Minimum T-Joins and Signed-Circuit Covering

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Abstract

Let G be a graph and T be a vertex subset of G with even cardinality. A T-join of G is a subset J of edges such that a vertex of G is incident with an odd number of edges in J if and only if the vertex belongs to T. Minimum T-joins have many applications in combinatorial optimizations. In this paper, we show that a minimum T-join of a connected graph G has at most $|E(G)| - \frac{1}{2}|E(\widehat{G})|$ edges where \widehat{G} is the maximum bidegeless subgraph of G. Further, we are able to use this result to show that every flow-admissible signed graph (G, σ) has a signed-circuit cover with length at most $\frac{19}{6}|E(G)|$.

Keywords: T-joins, circuit covers, signed graphs,

1 Introduction

In this paper, a graph may have multiple edges and loops. A loop is also treated as an edge. We follow the notation from the book [25]. A cycle is a connected 2-regular graph. An even graph is a graph in which every vertex has even degree. An Eulerian graph is a connected even graph. In a graph G, a circuit is the same as a cycle, which is minimal dependent set of the graphic matroid defined on G. A circuit cover C of a graph is a family of circuits which cover all edges of G. The length of a circuit cover is defined as $\ell(\mathbb{C}) = \sum_{C \in \mathbb{C}} |E(C)|$. It is a classic optimization problem initiated by Itai et. al. [18] to investigate circuit covers of graphs with shortest length. Thomassen [22] show that it is NP-complete to determine whether a bridgeless graph has a circuit cover with length at most k for a given integer k. A well-known conjecture, the Shortest Cycle Cover Conjecture, in this area was made by Alon and Tarsi [1] as follows.

Conjecture 1.1 (Alon and Tarsi [1]). Every bridgeless graph G has a circuit cover with length at most $\frac{7}{5}|E(G)|$.

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If the Shortest Cycle Cover Conjecture is true, then the bound given in the conjecture will be optimal as the shortest cycle cover of Petersen graph attains the bound. It is known that there connections of the Shortest Cycle Cover Conjecture to other problems in graph theory, such as, the Jeager's Petersen Flow Conjecture [14] and the Circuit Double Cover Conjecture (cf. [15]) due to Seymour [20] and Szekeres [19]. The best bound toward the Shortest Cycle Cover Conjecture is due to Bermond, Jackson and Jaeger [2], and independently due to Alon and Tarsi [1] as follows, which was further generalized by Fan [10] to weighted bridgeless graphs.

Theorem 1.2 (Bermond, Jackson and Jaeger [2], Alon and Tarsi [1]). Every bridgeless graph G has a circuit cover with length at most $\frac{5}{3}|E(G)|$.

The circuit cover problem is studied for matroids (cf. [12, 21]). Recently, a great deal of attention has been paid to the shortest circuit cover problem of signed graphs (or signed-graphical matroids) (cf. [4, 5, 16, 17, 23]).

A signed graph is a graph G associated with a mapping $\sigma : E(G) \to \{-1, +1\}$. A cycle of a signed graph (G, σ) is positive if it contains an even number of negative edges, and negative otherwise. A short barbell is a union of two negative cycles C_1 and C_2 such that $C_1 \cap C_2$ is a single vertex, and a long barbell consists of two disjoint negative cycles C_1 and C_2 joined by a minimal path P (which does not contain a subpath joining C_1 and C_2). A barbell of a signed graph (G, σ) could be a short barbell or a long barbell. A circuit or signed-circuit of a signed graph (G, σ) is a positive cycle or a barbell, which is a minimal dependent set in the signed graphic matroid associated with (G, σ) . A signed-circuit admits a nowhere-zero 3-flow [3]. A signed graph with a signed-circuit cover must admits a nowhere-zero integer flow. (For definition of nowhere-zero flow, readers may refer to [3, 25].) In fact, a signed graph has a signed-circuit cover if and only if it admits a nowhere-zero flow, so-called, flow-admissible [3].

It has been evident in [23] that a 3-connected signed-graph may not have a signed-circuit double cover, i.e., every edge is covered twice. This fact leads the current authors to ask, whether every flow-admissible signed graph (G, σ) has a signed-circuit cover with length at most 2|E(G)|? The first bound on the shortest circuit cover of signed graphs was obtained by Máčajová et. al. [17] as follows.

Theorem 1.3 (Máčajová, Raspaud, Rollová and Škoviera, [17]). Every flow-admissible signed graph (G, σ) has a signed-circuit cover with length at most 11|E(G)|.

The above result has been significently improved by Cheng et. al. [5] as follows.

Theorem 1.4 (Cheng, Lu, Luo and Zhang [5]). Every flow-admissible signed graph (G, σ) has a signedcircuit cover with length at most $\frac{14}{3}|E(G)|$.

The bound of Cheng et. al. has been further improved recently by Chen and Fan [4].

Theorem 1.5 (Chen and Fan [4]). Every flow-admissible signed graph (G, σ) has a signed-circuit cover with length at most $\frac{25}{6}|E(G)|$.

Recently, Kaiser et. al. [16] announce that every flow-admissible signed graph has a signed-circuit cover with length at most $\frac{11}{3}|E(G)|$. For a 2-edge-connected cubic signed graph (G, σ) , the present authors [23] show that (G, σ) has a signed-circuit cover with length less than $\frac{26}{9}|E(G)|$.

In this paper, we investigate the minimum T-join and the shortest signed-circuit cover problem. We show that a minimum T-join of a connected graph G has at most $|E(G)| - \frac{1}{2}|E(\widehat{G})|$ edges where \widehat{G} is the maximal bridgeless subgraph of G. The bound is tight. Further, we are able to use the bound of the size of a minimum T-join to show the following result, which improves previous bounds on the length of shortest signed-circuit cover.

Theorem 1.6. Every flow-admissible signed graph (G, σ) has a signed-circuit cover with length less than $\frac{19}{6}|E(G)|$.

Particularly, if (G, σ) is 2-edge-connected and has even negativeness (a detailed definition is given in the next section), then (G, σ) has a signed-circuit cover with length less than $\frac{8}{3}|E(G)|$.

2 Circuit covers and *T*-joins

Let G be a graph. The maximal bridgeless subgraph of G is denoted by \widehat{G} . If G is a bridgeless graph, then $\widehat{G} = G$. A *circuit k-cover* of a graph G is a family of circuits which covers every edge of G exactly k times.

Theorem 2.1 (Bermond, Jackson and Jeager [2]). Every bridgeless graph G has a 4-cover by 7 evensubgraphs.

Let G be a graph and T be a set of vertices of G. A T-join J of G with respect to T is a subset of edges of G such that $d_J(v) \equiv 1 \pmod{2}$ if and only if $v \in T$, where $d_J(v)$ is the number of edges of J incident with the vertex v. A T-join in a connected graph G is easy to construct: partition T into |T|/2pairs of vertices; and, for each pair of vertices, find a path joining them; and then the symmetric difference of all edge sets of these |T|/2 paths generates a T-join of G. On the other hand, the subgraph induced by a T-join always contains |T|/2 paths joining |T|/2 pairs of vertices of T which form a partition of T. The symmetric difference of a T-join and the edge set of an even-subgraph is another T-join of G.

A *T*-join is minimum if it has minimum number of edges among all *T*-joins. A minimum *T*-join *J* induces an acyclic subgraph of *G* (otherwise, the symmetric difference of *J* and a cycle induced by edges of *J* is a smaller *T*-join of *G*). There is a strong polynomial-time algorithm to find a minimum *T*-join in a given graph *G* (see [7]). The minimum *T*-join problem and its variations have applications to many other problems in combinatorial optimization and graph theory, for example, the Chinese postman problem [7], Traveling Salesman Problem (TSP) [6] and others (see [11, 13]).

Theorem 2.2. Let G be a connected graph and T be a set of vertices of even size. Then a minimum T-join of G has at most $|E(G)| - \frac{1}{2}|E(\widehat{G})|$ edges.

Proof. Let G be a connected graph and J be a minimum T-join of G. Note that \widehat{G} is bridgeless. By Lemma 2.1, \widehat{G} has 7-even-subgraph 4-cover $\{H_1, H_2, \ldots, H_7\}$. Since each H_i is an even-subgraph, the symmetric difference $H_i \oplus J$ is still a T-join of G, denoted by J_i for $i \in \{1, 2, \ldots, 7\}$. Let $\mathcal{J} = \{J, J_1, \ldots, J_7\}$. By the choice of J, it holds that $|J| \leq |J_i|$ for $i \in \{1, 2, \ldots, 7\}$.

Let e be an edge of \widehat{G} . If $e \notin J$, then $e \in J_i$ if and only if $e \in E(H_i)$ for $i \in \{1, \ldots, 7\}$. Hence e is covered exactly four times by the family of T-joins \mathcal{J} if $e \notin J$. If $e \in J$, then $e \in J_i$ if and only if $e \notin E(H_i)$ for $i \in \{1, \ldots, 7\}$. Since $\{H_1, \ldots, H_7\}$ covers e exactly four times, there are exactly three T-joins in $\{J_1, \ldots, J_7\}$ containing e. Together with J, the family of T-joins \mathcal{J} covers e exactly four times. Therefore, \mathcal{J} covers every edge e of \widehat{G} exactly four times.

For an edge $e \in E(G) \setminus E(\widehat{G})$, it is covered by \mathcal{J} at most eight times. Hence, J covers every edge of \widehat{G} exactly four times and other edges of G at most eight times. Therefore,

$$|J| \le \frac{1}{8} \Big(|J| + \sum_{i=1}^{7} |E(J_i)| \Big) \le \frac{1}{8} \Big(4|\widehat{G}| + 8(|E(G)| - |E(\widehat{G})|) \Big)$$
$$= |E(G)| - \frac{1}{2} |E(\widehat{G})|.$$

This completes the proof.

For 2-edge-connected graphs, the following result is a direct corollary of Theorem 2.2.

Corollary 2.3. Let G be a 2-edge-connected graph and T be an even subset of vertices. Then G has a T-join of size at most |E(G)|/2.

Remark. The bounds in Theorem 2.2 and Corollary 2.3 are sharp. For example, G is a tree with all even-degree vertices having degree two and T is the set of all odd-degree vertices, or G is an even cycle and T consists of the two antipodal vertices.

In the following, Theorem 2.2 and Corollary 2.3 will be applied to prove technical lemmas on signedcircuit covers, which are particularly useful in the proofs of our main results.

For a signed graph (G, σ) , denote the set of all negative edges of (G, σ) by $E^{-}(G, \sigma)$ and the set of all positive edges by $E^{+}(G, \sigma)$. We always use G^{+} to denote the subgraph of G induced by all edges in $E^{+}(G, \sigma)$. Then $\widehat{G^{+}}$ stands for the maximal bridgeless subgraph of G^{+} .

Lemma 2.4. Let (G, σ) be a signed graph such that G^+ is connected. For any $S \subseteq E^-(G, \sigma)$, the signed graph (G, σ) has an even-subgraph H such that

$$S \subseteq H \subseteq G^+ \cup S$$
 and $|E(H)| \le |E(G)| - \frac{1}{2}|E(\widehat{G^+})|.$

Proof. Let (G, σ) is a signed graph with G^+ being connected. For any $S \subseteq E^-(G, \sigma)$, let G[S] be the subgraph G[S] of G induced by edges in S. Let T be the set of all vertices of odd-degree in G[S], which has an even number of vertices. Let J be a minimum T-join of G^+ with respect to T. By Theorem 2.2, it follows that $|J| \leq |E(G^+)| - \frac{1}{2}|E(\widehat{G^+})|$.

Let $H = G[S] \cup J$. Then H is an even-subgraph which satisfies

$$S \subseteq H \subseteq G^+ \cup S.$$

Since $S \cap E(G^+) = \emptyset$, it follows

$$|E(H)| = |E(G[S] \cup J)| = |S| + |J| \le |S| + |E(G^+)| - \frac{1}{2}|E(\widehat{G^+})| \le |E(G)| - \frac{1}{2}|E(\widehat{G^+})|.$$

This completes the proof.

Now, we are going to define a function on signed graphs, which plays a key role in our proofs. Let \mathcal{B} be the union of all barbells of (G, σ) . For any barbell $B \in \mathcal{B}$, recall that \widehat{B} is the maximum bridgeless subgraph of B, which is the union of two negative cycles of B. Define

$$\tau(G,\sigma) = \begin{cases} |E(G)| & \text{if } \mathcal{B} = \emptyset;\\ \min\left\{ |E(\widehat{B})| \mid B \in \mathcal{B} \right\} & otherwise. \end{cases}$$

The following result connects the function $\tau(G, \sigma)$ and signed-circuit cover, which is the key lemma to prove our main theorem.

Lemma 2.5. Let (G, σ) be a 2-edge-connected signed graph such that G^+ is connected. If $S \subseteq E^-(G, \sigma)$ has an even number of edges, then (G, σ) has a family \mathfrak{F} of signed-circuits such that:

- (i) \mathcal{F} covers S and all 2-edge-cuts containing an edge from S;
- (ii) the length of \mathcal{F} satisfies

$$\ell(\mathfrak{F}) \leq \frac{1}{2} \big(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma) \big);$$

(iii) each negative loop of S (if exists) is contained in exactly one barbell of \mathcal{F} .

Proof. By Lemma 2.4, (G, σ) has an even-subgraph H such that

$$S \subseteq H \subseteq G^+ \cup S$$
 and $|E(H)| \le |E(G)| - \frac{1}{2}|E(\widehat{G^+})|.$

Note that, for a 2-edge-cut R of G, either $R \cap E(H) = \emptyset$ or $R \subset E(H)$ because H is an even-subgraph. So all edges of 2-edge-cuts containing an edge from S belong to E(H). For (i), it suffices to show that (G, σ) has a family of signed-circuits covering all edges of H.

Since H is an even-subgraph, it has a cycle decomposition, i.e., its edges can be decomposed into edge-disjoint cycles. Choose a cycle decomposition \mathcal{C} of H such that the number of positive cycles in \mathcal{C} is maximum over all cycle decompositions of H. Then for any two negative cycles C and C' of \mathcal{C} , we have $|V(C) \cap V(C')| \leq 1$. Otherwise $C \cup C'$ contains a positive cycle and can be decomposed into a family of cycles \mathcal{C}' , in which at least one cycle is positive. Then $\mathcal{C}' \cup (\mathcal{C} \setminus \{C, C'\})$ is a cycle decomposition of H which has more positive cycles than \mathcal{C} , contradicting the choice of \mathcal{C} . For two negative cycles C and C' with exactly one common vertex, the union $C \cup C'$ is a short barbell of (G, σ) . Let $\mathcal{F} =$ $\{D_1, D_2, \ldots, D_m, C_1, C_2, \ldots, C_{2k-1}, C_{2n}\}$ such that D_i for $i \in \{1, \ldots, m\}$ is either a positive cycle of \mathcal{C} or a short barbell consisting of two negative cycles of \mathcal{C} with exactly one common vertex, and C_i for $i \in \{1, \ldots, 2n\}$ is a negative cycle of \mathcal{C} . (Note that \mathcal{F} has an even number of negative cycle because |S| is even.) By the above choices, we have $C_i \cap C_j = \emptyset$ for $i, j \in \{1, \ldots, 2n\}$ and $i \neq j$.

If n = 0, then \mathcal{F} is a signed-circuit cover of H with length

$$\begin{split} \ell(\mathcal{F}) &= |E(H)| \le |E(G)| - \frac{1}{2} |E(\widehat{G^+})| = \frac{1}{2} \Big(3|E(G)| - |E(\widehat{G^+})| - |E(G)| \Big) \\ &\le \frac{1}{2} \Big(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma) \Big). \end{split}$$

The last inequality above follows from the fact that $\tau(G, \sigma) \leq |E(G)|$. So \mathcal{F} is a family of signed-circuits of the type we seek.

So, in the following, assume that $n \ge 1$. Let $Q = \bigcup_{C_i \in \mathcal{F}} C_i$, and let G' be the resulting graph obtained from G by contracting each cycle C_i for $i \in \{1, 2, ..., 2n\}$ to a single vertex u_i . Then $E(G') = E(G) \setminus (\bigcup_{C_i \in \mathcal{F}} E(C_i)) = E(G) \setminus E(Q)$.

Let $U = \{u_1, u_2, \ldots, u_{2n}\}$ and J be a minimum T-join of G' with respect to U. In the graph G', the subgraph induced by J has n edge-disjoint paths P_1, P_2, \ldots, P_n such that each P_k joins two distinct vertices from U. By Corollary 2.3, it follows that

$$|J| \le \frac{1}{2} |E(G')| = \frac{1}{2} (|E(G)| - |E(Q)|).$$
(1)

Let J_0 be the subgraph of G induced by edges in J. Then each endvertex u_i of n edge-disjoint paths of J_0 becomes a vertex $v_i \in C_i$ of G. Let $U' = \{v_1, v_2, ..., v_{2n}\}$ where $v_i \in C_i$ for $i \in \{1, 2, ..., 2n\}$. Note that, the edges of each P_k with $k \in \{1, ..., n\}$ may induce a disconnected graph in G consisting of subpaths of P_k . Then, for each path P_k , join all subpaths of $P_k \cap J_0$ by using a segment from each cycle C_i corresponding to an intermediate vertex $v_i \in U'$ of P_k , and the edge set J_1 of the resulting subgraph is a T-join of G with respect to U' satisfying $J_1 \cap E(G') = J$ and $J_1 \subseteq J \cup E(Q)$.

Among all T-joins J_1 of G with respect to U' such that $J_1 \subseteq J \cup E(Q)$ and $J_1 \cap E(G') = J$, let J' be a such T-join with $|J' \cap E(H)|$ being minimum. If J' induces a cycle, let C be a such cycle. If $E(C) \subseteq E(Q)$, then $J' \setminus E(C)$ is still a T-join of G with respect to U' and has fewer edges than J' which contradicts that $|J' \cap E(H)|$ is minimum. If $E(C) \setminus E(Q) \neq \emptyset$, then contracting edges of C from E(Q) generates a cycle of J_0 which contradicts that J is a minimum T-join of G'. Hence J' induces an acyclic subgraph of G.

Claim: $|J' \cap E(Q)| \leq \frac{1}{2}(|E(Q)| - \tau(G, \sigma)).$

Proof of the claim. The subgraph induced by J' in G is acyclic and hence has at least two vertices of degree 1. Since J' is a T-join of G with respect to $\{v_1, \ldots, v_{2n}\}$. We may assume that $d_{J'}(v_1) = 1$ and $d_{J'}(v_2) = 1$ (relabelling if necessary). So $J' \cap E(C_i) = \emptyset$ for $i \in \{1, 2\}$. Note that C_1 and C_2 are disjoint. Since G is 2-edge-connected, G has a path joining C_1 and C_2 and hence G has a barbell containing C_1 and C_2 . It follows that $\tau(G, \sigma) \leq |E(C_1)| + |E(C_2)|$.

Let $Q' = Q \setminus (C_1 \cup C_2)$ and let $J'' = J' \oplus E(Q')$, the symmetric difference of J' and Q'. Since $Q' \subseteq Q$ and $E(G') = E(G) \setminus E(Q)$, it follows that J'' is a *T*-join of *G* with respect to *U'* such that $|J'' \cap E(G')| = |J' \cap E(G')| = |J \cap E(G')|$. By the choice of J', we have $|J' \cap E(Q)| \leq |J'' \cap E(Q)|$. Furthermore, $J'' \cap E(C_i) = J' \cap E(C_i) = \emptyset$ for $i \in \{1, 2\}$. Therefore, $J' \cap E(Q)$ and $J'' \cap E(Q)$ form a partition of E(Q'). Hence,

$$|J' \cap E(Q)| \le \frac{1}{2}(|J' \cap E(Q)| + |J'' \cap E(Q)|) = \frac{1}{2}|E(Q')|$$
$$= \frac{1}{2}(|E(Q)| - (|E(C_1)| + |E(C_2)|))$$
$$\le \frac{1}{2}(|E(Q)| - \tau(G, \sigma)).$$

This completes the proof of the claim.

Note that the subgraph induced by J' in G has n edge-disjoint paths joining n pairs of vertices of U', and these n pairs of vertices form a partition of U'. So these n edge-disjoint paths together with the negative cycles C_i 's containing the endvertices of the paths form n barbells, denoted by B_1, B_2, \ldots, B_n . Let $\mathcal{F}' = \{D_1, D_2, \ldots, D_m, B_1, B_2, \ldots, B_n\}$. Then \mathcal{F}' is a signed-circuit cover of H. So the conclusion (i) follows. By the claim and (1),

$$\begin{split} |J'| &= |J' \cap E(G')| + |J' \cap E(Q)| = |J| + |J' \cap E(Q)| \\ &\leq \frac{1}{2}(|E(G)| - |E(Q)|) + \frac{1}{2}(|E(Q)| - \tau(G,\sigma)) \\ &= \frac{1}{2}(|E(G)| - \tau(G,\sigma)). \end{split}$$

So the length of \mathcal{F}' satisfies

$$\ell(\mathcal{F}') = |E(H)| + |J'| \le |E(H)| + \frac{1}{2}(|E(G)| - \tau(G,\sigma)) \le \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)),$$

which completes the proof of (ii).

If (G, σ) has a negative loop, then the negative loop is not contained in J' by the minimality of J'. So a negative loop (if exists) is contained in exactly one barbell of \mathcal{F}' , and (iii) follows.

3 Shortest signed-circuit covers

Let (G, σ) be a signed graph. An edge cut R of (G, σ) is a minimal set of edges whose removal disconnects G. A switching operation ζ on R is a mapping $\zeta : E(G) \to \{-1, 1\}$ such that $\zeta(e) = -1$ if $e \in R$ and $\zeta(e) = 1$, otherwise. Two signatures σ and σ' are equivalent if there exists an edge cut R such that $\sigma(e) = \zeta(e) \cdot \sigma'(e)$ where ζ is the switching operation on R [24]. The negativeness of a signed graph (G, σ) is the smallest number of negative edges over all equivalent signatures of σ , denoted by $\epsilon(G, \sigma)$. A signed graph is balanced if $\epsilon(G, \sigma) = 0$. In other words, a balanced signed graph is equivalent to a graph (a signed graph without negative edges). It is known that a signed graph (G, σ) has a circuit cover if and only if $\epsilon(G, \sigma) \neq 1$ and every cut-edge of G does not separate a balanced component, so-called sign-bridgeless (cf. [5, 17]). The length of a shortest signed-circuit cover of a signed graph (G, σ) is denoted by $scc(G, \sigma)$. The following are some usefully observations given in [23].

Observation 3.1. Let (G, σ) be a flow-admissible signed graph and σ' be an equivalent signature of σ . Then $scc(G, \sigma) = scc(G, \sigma')$.

By Observation 3.1, for shortest circuit cover problem, it is sufficient to consider all flow-admissible signed graphs (G, σ) with $\epsilon(G, \sigma)$ negative edges.

Observation 3.2. Let (G, σ) be a signed graph with $|E^{-}(G, \sigma)| = \epsilon(G, \sigma)$. Then every edge cut contains at most half the number of edges being negative.

By Observation 3.2, G^+ is connected if (G, σ) is connected and has minimum number of negative edges over all equivalent signatures, i.e., $|E^-(G, \sigma)| = \epsilon(G, \sigma)$. Before proceed to prove our main result, we have the following result for 2-edge-connected signed graphs with even negativeness. **Theorem 3.3.** Let (G, σ) be a 2-edge-connected signed graph with even negativeness. Then (G, σ) has a signed-circuit cover with length less than $\frac{8}{3}|E(G)|$.

Proof. By Observation 3.1, without loss of generality, we may assume that (G, σ) has minimum number of negative edges, i.e., $|E^{-}(G, \sigma)| = \epsilon(G, \sigma) \equiv 0 \pmod{2}$. If $\epsilon(G, \sigma) = 0$, the result follows from Theorem 1.2. So, in the following, assume that $|E^{-}(G, \sigma)| = \epsilon(G, \sigma) \ge 2$.

By Observation 3.2, G^+ is connected. By Lemma 2.5, (G, σ) has a family of signed-circuits \mathcal{F}_1 covering all edges in $E(G) \setminus E(\widehat{G^+})$ with length

$$\ell(\mathcal{F}_1) \le \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)) \le \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})|).$$

By Theorem 1.2, $\widehat{G^+}$ has a circuit cover \mathcal{F}_2 with length $\ell(\mathcal{F}_2) \leq \frac{5}{3} |E(\widehat{G^+})|$. Note that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a signed-circuit cover of (G, σ) . Since $|E(\widehat{G^+})| \leq |E(G)| - |E^-(G, \sigma)|$, it follows that

$$\begin{split} \ell(\mathcal{F}) &\leq \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) \leq \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})|) + \frac{5}{3}|E(\widehat{G^+})| \\ &= \frac{3}{2}|E(G)| + \frac{7}{6}|E(\widehat{G^+})| < \frac{3}{2}|E(G)| + \frac{7}{6}|E(G)| \\ &= \frac{8}{3}|E(G)|. \end{split}$$

This completes the proof.

In the following, we are going to prove our main result, Theorem 1.6. First, we need a variation of Lemma 2.5 as follows.

Lemma 3.4. Let (G, σ) be a 2-edge-connected flow-admissible signed graph with a negative loop e such that G^+ is connected. Then, for any integer $t \in \{1, 2\}$, there is a family of circuits \mathfrak{F}_t of (G, σ) such that: (i) \mathfrak{F}_t covers all edges of $E(G) \setminus E(\widehat{G^+})$;

(ii) the length of \mathcal{F}_t satisfies

$$\ell(\mathcal{F}) \le 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})|$$

(iii) e is contained in exactly t barbells of \mathcal{F} .

Proof. Since (G, σ) is flow-admissible and e is a negative loop, $(G - e, \sigma)$ has a negative edge e' and hence has a negative cycle (for example, a cycle in $G^+ \cup e'$ containing e'). Let C_e be a negative cycle of $(G - e, \sigma)$ with minimum number of edges and let B_e be a barbell consisting of e and C_e with minimum length.

Claim. $|E(B_e)| \leq \frac{1}{2}(|E(G)| + \tau(G, \sigma)).$

Proof of the claim. Let B be a barbell of (G, σ) such that $|E(\widehat{B})| = \tau(G, \sigma)$. Suppose C_1 and C_2 are two negative cycles contained in B.

The minimality C_e implies that $|E(C_e)| \le \max\{|E(C_1)|, |E(C_2)|\}$. Note that $1 \le \min\{|E(C_1)|, |E(C_2)|\}$. Therefore,

$$\tau(G,\sigma) \le |E(B_e)| = |E(C_e)| + 1 \le |E(C_1)| + |E(C_2)| = \tau(G,\sigma)$$

Hence, $|E(\widehat{B}_e)| = \tau(G, \sigma)$. Since (G, σ) is 2-edge-connected, there exist two edge-disjoint minimal paths P_1 and P_2 , joining e and C_e . So

$$\begin{split} |E(B_e)| &\leq \frac{1}{2} (|E(P_1 \cup \widehat{B_e})| + |E(P_2 \cup \widehat{B_e})|) \\ &= \frac{1}{2} ((|E(P_1)| + |E(P_2)| + |\widehat{B_e}|) + |\widehat{B_e}|) \\ &\leq \frac{1}{2} (|E(G)| + |E(\widehat{B_e})|) = \frac{1}{2} (|E(G)| + \tau(G, \sigma)). \end{split}$$

This completes the proof of the claim.

If $|E^{-}(G, \sigma)|$ is even, by Lemma 2.5, (G, σ) has a family of signed-circuits \mathcal{F}_{1} which satisfies (i) covering $E^{-}(G, \sigma)$ and all 2-edge-cuts containing a negative edge, and hence covering all edges of $E(G) \setminus E(\widehat{G^{+}})$; (ii) having length

$$\ell(\mathcal{F}_1) \le \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)) \le 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})|;$$

and (iii) e is contained in exactly one barbell of \mathcal{F}_1 . So Lemma 3.4 follows if t = 1. Now, assume that t = 2. By the claim, let $\mathcal{F}_2 = \mathcal{F}_1 \cup \{B_e\}$ which is a family of signed-circuits satisfying (i) and having length

$$\ell(\mathcal{F}_2) \le \ell(\mathcal{F}_1) + |E(B_e)| \\ \le \frac{1}{2} (3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)) + \frac{1}{2} (|E(G)| + \tau(G,\sigma)) \\ = 2|E(G)| - \frac{1}{2} |E(\widehat{G^+})|.$$

Then \mathcal{F}_2 is a family of signed-circuits of the type we seek.

In the following, assume that $|E^{-}(G,\sigma)|$ is odd. Let $S_1 = E^{-}(G,\sigma) \setminus e$ and let $S_2 = E^{-}(G,\sigma) \setminus e'$ where e' is a negative edge in C_e . For each S_t with $t \in \{1,2\}$, by Lemma 2.5, (G,σ) has a family of signed-circuits, denoted by \mathcal{F}_{S_t} , which covers S_t and all 2-edge-cuts containing an edge in S_t and has length

$$\ell(\mathfrak{F}_{S_t}) \leq \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)).$$

Particularly, the loop e is not covered by \mathcal{F}_{S_1} but is contained in exactly one barbell of \mathcal{F}_{S_2} .

Let $\mathcal{F}_t = \mathcal{F}_{S_t} \cup \{B_e\}$ for $t \in \{1, 2\}$. Then \mathcal{F}_t with $t \in \{1, 2\}$ covers $E^-(G, \sigma)$ and all 2-edge-cuts containing an negative edge (a 2-edge-cut containing e' is covered by B_e). Hence \mathcal{F}_t covers all edges of $E(G) \setminus E(\widehat{G^+})$. By the claim, the length of \mathcal{F}_t with $t \in \{1, 2\}$ satisfies

$$\begin{split} \ell(\mathcal{F}_t) &= \ell(\mathcal{F}_{S_t}) + |E(B_e)| \\ &\leq \frac{1}{2} \left(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma) \right) + \frac{1}{2} (|E(G)| + \tau(G,\sigma)) \\ &= 2|E(G)| - \frac{1}{2} |E(\widehat{G^+})|. \end{split}$$

Note that e is contained in exactly t barbells of \mathcal{F}_t for $t \in \{1, 2\}$. This completes the proof.

Lemma 3.5. Let (G, σ) be a 2-edge-connected signed graph such that $|E^{-}(G, \sigma)| \ge 2$ and G^{+} is connected. Then (G, σ) has a signed-circuit D such that D contains a negative edge in its cycle and

$$|E(D)| \le \frac{1}{2}(\tau(G,\sigma) + |E(G)|).$$

Proof. First assume that (G, σ) does not contain a barbell. Then $\tau(G, \sigma) = |E(G)|$. Let $e, e' \in E^-(G, \sigma)$ and $S = \{e, e'\}$. By Lemma 2.4, there exists an even subgraph H such that $S \subseteq E(H) \subseteq G^+ \cup S$ and

$$|E(H)| \le |E(G)| - \frac{1}{2}|E(\widehat{G^+})|.$$

Let C_e and $C_{e'}$ be two cycle of H containing e and e', respectively. Then either $C_e = C_{e'}$ or $|V(C_e \cap C_{e'})| \ge 2$. Otherwise, (G, σ) has a barbell, which contradicts to the assumption. No matter $C_e = C_{e'}$ or $|V(C_e \cap C_{e'})| \ge 2$, H has a cycle containing both e and e', denoted by D. Since $D \subseteq H \subseteq G^+ \cup S$, D has exactly two negative edges and hence is a positive cycle (a signed-circuit) of (G, σ) . Furthermore,

$$|E(D)| \le |E(H)| \le |E(G)| - \frac{1}{2}|E(\widehat{G^+})| \le |E(G)| = \frac{1}{2}(\tau(G,\sigma) + |E(G)|).$$

In the following, assume that (G, σ) does have a barbell. If (G, σ) itself is a short barbell, then let $D = (G, \sigma)$ and $\tau(G, \sigma) = |E(D)|$. Hence $|E(D)| = |E(G)| = \frac{1}{2}(\tau(G, \sigma) + |E(G)|)$ and the lemma holds trivially. Therefore, assume that (G, σ) is not a barbell. Then $\tau(G, \sigma) < |E(G)|$.

Among all barbells B of (G, σ) with $|E(\widehat{B})| = \tau(G, \sigma)$, let D be a such barbell with minimum number of edges. Then D has a negative cycle which contains a negative edge. Let P be the path of D joining the two negative cycles. By the minimality of D, the path P has the shortest length among all minimal paths joining the two cycles of D. Since (G, σ) is 2-edge-connected, there exists two edge-disjoint minimal paths P_1 and P_2 joining the two negative cycles of D. Then $|E(D)| \leq |E(\widehat{D})| + |E(P_i)|$ for $i \in \{1, 2\}$. By the minimality of P,

$$|E(D)| \le \frac{1}{2} \Big(2|E(\widehat{D})| + |E(P_1)| + |E(P_2)| \Big) \le \frac{1}{2} \Big(|E(\widehat{D})| + |E(G)| \Big) = \frac{1}{2} (\tau(G, \sigma) + |E(G)|).$$

This completes the proof.

Theorem 3.6. Let (G, σ) be a flow-admissible signed graph. Then (G, σ) has a family of circuits \mathcal{F} covering all edges in $E(G) \setminus E(\widehat{G^+})$ such that every negative loop is covered at most twice and its length satisfies

$$\ell(\mathfrak{F}) \le 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})| + b(G,\sigma),$$

where $b(G, \sigma)$ is the number of cut-edges of (G, σ) .

Proof. Note that, the theorem is ture if it holds for every connected component of (G, σ) . So, without loss of generality, assume that (G, σ) is connected. By Observation 3.1, we may assume that (G, σ) has $|E^{-}(G, \sigma)| = \epsilon(G, \sigma)$. If $\epsilon(G, \sigma) = 0$, then $E(G) \setminus E(G^{+}) = \emptyset$ and the result holds trivially. So assume that $\epsilon(G, \sigma) \neq 0$. Since (G, σ) is flow-admissible, it follows that $|E^{-}(G, \sigma)| \geq 2$.

We apply induction on $b(G, \sigma)$ to prove the theorem. First, we verify the base case $b(G, \sigma) = 0$. In other words, (G, σ) is 2-edge-connected.

If $\epsilon(G, \sigma)$ is even, by Lemma 2.5, (G, σ) has a family of signed-circuits \mathcal{F} covering all 2-edge-cut containing a negative edge such that each negative loop is covered exactly once and

$$\ell(\mathcal{F}) \le \frac{1}{2}(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma)) \le 2|E(G)| - \frac{1}{2}\tau(G,\sigma).$$

Then \mathcal{F} is a family of signed-circuits of the type desired. So assume that $\epsilon(G, \sigma)$ is odd and $\epsilon(G, \sigma) \geq 3$. By Lemma 3.5, (G, σ) has a signed-circuit D such that D has a cycle containing a negative edge and

$$|E(D)| \le \frac{1}{2}(\tau(G,\sigma) + |E(G)|).$$

Let e be a negative edge which is contained in a cycle of D, and let $S = E^{-}(G, \sigma) \setminus e$. By Lemma 2.5, (G, σ) has a family of signed-circuits \mathcal{F}' covering S and all 2-edge-cuts containing an edge from S such that each negative loop of S is covered exactly once and

$$\ell(\mathcal{F}') \leq \frac{1}{2} \left(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma) \right).$$

Let $\mathcal{F} = \mathcal{F}' \cup \{D\}$. Since $S \cup \{e\} = E^-(G, \sigma)$, it follows from Observation 3.2 that every positive edge in $E(G) \setminus \widehat{E(G^+)}$ is contained in a 2-edge-cut which contains an edge from $E^-(G, \sigma)$. Therefore, \mathcal{F} is a family of circuits of (G, σ) covering all edges of $E(G) \setminus \widehat{E(G^+)}$ such that

$$\begin{split} \ell(\mathcal{F}) &= \ell(\mathcal{F}') + |E(D)| \\ &\leq \frac{1}{2} \big(3|E(G)| - |E(\widehat{G^+})| - \tau(G,\sigma) \big) + \frac{1}{2} (\tau(G,\sigma) + |E(G)|) \\ &= 2|E(G)| - \frac{1}{2} |E(\widehat{G^+})|. \end{split}$$

Note that every negative loop is covered by exactly one barbell of \mathcal{F}' , and D is a signed-circuit. Hence \mathcal{F} covers every negative loop at most twice. So the theorem holds if $b(G, \sigma) = 0$.

So, in the following, assume that $b(G, \sigma) \neq 0$ and the theorem holds for all flow-admissible signed graph with at most $b(G, \sigma) - 1$ cut-edges.

Since $b(G, \sigma) \neq 0$, the graph G has a cut-edge. Let uv be a cut-edge of G such that $G \setminus uv$ consists of two components Q_1 and Q_2 , one of which, say Q_2 , contains no cut-edges. Without loss of generality, assume that $u \in V(Q_1)$ and $v \in V(Q_2)$. Since (G, σ) is flow-admissible, both Q_1 and Q_2 are not balanced and hence contain a negative edge. Hence Q_2 is either a negative loop or 2-edge-connected.

Let (G_1, σ) be the resulting signed graph constructed from (Q_1, σ) by adding a negative loop e_1 attached to u. Then (G_1, σ) is flow-admissible and $b(G_1, \sigma) = b(G, \sigma) - 1 < b(G, \sigma)$. By induction hypothesis, (G_1, σ) has a family of signed-circuits \mathcal{F}_1 covering all edges in $E(G_1) \setminus E(\widehat{G_1}^+)$ such that

$$\ell(\mathfrak{F}_1) \le 2|E(G_1)| - \frac{1}{2}|E(\widehat{G_1^+})| + b(G_1,\sigma)$$

and every negative loop is contained in at most two barbells of \mathcal{F}_1 . Assume that e_1 is contained in t barbells of \mathcal{F}_1 with $t \in \{1, 2\}$.

Let (G_2, σ) be the resulting graph constructed from (Q_2, σ) by attaching a negative loop e_2 to v. Then (G_2, σ) is flow-admissible and 2-edge-connected. By Lemma 3.4, (G_2, σ) has a family of signed-circuits \mathcal{F}_2 covering all edges in $E(G_2) \setminus E(\widehat{G_2^+})$ such that

$$\ell(\mathfrak{F}_2) \le 2|E(G_2)| - \frac{1}{2}|E(\widehat{G_2^+})|$$

and e_2 is contained in exactly t barbells of \mathcal{F}_2 .

Let B'_i with $1 \leq i \leq t$ be the barbells of \mathcal{F}_1 containing e_1 , and B''_i with $1 \leq i \leq t$ be the barbells of \mathcal{F}_2 containing e_2 . (Note that, $t \leq 2$.) Let $B_i = (B'_i \setminus e_1) \cup (B''_i \setminus e_2) \cup uv$ which is a barbell of (G, σ) for $1 \leq i \leq t$, and let $\mathcal{F} = (\mathcal{F}_1 \setminus \{B'_i | 1 \leq i \leq t\}) \cup (\mathcal{F}_2 \setminus \{B''_i | 1 \leq i \leq t\}) \cup \{B_i | 1 \leq i \leq t\}$. Since $|E(B_i)| = |E(B'_i)| + |E(B''_i)| - 1$ for each i with $1 \leq i \leq t$. Then

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) - t \le \left(2|E(G_1)| - \frac{1}{2}|\widehat{E(G_1)}| + b(G_1,\sigma)\right) + \left(2|E(G_2)| - \frac{1}{2}|\widehat{E(G_2)}|\right) - t.$$
(2)

Since $\widehat{G_1^+} \cup \widehat{G_2^+} = \widehat{G^+}$ and $\widehat{G_1^+} \cap \widehat{G_2^+} = \emptyset$, we have $|E(\widehat{G_1^+})| + |E(\widehat{G_2^+})| = |E(\widehat{G^+})|$. Note that $b(G_1, \sigma) = b(G, \sigma) - 1$ and $|E(G_1)| + |E(G_2)| = |E(G)| + 1$. It follows from (2) that

$$\ell(\mathcal{F}) \le 2(|E(G)|+1) - \frac{1}{2}|E(\widehat{G^+})| + (b(G,\sigma)-1) - t \le 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})| + b(G,\sigma).$$

Note that a negative loop of (G, σ) (if exists) is contained in one or two barbells of either \mathcal{F}_1 or \mathcal{F}_2 (but not both). Hence a negative loop of (G, σ) is contained in either one or two barbells in \mathcal{F} . This completes the proof.

Now, we are ready to prove our main result, Theorem 1.6.

Proof of Theorem 1.6. Let (G, σ) be a flow-admissible signed graph. By observation 3.1, we may assume that $|E^{-}(G, \sigma)| = \epsilon(G, \sigma)$. If $\epsilon(G, \sigma) = 0$, then the results follows directly from Theorem 1.2. So assume that $\epsilon(G, \sigma) \ge 2$. By Observation 3.2, G^+ is connected.

By Theorem 3.6, (G, σ) has a family of signed-circuits \mathcal{F}_1 covering all edges in $E(G) \setminus E(\widehat{G^+})$ with length

$$\ell(\mathfrak{F}_1) \le 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})| + b(G,\sigma),$$

where $b(G, \sigma)$ is the number of cut-edges of (G, σ) . By Theorem 1.2, the subgraph $\widehat{G^+}$ has a signed-circuit cover \mathcal{F}_2 with length $\ell(\mathcal{F}_2) \leq \frac{5}{3} |E(\widehat{G^+})|$. Therefore, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a signed-circuit cover of (G, σ) , and

$$\begin{split} \ell(\mathcal{F}) &= \ell(\mathcal{F}_1) + \ell(\mathcal{F}_2) \leq 2|E(G)| - \frac{1}{2}|E(\widehat{G^+})| + b(G,\sigma) + \frac{5}{3}|E(\widehat{G^+})| \\ &\leq 2|E(G)| + \frac{7}{6}|E(\widehat{G^+})| + b(G,\sigma) \\ &< 2|E(G)| + \frac{7}{6}|E(G)| = \frac{19}{6}|E(G)|. \end{split}$$

This completes the proof of Theorem 1.6.

Remark. Let (P_{10}, σ) be the signed graph with P_{10} being the Petersen graph and $E^-(P_{10}, \sigma)$ inducing a 5-cycle. Máčajová et. al. [17] show that a shortest circuit cover of (P_{10}, σ) has length exactly $\frac{5}{3}|E(P_{10})|$. The optimal upper bound for the shortest signed-circuit cover remains to be investigated.

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