PACKING AND COVERING INDUCED SUBDIVISIONS

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ABSTRACT. A class $\mathcal{F}$ of graphs has the induced Erdős-Pósa property if there exists a function f such that for every graph G and every positive integer k, G contains either k pairwise vertex-disjoint induced subgraphs that belong to $\mathcal{F}$, or a vertex set of size at most $f(k)$ hitting all induced copies of graphs in $\mathcal{F}$. Kim and Kwon (SODA'18) showed that for a cycle C_ℓ of length ℓ, the class of C_ℓ-subdivisions has the induced Erdős-Pósa property if and only if $\ell \leq 4$. In this paper, we investigate whether or not the class of H-subdivisions has the induced Erdős-Pósa property for other graphs H.

We completely settle the case when H is a forest or a complete bipartite graph. Regarding the general case, we identify necessary conditions on H for the class of H-subdivisions to have the induced Erdős-Pósa property. For this, we provide three basic constructions that are useful to prove that the class of the subdivisions of a graph does not have the induced Erdős-Pósa property. Among remaining graphs, we prove that if H is either the diamond, the 1-pan, or the 2-pan, then the class of H-subdivisions has the induced Erdős-Pósa property.

1. INTRODUCTION

All graphs in this paper are finite and without loops or parallel edges. In this paper we are concerned with the induced version of the Erdős-Pósa property. This property expresses a duality between invariants of packing and covering related to a class of graphs. Its name originates from the following result.

Theorem 1.1 (Erdős-Pósa Theorem, [EP62])**.** *There is a function $f(k) = \mathcal{O}(k \log k)$ such that for every graph G and every positive integer k, G contains either k vertex-disjoint cycles, or a vertex set X of size at most $f(k)$ such that $G - X$ has no cycle.*

In general, we say that a class of graphs has the *Erdős-Pósa property* if a similar statement holds: either we can find in a graph many occurrences of members of the class, or we hit them all with a small number of vertices. Since the proof of Theorem 1.1 by Paul Erdős and Lajós Pósa, the line of research of identifying new classes that have the Erdős-Pósa property has been very active (see surveys [Ree97, RT17]). These results are not only interesting because they express some duality between two parameters: they can also be used to design algorithms (see e.g. [Tho88, FLM$^+$16, CRST17]).

Several authors attempted to extend Theorem 1.1 in various directions. One of them is to consider long cycles, i.e. cycles of length at least ℓ for some fixed integer $\ell \geq 3$.

Theorem 1.2 ([MNŠW17], see also [BBR07, FH14])**.** *There is a function $f(k, \ell) = \mathcal{O}(k\ell + k \log k)$ such that for every graph G and every positive integer k, G contains either k vertex-disjoint cycles of length at least ℓ, or a vertex set X of size at most $f(k, \ell)$ such that $G - X$ has no such cycle.*

As every cycle contains an induced cycle, Theorem 1.1 also holds if one replaces *cycle* with *induced cycle* in its statement. This is not so clear with Theorem 1.2 since a long cycle in a graph does not always contain a long induced cycle. In [JP17], Jansen and Ma. Pilipczuk asked whether Theorem 1.2 holds for induced cycles of length at least ℓ for $\ell = 4$. This was recently proved to be true by Kim and the first author:

Theorem 1.3 ([KK17]). *There is a function $f(k) = \mathcal{O}(k^2 \log k)$ such that, for every graph G and every positive integer k, either G has k vertex-disjoint induced cycles of length at least 4, or it contains a vertex set X of size at most $f(k)$ such that $G - X$ has no such cycle.*

They also showed that the Erdős-Pósa type statement in Theorem 1.3 cannot be extended to induced cycles of length at least ℓ for fixed $\ell > 4$ (even with a different order of magnitude for f).

The aim of this paper is to investigate if a statement as that of Theorem 1.3 holds for other induced structures. In order to present it formally, we introduce some notions. A class $\mathcal{F}$ of graphs has the *induced Erdős-Pósa property* if there exists a *bounding function* $f \colon \mathbb{N} \to \mathbb{R}$ such that for every graph G and every positive integer k, G contains either k pairwise vertex-disjoint induced subgraphs that belong to $\mathcal{F}$, or a vertex set of size at most $f(k)$ hitting all induced copies of graphs in $\mathcal{F}$.[1]

A *subdivision of H* (H-*subdivision* for short) is a graph obtained from H by subdividing some of its edges. A subgraph of a graph G is called an *induced subdivision of H* (or *induced H-subdivision*) if it is an induced subgraph of G that is a subdivision of H. For graphs H and G, we denote by $\nu_H(G)$ the maximum size of a collection of vertex-disjoint induced subdivisions of H in G (called *packing*). We denote by $\tau_H(G)$ the minimum size of a subset $X \subseteq V(G)$ (called *hitting set*) such that $G - X$ has no induced subdivision of H. By definition, the class of H-subdivisions has the induced Erdős-Pósa property if there is a bounding function $f \colon \mathbb{N} \to \mathbb{R}$ such that $\tau_H(G) \leq f(\nu_H(G))$ for every graph G. To avoid a long terminology, we allow to say that H-subdivisions have the induced Erdős-Pósa property.

Theorem 1.3 can be reformulated in terms of the induced Erdős-Pósa property of C_4-subdivisions. The authors of [KK17] noted that it is an interesting topic to investigate the induced Erdős-Pósa property of subdivisions of other graphs.

In this paper, we determine whether H-subdivisions have the induced version of the Erdős-Pósa property or not, for various graphs H. We note that the classic (i.e. non-induced) Erdős-Pósa property of subdivisions has been investigated before [Tho88, Liu17, BH17, MNŠW17].

Our results. Towards a classification of graphs based on the induced Erdős-Pósa property of their subdivisions, we consider several simple extensions of cycles, depicted in Figure 1 (see Section 2 for a formal definition).

Theorem 1.4. *If H is either the diamond, the 1-pan, or the 2-pan, then H-subdivisions have the induced Erdős-Pósa property with a polynomial bounding function.*

For 1- and 2-pans, we furthermore give a polynomial-time algorithm that constructs a packing or a hitting set of bounded size. For diamond, we give an algorithm that

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[1]We decided to use this terminology because the classic Erdős-Pósa property considers the subgraph relation as a containment relation and therefore the sentence "induced H-subdivisions have the Erdős-Pósa property" might be confusing.
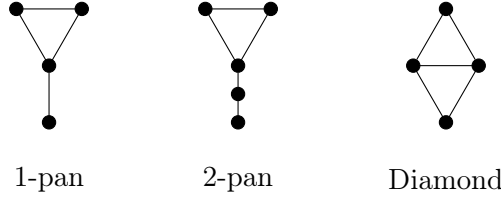
FIGURE 1. The graphs mentioned in the statement of Theorem 1.4.

runs in time $k^{\mathcal{O}(k)} \cdot |G|^c$ for some constant c. This will be explicitly mentioned in the statement of the theorems for each of the graphs.

We then give negative results for graphs satisfying certain general properties.

Theorem 1.5. *Let H be a graph which satisfies one of the following:*

    (1) H is a forest and two vertices of degree at least 3 lie in the same connected component;

    (2) H contains an induced cycle of length at least 5;

    (3) H contains a cycle C and two adjacent vertices having no neighbors in C;

    (4) H contains a cycle C and three vertices having no neighbors in C;

    (5) $H = K_{2,n}$ with $n \geq 3$;

    (6) H is not planar.

Then H-subdivisions do not have the induced Erdős-Pósa property.

We remark that if a forest F has no two vertices of degree at least 3 in the same connected component, then F is a disjoint union of subdivisions of stars, and therefore every subdivision of F contains F as an induced subgraph. Thus, for such a forest F, F-subdivisions trivially have the induced Erdős-Pósa property (see Lemma 3.3). By item (1) of Theorem 1.5, other forests F will not satisfy this property. Therefore, we can focus on graphs containing a cycle.

The rest of Theorem 1.5 provides, for graphs that have cycles, necessary conditions for their subdivisions to have the induced Erdős-Pósa property. By combining together Theorems 1.3, 1.4, and 1.5, we obtain the following dichotomies.

Corollary 1.6.

    (1) Let F be a forest. Then F-subdivisions have the induced Erdős-Pósa property if and only if each connected component of F has at most one vertex of degree more than 2.

    (2) Let n, m be positive integers with $n \leq m$. Then $K_{n,m}$-subdivisions have the induced Erdős-Pósa property if and only if $n \leq 1$ or $m \leq 2$.

    (3) Let n be a positive integer. Then the subdivisions of the n-pan have the induced Erdős-Pósa property if and only if $n \leq 2$.

We also show that if H has a cycle and no vertex of degree more than 3, then there is no $o(k \log k)$ bounding function for H (Theorem 7.1).

Our techniques. To obtain negative results, we describe three constructions. Their common point is that any induced subdivision of the considered graph H that they contain has a strictly constrained position. This will ensure that two distinct induced

H-subdivisions meet. On the other hand, they contain several distinct induced H-subdivisions, so one needs many vertices to hit them all. Compared to their non-induced counterparts, induced subdivisions of a fixed graph do not appear in very dense subgraphs such as cliques, which leaves more freedom in the design of our constructions.

The proofs of the positive results for 1-pan and 2-pan start similarly. We first show that if S is a smallest 1-pan- (resp. 2-pan-) subdivision in G that hits all other induced subdivisions of the 1-pan (resp. 2-pan), then either G contains k pairwise vertex-disjoint induced subdivisions of the 1-pan, or all the 1-pan-(resp. 2-pan-)subdivisions can be hit with $\mathcal{O}(k \log k)$ vertices. By applying inductively this result we can conclude that the subdivisions of the 1-pan (resp. 2-pan) have the induced Erdős-Pósa property with gap $\mathcal{O}(k^2 \log k)$. The proof for the diamond is much more involved.

Organization of the paper. We introduce the definitions and some of the external results that we need in Section 2. In sections from 3 to 7, we present general constructions and use them to obtain the negative results contained in this paper. The positive results about pans and diamonds are proved in Sections 8 and 9, respectively. We conclude in Section 10 with remarks and possible directions for future research. The sections from 3 to 9 are mostly independent and can be read separately.

2. Preliminaries

Basics. We denote by $\mathbb{R}$ the set of reals and by $\mathbb{N}$ the set of non-negative integers. For an integer $p \geq 1$, we denote by $\mathbb{N}_{\geq p}$ the set of integers greater than or equal to p. Let G be a graph. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. For every $X \subseteq V(G)$ (resp. $X \subseteq E(G)$), we denote by $G - X$ the graph obtained by removing all vertices (resp. edges) of X from G. For every $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. We use $|G|$ as a shorthand for $|V(G)|$.

For two graphs G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

For a subgraph H of a graph G and $v \in V(G) \setminus V(H)$, we say that v *dominates* H if v is adjacent to every vertex of H. In this case, we also say that v is a H-dominating vertex.

For $v, w \in V(G)$, we denote by $\operatorname{dist}_G(v, w)$ the number of edges of a shortest path from v to w in G; that is the *distance* between v and w in G. This notation is extended to sets $V, W \subseteq V(G)$ as $\operatorname{dist}_G(V, W) = \min_{(v,w) \in V \times W} \operatorname{dist}_G(v, w)$. When V consists of one vertex v, we write $\operatorname{dist}_G(v, W)$ for $\operatorname{dist}_G(\{v\}, W)$. Let $r \in \mathbb{N}_{\geq 1}$. The *r-neighborhood* of a subset $S \subseteq V(G)$, that we denote by $N_G^r[S]$, is the set of all vertices w such that $\operatorname{dist}_G(w, S) \leq r$.

Subdivisions. We recall the definition of subdivision given in the introduction. The operation of *subdividing* an edge e of a graph G adds a new vertex of degree 2 adjacent to the endpoints of e and removes e. We say that a graph H' is a subdivision of a graph H if H' can be obtained from H by a sequence of edge subdivisions. We also say that G has an *induced subdivision* of H, or shortly an induced H-subdivision, if there is an induced subgraph of G that is isomorphic to an H-subdivision.

Models. We formulate an induced subdivision of H in a graph G as a mapping from H to G. A *model* of H in G is an injective function φ with domain $V(H) \cup E(H)$ that maps vertices of H to vertices of G and edges of H to induced paths of G such that:

- for every edge $uv \in E(H)$, $\varphi(u)$ and $\varphi(v)$ are the endpoints of $\varphi(uv)$;
- for distinct edges $u_1v_1, u_2v_2 \in E(H)$, there is no edge between $V(\varphi(u_1v_1)) \setminus \{\varphi(u_1), \varphi(v_1)\}$ and $V(\varphi(u_2v_2)) \setminus \{\varphi(u_2), \varphi(v_2)\}$;
- for every edge $uv \in E(H)$ and a vertex $w \in V(H) \setminus \{u, v\}$, $\varphi(w)$ has no neighbors in $V(\varphi(uv)) \setminus \{\varphi(u), \varphi(v)\}$;
- for two non-adjacent vertices $u, v \in V(H)$, $\varphi(u)\varphi(v) \notin E(G)$.

It is easy to see that a graph G contains an induced subdivision of a graph H if and only if there is a model of H in G. We will often used this fact implicitly. If H' is an induced subgraph of H, then we denote by $\varphi(H')$ the subgraph of G induced by $\bigcup_{v \in V(H')} V(\varphi(v))$.

Small graphs. We denote by K_n the complete graph on n vertices, and by C_n the cycle on n vertices. The *claw* is the complement of the disjoint union of K_3 and K_1. The *n-pan* is the graph obtained from the disjoint union of a path on n edges and K_3 by identifying one endpoint of the path with a vertex in K_3. The *diamond* is the graph obtained by removing an edge in K_4.

General tools. We collect in this subsection general results that will be used in the paper. The first one is a lemma used by Simonovits [Sim67] to give a new proof of Theorem 1.1.

Lemma 2.1 ([Sim67], see also [Die10, Lemma 2.3.1]). *Let k be a positive integer. If a cubic multigraph has at least $24k \log k$ vertices, then it contains at least k vertex-disjoint cycles. Furthermore, such k cycles can be found in time $\mathcal{O}(|G|^3)$.*

The second one is related to an Erdős-Pósa type problem about paths intersecting a fixed set of vertices. Let G be a graph and $A \subseteq V(G)$. An *A-path* in G is a path with both end vertices in A and all internal vertices in $V(G) \setminus A$. An *A-cycle* is a cycle containing at least one vertex of A.

Theorem 2.2 ([Gal64]). *Let G be a graph, $A \subseteq V(G)$, and k be a positive integer. Then one can find in time $\mathcal{O}(k|G|^2)$ either k vertex-disjoint A-paths, or a vertex set X of G with $|X| \le 2k - 2$ such that $G - X$ has no A-paths.*

We use the regular partition lemma introduced in [CKOW17]. For a sequence $(A_1, \dots, A_\ell)$ of finite subsets of an interval $I \subseteq \mathbb{R}$, a partition $\{I_1, \dots, I_k\}$ of I into intervals is called a *regular partition* of I with respect to $(A_1, \dots, A_\ell)$ if each $i \in \{1, \dots, k\}$ satisfies one of the following:

(RP1) $A_1 \cap I_i = A_2 \cap I_i = \cdots = A_\ell \cap I_i \ne \emptyset$.

(RP2) $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \dots, \ell\}$ with $j < j'$, $\max(A_j \cap I_i) < \min(A_{j'} \cap I_i)$.

(RP3) $|A_1 \cap I_i| = |A_2 \cap I_i| = \cdots = |A_\ell \cap I_i| > 0$, and for all $j, j' \in \{1, \dots, \ell\}$ with $j < j'$, $\max(A_{j'} \cap I_i) < \min(A_j \cap I_i)$.

The number of parts k is called the *order* of the regular partition. The following can be obtained using multiple applications of the Erdős-Szekeres Theorem [ES87].

Lemma 2.3 (Regular partition lemma [CKOW17]). *Let $I \subseteq \mathbb{R}$ be an interval. There exists a function N such that for all positive integers n, ℓ, the value $N = N(n, \ell)$ satisfies the following. For every sequence $(A_1, \dots, A_N)$ of n-element subsets of I, there exist a subsequence $(A_{j_1}, \dots, A_{j_\ell})$ of $(A_1, \dots, A_N)$ and a regular partition of I with respect to $(A_{j_1}, \dots, A_{j_\ell})$ that has order at most n.*

We will use this lemma with $n \in \{2, 3, 4\}$. We note that for fixed n, the function $N(n, \ell)$ is a polynomial function in ℓ, but the known upper bound following from the proof of the result is big. For instance, $N(2, \ell) = \mathcal{O}(\ell^{40})$ and $N(3, \ell) = \mathcal{O}(\ell^{968})$.

3. NEGATIVE RESULTS USING THE TRIANGLE-WALL

We prove in this section items (1) and (5) of Theorem 1.5. The cornerstone of both proofs is the use of a triangle-wall, which we describe hereafter. These results are then used to show the dichotomies (1) and (2) of Corollary 1.6.

Let $n \in \mathbb{N}_{\geq 2}$. The n-*garland* is the graph obtained from the path $x_0 y_0 \ldots x_i y_i \ldots x_{n-1} y_{n-1}$ by adding, for every $i \in \{0, \ldots, n-1\}$, a new vertex z_i adjacent to x_i and y_i. Let $Q_0, \ldots, Q_{n-1}$ be n copies of the $2n$-garland. For every $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, 2n-1\}$, we respectively denote by x_j^i, y_j^i, z_j^i the copies of the vertices x_j, y_j, z_j in Q_i. The graph Γ_n (informally called *triangle-wall*) is constructed from the disjoint union of $Q_0, \ldots, Q_{n-1}$ as follows:

- for every even $i \in \{0, \ldots, n-1\}$ and every odd $j \in \{0, \ldots, 2n-1\}$, we add an edge between the vertex z_j^i and the vertex z_j^{i+1};
- for every odd $i \in \{0, \ldots, n-1\}$ and every even $j \in \{0, \ldots, 2n-1\}$, we add an edge between the vertex z_j^i and the vertex z_j^{i+1}.
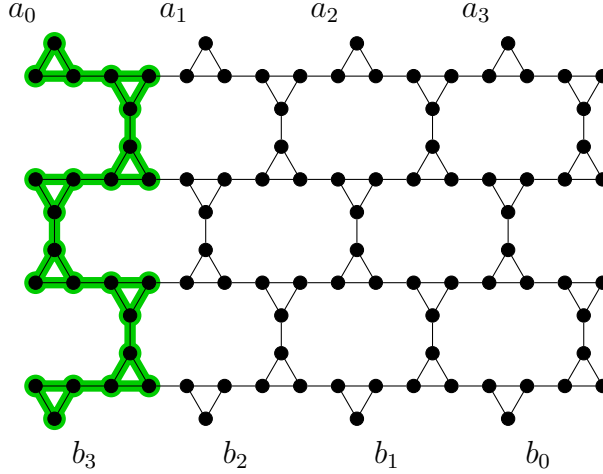

FIGURE 2. The graph Γ_4, with C_0 depicted in green.

For $i \in \{0, \ldots, n-1\}$, we set

$$a_i = z_{2i}^0, \qquad b_i = \begin{cases} z_{2(n-i-1)}^p & \text{if } n \text{ is even,} \\ z_{2(n-i)-1}^p & \text{otherwise.} \end{cases},$$

$$\text{and } C_i = \bigcup_{j=0}^{n-1} \{x_{2i}^j, y_{2i}^j, z_{2i}^j, x_{2i+1}^j, y_{2i+1}^j, z_{2i+1}^j\}.$$

Intuitively, the Q_i's form the rows of Γ_n and the C_i's its columns. The graph Γ_4 is depicted in Figure 2. Observe that there is no induced claw in Γ_n.

3.1. **Complete bipartite patterns.** In this section, we complete the study of the induced Erdős-Pósa property of subdivisions of complete bipartite graphs by proving item (2) of Corollary 1.6.

Lemma 3.1 (item (5) of Theorem 1.5). *For every integer $r \geq 3$, the subdivisions of $K_{2,r}$ do not have the induced Erdős-Pósa property.*

Proof. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq 1}}$ of graphs such that $\nu_{K_{2,r}}(G_n) = \mathcal{O}(1)$ while $\tau_{K_{2,r}}(G_n) = \Omega(n)$.

For $n \in \mathbb{N}_{\geq 1}$, the graph G_n is obtained as follows from the graph Γ_{rn}:

- for every $i \in \{0, \ldots, n-1\}$, add a new vertex u_i and make it adjacent to $a_{ir}, a_{ir+1}, \ldots, a_{(i+1)r-1}$;
- for every $i \in \{0, \ldots, n-1\}$, add a new vertex v_i and make it adjacent to $b_{ir}, b_{ir+1}, \ldots, b_{(i+1)r-1}$;
- for every distinct $i, j \in \{0, \ldots, n-1\}$, add all possible edges between $\{u_i, v_i\}$ and $\{u_j, v_j\}$ (thus, u_i and v_j are not adjacent iff $i = j$).

Let us show that $\nu_{K_{2,r}}(G_n) \leq 1$. For this we consider a model φ of $K_{2,r}$ in G_n. Notice that $\varphi(K_{2,r})$ has exactly two vertices of degree r, and that they are not adjacent. As each of them has $r \geq 3$ pairwise non-adjacent neighbors, we observe that none of these two vertices of degree r belongs to the subgraph Γ_{rn} of G_n. Furthermore, as they are not adjacent, one is u_i and the other one v_i, for some $i \in \{0, \ldots, n-1\}$.

Let us now focus on the (u_i, v_i)-paths of $\varphi(K_{2,r})$. Observe there is no edge between the internal vertices of two distinct (u_i, v_i)-paths of $\varphi(K_{2,r})$. Therefore if two such paths contain a vertex of $\{u_j, v_j : j \in \{0, \ldots, n-1\} \setminus \{i\}\}$ each, one of these vertices is u_j and the other one is v_j, for some $j \in \{0, \ldots, n-1\}$. As $r \geq 3$, we deduce that at least one (u_i, v_i)-path of $\varphi(K_{2,r})$ does not contain any vertex of $\{u_j, v_j : j \in \{0, \ldots, n-1\} \setminus \{i\}\}$. The set of internal vertices of this path lies in Γ_{rn} and connects a neighbor of u_i to a neighbor of v_i, i.e. a vertex of $\{a_{ir}, a_{ir+1}, \ldots, a_{(i+1)r-1}\}$ to a vertex of $\{b_{ir}, b_{ir+1}, \ldots, b_{(i+1)r-1}\}$.

We just proved that every induced subdivision of $K_{2,r}$ in G_n contains a path of the subgraph Γ_{rn} connecting a vertex of $\{a_{ir}, a_{ir+1}, \ldots, a_{(i+1)r-1}\}$ to a vertex of $\{b_{ir}, b_{ir+1}, \ldots, b_{(i+1)r-1}\}$, for some $i \in \{1, \ldots, n\}$. As every two such paths meet for different values of i, we deduce $\nu_{K_{2,r}}(G_n) \leq 1$.

We now show that when $n \geq 2r$, $\tau_{K_{2,r}}(G_n) \geq \frac{n}{2}$. For this we consider the sets defined for every $i \in \{0, \ldots, n-1\}$ as follows:

$$C_i^+ = \{u_i, v_{n-i-1}\} \cup \bigcup_{j=(i+1)r-1}^{ir} C_j.$$

The set C_i^+ contains u_i, v_{n-i-1} and the vertices that are, intuitively, in the columns that are between them. Let X be a subset of $V(G_n)$ with $\lceil \frac{n}{2} \rceil - 1$ vertices. Observe that for distinct $i, j \in \{1, \ldots, n\}$, the sets C_i^+ and C_j^+ are disjoint. Hence the $\lceil n/2 \rceil$ sets

$$\left\{ C_i^+ \cup C_{n-i-1}^+, \ i \in \left\{0, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}\right\}$$

are disjoint. As $|X| = \lceil \frac{n}{2} \rceil - 1$, we have $X \cap (C_i^+ \cup C_{n-i-1}^+) = \emptyset$ for some $i \in \left\{0, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}$.

Recall that Γ_n is composed of n disjoint garlands $Q_0, \ldots Q_{n-1}$ connected together. As $|X| < \frac{n}{2}$ and $r \le \frac{n}{2}$, some r of these garlands do not intersect X. Let $\pi \colon \{1, \ldots, r\} \to \{1, \ldots, n\}$ be an increasing function such that $Q_{\pi(i)} \cap X = \emptyset$, for every $i \in \{1, \ldots, r\}$. Then there is a collection of disjoint paths $P_1, \ldots, P_r$ of $G - X$ so that P_j connects a_{ir+j} to $b_{(i+1)r-1-j}$, for every $j \in \{1, \ldots, r\}$. The path P_j can be obtained by following in $G_n[C_{ir+j}]$ (intuitively, the j-th column of C_i^+) a shortest path from a_{ir+j} to $Q_{\pi(r-1-j)}$, then chordlessly following $Q_{\pi(r-1-j)}$ up to a vertex of $C_{(i+1)r-1-j}$ and finally following a chordless path to $b_{(i+1)r-1-j}$ in $G_n[C_{(i+1)r-1-j}]$. We deduce that $G - X$ contains a model of $K_{2,r}$. As this argument holds for every $X \subseteq V(G)$ such that $|X| \le \lceil \frac{n}{2} \rceil - 1$, we deduce that $\tau_{K_{2,r}}(G_n) \ge \frac{n}{2}$.

Therefore $\nu_{K_{2,r}}(G_n) = \mathcal{O}(1)$ and $\tau_{K_{2,r}}(G_n) = \Omega(n)$. This concludes the proof. $\qquad\square$

Corollary 3.2 (item (2) of Corollary 1.6). *Let r, r' be positive integers with $r \le r'$. Then the $K_{r,r'}$-subdivisions have the induced Erdős-Pósa property if and only if $r \le 1$ or $r' \le 2$.*

Proof. When $r = 1$, the result holds by Lemma 3.3 (to be proved in the next section). When $r = r' = 2$, $K_{2,r}$ is a cycle on four vertices and the result holds by Theorem 1.3. When $r = 2$ and $r' \ge 3$, the result follows from Lemma 3.1. In the case where $r \ge 3$, then $K_{r,r'}$ is not planar and the result follows from Lemma 6.2. $\qquad\square$

3.2. **Acyclic patterns.** We show in this section that if a graph is acyclic, then its subdivisions either have the induced Erdős-Pósa property with a linear bounding function (Lemma 3.3), or they do not have the induced Erdős-Pósa property (Lemma 3.4). This proves item (1) of Corollary 1.6.

Lemma 3.3. *Let H be a graph whose connected components are paths or subdivided stars. Then the subdivisions of H have the induced Erdős-Pósa property with a bounding function of order $\mathcal{O}(k)$.*

Proof. We show that for every graph G we have $\tau_H(G) \le \nu_H(G) \cdot |H|$. Towards a contradiction we assume that the above statement has a counterexample G, that we choose to have the minimum number of vertices. Clearly $\nu_H(G) \ge 1$. Let M be an induced subdivision of H with minimum number of vertices. As the connected components of H are paths and subdivided stars, M is a copy of H. Hence $|M| = |H|$. By minimality of G, we have $\tau_H(G - V(M)) \le \nu_H(G - V(M)) \cdot |H| \le (\nu_H(G) - 1) \cdot |H|$. We deduce

$$\begin{aligned} \tau_H(G) &\le \tau_H(G - V(M)) + |M| \\ &\le (\nu_H(G) - 1)|H| + |H| \\ &= \nu_H(G) \cdot |H|. \end{aligned}$$

This contradicts the definition of G. $\qquad\square$

Lemma 3.4 (item (1) of Theorem 1.5). *Let H be a forest, one connected component of which contains at least two vertices of degree at least 3. Then the subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. Our goal is to construct an infinite family of graphs $(G_n)_{n \in \mathbb{N}_{\ge 1}}$ such that $\nu_H(G_n) = \mathcal{O}(1)$ while $\tau_H(G_n) = \Omega(n)$. Let J be a connected component of H which contains at least two vertices of degree at least 3 and let u and u' be two such vertices, that are at

minimal distance of each other. Let $\{v, v'\}$ be an edge of the path connecting u to u', with the convention that u is closer to v than to v'. We call T the connected component of $H - \{vv'\}$ that contains v, T' that that contains v', and D the union of the remaining connected components. We also call F (resp. F') the graph obtained from T (resp. T') by removing the vertices of the path from u to v (resp. u' to v'). See Figure 3 for an illustration.
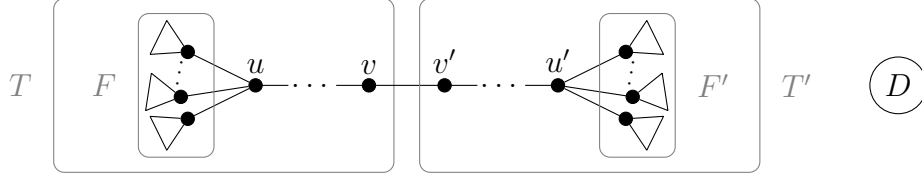
FIGURE 3. The forest H.

Recall that the triangle-wall Γ_n is defined in Section 2 and has special vertices called a_i and b_i. For every $n \in \mathbb{N}_{\geq 1}$, we construct the graph G_n from a copy of Γ_n as follows:

(1) for every $i \in \{0, \ldots, n-1\}$, we add a new copy T_i of T and add an edge from the copy of v to a_i, and let F_i be the copy of F in T_i;

(2) for every $i \in \{0, \ldots, n-1\}$, we add a new copy T_i' of T' and add an edge from the copy of v' to b_i, and let F_i' be the copy of F in T_i';

(3) for every $i \in \{0, \ldots, n-1\}$, we add a new copy D_i of D;

(4) for every distinct $i, j \in \{0, \ldots, n-1\}$, we add all edges between $F_i \cup D_i$ and $F_j \cup D_j$, between $F_i' \cup D_i$ and $F_j' \cup D_j$ and between $F_i \cup D_i$ and $F_j' \cup D_j$.
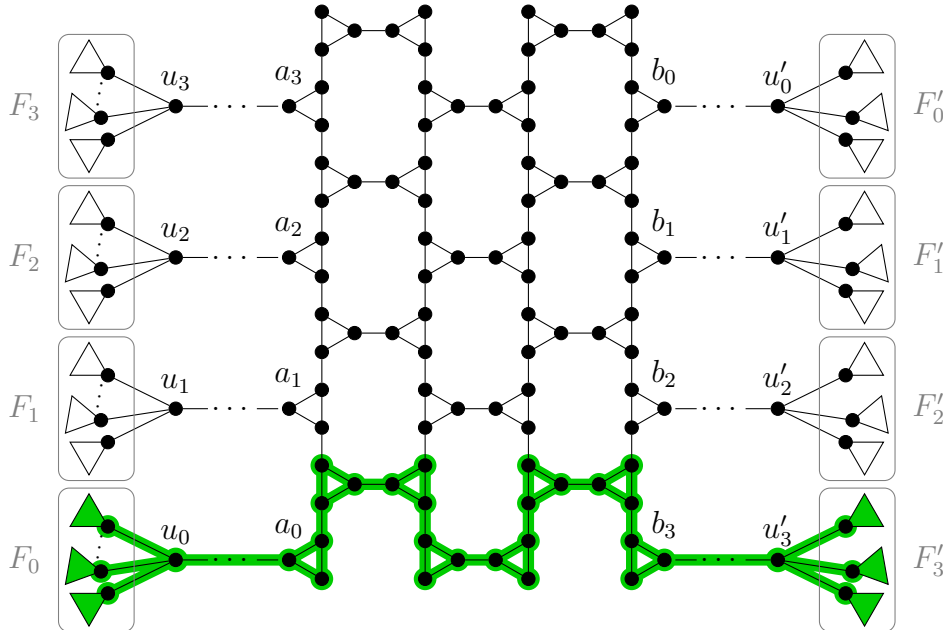
FIGURE 4. The graph G_4. The D_i's and the edges between the $V(F_i) \cup V(F_i')$'s (framed) are not depicted.

Let $n \in \mathbb{N}$ and let us show that every induced subdivision of H in G_n has a very restricted position. For convenience we refer to the copy of Γ_n in G_n as Γ_n and if w is

a vertex of H, we refer by w_i to its copy in T_i, T_i', or D_i. We denote by P_{xy} the unique path of T (resp. T') with endpoints $x, y \in V(T)$ (resp. $x, y \in V(T')$) and similarly for $P_{x_i y_i}$ in T_i and T_i'.

Let φ be a model of H in G_n.

Claim 3.5. *There is an integer $i \in \{0, \dots, n-1\}$ such that every vertex of degree at least 3 of $\varphi(H)$ belongs to $V(T_i) \cup V(T_i') \cup V(D_i)$.*

Proof. Let x, y be two vertices of degree 3 in $\varphi(H)$. As mentioned above, these vertices do not belong to Γ_n. Let us assume towards a contradiction that for distinct $i, j \in \{0, \dots, n-1\}$ we have

$$x \in V(T_i) \cup V(T_i') \cup V(D_i) \quad \text{and} \quad y \in V(T_j) \cup V(T_j') \cup V(D_j).$$

The first case we consider is when $x \in \{u_i, u_i'\}$. Then at most one of its neighbors (in $\varphi(H)$) belongs to $P_{u_i v_i}$ or $P_{u_i' v_i'}$ so x has two neighbors x_1 and x_2 that belong to $V(F_i) \cup V(F_i') \cup V(D_i)$. If additionally $y \in \{u_j, u_j'\}$, then we can similarly deduce that it has (in $\varphi(H)$) two neighbors that belong to $V(F_j) \cup V(F_j')$. By construction they are both adjacent to x_1 and x_2 and form together with them an induced C_4, a contradiction. We deduce $y \notin \{u_j, u_j'\}$, i.e. $y \in V(F_j) \cup V(F_j')$. But then y is adjacent to x_1 and x_2, which again forms an induced C_4 and hence is not possible. Notice that the case where $x \notin \{u_i, u_i'\}$ and $y \in \{u_j, u_j'\}$ is symmetric.

The only remaining case to consider is then when $x \notin \{u_i, u_i'\}$ and $y \notin \{u_j, u_j'\}$. Then $xy \in E(\varphi(H))$. At most one neighbor of x (in $\varphi(H)$) is u_i, so x has a neighbor $x_1 \neq y$ in $V(F_k) \cup V(F_k') \cup V(D_k)$ for some $k \in \{1, \dots, n\}$. As this neighbor cannot be adjacent to y without creating a cycle, it belongs in fact to $V(F_j) \cup V(F_j') \cup V(D_j)$. Symmetrically, y has a neighbors distinct from x in $V(F_i) \cup V(F_i') \cup V(D_i)$. But then $G_n[\{x, x_1, y, y_1\}]$ contains a cycle, a contradiction. This concludes the proof. ◇

Let i be as in the statement of Claim 3.5. Observe that $\varphi(H)$ contains no vertex of $V(T_j) \cup V(T_j') \cup V(D_j)$ for $j \in \{0, \dots, n-1\} \setminus \{i\}$, otherwise it would contain a cycle:

$$V(\varphi(H)) \subseteq V(T_i) \cup V(T_i') \cup V(D_i) \cup V(\Gamma_n).$$

By construction $|E(T_i \cup T_i' \cup D_i)| = |E(H)| - 1$. We deduce that $\varphi(H)$ contains a path Q from a vertex x of T_i to a vertex y of T_i'. As Q does not intersect $V(T_j) \cup V(T_j') \cup V(D_j)$ for $j \in \{0, \dots, n-1\} \setminus \{i\}$, a subpath of it links a_i to b_i in Γ_n.

We proved that if there is an induced subdivision of H in G_n, it contains an path of Γ_n from a_i to b_i for some $i \in \{0, \dots, n-1\}$. Notice that two such paths meet for distinct values of i. Consequently, $\nu_H(G_n) \leq 1$. On the other hand, $G_p[V(T_i) \cup V(P) \cup V(T_i') \cup V(D_i)]$ is an induced subdivision of H for every $i \in \{1, \dots, n\}$ and every chordless path P of Γ_n connecting a_i to b_i. Hence $\nu_H(G_n) = 1$.

We now show that $\tau_H(G_n) = \Omega(n)$.

Claim 3.6. *For every $n \in \mathbb{N}$, $\tau_H(G_n) \geq \frac{n}{2}$.*

Proof. This proof is similar to the end of the proof of Lemma 3.1. Let $n \in \mathbb{N}$. We consider a set X of $\lceil \frac{n}{2} \rceil - 1$ vertices of G_n and show that $G_n - X$ contains a induced subdivision of H. For every $i \in \{0, \dots, n-1\}$, we set

$$C_i^+ = V(T_i) \cup V(D_i) \cup V(T_{p-i-1}') \cup C_i.$$

Intuitively C_i contains the vertices of the copies T_i and T_{p-i+1}, a path linking them, and D_i; see Figure 4 for a depiction of C_1 in G_4 (in green). As in the proof of Lemma 3.1 we

deduce the existence of an integer i such that none of C_i^+ and C_{n-i-1}^+ intersects X. As $|X| < n$, for some $j \in \{0, \ldots, n-1\}$ the garland Q_j (which is a subgraph of Γ_n) does not contain a vertex of X. Besides, any of these garlands intersect each of $C_0, \ldots, C_{n-1}$. We deduce that in $G - X$, C_i and C_{n-i-1} (which by definition of i do not intersect X) belong to the same connected component. An induced subdivision of H can then be found by connecting the vertex a_i of T_i to the vertex b_i of T_{n-i-1} using a chordless path. As this argument works for every $X \subseteq V(G)$ such that $|X| \leq \lceil \frac{n}{2} \rceil - 1$, we deduce that $\tau_H(G_n) \geq \frac{n}{2}$. $\diamond$

We constructed a family $(G_n)_{n \in \mathbb{N}}$ of graphs such that $\nu_H(G_n) = \mathcal{O}(1)$ and $\tau_H(G_n) = \Omega(n)$. This proves that subdivisions of H do not have the induced Erdős-Pósa property. $\square$

4. Negative results using hypergraphs

We prove in this section negative results about the induced Erdős-Pósa property of the subdivisions of graphs containing long cycles and of graphs that have an edge away from a cycle, which are respectively items (2) and (3) of Theorem 1.5. We start with an easy lemma.

Lemma 4.1. *Let H be a graph containing a cycle C, let H' be a subdivision of H, and let e be an edge of H' contained in the subdivision of C. Then $H' - \{e\}$ does not contain an induced subdivision of H.*

Proof. For a graph G, let $c(G)$ denote the number of distinct (and not disjoint) cycles in G. Observe that there is a bijection between the cycles of G and those of a subdivision G' of G, hence $c(G) = c(G')$. Besides, if G' is a subgraph of G, then every cycle of G' is a cycle of G, hence $c(G') \leq c(G)$.

As e belongs to a cycle, $c(H' - \{e\}) < c(H') = c(H)$. If $H' - \{e\}$ contained a subdivision H'' of H we would have the following contradiction:

$$c(H) = c(H'') \leq c(H' - \{e\}) < c(H') = c(H). \qquad \square$$

Our proofs rely on a suitable modification of the construction given in [KK17] to show that for $\ell \geq 5$, C_ℓ-subdivisions have no induced Erdős-Pósa property.

A pair (X, E) consisting of a set X and a family E of non-empty subsets of X is called a *hypergraph*. Each element in E is called a *hyperedge*, and for a hypergraph H, let $E(H)$ denote the set of hyperedges in H. A subset Y of X is called a *hitting set* if for every $F \in E$, $Y \cap F \neq \emptyset$. For positive integers a, b with $a \geq b$, let $U_{a,b}$ be the hypergraph (X, E) such that $X = \{1, \ldots, a\}$ and E is the set of all subsets of X of size b. It is not hard to observe that for every positive integer k, every two hyperedges of $U_{2k-1,k}$ intersect and that the minimum size of a hitting set of $U_{2k-1,k}$ is precisely k.

The bipartite graph UB_k with bipartition (A, B) is defined as follows:

- $A = \{1, \ldots, 2k-1\}$;
- for every hyperedge $F = \{a_1, a_2, \ldots, a_k\}$ of $U_{2k-1,k}$ with $a_1 < \cdots < a_k$, we add fresh vertices $p_1^F, \ldots, p_{k+1}^F$ to B and the edges of the path $P_F = p_1^F a_1 p_2^F a_2 \cdots p_k^F a_k p_{k+1}^F$.

We also set $\mathcal{P}(UB_k) = \{P_F : F \in E(U_{2k-1,k})\}$.

We are ready to prove one of the main results of this section.

Theorem 4.2 (item (2) of Theorem 1.5)**.** *If a graph H contains an induced cycle of length at least 5, then the subdivisions of H have no induced Erdős-Pósa property.*

Proof. Let H be a graph containing an induced cycle C of length at least 5, and let $m := |H|$. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq m}}$ of graphs such that $\nu_H(G_n) = 1$ while $\tau_H(G_n) \geq n$ for every $n \in \mathbb{N}_{\geq m}$.

Let $n \geq m$ be an integer and let us fix an edge uv of C. Let H' be the graph obtained from H by subdividing each edge into a path of length $2n$. In particular, we subdivide uv into $uw_1w_2 \cdots w_{2n-1}v$. We construct a graph $G = G_n$ from UB_n as follows:

- for each path P of $\mathcal{P}(UB_n)$, we take a copy H_P of H' and identify the copy of $uw_1w_2 \cdots w_{2n-1}w$ with the path P, in the same order
- then we add all possible edges between $V(H_P) \setminus A$ and $V(H_{P'}) \setminus A$ for two distinct paths P and P' in $\mathcal{P}(UB_n)$.

Note that A is still an independent set in G. Clearly, each H_P is an induced subdivision of H.

Claim 4.3. *Every induced subdivision of H in G is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$, and contains the path P.*

Proof. Let φ be a model of H in G and $C' := \varphi(C)$.

We first show that C' is contained in H_P for some $P \in \mathcal{P}(UB_n)$. Since A is an independent set, C' contains a vertex v of $H_P - A$ for some $P \in \mathcal{P}(UB_n)$. Suppose for contradiction that C' is not contained in H_P. Then C' also contains a vertex v' of $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n)$, $P' \neq P$. By construction v is adjacent to v'.

As C' contains no triangle, C' does not contain any vertex in $H_{P''}$ for $P'' \in \mathcal{P}(UB_n) \setminus \{P, P'\}$.

We now analyze the order of $C'[A]$. Let us observe that if I is an independent set of a cycle M, we have $|N_M(I)| \geq |I|$ since we can define an injective mapping from I to $V(M) \setminus I$ by mapping $v \in V(T)$ to its left neighbor. Since $C'[V(C') \cap A]$ is an independent set in G, the previous observation implies that $|C' - A| \geq |C'[V(C') \cap A]|$.

Suppose $C' - (\{v, v'\} \cup A)$ has two vertices w and w'. Then either C' has a vertex with three neighbors in C' (when both w and w' belong to one of $H_P - A$ and $H_{P'} - A$), or it contains an induced C_4 (when $\{w, w'\}$ intersects both $V(H_P - A)$ and $V(H_{P'} - A)$). This is not possible as C' is an induced cycle of length at least 5. We deduce that $C' - A$ has at most three vertices.

Observe that if $|C' - A| = |V(C') \cap A| = 3$, then C' is a C_6 and $C' - A$ is an independent set. This is not possible as $vv' \in E(C' - A)$, hence $|C'| \leq 5$. Since $V(C') \cap A$ is an independent set in C', we deduce $|V(C') \cap A| \leq 2$.

Now, if $|V(C') \cap A| = 2$, then $C' - A$ has three vertices and only one edge, contradicting the fact that it intersects both $H_P - A$ and $H_{P'} - A$. Consequently, C' has length 4, which is a contradiction.

We conclude that C' is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$. Note that every cycle in H_P contains more than m vertices of $H_P - V(P)$, by construction. Therefore, if $\varphi(H)$ contains a vertex of $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n)$, $P' \neq P$, then this vertex should have degree at least m in $\varphi(H)$, which cannot happen in an induced subdivision of H. Thus, all other vertices are also contained in $V(H_P) \cup A$, which proves the first part of the claim. Since P is a part of the subdivisions of the cycle of H in H_P, by Lemma 4.1, the subdivision of H should contain the path P. $\diamond$

Since every induced subdivision of H in G contains a path in $\mathcal{P}(UB_n)$, no two induced subdivisions of H in G are vertex-disjoint, hence $\nu_H(G) = 1$. For every vertex subset

S of size at most $n - 1$, there exists $P \in \mathcal{P}(G)$ such that H_P contains no vertex of S. Therefore $\tau_H(G) \geq n$, as required. $\qquad\square$

Using the same construction, we can also prove the following.

Theorem 4.4 (item (3) of Theorem 1.5). *Let H be a graph containing a cycle C and two adjacent vertices having no neighbors in C. Then the subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. We use the same construction in Theorem 4.2. Similarly, we claim that every induced subdivision of H in G is contained in $V(H_P) \cup A$ for some $P \in \mathcal{P}(UB_n)$.

Let φ be a model of H in G and let $C' := \varphi(C)$. Let vw be an edge of $\varphi(H)$ having no neighbors in C'. Such an edge exists by the assumption that H contains two adjacent vertices having no neighbors in C.

As A is independent, one of v and w is not contained in A. Without loss of generality, we assume $v \in V(H_P) \setminus A$ for some $P \in \mathcal{P}(UB_n)$. Since v has no neighbor in C', C' should be contained in $V(H_P) \cup A$. By construction, C' has at least $|H|$ vertices in $H_P - A$. Thus, if $\varphi(H)$ contained a vertex in $H_{P'} - A$ for some $P' \in \mathcal{P}(UB_n) \setminus \{P\}$, then it would have degree at least $|H|$, which is not possible. This implies that $V(\varphi(H)) \subseteq V(H_P) \cup A$, and again by Lemma 4.1, $\varphi(H)$ contains P. The remaining part is the same as in Theorem 4.2. $\qquad\square$

5. Negative results using the semi-grid

In this section, we show that if H contains a cycle and three vertices that have no neighbors in C, then H-subdivisions have no induced Erdős-Pósa property. For example, we may consider a graph F that is the disjoint union of C_3 and three isolated vertices. Clearly, triangle-walls contain F as an induced subgraph. Also, in the construction based on hypergraphs in Section 4, we have an independent set A, and we can choose some C_4 by picking two vertices from each of $V(H_P) \setminus A$ and $V(H_{P'}) \setminus A$ for some distinct $P, P' \in \mathcal{P}(UB_n)$, and then choose three vertices in A having no neighbors in C_4 (for this, we choose P, P' so that $A \setminus (V(P) \cup V(P'))$ has 3 vertices). Therefore, the constructions presented in the previous sections do not seem helpful to deal with this case. Let us introduce a new one.

For $n \in \mathbb{N}_{\geq 3}$, we define the *semi-grid* SG_n of order n as follows:

(1) $V(SG_n) = \{v_{i,j} : i, j \in \{1, \ldots, n\}, i \geq j\}$;
(2) for every $i \in \{1, \ldots, n\}$, we add the edges of the path P_i, defined as the concatenation of $v_{i,1} \cdots v_{i,i}$ and $v_{i,i} \cdots v_{n,i}$;
(3) additionally, we add an edge between two vertices v and w if there is no $i \in \{1, \ldots, n\}$ such that $v, w \in V(P_i)$.

An example is depicted in Figure 5.

Observe that SG_n satisfies the following, for every $i, j \in \{1, \ldots, n\}$ with $i < j$:

- P_i is an induced path;
- for $i, j \in \{1, 2, \ldots n\}$ with $i < j$, $V(P_i) \cap V(P_j) = \{v_{j,i}\}$;
- every vertex of SG_n belongs to at most two paths of $P_1, \ldots, P_n$;
- $V(SG_n) \setminus V(P_i)$ has at least $n - 2$ neighbors in P_i (this is because a vertex in $V(SG_n) \setminus V(P_i)$ belongs to at most two paths of $P_1, \ldots, P_n$, which may contain non-neighbors of v on their intersections with P_i).
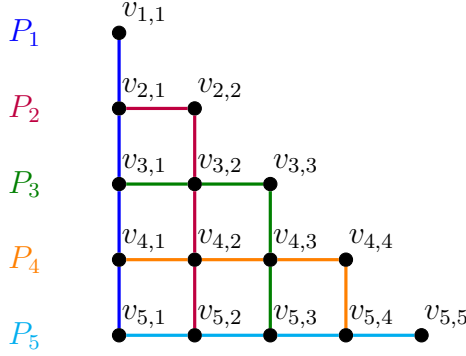
FIGURE 5. The graph SG_5. The P_i's are drawn with different colors. The edges added in step (3) are not depicted.

Theorem 5.1 (item (4) of Theorem 1.5). *Let H be a graph that contains a cycle C and three vertices having no neighbors in C. Then the subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. Let $m := |H|$. We construct an infinite family $(G_n)_{n \in \mathbb{N}_{\geq m+2}}$ of graphs such that $\nu_H(G_n) = 1$ while $\tau_H(G_n) \geq \frac{n}{2}$ for every $n \in \mathbb{N}_{\geq m+2}$.

Let us fix an edge uv of C. Let $n \geq m + 2$ be an integer and let H' be the graph obtained from H by subdividing each edge into a path of length $n + 1$. In particular, we subdivide the edge uv into $uw_1w_2 \cdots w_nv$. The graph $G = G_n$ is constructed from SG_n as follows:

- we take copies $H_1, H_2, \ldots, H_n$ of H' and for each $j \in \{1, \ldots, n\}$, we identify the copy of $w_1w_2 \cdots w_n$ in H_j with the path P_j;
- then we add all possible edges between $V(H_j) \setminus V(P_j)$ and $V(G) \setminus V(H_j)$.

Clearly, each H_j is an induced subdivision of H. We show that every induced subdivision of H in G is H_j for some $j \in \{1, \ldots, n\}$.

Claim 5.2. *Every induced subdivision of H in G is contained in H_j for some $j \in \{1, \ldots, n\}$ and contains the path P_j.*

Proof. Let φ be a model of H in G, and let $F := \varphi(H)$ and $C' := \varphi(C)$, and $Z = \{z_1, z_2, z_3\}$ be a set of three vertices in F having no neighbors in C'. Such a set Z exists because H contains three vertices having no neighbors in C.

We consider two cases depending on whether Z contains a vertex of $V(H_j) \setminus V(P_j)$ for some $j \in \{1, \ldots, n\}$, or not.

(*First case: Z contains a vertex of $V(H_j) \setminus V(P_j)$ for some $j \in \{1, 2, \ldots, n\}$.*)

By construction, for all $a \in V(H_j) \setminus V(P_j)$ and $b \in V(G) \setminus V(H_j)$, a is adjacent to b. As Z has no neighbors in C', C' is contained in H_j. Furthermore, since P_j is a path, C' contains a vertex of $V(H_j) \setminus V(P_j)$. It implies that Z is also contained in H_j.

First assume that C' contains P_j. In this case, every vertex in $V(G) \setminus V(H_j)$ has at least $n - 2 \geq m$ neighbors in P_j. As the maximum degree of H is less than m, no vertex of F belongs to $V(G) \setminus V(H_j)$. Hence F is an induced subgraph of H_j. By assumption, F contains P_j so we are done.

In the remaining case, C' does not contain P_j. As $V(C') \subseteq V(H_j)$ and P_j is a path, we deduce that all vertices of C' belong to $V(H_j) \setminus V(P_j)$. By construction every vertex

in $V(G) \setminus V(H_j)$ dominates C'. Since C' contains at least m vertices while H has maximum degree less than m, all other vertices of F are contained in H_j. If F does not contain an edge in P_j, then F contains no induced subdivision of H by Lemma 4.1. Therefore, F contains P_j, as required.

(*Second case:* $Z \subseteq V(SG_n)$.)

First assume that z_1, z_2, z_3 are all contained in P_i for some $i \in \{1, \ldots, n\}$. We can observe that every vertex of $V(G) \setminus V(P_i)$ has a neighbor in $\{z_1, z_2, z_3\}$. Thus, C' is an induced subgraph of H_i. Since P_i is a path and z_1 is already in P_i, C' is contained in $V(H_i) \setminus V(P_i)$. Since C' contains at least m vertices and every vertex in $V(G) \setminus V(H_i)$ dominates C', all other vertices of F are contained in H_i. If F does not contain an edge in P_i, then F contains no induced subdivision of H by Lemma 4.1. Therefore, F contains P_i.

We can assume that two of z_1, z_2, z_3 are not contained in the same path P_i of SG_n. Without loss of generality, we assume that $z_1 \in V(P_a) \cap V(P_{a'})$ and $z_2 \in V(P_b) \cap V(P_{b'})$ such that

- $\{a, a'\} \cap \{b, b'\} = \emptyset$,
- if $a = a'$, then $z_1 := v_{a,a}$,
- if $b = b'$, then $z_2 := v_{b,b}$.

Therefore the four vertices $u_{a,b} \in V(P_a) \cap V(P_b)$, $u_{a,b'} \in V(P_a) \cap V(P_{b'})$, $u_{a',b} \in V(P_{a'}) \cap V(P_b)$, and $u_{a',b'} \in V(P_{a'}) \cap V(P_{b'})$ are the only vertices in G that are not adjacent to both z_1 and z_2. Thus, $V(C') \subseteq \{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\}$.

Observe that z_3 is contained in at least one and at most two paths of $P_a, P_{a'}, P_b, P_{b'}$. If z_3 is contained in exactly one path of them, say Q, then z_3 is adjacent to the vertices of $\{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\} \setminus V(Q)$. But every cycle in $\{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\}$ contains a vertex of $\{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\} \setminus V(Q)$, a contradiction. We may assume that z_3 is contained in two paths of them, say P_x and P_y. Then $z_3 = u_{x,y}$, and the other three vertices form a triangle C'. But then z_3 is adjacent to the vertex in $\{u_{a,b}, u_{a',b}, u_{a,b'}, u_{a',b'}\} \setminus (V(P_x) \cup V(P_y))$, a contradiction.

This concludes the proof of the claim. $\diamond$

By Claim 5.2, every induced subdivision of H in G contains P_j for some $j \in \{1, \ldots, n\}$. Thus, two induced subdivisions of H always intersect, and hence $\nu_H(G) = 1$. Let S be a vertex subset of size less than $\frac{n}{2}$. Clearly each vertex of S hits at most two models in the graph. Thus, S hits less than $2(\frac{n}{2}) = n$ models, and $G - S$ contains an induced subdivision of H. This implies that $\tau_H(G) \geq \frac{n}{2}$, as required. $\square$

6. Negative results for non-planar graphs

We prove here item (6) of Theorem 1.5 (which is Lemma 6.2): subdivisions of a non-planar graph never have the induced Erdős-Pósa property.

We use the notion of *Euler genus* of a graph G. The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [MT01] for more details on graph embedding. The *Euler genus* $\gamma(G)$ of a graph G is the minimum Euler genus of a surface where G can be embedded.

We will adapt the proof of [RT17, Lemma 5.2] (similar to that of [RS86, Lemma 8.14] but dealing with subdivisions) to the setting of induced subdivisions.

Lemma 6.1 (from the proof of [RT17, Lemma 5.2]). *For every non-planar graph H, there is a family of graphs $(G_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,*

(i) *$\gamma(G_n) = \gamma(H)$; and*

(ii) *$G_n - X$ contains a subdivision of H, for every $X \subseteq V(G_n)$ with $|X| < n$.*

Lemma 6.2. *For every non-planar graph H, the subdivisions of H do not have the induced Erdős-Pósa property.*

Proof. We construct a family of graphs $(\dot{G}_n)_{n\in\mathbb{N}}$ such that $\nu_H(\dot{G}_n) = 1$ and $\tau_H(\dot{G}_n) \geq n$, for every $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, let G_n be the graph of Lemma 6.1 and let $\dot{G}_n$ be the graph obtained by subdividing once every edge of G_n. Notice that for every subdivision G' of a graph G we have $\gamma(G) = \gamma(G')$ (an embedding of one in a given surface can straightforwardly be deduced from an embedding of the other). Hence $\gamma(\dot{G}_n) = \gamma(G_n) = \gamma(H)$. As in the proof of [RT17, Lemma 5.2]), we observe that since H is not planar the disjoint union of two subdivisions of H has Euler genus larger than that of H (see [BHK62]). Therefore $\dot{G}_n$ does not contain two disjoint (induced) subdivisions of H: $\nu_H(\dot{G}_n) \leq 1$. For every subgraph J of G_n, the subgraph of $\dot{G}_n$ obtained by subdividing once every edge of J is induced. Hence property (ii) of $(G_n)_{n\in\mathbb{N}}$ implies that for every $X \subseteq V(\dot{G}_n)$ with $|X| < n$ the graph $\dot{G}_n - X$ contains an induced subdivision of H. In other words $\tau_H(\dot{G}_n) \geq n$.   □

7. A LOWER-BOUND ON THE BOUNDING FUNCTION

We proved in the previous sections that subdivisions of graphs with certain properties do not have the induced Erdős-Pósa property. In this section we give a negative result of a different type by proving that the bounding function for the induced Erdős-Pósa property of subdivisions cannot be too small. Our proof is an adaptation to the induced setting of an observation already known for the non-induced Erdős-Pósa property.

We first need a few more definitions. For every graph G, the *girth* of G, denoted by $\mathbf{girth}(G)$ is the minimum order of a cycle in G. The treewidth of G, denoted by $\mathbf{tw}(G)$ is a graph invariant that can be defined using tree-decompositions. We avoid the technical definition here and only state the two well-known properties of treewidth that we use:

- deleting a vertex or an edge in a graph decreases its treewidth by at most one;
- for every planar graph H of maximum degree 3, there is a constant $c \in \mathbb{N}$ such that every graph of treewidth at least c contains a subdivision of H (Grid Minor Theorem, [RS86]).

We refer the reader to [Die10] for an introduction to treewidth.

Theorem 7.1. *Let H be a graph that has a cycle and no vertex of degree more than 3. There is no function $f(k) = o(k \log k)$ such that subdivisions of H have the induced Erdős-Pósa property with bounding function f.*

Proof. When H is not planar, the result follows from Lemma 6.2. We therefore assume for now that H is planar. By the Grid Minor Theorem, there is a constant c such that every graph G satisfying $\mathbf{tw}(G) \geq c$ contains a subdivision of H. We will construct sequences $(G_n)_{n\in\mathbb{N}}$ (graphs) and $(k_n)_{n\in\mathbb{N}}$ (integers) such that $\nu_H(G_n) = \mathcal{O}(k_n)$ while $\tau_H(G_n) = \Omega(k_n \log k_n)$.

We start with an infinite family $(R_n)_{n \in \mathbb{N}}$ of 3-regular graphs (of increasing order), called Ramanujan graphs, whose existence is proved in [Mor94, Theorem 5.13]. These graphs have the following properties:

(1) $\forall n \in \mathbb{N}$, $\mathbf{girth}(R_n) \geq \frac{2}{3} \log |R_n|$ ([Mor94, Theorem 5.13]);

(2) $\exists n' \in \mathbb{N}$, $\exists \alpha \in \mathbb{R}_{>0}$, $\forall n \in \mathbb{N}$, $n \geq n' \Rightarrow \mathbf{tw}(R_n) \geq \alpha |R_n|$ (see [BEM+04, Corollary 1]).

Recall that $n \mapsto |R_n|$ is increasing. We denote by n'' the minimum integer that is larger than n' and such that $|R_n| \geq \frac{c}{\alpha}$ for every $n \geq n''$. For every integer $n \geq n''$, we define k_n as the maximum positive integer such that

$$\frac{1}{\alpha} (c + k_n \log k_n) \leq |R_n|$$

Such a value exists by definition of n''. Notice that $n \mapsto k_n$ is non-decreasing and is not upper-bounded by a constant. Observe that for every $n \geq n''$, $\mathbf{tw}(R_n) \geq c + k_n \log k_n$.

Let us define, for every integer $n \in \mathbb{N}$, G_n as the graph obtained from R_n by subdividing once every edge. We then have, for every $n \geq n''$:

$$|G_n| = \frac{5}{2} \cdot |R_n|$$

$$(1) \qquad \geq \frac{5}{2\alpha}(c + k_n \log k_n) \qquad\qquad \text{and}$$

$$\mathbf{girth}(G_n) = 2 \cdot \mathbf{girth}(R_n)$$

$$(2) \qquad \geq \frac{4}{3} \cdot \log \left(\frac{2}{5} \cdot |G_n| \right).$$

Claim 7.2. *For every integer $n \geq n''$, $\boldsymbol{\tau}_H(G_n) \geq k_n \log k_n$.*

Proof. Let $X \subseteq V(G_n)$ be such that $|X| < k_n \log k_n$. We show that $G_n - X$ contains an induced subdivision of H. Recall that each vertex of degree 2 of G_n was obtained by subdividing an edge of R_n. We define X^e as the set of edges of R_n corresponding to the vertices of degree 2 in X and set $X^v = X \cap V(R_n)$. As the deletion of an edge or a vertex in a graph decreases its treewidth by at most one, the graph obtained from R_n by deleting X^v and X^e has treewidth at least c. By definition of c, this graph contains a subdivision S of H. Notice that the corresponding subdivision of H in G_n (i.e. that obtained by subdividing once every edge of S) is induced and does not contain any vertex of X. As this holds for every subset of $V(G_n)$ with less than $k_n \log k_n$ vertices, we deduce that $\boldsymbol{\tau}_H(G_n) \geq k_n \log k_n$. $\diamond$

Claim 7.3. *There is a $n''' \in \mathbb{N}$, such that for every $n \geq n'''$, $\boldsymbol{\nu}_H(G_n) < k_n$.*

Proof. Let $n \geq n''$ and let us assume that G contains k_n disjoint induced subdivisions of H. As H has a cycle, the order of each of these subdivisions is at least the girth of G. We deduce:

$$|G_n| \geq k_n \cdot \mathbf{girth}(G_n)$$

$$\geq k_n \cdot \frac{4}{3} \cdot \log \left(\frac{2}{5} \cdot |G_n| \right) \qquad\qquad \text{by (2)}$$

$$\geq k_n \cdot \frac{4}{3} \cdot \log \left(\frac{1}{\alpha}(c + k_n \log k_n) \right) \qquad\qquad \text{by (1)}.$$

On the other hand, from the maximality of k_n we get:

$$|G_n| < \frac{5}{2\alpha}(c + k_n \log k_n).$$

Combining these bounds together we obtain:

$$(3) \qquad k_n \cdot \frac{4}{3} \cdot \log\left(\frac{1}{\alpha}(c + k_n \log k_n)\right) < \frac{5}{2\alpha}(c + k_n \log k_n).$$

The left-hand side of (3) is a $\Omega(k_n \log(k_n \log k_n))$ while its right-hand side is a $\mathcal{O}(k_n \log k_n)$. As $n \mapsto k_n$ is not upper-bounded by a constant, there is a positive integer $n''' \geq n''$ such that for every $n \geq n'''$, (3) does not hold. Therefore, when $n \geq n'''$ we have $\nu_H(G_n) < k_n$. $\diamond$

The sequences $(G_n)_{n \in \mathbb{N}}$ and $(k_n)_{n \in \mathbb{N}}$ have the property that $\tau_H(G_n) = \Omega(k_n \log k_n)$ (Claim 7.2) while $\nu_H(G_n) = \mathcal{O}(k_n)$ (Claim 7.3), as required.

$\square$

8. Subdivisions of 1-pan or 2-pan have the induced Erdős-Pósa property

Recall that for $p \in \mathbb{N}_{\geq 1}$, the p-pan is the graph obtained by adding an edge between a triangle and an end vertex of a path on p vertices. We show in this section that for every $p \in \{1, 2\}$, the subdivisions of the p-pan have the induced Erdős-Pósa property with bounding function $\mathcal{O}(k^2 \log k)$ (Theorem 8.2 and Theorem 8.7). For $p \geq 3$, the p-pan has an edge that has no neighbor in its triangle, and by Theorem 4.4, the subdivisions of the p-pan do not have the induced Erdős-Pósa property. This proves the dichotomy (3) of Corollary 1.6. Furthermore, the proofs of our positive results yield polynomial-time algorithms for finding a packing of induced subdivisions of the p-pan or a hitting set of size $\mathcal{O}(k^2 \log k)$.

The proofs for 1-pan and 2-pan start similarly. Let $p \in \{1, 2\}$ and let G and H be graphs such that:

- H is an induced subdivision of the p-pan in G with minimum number of vertices,
- $G - V(H)$ has no induced subdivision of the p-pan.

With these assumptions, we will show that for every $k \in \mathbb{N}$, G contains either k pairwise vertex-disjoint induced subdivisions of the p-pan, or a vertex set S of size $\mathcal{O}(k \log k)$ hitting all induced subdivisions of the p-pan. By applying inductively this result we can conclude that a graph contains either k pairwise vertex-disjoint induced subdivisions of the p-pan, or a vertex set of size $\mathcal{O}(k^2 \log k)$ hitting all induced subdivisions of the p-pan.

The following algorithm to find an induced subdivision of the p-pan will be necessary.

Lemma 8.1. *Let p be a positive integer. Given a graph G, one can find an induced subdivision of the p-pan with minimum number of vertices, if one exists, in time $\mathcal{O}(|G|^{p+5})$.*

Proof. We first describe how, given vertices $v_1, v_2, \ldots, v_p$ and w_1, w_2, w_3, one can in $\mathcal{O}(|G|^2)$ steps determine whether there exists an induced subdivision of the p-pan H such that

- $\{v_1, v_2 \ldots, v_p, w_1, w_2, w_3\} \subseteq V(H)$,
- $v_1 \ldots v_p w_2$ is an induced path of G,
- both w_1 and w_3 are adjacent to w_2,
- $H - \{v_1, v_2, \ldots, v_p, w_2\}$ is an induced path of G from w_1 to w_3,
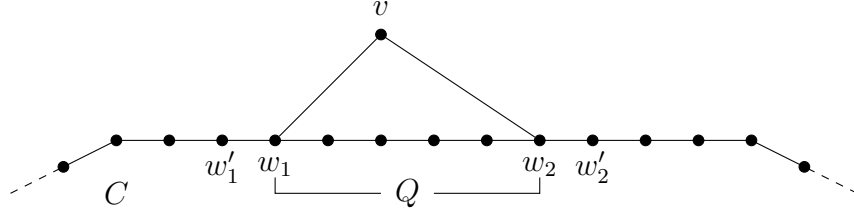
FIGURE 6. Two neighbors w_1 and w_2 of v with minimum $\mathrm{dist}_C(w_1, w_2)$ in Lemma 8.3.

and output such an induced subdivision with minimum order if one exists. Since it is an induced subdivision of the p-pan, there should not exist an edge between $\{v_1, v_2, \ldots, v_p\}$ and the induced path $H - \{v_1, v_2, \ldots, v_p, w_2\}$, and furthermore, there should not exist an edge between w_2 and the internal vertices of the path.

We may check the first three conditions in time $\mathcal{O}(p^2)$. If one of them is not satisfied, then we answer negatively.

We then compute a shortest path P from w_1 to w_3 in

$$G - \{v_1, v_2, \ldots, v_p, w_2\} - (N_G(\{v_1, v_2, \ldots, v_p, w_2\}) \setminus \{w_1, w_3\}).$$

It can be done in time $|G|^2$, for instance using Dijkstra's algorithm. If such a path does not exist, then we answer negatively. Otherwise, $G[\{v_1, v_2, \ldots, v_p, w_2\} \cup V(P)]$ is the desired induced subdivision of the p-pan.

We apply the above procedure for every choice of $p + 3$ vertices of G. In the end, we output an induced subdivision of the p-pan of minimum order among those returned, if any. Overall this takes $\mathcal{O}(|G|^{p+5})$ steps. If S is an induced subdivision of the p-pan in G, then one choice of $v_1 v_2 \cdots v_p w_2$ corresponds to the path of length p of S and w_1 and w_3 to the two neighbors of w_2 on the cycle. A shortest path from w_1 to w_3 yields an induced subdivision of the p-pan of order at most $|S|$. This guarantees that the algorithm outputs an induced subdivision of the p-pan with minimum order. □

8.1. **On induced subdivisions of the 1-pan.** The aim of this section is to prove the following theorem.

Theorem 8.2. *There is a polynomial-time algorithm which, given a graph G and a positive integer k, finds either k vertex-disjoint induced subdivisions of the 1-pan in G or a vertex set of size $\mathcal{O}(k^2 \log k)$ hitting every induced subdivision of the 1-pan in G.*

Let us refer to an induced subdivision of the 1-pan as a pair (v, C), where v is the vertex of degree one and C is the cycle. We start with structural lemmas.

Lemma 8.3. *Let G be a graph and let $H = (u, C)$ be an induced subdivision of the 1-pan in G with minimum number of vertices. If $|C| \geq 5$, then every vertex of $V(G) \setminus V(H)$ has at most one neighbor in C.*

Proof. Suppose for contradiction that there exists a vertex $v \in V(G) \setminus V(H)$ having at least two neighbors in C. We choose two neighbors w_1, w_2 of v with minimum $\mathrm{dist}_C(w_1, w_2)$. Let Q be a shortest path from w_1 to w_2 in C. See Figure 6 for an illustration. Then $G[V(Q) \cup \{v\}]$ is an induced cycle, and it is strictly shorter than C, as $|C| \geq 5$ and

$$|Q| + 1 \leq \frac{|C|}{2} + 2 < |C|.$$

If $V(C) = V(Q)$, then w_1 is adjacent to w_2, and by the choice of Q, Q also has length 1. This is not possible. So we know that $C - V(Q)$ has at least one vertex.

For each $i \in \{1, 2\}$, let w_i' be the neighbor of w_i in C, which is not on the path Q. If $w_1' = w_2'$, then Q has length at most 2, and $|C| \leq 4$, a contradiction. So, we may assume that $w_1' \neq w_2'$.

Since H is an induced subdivision with minimum number of vertices, for each $i \in \{1, 2\}$, $G[V(Q) \cup \{v, w_i'\}]$ is not an induced subdivision of the 1-pan. It implies that v is adjacent to both w_1' and w_2'. Note that w_1' is not adjacent to w_2'; otherwise, $\mathrm{dist}_C(w_1, w_2) \leq \mathrm{dist}_C(w_1', w_2') = 1$ and we would have $|C| \leq 4$, a contradiction. Therefore, $G[\{v, w_1, w_1', w_2'\}]$ is isomorphic to the 1-pan, contradicting the assumption that H is an induced subdivision with minimum number of vertices.

We conclude that every vertex in $V(G) \setminus V(H)$ has at most one neighbor in C. $\square$

Suppose (u, C) is an induced subdivision of the 1-pan in a graph G. In the next lemmas, we explain how to extract many induced subdivisions of the 1-pan from a set of $V(C)$-paths of $G - E(C)$. For an induced cycle U of G and $q \in V(G) \setminus V(U)$, we call (q, U) a *good pair* if $G[\{q\} \cup V(U)]$ contains an induced subdivision of the 1-pan.

Lemma 8.4. *Let $C = v_1 v_2 \cdots v_m v_1$ be an induced cycle of length at least 4 in a graph G, and let $v \in V(G) \setminus V(C)$ such that v is adjacent to v_3, and non-adjacent to v_1, v_2, v_4. Then $G[V(C) \cup \{v\}]$ contains an induced subdivision of the 1-pan.*

Proof. If v has no neighbors in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$, then it is clear. We may assume v has a neighbor in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$. We choose a neighbor v_i of v in $V(C) \setminus \{v_1, v_2, v_3, v_4\}$ with minimum i. Then $G[\{v, v_2, v_3, v_4, \ldots, v_i\}]$ is an induced subdivision of the 1-pan in G. $\square$

Lemma 8.5. *Let $k \geq 2$ be an integer, let G be a graph, and (v, C) be an induced subdivision of the 1-pan in G with minimum number of vertices. Given a set $\mathcal{P}$ of vertex-disjoint $V(C)$-paths in $G - E(C)$ with $|\mathcal{P}| \geq 108k \log k$, one can find k vertex-disjoint induced subdivisions of the 1-pan in time $\mathcal{O}(|G|^3)$.*

Proof. The existence of such a set $\mathcal{P}$ guarantees that C has length at least 5.

For each $P \in \mathcal{P}$, let $\mathrm{end}(P)$ be the set of end vertices of P. We construct a subset $\mathcal{P}'$ of $\mathcal{P}$ with the following property:

- $\forall P_1, P_2 \in \mathcal{P}'$, $\mathrm{dist}_C(\mathrm{end}(P_1), \mathrm{end}(P_2)) \geq 3$,
- $|\mathcal{P}'| \geq 12k \log k$.

This can be done by repeatedly choosing a path P in $\mathcal{P}$ and discarding from $\mathcal{P}$ all paths that have an endpoint at distance at most 4 from one of $\mathrm{end}(P)$ in C. For each path added to $\mathcal{P}'$, at most 8 are discarded, hence $|\mathcal{P}'| \geq |\mathcal{P}|/9 \geq 12k \log k$.

We consider the subgraph H on the vertex set $V(C) \cup (\bigcup_{P \in \mathcal{P}'} V(P))$ and edge set $E(C) \cup (\bigcup_{P \in \mathcal{P}'} E(P))$. This graph has maximum degree 3 and has at least $24k \log k$ vertices of degree 3, as each path of $\mathcal{P}'$ contributes for two vertices of degree 3.

According to Lemma 2.1, one can in time $|G|^3$ construct a set $\mathcal{Q}'$ of k vertex-disjoint cycles in H. Observe that C intersects all other cycles, and thus, C is not contained in $\mathcal{Q}'$. Note that for each cycle U of $\mathcal{Q}'$, $G[V(U)]$ contains an induced cycle that has at least one edge of C. This is always possible: in the cycle U, every chord e divides the cycle into two paths, one of which, together with e, is again a cycle containing an edge of C and less chords. Let $\mathcal{Q}$ be a collection of k resulting induced cycles of G obtained from $\mathcal{Q}'$.
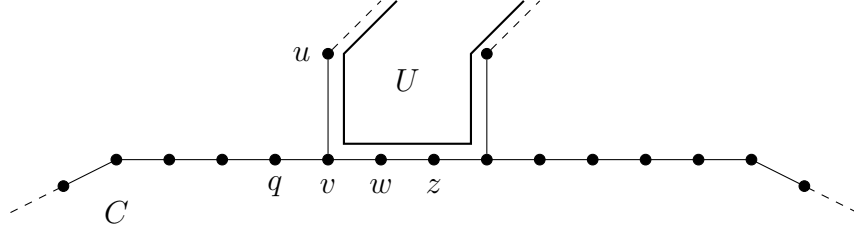
FIGURE 7. Four consecutive vertices u, v, w, z of U and the neighbor q of v in C other than w, described in Lemma 8.5. Since U is an induced cycle, such a neighbor q exists.

For each cycle $U \in \mathcal{Q}$, there are four consecutive vertices u, v, w, z of U such that $vw \in E(C) \cap E(U)$ and $u \notin V(C)$. See Figure 7 for an illustration. Note that the neighbor of v in C other than w is not contained in U, because U has no vertex of degree 3 in G. Let q be the neighbor of v in C other than w. Observe that q is adjacent to v but has no neighbors in $\{u, w, z\}$ by Lemma 8.3. Indeed, if z is in C, then q is not adjacent to z because $|C| \geq 5$, and if z is not in C, then q is not adjacent to z because q has a neighbor w in C. Therefore, $G[\{q\} \cup V(U)]$ contains an induced subdivision of the 1-pan by Lemma 8.4, that is, (q, U) is a good pair.

Following the above procedure, for each $U_i \in \mathcal{Q}$, we choose a vertex q_i as q. Observe that for distinct cycles $U_i, U_j \in \mathcal{Q}$, $q_i \neq q_j$ because otherwise

$$\text{dist}_C(V(U_i) \cap V(C), V(U_j) \cap V(C)) \leq 2,$$

and it implies that for some $P, P' \in \mathcal{P}'$,

$$\text{dist}_C(\text{end}(P), \text{end}(P')) \leq 2.$$

It contradicts our choice of $\mathcal{P}'$. Therefore, $q_1, q_2, \ldots, q_k$ are distinct vertices, and

$$(q_1, U_1), (q_2, U_2), \ldots, (q_k, U_k)$$

are pairwise disjoint k good pairs. Using Lemma 8.4, one can output k vertex-disjoint induced subdivisions of the 1-pan in linear time. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 8.2. If $k = 1$, then there is nothing to show. We assume that $k \geq 2$.

We construct sequences of graphs $G_1, \ldots, G_{\ell+1}$ and $F_1, \ldots, F_\ell$ with maximum ℓ such that

- $G_1 = G$,
- for every $i \in \{1, \ldots, \ell\}$, F_i is an induced subdivision of the 1-pan in G_i with minimum number of vertices,
- for every $i \in \{1, \ldots, \ell\}$, $G_{i+1} = G_i - V(F_i)$.

Such a sequence can be constructed in polynomial time repeatedly applying Lemma 8.1. If $\ell \geq k$, then we have found a packing of k induced subdivisions of the 1-pan. Hence, we may assume that $\ell \leq k - 1$.

Let $\mu_k := 216k \log k + 12k - 11$. The rest of the proof relies on the following claim.

Claim 8.6. *Let $j \in \{1, \ldots, \ell+1\}$. One can find in polynomial time either k vertex-disjoint induced subdivisions of the 1-pan, or a vertex set X_j of G_j of size at most $(\ell + 1 - j)\mu_k$ such that $G_j - X_j$ has no induced subdivision of the 1-pan.*

Proof. We prove the claim by induction for $j = \ell + 1$ down to $j = 1$. The claim trivially holds for $j = \ell + 1$ with $X_{\ell+1} = \emptyset$ because $G_{\ell+1}$ has no induced subdivision of the 1-pan. Let us assume that for some $j \le \ell$, we obtained a required vertex set X_{j+1} of G_{j+1} of size at most $(\ell - j)\mu_k$. Then in $G_j - X_{j+1}$, F_j is an induced subdivision of the 1-pan with minimum number of vertices. If F_j has at most 5 vertices, then we set $X_j := X_{j+1} \cup V(F_j)$. Clearly, $|X_j| \le (\ell - j + 1)\mu_k$. We may thus assume that F_j has at least 6 vertices. Let $F_j := (u, C)$. Observe that $|C| \ge 5$.

We first apply Gallai's A-path Theorem (Theorem 2.2) with $A = V(C)$ for finding at least $108k \log k$ pairwise vertex-disjoint $V(C)$-paths in $(G_j - X_{j+1}) - E(C)$. Assume it outputs such $V(C)$-paths. Then, by applying Lemma 8.5 to $G_j - X_{j+1}$ and $V(C)$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 1-pan. Thus, we may assume that Theorem 2.2 outputs a vertex set S of size at most $216k \log k$ hitting all $V(C)$-paths in $(G_j - X_{j+1}) - E(C)$.

Now, we consider the graph $G'_j := G_j - (X_{j+1} \cup S \cup \{u\})$. Suppose G'_j contains an induced subdivision $Q = (d, D)$ of the 1-pan. Then $G'_j[V(C) \cap V(Q)]$ is connected; otherwise, G'_j contains a $V(C)$-path in $(G_j - X_{j+1}) - E(C)$, contradicting with that S hits all such $V(C)$-paths. Furthermore, $G'_j[V(C) \cap V(Q)]$ contains no edge of D; otherwise, we also have a $V(C)$-path in $(G_j - X_{j+1}) - E(C)$. Thus, we have that $|V(C) \cap V(Q)| \le 2$.

To hit such remaining induced subdivisions of the 1-pan, we recursively construct sets $U \subseteq V(C) \setminus S$ and $\mathcal{J}$ as follows. At the beginning, set $U := \emptyset$ and $\mathcal{J} := \emptyset$. For every set of four consecutive vertices v_1, v_2, v_3, v_4 of C with $v_2, v_3 \notin U$ and $v_2, v_3 \notin S$, we test whether $G'_j[(V(G'_j) \setminus V(C)) \cup \{v_2, v_3\}]$ contains an induced subdivision H of the 1-pan, and if so, add vertices v_1, v_2, v_3, v_4 to U, and add H to $\mathcal{J}$ and increase the counter by 1. If the counter reaches k, then we stop.

Observe that graphs in $\mathcal{J}$ are pairwise vertex-disjoint; otherwise, we can find a $V(C)$-path in $(G_j - X_{j+1}) - E(C)$, a contradiction. Thus, if the counter reaches k, we can output k pairwise vertex-disjoint induced subdivisions of the 1-pan in polynomial time.

Assume that the counter does not reach k. Then we have $|U| \le 4(k - 1)$. We take the 1-neighborhood U' of U in C, and observe that $|U'| \le 12(k - 1)$. We claim that $G'_j - U'$ has no induced subdivision of the 1-pan. Suppose for contradiction that there is an induced subdivision F of the 1-pan in $G'_j - U_j$. The intersection of F on C is a set of at most two consecutive vertices, say T. In case when $|T| = 1$, neighbors of T in C are not contained in U, as U' is the 1-neighborhood of U in C. Thus, we can increase the counter by adding this induced subdivision to $\mathcal{J}$, a contradiction. Therefore, $G'_j - U'$ has no induced subdivision of the 1-pan, and

$$X_j := X_{j+1} \cup (S \cup \{u\} \cup U')$$

satisfies the claim. $\diamond$

The result follows from the claim with $j = 1$. $\square$

8.2. On induced subdivisions of the 2-pan.
This section is devoted to the proof of the following result.

Theorem 8.7. *There is a polynomial-time algorithm which, given a graph G and a positive integer k, finds either k vertex-disjoint induced subdivisions of the 2-pan in G or a vertex set of size $\mathcal{O}(k^2 \log k)$ hitting every induced subdivision of the 2-pan in G.*
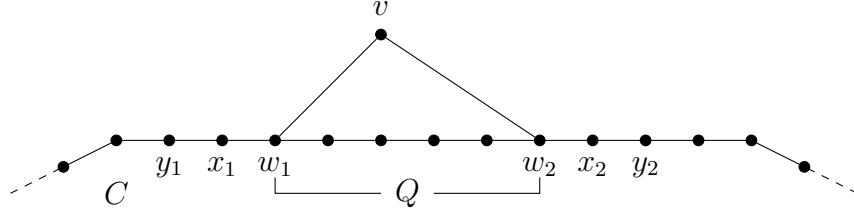
FIGURE 8. Two neighbors w_1 and w_2 of v with minimum $\mathrm{dist}_C(w_1, w_2)$ in Lemma 8.8.

We refer to an induced subdivision of the 2-pan as a tuple (v_1, v_2, C), where v_1, v_2 are respectively the vertices of degree one and two that are not contained in the cycle, and C is the cycle.

As in the 1-pan case, we first obtain a structural property.

Lemma 8.8. *Let G be a graph and let $H = (v_1, v_2, C)$ be an induced subdivision of the 2-pan in G with minimum number of vertices such that $|C| \geq 11$. For every vertex v in $V(G) \setminus V(H)$, either v has at most one neighbor in C or it dominates C.*

Proof. Suppose there exists a vertex $v \in V(G) \setminus V(H)$ such that it has at least two neighbors in C and has a non-neighbor z. We choose two neighbors w_1, w_2 of v and a non-neighbor z such that

    (1) $\mathrm{dist}_C(w_1, w_2)$ is minimum,
    (2) subject to (1), $\mathrm{dist}_C(w_1, z) + \mathrm{dist}_C(w_2, z)$ is minimum.

Let Q be a shortest path of C from w_1 to w_2. Then $G[V(Q) \cup \{v\}]$ is an induced cycle, and it is strictly shorter than C, because $|Q| + 1 \leq |C|/2 + 2 < |C|$.

For each $i \in \{1, 2\}$, let x_i be the neighbor of w_i in C, which is not on the path Q, and let y_i be the neighbor of x_i in C other than w_i. See Figure 8 for an illustration. Note that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$ and furthermore, there are no edges between $\{x_1, y_1\}$ and $\{x_2, y_2\}$; otherwise, C would have length at most 10, a contradiction.

Observe that for each $i \in \{1, 2\}$, v has a neighbor in $\{x_i, y_i\}$; otherwise, $G[V(Q) \cup \{v, x_i, y_i\}]$ is an induced subdivision of the 2-pan, which has smaller number of vertices than H.

We claim that v is complete to $\{x_1, y_1, x_2, y_2\}$. Suppose not. Without loss of generality, we may assume that v has a non-neighbor in $\{x_1, y_1\}$. Since v has a neighbor in $\{x_2, y_2\}$, there is an induced cycle C' in $G[\{v, w_2, x_2, y_2\}]$ that contains v. Depending which of x_1, y_1 is not adjacent to v, one of (x_1, y_1, C') and (y_1, x_1, C') is an induced subdivision of the 2-pan with less vertices than H. This is a contradiction, and consequently, v is complete to $\{x_1, y_1, x_2, y_2\}$.

By the choice of w_1, w_2, this also implies that w_1 and w_2 are adjacent. But then $z \in V(C) \setminus \{x_1, y_1, w_1, w_2, x_2, y_2\}$, and z is closer on C to one of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ than to $\{w_1, w_2\}$. As this contradicts the choice of w_1, w_2, z, we conclude that v dominates C. ☐

The following is a 2-pan counterpart of Lemma 8.4.

Lemma 8.9. *Let $C = v_1 v_2 \cdots v_m v_1$ be an induced cycle of length at least 4 in a graph G and $w_1, w_2, w_3, w_4 \in V(G) \setminus V(C)$ such that*

    (1) $v_2 w_1 w_2 w_3 w_4$ is an induced path,

(2) v_2w_1 *is the only edge between* $\{v_1, v_2, v_3, v_4\}$ *and* $\{w_1, w_2, w_3, w_4\}$, *and*
(3) *each vertex in* C *has at most one neighbor in* $\{w_1, w_2, w_3, w_4\}$.
Then $G[V(C) \cup \{w_1, w_2, w_3, w_4\}]$ *contains an induced subdivision of the* 2-*pan.*

Proof. If v_2w_1 is the only edge between $\{w_1, w_2\}$ and C, then (w_1, w_2, C) is an induced subdivision of the 2-pan. Thus, we may assume that C has length at least 5, and w_1 or w_2 has a neighbor in $C - \{v_1, v_2, v_3, v_4\}$. We choose $i \in \{5, \dots, m\}$ such that:

- v_i has a neighbor in $\{w_1, w_2\}$; and
- no internal vertex of the path $P = v_i v_{i+1} \dots v_m v_1 v_2$ has a neighbor in $\{w_1, w_2\}$.

We may assume that $i = 5$; otherwise, the induced cycle in $G[V(P) \cup \{w_1, w_2\}]$ that contains v_i together with v_3, v_4 forms an induced subdivision of the 2-pan. On the other hand, we can observe that P contains at most 2 internal vertices; otherwise the induced cycle in $G[\{w_1, w_2, v_2, v_3, v_4, v_5\}]$ that contains v_i together with two internal vertices in P forms an induced subdivision of the 2-pan.

We distinguish cases depending the number of internal vertices of P.

First case: P has a unique internal vertex, i.e. $m = 5$. As v_5 has a neighbor in $\{w_1, w_2\}$, it has none in $\{w_3, w_4\}$ by our assumption. Therefore, $G[V(P) \cup \{w_1, w_2, w_3, w_4\}]$ contains an induced subdivision of the 2-pan.

Second case: P has exactly 2 internal vertices, i.e. $m = 6$. Note that v_6 has no neighbors in $\{w_1, w_2\}$. If v_6 has a neighbor in $\{w_3, w_4\}$, then the induced cycle in $G[\{v_6, v_1, v_2, w_1, w_2, w_3, w_4\}]$ that contains v_2 together with v_3, v_4 induces a subdivision of the 2-pan. Hence we may assume that v_6 has no neighbors in $\{w_1, w_2, w_3, w_4\}$. Since v_5 has no neighbor in $\{w_3, w_4\}$ (as above), $G[V(P) \cup \{w_1, w_2, w_3, w_4\}]$ contains an induced subdivision of the 2-pan.

This concludes the lemma. $\qquad\qquad\square$

Lemma 8.10. *Let k be a positive integer, G be a graph, (v_1, v_2, C) be an induced subdivision of the* 2-*pan in G with minimum number of vertices, and D be the set of all vertices dominating C. Given a set $\mathcal{P}$ of vertex-disjoint $V(C)$-paths in $G - E(C) - D$ with $|\mathcal{P}| \geq 396k \log k$, one can find in polynomial time k vertex-disjoint induced subdivisions of the* 2-*pan.*

Proof. For each $P \in \mathcal{P}$, let $\text{end}(P)$ be the set of end vertices of P. We construct a subset $\mathcal{P}'$ of $\mathcal{P}$ with the following property:

$$\forall P_1, P_2 \in \mathcal{P}', \ \text{dist}_C(\text{end}(P_1), \text{end}(P_2)) \geq 9.$$

This can be done by repeatedly choosing a path P in $\mathcal{P}$ and discarding from $\mathcal{P}$ all paths Q that have an endpoint at distance at most 8 from one of P. For each path added to $\mathcal{P}'$, at most 32 are discarded, hence $|\mathcal{P}'| \geq |\mathcal{P}|/33 \geq 12k \log k$.

We consider the subgraph H on the vertex set $V(C) \cup (\bigcup_{P \in \mathcal{P}'} V(P))$ and edge set $E(C) \cup (\bigcup_{P \in \mathcal{P}'} E(P))$. This graph has maximum degree 3 and has at least $24k \log k$ vertices of degree 3, as each path of $\mathcal{P}'$ contributes for two vertices of degree 3. According to Lemma 2.1, one can in polynomial time construct a set $\mathcal{Q}$ of k vertex-disjoint cycles of C. Observe that C intersects all other cycles, and thus, C is not contained in $\mathcal{Q}$.

As in the proof of Lemma 8.5 we note that for each cycle U of $\mathcal{Q}$, $G[V(U)]$ has an induced cycle containing at least one edge of C. Let $\mathcal{Q}'$ be a collection of k resulting induced $V(C)$-cycles obtained from $\mathcal{Q}$.
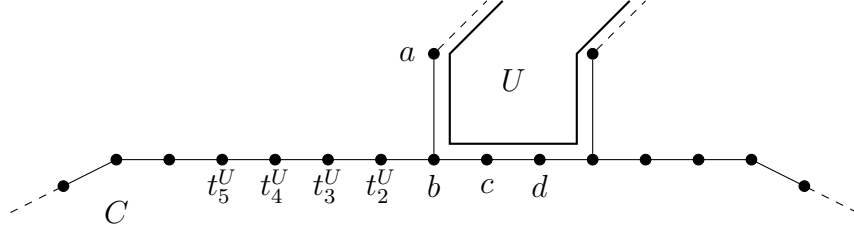
FIGURE 9. A cycle U in $\mathcal{Q}'$ and four selected vertices in Lemma 8.10.

For each cycle U in $\mathcal{Q}'$, we want to find 4 consecutive vertices of C that are not contained in $\bigcup_{F \in \mathcal{Q}'} (V(F) \cap V(C))$ such that the last vertex has an edge of C to U. See Figure 9 for an illustration. Since $|\mathcal{Q}'| \geq 2$ and each cycle of $\mathcal{Q}'$ intersects C, there exist another cycle $U' \in \mathcal{Q}' \setminus \{U\}$ and a subpath $T = t_1 t_2 \cdots t_x$ of C such that

- $t_1 \in V(U)$ and $t_x \in V(U')$,
- $t_2, \ldots, t_{x-1} \notin \bigcup_{F \in \mathcal{Q}'} V(F)$.

For $i \in \{1, 2, 3, 4, 5\}$, we assign $t_i^U := t_i$.

We claim that $G[V(U) \cup \{t_i^U : 2 \leq i \leq 5\}]$ contains an induced subdivision of the 2-pan. Let $abcd$ be the subpath of U such that $b = t_1^U$ and $bc \in E(C) \cap E(U)$. By Lemma 8.8 and since we work in $G - D$, every vertex of U has at most one neighbor in C. In particular, bt_2^U is the only edge between $\{a, b, c, d\}$ and $\{t_i^U : 2 \leq i \leq 5\}$. Therefore, one can find an induced subdivision of the 2-pan in $G[V(U) \cup \{t_i^U : 2 \leq i \leq 5\}]$ using Lemma 8.9.

From the choice of $\mathcal{P}'$, the sets in $\{\{t_i^U : 2 \leq i \leq 5\} : U \in \mathcal{Q}'\}$ are pairwise disjoint. Hence by applying Lemma 8.9 as above for every $U \in \mathcal{Q}'$ we obtain in polynomial time a collection of k vertex-disjoint induced subdivisions of the 2-pan.

This concludes the proof. $\qquad\square$

The last thing to show is that in fact, an induced subdivision of the 2-pan never contains a vertex dominating C.

Lemma 8.11. *Let k be a positive integer, let G be a graph, (v_1, v_2, C) be an induced subdivision of the 2-pan in G with minimum number of vertices and $|C| \geq 11$, and D be the set of all vertices dominating C. Every induced subdivision of the 2-pan in $G - \{v_1, v_2\}$ has no C-dominating vertices.*

Proof. Suppose that there exists an induced subdivision H of the 2-pan in $G - \{v_1, v_2\}$ containing a C-dominating vertex v. We prove two claims.

Claim 8.12. *There is no induced path $p_1 p_2 p_3$ in $H - v$ such that $p_1 \in V(C)$ and $p_2, p_3 \in V(G) \setminus (V(C) \cup D)$.*

Proof. Let q_1, q_2 be the neighbors of p_1 in C. As v is C-dominating, for each $i \in \{1, 2\}$, vp_1q_iv is a triangle. By minimality of (v_1, v_2, C) and the fact that $|C| \geq 11$, we deduce that q_i does not belong to H. For the same reason, v has no neighbor in $\{p_2, p_3\}$.

Let $i \in \{1, 2\}$. Consider the subgraph induced by $\{v, q_i, p_1, p_2, p_3\}$. As above the graph $G[\{v, q_i, p_1, p_2, p_3\}]$ is not a 2-pan. So q_i is adjacent to either p_2 or p_3. Since $p_2 \notin D$, p_2 has exactly one neighbor in C by Lemma 8.8 (which is p_1) and thus it is not adjacent to q_i. Therefore, both q_1 and q_2 are adjacent to p_3. By Lemma 8.8, p_3

is C-dominating, and thus p_3 is adjacent to p_1. This contradicts the assumption that $p_1p_2p_3$ is an induced path. We conclude that there is no such an induced path.　　◇

A similar observation can be made for a path continuing from a C-dominating vertex.

Claim 8.13. *There is no induced path wvp_1p_2 in H such that $w \in V(C)$ and $p_1, p_2 \in V(G) \setminus (V(C) \cup D)$.*

Proof. Let q_1, q_2 be the neighbors of w in C. As above we can deduce that none of q_1 and q_2 belongs to H. Also, for every $i \in \{1, 2\}$, $G[\{v, w, q_i, p_1, p_2\}]$ is not a 2-pan, so q_i is adjacent to either p_1 or p_2. If both q_1 and q_2 are adjacent to p_1 (or p_2), then by Lemma 8.8, p_1 (or p_2) is C-dominating, a contradiction. Therefore, we may assume, without loss of generality, that q_1 is adjacent to p_1, but not to p_2, and q_2 is adjacent to p_2 but not to p_1. Then the cycle $p_1p_2q_2wq_1$ with two more vertices in C induces a 2-pan, which is smaller than (v_1, v_2, C), a contradiction. We conclude that there is no such an induced path.　　◇

Observe that $|V(H) \cap V(C)| \leq 3$, otherwise the C-dominating vertex in H has degree 4 in H. Also, since H contains a vertex of C, we have $|V(H) \cap D| \leq 3$. Furthermore, $|V(H) \cap V(C)| \geq 2$ and $|V(H) \cap D| \geq 2$, then H contains an induced subgraph isomorphic to C_4, a contradiction. Thus, we may assume that $|V(H) \cap V(C)| + |V(H) \cap D| \leq 4$.

So $H - (V(C) \cup D)$ has at least $11 - 4 = 7$ vertices divided into at most 5 connected components. Thus, one of connected components of $H - (V(C) \cup D)$ contains an edge x_1x_2 whose one end vertex, say x_1, has a neighbor in $V(H) \cap (V(C) \cup D)$. We call x a neighbor of x_1 in $V(H) \cap (V(C) \cup D)$.

If $x \in V(C)$, then x is not adjacent to x_2; otherwise, H contains an induced subgraph isomorphic to C_3. Then xx_1x_2 is an induced path, and this is a contradiction by Claim 8.12. So, we may assume that $x \in D$. Note that similarly x is not adjacent to x_2, and since H does not contain v_1, v_2, H contains a vertex of C, say y. Then y has no neighbors in $\{x_1, x_2\}$; otherwise, $G[\{v, y, x_1, x_2\}]$ contains C_3 or C_4, a contradiction. But then yvx_1x_2 is an induced path, and this is a contradiction by Claim 8.13.

We conclude that there is no such an induced subdivision of the 2-pan.　　□

We can now prove Theorem 8.7.

Proof of Theorem 8.7. As in the proof for 1-pans, we construct a maximal sequence of graphs $G_1, \ldots, G_{\ell+1}$ and $F_1, \ldots, F_\ell$ such that

- $G_1 = G$;
- for each $i \in \{1, \ldots, \ell\}$, F_i is an induced subdivision of the 2-pan in G_i with minimum number of vertices; and
- for each $i \in \{1, \ldots, \ell\}$, $G_{i+1} = G_i - V(F_i)$.

Such a sequence can be constructed in polynomial time by repeatedly applying Lemma 8.1. If $\ell \geq k$, then we have found a packing of k induced subdivisions of the 2-pan. Hence, we assume in the sequel that $\ell \leq k - 1$. Let $\mu_k := 792k \log k + 25k - 23$.

Claim 8.14. *Let $j \in \{1, \ldots, \ell+1\}$. One can find in polynomial time either k vertex-disjoint induced subdivisions of the 2-pan, or a vertex set X_j of G_j of size at most $(\ell + 1 - j)\mu_k$ such that $G_j - X_j$ has no induced subdivision of the 2-pan.*

Proof. We prove claim by induction for $j = \ell + 1$ down to $j = 1$. The claim trivially holds for $j = \ell + 1$ with $X_{\ell+1} = \emptyset$ because $G_{\ell+1}$ has no induced subdivision of the

2-pan. Let us assume that for some $j \leq \ell$, we obtained a required vertex set X_{j+1} of G_{j+1} of size at most $(\ell - j)\mu_k$. Then in $G_j - X_{j+1}$, F_j is an induced subdivision of the 2-pan with minimum number of vertices. If F_j has less than 13 vertices, then we set $X_j := X_{j+1} \cup V(F_j)$. Clearly, $|X_j| \leq (\ell - j + 1)\mu_k$. We may assume F_j has at least 13 vertices. Let $F_j := (u_1, u_2, C)$, and let D be the set of vertices in $G_j - X_{j+1}$ that dominate C. Note that C has length at least 11.

According to Lemma 8.11, there are no induced subdivisions of the 2-pan intersecting D. Therefore, we can ignore the vertex set D.

We apply Gallai's A-path Theorem (Theorem 2.2) with $A = V(C)$ for finding at least $396k \log k$ pairwise vertex-disjoint $V(C)$-paths in $G_j - (D \cup X_{j+1} \cup \{u_1, u_2\}) - E(C)$. Assume that it outputs such $V(C)$-paths. Then, by applying Lemma 8.10 to $G_j - X_{j+1}$ and $V(C)$, one can find in polynomial time k vertex-disjoint induced subdivisions of the 2-pan. Thus, we may assume that Theorem 2.2 outputs a vertex set S of size at most $792k \log k$ hitting all $V(C)$-paths in $G_j - (D \cup X_{j+1} \cup \{u_1, u_2\}) - E(C)$.

Now, we consider the graph $G'_j := G_j - (X_{j+1} \cup D \cup S \cup \{u_1, u_2\})$. Suppose G'_j contains an induced subdivision $Q = (w_1, w_2, W)$ of the 2-pan. Then $G'_j[V(C) \cap V(Q)]$ is connected; otherwise, G'_j contains a $V(C)$-path in $(G_j - D - X_{j+1}) - E(C)$, contradicting the fact that S hits all such $V(C)$-paths. Furthermore, $G'_j[V(C) \cap V(Q)]$ contains no edge of W; otherwise, we also have a $V(C)$-path in $(G_j - D - X_{j+1}) - E(C)$. Thus, we have that $|V(C) \cap V(Q)| \leq 3$.

We recursively construct sets $U \subseteq V(C)$ and $\mathcal{J}$ as follows. We start with $U := \emptyset$ and $\mathcal{J} := \emptyset$. For every set of five consecutive vertices v_1, v_2, v_3, v_4, v_5 of C with $v_2, v_3, v_4 \notin U$, we test whether $G'_j[(V(G'_j) \setminus V(C)) \cup \{v_2, v_3, v_4\}]$ contains an induced subdivision H of the 2-pan, and if so, add vertices v_1, v_2, v_3, v_4, v_5 to U, and add H to $\mathcal{J}$ and increase the counter by 1. If the counter reaches k, then we stop. Note that graphs in $\mathcal{J}$ are pairwise vertex-disjoint; otherwise, we can find a $V(C)$-path, a contradiction. Thus, if the counter reaches k, we can output k pairwise vertex-disjoint induced subdivisions of the 2-pan.

Assume the counter does not reach k. Then we have $|U| \leq 5(k - 1)$. We take the 2-neighborhood U' of U in C, and observe that $|U'| \leq 25(k-1)$. We claim that $G'_j - U'$ has no induced subdivision of the 2-pan. Suppose for contradiction that there is an induced subdivision F of the 2-pan in $G'_j - U_j$. The intersection of F on C is a set of at most three consecutive vertices, say T.

Since $\text{dist}_C(T, U) \geq 2$ by the construction of U', there exists three consecutive vertices z_1, z_2, z_3 of C not containing U, such that $G'_j[(V(G'_j) \setminus V(C)) \cup \{z_1, z_2, z_3\}]$ contains an induced subdivision of the 2-pan. It implies that we can increase the counter by adding this induced subdivision to $\mathcal{J}$, a contradiction. Therefore, $G'_j - U'$ has no induced subdivision of the 2-pan.

Observe that

$$|X_j| \leq |X_{j+1} \cup S \cup \{u_1, u_2\} \cup U'| \leq (\ell - j)\mu_k + 792k \log k + 2 + 25(k-1) \leq (\ell - j + 1)\mu_k.$$

So, $X_j := X_{j+1} \cup S \cup \{u_1, u_2\} \cup U'$ satisfies the claim. $\diamond$

The result follows from the claim with $j = 1$. $\qquad\square$

9. Subdivisions of the diamond have the induced Erdős-Pósa property

In this section, we prove that the subdivisions of the diamond have the induced Erdős-Pósa property.

Theorem 9.1. *There exists a polynomial function $g : \mathbb{N} \to \mathbb{N}$ satisfying the following. Given a graph G and a positive integer k, one can in time $\mathcal{O}(kN(3, 3k)^{3k} + k^2|G|^7)$ output either k vertex-disjoint induced subdivisions of the diamond, or a vertex set of size at most $g(k)$ hitting every induced subdivision of the diamond.*

Remark that while the bounding function g that we obtained is polynomial, our upper-bound on its order is large; $g(k) = \mathcal{O}(N(4, 3k))$, where N is the function of Lemma 2.3.

We follow a similar line of proofs as in the previous section. We first deal with the case where the considered graph G has an induced subdivision H of the diamond and $G - V(H)$ is H-induced-subdivision-free (here we do not need the minimality of a model). Observe that a subdivision of the diamond consists of three internally disjoint paths between two distinct vertices. This simple observation allows us to focus on the following sightly easier setting: given a graph G and an induced path Q, we aim at finding either many vertex-disjoint induced subdivisions of the diamond or a small vertex set hitting all the induced subdivisions of the diamond that meet Q.

For a vertex subset S of a graph G and $v \in V(G) \setminus S$, a path is called a (v, S)-*path*, if it starts with v and ends at a vertex in S and contains no other vertices in S.

We will use the Erdős-Pósa property of A-ℓ-combs, recently developed by Bruhn, Heinlein, and Joos [BHJ18], to first exclude induced diamond subdivisions of a special type. Since we only use it with $\ell = 2$, we name it A-*claw* for simplicity. Given a graph G and a vertex subset A, we say that a subgraph H of G is an A-*claw* if

- H contains a vertex $v \in V(G) \setminus A$,
- H consists of three (v, A)-paths P_1, P_2, P_3 such that $V(P_i) \cap V(P_j) = \{v\}$, for every distinct $i, j \in \{1, 2, 3\}$.

For convenience, if H is a subgraph of G, then we refer to $V(H)$-claws as H-*claws*. The *leaves* of an A-claw are its vertices in A.

9.1. **Base polynomial-time algorithms.** In this section we present algorithms to find (collections of) induced subdivisions of the diamond and claws. We first show that one can detect an induced subdivision of the diamond in polynomial time. The following lemma is useful.

Lemma 9.2. *Given a graph G, an induced path Q, and a Q-claw F, one can find in time $\mathcal{O}(|G|^3)$ an induced subdivision of the diamond in $G[V(F) \cup V(Q)]$.*

Proof. We prove by induction on $|V(F) \cup V(Q)|$ that one can find an induced subdivision of the diamond in $G[V(F) \cup V(Q)]$ in $|V(F) \cup V(Q)| \cdot \mathcal{O}(|E(G)|)$ steps.

Let v be the vertex of $F - V(Q)$ that is connected via the paths P_1, P_2, P_3 to Q, forming the Q-claw F. Without loss of generality, we assume that the end vertices of P_1, P_2, P_3 appear in Q in this order and that P_1 and P_3 meet Q on its endpoints. For each $i \in \{1, 2\}$, let Q_i be the subpath of Q from the end vertex of P_i to the end vertex of P_{i+1}. Let $H := G[V(F) \cup V(Q)]$.

If H contains no edges other than those in $P_1 \cup P_2 \cup P_3 \cup Q$, then clearly, it is an induced subdivision of the diamond. Besides, we may assume that each of P_1, P_2, P_3
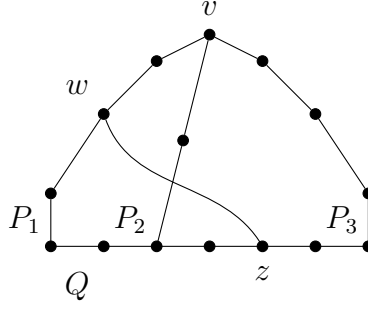
FIGURE 10. An edge wz where w is an internal vertex of P_1 and z is an internal vertex of $Q_2 \cup P_3$ in Lemma 9.2.

is an induced path; otherwise we could shorten it and apply the induction hypothesis. Therefore, we may assume that H contains an edge wz, which does not have both endpoints in one of P_1, P_2, P_3, or Q.

Suppose w is an internal vertex of P_1 and z is an internal vertex of $P_2 \cup Q_1$. Then from z, there are three paths to P_1, where the end vertices of two paths are the end vertices of P_1. As P_1 is an induced path, by induction hypothesis, one can find an induced subdivision of the diamond in $G[V(P_1 \cup P_2 \cup Q_1)]$ in time $|V(P_1 \cup P_2 \cup Q_1)| \cdot \mathcal{O}(E(G))$.

We thus may assume that there is no such an edge. We can apply the same argument for all pairs $(P_2, P_1 \cup Q_1), (P_2, P_3 \cup Q_2), (P_3, P_2 \cup Q_2)$.

Since Q is an induced path, one of w and z is contained in $F - V(Q)$. If $w \in V(P_2) \setminus V(Q)$, then z is contained in $P_1 \cup P_3 \cup Q - V(P_2)$, which is not possible by the previous argument. Therefore, we may assume that w is an internal vertex of P_1 without loss of generality, and z is an internal vertex of $P_3 \cup Q_2$. See Figure 10 for an illustration.

Let r be the neighbor of v in P_2, and let P_1' be the subpath of P_1 from v to w. Note that $G[V(P_1') \cup \{r\}]$ is an induced path, and there are three paths from z to P_1', namely, two paths along $P_2 \cup P_3 \cup Q_2$ and wz. Since the union of those paths does not contain the end vertex of P_1 in Q, we obtained a smaller induced subgraph graph satisfying the premisses of the lemma. By the induction hypothesis, one can find an induced subdivision of the diamond in time $|V(P_1' \cup P_2 \cup P_3 \cup Q_2)| \cdot \mathcal{O}(E(G))$.

This completes the proof. □

Proposition 9.3. *Given a graph G, one can test in time $\mathcal{O}(|G|^7)$ whether G contains an induced subdivision of the diamond, and output one if exists.*

Proof. Let us consider 4 vertices a, b, c, and d such that abc is an induced path and d is adjacent to neither a nor c.

Using Menger's theorem, we can check in $\mathcal{O}(|G|^3)$-time whether there are three internally vertex-disjoint paths from d to a, b, c. If such paths do not exist, then there is no induced subdivision H such that

- b, d are vertices of degree 3 in H, and
- a, c are neighbors of b in H.

On the other hand, if the 3 paths exist, then by Lemma 9.2, we can detect such an induced subdivision of the diamond in $\mathcal{O}(|G|^3)$ steps. By iterating over all possible choices of vertices a, b, c, and d, we get a total running time is $\mathcal{O}(|G|^7)$. □

Now we prove that every large enough collection of Q-claws contains many disjoint induced subdivisions of the diamond.

Lemma 9.4. *Let G be a graph and let Q be an induced path of G. Given a set of $N(3, 3k)$ vertex-disjoint Q-claws, one can find k vertex-disjoint induced subdivisions of the diamond in time $\mathcal{O}(N(3, 3k)^{3k} + |G|^3)$.*

Proof. Let $Q = v_1 v_2 \cdots v_m$, and $I = \{1, \ldots, m\}$. Let $T_1, T_2, \ldots, T_{N(3,3k)}$ be a given set of pairwise vertex-disjoint Q-claws. For each $i \in \{1, \ldots, N(3, 3k)\}$, let $A_i := \{j \in \mathbb{N} : v_j \in V(T_i)\}$. By definition, $|A_i| = 3$.

We apply the regular partition lemma (Lemma 2.3) to A_i's with $n = 3$. Then there exist a subsequence $(A_{c_1}, \ldots, A_{c_{3k}})$ of $(A_1, \ldots, A_{N(3,3k)})$ and a regular partition of I with respect to $(A_{c_1}, \ldots, A_{c_{3k}})$ that has order at most 3. Since given Q-claws are pairwise vertex-disjoint, in each part, restrictions of $A_{c_1}, \ldots, A_{c_{3k}}$ are not the same (i.e. item (1) of the definition of a regular partition does not hold). Algorithmically, the regular partition can be found in $\mathcal{O}(N(3, 3k)^{3k})$-time by simply iterating over possible $3k$-tuples until one forming a regular partition is found.

Suppose first that the order of the partition is 1. Then one of the following holds:

- For all $j, j' \in \{1, \ldots, 3k\}$ with $j < j'$, $\max(A_{c_j}) < \min(A_{c_{j'}})$.
- For all $j, j' \in \{1, \ldots, 3k\}$ with $j < j'$, $\max(A_{c_{j'}}) < \min(A_{c_j})$.

For each $i \in \{1, \ldots, 3k\}$, let Q_i be the minimal subpath of Q containing the vertices in $\{v_j : j \in A_{c_i}\}$. Observe that the Q_i's are disjoint and that T_{c_i} is a Q_i-claw, for every $i \in \{1, \ldots, 3k\}$. Using Lemma 9.2, one can find in time $\mathcal{O}(|G|^3)$ an induced subdivision of the diamond in each $G[V(T_{c_i}) \cup V(Q_i)]$, so we are done.

Suppose the order of the partition is 2 or 3. Let I_1 and I_2 be the first two parts in the partition, and we may assume that A_{c_i} has an element in each of the two parts. For each $i \in \{1, \ldots, 3k - 2\}$, let S_i be the minimal subpath of Q containing the vertices in

$$\{v_j : j \in (A_{c_i} \cup A_{c_{i+1}} \cup A_{c_{i+2}}) \cap I_1\},$$

and let R_i be the minimal subpath of Q containing the vertices in

$$\{v_j : j \in (A_{c_i} \cup A_{c_{i+1}} \cup A_{c_{i+2}}) \cap I_2\}.$$

Then subgraphs in

$$\{S_{3\ell-2} \cup R_{3\ell-2} \cup T_{c_{3\ell-2}} \cup T_{c_{3\ell-1}} \cup T_{c_{3\ell}} : 1 \leq \ell \leq k\}$$

are pairwise vertex-disjoint.

Furthermore, since $S_{3\ell-2}$ is an induced path, and there are three internally vertex-disjoint paths from the intersection of $R_{3\ell-2}$ and $T_{c_{3\ell-1}}$ to $S_{3\ell-2}$ in $T_{c_{3\ell-2}} \cup T_{c_{3\ell-1}} \cup T_{C_{3\ell}} \cup S_{3\ell-2} \cup R_{3\ell-2}$, one can output in time $\mathcal{O}(|G|^3)$ an induced subdivision of the diamond in the subgraph, using Lemma 9.2. Therefore, we obtain k pairwise vertex-disjoint induced subdivisions of the diamond, as required.   □

In [BHJ18], Bruhn, Heinlein, and Joos show that A-claws have the (non-induced) Erdős-Pósa property. Their proof is unfortunately not written algorithmically, however it can easily be turned into a polynomial-time algorithm. For completeness, we give an algorithmic version hereafter.

Lemma 9.5. *Let T be a tree with no vertices of degree 2. If L is the set of all leaves of T, then $|L| \geq |V(T) \setminus L| - 2$.*

Proof. We note that

- $|L| + |V(T) \setminus L| = |T| = |E(T)| - 1$, and
- $|L| + 3|V(T) \setminus L| \leq \sum_{t \in V(T)} \deg_T(t) = 2|E(T)|$.

Combining the two equations, we have that $|A| \geq |V(T) \setminus L| - 2$, as required. □

Lemma 9.6. *Let F be a forest of maximum degree 3 and where each component has a degree-3 vertex. Let L be the set of its leaves, and k be a positive integer.*

(1) *If F has at least $6k$ leaves, then in time $\mathcal{O}(k|F|)$ one can find k pairwise vertex-disjoint L-claws in F.*

(2) *If F contains less than $6k$ leaves and less than k connected components, then F contains less than $14k$ vertices of degree 1 or 3.*

Proof. Proof of (1). In this proof we call *good forest* every forest of maximum degree 3 and where each component has a degree-3 vertex. We prove by induction on k. If $k = 1$, then there is at least one component containing an L-claw and (1) holds. So, we may assume that $k \geq 2$ and suppose that the statement holds for smaller values of k. Observe if F has a vertex of degree 2, one can delete it and add an edge between its neighbors without changing the number of leaves neither the existence of L-claws. Therefore we may assume that F has no degree-2 vertex.

Let us fix a root node in F and call T the component containing it. We take a furthest node v from the root that is not a leaf. By our assumption above, it has exactly two children. Such a node can be found in linear time using Breadth First Search. If v is the root of T, then T is an L-claw, and since T has maximum degree 3, the remaining part of F contains at least $6k - 3$ leaves. Clearly $F - V(T)$ is a good forest. Thus, by the induction hypothesis, we can obtain $k - 1$ pairwise vertex-disjoint L-claws in $F - V(T)$ in time $\mathcal{O}((k - 1)|F|)$. Together with T, they form a collection of k vertex-disjoint L-claws in F, that we found in time $\mathcal{O}(k|F|)$. Thus in the sequel we may assume that v is not the root.

Let w be the parent v and let T_w be the subtree of T that is induced by the set of all descendants of w (including w). Observe that T_w has at least 3 leaves, hence it contains an L-claw. This claw can be found in linear time using Breadth First Search.

We distinguish two cases:

Case 1: w is the root. Then $T = T_w$ has at most 6 leaves. Clearly $F - T$ is a good forest with at least $6k - 6$ leaves.

Case 2: w is not the root. Then T_w has at most 4 leaves. Let x be the vertex of $T - V(T_w)$ at minimum distance from v. Let P be the path of T from x to w and let $T^- = T - (V(T_w) \cup V(P) \setminus \{x\})$. Then T^- is a good forest with at least $6k - 6$ leaves.

In each case $F - T_w$ contains a good subforest with at least $6(k - 1)$ leaves (each belonging to L) and which can be found in linear time. Considering the aforementioned claw present in T_w and applying the induction hypothesis to $F - T_w$, we finally obtain as above k pairwise vertex-disjoint L-claws in time $\mathcal{O}(k|T|)$, as required.

Proof of (2). Let n be the number of components of F. By the assumption, we have $n < k$. Let $T_1, T_2, \ldots, T_n$ be the connected components of F, and for each $i \in \{1, 2, \ldots, n\}$, let m_i be the number of vertices of degree 3 in T_i and let ℓ_i be the number of vertices of degree 1 in T_i. By Lemma 9.5, we know that $\ell_i \geq m_i - 2$. Since $\sum_{i=1}^{n} \ell_i < 6k$, we have $\sum_{i=1}^{n} m_i \leq \sum_{i=1}^{n} \ell_i + 2n < 6k + 2k = 8k$. Thus, we have $\sum_{i=1}^{n} (\ell_i + m_i) < 14k$. □

Proposition 9.7. *Given a graph G, a subset $A \subseteq V(G)$, and a positive integer k, one can in time $\mathcal{O}(|G|^4)$ output either k pairwise vertex-disjoint A-claws, or a vertex set S of size $14k$ hitting all A-claws.*

Proof. Let $S_0 := \emptyset$, and F_0 be the empty graph. Let us apply the following algorithm. We start with $i = 1$ and, while $G - S_{i-1}$ contains an A-claw, we do the following:

(1) let X be an A-claw of $G - S_{i-1}$;
(2) if X intersects $F_{i-1} - S_{i-1}$, define F_i as the union of F_{i-1} and a path from $F_{i-1} - S_{i-1}$ to A (that is contained in X);
(3) otherwise, define F_i as the disjoint union of X and F_{i-1};
(4) let S_i be the set of all vertices of degree 1 or 3 in F_i;
(5) increment i.

Let n be the maximum value for which F_i and S_i are defined. Note that one can find an A-claw by guessing the vertex of degree 3 and then testing whether there are three paths from it to A using Menger's theorem. Thus, checking the condition of the while loop and step (1) can be performed in time $\mathcal{O}(|G|^3)$, and we may construct the sequences $F_1, \ldots, F_n$ and $S_1, \ldots, S_n$ in time $\mathcal{O}(|G|^4)$. Let $F := F_n$ and $S := S_n$. By construction, we know that $G - S$ has no A-claws and F is a forest with all its leaves in A, maximum degree 3, and where every component has a degree-3 vertex. If $|S| \leq 14k$, then we are done. We may assume that $|S| > 14k$. In particular $|F| > 14k$.

If F contains k connected components, then we can simply find one A-claw from each connected component, and output k vertex-disjoint A-claws. We may assume that F contains less than k connected components. Then by (2) of Lemma 9.6, F contains at least $6k$ leaves, and by (1) of the same lemma, one can find k vertex-disjoint A-claws in time $\mathcal{O}(k|G|) = \mathcal{O}(|G|^2)$. $\square$

9.2. **Structural lemmas.** For a set $A \subseteq V(G)$, a *Tutte bridge* of A in G is a subgraph of G consisting of one component C of $G - A$ and all edges joining C and A and all vertices of A incident with those edges. We discuss in this section under which conditions a Tutte bridge can be used to construct an induced subdivision of the diamond.

Lemma 9.8. *Let Q be an induced path in a graph G and H be a Tutte bridge of $V(Q)$ in G such that $|V(H) \cap V(Q)| \geq 3$. Then H contains a Q-claw.*

Proof. Let a, b, c be three distinct vertices in $V(H) \cap V(Q)$, and let a', b', c' be their (not necessarily distinct) neighbors in $H - V(Q)$, respectively. It is easy to check that a subgraph-minimal tree of $H - V(Q)$ spanning a', b', and c' form (together with a, b, and c) a Q-claw. Such a subgraph exists since $H - V(Q)$ is connected. $\square$

Recall that our intermediate goal is to prove an induced Erdős-Pósa type result for subdivisions of the diamond intersecting a given path Q of a graph G. We saw in the previous section that Q-claws can be used to construct induced subdivisions of the diamond (Lemma 9.2), and how to deal with those (Proposition 9.7). So we may now focus on the case where there is no A-claw. In particular, Lemma 9.8 allows us to assume that there is no Tutte bridge H of $V(Q)$ such that $|V(H) \cap V(Q)| \geq 3$. The following lemma shows that if a Tutte bridge containing a cycle C connected to Q via two disjoint paths (with some additional properties), then one can also find an induced subdivision of the diamond. Note that first two conditions do not always imply the existence of an induced subdivision of the diamond: if ab is an edge that is a component of $G - V(Q)$

and $N(a) \cap V(Q) = N(b) \cap V(Q) = \{p, q\}$ for some two consecutive vertices p, q in Q, then this Tutte bridge induces a K_4, that does not contain an induced subdivision of the diamond. To avoid this, we require that the two vertices in $V(H) \cap V(Q)$ are not consecutive.

Lemma 9.9. *Let Q be an induced path in a graph G and H be a Tutte bridge of $V(Q)$ in G such that:*

(1) $V(H) \cap V(Q) = \{v_1, v_2\}$ for some non-adjacent vertices v_1, v_2 in Q; and

(2) some cycle of H is connected to $\{v_1, v_2\}$ via two vertex-disjoint (possibly empty) paths,

then $G[V(H) \cup V(Q')]$ contains an induced subdivision of the diamond, where Q' is the subpath of Q from v_1 to v_2.

Proof. Towards a contradiction, we assume that the statement does not hold and we consider, among all graphs G, all induced paths Q of G and all Tutte bridges H of $V(Q)$, a triple (G, Q, H) such that H has minimum number of vertices.

Let C be a cycle of H as in the statement and let P_1 and P_2 be the two paths from $V(H) \cap V(Q)$ to C such that $v_1 \in V(P_1)$ and $v_2 \in V(P_2)$. Recall that v_1 is not adjacent to v_2. By minimality, we may assume that $V(H) = V(C) \cup V(P_1) \cup V(P_2)$ and that P_1 and P_2 are induced, otherwise we could remove vertices or take shorter paths. Let w_1 and w_2 be the end vertices of P_1 and P_2 on C, respectively, and let Q_1 and Q_2 be the two subpaths from w_1 to w_2 in C.

If the subgraph $C \cup P_1 \cup P_2$ is induced in G, then $G[V(H) \cup V(Q')]$ is an induced subdivision of the diamond and we are done. So we now consider all possible ways this subgraph can be non-induced.

We first show that C is an induced cycle. If w_1 is adjacent to w_2, then C is induced; because otherwise we could take an induced cycle in $G[V(C)]$ containing the edge $w_1 w_2$, which would be shorter than C, a contradiction. In the case where $w_1 w_2 \notin E(G)$, we may assume that each of Q_1 and Q_2 is induced, otherwise, we could find a shorter cycle. Now, observe than if C has a chord e from an internal vertex z_1 of Q_1 to an internal vertex z_2 of Q_2, then Q_1 together with e forms a Q_2-claw and thus contain an induced subdivision of the diamond, according to Lemma 9.2. Therefore C is indeed induced.

Observe that there are no edges between $H - V(Q)$ and $Q - \{v_1, v_2\}$, because of the condition that $|V(H) \cap V(Q)| = 2$. Also, P_i has no neighbors in $P_{3-i} - w_{3-i}$ for each $i \in \{1, 2\}$, otherwise we could find a smaller cycle such as C. Therefore, we may assume that there is an edge between some vertex $y \in V(P_i - w_i)$ and some internal vertex of Q_j, for some $i, j \in \{1, 2\}$. Then H contains a P_i-claw constructed as follows. The degree-3 vertex of the claw is y; it is connected to P_i using the aforementioned edge (first path), the subpath of Q_j connecting y to w_1 (second path) and the rest of Q_j together with P_{3-i} and Q' (third path). By to Lemma 9.2 $G[V(H) \cup V(Q')]$ then contains an induced subdivision of the diamond, as required. □

Te following lemma handles a remaining case that is not covered by Lemma 9.9.

Lemma 9.10. *Let Q be an induced path in a graph G and H_1, H_2 be two Tutte bridges of $V(Q)$ in G such that:*

- *$|V(H_i) \cap V(Q)| = 2$ for each i,*
- *Q_1 and Q_2 share at least one edge, where Q_i is the minimal subpath of Q containing $V(H_i) \cap V(Q)$, for every $i \in \{1, 2\}$.*

Then $G[V(H_1) \cup V(H_2) \cup V(Q_1) \cup V(Q_2)]$ contains an induced subdivision of the diamond.

Proof. Let $Q = v_1 v_2 \cdots v_m$, and let v_{a_i}, v_{b_i} be the end vertices of Q_i such that $a_i < b_i$, for every $i \in \{1, 2\}$. Without loss of generality, we may assume that $a_1 \leq a_2$. As Q_1 and Q_2 share an edge, $b_1 \geq a_2 + 1$.

Let x_1 and y_1 be (possibly identical) neighbors of v_{a_1} and v_{b_1} in $H_1 - V(Q)$, respectively, and let R_1 be a path from x_1 to y_1 in $H_1 - V(Q)$. Similarly, let x_2 and y_2 be neighbors of v_{a_2} and v_{b_2} in $H_2 - V(Q)$, respectively, and let R_2 be a shortest path from x_2 to y_2 in $H_2 - V(Q)$. Let R be the subpath of Q from v_{a_1} to v_{a_2}.

Observe that $G[V(R) \cup \{x_1, x_2\}]$ is an induced path from x_1 to x_2, because $V(H_i) \cap V(Q)$ is exactly $\{a_i, b_i\}$ for every $i \in \{1, 2\}$. Let $j = \min\{b_1, b_2\}$. It is easy to see that there are three paths from v_j to the induced path $G[V(R) \cup \{x_1, x_2\}]$, namely, a subpath of Q from v_{a_2} to v_j, and two paths along R_1 and R_2. Thus, by Lemma 9.2, $G[V(R_1) \cup V(R_2) \cup V(Q_1) \cup V(Q_2)]$ contains an induced subdivision of the diamond. $\square$

9.3. **The main proof.** We can now describe the main proof of this section. The following proposition asserts that the subdivisions of the diamond intersecting a given induced path have the induced Erdős-Pósa property.

Lemma 9.11. *Let G be a connected graph and $v, w \in V(G)$ be non-adjacent vertices such that $G - \{v, w\}$ is connected. One can in time $\mathcal{O}(|G|^2)$ whether there is a cycle C with two vertex-disjoint paths from $\{v, w\}$ to C.*

Proof. We take a block-cut decomposition of G. Let B_v and B_w be the blocks of G containing v and w, respectively. Let $B_v = B_1 - B_2 - \cdots - B_m = B_w$ be the sequence of blocks of G in the block-cut decomposition of G.

Suppose there is $i \in \{1, \ldots, m\}$ such that B_i contains a cycle C. Note that for any two vertices a, b in B_i, there are two vertex-disjoint paths from C to $\{a, b\}$, as B_i is 2-connected. If $B_v = B_w$, then there are two vertex-disjoint paths from C to $\{v, w\}$ directly. Otherwise, along the cut vertices connecting blocks of $B_1, \ldots, B_m$, we may find two vertex-disjoint paths from C to $\{v, w\}$ in G.

Thus, we may assume that there is no $i \in \{1, \ldots, m\}$ such that B_i contains a cycle. In this case, for every cycle C, there is a cut vertex separating C and $\{v, w\}$. So, we can deduce that there is no cycle C with two vertex-disjoint paths from $\{v, w\}$ to C. $\square$

Proposition 9.12. *There exists a polynomial function $g_2 : \mathbb{N} \to \mathbb{N}$ satisfying the following. Given a graph G, an induced path P of G, and a positive integer k, one can in time $\mathcal{O}(N(3, 3k)^{3k} + k|G|^7)$ output either k vertex-disjoint induced subdivisions of the diamond, or a vertex set of size at most $g_2(k)$ hitting all the induced subdivisions of the diamond that intersect P.*

Proof. Let $P = v_1 v_2 \cdots v_m$, and $I = \{1, \ldots, m\}$.

We first apply the A-claw lemma to P. By Proposition 9.7, one can in time $\mathcal{O}(|G|^4)$ output either $N(3, 3k)$ pairwise vertex-disjoint P-claws, or a vertex set of size at most $14N(3, 3k)$ hitting all P-claws. In the former case, we use Lemma 9.4 to obtain k pairwise vertex-disjoint induced subdivisions of the diamond in time $\mathcal{O}(N(3, 3k)^{3k} + |G|^3)$ and we are done.

So we may assume that G contains a vertex subset X_1 of size at most $14N(3, 3k)$ such that $G - X_1$ has no P-claws. Let $G_1 := G - X_1$ and $P_1 = P - X_1$. By Lemma 9.8, G_1 contains no Tutte bridge of $V(P_1)$ such that $|V(H) \cap V(P_1)| \geq 3$.

Now, we greedily construct a maximal set $\mathcal{U}$ of pairwise vertex-disjoint Tutte bridges H of $V(P_1)$ in G_1 such that:

- $|V(H) \cap V(P_1)| = 2$,
- $V(H) \cap V(P_1)$ are not consecutive vertices of P, and
- H contains a cycle C and two vertex-disjoint paths from C to $V(H) \cap V(P_1)$ in H.

Since there are at most $|G|$ connected components of $G_1 - V(P_1)$, one can find such a set by considering each connected component of $G_1 - V(P_1)$ and then testing whether the corresponding Tutte bridge satisfies the conditions. The last condition can be checked in time $\mathcal{O}(|H|^2)$ using Lemma 9.11.

Suppose $|\mathcal{U}| \geq N(2, 3k)$. In this case, we apply the regular partition lemma (Lemma 2.3) with $n = 2$ and obtain k vertex-disjoint induced subdivisions of the diamond, with the whole path P. Following the same line of proof as in Lemma 9.4, if the order of the resulting partition is 2, then we can output k pairwise vertex-disjoint induced subdivisions of the diamond in time $\mathcal{O}(|G|^3)$. When the order of the partition is 1, Lemma 9.9 implies that there is an induced subdivision of the diamond in H together with the minimal subpath of P containing $V(H) \cap V(P)$, for every Tutte bridge H of the subset of $\mathcal{U}$ given by the regular partition lemma. So in this case, using Proposition 9.3, one can construct k vertex-disjoint induced subdivisions of the diamond in time $\mathcal{O}(k|G|^7)$.

Otherwise, let $X_2 := \bigcup_{H \in \mathcal{U}}(V(H) \cap V(P))$. Then we have that $|X_2| \leq 2N(2, 3k)$ and X_2 hits all Tutte bridges of $V(P_1)$ in G_1 satisfying the three conditions above. Let $G_2 := G_1 - X_2$ and $P_2 := P_1 - X_2$.

In the next step, we greedily build a maximal set $\mathcal{W}$ of pairwise vertex-disjoint pairs of Tutte bridges (H_1, H_2) of $V(P_2)$ in G_2 such that:

- $|V(H_i) \cap V(P_2)| = 2$,
- Q_1 and Q_2 share an edge, where Q_i is a minimal subpath of P containing $V(H_i) \cap V(P)$.

We can construct $\mathcal{W}$ by considering all pairs of connected components of $G_2 - V(P_2)$.

Suppose $|\mathcal{W}| \geq 4N(4, 3k)$. Let v_{a_i}, v_{b_i} be the vertices of $V(H_i) \cap V(P)$ such that $a_i < b_i$. Note that H_1 might intersect H_2, and therefore, there are four types of a pair (H_1, H_2), depending on whether $v_{a_1} = v_{a_2}$ and $v_{b_1} = v_{b_2}$. As $|\mathcal{W}| \geq 4N(4, 3k)$, there is a subset $\mathcal{W}_1$ of $\mathcal{W}$ of size at least $N(4, 3k)$ which consists of pairs of the same type.

We apply the regular partition lemma with n equal to the size of $(V(H_1) \cup V(H_2)) \cap V(P)$ for pairs in $\mathcal{W}_1$. Similar to the previous case, when the order of the partition is larger than 1, then by the same line of proofs as in Lemma 9.4, one can find in polynomial time k pairwise vertex-disjoint induced subdivisions of the diamond. When the order of the partition is 1, Lemma 9.10 implies that H_1, H_2, together with the minimal subpath of P containing $(V(H_1) \cup V(H_2)) \cap V(P)$ contains an induced subdivision of the diamond. Thus, using Proposition 9.3, one can construct k vertex-disjoint induced subdivisions of the diamond in time $\mathcal{O}(k|G|^7)$.

Otherwise, let $X_3 := \bigcup_{(H_1, H_2) \in \mathcal{W}}((V(H_1) \cup V(H_2)) \cap V(P))$. Then we have that $|X_3| \leq 16N(4, 3k)$ and X_3 hits all pairs of Tutte-bridges satisfying the above conditions. Let $G_3 := G_2 - X_3$ and $P_3 := P_2 - X_3$.

Now, we claim that the remaining induced subdivisions of the diamond have restricted positions.

Claim 9.13. *Let X be an induced subdivision of the diamond in G_3 that intersects P_3. Then we have*

- *$X - V(P_3)$ has one component and $|V(X) \cap V(P_3)| \leq 2$, and*
- *if $|V(X) \cap V(P_3)| = 2$, then the two vertices in $V(X) \cap V(P_3)$ are adjacent.*

Proof. Suppose that $|V(X) \cap V(P_3)| > 1$. Observe that if $X - V(P_3)$ intersects some Tutte bridge F of $V(P_3)$, then $|V(F) \cap V(P_3)| = 2$. The upper-bound holds because otherwise H would have been considered when constructing X_1, while the lower-bound holds as we assume $|V(X) \cap V(P_3)| > 1$, and X is connected.

Also, if $X - V(P_3)$ intersects exactly one Tutte bridge F of $V(P_3)$, then F contains a cycle where there are two vertex-disjoint paths from the cycle to $V(F) \cap V(P_3)$. So, in that case, the two vertices of $V(F) \cap V(P_3)$ are consecutive (otherwise we would have considered F when constructing X_2) and we are done.

Thus, we may assume that there are at least two Tutte bridges of $V(P_3)$ intersecting X outside P_3. We obtain a contradiction as follows. Let $F_1, F_2, \ldots, F_n$ be the set of Tutte bridges of $V(P_3)$ such that $F_i - V(P_3)$ contains a vertex of X. For each $i \in \{1, \ldots, n\}$, let Q_i be the minimal subpath of P_3 containing the two vertices of $V(F_i) \cap V(P_3)$. Observe that no two such paths share an edge, because the corresponding bridges have been handled when constructing X_3. On the other hand if no two paths in $Q_1, \ldots, Q_n$ share an edge, then it is easy to see that X has a cut vertex, a contradiction. $\diamond$

By Claim 9.13 if G_3 contains an induced subdivision X of the diamond, then $V(X) \cap V(P)$ consists of at most two consecutive vertices of P. Therefore, we can find in polynomial time either k pairwise vertex-disjoint induced subdivisions of the diamond, or a vertex set X_4 of size at most $2k$ hitting all remaining induced subdivisions of the diamond. In the latter case, we obtain a hitting set of size at most

$$|X_1 \cup X_2 \cup X_3 \cup X_4| \leq 14N(3, 3k) + N(2, 3k) + 16N(4, 3k) + 2k.$$

So, the function $g_2(k) = 14N(3, 3k) + N(2, 3k) + 16N(4, 3k) + 2k$ satisfies the statement. $\square$

Now, we prove the second intermediate proposition.

Proposition 9.14. *There exists a polynomial function $g_1 : \mathbb{N} \to \mathbb{N}$ satisfying the following. Given a graph G, an induced subdivision H of the diamond in G, and a positive integer k such that*

- *$G - V(H)$ has no induced subdivision of the diamond,*

then one can in time $\mathcal{O}(N(3, 3k)^{3k} + k|G|^7)$ output either k vertex-disjoint induced subdivisions of the diamond, or a vertex set of size at most $g_1(k)$ hitting every induced subdivision of the diamond.

Proof. We set $g_1(k) := 3g_2(k)$. Let P_1, P_2, P_3 be the three paths forming H. Since every induced subdivision of the diamond in G intersects H, it intersects at least one of P_1, P_2, P_3. So, by applying Proposition 9.12 to each of P_1, P_2, P_3, we can output in time $\mathcal{O}(N(3, 3k)^{3k} + k|G|^7)$ either k vertex-disjoint induced subdivisions of the diamond, or a vertex set of size at most $g_1(k) = 3g_2(k)$ hitting every induced subdivision of the diamond. $\square$

Proof of Theorem 9.1. We assign $g(k) := kg_1(k)$. We can prove Theorem 9.1 using Proposition 9.14, with exactly the same argument in the proof for 1-pan (Theorem 8.2). $\square$

10. Concluding remarks and open problems

In this paper, we investigated the induced Erdős-Pósa property of subdivisions beyond known results about cycles and obtained both positive and negative results. We note that our positive results for pans come with polynomial-time algorithms that output either a large packing of induced subdivisions of the considered graph H, or a small hitting set. These can be directly used to design approximation algorithms for computing the maximum size of a packing of induced subdivisions of H and the minimum size of a hitting set (as in [CRST17] for instance). For 1-pans and 2-pans, this gives a polynomial-time $\mathcal{O}(\mathsf{OPT}\log\mathsf{OPT})$-approximation. On the other hand, our negative results cover a vast class of graphs.

The most general open problem on the topic discussed in this paper is to characterize the graphs H whose subdivisions have the induced Erdős-Pósa property. According to Theorem 1.5, every graph H for which the question is open satisfies the following:

- H is planar and has a cycle C;
- every induced cycle of H is a C_3 or a C_4;
- let $\bar{N}(C)$ denote the vertices of H that are not adjacent to C (equivalently, vertices at distance at least 2 from C), then $|\bar{N}(C)| \leq 2$ and in the case of equality, the two vertices of $\bar{N}$ are independent.

The study of subdivisions of specific graphs is an intermediate step towards this goal. A direction of research towards the aforementioned characterisation would be to investigate whether the Erdős-Pósa property is inherited by induced subdivisions. Formally, is it true that if the induced Erdős-Pósa property holds for subdivisions of some graph H, then it also holds for the subdivisions of every graph H' contained as an induced subdivision in H?

Observe that the constructions we used in our counterexamples contain arbitrarily large complete subgraphs. Therefore the landscape of the induced Erdős-Pósa property of subdivisions might be much different if one restricts their attention to graphs excluding a dense subgraph. In this direction, Weißauer recently proved that for every $s, \ell \in \mathbb{N}$, subdivisions of C_ℓ have the induced Erdős-Pósa property in $K_{s,s}$-subgraph-free graphs [Wei18]. This contrasts with the general case where subdivisions of C_ℓ stop having the induced Erdős-Pósa property from $\ell = 5$ (see the note below Theorem 1.3).

Another line of research in the study of the Erdős-Pósa property of graph classes is to optimize the bounding function. We note that all our positive results hold with a polynomial bounding function. On the other hand, we obtained in Theorem 7.1 a lower bound of $\Omega(k\log k)$ for non-acyclic subcubic graphs. We do not expect our upper-bounds to be tight and it is an open question to find the correct order of magnitude of the bounding functions for the graphs we considered. In this direction it is also open to determine the correct order of magnitude of the bounding function in Theorem 1.3.

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