APPROXIMATE CLOAKING FOR TIME-DEPENDENT MAXWELL EQUATIONS VIA TRANSFORMATION OPTICS

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Abstract. We study approximate cloaking using transformation optics for electromagnetic waves in the time domain. Our approach is based on estimates of the degree of visibility in the frequency domain for all frequencies in which the frequency dependence is explicit. The difficulty and the novelty analysis parts are in the low and high frequency regimes. To this end, we implement a variational technique in the low frequency domain, and multiplier and duality techniques in the high frequency domain. Our approach is inspired by the work of Nguyen and Vogelius on the wave equation.

Key words. cloaking, transformation optics, Maxwell's equations, radiation condition.

AMS subject classification. 35A15, 35B40, 35F10, 78A40, 78M30.

1. Introduction and statements of results

Cloaking via transformation optics was introduced by Pendry, Schurig, and Smith [42] for the Maxwell system and by Leonhardt [22] in the geometric optics setting. The idea is to use the invariance of Maxwell equations under a change of variables. They used a singular change of variables that blows up a point into a cloaked region. The same transformation was used by Greenleaf, Lassas, and Uhlmann in an inverse context [12]. However, the singular nature of the cloaks presents various difficulties in practice as well as in theory: (1) they are hard to fabricate and (2) in certain cases, the correct definition (and therefore the properties) of the corresponding electromagnetic fields is an issue. To avoid using the singular structure, various regularized schemes have been proposed. One of them was suggested by Kohn, Shen, Vogelius, and Weinstein [19] in which they used a transformation which blows up a small ball of radius ρ instead of a point into the cloaked region. Other, related regularizations schemes have also been proposed [44, 13]. It is worth mentioning that there are other techniques for cloaking, some of which use negative index materials such as cloaking using complementary media, see, e.g., [20, 29] and cloaking via localized resonance, see, e.g., [30] (see also [41, 24, 28]).

Approximate cloaking using transformation optics for the acoustic setting has been investigated in the last fifteen years. In the frequency domain, if an appropriate or a fixed lossy layer (damping layer) is implemented between the transformation cloak and the cloaked region, then cloaking is achieved, and the degree of visibility is of the order ρ in three dimensions and $1/|\ln \rho|$ in two dimensions, see [19, 26] respectively. Without such a lossy layer, the phenomena are more complex and have been investigated in more depth [27]. In this setting, there are two distinct situations: resonant and non-resonant. In the non-resonant case, cloaking is achieved with the same degree of visibility; however, the field inside the cloaked region might depend on the field outside (cloaking vs shielding). In the resonant case, the energy inside the cloaked region can blow up, and cloaking might not be achieved. Different cloaking aspects related to the Helmholtz equation such as zero frequency context and the enhancement, have been studied [19, 33, 2, 14, 16] and references therein. There are much less rigorous works in the time domain. Cloaking using transformation optics for the wave equation was established in which a lossy layer is also used [35], and in which the dispersion of the transformation cloak using the Drude-Lorentz model is accounted and a fixed lossy layer is used [36], in this direction.

In the electromagnetic time harmonic context, the situation on one hand shares some common features with the scalar case and on the other hand has some distinct figures, see [39]. In the non-resonant electromagnetic case, without sources inside the cloaked region, it is shown that cloaking is achieved and the degree of visibility is of the order ρ^3. In the resonant electromagnetic case, in contrast to the scalar case, cloaking is always achieved even if the energy inside the cloaked region might blow up. Moreover, the degree of visibility varies between the non-resonant and resonant cases. Other works on cloaking for the Maxwell equations in the time harmonic regime can be found in [11, 47, 48, 3, 9, 21] and references therein.

This paper is devoted to cloaking using transformation optics for the Maxwell equations in the *time* domain. We use the regularization transformation instead of the singular one for the starting point, which is necessary for viewing previous results in the time harmonic regime. Concerning the analysis, we first transform the Maxwell equations in the time domain into a family of the Maxwell equations in the time harmonic regime by taking the Fourier transform of the solutions with respect to time. After obtaining appropriate estimates on the near invisibility of the Maxwell equations in the time harmonic regime, we simply invert the Fourier transform. This idea has its roots in the work of Nguyen and Vogelius [35] (see also [36]) in the context of acoustic cloaking and was used to study impedance boundary conditions in the time domain [38] and cloaking for the heat equation [32]. To implement this idea, the heart of the matter is to obtain the degree of visibility in which the dependence on frequency is *explicit* and well controlled. The analysis involves a variational method, a multiplier technique, and a duality argument in different ranges of frequency. An intriguing fact about the Maxwell equations in the time harmonic regime worth mentioned is that the multiplier technique does not fit well for the purposes of cloaking in the very high frequency regime, and a duality argument is involved instead. Another key technical point is the proof of the radiating condition for the Fourier transform in time of the weak solutions of the general Maxwell equations, a fact which is interesting in itself. Note that after a change of variables, the study of the cloaking effect can be derived from the study of the effect of a small inclusion which is known when the coefficients inside the small inclusions are fixed (or has a finite range), generally for a fixed frequency, see, e.g., [4, 45]. Nevertheless, the situation in the context of cloaking is non-standard since the coefficients inside the small inclusion blow up as the diameter goes to 0.

Let us now describe the problem in more detail. For simplicity, we suppose that the cloaking device occupies the annular region $B_2 \setminus B_{1/2}$ and the cloaked region is the ball $B_{1/2}$ in $\mathbb{R}^3$ in which the permittivity and the permeability are given by two 3×3 matrices ε_O, μ_O, respectively. In this paper, for $r > 0$, we denote B_r as the ball centered at the origin and of radius r. Throughout this paper, we assume that, in $B_{1/2}$,

$$\varepsilon_O,\ \mu_O \text{ are real, symmetric,} \tag{1.1}$$

and uniformly elliptic, i.e.,

$$\frac{1}{\Lambda}|\xi|^2 \le \langle \varepsilon_O(x)\xi, \xi\rangle, \langle \mu_O(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^3, \tag{1.2}$$

for a.e. $x \in B_{1/2}$ and for some $\Lambda \ge 1$. We also assume ε_O, μ_O are piecewise C^1 to ensure the uniqueness of solutions via the unique continuation principle (see [40, 5], and also [43]).

Let $\rho \in (0, 1)$ and let $F_\rho : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F_\rho(x) = \begin{cases} x & \text{in } \mathbb{R}^3 \setminus B_2, \\ \left(\dfrac{2-2\rho}{2-\rho} + \dfrac{|x|}{2-\rho}\right)\dfrac{x}{|x|} & \text{in } B_2 \setminus B_\rho, \\ \dfrac{x}{\rho} & \text{in } B_\rho. \end{cases} \tag{1.3}$$

The cloaking device in $B_2 \setminus B_{1/2}$ constructed via the transformation optics technique is characterized by the triple of permittivity, permeability, and conductivity and contains two layers. The first one in $B_2 \setminus B_1$ that comes from the transformation technique using the map F_ρ is

$$(F_{\rho *} I, F_{\rho *} I, 0)$$

and the second one in $B_1 \setminus B_{1/2}$, which is a fixed lossy layer, is

$$(I, I, 1).$$

Here and in what follows, for a diffeomorphism F and a matrix-valued function A, one denotes

$$F_* A := \frac{DFADF^T}{|\det DF|} \circ F^{-1}. \tag{1.4}$$

Remark 1.1. Different fixed lossy layer can be used. However, to simplify the notations and to avoid several unnecessary technical points, the triple $(I, I, 1)$ is considered.

Assume that the medium is homogeneous outside the cloaking device and the cloaked region. In the presence of the cloaked object and the cloaking device, the medium in the whole space $\mathbb{R}^3$ is described by the triple $(\varepsilon_c, \mu_c, \sigma_c)$ given by

$$(\varepsilon_c, \mu_c, \sigma_c) = \begin{cases} (I, I, 0) & \text{in } \mathbb{R}^3 \setminus B_2, \\ (F_{\rho *} I, F_{\rho *} I, 0) & \text{in } B_2 \setminus B_1, \\ (I, I, 1) & \text{in } B_1 \setminus B_{1/2}, \\ (\varepsilon_O, \mu_O, 0) & \text{in } B_{1/2}. \end{cases} \tag{1.5}$$

Let $\mathcal{J}$ represent a charge density. We assume that

$$\mathcal{J} \in L^1([0, \infty); [L^2(\mathbb{R}^3)]^3) \text{ with } \operatorname{supp} \mathcal{J} \subset [0, T] \times (B_{R_0} \setminus B_2), \text{ for some } T > 0, R_0 > 2, \tag{1.6}$$

and

$$\operatorname{div} \mathcal{J} = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}^3. \tag{1.7}$$

With the cloaking device and the cloaked object, the electromagnetic wave generated by $\mathcal{J}$ with zero data at time 0 is the unique weak solution $(\mathcal{E}_c, \mathcal{H}_c) \in L^\infty_{\text{loc}}([0, \infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \varepsilon_c \dfrac{\partial \mathcal{E}_c}{\partial t} = \nabla \times \mathcal{H}_c - \mathcal{J} - \sigma_c \mathcal{E}_c & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mu_c \dfrac{\partial \mathcal{H}_c}{\partial t} = -\nabla \times \mathcal{E}_c & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathcal{E}_c(0, \cdot) = \mathcal{H}_c(0, \cdot) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.8}$$

In the homogeneous space, the field generated by $\mathcal{J}$ with zero data at time 0 is the unique weak solution $(\mathcal{E}, \mathcal{H}) \in L^\infty_{\text{loc}}([0, \infty), [L^2(\mathbb{R}^3)]^6)$ to the system

$$\begin{cases} \dfrac{\partial \mathcal{E}}{\partial t} = \nabla \times \mathcal{H} - \mathcal{J} & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \dfrac{\partial \mathcal{H}}{\partial t} = -\nabla \times \mathcal{E} & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ \mathcal{E}(0, \cdot) = \mathcal{H}(0, \cdot) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.9}$$

The meaning of weak solutions, in a slightly more general context, is as follows.

Definition 1.1. *Let ε, μ, $\in [L^\infty(\mathbb{R}^3)]^{3\times 3}$, $\sigma_m, \sigma_e \in L^\infty(\mathbb{R}^3)$ be such that ε and μ are real, symmetric, and uniformly elliptic in $\mathbb{R}^3$, and σ_m and σ_e are real and nonnegative in $\mathbb{R}^3$, and let $f_e, f_m \in L^1_{\rm loc}([0,\infty); [L^2(\mathbb{R}^3)]^3)$. A pair $(\mathcal{E},\mathcal{H}) \in L^\infty_{\rm loc}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ is called a weak solution of*

$$\begin{cases} \varepsilon \dfrac{\partial \mathcal{E}}{\partial t} = \nabla \times \mathcal{H} - \sigma_e \mathcal{E} + f_m & \text{in } (0,+\infty)\times\mathbb{R}^3, \\ \mu \dfrac{\partial \mathcal{H}}{\partial t} = -\nabla \times \mathcal{E} - \sigma_m \mathcal{H} + f_e & \text{in } (0,+\infty)\times\mathbb{R}^3, \\ \mathcal{E}(0,\cdot) = 0; \mathcal{H}(0,\cdot) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.10}$$

if

$$\begin{cases} \dfrac{d}{dt}\langle \varepsilon \mathcal{E}(t,.), E\rangle + \langle \sigma_e \mathcal{E}(t,.), E\rangle - \langle \mathcal{H}(t,.), \nabla \times E\rangle = \langle f_m(t,.), E\rangle, \\ \dfrac{d}{dt}\langle \mu \mathcal{H}(t,.), H\rangle + \langle \sigma_m \mathcal{H}(t,.), H\rangle + \langle \mathcal{E}(t,.), \nabla \times H\rangle = \langle f_e(t,.), H\rangle, \end{cases} \quad \text{for } t > 0, \tag{1.11}$$

for all $(E,H) \in [H(\mathrm{curl}, \mathbb{R}^3)]^2$, and

$$\mathcal{E}(0,.) = \mathcal{H}(0,.) = 0 \text{ in } \mathbb{R}^3. \tag{1.12}$$

Some comments on Definition 1.1 are in order. System (1.11) is understood in the distributional sense. Initial condition (1.12) is understood as

$$\langle \varepsilon \mathcal{E}(0,.), E\rangle = \langle \mu \mathcal{H}(0,.), H\rangle = 0 \quad \text{for all } (E,H) \in [H(\mathrm{curl}, \mathbb{R}^3)]^2. \tag{1.13}$$

From (1.11), one can check that

$$\langle \varepsilon \mathcal{E}(t,.), E\rangle, \langle \mu \mathcal{H}(t,.), H\rangle \in W^{1,1}_{\rm loc}([0,+\infty)).$$

This in turn ensures the trace sense in (1.13).

Concerning the well-posedness of (1.10), we have, see, e.g., [37, Theorem 3.1],

Proposition 1.1. *Let $f_e, f_m \in L^1_{\rm loc}([0,\infty); [L^2(\mathbb{R}^3)]^3)$. There exists a unique weak solution $(\mathcal{E},\mathcal{H}) \in L^\infty_{\rm loc}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ of (1.10). Moreover, for $T > 0$, the following estimate holds*

$$\int_{\mathbb{R}^3} |\mathcal{E}(t,x)|^2 + |\mathcal{H}(t,x)|^2 dx \le C \left(\int_0^t \left\| \big(f_e(s,.), f_m(s,.)\big) \right\|_{L^2(\mathbb{R}^3)} ds \right)^2 \quad \text{for } t \in [0,T], \tag{1.14}$$

for some positive constant C depending only on the ellipticity of ε and μ.

Remark 1.2. We emphasize here that the constant C in Proposition 1.1 is independent of T. This fact is later used in the proof of the radiating condition. In [37], the authors considered dispersive materials and also dealt with Maxwell equations which are non-local in time.

We are ready to state the main result of the paper that is proved in Section 3.

Theorem 1.1. *Let $\rho \in (0,1)$, $T > 0$ and let $(\mathcal{E}_c,\mathcal{H}_c), (\mathcal{E},\mathcal{H}) \in L^\infty_{\rm loc}([0,\infty), [L^2(\mathbb{R}^3)]^6)$ be the unique solutions to systems (1.8) and (1.9), respectively. Assume (1.6) and (1.7). Then, for $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$,*

$$\|(\mathcal{E}_c,\mathcal{H}_c) - (\mathcal{E},\mathcal{H})\|_{L^\infty((0,T); L^2(K))} \le C T \rho^3 \|\mathcal{J}\|_{H^{11}((0,\infty); [L^2(\mathbb{R}^3)]^3)}, \tag{1.15}$$

for some positive constant C depending only on K, R_0.

Remark 1.3. Assertion (1.15) is optimal since it gives the same degree of visibility as in the frequency domain in [39] where the optimality is established.

Remark 1.4. Estimate (1.15) requires that $\mathcal{J}$ is regular. The condition of the regularity of $\mathcal{J}$ is not optimal, and this optimality would be studied elsewhere.

Our approach is inspired by the work of Nguyen and Vogelius [35] (see also [36, 38]), where they studied approximate cloaking for the acoustic setting in the time domain. The main idea can be briefly described as follows. We first transform the time-dependent Maxwell systems into a family of the time-harmonic Maxwell systems by taking the Fourier transform of the solutions with respect to time. After obtaining the appropriate degree of near invisibility for the Maxwell equations in the time harmonic regime, where the frequency dependence is *explicit*, we simply invert the Fourier transform. The analysis in the frequency domain ω (in Section 2) can be divided into three steps that deal with frequencies in low and moderate $(0 < \omega < 1)$, moderate and high $(1 < \omega < 1/\rho)$, high and very high $(\omega > 1/\rho)$ regimes. The analysis in the low and moderate frequency regime (in Section 2.1) is based on a variational approach. In comparison with [39], one needs to additionally derive an estimate for small frequency in which the frequency dependence is explicit. In the moderate and high frequency regime, to obtain appropriate estimates, we use the multiplier technique and the test functions are inspired from the scalar case due to Morawetz (see [25]). The analysis in the moderate and high frequency regime is given in Section 2.2. There is a significant difference between the scalar case and the Maxwell vectorial case. It is known in the scalar case that one can control the normal derivative of a solution to the exterior Helmholtz equation in homogeneous medium by its value on the boundary of a convex, bounded subset of $\mathbb{R}^3$. However, in contrast with the scalar case, one cannot either use tangential components of the electromagnetic fields to control the normal component in the same Sobolev norms and conversely. This fact can be seen from the explicit solutions outside a unit ball of Maxwell equations (see, e.g., [17, Theorem 2.50]). This is the reason for which we cannot use the multiplier technique in the very high frequency regime and again reveals the distinct structure of Maxwell equations in the time harmonic regime as compared to the Helmholtz equations. The analysis in the high and very high frequency regime in Section 2.3 is based on the duality method inspired from [23]. The proof of Theorem 1.1 based on the frequency analysis is given in Section 3. A key technical point required for the analysis in the frequency domain is the establishment of the radiation condition for the Fourier transform with respect to time of the solutions of Maxwell equations. The rigorous proof on the radiation condition in a general setting is new to our knowledge and is interesting in itself.

The paper is organized as follows. Section 2 is devoted to the estimates for Maxwell's equations in frequency domain. Section 3 gives the proof of Theorem 1.1. The assertion on the radiation condition is also stated and proved there.

2. Frequency analysis

In this section, we provide estimates to assess the degree of visibility in the frequency domain. We first recall some notations. Let U be a smooth open subset of $\mathbb{R}^3$. We denote

$$H(\mathrm{curl}, U) := \Big\{\phi \in [L^2(U)]^3 : \nabla \times \phi \in [L^2(U)]^3\Big\},$$

$$H(\mathrm{div}, U) := \Big\{\phi \in [L^2(U)]^3 : \mathrm{div}\, \phi \in L^2(U)\Big\}.$$

We also use the notations $H_{\mathrm{loc}}(\mathrm{curl}, U)$ and $H_{\mathrm{loc}}(\mathrm{div}, U)$ with the usual convention.

Given $\mathbb{J} \in [L^2(\mathbb{R}^3)]^3$ with compact support, let $(\mathbb{E}, \mathbb{H}) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ and $(\mathbb{E}_\rho, \mathbb{H}_\rho) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ $(\rho > 0)$ be the corresponding unique radiating solutions of the following systems

$$\begin{cases} \nabla \times \mathbb{E} = i\omega \mathbb{H} & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbb{H} = -i\omega \mathbb{E} + \mathbb{J} & \text{in } \mathbb{R}^3, \end{cases} \tag{2.1}$$

and

$$(2.2)\qquad \begin{cases} \nabla \times \mathbb{E}_\rho = i\omega\mu_\rho \mathbb{H}_\rho & \text{in } \mathbb{R}^3, \\ \nabla \times \mathbb{H}_\rho = -i\omega\varepsilon_\rho \mathbb{E}_\rho + \sigma_\rho \mathbb{E}_\rho + \mathbb{J} & \text{in } \mathbb{R}^3. \end{cases}$$

Here, for $\rho > 0$,

$$(2.3)\qquad (\varepsilon_\rho, \mu_\rho, \sigma_\rho) = \begin{cases} (I, I, 0) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ (\rho^{-1} I, \rho^{-1} I, \rho^{-1}) & \text{in } B_\rho \setminus B_{\rho/2}, \\ ({F_\rho^{-1}}_* \varepsilon_O, {F_\rho^{-1}}_* \mu_O, 0) & \text{in } B_{\rho/2}. \end{cases}$$

Recall that for $\omega > 0$, a solution $(E, H) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus B_R)]^2$, for some $R > 0$, of the Maxwell equations

$$\begin{cases} \nabla \times E = i\omega H & \text{in } \mathbb{R}^3 \setminus B_R, \\ \nabla \times H = -i\omega E & \text{in } \mathbb{R}^3 \setminus B_R \end{cases}$$

is called radiating if it satisfies one of the (Silver-Müller) radiation conditions

$$(2.4)\qquad H \times x - |x| E = O(1/|x|) \quad \text{or} \quad E \times x + |x| H = O(1/|x|) \text{ as } |x| \to +\infty.$$

Here and in what follows, for $\alpha \in \mathbb{R}$, $O(|x|^\alpha)$ denotes a quantity whose norm is bounded by $C|x|^\alpha$ for some constant $C > 0$.

Throughout this section, we assume

$$(2.5)\qquad \operatorname{div} \mathbb{J} = 0 \quad \text{and} \quad \operatorname{supp} \mathbb{J} \subset B_{R_0} \setminus B_2,$$

for some $R_0 > 2$. One sees later (in Section 3) that if $(\hat{\mathcal{E}}_c, \hat{\mathcal{H}}_c)$ and $(\hat{\mathcal{E}}, \hat{\mathcal{H}})$ are the corresponding Fourier transform with respect to t of $(\mathcal{E}_c, \mathcal{H}_c)$ and $(\mathcal{E}, \mathcal{H})$ in (1.8)-(1.9) and if one defines $(\hat{\mathcal{E}}_\rho, \hat{\mathcal{H}}_\rho) = (DF_\rho^T \hat{\mathcal{E}}_c, DF_\rho^T \hat{\mathcal{H}}_c) \circ F_\rho$ in $\mathbb{R}^3$ then $(\hat{\mathcal{E}}, \hat{\mathcal{H}})$ and $(\hat{\mathcal{E}}_\rho, \hat{\mathcal{H}}_\rho)$ satisfy (2.1) and (2.2) respectively (for some $\mathbb{J}$). This is the motivation for the introduction of $(\mathbb{E}, \mathbb{H})$ and $(\mathbb{E}_\rho, \mathbb{H}_\rho)$.

The goal of this section is to derive estimates for $(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})$ in which the dependence on the frequency ω and ρ is explicit. More precisely, we establish the following three results.

Proposition 2.1. *Let $0 < \rho < \rho_0$ and $0 < \omega < \omega_0$. We have*

$$(2.6)\qquad \|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \omega^{-1} \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)},$$

for some positive constant C_R depending only on R_0, R, ω_0, and ρ_0.

Proposition 2.2. *Let $0 < \rho < \rho_0$ and $0 < \omega_0 \le \omega \le \omega_1 \rho^{-1}$, and assume that ρ_0 is small enough and ω_0 is large enough. We have, for $R > 2$,*

$$(2.7)\qquad \|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \omega^3 \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)},$$

for some positive constant C_R depending only on R, R_0, ω_0, and ω_1.

Proposition 2.3. *Let $0 < \rho < 1$, $\omega_1 > 0$, and $\omega > \omega_1 \rho^{-1}$. We have, for $R > 2$,*

$$(2.8)\qquad \|(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})\|_{L^2(B_R \setminus B_2)} \le C_R \omega^{17/2} \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)},$$

for some positive constant C_R depending only on R_0, R, and ω_1.

To motivate the analysis in this section, we define

$$(2.9)\qquad (\mathbf{E}_\rho, \mathbf{H}_\rho) = \begin{cases} (\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H}) & \text{in } \mathbb{R}^3 \setminus B_\rho, \\ (\mathbb{E}_\rho, \mathbb{H}_\rho) & \text{in } B_\rho, \end{cases}$$

and set

$$(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) = (\mathbf{E}_\rho, \mathbf{H}_\rho)(\rho\,\cdot\,) \text{ in } \mathbb{R}^3. \tag{2.10}$$

Then, $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) \in [L^2_{\mathrm{loc}}(\mathbb{R}^3)]^6$ with $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho) \in \cap_{R>1} H(\mathrm{curl}, B_R \setminus \partial B_1)$ is the unique radiating solution of

$$\begin{cases} \nabla \times \widetilde{\mathbf{E}}_\rho = i\omega\tilde{\mu}_\rho \widetilde{\mathbf{H}}_\rho & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ \nabla \times \widetilde{\mathbf{H}}_\rho = -i\omega\tilde{\varepsilon}_\rho \widetilde{\mathbf{E}}_\rho + \tilde{\sigma}_\rho \widetilde{\mathbf{E}}_\rho & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ [\widetilde{\mathbf{E}}_\rho \times \nu] = -\mathbb{E}(\rho\,\cdot\,) \times \nu & \text{on } \partial B_1, \\ [\widetilde{\mathbf{H}}_\rho \times \nu] = -\mathbb{H}(\rho\,\cdot\,) \times \nu & \text{on } \partial B_1, \end{cases} \tag{2.11}$$

where

$$(\tilde{\varepsilon}_\rho, \tilde{\mu}_\rho, \tilde{\sigma}_\rho) := \begin{cases} (\rho I, \rho I, 0) & \text{in } \mathbb{R}^3 \setminus B_1, \\ (I, I, 1) & \text{in } B_1 \setminus B_{1/2}, \\ (\varepsilon_O, \mu_O, 0) & \text{in } B_{1/2}. \end{cases} \tag{2.12}$$

Here and in what follows for a smooth, bounded, open subset D of $\mathbb{R}^3$, we denote $[u] := u|_{\mathrm{ext}} - u|_{\mathrm{int}}$ on ∂D for an appropriate (vectorial) function u.

We will study (2.11) and using this to derive estimates for $(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H})$ in the following three subsections.

2.1. **Low and moderate frequency analysis - Proof of Proposition 2.1.** This section is devoted to the proof of Proposition 2.1 and contains two subsections. In the first subsection, we present several useful lemmas and the proof of Proposition 2.1 is given in the second subsection.

2.1.1. *Some useful lemmas.* We first recall the following known result which is the basic ingredient for the variational approach.

Lemma 2.1. *Let D be a smooth, bounded, open subset of $\mathbb{R}^3$ and let ϵ be a measurable, symmetric, uniformly elliptic, matrix-valued function defined in D. Assume that one of the following two conditions holds:*

i) $(u_n)_{n\in\mathbb{N}} \subset H(\mathrm{curl}, D)$ *is a bounded sequence in* $H(\mathrm{curl}, D)$ *such that*

$\big(\operatorname{div}(\epsilon u_n)\big)_{n\in\mathbb{N}}$ *converges in* $H^{-1}(D)$ *and* $\big(u_n \times \nu\big)_{n\in\mathbb{N}}$ *converges in* $H^{-1/2}(\partial D)$.

ii) $(u_n)_{n\in\mathbb{N}} \subset H(\mathrm{curl}, D)$ *is a bounded sequence in* $H(\mathrm{curl}, D)$ *such that*

$\big(\operatorname{div}(\epsilon u_n)\big)_{n\in\mathbb{N}}$ *converges in* $L^2(D)$ *and* $\big((\epsilon u_n) \cdot \nu\big)_{n\in\mathbb{N}}$ *converges in* $H^{-1/2}(\partial D)$.

There exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ which converges in $[L^2(D)]^3$.

The conclusion of Lemma 2.1 under condition $i)$ is [31, Lemma 1] and has its roots in [15, 8, 46]. The conclusion of Lemma 2.1 under condition $ii)$ can be obtained in the same way.

In what follows, the following notations are used

$$H^{-1/2}(\operatorname{div}_\Gamma, \Gamma) := \Big\{ \phi \in [H^{-1/2}(\Gamma)]^3;\ \phi \cdot \nu = 0 \text{ and } \operatorname{div}_\Gamma \phi \in H^{-1/2}(\Gamma) \Big\},$$

$$\|\phi\|_{H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)} := \|\phi\|_{H^{-1/2}(\Gamma)} + \|\operatorname{div}_\Gamma \phi\|_{H^{-1/2}(\Gamma)}.$$

We have

Lemma 2.2. *Let $0 < \omega < \omega_0$ and D be a simply connected, bounded, open subset of $\mathbb{R}^3$ of class C^1, and denote $\Gamma = \partial D$. Let $h \in H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $E \in H(\operatorname{curl}, D)$. We have*

$$\left| \int_\Gamma \bar{E} \cdot h \, ds \right| \le C \Big(\omega \|E\|_{L^2(D)} + \|\nabla \times E\|_{L^2(D)} \Big) \Big(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1} \| \operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)} \Big), \tag{2.13}$$

for some positive constant C depending only on D and ω_0.

Here and in what follows, $\bar{u}$ denotes the complex conjugate of u.

Proof. Let $(E^0, H^0) \in [H(\operatorname{curl}, D)]^2$ be the unique solution to

$$\begin{cases} \nabla \times E^0 = i\omega(1+i)H^0 & \text{in } D, \\ \nabla \times H^0 = -i\omega(1+i)E^0 & \text{in } D, \\ E^0 \times \nu = h & \text{on } \Gamma. \end{cases}$$

We prove by contradiction that

$$\|(E^0, H^0)\|_{L^2(D)} \le C \left(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1} \| \operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)} \right) \tag{2.14}$$

for some positive constant C depending only on ω_0. Assume that there exist sequences $((E_n, H_n)) \subset [H(\operatorname{curl}, D)]^2$, $(\omega_n) \subset (0, \omega_0)$ and $(h_n) \subset H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ such that

$$\|(E_n, H_n)\| = 1 \text{ for all } n, \tag{2.15}$$

$$\|h_n\|_{H^{-1/2}(\Gamma)} + \omega_n^{-1} \| \operatorname{div}_\Gamma h_n\|_{H^{-1/2}(\Gamma)} \text{ converges to } 0, \tag{2.16}$$

and

$$\begin{cases} \nabla \times E_n = i\omega_n(1+i)H_n & \text{in } D, \\ \nabla \times H_n = -i\omega_n(1+i)E_n & \text{in } D, \\ E_n \times \nu = h_n & \text{in } \Gamma. \end{cases} \tag{2.17}$$

Without loss of generality, one can assume that $\omega_n \to \omega^*$. Applying Lemma 2.1, one might assume that (E_n, H_n) converges to some $(E, H) \in [L^2(D)]^6$. We only consider the case $\omega_* = 0$, the case where $\omega_* > 0$ is standard. Then

$$\begin{cases} \nabla \times E = 0 & \text{in } D, \\ \operatorname{div} E = 0 & \text{in } D, \\ E \times \nu = 0 & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} \nabla \times H = 0 & \text{in } D, \\ \operatorname{div} H = 0 & \text{in } D, \\ H \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

We also have, for each connected component Γ_j of Γ,

$$\int_{\Gamma_j} E \cdot \nu \, ds = \lim_{n \to \infty} \int_{\Gamma_j} E_n \cdot \nu \, ds = \lim_{n \to \infty} \Big[\frac{1}{-i\omega_n(1+i)} \int_{\Gamma_j} (\nabla \times H_n) \cdot \nu \, ds \Big] = 0.$$

Since D is simply connected, it follows (see, e.g., [10, Theorems 2.9 and 3.1]) that $E = \nabla \times \xi_E$ and $H = \nabla \xi_H$ for some ξ_E, $\xi_H \in H^1(D)$. We derive from the systems of E and H that

$$\int_D |\nabla \times \xi_E|^2 \, dx = 0 \quad \text{and} \quad \int_D |\nabla \xi_H|^2 \, dx = 0.$$

This yields that $E = H = 0$ in D. We have a contradiction. Therefore, (2.14) is proved.

We have

$$\int_\Gamma \bar{E}\cdot h\,ds = \int_\Gamma \bar{E}\cdot (E^0\times\nu)\,ds = \int_D (\nabla\times\bar{E})\cdot E^0\,dx - \int_D \bar{E}\cdot(\nabla\times E^0)\,dx \text{ (integration by parts)}$$
$$= \int_D (\nabla\times\bar{E})\cdot E^0\,dx - i\omega(1+i)\int_D \bar{E}\cdot H^0\,dx.$$

It follows from Hölder's inequality and (2.14) that

$$\left|\int_\Gamma \bar{E}\cdot h\,ds\right| \le \Big(\omega\|E\|_{L^2(D)} + \|\nabla\times E\|_{L^2(D)}\Big)\|(E^0,H^0)\|_{L^2(D)}$$
$$\le C\Big(\omega\|E\|_{L^2(D)} + \|\nabla\times E\|_{L^2(D)}\Big)\Big(\|h\|_{H^{-1/2}(\Gamma)} + \omega^{-1}\|\operatorname{div}_\Gamma h\|_{H^{-1/2}(\Gamma)}\Big),$$

which is (2.13). □

The following simple result is used in our analysis.

Lemma 2.3. *Let D be a C^1 bounded open subset of $\mathbb{R}^3$ and denote $\Gamma=\partial D$. Let $h\in H^{-1/2}(\operatorname{div}_\Gamma,\Gamma)$ and $u\in H(\operatorname{curl},D)$. We have*

$$\left|\int_\Gamma \bar{u}\cdot h\right| \le C\|u\|_{H(\operatorname{curl},D)}\|h\|_{H^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}. \tag{2.18}$$

for some positive constant C independent of h and u.

Proof. The result is standard. For the convenience of the reader, we present the proof. By the trace theory, see, e.g., [1, 6], there exists $\phi\in H(\operatorname{curl},D)$ such that

$$\phi\times\nu = h \text{ on } \Gamma \quad \text{ and } \quad \|\phi\|_{H(\operatorname{curl},D)} \le C\|h\|_{H^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}$$

for some positive constant C depending only on D. Then, by integration by parts, we have

$$\int_\Gamma \bar{u}\cdot h = \int_\Gamma \bar{u}\cdot(\phi\times\nu) = \int_D \nabla\times\bar{u}\cdot\phi - \int_D \bar{u}\cdot\nabla\times\phi.$$

The conclusion follows by Hölder's inequality. □

We next present an estimate for the exterior domain in the small and moderate frequency regime.

Lemma 2.4. *Let $R_0>2$, $0<k<k_0$, and $D\subset B_1$ be a smooth open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3\setminus D$ is connected. Let $(f_1,f_2)\in[L^2(\mathbb{R}^3)]^6$ with support in $B_{R_0}\setminus D$, and assume that $(E,H)\in[\cap_{R>1}H(\operatorname{curl},B_R\setminus D)]^2$ is a radiating solution of*

$$\begin{cases} \nabla\times E = ikH + f_1 & \text{in } \mathbb{R}^3\setminus\bar{D},\\ \nabla\times H = -ikE + f_2 & \text{in } \mathbb{R}^3\setminus\bar{D}. \end{cases}$$

We have, for $R>2$,

$$\|(E,H)\|_{L^2(B_R\setminus D)} \le C_R\Big(\|(E\times\nu,H\times\nu)\|_{H^{-1/2}(\partial D)} + \|(f_1,f_2)\|_{L^2} + k^{-1}\|(\operatorname{div} f_1,\operatorname{div} f_2)\|_{L^2}\Big), \tag{2.19}$$

for some positive constant C_R depending only on D, k_0, R_0, and R.

Proof. By the Stratton-Chu formula, we have, for $x \in \mathbb{R}^3$ with $|x| > R_0 + 1$,

$$E(x) = \int_{\partial B_{R_0+1/2}} \nabla_x G_k(x,y) \times \big(\nu(y) \times E(y)\big) dy$$
$$+ ik \int_{\partial B_{R_0+1/2}} \nu(y) \times H(y) G_k(x,y) dy - \int_{\partial B_{R_0+1/2}} \nu(y) \cdot E(y) \nabla_x G_k(x,y) dy,$$

and

$$H(x) = \int_{\partial B_{R_0+1/2}} \nabla_x G_k(x,y) \times \big(\nu(y) \times H(y)\big) dy$$
$$- ik \int_{\partial B_{R_0+1/2}} \nu(y) \times E(y) G_k(x,y) dy - \int_{\partial B_{R_0+1/2}} \nu(y) \cdot H(y) \nabla_x G_k(x,y) dy,$$

where

$$G_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \text{ for } x \neq y. \tag{2.20}$$

It follows that, for $R > R_0 + 1$,

$$\|(E,H)\|_{L^2(B_R \setminus D)} \le C_R \|(E,H)\|_{L^2(B_{R_0+1} \setminus D)}. \tag{2.21}$$

Hence, it suffices to prove (2.19) for $R = R_0 + 1$ by contradiction. Assume that there exist sequences $(k_n) \subset (0, k_0)$, $\big((f_{1,n}, f_{2,n})\big) \subset [L^2(\mathbb{R}^3 \setminus D)]^6$ with support in $B_{R_0} \setminus D$, and $\big((E_n, H_n)\big) \subset [\cap_{R>1} H(\operatorname{curl}, B_R \setminus D)]^2$ such that $\|(E_n, H_n)\|_{L^2(B_{R_0+1} \setminus D)} = 1$,

$$\lim_{n \to +\infty} \Big(\|(E_n \times \nu, H_n \times \nu)\|_{H^{-1/2}(\partial D)} + \|(f_{1,n}, f_{2,n})\|_{L^2} + k_n^{-1} \|(\operatorname{div} f_{1,n}, \operatorname{div} f_{2,n})\|_{L^2} \Big) = 0, \tag{2.22}$$

and

$$\begin{cases} \nabla \times E_n = ik_n H_n + f_{1,n} & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H_n = -ik_n E_n + f_{2,n} & \text{in } \mathbb{R}^3 \setminus \bar{D}. \end{cases}$$

Without loss of generality, one might assume that $k_n \to k_*$ as $n \to +\infty$. Using Lemma 2.1, (2.21), and (2.22), one can assume that (E_n, H_n) converges to (E, H) in $[L^2(B_R \setminus D)]^6$. We first consider the case $k_* = 0$. We have

$$\begin{cases} \nabla \times E = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ E \times \nu = 0 & \text{on } \partial D, \end{cases} \qquad \begin{cases} \nabla \times H = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ H \times \nu = 0 & \text{on } \partial D, \end{cases} \tag{2.23}$$

$$\operatorname{div} E = 0 \text{ in } \mathbb{R}^3 \setminus \bar{D} \quad \operatorname{div} H = 0 \text{ in } \mathbb{R}^3 \setminus \bar{D}, \tag{2.24}$$

and

$$|E(x)| = O(|x|^{-2}) \text{ and } \quad |H(x)| = O(|x|^{-2}) \text{ for large } x. \tag{2.25}$$

Assertion (2.25) can be derived again from the Stratton-Chu formula using the fact that $\lim_{n \to +\infty} k_n = 0$. It follows from (2.23), (2.24), and (2.25) that (see, e.g., [39, Lemma 3.5], [10, Chapter I])

$$E = H = 0 \text{ in } \mathbb{R}^3 \setminus D.$$

We have a contradiction with the fact $\|(E_n, H_n)\|_{L^2(B_{R_0+1} \setminus D)} = 1$.

We next consider the case $k_* > 0$. In this case, we have (E,H) satisfies the radiating condition and

$$\begin{cases} \nabla \times E = ik_* H & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H = -ik_* E & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ E \times \nu = H \times \nu = 0 & \text{on } \partial D. \end{cases}$$

One also reaches $(E,H) = (0,0)$ in $\mathbb{R}^3 \setminus D$ and obtains a contradiction. □

In the same spirit, we have

Lemma 2.5. *Let $0 < \rho < \rho_0$, $0 < \omega < \omega_0$, $1/2 < r < 1$, and $R_0 > 2$. Let $h = (h_1, h_2) \in [H^{-1/2}(\operatorname{div}_{\partial B_1}, \partial B_1)]^2$. Assume that $(E,H) \in [L^2_{\rm loc}(\mathbb{R}^3 \setminus B_r)]^6$ with $(E,H) \in [\cap_{R>1} H(\operatorname{curl}, (B_R \setminus B_r) \setminus \partial B_1)]^2$ is a radiating solution of*

$$\begin{cases} \nabla \times E = i\omega \tilde{\mu}_\rho H & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ \nabla \times H = -i\omega \tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ [E \times \nu] = h_1, [H \times \nu] = h_2 & \text{on } \partial B_1. \end{cases}$$

We have, for $R > 2$,

$$\|(E,H)\|_{L^2(B_R \setminus B_r)} \le C_R \Big(\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1, h_2)\|_{H^{-1/2}(\partial B_1)} + \omega^{-1} \|(\operatorname{div}_{\partial B_1} h_1, \operatorname{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big),$$

for some positive constant C_R independent of (h_1,h_2), (f_1,f_2), ρ, and ω.

Proof. As argued in the proof of Lemma 2.4, by Stratton-Chu's formulas, it suffices to prove

$$\|(E,H)\|_{L^2(B_2 \setminus B_r)} \le C_R \Big(\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1, h_2)\|_{H^{-1/2}(\partial B_r)} + \omega^{-1} \|(\operatorname{div}_{\partial B_1} h_1, \operatorname{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big),$$

by contradiction. Assume that there exist sequences $(\omega_n) \subset (0, \omega_0)$, $\big((h_{1,n}, h_{2,n})\big) \subset [H^{-1/2}(\operatorname{div}_\Gamma, \partial B_1)]^2$, $\big((f_{1,n}, f_{2,n})\big) \subset L^2(\mathbb{R}^3 \setminus B_r)$ with support in $B_1 \setminus B_r$, and $\big((E_n, H_n)\big) \subset [\cap_{R>1} H(\operatorname{curl}, B_R \setminus D)]^2$ such that

$$\|(E_n, H_n)\|_{L^2(B_2 \setminus B_r)} = 1, \tag{2.26}$$

$$\lim_{n \to +\infty} \Big(\|(E_n \times \nu, H_n \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_{1,n}, h_{2,n})\|_{H^{-1/2}(\partial B_r)} + \omega_n^{-1} \|(\operatorname{div}_{\partial B_1} h_{1,n}, \operatorname{div}_{\partial B_1} h_{2,n})\|_{H^{-1/2}(\partial B_1)} \Big) = 0, \tag{2.27}$$

and

$$\begin{cases} \nabla \times E_n = i\omega_n \tilde{\mu}_{\rho_n} H_n & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ \nabla \times H_n = -i\omega_n \tilde{\varepsilon}_{\rho_n} E_n + \tilde{\sigma}_{\rho_n} E_n & \text{in } (\mathbb{R}^3 \setminus \bar{B}_r) \setminus \partial B_1, \\ [E_n \times \nu] = h_{1,n}, [H_n \times \nu] = h_{2,n} & \text{on } \partial B_1. \end{cases}$$

Without loss of generality, one might assume that $\omega_n \to \omega_*$ and $\rho_n \to \rho_*$ as $n \to +\infty$. We first consider the case $\rho_* = 0$. Since, as $n \to +\infty$,

$$(-i\omega_n + 1) E_n \cdot \nu|_{int} = -i\omega_n \rho_n E_n \cdot \nu|_{ext} - \operatorname{div}_{\partial B_1} h_{2,n} \to 0 \text{ in } H^{-1/2}(\partial B_1)$$

and

$$H_n \cdot \nu|_{int} = \rho_n H_n \cdot \nu|_{ext} - (i\omega_n)^{-1} \operatorname{div}_{\partial B_1} h_{1,n} \to 0 \text{ in } H^{-1/2}(\partial B_1),$$

using (2.27) and applying Lemma 2.1, one can assume that (E_n, H_n) converges to (E, H) in $L^2(B_1 \setminus B_r)$. Moreover,

$$\begin{cases} \operatorname{div} E = \operatorname{div} H = 0 & \text{in } B_1 \setminus \bar{B}_r, \\ E \times \nu = H \times \nu = 0 & \text{on } \partial B_r, \\ E \cdot \nu = H \cdot \nu = 0 & \text{on } \partial B_1. \end{cases}$$

It follows that $(E, H) = (0, 0)$ in $B_1 \setminus B_r$. We derive that

$$\lim_{n \to +\infty} \|(E_n, H_n)\|_{L^2(B_1 \setminus B_r)} = 0 \tag{2.28}$$

and, by [15, Lemma A1],

$$\lim_{n \to +\infty} \|(E_n \times \nu, H_n \times \nu)|_{\text{int}}\|_{H^{-1/2}(\partial B_1)} = 0.$$

This yields

$$\lim_{n \to +\infty} \|(E_n \times \nu, H_n \times \nu)|_{\text{ext}}\|_{H^{-1/2}(\partial B_1)} = 0. \tag{2.29}$$

This in turn implies, by Lemma 2.4, that

$$\lim_{n \to +\infty} \|(E_n, H_n)\|_{L^2(B_2 \setminus B_1)} = 0. \tag{2.30}$$

Combining (2.26), (2.28), and (2.30), we obtain a contradiction.

We next consider the case $\rho_* > 0$. The proof in this case is similar to the one in Lemma 2.4 and omitted (see also [31, Lemma 4] for the case $\omega_* > 0$). □

Remark 2.1. The proof gives the following slightly sharper estimate (for small ω):

$$\begin{aligned} \|(E, H)\|_{L^2(B_R \setminus B_r)} \le C_R \Big(&\|(E \times \nu, H \times \nu)\|_{H^{-1/2}(\partial B_r)} + \|(h_1, h_2)\|_{H^{-1/2}(\partial B_r)} \\ &+ \|(\omega^{-1} \operatorname{div}_{\partial B_1} h_1, \operatorname{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big). \end{aligned} \tag{2.31}$$

We are ready to give the main result of this section:

Lemma 2.6. *Let $0 < \rho < \rho_0$ and $0 < \omega < \omega_0$, and let $h_1, h_2 \in H^{-1/2}(\operatorname{div}_{\partial B_1}, \partial B_1)$. Let $(E_\rho, H_\rho) \in [\cap_{R>1} H(\operatorname{curl}, B_R \setminus \partial B_1)]^2$ be the unique radiating solution of*

$$\begin{cases} \nabla \times E = i\omega \tilde{\mu}_\rho H & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ \nabla \times H = -i\omega \tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3 \setminus B_1, \\ [E \times \nu] = h_1,\ [H \times \nu] = h_2 & \text{on } \partial B_1. \end{cases} \tag{2.32}$$

We have

$$\|(E, H)\|_{L^2(B_2 \setminus B_{2/3})} \le C \Big(\|(h_1, h_2)\|_{H^{-1/2}(\partial B_1)} + \omega^{-1} \|(\operatorname{div}_{\partial B_1} h_1, \operatorname{div}_{\partial B_1} h_2)\|_{H^{-1/2}(\partial B_1)} \Big),$$

for some positive constant C depending only on ρ_0 and ω_0.

Proof. Multiplying the first equation of (2.32) by $\tilde{\mu}_\rho^{-1}\nabla \times \bar{E}$ and integrating over $B_R \setminus \partial B_1$, we have, for $R > 1$,

$$\int_{B_R\setminus\partial B_1} \tilde{\mu}_\rho^{-1}\nabla \times E \cdot \nabla \times \bar{E}\, dx = i\omega \int_{B_R\setminus\partial B_1} H \cdot \nabla \times \bar{E}\, dx$$

$$= i\omega \int_{B_R\setminus\partial B_1} (-i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E) \cdot \bar{E}\, dx + i\omega \int_{\partial B_R} (H \times \nu) \cdot \bar{E}\, dx$$

$$- i\omega \int_{\partial B_1} (H \times \nu)|_{\text{ext}} \cdot \bar{E}|_{\text{ext}} - (H \times \nu)|_{\text{int}} \cdot \bar{E}|_{\text{int}}.$$

Using the definition of $\tilde{\sigma}_\rho$ and considering the imaginary part, we have

$$\int_{B_1\setminus B_{1/2}} |E|^2\, dx = \Re\left(\int_{\partial B_1} h_2 \cdot \bar{E}|_{\text{ext}}\, dx - \bar{h}_1 \cdot H|_{\text{int}}\, dx\right) - \Re\int_{\partial B_R} (H \times \nu) \cdot \bar{E}\, dx. \tag{2.33}$$

Letting $R \to \infty$ and using the radiation condition, we derive from (2.33) that

$$\begin{aligned}\int_{B_1\setminus B_{1/2}} |E|^2\, dx &\le \left|\int_{\partial B_1} h_2 \cdot \bar{E}|_{\text{ext}} - \bar{h}_1 H|_{\text{int}}\, ds\right| \\ &\le \left|\int_{\partial B_1} h_2 \cdot \bar{E}|_{\text{ext}}\right| + \left|\int_{\partial B_1} \bar{h}_1 \cdot H|_{\text{ext}}\, ds\right| + \left|\int_{\partial B_1} (\bar{h}_1 \times \nu) \cdot h_2\, ds\right|.\end{aligned} \tag{2.34}$$

Applying Lemma 2.2 with $D = B_2 \setminus B_1$, we have

$$\left|\int_{\partial B_1} h_2 \cdot \bar{E}|_{\text{ext}}\, ds\right| \le C\omega\|(E,H)\|_{L^2(B_2\setminus B_1)}\Big(\|h_2\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|\operatorname{div}_\Gamma h_2\|_{H^{-1/2}(\partial B_1)}\Big) \tag{2.35}$$

and

$$\left|\int_{\partial B_1} h_1 \cdot \bar{H}|_{\text{ext}}\, ds\right| \le C\omega\|(E,H)\|_{L^2(B_2\setminus B_1)}\Big(\|h_1\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|\operatorname{div}_\Gamma h_1\|_{H^{-1/2}(\partial B_1)}\Big). \tag{2.36}$$

Applying Lemma 2.3, we obtain

$$\left|\int_{\partial B_1} (\bar{h}_1 \times \nu) \cdot h_2\, ds\right| \le C\|(h_1,h_2)\|^2_{H^{-1/2}(\operatorname{div}_{\partial B_1}, \partial B_1)}. \tag{2.37}$$

Denote

$$M = \|(h_1,h_2)\|_{H^{-1/2}(\partial B_1)} + \omega^{-1}\|(\operatorname{div}_\Gamma h_1, \operatorname{div}_\Gamma h_2)\|_{H^{-1/2}(\partial B_1)}.$$

Combining (2.34), (2.35), (2.36) and (2.37) yields

$$\int_{B_1\setminus B_{1/2}} |E|^2\, dx \le C\Big(\omega M\|(E,H)\|_{L^2(B_2\setminus B_1)} + M^2\Big). \tag{2.38}$$

From the equations of (E,H) in $B_1 \setminus B_{1/2}$, we have

$$\Delta E + \omega^2 E - i\omega E = 0 \text{ in } B_1 \setminus B_{1/2}.$$

It follows from (2.38) that

$$\|E\|^2_{L^2(\partial B_{2/3})} + \|\nabla E\|^2_{L^2(\partial B_{2/3})} \le C\Big(\omega M\|(E,H)\|_{L^2(B_2\setminus B_1)} + M^2\Big), \tag{2.39}$$

which yields

$$\|(E,H)\|^2_{L^2(\partial B_{2/3})} \le C\Big(\omega^{-1} M\|(E,H)\|_{L^2(B_2\setminus B_1)} + \omega^{-2} M^2\Big). \tag{2.40}$$

Using (2.40) and applying Lemma 2.5 with $r = 2/3$, we derive that

$$\|(E,H)\|^2_{L^2(B_R\setminus B_{2/3})} \le C\Big(\omega^{-1}M\|(E,H)\|_{L^2(B_2\setminus B_1)} + \omega^{-2}M^2\Big),$$

and the conclusion follows. □

We end this subsection with

Lemma 2.7. *Let $0 < \rho < 1$ and $\rho\omega < k_0$, and let $D \subset B_1$ be a smooth, open subset of $\mathbb{R}^3$. Assume that $(E,H) \in [\cap_{R>2} H(\mathrm{curl}, B_R \setminus D)]^2$ is a radiating solution to the system*

$$\begin{cases} \nabla\times E = i\omega\rho H & \text{in } \mathbb{R}^3\setminus D,\\ \nabla\times H = -i\omega\rho E & \text{in } \mathbb{R}^3\setminus D. \end{cases}$$

We have, for $R \ge 1$ and $x \in B_{3R/\rho}\setminus B_{2R/\rho}$,

$$|E(x), H(x)| \le C_R\rho^3(\omega^2+1)\|(E,H)\|_{L^2(B_2\setminus D)}, \tag{2.41}$$

for some positive constant C depending only on k_0 and R.

Proof. We only prove (2.41) for E, the proof H is similar. By Stratton-Chu's formula, we have, for $x \in \mathbb{R}^3\setminus\bar{B}_1$,

$$E(x) = \int_{\partial B_1} \nabla_x G_k(x,y)\times\big(\nu(y)\times E(y)\big)dy \\ + i\omega\rho\int_{\partial B_1}\nu(y)\times H(y)G_k(x,y)dy - \int_{\partial B_1}\nu(y)\cdot E(y)\nabla_x G_k(x,y)dy, \tag{2.42}$$

where $k = \omega\rho$ and G_k is given in (2.20).

Let $(\widetilde{E},\widetilde{H}) \in [H(\mathrm{curl}, B_1)]^2$ be the unique solution to the system

$$\begin{cases} \nabla\times\widetilde{E} = i\omega\rho(1+i)\widetilde{H} & \text{in } B_1,\\ \nabla\times\widetilde{H} = -i\omega\rho(1+i)\widetilde{E} & \text{in } B_1,\\ \widetilde{E}\times\nu = E\times\nu & \text{on } \partial B_1. \end{cases} \tag{2.43}$$

By a contradictory argument, see, e.g., [39] (see also the proof of Lemma 2.6), we obtain

$$\|(\widetilde{E},\widetilde{H})\|_{L^2(B_1)} \le C\|E\times\nu_{\mathrm{ext}}, H\cdot\nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)}. \tag{2.44}$$

Since

$$\left|\int_{\partial B_1} E\times\nu\,ds\right| = \left|\int_{\partial B_1}\widetilde{E}\times\nu\,ds\right| = \left|\int_{B_1}\nabla\times\widetilde{E}\,dx\right| = \left|\int_{B_1}\omega\rho(1+i)\widetilde{H}dx\right|,$$

we obtain

$$\left|\int_{\partial B_1} E\times\nu\,ds\right| \le C\omega\rho\|(E,H)\|_{L^2(B_2\setminus D)}. \tag{2.45}$$

Similarly, we have

$$\left|\int_{\partial B_1} H\times\nu\,ds\right| \le C\omega\rho\|(E,H)\|_{L^2(B_2\setminus D)}. \tag{2.46}$$

One has

$$\int_{\partial B_1}\nu\cdot E\,ds = \frac{1}{i\omega\rho}\int_{\partial B_1}\nu\cdot\nabla\times H\,ds = 0. \tag{2.47}$$

Rewrite (2.42) under the form

$$
\begin{aligned}
&E(x) = \\
&\int_{\partial B_1} \nabla_x G_k(x,0) \times \big(\nu(y) \times E(y)\big) dy + \int_{\partial B_1} \big(\nabla_x G_k(x,y) - \nabla_x G_k(x,0)\big) \times \big(\nu(y) \times E(y)\big) dy \\
&+ ik \int_{\partial B_1} \nu(y) \times H(y) G_k(x,0) dy + ik \int_{\partial B_1} \nu(y) \times H(y) \big(G_k(x,y) - G_k(x,0)\big) dy \\
&- \int_{\partial B_1} \nu(y) \cdot E(y) \nabla_x G_k(x,0) dy - \int_{\partial B_1} \nu(y) \cdot E(y) \big(\nabla_x G_k(x,y) - \nabla_x G(x,0)\big) dy.
\end{aligned}
$$

Using the facts, for $|x| \in (2R/\rho, 3R/\rho)$ and $y \in \partial B_1$,

$$
|G_k(x,y) - G_k(x,0)| \le C(1+\omega)\rho^2, \quad |\nabla G_k(x,y) - \nabla G_k(x,0)| \le C(1+\omega^2)\rho^3,
$$

$$
\|E\|_{L^2(\partial B_1)} \le C\|E\|_{L^2(B_2 \setminus D)}, \quad \text{and} \quad \|H\|_{L^2(\partial B_1)} \le C\|H\|_{L^2(B_2 \setminus D)},
$$

we derive the conclusion from (2.45), (2.46), and (2.47). □

2.1.2. *Proof of Proposition 2.1.* Applying Lemma 2.6 to $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)$, defined in (2.11), we have

$$
\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2 \setminus B_1)} \le C\omega^{-1} \|\big(\mathbb{E}(\rho\, .), \mathbb{H}(\rho\, .)\big)\|_{L^2(\partial B_1)}. \tag{2.48}
$$

Since $\operatorname{div} \mathbb{J} = 0$, we have

$$
\Delta \mathbb{E} + \omega^2 \mathbb{E} = -i\omega \mathbb{J} \text{ in } \mathbb{R}^3.
$$

It follows that, for $x \in B_2$,

$$
\mathbb{E}(x) = -i\omega \int_{\mathbb{R}^3} \mathbb{J}(y) G_\omega(x,y)\, dy \quad \text{and} \quad \mathbb{H}(x) = -\nabla_x \times \int_{\mathbb{R}^3} \mathbb{J}(y) G_\omega(x,y)\, dy. \tag{2.49}
$$

This yields

$$
\|\big(\mathbb{E}(\rho.), \mathbb{H}(\rho.)\big)\|_{L^\infty(\partial B_1)} \le C\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.50}
$$

From (2.48) and (2.50), we obtain

$$
\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2 \setminus B_1)} \le C\omega^{-1}\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.51}
$$

Applying Lemma 2.7 to $(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)$, we have, for $x \in B_{3r/\rho} \setminus B_{2r/\rho}$,

$$
\Big|\big(\widetilde{\mathbf{E}}_\rho(x), \widetilde{\mathbf{H}}_\rho(x)\big)\Big| \le C_r \omega^{-1} \rho^3 \|\mathbb{J}\|_{L^2(\mathbb{R}^3)} \text{ for } r > 1/2,
$$

Since $(\mathbb{E}_\rho, \mathbb{H}_\rho) - (\mathbb{E}, \mathbb{H}) = (\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)(\rho^{-1}\, \cdot)$ in $\mathbb{R}^3 \setminus B_2$, the conclusion follows. □

2.2. **Moderate and high frequency analysis - Proof of Proposition 2.2.** This section contains two subsections. In the first, we present several lemmas used in the proof of Proposition 2.2 and in the second, the proof of Proposition 2.2 is given.

2.2.1. *Some useful lemmas.* The main result of this subsection is Lemma 2.9 which is analogous to Lemma 2.5 though for the moderate and high frequency regime. We begin with

Lemma 2.8. *Let $\omega > \omega_0$, and let Ω be a* **convex***, bounded subset of $\mathbb{R}^3$ of class C^1. Let $j \in H(\operatorname{div}, \Omega)$, and let $u \in H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$ be such that*

$$\nabla \times (\nabla \times u) - \omega^2 u = j \text{ in } \Omega, \tag{2.52}$$

and $u \cdot \nu$, $(\nabla \times u) \cdot \nu \in L^2(\partial\Omega)$. Then

$$\begin{aligned} &\|(\omega u \times \nu, (\nabla \times u) \times \nu)\|_{L^2(\partial\Omega)} \\ &\quad \le C\Big(\|(\omega u, \nabla \times u)\|_{L^2(\Omega)} + \|(\omega u \cdot \nu, (\nabla \times u) \cdot \nu)\|_{L^2(\partial\Omega)} + \|j\|_{L^2(\Omega)} + \omega^{-1}\|\operatorname{div} j\|_{L^2(\Omega)}\Big), \end{aligned} \tag{2.53}$$

for some positive constant C depending only on Ω and ω_0.

Proof. The analysis is based on the multiplier technique. We first consider $\operatorname{div} j = 0$. Multiplying (2.52) by $(\nabla \times \bar{u}) \times x$ and integrating over Ω, we obtain

$$\int_\Omega j \cdot (\nabla \times \bar{u}) \times x \, dx = \int_\Omega \nabla \times (\nabla \times u) \cdot (\nabla \times \bar{u}) \times x \, dx - \omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx. \tag{2.54}$$

Set

$$I_1 := -\omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx \quad \text{ and } \quad I_2 := \int_\Omega \nabla \times (\nabla \times u) \cdot (\nabla \times \bar{u}) \times x \, dx.$$

We have

$$\begin{aligned} I_1 &= -\omega^2 \int_\Omega u \cdot (\nabla \times \bar{u}) \times x \, dx = \omega^2 \int_\Omega (\nabla \times \bar{u}) \cdot (u \times x) \, dx \\ &= \omega^2 \int_\Omega \bar{u} \cdot \nabla \times (u \times x) \, dx - \omega^2 \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \quad \text{(by integration by parts).} \end{aligned}$$

Recall that, for all $v \in [H^1(\Omega)]^3$,

$$\nabla \times (v \times x) = -x \times (\nabla \times v) + v + \nabla(v \cdot x) - x \operatorname{div} v \quad \text{ in } \Omega. \tag{2.55}$$

Using (2.55) and the fact $\operatorname{div} u = \operatorname{div} j = 0$ in Ω, we derive that

$$\begin{aligned} I_1 &= -\omega^2 \int_\Omega \bar{u} \cdot \big[x \times (\nabla \times u)\big] \, dx + \omega^2 \int_\Omega |u|^2 \, dx \\ &\qquad + \omega^2 \int_\Omega \bar{u} \cdot \nabla(u \cdot x) \, dx - \omega^2 \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds \\ &= -\overline{I_1} + \omega^2 \left[\int_\Omega |u|^2 \, dx + \int_{\partial\Omega} (\bar{u} \cdot \nu)(u \cdot x) \, ds - \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds\right]. \end{aligned}$$

This implies

$$\Re I_1 = \frac{\omega^2}{2} \left(\int_\Omega |u|^2 \, dx + \int_{\partial\Omega} (\bar{u} \cdot \nu)(u \cdot x) \, ds - \int_{\partial\Omega} (\bar{u} \times \nu) \cdot (u \times x) \, ds\right). \tag{2.56}$$

Similarly, we have

$$\Re I_2 = \frac{1}{2} \left(\int_\Omega |\nabla \times u|^2 \, dx + \int_{\partial\Omega} (\nabla \times \bar{u} \cdot \nu)(\nabla \times u \cdot x) \, ds - \int_{\partial\Omega} \big((\nabla \times \bar{u}) \times \nu\big) \cdot \big((\nabla \times u) \times x\big) \, ds\right). \tag{2.57}$$

Combining (2.54), (2.56), and (2.57) yields

$$\int_\Omega \omega^2|u|^2 + |\nabla \times u|^2\,dx - \int_{\partial\Omega} \omega^2(\bar{u}\times\nu)\cdot(u\times x) + ((\nabla\times\bar{u})\times\nu)\cdot((\nabla\times u)\times x)\,ds \\ + \int_{\partial\Omega} \omega^2(\bar{u}\cdot\nu)(u\cdot x) + (\nabla\times\bar{u}\cdot\nu)(\nabla\times u\cdot x)\,ds = 2\Re\left\{\int_\Omega j\cdot(\nabla\times\bar{u})\times x\,dx\right\}. \tag{2.58}$$

This implies (2.53) in the case where $\operatorname{div} j = 0$ in Ω.

We next consider an arbitrary $\operatorname{div} j$. Let $\phi \in H^1_0(\Omega)$ be the unique solution of

$$\Delta\phi = \operatorname{div} j \quad \text{ in } \Omega.$$

It is clear that

$$\|\phi\|_{H^1(\Omega)} \le C\|j\|_{L^2(\Omega)} \tag{2.59}$$

and

$$\|\nabla\phi\times\nu\|_{L^2(\partial\Omega)} \le C\|\phi\|_{H^2(\Omega)} \le C\|\operatorname{div} j\|_{L^2(\Omega)}, \tag{2.60}$$

for some positive constant C depending only on Ω. Set

$$\tilde{u} = u - \omega^{-2}\nabla\phi \text{ in } \Omega. \tag{2.61}$$

We have

$$\nabla\times\nabla\times\tilde{u} - \omega^2\tilde{u} = j - \nabla\phi \text{ in } \Omega.$$

Since $\operatorname{div}(j - \nabla\phi) = 0$ in Ω, applying the previous case to $\tilde{u}$, we obtain the conclusions from (2.59), (2.60), and (2.61). □

As a consequence of Lemma 2.8, one has

Corollary 2.1. *Let $\omega > \omega_0$. Let $j \in H(\operatorname{div}, B_1 \setminus B_{3/4})$, and let $(E, H) \in [H(\operatorname{curl}, B_1 \setminus B_{3/4})]^2$ be such that $E\cdot\nu, H\cdot\nu \in [L^2(\partial B_1)]^3$. Assume that*

$$\begin{cases} \nabla\times E = i\omega H & \text{in } B_1\setminus B_{3/4}, \\ \nabla\times H = -i\omega E + j & \text{in } B_1\setminus B_{3/4}, \end{cases} \quad \text{and} \quad \operatorname{div} j = 0 \text{ in } B_1\setminus B_{3/4}.$$

We have

$$\|(E\times\nu, H\times\nu)\|_{L^2(\partial B_1)} \le C\Big(\|(E,H)\|_{L^2(B_1\setminus B_{3/4})} + \|(E\cdot\nu, H\cdot\nu)\|_{L^2(\partial B_1)} + \|j\|_{L^2(B_1\setminus B_{3/4})}\Big),$$

for some positive constant C depending only on ω_0.

Proof. Let $0 \le \phi \le 1$ be a smooth function in B_1 such that $\phi(x) = 0$ in $B_{4/5}$ and $\phi(x) = 1$ in $B_1 \setminus B_{5/6}$. Extend u and j by 0 in $B_{3/4}$, and set $u = \phi E$ in B_1. Then

$$\nabla\times\nabla\times u - \omega^2 u = i\omega\phi j + \nabla\times(\nabla\phi\times E) + \nabla\phi\times(\nabla\times E) \text{ in } B_1. \tag{2.62}$$

Since $\Delta E + \omega^2 E = i\omega j$ in $B_1 \setminus B_{3/4}$, we have

$$\|\nabla E\|_{L^2(B_{5/6}\setminus B_{4/5})} \le C\omega\Big(\|E\|_{L^2(B_1\setminus B_{3/4})} + \|j\|_{L^2(B_1\setminus B_{3/4})}\Big). \tag{2.63}$$

Applying Lemma 2.8 and using (2.62) and (2.63), one obtains the conclusion. □

The main result of this section is the following lemma, which is a variant of Lemma 2.6 in the case where $\omega_0 < \omega < \omega_1\rho^{-1}$.

Lemma 2.9. *Let $0 < \rho < \rho_0$ and $0 < \omega_0 < \omega < \omega_1/\rho$. Suppose that $h_1, h_2 \in L^2(\operatorname{div}_\Gamma, \partial B_1)$, and let $(E, H) \in [\cap_{R>1} H(\operatorname{curl}, B_R \setminus \partial B_1)]^2$ be the unique radiating solution to the system*

$$\begin{cases} \nabla \times E = i\omega\tilde{\mu}_\rho H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3, \\ [E \times \nu] = h_1, \; [H \times \nu] = h_2 & \text{on } \partial B_1. \end{cases}$$

We have, if ρ_0 is small enough and ω_0 is large enough, that

$$\|(E \times \nu, H \times \nu)_{\rm int}\|_{L^2(\partial B_1)} \le C\Big(\|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega^{-1}\|(\operatorname{div}_{\partial B_1} h_1, \operatorname{div}_{\partial B_1} h_2)\|_{L^2(\partial B_1)}\Big),$$

for some positive constant C depending only on ω_0, ω_1, and ρ_0.

Proof. Applying Corollary 2.1, we have

$$\|(E \times \nu|_{\rm int}, H \times \nu|_{\rm int})\|_{L^2(\partial B_1)} \le C\Big(\|(E, H)\|_{L^2(B_1 \setminus B_{3/4})} + \|(E\cdot\nu, H\cdot\nu)|_{\rm int}\|_{L^2(\partial B_1)}\Big). \tag{2.64}$$

One has, see, e.g., [8],

$$\|(E\cdot\nu, H\cdot\nu)|_{\rm ext}\|_{L^2(\partial B_1)} \le C\Big(\|(E\times\nu, H\times\nu)|_{\rm ext}\|_{L^2(\partial B_1)} + \|(E, H)\|_{L^2(B_2\setminus B_1)} + \|(E, H)\|_{L^2(\partial B_2)}\Big).$$

Applying Lemma 2.4 for (E, H) in $\mathbb{R}^3 \setminus B_1$, we obtain

$$\|(E, H)\|_{L^2(B_2\setminus B_1)} + \|(E, H)\|_{L^2(\partial B_2)} \le C\|(E \times \nu, H \times \nu)|_{\rm ext}\|_{L^2(\partial B_1)}.$$

It follows that

$$\|(E\cdot\nu, H\cdot\nu)|_{\rm ext}\|_{L^2(\partial B_1)} \le C\|(E \times \nu, H \times \nu)|_{\rm ext}\|_{L^2(\partial B_1)}. \tag{2.65}$$

Since

$$\big(1 - (i\omega)^{-1}\big)E\cdot\nu|_{\rm int} = \rho E\cdot\nu|_{\rm ext} + \frac{1}{i\omega}\operatorname{div}_{\partial B_1} h_2 \quad \text{and} \quad H\cdot\nu|_{\rm int} = \rho H\cdot\nu|_{\rm ext} - \frac{1}{i\omega}\operatorname{div}_{\partial B_1} h_1,$$

we derive from (2.65) that

$$\|(E\cdot\nu, H\cdot\nu)|_{\rm int}\|_{L^2(\partial B_1)} \le C\Big(\rho\|(E \times \nu, H \times \nu)|_{\rm ext}\|_{L^2(\partial B_1)} + \omega^{-1}\|\operatorname{div}_{\partial B_1}(h_1, h_2)\|_{L^2(\partial B_1)}\Big).$$

From the transmission conditions on ∂B_1, we deduce that

$$\begin{aligned} &\|(E\cdot\nu, H\cdot\nu)|_{\rm int}\|_{L^2(\partial B_1)} \\ &\quad \le C\Big(\rho\|(E \times \nu, H \times \nu)|_{\rm int}\|_{L^2(\partial B_1)} + \rho\|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega^{-1}\|\operatorname{div}_{\partial B_1}(h_1, h_2)\|_{L^2(\partial B_1)}\Big). \end{aligned} \tag{2.66}$$

On the other hand, as in (2.34), we have

$$\begin{aligned} \int_{B_1\setminus B_{1/2}} |E|^2\,dx &\le \left|\int_{\partial B_1} h_2\cdot\bar{E}|_{\rm ext} - \bar{h}_1 H|_{\rm int}\,ds\right| \\ &\le C\Big(\omega_0^2\|(h_1, h_2)\|^2_{L^2(\partial B_1)} + \omega_0^{-2}\|(E\times\nu, H\times\nu)|_{\rm ext}\|^2_{L^2(\partial B_1)}\Big). \end{aligned} \tag{2.67}$$

Since $\Delta E + \omega^2 E - i\omega E = 0$ in $B_1 \setminus B_{1/2}$, it follows that

$$\int_{B_{3/4}\setminus B_{2/3}} |E|^2 + \omega^{-2}|\nabla E|^2\,dx \le C\Big(\omega_0^2\|(h_1, h_2)\|^2_{L^2(\partial B_1)} + \omega_0^{-2}\|(E\times\nu, H\times\nu)|_{\rm ext}\|^2_{L^2(\partial B_1)}\Big). \tag{2.68}$$

An integration by parts yields, for $2/3 < r < 3/4$, that

$$\omega^2 \int_{B_1 \setminus B_r} |H|^2 \, dx - \omega^2 \int_{B_1 \setminus B_r} |E|^2 \, dx = \Re \Big\{ i\omega \int_{\partial B_1} \bar{E}|_{\rm int} (H \times \nu|_{\rm int}) \, ds - i\omega \int_{\partial B_r} \bar{E}|_{\rm int} (H \times \nu|_{\rm int}) \, ds \Big\}. \tag{2.69}$$

Combining (2.67), (2.68), and (2.69) yields

$$\|(E, H)\|_{L^2(B_1 \setminus B_{3/4})} \le C \Big(\omega_0 \|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega_0^{-1} \|(E \times \nu, H \times \nu)|_{\rm int}\|_{L^2(\partial B_1)} \Big). \tag{2.70}$$

From (2.64), (2.66), and (2.70), one obtains that, for ρ small enough,

$$\|(E \times \nu|_{\rm int}, H \times \nu|_{\rm int})\|_{L^2(\partial B_1)} \le C \Big(\omega_0 \|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega_0^{-1} \|(E \times \nu, H \times \nu)|_{\rm int}\|_{L^2(\partial B_1)} + \omega^{-1} \| \operatorname{div}_{\partial B_1} (h_1, h_2) \|_{L^2(B_1)} \Big).$$

This implies

$$\|(E \times \nu|_{\rm int}, H \times \nu|_{\rm int})\|_{L^2(\partial B_1)} \le C \Big(\|(h_1, h_2)\|_{L^2(\partial B_1)} + \omega^{-1} \| \operatorname{div}_{\partial B_1} (h_1, h_2) \|_{L^2(B_1)} \Big),$$

for ω_0 large enough and ρ small enough. □

2.2.2. *Proof of Proposition 2.2.* Since $\omega > \omega_0$ is large, by (2.49), one has

$$\|\mathbb{E}(\rho.), \mathbb{H}(\rho.)\|_{L^2(\partial B_1)} + \omega^{-1} \| \operatorname{div}_{\partial B_1} (\mathbb{E}(\rho.) \times \nu, \operatorname{div}_{\partial B_1} \mathbb{H}(\rho.) \times \nu) \|_{L^2(\partial B_1)} \le C \omega \|\mathbb{J}\|_{L^2(\mathbb{R}^3)}.$$

Applying Lemma 2.9, we obtain

$$\|(\widetilde{\mathbf{E}}_\rho, \widetilde{\mathbf{H}}_\rho)\|_{L^2(B_2 \setminus B_1)} \le C \omega \|\mathbb{J}\|_{L^2(\mathbb{R}^3)}.$$

The conclusion now follows from Lemma 2.7. □

2.3. **High and very high frequency analysis - Proof of Proposition 2.3.** This section contains two subsections. In the first, we present several lemmas used in the proof of Proposition 2.3 and in the second, the proof of Proposition 2.3 is given.

2.3.1. *Some useful lemmas.* We begin this section with a trace-type result for Maxwell's equations in a bounded domain. The analysis is based on a dual argument, see, e.g., [23, 7]). In this subsection, D denotes a smooth, bounded, open subset of $\mathbb{R}^3$.

Lemma 2.10. *Let $\omega > \omega_0 > 0$ and $f \in H(\operatorname{div}, D)$. Assume that $(E, H) \in [H(\operatorname{curl}, D)]^2$ satisfies the equations*

$$\begin{cases} \nabla \times E = i\omega H & \text{in } D, \\ \nabla \times H = -i\omega E + f & \text{in } D. \end{cases} \tag{2.71}$$

Then

$$\|E\|_{H^{-1/2}(\partial D)} + \omega \|H \times \nu\|_{H^{-3/2}(\partial D)} \le C \Big(\omega^2 \|E\|_{L^2(D)} + \omega \|f\|_{L^2(D)} + \omega^{-1} \| \operatorname{div} f\|_{L^2(D)} \Big),$$

for some positive constant C depending only on D and ω_0.

Remark 2.2. It is crucial to our analysis that the constant C is independent of ω.

Proof. We have, from (2.71),

$$\Delta E+\omega^2 E=\nabla(\operatorname{div} E)-\nabla\times(\nabla\times E)+\omega^2 E=\frac{1}{i\omega}\nabla(\operatorname{div} f)-i\omega f \text{ in } D. \tag{2.72}$$

Fix $\phi\in[H^{1/2}(\partial D)]^3$ (arbitrary). By the trace theory, see, e.g., [10, Theorem 1.6], there exists $\xi\in[H^2(D)]^3$ such that

$$\xi=0 \text{ on } \partial D, \quad \frac{\partial\xi}{\partial\nu}=\phi \text{ on } \partial D, \tag{2.73}$$

and

$$\|\xi\|_{H^2(D)}\le C\|\phi\|_{H^{1/2}(\partial D)}. \tag{2.74}$$

Here and in what follows, C denotes a positive constant depending only on D and ω_0. Multiplying (2.72) by ξ and integrating by parts, we obtain

$$\int_D(\Delta\xi+\omega^2\xi)E-\int_{\partial D}E\phi=\int_D(\Delta E+\omega^2 E)\xi=\int_D-\frac{1}{i\omega}\operatorname{div} f\operatorname{div}\xi-i\omega f\xi. \tag{2.75}$$

We derive from (2.73), (2.74), and (2.75) that

$$\left|\int_{\partial D}E\phi\,ds\right|\le C\Big(\omega^2\|E\|_{L^2(D)}+\omega\|f\|_{L^2(D)}+\omega^{-1}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\phi\|_{H^{1/2}(\partial D)},$$

which implies, since ϕ is arbitrary,

$$\|E\|_{H^{-1/2}(\partial D)}\le C\Big(\omega^2\|E\|_{L^2(D)}+\omega\|f\|_{L^2(D)}+\omega^{-1}\|\operatorname{div} f\|_{L^2(D)}\Big). \tag{2.76}$$

It remains to prove

$$\|H\times\nu\|_{H^{-3/2}(\partial D)}\le C\Big(\omega\|E\|_{L^2(D)}+\|f\|_{L^2(D)}+\omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big). \tag{2.77}$$

Fix $\varphi\in H^{3/2}(\partial D)$ (arbitrary), consider an extension of φ in D such that its $H^2(D)$-norm is bounded by $C\|\varphi\|_{H^{3/2}(\partial D)}$, and still denote this extension by φ. Such an extension exists by the trace theory, see, e.g., [10, Theorem 1.6]. We have

$$\int_{\partial D}H\times\nu\cdot\varphi\,ds=\int_D\Big(\nabla\times\varphi\cdot H-\nabla\times H\cdot\varphi\Big)\,dx. \tag{2.78}$$

Since

$$\begin{aligned}\left|\int_D\nabla\times\varphi\cdot H\,dx\right|&=\omega^{-1}\left|\int_D\nabla\times\varphi\cdot\nabla\times E\,dx\right|\\&=\omega^{-1}\left|\int_D\nabla\times(\nabla\times\varphi)\cdot E\,dx+\int_{\partial D}E\cdot[(\nabla\times\varphi)\times\nu)]\,ds\right|,\end{aligned}$$

and $\nabla\times H=i\omega E+f$, it follows from (2.76) that

$$\Big|\int_D\nabla\times\varphi\cdot H\,dx\Big|\le C\Big(\omega\|E\|_{L^2(D)}+\|f\|_{L^2(D)}+\omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)} \tag{2.79}$$

and

$$\Big|\int_D\nabla\times H\cdot\varphi\,dx\Big|\le C\Big(\omega\|E\|_{L^2(D)}+\|f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)}. \tag{2.80}$$

Combining (2.78), (2.79), and (2.80) yields

$$\left|\int_{\partial D}H\times\nu\cdot\varphi\,ds\right|\le C\Big(\omega\|E\|_{L^2(D)}+\|f\|_{L^2(D)}+\omega^{-2}\|\operatorname{div} f\|_{L^2(D)}\Big)\|\varphi\|_{H^{3/2}(\partial D)}.$$

Since φ is arbitrary, assertion (2.77) follows. The proof is complete. □

Using Lemma 2.10, we establish the following Lemma, which is the main result of this subsection.

Lemma 2.11. *Let $\omega > \omega_1 > 0$, $0 < \rho < 1$, and assume that $\omega\rho > \omega_1$. Given $h_1, h_2 \in H^{3/2}(\mathrm{div}_\Gamma, \partial B_1)$, let $(E, H) \in [\cap_{R>1} H(\mathrm{curl}, B_R \setminus \partial B_1)]^2$ be the unique radiating solution of*

$$\begin{cases} \nabla \times E = i\omega\tilde{\mu}_\rho H & \text{in } \mathbb{R}^3, \\ \nabla \times H = -i\omega\tilde{\varepsilon}_\rho E + \tilde{\sigma}_\rho E & \text{in } \mathbb{R}^3, \\ [E \times \nu] = h_1, [H \times \nu] = h_2 & \text{on } \partial B_1. \end{cases}$$

We have

$$\|E \times \nu|_{\mathrm{ext}}\|_{H^{-1/2}(\partial B_1)} + \omega\|H \times \nu|_{\mathrm{ext}}\|_{H^{-3/2}(\partial B_1)} \le C\Big(\omega^4\|h_2\|_{H^{1/2}(\partial B_1)} + \omega^3\|h_1\|_{H^{3/2}(\partial B_1)}\Big),$$

for some positive constant C depending only on ω_1.

Proof. As in (2.34), we have

$$\int_{B_1 \setminus B_{1/2}} |E|^2\, dx \le \left| \int_{\partial B_1} h_2 \cdot \bar{E}|_{\mathrm{ext}} - \bar{h}_1 H|_{\mathrm{int}}\, ds \right|.$$

This implies

$$\int_{B_1 \setminus B_{1/2}} |E|^2\, dx \le \|h_2\|_{H^{1/2}(\partial B_1)} \|E|_{\mathrm{int}}\|_{H^{-1/2}(\partial B_1)} + \|h_1\|_{H^{3/2}(\partial B_1)} \|H \times \nu|_{\mathrm{int}}\|_{H^{-3/2}(\partial B_1)} + \|h_2\|^2_{L^2(\partial B_1)}. \tag{2.81}$$

Applying Lemma 2.10 to (E, H) with $f = E$ in $B_1 \setminus B_{1/2}$, we have

$$\|E|_{\mathrm{int}}\|_{H^{-1/2}(\partial B_1)} + \omega\|H \times \nu\|_{H^{-3/2}(\partial B_1)} \le C\omega^2 \|E\|_{L^2(B_1 \setminus B_{1/2})}.$$

It follows from (2.81) that

$$\|E\|_{L^2(B_1 \setminus B_{1/2})} \le C\Big(\omega^2 \|h_2\|_{H^{1/2}(\partial B_1)} + \omega\|h_1\|_{H^{3/2}(\partial B_1)}\Big).$$

Applying Lemma 2.10 to (E, H) with $f = E$ in $B_1 \setminus B_{1/2}$ again, one has

$$\|E \times \nu|_{\mathrm{int}}\|_{H^{-1/2}(\partial B_1)} + \omega\|H \times \nu|_{\mathrm{int}}\|_{H^{-3/2}(\partial B_1)} \le C\Big(\omega^4\|h_2\|_{H^{1/2}(\partial B_1)} + \omega^3\|h_1\|_{H^{3/2}(\partial B_1)}\Big).$$

Using the transmission condition at ∂B_1, one reaches the conclusion. $\square$

We end this subsection by a simple consequence of Stratton-Chu's formula.

Lemma 2.12. *Let $0 < \rho < 1$, $\omega > \omega_1 > 0$ be such that $\omega\rho > \omega_1$, and let $D \subset B_1$. Assume that $(E, H) \in \big[H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus D)\big]^2$ is a radiating solution to the Maxwell equations*

$$\begin{cases} \nabla \times E = i\omega\rho H & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ \nabla \times H = -i\omega\rho E & \text{in } \mathbb{R}^3 \setminus \bar{D}. \end{cases}$$

We have

$$|E(x)| \le \frac{C|\omega\rho|^{3/2}}{|x|}\|E \times \nu\|_{H^{-1/2}(\partial D)} + \frac{C|\omega\rho|^{5/2}}{|x|}\|H \times \nu\|_{H^{-3/2}(\partial D)} \text{ for } x \in B_{3/\rho} \setminus B_{1/\rho},$$

for some positive constant C independent of x, ω, and ρ.

2.3.2. *Proof of Proposition 2.3.* Apply Lemma 2.11, we have

$$\|\widetilde{\mathbf{E}}_\rho \times \nu\|_{H^{-1/2}(B_2\setminus B_1)} + \omega\|\widetilde{\mathbf{H}}_\rho \times \nu\|_{H^{-3/2}(B_2\setminus B_1)} \leq C\omega^3\|\mathbb{E}(\rho\,\cdot) \times \nu\|_{H^{3/2}(\partial B_1)} + C\omega^4\|\mathbb{H}(\rho\,\cdot) \times \nu\|_{H^{1/2}(\partial B_1)}. \tag{2.82}$$

Since $\omega > \omega_0$, which is large, by (2.49), one has

$$\omega^3\|\mathbb{E}(\rho\,\cdot) \times \nu\|_{H^{3/2}(\partial B_1)} + \omega^4\|\mathbb{H}(\rho\,\cdot) \times \nu\|_{H^{1/2}(\partial B_1)} \leq C\omega^6\rho^{1/2}\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}. \tag{2.83}$$

Applying Lemma 2.12, we derive from (2.82) and (2.83) that

$$\|\widetilde{\mathbf{E}}_\rho\|_{L^2(B_3\setminus B_{1/2})} \leq C\omega^{15/2}\rho^3\|\mathbb{J}\|_{L^2(\mathbb{R}^3)},$$

which yields

$$\|\widetilde{\mathbf{H}}_\rho\|_{L^2(B_2\setminus B_1)} \leq C\omega^{17/2}\rho^3\|\mathbb{J}\|_{L^2(\mathbb{R}^3)}.$$

The proof is complete. □

3. Proof of Theorem 1.1

To implement the analysis in the frequency domain, let us introduce the notation for the Fourier transform with respect to t:

$$\hat{u}(\omega, x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(t,x)e^{i\omega t}\,dt, \tag{3.1}$$

for an appropriate function $u \in L^\infty_{\text{loc}}([0,+\infty), L^2(\mathbb{R}^3))$; here we extend u by 0 for $t < 0$.

The starting point of the frequency analysis is based on the following result:

Proposition 3.1. *Let $f_e, f_m \in L^2\big([0,\infty); [L^2(\mathbb{R}^3)]^3\big) \cap L^1\big([0,\infty); [L^2(\mathbb{R}^3)]^3\big)$. Let $(\mathcal{E}, \mathcal{H}) \in L^\infty_{\text{loc}}\big([0,+\infty), [L^2(\mathbb{R}^3)]^6\big)$ be the unique weak solution of (1.10). Assume that there exists $R_0 > 0$ such that $\operatorname{supp} f_e(t,\cdot)$, $\operatorname{supp} f_m(t,\cdot)$, $\operatorname{supp}\sigma_e$, $\operatorname{supp}\sigma_m \subset B_{R_0}$ for $t > 0$. Then, for almost every $\omega > 0$, $(\hat{\mathcal{E}}, \hat{\mathcal{H}})(\omega, .) \in [H_{\text{loc}}(\text{curl}, \mathbb{R}^3)]^2$ is the unique,* **radiating** *solution to the system*

$$\begin{cases} \nabla \times \hat{\mathcal{E}}(\omega,.) = i\omega\mu\hat{\mathcal{H}}(\omega,.) - \sigma_m\hat{\mathcal{H}}(\omega,\cdot) + \hat{f}_e(\omega,\cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{\mathcal{H}}(\omega,.) = -i\omega\varepsilon\hat{\mathcal{E}}(\omega,.) + \sigma_e\hat{\mathcal{E}}(\omega,.) - \hat{f}_m(\omega,.) & \text{in } \mathbb{R}^3. \end{cases} \tag{3.2}$$

Proof. Let $(\mathcal{E}_\delta, \mathcal{H}_\delta) \in L^\infty_{\text{loc}}\big([0,\infty), [L^2(\mathbb{R}^3)]^6\big)$ be the unique weak solution to

$$\begin{cases} \varepsilon\dfrac{\partial\mathcal{E}_\delta}{\partial t} = \nabla \times \mathcal{H}_\delta - \sigma_e\mathcal{E}_\delta - \delta\mathcal{E}_\delta + f_m & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \mu\dfrac{\partial\mathcal{H}_\delta}{\partial t} = -\nabla \times \mathcal{E}_\delta - \sigma_m\mathcal{H}_\delta - \delta\mathcal{H}_\delta + f_e & \text{in } (0,+\infty) \times \mathbb{R}^3, \\ \mathcal{E}_\delta(0,) = 0; \mathcal{H}_\delta(0,) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

By the standard Galerkin approach (see e.g., [34]), one can prove that

$$\delta\int_0^{+\infty}\int_{\mathbb{R}^3} |\mathcal{E}_\delta(s,x)|^2 + |\mathcal{H}_\delta(s,x)|^2\,dx\,ds \leq C\|(f_e, f_m)\|^2_{L^2(\mathbb{R}_+,\mathbb{R}^3)},$$

for some positive constant independent of δ and (f_e, f_m). Hence $\mathcal{E}_\delta, \mathcal{H}_\delta \in L^2\big((0,\infty); [L^2(\mathbb{R}^3)]^3\big)$, and thus $\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta \in L^2\big((0,\infty); [L^2(\mathbb{R}^3)]^3\big)$ by Parserval's theorem. It follows, for a.e. $\omega > 0$, that $(\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta) \in H(\text{curl}, \mathbb{R}^3)$ is the unique solution to

$$\begin{cases} \nabla \times \hat{\mathcal{E}}_\delta(\omega,.) = i\omega\mu\hat{\mathcal{H}}_\delta(\omega,.) - (\sigma_m + \delta)\hat{\mathcal{H}}_\delta(\omega,.) + \hat{f}_e(\omega,\cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \hat{\mathcal{H}}_\delta(\omega,.) = -i\omega\varepsilon\hat{\mathcal{E}}_\delta(\omega,.) + (\sigma_e + \delta)\hat{\mathcal{E}}_\delta(\omega,.) - \hat{f}_m(\omega,.) & \text{in } \mathbb{R}^3. \end{cases} \tag{3.3}$$

For $0 < \omega_1 < \omega < \omega_2 < \infty$, one can check that the solution of (3.3) satisfies

$$\|(\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta)(\omega, .)\|_{H(\mathrm{curl}, B_R)} \le C\|(\hat{f}_e, \hat{f}_m)(\omega, .)\|_{L^2(\mathbb{R}^3)} \le C\|(f_e, f_m)\|_{L^1((0,\infty), L^2(\mathbb{R}^3))}, \tag{3.4}$$

for some positive constant C depending only on ε, μ, R, ω_1, and ω_2. Letting $\delta \to 0$ and using the limiting absorption principle, see e.g., [31, (2.28) and the following paragraph], one derives that

$$(\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta)(\omega,) \rightharpoonup (\mathcal{E}_0, \mathcal{H}_0)(\omega, .) \text{ weakly in } [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2 \text{ as } \delta \to 0, \tag{3.5}$$

where $(\mathcal{E}_0, \mathcal{H}_0)(\omega, .) \in [H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)]^2$ is the unique, radiating solution to the system

$$\begin{cases} \nabla \times \mathcal{E}_0(\omega, .) = i\omega\mu\mathcal{H}_0(\omega, .) - \sigma_m \mathcal{H}_0 + \hat{f}_e(\omega, \cdot) & \text{in } \mathbb{R}^3, \\ \nabla \times \mathcal{H}_0(\omega, .) = -i\omega\varepsilon\mathcal{E}_0(\omega, .) + \sigma_e\mathcal{E}_0(\omega, .) - \hat{f}_m(\omega, .) & \text{in } \mathbb{R}^3. \end{cases}$$

From (3.4) and (3.5), we have

$$(\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta) \to (\mathcal{E}_0, \mathcal{H}_0) \text{ in the distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3 \text{ as } \delta \to 0. \tag{3.6}$$

We claim that

$$(\hat{\mathcal{E}}_\delta, \hat{\mathcal{H}}_\delta) \to (\hat{\mathcal{E}}, \hat{\mathcal{H}}) \text{ in the distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3, \tag{3.7}$$

and the conclusion follows from (3.6) and (3.7).

It remains to prove (3.7). Let $\phi \in \big[C_c^\infty\big((0,\infty) \times \mathbb{R}^3\big)\big]^3$. We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} (\hat{\mathcal{E}}_\delta(\omega, x) - \hat{\mathcal{E}}(\omega, x))\bar{\phi}(\omega, x)\, dx d\omega = \int_{\mathbb{R}}\int_{\mathbb{R}^3} (\mathcal{E}_\delta(t, x) - \mathcal{E}(t, x))\bar{\check{\phi}}(t, x)\, dx dt.$$

We derive that, by applying Proposition 1.1 to $(\mathcal{E}_\delta - \mathcal{E}, \mathcal{H}_\delta - \mathcal{H})$,

$$\|\mathcal{E}_\delta(t, .) - \mathcal{E}(t, .)\|_{L^2(\mathbb{R}^3)} \le C\delta \int_0^t \|(\mathcal{E}(s, .), \mathcal{H}(s, .))\|_{L^2(\mathbb{R}^3)}\, ds \text{ for } t > 0,$$

and, by applying Proposition 1.1 for $(\mathcal{E}, \mathcal{H})$,

$$\|(\mathcal{E}(t, .), \mathcal{H}(t, .))\|_{L^2(\mathbb{R}^3)} \le C\|(f_e, f_m)\|_{L^1((0,\infty), [L^2(\mathbb{R}^3)]^6)} \text{ for } t > 0.$$

It follows that

$$\|\mathcal{E}_\delta(t, .) - \mathcal{E}(t, .)\|_{L^2(\mathbb{R}^3)} \le C\delta t. \tag{3.8}$$

From (3.8), we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} (\mathcal{E}_\delta(t, x) - \mathcal{E}(t, x))\bar{\check{\phi}}(t, x)\, dx dt \le C\delta \int_{\mathbb{R}} t\|\check{\phi}(t, .)\|_{L^2(\mathbb{R}^3)}\, dt. \tag{3.9}$$

From (3.9) and the fast decay property of $\check{\phi}$, we derive that

$$\hat{\mathcal{E}}_\delta \to \hat{\mathcal{E}} \text{ in the distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3.$$

Similarly, one can prove that

$$\hat{\mathcal{H}}_\delta \to \hat{\mathcal{H}} \text{ in the distributional sense in } \mathbb{R}_+ \times \mathbb{R}^3.$$

The proof is complete. □

We are ready to give

Proof of Theorem 1.1. Fix $K \subset\subset \mathbb{R}^3 \setminus \bar{B}_1$ and $T > 0$. Using the fact that $\hat{\mathcal{E}}_c(-k, x) = \overline{\hat{\mathcal{E}}_c}(k, x)$ and $\hat{\mathcal{E}}(-k, x) = \bar{\hat{\mathcal{E}}}(k, x)$ for $k > 0$, one has, for $0 < t < T$,

$$\|\mathcal{E}_c(t, \cdot) - \mathcal{E}(t, \cdot)\|_{L^2(K)} \le \int_0^T \|\partial_t\mathcal{E}_c(t, \cdot) - \partial_t\mathcal{E}(t, \cdot)\|_{L^2(K)} \le T\int_0^\infty \omega\|\hat{\mathcal{E}}_c(\omega, \cdot) - \hat{\mathcal{E}}(\omega, \cdot)\|_{L^2(K)} d\omega. \tag{3.10}$$

We have, by Proposition 2.1,

$$\int_0^1 \omega \|\hat{\mathcal{E}}_c(\omega,.) - \hat{\mathcal{E}}(\omega,.)\|_{L^2(K)} d\omega \le C \int_0^1 \rho^3 \|\hat{\mathcal{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)} d\omega \le C\rho^3 \|\mathcal{J}\|^2_{L^2(\mathbb{R};L^2(\mathbb{R}^3))}, \tag{3.11}$$

by Proposition 2.2 (here to simplify the notations we assume that $\omega_0 = 1$),

$$\int_1^{1/\rho} \omega \|\hat{\mathcal{E}}_c(\omega,.) - \hat{\mathcal{E}}(\omega,.)\|_{L^2(K)} d\omega \le C\rho^3 \int_1^{1/\rho} \omega^4 \|\hat{\mathcal{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)} d\omega, \tag{3.12}$$

and, by Proposition 2.3,

$$\int_{1/\rho}^{+\infty} \omega \|\hat{\mathcal{E}}_c(\omega,.) - \hat{\mathcal{E}}(\omega,.)\|_{L^2(K)} d\omega \le C\rho^3 \int_{\frac{1}{\rho}}^{+\infty} \omega^{19/2} \|\hat{\mathcal{J}}(\omega,.)\|_{L^2(\mathbb{R}^3)} d\omega. \tag{3.13}$$

A combination of (3.12) and (3.13) yields

$$\begin{aligned} \int_1^{\infty} \omega \|\hat{\mathcal{E}}_c(\omega,.) - \hat{\mathcal{E}}(\omega,.)\|_{L^2(K)} d\omega &\le C\rho^3 \int_1^{+\infty} \frac{1}{\omega} \|\widehat{\partial_t^{(11)} \mathcal{J}}(\omega,\cdot)\|_{L^2(\mathbb{R}^3))}\, d\omega \\ &\le C\rho^3 \|\mathcal{J}\|_{H^{11}(\mathbb{R},L^2(\mathbb{R}^3))} \end{aligned} \tag{3.14}$$

We derive from (3.10), (3.11), and (3.14) that, for $0 < t < T$,

$$\|\mathcal{E}_c(t,\cdot) - \mathcal{E}(t,\cdot)\|_{L^2(K)} \le CT\rho^3 \|\mathcal{J}\|_{H^{11}(\mathbb{R},L^2(\mathbb{R}^3))}.$$

The proof is complete. □

References

[1] A. Alonso and A. Valli, *Some remarks on the characterization of the space of tangential traces of H(rot; ...) and the construction of an extension operator*, Manuscripta mathematica **89** (1996), 159–178.

[2] H. Ammari, H. Kang, H. Lee, M. Lim, *E*nhancement of near-cloaking. Part II: The Helmholtz equation, Comm. Math. Phys. **317** (2013), 485–502.

[3] H. Ammari, H. Kang, H. Lee, M. Lim, S. Yu, *Enhancement of Near Cloaking for the Full Maxwell Equations*, SIAM J. Appl. Math. **73** (2013), 2055–2076.

[4] H. Ammari, M. Vogelius, D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations*, J. Math. Pures Appl. **80** 2001, 769–814.

[5] J. Ball, Y. Capdeboscq, and B. Tsering-Xiao, *On uniqueness for time harmonic anisotropic Maxwell's equations with piecewise regular coefficients*, Math. Models Methods Appl. Sci. **22** (2012), 1250036.

[6] A. Buffa, M. Costabel, and D. Sheen, *On traces for $H(\mathrm{curl}, \Omega)$ in Lipschitz domains*, J. Math. Anal. Appl. **276** (2002), 845–867.

[7] Y. Capdeboscq, G. S. Alberti, *Lectures on Elliptic Methods for Hybrid Inverse Problems*, Cours spécialisés, collection SMF **25** (2018).

[8] M. Costabel, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, Math. Methods Appl. Sci. **12** (1990), 365–368.

[9] Y. Deng, H. Liu, and G. Uhlmann, *F*ull and partial cloaking in electromagnetic scattering, Arch. Ration. Mech. Anal. **223** (2017), 265–299.

[10] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations, theory and algorithms*, Springer-Verlag, Berlin, 1986.

[11] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Full-wave invisibility of active devices at all frequencies*, Comm. Math. Phys. **275** (2007), 749–789.

[12] A. Greenleaf, M. Lassas, and G. Uhlmann, *On nonuniqueness for Calderon's inverse problem*, Math. Res. Lett. **10** (2003), 685–693.

[13] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann, *Improvement of cylindrical cloaking with the SHS lining*, Opt. Exp. **15** (2007), 12717–12734.

[14] R. Griesmaier and M. S. Vogelius *Enhanced approximate cloaking by optimal change of variables*, Inverse Problems **30** (2014), 035014.

[15] H. Haddar, P. Joly, and H-M. Nguyen, *Generalized impedance boundary conditions for scattering problems from strongly absorbing obstacles: the case of Maxwell's equations*, Math. Models Methods Appl. Sci. **18** (2008), 1787 –1827.

[16] H. Heumann and M.S. Vogelius, *Analysis of an enhanced approximate cloaking scheme for the conductivity problem*, Asymptot. Anal. **87** (2014), 22–246.

[17] A. Kirsch and F. Hettlich, *The mathematical theory of time-harmonic Maxwell's equations, expansion, integral, and variational methods*, Springer, 2015.

[18] R. V. Kohn, D. Onofrei, M. S. Vogelius, and M. I. Weinstein, *Cloaking via change of variables for the Helmholtz equation*, Comm. Pure Appl. Math. **63** (2000), 973–1016.

[19] R. V. Kohn, H. Shen, M.S. Vogelius, and M. I. Weinstein, *Cloaking via change of variables in electric impedance tomography*, Inverse Problem **24** (2008), 015–016.

[20] Y. Lai, H. Chen, Z. Zhang, and C. T. Chan, *Complementary media invisibility cloak that cloaks objects at a distance outside the cloaking shell*, Phys. Rev. Lett. **102** (2009), 093901.

[21] M. Lassas and T. Zhou, *The blow-up of electromagnetic fields in 3-dimensional invisibility cloaking for Maxwell's equations*, SIAM J. Appl. Math. **76** (2016), 457–478.

[22] U. Leonhardt, *Optical conformal mapping*, Science **312** (2006), 1777–1780.

[23] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications* (3 volumes), Springer, 1972.

[24] G. W. Milton and N. A. Nicorovici, *On the cloaking effects associated with anomalous localized resonance*, Proc. R. Soc. Lond. Ser. A **462** (2006), 3027–3059.

[25] C. S. Morawetz and D. Ludwig, *An inequality for the reduced wave operator and the justification of geometrical optics*, Comm. Pure Appl. Math. **21** (1968), 187–203.

[26] H-M. Nguyen, *Cloaking via change of variables for the Helmholtz equation in the whole space*, Com. Pure Appl. Math. **63** (2010), 1505–1524.

[27] H-M. Nguyen, *Approximate cloaking for the Helmholtz equation via transformation optics and consequences for perfect cloaking*, Comm. Pure Appl. Math. **65** (2012), 155–186.

[28] H-M. Nguyen, *Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime*, J. Eur. Math. Soc. (JEMS) **17** (2015), 1327–1365.

[29] H-M. Nguyen, *Cloaking using complementary media in the quasistatic regime*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), 1509–1518.

[30] H-M. Nguyen, *Cloaking an arbitrary object via anomalous localized resonance: the cloak is independent of the object* SIAM J. Math. Anal. **49** (2017) 3208–3232.

[31] H-M. Nguyen, *Superlensing using complementary media and reflecting complementary media for electromagnetic waves*, Adv. Nonlinear Anal., https://doi.org/10.1515/anona-2017-0146.

[32] H-M. Nguyen and T. Nguyen, *Approximate cloaking for the heat equation via transformation optics*, submitted, 2018.

[33] H-M. Nguyen and M. Vogelius, *A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 2283–2315.

[34] H-M. Nguyen and M. S. Vogelius, *Full range scattering estimates and their application to cloaking*, M.S. Arch. Rational Mech. Anal. **203** (2012), 769–807.

[35] H-M. Nguyen and M. S. Vogelius, *Approximate cloaking for the wave equation via change of variables*, SIAM J. Math. Anal. **44** (2012), 1894–1924.

[36] H-M Nguyen and M.S. Vogelius, *Approximate cloaking for the full wave equation via change of variables: The Drude Lorentz model*, J. Math. Pures Appl. **106** (2016), 797-836.

[37] H.-M. Nguyen and V. Vinoles, *Electromagnetic wave propagation in media consisting of dispersive metamaterials*, C. R. Math. Acad. Sci. Paris **356** (2018), 757–775.

[38] H.-M. Nguyen and L. V. Nguyen. *Generalized impedance boundary conditions for strongly absorbing obstacle: The full wave equation*, Math. Models Methods Appl. Sci. **25** (2015), 1927-1960, .

[39] H.-M. Nguyen and L. Tran, *Approximate cloaking for electromagnetic waves via transformation optics: cloaking vs infinite energy*, submitted.

[40] T. Nguyen, J. N. Wang, *Quantitative uniqueness estimate for the Maxwell system with Lipschitz anisotropic media*, Proc. Am. Math. Soc. **140** (2012), 595–605.

[41] N. A. Nicorovici, R. C. McPhedran, and G. M. Milton, *Optical and dielectric properties of partially resonant composites*, Phys. Rev. B **49** (1994), 8479–8482.
[42] J. B. Pendry, D. Schurig, D. R. Smith *Controlling electromagnetic fields*, Science **321** (2006), 1780-1782.
[43] M. H. Protter, *Unique continuation principle for elliptic equations*, Trans. Am. Math. Soc. **95** (1960), 81–91.
[44] Z. Ruan, M. Yan, C. M. Neff, and M. Qiu, *Ideal cylindrical cloak: Perfect but sensitive to tiny perturbations*, Phys. Rev. Lett. **99** (2007), 113903.
[45] M. S. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter*, M2AN Math. Model. Numer. Anal. **34** (2000), 723–748.
[46] Ch. Weber, *A local compactness theorem for Maxwell's equations* Math. Methods Appl. Sci. **2** (1980), 12?25.
[47] R. Weder, *A rigorous analysis of high-order electromagnetic invisibility cloaks*, J. Phys. A: Math. Theor. **41** (2008), 065207.
[48] R. Weder, *The boundary conditions for point transformed electromagnetic invisibility cloaks*, J. Phys. A: Math. Theor. **41** (2008), 415401.

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