribution. For this, let ϕ_0 be the density of a standard Gaussian distribution and let $\psi := \lambda\psi_{\xi_1} + (1 - \lambda)\psi_{\xi_2}$ be a 2-mixture of local distributions of order 2, where $\psi_{\xi_j} = \phi_{\xi_j} + \alpha_{j1}\phi'_{\xi_j} + \alpha_{j2}\phi''_{\xi_j}$. We choose the parameters as follows: $(\xi_1, \alpha_{11}, \alpha_{12}) = (-1, 0.1, 0.4)$, $(\xi_2, \alpha_{21}, \alpha_{22}) = (2, -0.2, 0.6)$ and $\lambda = 0.6$. Note that the ψ_{ξ_j} are non-negative for this choice of α_{ji} , so they are indeed probability density functions; see [Mar02, Example 4]. We create a sample of size 20,000 from this probability distribution using Mathematica [Wol18] and compute the empirical moments of that sample. Using that, we derive the empirical moments of the underlying 2-mixture of local Dirac distributions as explained in Remark 6.5 and then apply Algorithm 2 to infer the parameters. We obtain the following values:

$$\begin{aligned} \xi_1 &= -0.98121, & \alpha_{11} &= 0.14076, & \alpha_{12} &= 0.39268, & \lambda &= 0.59457, \\ \xi_2 &= 1.95600, & \alpha_{21} &= -0.20486, & \alpha_{22} &= 0.62641. \end{aligned}$$

Note that for this process the distribution ϕ_0 is assumed to be known. In particular, we need to know its standard deviation or have a way of estimating it. The reconstructed parameters only provide a rough approximation to the original parameters. Increasing the sample size can give a better approximation. The original distribution and the reconstructed one are shown in Figure 3.

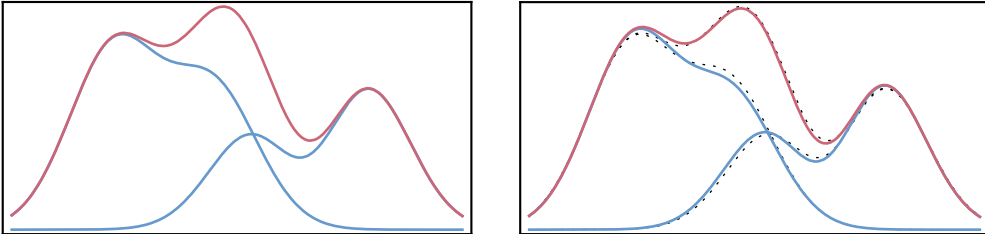


FIGURE 3. The 2-mixture of local Gaussian distributions of Example 6.6 on the left, as well as its reconstruction on the right; the dotted lines are the original distribution for comparison.

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