SIAM J. CONTROL OPTIM.
Vol. 58, No. 4, pp. 2078–2113

MEAN-FIELD LEADER-FOLLOWER GAMES WITH TERMINAL STATE CONSTRAINT*

GUANXING FU† AND ULRICH HORST‡

Abstract. We analyze linear McKean–Vlasov forward-backward SDEs arising in leader-follower games with mean-field type control and terminal state constraints on the state process. We establish an existence and uniqueness of solutions result for such systems in time-weighted spaces as well as a convergence result of the solutions with respect to certain perturbations of the drivers of both the forward and the backward component. The general results are used to solve a novel single player model of portfolio liquidation under market impact with expectations feedback as well as a novel Stackelberg game of optimal portfolio liquidation with asymmetrically informed players.

Key words. mean-field control, Stackelberg game, mean-field game with a major player, McKean–Vlasov FBSDE, portfolio liquidation, singular terminal constraint

AMS subject classifications. 93E20, 91B70, 60H30

DOI. 10.1137/19M1241878

1. Introduction and overview.

Mean field games (MFGs) are a powerful tool to analyze strategic interactions in large populations when each individual player has only a small impact on the behavior of other players. Introduced independently by Huang, Malhamé, and Caines [31] and Lasry and Lions [37], MFGs have received considerable attention in the probability and stochastic control literature in the last decade. A probabilistic approach to solving MFGs was introduced by Carmona and Delarue in [13]. Using a maximum principle of Pontryagin type, they showed that solving the MFG reduces to solving a McKean–Vlasov forward-backward SDE (FBSDE) of the form

$$
\begin{cases} dX_t = b(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt + \sigma\, dW_t, \\ -dY_t = h(t, X_t, Y_t, \mathcal{L}(X_t, Y_t))\, dt - Z_t\, dW_t, \\ X_0 = \chi,\ Y_T = l(X_T, \mathcal{L}(X_T)), \end{cases} \tag{1.1}
$$

where X is the state of the representative player, Y is the adjoint variable, and $\mathcal{L}(\cdot)$ denotes the law of a stochastic process. In MFGs with common noise [2, 3], the dependence of the coefficients on the law of the process (X, Y) is of conditional form. FBSDEs of the form (1.1) also arise in mean-field control (MFC) problems [1, 4, 14] and in MFGs with a major player [9, 10, 16] when formulating stochastic maximum principles. Different types of MFGs with a major player have been considered in the literature. Nash equilibria in games between many small players and a single major player have been analyzed in, e.g., [12, 15, 29, 30, 39]. Leader-follower ("Stackelberg") games of mean-field type between a major and many minor players have been studied

*Received by the editors January 30, 2019; accepted for publication (in revised form) April 3, 2020; published electronically July 27, 2020.
https://doi.org/10.1137/19M1241878
Funding: This work was financially supported by the TRCRC 190, Rationality and Competition: The Economic Performance of Individuals and Firms. The first author was supported by a Tier 2 grant, Nonstandard BSDEs in Mathematical Finance: Theory, Application, Numerical Methods.
†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (fuguanxing725@gmail.com).
‡Department of Mathematics, and School of Business and Economics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany (horst@math.hu-berlin.de).

in, e.g., [9, 10]. These games can be viewed as leader-follower games between the major player and a representative minor player in which the leader's (major player's) optimization problem is a MFC control problem where the state dynamics follows a controlled FBSDE that characterizes the follower's (minor player's) optimal response to the leader's control. A very different class of leader-follower games, namely principal-agent games, has been considered in, e.g., [19, 20, 21, 22]. In these models, optimal contracts can be characterized by (F)BSDE without mean-field terms. A mean-field principal-agent model has been studied in the recent work by [23].

1.1. McKean–Vlasov FBSDE with terminal state constraint. In this paper, we study a novel class of leader-follower games with asymmetrically informed players and terminal state constraint on the state processes in which both the leader and the follower solve mean-field control problems. Our games naturally arise in Stackelberg games of optimal portfolio liquidation.

Formulating a novel stochastic maximum principle we show that the analysis of the leader-follower game reduces to solving linear McKean–Vlasov FBSDEs of the form

$$
\left\{\begin{aligned} dQ_t &= \left(-\Lambda^1_t R_t - \Lambda^2_t \mathbb{E}\left[\gamma_t Q_t \middle|\, \mathcal{F}^0_t\right] + \overline{f}_t\right)\, dt,\\ -dR_t &= \left(\Lambda^4_t Q_t + \Lambda^3_t \mathbb{E}[\zeta_t R_t | \mathcal{F}^0_t] + \Lambda^5_t \mathbb{E}[\varrho_t Q_t|\mathcal{F}^0_t] + \overline{g}_t\right)\, dt - Z_t\, dW_t,\\ Q_0 &= \chi,\ Q_T = 0, \end{aligned}\right. \tag{1.2}
$$

with given initial and terminal conditions for the forward and unspecified terminal condition for the backward process. Here, $W = (\overline{W}, W^0)$ is a multidimensional Brownian motion generating the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t\geq 0}$ is the filtration generated by W^0. The special case $\Lambda^2 = \Lambda^3 = \Lambda^5 = \overline{f} = \overline{g} = 0$ arises in the single player portfolio liquidation models under market impact studied in, e.g., [5, 27]. The special case $\Lambda^2 = \Lambda^5 = \overline{f} = \overline{g} = 0$ was recently analyzed in [25] in the framework of a MFG of optimal portfolio liquidation. In such models, both the initial and the terminal condition of the state sequence are given. The terminal state constraint on the state process results in a singular terminal value of the value function and hence an unspecified terminal value of the adjoint equation arising in the stochastic maximum principle, which is usually given by the derivative of the terminal payoff. A class of stochastic optimal control problems with the terminal states being constrained to a convex set was studied by [33]. They assumed a strict invertibility of the diffusion term with respect to the control. This assumption is not satisfied in portfolio liquidation models where the state dynamic is degenerate.

We prove a general existence and uniqueness of solutions result for the system (1.2) under boundedness assumptions on the model parameters that allows us to solve single player portfolio liquidation problems with private information and expectations feedback. The existence and uniqueness result is complemented by a convergence result for the solution of (1.2) with respect to the parameters $(\overline{f}, \overline{g})$ that allows us to formulate a stochastic maximum principle for leader-follower games of portfolio liquidation with asymmetrically informed players.

The existence and uniqueness of solutions to (1.2) is obtained via two nested continuation arguments. Standard continuation methods for McKean–Vlasov FBSDEs established in, e.g., [3, 11] do not apply to the system (1.2), due to the unknown terminal value of the backward process. In order to overcome this problem we make a linear ansatz $R = AQ + H$, from which we derive an exogenous BSDE with singular terminal condition for the process A, and a BSDE with known asymptotic behavior at the terminal time for the process H. The driver of the latter BSDE depends on the unbounded process A. The nature of the FBSDE for (Q, H) is different from [25] where a similar ansatz gave a BSDE with known terminal condition. Analyzing simultaneously the triple (Q, H, R) allows us to prove the fixed-point condition arising in the application of the continuation method in a suitable space.

Our second main result is a convergence result for the solution (Q, R) to the system (1.2) with respect to the "input" $(\overline{f}, \overline{g})$. Our convergence is not in the L^2 sense as in the standard FBSDE literature [38, 42] but rather in the L^ν $(1 < \nu < 2)$ sense. Specifically, we consider the convergence of the solutions (Q^n, R^n) to a penalized version of (1.2) under a uniform L^2 boundedness assumption on the sequence $(\overline{f}^n, \overline{g}^n)$. For such inputs a result of Komlós [35] guarantees the Cesaro convergence of $(\overline{f}^n, \overline{g}^n)$ along a subsequence in L^ν $(1 < \nu < 2)$. We prove the convergence of the solutions in the same sense. To this end, we define auxiliary processes to decouple the system (1.2) and then show that these processes solve the system (1.2) in the right spaces. The convergence result then follows from the previously established uniqueness result.

1.2. Applications to optimal portfolio liquidation. Models of optimal portfolio liquidation have received substantial attention in the financial mathematics and stochastic control literature in recent years; see [5, 26, 27, 36, 41] among many others. In such models, the controlled state sequence typically follows a dynamic of the form

$$X_t = x - \int_0^t \xi_s \, ds,$$

where $x \in \mathbb{R}$ is the initial portfolio and ξ is the trading rate. The set of admissible controls is confined to those processes ξ that satisfy almost surely the liquidation constraint $X_T = 0$. It is typically assumed that the unaffected price process against which the trading costs are benchmarked follows some Brownian martingale S and that the trader's transaction price is given by

$$\widetilde{S}_t = S_t - \int_0^t \kappa_s \xi_s \, ds - \eta_t \xi_t.$$

The integral term accounts for permanent price impact; the term $\eta_t \xi_t$ accounts for instantaneous impact that does not affect future transactions. The trader's objective is then to minimize the cost functional

$$J(\xi) = \mathbb{E}\left[\int_0^T \left(\kappa_s \xi_s X_s + \eta_s |\xi_s|^2 + \lambda_s |X_s|^2\right) ds\right]$$

over all admissible liquidation strategies. We refer to [5, 27] for an interpretation of the processes η, κ, λ.

1.2.1. Single player model with expectations feedback. Standard portfolio liquidation models assume that a trader's permanent price impact is driven by his observable transactions. If the transactions are not directly observable, then it is natural to assume that the permanent impact is driven by the market's expectation about the trader's transactions as in [1, 6], given the publicly observable information.

In section 3 we solve a single player liquidation model with expectations feedback where uncertainty is generated by the multidimensional Brownian motion $W = (\overline{W}, W^0)$. The Brownian motion W^0 describes a commonly observed random factor that drives market dynamics; the Brownian motion $\overline{W}$ is private information to the trader. Specifically, we assume that the trader's transaction price is given by

$$\widetilde{S}_t = S_t - \int_0^t \left\{ \kappa_s \mathbb{E}[\xi_s|\mathcal{F}_s^0] + \widetilde{g}_s \right\} ds - \eta_t \xi_t, \tag{1.3}$$

where S is an $\mathbb{F}^0$ martingale, $\mathbb{E}[\xi_s|\mathcal{F}_s^0]$ is the market's expectation about the trader's strategy, and $\widetilde{g}$ is an $\mathbb{F}^0$-adapted process that will be endogenized in the next subsection. Assuming a standard quadratic running cost function as in [5, 27], the objective of the trader is then to minimize the functional

$$J(\xi) = \mathbb{E}\left[\int_0^T \kappa_t X_t \mathbb{E}[\xi_t|\mathcal{F}_t^0] + \widetilde{g}_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2 \, dt \right], \tag{1.4}$$

subject to the state dynamics

$$\begin{aligned} dX_t &= -\xi_t \, dt, \\ X_0 &= x, \quad X_T = 0. \end{aligned} \tag{1.5}$$

We allow the cost coefficients to be private information, i.e., to be $\mathbb{F}$ adapted. This justifies the conditional expectation term in the price dynamics. A standard stochastic maximum principle suggests that the optimal strategy is given by

$$\xi_t^* = \frac{Y_t - \mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]}{2\eta_t}, \tag{1.6}$$

where X is the portfolio process, Y is the adjoint variable, and (X, Y) solves (1.2) with $\overline{f} = 0$, $\overline{g} = \widetilde{g}$:

$$\begin{cases} dX_t = \; -\dfrac{Y_t - \mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]}{2\eta_t} \, dt, \\ -dY_t = \left(\kappa_t \mathbb{E}\left[\dfrac{Y_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] - \kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0] + 2\lambda_t X_t + \widetilde{g}_t \right) dt - Z_t \, dW_t, \\ X_0 = x, \; X_T = \; 0. \end{cases} \tag{1.7}$$

If the terms $\mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]$ and $\kappa_t \mathbb{E}\left[\frac{1}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]$ drop out of the FBSDE system, then the system reduces to that arising in the MFG analyzed in [25]. In the next subsection we introduce a model extension where the privately informed trader is the follower in a Stackelberg game of optimal portfolio liquidation.

1.2.2. Mean-field type Stackelberg game with asymmetric information.

In section 4 we solve a Stackelberg game of optimal portfolio liquidation with asymmetrically informed players. The leader (she) has the first-mover advantage while the follower (he) has an informational advantage.

We assume again that uncertainty is generated by the multidimensional Brownian motion $W = (\overline{W}, W^0)$ and that W^0 describes a commonly observed market factor while $\overline{W}$ is private information to the follower. For a given $\mathbb{F}^0$-adapted strategy ξ^0 of the Stackelberg leader, we assume that the follower's liquidation problem is the same as in the previous subsection with

$$\widetilde{g} = \widetilde{\kappa}^0 \xi^0$$

for some $\mathbb{F}^0$-adapted process $\widetilde{\kappa}^0$ that measures the impact of the leader on the follower's transaction price.[1] Let $\xi^*(\cdot)$ be the follower's optimal response function to the leader's strategy and put $\mu^* = \mathbb{E}[\xi^*(\cdot)|\mathcal{F}^0]$. Following the standard approach we assume that the leader's transaction price is

$$\widetilde{S}^0_t = S_t - \int_0^t \overline{\kappa}^0_s \mu^*_s \, ds - \int_0^t \kappa^0_s \xi^0_s \, ds - \eta^0_t \xi^0_t \tag{1.8}$$

for $\mathbb{F}^0$-adapted coefficients $\eta^0, \kappa^0, \overline{\kappa}^0$. The difference is that now the leader controls the transaction price both directly and indirectly through the dependence of the follower's optimal response on her trading strategy.

We assume that the leader's cost functional is given by

$$J^0(\xi^0) = \mathbb{E}\left[\int_0^T \left(\overline{\kappa}^0_t \mu^*_t X^0_t + \kappa^0_t X^0_t \xi^0_t + \eta^0_t (\xi^0_t)^2 + \lambda^0_t (X^0_t)^2 + \overline{\lambda}_t (\mu^*_t)^2\right) dt\right], \tag{1.9}$$

where X^0 denotes her portfolio process and $\lambda^0, \overline{\lambda}$ are $\mathbb{F}^0$-adapted. The additional cost term $\overline{\lambda}_t (\mu^*_t)^2$ serves two purposes. Mathematically, it guarantees that the optimization problem is convex if $\overline{\lambda}_t$ is large enough; economically it prevents the followers from excessive liquidity provision in equilibrium.[2]

The leader's control problem is a MFC problem with state process (X^0, X, Y), where (X, Y) solves the FBSDE (1.7) with $\widetilde{g} = \widetilde{\kappa}^0 \xi^0$ and

$$\begin{aligned} dX^0_t &= -\xi^0_t \, dt, \\ X^0_0 &= x, \ X^0_T = 0. \end{aligned} \tag{1.10}$$

In particular, the dynamics of the controlled state processes in the leader's problem follow an ODE-FBSDE system rather than an SDE as is usually the case in single player optimization problems. This renders the analysis of the leader's problem challenging when one imposes a strict liquidation constraint. In order to overcome this problem we combine two methods that have previously been applied to solve liquidation problems, namely solving an FBSDE system with unknown terminal condition on the backward component of the form (1.2) and the penalization approach where the liquidation constraint is replaced by an increasing penalization of open positions at the terminal time. The latter leads to a sequence of FBSDE systems of the form (1.2) where the terminal condition on the forward process is replaced by a terminal condition on the backward process that reflects the penalization. The solutions to the unconstrained problems converge to the solution of the corresponding constrained problem under suitable conditions.

Having solved the follower's problem by solving a system of the form (1.2), by using the penalization approach we then solve a family of unconstrained problems of the leader whose optimal strategies converge to some limit $\xi^{0,*}$ in a Cesaro sense. Subsequently, we prove that the limit admits the representation

$$\xi^{0,*}_t = \frac{p_t + \mathbb{E}[\widetilde{\kappa}^0_t q_t | \mathcal{F}^0_t] - \kappa^0_t X^{0,*}_t}{2\eta^0_t} \tag{1.11}$$

[1]It is implicitly assumed that ξ^0 is observable to the follower. This seems customary in Stackelberg games. Relaxing this assumption of "sunshine trading" would result in a leader-follower game with incomplete information that would be even more complex to study.

[2]We thank C.A. Lehalle for providing this interpretation.

in terms of the state equation (1.10) and the adjoint equations

(1.12)

$$-dp_t = \left(\overline{\kappa}_t^0 \mathbb{E}\left[\frac{Y_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] - \overline{\kappa}_t^0 \mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0] + \kappa_t^0 \xi_t^{0,*} + 2\lambda_t^0 X_t^{0,*}\right) dt - Z_t\, dW_t^0$$

and

(1.13)

$$\begin{cases} -dq_t = \left(-\dfrac{r_t}{2\eta_t} - \mathbb{E}\left[\kappa_t q_t|\mathcal{F}_t^0\right]\dfrac{1}{2\eta_t} + \overline{f}_t\right) dt, \\ -dr_t = \left(-2\lambda_t q_t + \kappa_t \mathbb{E}\left[\dfrac{r_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] + \kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}[\kappa_t q_t|\mathcal{F}_t^0] + \overline{g}_t\right) dt - Z_t\, dW_t, \\ \quad q_0 = 0,\ q_T = 0, \end{cases}$$

where

$$\overline{f}_t = \frac{\overline{\kappa}_t^0 X_t^{0,*}}{2\eta_t} + \frac{\overline{\lambda}_t}{\eta_t}\mathbb{E}\left[\frac{Y_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] - \frac{\overline{\lambda}_t}{\eta_t}\mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]$$

and

$$\begin{aligned}\overline{g}_t =& -\kappa_t \mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\overline{\kappa}_t^0 X_t^{0,*} - 2\overline{\lambda}_t \kappa_t \mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\left(\mathbb{E}\left[\frac{Y_t}{2\eta_t}\middle|\mathcal{F}_t^0\right]\right.\\ &\left. - \mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]\right).\end{aligned}$$

The system (1.13) is again a special case of (1.2); it depends on the previously constructed process $\xi^{0,*}$. We finally prove a novel maximum principle that states that if for some process $\xi^{0,*}$ the system (1.10), (1.12), and (1.13) has a solution, then $\xi^{0,*}$ is an optimal strategy for the leader. Here, p is the adjoint variable to the ODE component of the state process and (q, r) are the adjoint variables to the FBSDE component of the state process.

To the best of our knowledge, no numerical methods for simulating the mean-field FBSDEs arising in our Stackelberg game are yet available. In order to get some quantitative insight into the equilibrium dynamics, we therefore simulate a deterministic benchmark model with constant coefficients. In this case, our conditional mean-field FBSDEs reduce to deterministic forward-backward ODEs for which numerical methods exist. Our simulations suggest that the solution to the Stackelberg game is very different from the solution to single player models. In particular, beneficial round-trips may exist for the follower. This is not the case in deterministic single player models; in the Stackelberg game the follower may act as a liquidity provider for the leader. Furthermore, depending on the strength of interaction the presence of the follower may (or may not) reduce the leader's trading cost.

Remark 1.1. A special case of the system (1.5), (1.7), (1.10), (1.12), and (1.13) arises in MFGs of optimal portfolio liquidation between a major and many minor players. Thus, as a byproduct we obtain an extension of the MFG in [25] to a MFG with a major player. A related model without liquidation constraint and without any feedback of the major player's strategy on the minor players' transaction price has been considered in, e.g., [24, 32]. MFGs of optimal trading with incomplete information but without the strict liquidation constraint have also been considered in [17, 18]. The incompleteness of information arises from an unobservable latent process. The nature of the FBSDEs arising in our work and [17, 18] is very different:

the mean-field terms in [17, 18] are exogenous (see [17, equation (3.16)] and [18, equation (3.11)]) while the mean-field term in our work is endogenous. Moreover, in [17, 18] there are multiple groups of traders with heterogeneous beliefs (as in [7]),which is not the case in our model.

The rest of this paper is organized as follows. Our general existence, uniqueness, and convergence results for the FBSDE (1.2) are established in section 2. The MFC problem and the Stackelberg game of optimal portfolio liquidation introduced above are solved in sections 3 and 4, respectively. Our numerical simulations are reported in section 4.3.

Notation and conventions. Throughout, we work on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which there exist two independent Brownian motions W^0 and $\overline{W}$. We denote by $\mathbb{F}^0 = (\mathcal{F}^0_t)_{0\le t\le T}$ and $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the filtrations generated by W^0 and W, augmented by the $\mathbb{P}$ null sets, respectively, where $W = (\overline{W}, W^0)$. For a subspace $\mathbb{I} \subseteq \mathbb{R}$ and a filtration $\mathbb{G}$, we introduce the following spaces:

$$
\begin{aligned}
L^0_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) =&\{X : X : [0,T]\times\Omega \to \mathbb{I} \text{ and } X \text{ is } \mathbb{G} \text{ progressively measurable} \\
&\quad \text{and } \mathbb{I} \text{ valued}\}\\
L^k_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) =& \left\{X \in L^0_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) : \mathbb{E}\left[\int_0^T |X_t|^k\,dt\right] < \infty\right\},\ k \ge 1,\\
L^\infty_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) =& \left\{X \in L^0_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) : \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |X_t(\omega)| < \infty\right\}.
\end{aligned}
$$

The space $L^0_{\mathbb{G}}([0,T)\times\Omega;\mathbb{I})$ is defined similarly as $L^0_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I})$. For $k \ge 1$, the space $L^k_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I})$ is equipped with the norm $\|X\|_{L^k} = \big(\mathbb{E}\big[\int_0^T |X_t|^k\,dt\big]\big)^{1/k}$. The spaces

$$
\begin{aligned}
S^2_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) &= \left\{X \in L^0_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I}) : \mathbb{E}\left[\sup_{0\le t\le T}|X_t|^2\right] < \infty\right\},\\
S^{2,-}_{\mathbb{G}}([0,T)\times\Omega;\mathbb{I}) &= \left\{X \in L^0_{\mathbb{G}}([0,T)\times\Omega;\mathbb{I}) : \sup_{\epsilon>0}\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|X_t|^2\right] \le C\right\}
\end{aligned}
$$

are equipped with the respective norms

$$
\|X\|_{S^2} = \left(\mathbb{E}\left[\sup_{0\le t\le T}|X_t|^2\right]\right)^{1/2},\ \ \|X\|_{S^{2,-}} = \sup_{\epsilon\ge 0}\left(\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|X_t|^2\right]\right)^{1/2}.
$$

For $\beta \in \mathbb{R}$ we introduce the space

$$
\mathcal{H}_\beta([0,T]\times\Omega;\mathbb{I}) = \left\{X \in \mathbb{S}^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{I}) : \mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{X_t}{(T-t)^\beta}\right|^2\right] < \infty\right\}
$$

with

$$
\|X\|_\beta = \left(\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{X_t}{(T-t)^\beta}\right|^2\right]\right)^{1/2}.
$$

Finally, we put

$$
L^{2,-}_{\mathbb{G}}([0,T)\times\Omega;\mathbb{I}) = \left\{X \in L^0_{\mathbb{G}}([0,T)\times\Omega;\mathbb{I}) :\ \text{for each}\ \epsilon > 0\ \mathbb{E}\left[\int_0^{T-\epsilon}|X_t|^2\,dt\right] < \infty\right\}.
$$

For $\phi \in L^\infty_{\mathbb{G}}([0,T]\times\Omega;\mathbb{I})$, we denote by $\|\phi\| = \text{esssup}_{(t,\omega)\in[0,T]\times\Omega}|\phi(t,\omega)|$ and $\phi_\star = \text{essinf}_{(t,\omega)\in[0,T]\times\Omega}\phi(t,\omega)$ its upper and lower bounds, respectively. When $\mathbb{I}=\mathbb{R}$, for simplicity we will write $L^2_{\mathbb{F}}$ instead of $L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$; the same convention holds for other spaces. Finally, we adopt the convention that a positive constant C may vary from line to line but always depends on the underlying generic constants.

2. The McKean–Vlasov FBSDE.

In this section, we prove a general existence and uniqueness of solutions result (in a suitable space) for the FBSDE (1.2) along with the convergence result with respect to the processes $(\overline{f},\overline{g})$. We assume throughout that the system's coefficients satisfy the following assumption.

ASSUMPTION 2.1. (i) *The stochastic processes γ, ζ, ϱ, and Λ^i $(i=1,\dots,5)$ belong to $L^\infty_{\mathbb{F}}$.*

(ii) *There exist constants $\theta_i>0$ $(i=1,2)$ such that (recall that $\phi_\star$ denotes the lower bound of an $L^\infty_{\mathbb{F}}$ random variable)*

$$\left(\Lambda^1 - \frac{\|\gamma\||\Lambda^2|^2}{2\theta_1} - \frac{\|\Lambda^3\||\zeta|^2}{2\theta_2}\right)_\star > 0$$

and

$$\left(\Lambda^4 - \frac{\|\gamma\|\theta_1}{2} - \frac{\|\Lambda^3\|\theta_2}{2} - \|\Lambda^5\|\|\varrho\|\right)_\star > 0.$$

(iii) *The initial condition χ belongs to $L^2_{\mathbb{F}}$ and $(\overline{f},\overline{g})\in S^2_{\mathbb{F}}\times L^2_{\mathbb{F}}$.*

The linear ansatz $R=AQ+H$ on $[0,T)$ results in the following FBSDE for the triple (Q,H,R):

$$\tag{2.1} \begin{cases} dQ_t = \left(-\Lambda^1_t R_t - \Lambda^2_t \mathbb{E}\left[\gamma_t Q_t\middle|\mathcal{F}^0_t\right] + \overline{f}_t\right)dt,\\ -dH_t = \left(-\Lambda^1_t A_t H_t - \Lambda^2_t A_t \mathbb{E}[\gamma_t Q_t|\mathcal{F}^0_t] + A_t\overline{f}_t + \Lambda^3_t\mathbb{E}[\zeta_t R_t|\mathcal{F}^0_t]\right.\\ \qquad\qquad \left. +\Lambda^5_t\mathbb{E}[\varrho_t Q_t|\mathcal{F}^0_t] + \overline{g}_t\right)dt - Z^H_t\,dW_t,\\ -dR_t = \left(\Lambda^4_t Q_t + \Lambda^3_t\mathbb{E}[\zeta_t R_t|\mathcal{F}^0_s] + \Lambda^5_t\mathbb{E}[\varrho_t Q_t|\mathcal{F}^0_t] + \overline{g}_t\right)dt - Z^R_t\,dW_t,\\ R = AQ+H,\ t\in[0,T),\\ Q_0=\chi,\ Q_T=0, \end{cases}$$

where A satisfies the singular BSDE

$$\tag{2.2} -dA_t = \left(\Lambda^4_t - \Lambda^1_t A^2_t\right)dt - Z^A_t\,dW_t, \quad \lim_{t\nearrow T} A_t = \infty.$$

The singular terminal condition on A results from the terminal state constraint on Q. By Assumption 2.1(i,ii) both Λ^1 and $(\Lambda^1)^{-1}$ are bounded. Thus, it follows from [27, Proposition 6.1 and Theorem 6.3] that there exists a *unique* pair $(A,Z^A)\in\mathcal{H}_{-1}\times L^{2,-}_{\mathbb{F}}$ that satisfies (2.2) as well as the following estimate:

$$\tag{2.3} 0 < \frac{1}{\mathbb{E}\left[\int_t^T \Lambda^1_u\,du\middle|\mathcal{F}_t\right]} \le A_t \le \frac{1}{(T-t)^2}\mathbb{E}\left[\int_t^T \frac{1}{\Lambda^1_u} + (T-u)^2\Lambda^4_u\,du\middle|\mathcal{F}_t\right].$$

It follows from (2.3) that A is nonnegative and that for all $0\le t_1<t_2\le T$,

$$\tag{2.4} e^{-\int_{t_1}^{t_2}\Lambda^1_s A_s\,ds} \le C\left(\frac{T-t_2}{T-t_1}\right)^\beta \le C\left(\frac{T-t_2}{T-t_1}\right)^\tau, \text{ where } \beta=\Lambda^1_\star/\|\Lambda^1\| \text{ and } 0\le\tau\le\beta.$$

2.1. Existence and uniqueness of solutions. In view of [25], we expect to find a solution (Q, H, R, Z^H, Z^R) to (2.1) such that $(Q, R) \in \mathcal{H}_\alpha \times L^2_{\mathbb{F}}$ for some $\alpha > 0$. Unlike in [25] the process H is only defined on $[0, T)$. The following argument shows that if we can find a solution such that $(Q, R) \in \mathcal{H}_\alpha \times L^2_{\mathbb{F}}$, then there exists a process $H \in S^{2,-}_{\mathbb{F}}$ that satisfies the second backward equation in (2.1). In fact, for any $0 \le t < T$, let us put

$$H_t = \mathbb{E}\left[\int_t^T e^{-\int_t^s \Lambda^1_u A_u \, du} K_s \, ds \middle| \mathcal{F}_t \right],$$

with

$$K_s = \left(-\Lambda^2_s A_s \mathbb{E}[\gamma_s Q_s | \mathcal{F}^0_s] + A_s \overline{f}_s + \Lambda^3_s \mathbb{E}[\zeta_s R_s | \mathcal{F}^0_s] + \Lambda^5_s \mathbb{E}[\varrho_s Q_s | \mathcal{F}^0_s] + \overline{g}_s \right).$$

The following argument shows that the conditional expectation is in fact well defined and that $H \in S^{2,-}_{\mathbb{F}}$. By (2.3) and (2.4),

$$\begin{aligned}
&\left| \int_t^T e^{-\int_t^s \Lambda^1_u A_u \, du} K_s \, ds \right| \\
&\le C \int_t^T \frac{(T-s)^{\beta-1}}{(T-t)^\beta} \left(\mathbb{E}[|Q_s| | \mathcal{F}^0_s] + |\overline{f}_s| \right) ds + C \int_0^T \left(\mathbb{E}[|R_s| + |Q_s| | \mathcal{F}^0_s] + |\overline{g}_s| \right) ds \\
&\le C \sup_{0 \le s \le T} \mathbb{E}[|Q_s| | \mathcal{F}^0_s] + C \sup_{0 \le s \le T} |\overline{f}_s| + C \int_0^T \left(\mathbb{E}[|R_s| + |Q_s| | \mathcal{F}^0_s] + |\overline{g}_s| \right) ds.
\end{aligned} \tag{2.5}$$

From this and Doob's maximal inequality, we obtain a constant $C > 0$ such that for any $\epsilon > 0$

$$\mathbb{E}\left[\sup_{0 \le t \le T-\epsilon} |H_t|^2 \right] \le C \left(\|Q\|^2_\alpha + \|\overline{f}\|^2_{S^2} + \|R\|^2_{L^2} + \|\overline{g}\|^2_{L^2} \right).$$

Moreover, the martingale representation theorem yields the existence of a process $Z^H \in L^{2,-}_{\mathbb{F}}$ such that (H, Z^H) satisfies the second equation in (2.1).

Remark 2.2. We notice that it is not clear if the limit $\lim_{t \to T} \int_t^T e^{-\int_t^s \Lambda^1_u A_u \, du} K_s \, ds$ exists. That is, from the definition of the process H it is not clear if the limit $\lim_{t \to T} H_t$ exists. This is why we consider $S^{2,-}_{\mathbb{F}}$ as the (canonical) state space for the process H.

In view of the preceding argument our goal is to establish the existence and uniqueness of a solution $(Q, H, R, Z^H, Z^R) \in \mathcal{H}_\alpha \times S^{2,-}_{\mathbb{F}} \times L^2_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}}$. To this end, we apply a nested continuation method to the system

$$\begin{cases}
dQ_t = \left(-\Lambda^1_t R_t - \Lambda^2_t \mathbb{E}\left[\gamma_t Q_t | \mathcal{F}^0_t \right] + \overline{f}_t \right) dt, \\
-dH_t = \left(-\Lambda^1_t A_t H_t - \Lambda^2_t A_t \mathbb{E}[\gamma_t Q_t | \mathcal{F}^0_t] + A_t \overline{f}_t + \mathfrak{p} \Lambda^3_t \mathbb{E}[\zeta_t R_t | \mathcal{F}^0_t] \right. \\
\qquad \left. + \mathfrak{p} \Lambda^5_t \mathbb{E}[\varrho_t Q_t | \mathcal{F}^0_t] + \overline{g}_t + f_t \right) dt - Z^H_t \, dW_t, \\
-dR_t = \left(\Lambda^4_t Q_t + \mathfrak{p} \Lambda^3_t \mathbb{E}[\zeta_t R_t | \mathcal{F}^0_s] + \mathfrak{p} \Lambda^5_t \mathbb{E}[\varrho_t Q_t | \mathcal{F}^0_t] + \overline{g}_t + f_t \right) dt - Z^R_t \, dW_t, \\
R = AQ + H, \ t \in [0, T), \\
Q_0 = \chi, \ Q_T = 0.
\end{cases} \tag{2.6}$$

In a first step, we prove the existence of a unique solution to the above system for $\mathfrak{p} = 0$. Subsequently, we show that the solution result extends to $\mathfrak{p} = 1$.

LEMMA 2.3. *If $\mathfrak{p} = 0$, then the FBSDE* (2.6) *has a solution in* $\mathcal{H}_\alpha \times S^{2,-}_{\mathbb{F}} \times L^2_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}}$ *for any* $f \in L^2_{\mathbb{F}}$, *where* $0 < \alpha < \beta$.

Proof. Notice that the system (2.6) is still coupled for $\mathfrak{p} = 0$. To solve it, we apply a continuation method to the following system:

$$\begin{cases} dQ_t = \left(-\Lambda^1_t R_t - \overline{\mathfrak{p}}\Lambda^2_t \mathbb{E}\left[\gamma_t Q_t | \mathcal{F}^0_t\right] + \overline{f}_t + b'_t\right) dt, \\ -dH_t = \left(-\Lambda^1_t A_t H_t - \overline{\mathfrak{p}}\Lambda^2_t A_t \mathbb{E}[\gamma_t Q_t|\mathcal{F}^0_t] + A_t\overline{f}_t + \overline{g}_t + f_t + f'_t\right) dt - Z^H_t\, dW_t, \\ -dR_t = \left(\Lambda^4_t Q_t + \overline{g}_t + f_t + f'_t - A_t b'_t\right) dt - Z^R_t\, dW_t, \\ R = AQ + H,\ t \in [0, T), \\ Q_0 = \chi,\ Q_T = 0. \end{cases} \tag{2.7}$$

Step 1. *For $\overline{\mathfrak{p}} = 0$, the system* (2.7) *is solvable in* $\mathcal{H}_\alpha \times S^{2,-}_{\mathbb{F}} \times L^2_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}}$ *for any* $(b', f') \in \mathcal{H}_\alpha \times \mathcal{H}_{\alpha-1}$.

If $\overline{\mathfrak{p}} = 0$, then the system (2.7) is decoupled, and we let H be

$$H_t = \mathbb{E}\left[\int_t^T e^{-\int_t^s \Lambda^1_u A_u\, du}\left(A_s\overline{f}_s + \overline{g}_s + f_s + f'_s\right) ds \middle| \mathcal{F}_t\right],\ 0 \le t < T. \tag{2.8}$$

Moreover, by the estimate (2.4) and Doob's maximal inequality, we have for any $\epsilon > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T-\epsilon} |H_t|^2\right] \le C\left(\|\overline{f}\|^2_{S^2} + \|\overline{g}\|^2_{L^2} + \|f\|^2_{L^2} + \|f'\|^2_{\alpha-1}\right), \tag{2.9}$$

where C is independent of ϵ. Thus, H belongs to $S^{2,-}_{\mathbb{F}}$. For each $\epsilon > 0$, martingale representation implies the existence of a unique $Z^H \in L^2_{\mathbb{F}}([0, T-\epsilon] \times \Omega; \mathbb{R})$ such that (H, Z^H) satisfies the second equation in (2.7). Uniqueness implies $Z^H \in L^{2,-}_{\mathbb{F}}$.

We now turn to the process Q. Taking $R = AQ + H$ into the SDE for Q yields

$$Q_t = \chi e^{-\int_0^t \Lambda^1_u A_u\, du} + \int_0^t e^{-\int_s^t \Lambda^1_u A_u\, du}\left(-\Lambda^1_s H_s + \overline{f}_s + b'_s\right) ds,\ 0 \le t \le T. \tag{2.10}$$

Using monotone convergence and the estimate (2.9) this implies

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{Q_t}{(T-t)^\alpha}\right|^2\right] \\ &\le C\left(\|\chi\|^2_{L^2} + \mathbb{E}\left[\left(\int_0^T \frac{|H_s|}{(T-s)^\alpha}\, ds\right)^2\right] + \|\overline{f}\|^2_{S^2} + \|b'\|^2_\alpha\right) \\ &= C\left(\|\chi\|^2_{L^2} + \lim_{\epsilon\searrow 0}\mathbb{E}\left[\left(\int_0^{T-\epsilon} \frac{|H_s|}{(T-s)^\alpha}\, ds\right)^2\right] + \|\overline{f}\|^2_{S^2} + \|b'\|^2_\alpha\right) \\ &\le C\left(\|\chi\|^2_{L^2} + \lim_{\epsilon\searrow 0}\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|H_t|^2\right] + \|\overline{f}\|^2_{S^2} + \|b'\|^2_\alpha\right) \\ &\le C\left(\|\chi\|^2_{L^2} + \|\overline{f}\|^2_{S^2} + \|\overline{g}\|^2_{L^2} + \|f\|^2_{L^2} + \|f'\|^2_{\alpha-1} + \|b'\|^2_\alpha\right). \end{aligned} \tag{2.11}$$

This shows that $Q \in \mathcal{H}_\alpha$.

Moreover, since $R_{T-\epsilon} = A_{T-\epsilon}Q_{T-\epsilon} + H_{T-\epsilon}$ is square integrable and because the coefficients in the third equation of (2.7) satisfy standard conditions for the solvability of BSDEs on $[0,T-\epsilon]$ there exists a unique $Z^R \in L^2_{\mathbb{F}}([0,T-\epsilon]\times\Omega;\mathbb{R})$ such that (R,Z^R) satisfy the third equation of (2.7). Uniqueness implies $Z^R \in L^{2,-}_{\mathbb{F}}$.

We now show that $R \in L^2_{\mathbb{F}}$. To do so, we use a similar argument as in [25]. Integration by parts for the product QR on $[0,T-\epsilon]$ yields

$$\begin{aligned} H_{T-\epsilon}Q_{T-\epsilon} \le A_{T-\epsilon}Q^2_{T-\epsilon} + H_{T-\epsilon}Q_{T-\epsilon} &= Q_{T-\epsilon}R_{T-\epsilon} \\ &= Q_0R_0 - \int_0^{T-\epsilon} \Lambda^4_t Q^2_t\,dt - \int_0^{T-\epsilon} Q_t\left(\overline{g}_t + f_t + f'_t - A_b b'_t\right)dt \\ &\quad + \int_0^{T-\epsilon} Q_t Z^R_t\,dW_t - \int_0^{T-\epsilon}\Lambda^1_t R^2_t\,dt + \int_0^{T-\epsilon} R_t(\overline{f}_t + b'_t)\,dt, \end{aligned}$$

which implies that

$$\begin{aligned} \int_0^{T-\epsilon}\left(\Lambda^4_tQ^2_t + \Lambda^1_tR^2_t\right)dt &\le |H_{T-\epsilon}Q_{T-\epsilon}| + |Q_0(A_0Q_0+H_0)| \\ &\quad + \left|\int_0^{T-\epsilon} Q_t\left(\overline{g}_t + f_t + f'_t - A_b b'_t\right)dt\right| \\ &\quad + \int_0^{T-\epsilon} Q_tZ^R_t\,dW_t + \left|\int_0^{T-\epsilon} R_t(\overline{f}_t+b'_t)\,dt\right|. \end{aligned} \tag{2.12}$$

Next, we show $\mathbb{E}\left[\int_0^{T-\epsilon} Q_tZ^R_t\,dW_t\right] = 0$. Indeed, by the BDG inequality

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le t\le T-\epsilon}\left|\int_0^t Q_sZ^R_s\,dW_s\right|\right] &\le C\mathbb{E}\left(\int_0^{T-\epsilon} Q^2_t(Z^R_t)^2\,dt\right)^{\frac{1}{2}} \\ &\le C\mathbb{E}\left[\sup_{0\le t\le T}|Q_t|\left(\int_0^{T-\epsilon}(Z^R_t)^2\,dt\right)^{\frac{1}{2}}\right] \\ &\le C\mathbb{E}\left[\sup_{0\le t\le T}|Q_t|^2\right] + C\mathbb{E}\left[\int_0^{T-\epsilon}|Z^R_t|^2\,dt\right] \\ &< \infty. \end{aligned} \tag{2.13}$$

For a localizing sequence of stopping times T_n it holds by dominated convergence that

$$\mathbb{E}\left[\int_0^{T-\epsilon} Q_tZ^R_t\,dW_t\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_0^{(T-\epsilon)\wedge T_n} Q_tZ^R_t\,dW_t\right] = 0.$$

Taking expectations on both sides of (2.12) we thus have

$$\begin{aligned} &\mathbb{E}\left[\int_0^{T-\epsilon}\left(\Lambda^4_tQ^2_t + \Lambda^1_tR^2_t\right)dt\right] \\ \le\ &\mathbb{E}\left[|H_{T-\epsilon}Q_{T-\epsilon}|\right] + \|A_0\|\mathbb{E}[Q_0^2] + \frac{1}{2}\mathbb{E}[Q_0^2] + \frac{1}{2}\mathbb{E}[H_0^2] \\ +\ &\delta\mathbb{E}\left[\int_0^T R^2_t\,dt\right] + \frac{1}{4\delta}\mathbb{E}\left[\int_0^T|\overline{f}_t + b'_t|^2\,dt\right] \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\left[\sup_{0\le t\le T}|Q_t|\int_0^T|\overline{g}_t+f_t+f_t'-A_tb_t'|\,dt\right]\qquad\text{(by Young's inequality)}\\
&\le\ \mathbb{E}\left[|H_{T-\epsilon}Q_{T-\epsilon}|\right]+C\mathbb{E}[Q_0^2]+\frac{1}{2}\mathbb{E}[H_0^2]+\delta\mathbb{E}\left[\int_0^T R_t^2\,dt\right]\\
&+\ C\mathbb{E}\left[\int_0^T|\overline{f}_t|^2+|b_t'|^2\,dt\right]\ \ (\|A_0\|\text{ is bounded by (2.3)})\\
&+\mathbb{E}\left[\sup_{0\le t\le T}|Q_t|\int_0^T\left|\frac{f_t'}{(T-t)^{\alpha-1}}(T-t)^{\alpha-1}+\frac{b_t'}{(T-t)^{\alpha}}(T-t)^{\alpha-1}\right|\,dt\right]\\
&+\ \mathbb{E}\left[\sup_{0\le t\le T}|Q_t|\int_0^T|\overline{g}_t+f_t|\,dt\right]\\
&\le\ \delta\mathbb{E}\left[\int_0^T|R_t|^2\,dt\right]+C\big(\mathbb{E}[Q_0^2]+\mathbb{E}[|H_0^2|]+\|Q\|_\alpha^2+\|\overline{g}\|_{L^2}^2+\|\overline{f}\|_{S^2}^2\\
&+\ \|f\|_{L^2}^2+\|f'\|_{\alpha-1}^2+\|b'\|_\alpha^2\big)+\mathbb{E}[|H_{T-\epsilon}Q_{T-\epsilon}|].
\end{aligned}
$$

Assumption 2.1(ii) implies $\Lambda_\star^1>0$ and $\Lambda_\star^4>0$. Thus, by taking $\delta=\Lambda_\star^1/2$ and taking $\epsilon\to0$, from (2.9) and (2.11) we get $R\in L_{\mathbb{F}}^2$.

Step 2. *If* (2.7) *admits a solution for some* $\overline{\mathfrak{p}}\in[0,1]$ *and for any* $(b',f')\in\mathcal{H}_\alpha\times\mathcal{H}_{\alpha-1}$*, then there exists a constant* $\overline{\mathfrak{d}}>0$*, which does not depend on* $\overline{\mathfrak{p}}$*,* b' *or* f'*, such that* $\overline{\mathfrak{p}}+\overline{\mathfrak{d}}\in[0,1]$ *and the same result holds for* $\overline{\mathfrak{p}}+\widehat{\mathfrak{d}}$ *for any* $\widehat{\mathfrak{d}}\in[0,\overline{\mathfrak{d}}]$*.*

For fixed $Q\in\mathcal{H}_\alpha$, since

$$
-\overline{\mathfrak{d}}\Lambda^2\mathbb{E}\left[\gamma Q|\,\mathcal{F}^0\right]+b'\in\mathcal{H}_\alpha,\quad -\overline{\mathfrak{d}}\Lambda^2A\mathbb{E}[\gamma Q|\mathcal{F}^0]+f'\in\mathcal{H}_{\alpha-1},
$$

there exists a solution $(\widetilde{Q},\widetilde{H},\widetilde{R},Z^{\widetilde{H}},Z^{\widetilde{R}})\in\mathcal{H}_\alpha\times S_{\mathbb{F}}^{2,-}\times L_{\mathbb{F}}^2\times L_{\mathbb{F}}^{2,-}\times L_{\mathbb{F}}^{2,-}$ to the following system:

$$
\text{(2.14)}\qquad
\begin{cases}
d\widetilde{Q}_t=\ \Big(-\Lambda_t^1\widetilde{R}_t-\overline{\mathfrak{p}}\Lambda_t^2\mathbb{E}\left[\gamma_t\widetilde{Q}_t\Big|\,\mathcal{F}_t^0\right]-\overline{\mathfrak{d}}\Lambda_t^2\mathbb{E}\left[\gamma_tQ_t|\,\mathcal{F}_t^0\right]+\overline{f}_t+b_t'\Big)\,dt,\\
-d\widetilde{H}_t=\Big(-\Lambda_t^1A_t\widetilde{H}_t-\overline{\mathfrak{p}}\Lambda_t^2A_t\mathbb{E}[\gamma_t\widetilde{Q}_t|\mathcal{F}_t^0]-\overline{\mathfrak{d}}\Lambda_t^2A_t\mathbb{E}[\gamma_tQ_t|\mathcal{F}_t^0]\\
\qquad\qquad+A_t\overline{f}_t+\overline{g}_t+f_t+f_t'\Big)\,dt-Z_t^{\widetilde{H}}\,dW_t,\\
-d\widetilde{R}_t=\ \Big(\Lambda_t^4\widetilde{Q}_t+\overline{g}_t+f_t+f_t'-A_tb_t'\Big)\,dt-Z_t^{\widetilde{R}}\,dW_t,\\
\widetilde{R}=A\widetilde{Q}+\widetilde{H},\ t\in[0,T),\\
\widetilde{Q}_0=\chi,\ \widetilde{Q}_T=0.
\end{cases}
$$

It remains to prove that the mapping $\Phi:\mathcal{H}_\alpha\to\mathcal{H}_\alpha$, $Q\mapsto\widetilde{Q}$ is a contraction when $\overline{\mathfrak{d}}$ is small enough and independent of $\overline{\mathfrak{p}}$, b', and f'. For any Q, $Q'\in\mathcal{H}_\alpha$, let $(\widetilde{Q},\widetilde{H},\widetilde{R},Z^{\widetilde{H}},Z^{\widetilde{R}})$ and $(\widetilde{Q}',\widetilde{H}',\widetilde{R}',Z^{\widetilde{H}'},Z^{\widetilde{R}'})$ be the corresponding solutions. Integration by parts for $(\widetilde{Q}-\widetilde{Q}')(\widetilde{R}-\widetilde{R}')$ on $[0,T-\epsilon]$ implies

$$\begin{aligned}
&(\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})(\widetilde{H}_{T-\epsilon}-\widetilde{H}'_{T-\epsilon}) \leq (\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})(A_{T-\epsilon}(\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})\\
&+(\widetilde{H}_{T-\epsilon}-\widetilde{H}'_{T-\epsilon}))\\
&=(\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})(\widetilde{R}_{T-\epsilon}-\widetilde{R}'_{T-\epsilon})\\
&= -\int_0^{T-\epsilon}\Lambda^4_t(\widetilde{Q}_t-\widetilde{Q}'_t)^2\,dt-\int_0^{T-\epsilon}\Lambda^1_t(\widetilde{R}_t-\widetilde{R}'_t)\,dt\\
&\quad -\int_0^{T-\epsilon}\overline{\mathfrak{p}}\Lambda^2_t(\widetilde{R}_t-\widetilde{R}'_t)\mathbb{E}[\gamma_t(\widetilde{Q}_t-\widetilde{Q}'_t)|\mathcal{F}^0_t]\,dt\\
&\quad -\int_0^{T-\epsilon}\overline{\mathfrak{d}}\Lambda^2_t(\widetilde{R}_t-\widetilde{R}'_t)\mathbb{E}[\gamma_t(Q_t-Q'_t)|\mathcal{F}^0_t]\,dt+\int_0^{T-\epsilon}(\widetilde{Q}_t-\widetilde{Q}'_t)(Z^{\widetilde{R}}_t-Z^{\widetilde{R}'}_t)\,dW_t.
\end{aligned}$$

The same argument as (2.13) implies the expectation of the stochastic integral is 0. Thus, by taking expectations on both sides and Young's inequality one has

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon}\Lambda^4_t(\widetilde{Q}_t-\widetilde{Q}'_t)^2\,dt\right]+\mathbb{E}\left[\int_0^{T-\epsilon}\Lambda^1_t(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]\\
&\leq \mathbb{E}|(\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})(\widetilde{H}_{T-\epsilon}-\widetilde{H}'_{T-\epsilon})|+\delta\mathbb{E}\left[\int_0^{T-\epsilon}(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]\\
&\quad+\frac{\|\Lambda^2\|^2\|\gamma\|^2\overline{\mathfrak{d}}}{4\delta}\mathbb{E}\left[\int_0^{T-\epsilon}(Q_t-Q'_t)^2\,dt\right]\\
&\quad+\mathbb{E}\left[\int_0^{T-\epsilon}\|\gamma\|\left(\frac{\theta_1}{2}(\widetilde{Q}_t-\widetilde{Q}'_t)^2+\frac{(\Lambda^2_t)^2(\widetilde{R}_t-\widetilde{R}'_t)^2}{2\theta_1}\right)\,dt\right],
\end{aligned}$$

which implies by rearranging terms

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon}\left(\Lambda^4_t-\frac{\theta_1\|\gamma\|}{2}\right)(\widetilde{Q}_t-\widetilde{Q}'_t)^2\,dt\right]\\
&+\mathbb{E}\left[\int_0^{T-\epsilon}\left(\Lambda^1_t-\frac{\|\gamma\|(\Lambda^2_t)^2}{2\theta_1}\right)(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]\\
&\leq \mathbb{E}|(\widetilde{Q}_{T-\epsilon}-\widetilde{Q}'_{T-\epsilon})(\widetilde{H}_{T-\epsilon}-\widetilde{H}'_{T-\epsilon})|+\delta\mathbb{E}\left[\int_0^{T-\epsilon}(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]\\
&+\frac{\|\Lambda^2\|^2\|\gamma\|^2\overline{\mathfrak{d}}}{4\delta}\mathbb{E}\left[\int_0^{T-\epsilon}(Q_t-Q'_t)^2\,dt\right].
\end{aligned}$$

Assumption 2.1(ii) implies $\Lambda^4_t-\frac{\theta_1\|\gamma\|}{2}\geq\left(\Lambda^4-\frac{\theta_1\|\gamma\|}{2}\right)_\star>0$ and $\Lambda^1_t-\frac{\|\gamma\|(\Lambda^2_t)^2}{2\theta_1}\geq\left(\Lambda^1-\frac{\|\gamma\|(\Lambda^2)^2}{2\theta_1}\right)_\star>0$. Thus, by choosing δ small enough and letting $\epsilon\to 0$ we have

$$\mathbb{E}\left[\int_0^T(\widetilde{Q}_s-\widetilde{Q}'_s)^2\,ds\right]+\mathbb{E}\left[\int_0^T(\widetilde{R}_s-\widetilde{R}'_s)^2\,ds\right]\leq C\overline{\mathfrak{d}}\mathbb{E}\left[\int_0^T(Q_t-Q'_t)^2\,dt\right]. \tag{2.15}$$

Considering the SDE for $\widetilde{Q}$ in terms of $\widetilde{R}$, by (2.15) we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|\widetilde{Q}_t-\widetilde{Q}'_t|^2\right]\le C\overline{\mathfrak{d}}\mathbb{E}\left[\int_0^T (Q_t-Q'_t)^2\,dt\right]. \tag{2.16}$$

Since $\widetilde{H}\in S^{2,-}_{\mathbb{F}}$, we have the following expression:

$$\begin{aligned}\widetilde{H}_t = \ & \mathbb{E}\bigg[\int_t^T e^{-\int_t^s \Lambda^1_u A_u\,du}\Big(-\overline{\mathfrak{p}}\Lambda^2_s A_s\mathbb{E}[\gamma_s\widetilde{Q}_s|\mathcal{F}^0_s]-\overline{\mathfrak{d}}\Lambda^2_s A_s\mathbb{E}[\gamma_s Q_s|\mathcal{F}^0_s] \\ & + A_s\overline{f}_s+\overline{g}_s+f_s+f'_s\Big)\,ds\Big|\mathcal{F}_t\bigg].\end{aligned} \tag{2.17}$$

From (2.17), Doob's maximal inequality and (2.16) yield that for any $\epsilon>0$

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|\widetilde{H}_t-\widetilde{H}'_t|^2\right]\\
&\le C\mathbb{E}\left\{\sup_{0\le t\le T}\left|\mathbb{E}\left[\int_t^T\frac{(T-s)^{\beta-1}}{(T-t)^\beta}\mathbb{E}[|\widetilde{Q}_s-\widetilde{Q}'_s||\mathcal{F}^0_s]\,ds\bigg|\,\mathcal{F}_t\right]\right|^2\right\}\\
&\quad + C\overline{\mathfrak{d}}\mathbb{E}\left\{\sup_{0\le t\le T}\left|\mathbb{E}\left[\int_t^T\frac{(T-s)^{\beta-1}}{(T-t)^\beta}\mathbb{E}[|Q_s-Q'_s||\mathcal{F}^0_s]\,ds\bigg|\,\mathcal{F}_t\right]\right|^2\right\}\\
&\le C\mathbb{E}\left\{\sup_{0\le t\le T}\left|\mathbb{E}\left[\sup_{0\le s\le T}\mathbb{E}[|\widetilde{Q}_s-\widetilde{Q}'_s||\mathcal{F}^0_s]\bigg|\,\mathcal{F}_t\right]\right|^2\right\}\\
&\quad + C\overline{\mathfrak{d}}\mathbb{E}\left\{\sup_{0\le t\le T}\left|\mathbb{E}\left[\sup_{0\le s\le T}\mathbb{E}[|Q_s-Q'_s||\mathcal{F}^0_s]\bigg|\mathcal{F}_t\right]\right|^2\right\}\\
&\le C\overline{\mathfrak{d}}\mathbb{E}\left[\int_0^T(Q_t-Q'_t)^2\,dt\right]+C\overline{\mathfrak{d}}\mathbb{E}\left[\sup_{0\le t\le T}\left|Q_t-Q'_t\right|^2\right],\end{aligned} \tag{2.18}$$

where C is independent of ϵ. Finally, considering the SDE for $\widetilde{Q}$ in terms of $\widetilde{H}$, by (2.16), (2.18), and the same argument as (2.11), we have

$$\|\widetilde{Q}-\widetilde{Q}'\|_\alpha\le C\overline{\mathfrak{d}}\|Q-Q'\|_\alpha.$$

Thus, when $\overline{\mathfrak{d}}$ is small enough, Φ is a contraction.

Step 3. Note that $\overline{\mathfrak{d}}$ does not depend on $\overline{\mathfrak{p}}$, b', or f'. Iterating the argument in Step 2 finitely often, we finally conclude that there is a solution to (2.7) with $\overline{\mathfrak{p}}=1$. The desired result then follows by setting $f'=b'=0$. ∎

THEOREM 2.4. *The FBSDE system* (2.1) *admits a unique solution* $(Q,H,R,Z^H,Z^R)\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}$, *where* $0<\alpha<\beta$; *the constant* β *was defined in* (2.4).

Proof. We first prove the existence of a solution. In a second step we prove the uniqueness of solutions.

Step 1. *Existence of a solution.* By Lemma 2.3, the FBSDE system (2.6) admits a solution $(Q,H,R,Z^H,Z^R)\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}$ when $\mathfrak{p}=0$, for any $f\in L^2_{\mathbb{F}}$. Hence, it remains to prove that if for some $\mathfrak{p}\in[0,1]$ the system (2.6) admits a solution for any $f\in L^2_{\mathbb{F}}$, then there exists a positive constant $\mathfrak{d}>0$ that is independent of $\mathfrak{p}$ and f such that $\mathfrak{p}+\mathfrak{d}\in[0,1]$ and the same result holds true for $\mathfrak{p}+\widehat{\mathfrak{d}}$ for any $\widehat{\mathfrak{d}}\in[0,\mathfrak{d}]$. The proof is similar to the proof of Lemma 2.3.

For any fixed $(Q,R,f)\in\mathcal{H}_\alpha\times L^2_{\mathbb{F}}\times L^2_{\mathbb{F}}$, we introduce the following system:

$$
\begin{cases}
d\widetilde{Q}_t = \left(-\Lambda^1_t\widetilde{R}_t-\Lambda^2_t\mathbb{E}\left[\gamma_t\widetilde{Q}_t\middle|\mathcal{F}^0_t\right]+\overline{f}_t\right)dt,\\
-d\widetilde{H}_t = \Big(-\Lambda^1_tA_t\widetilde{H}_t-\Lambda^2_tA_t\mathbb{E}[\gamma_t\widetilde{Q}_t|\mathcal{F}^0_t]+A_t\overline{f}_t+\mathfrak{p}\Lambda^3_t\mathbb{E}[\zeta_t\widetilde{R}_t|\mathcal{F}^0_t]\\
\qquad\qquad +\mathfrak{p}\Lambda^5_t\mathbb{E}[\varrho_t\widetilde{Q}_t|\mathcal{F}^0_t]+\overline{g}_t\Big)dt,\\
\qquad\qquad +\left(f_t+\mathfrak{d}\Lambda^3_t\mathbb{E}[\zeta_tR_t|\mathcal{F}^0_t]+\mathfrak{d}\Lambda^5_t\mathbb{E}[\varrho_tQ_t|\mathcal{F}^0_t]\right)dt-Z^{\widetilde{H}}_t\,dW_t,\\
-d\widetilde{R}_t = \Big(\Lambda^4_t\widetilde{Q}_t+\mathfrak{p}\Lambda^3_t\mathbb{E}[\zeta_t\widetilde{R}_t|\mathcal{F}^0_s]+\mathfrak{d}\Lambda^3_t\mathbb{E}[\zeta_tR_t|\mathcal{F}^0_s]+\mathfrak{p}\Lambda^5_t\mathbb{E}[\varrho_t\widetilde{Q}_t|\mathcal{F}^0_t]\\
\qquad\qquad +\mathfrak{d}\Lambda^5_t\mathbb{E}[\varrho_tQ_t|\mathcal{F}^0_t]+\overline{g}_t+f_t\Big)dt-Z^{\widetilde{R}}_t\,dW_t,\\
\widetilde{R}=A\widetilde{Q}+\widetilde{H},\ t\in[0,T),\\
\widetilde{Q}_0=\chi,\ \widetilde{Q}_T=0.
\end{cases}
\tag{2.19}
$$

Since $f+\mathfrak{d}\Lambda^3\mathbb{E}[\zeta R|\mathcal{F}^0]+\mathfrak{d}\Lambda^5\mathbb{E}[\varrho Q|\mathcal{F}^0]\in L^2_{\mathbb{F}}$, there exists a solution $(\widetilde{Q},\widetilde{H},\widetilde{R},Z^{\widetilde{H}},Z^{\widetilde{R}})$ $\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}$ by assumption. This defines a mapping

$$
\Phi:(Q,R)\in\mathcal{H}_\alpha\times L^2_{\mathbb{F}}\to(\widetilde{Q},\widetilde{R})\in\mathcal{H}_\alpha\times L^2_{\mathbb{F}}.
\tag{2.20}
$$

It is sufficient to prove the existence of a fixed point of Φ. To this end, for any $Q,\ Q'\in\mathcal{H}_\alpha,\ R,\ R'\in L^2_{\mathbb{F}}$, by integration by parts and using the same arguments leading to the estimate (2.15) we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T\left(\Lambda^1_t-\frac{\|\gamma\||\Lambda^2_t|^2}{2\theta_1}-\frac{\|\Lambda^3\||\zeta_t|^2}{2\theta_2}\right)(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]\\
&\quad+\mathbb{E}\left[\int_0^T\left(\Lambda^4_t-\frac{\theta_1\|\gamma\|}{2}-\frac{\theta_2\|\Lambda^3\|}{2}-\|\Lambda^5\|\|\varrho\|\right)(\widetilde{Q}_t-\widetilde{Q}'_t)^2\,dt\right]\\
&\le 2\delta\mathbb{E}\left[\int_0^T|\widetilde{Q}-\widetilde{Q}'_t|^2\,dt\right]+\frac{\mathfrak{d}\|\Lambda^5\|^2\|\varrho\|^2}{4\delta}\mathbb{E}\left[\int_0^T(Q_t-Q'_t)^2\,dt\right]\\
&\quad+\frac{\mathfrak{d}\|\Lambda^3\|^2\|\zeta\|^2}{4\delta}\mathbb{E}\left[\int_0^T(R_t-R'_t)^2\,dt\right].
\end{aligned}
\tag{2.21}
$$

Assumption 2.1(ii) implies $\Lambda^1_t-\frac{\|\gamma\||\Lambda^2_t|^2}{2\theta_1}-\frac{\|\Lambda^3\||\zeta_t|^2}{2\theta_2}\ge\left(\Lambda^1-\frac{\|\gamma\||\Lambda^2|^2}{2\theta}-\frac{\|\Lambda^3\||\zeta|^2}{2\theta_2}\right)_\star>0$ and $\Lambda^4_t-\frac{\theta_1\|\gamma\|}{2}-\frac{\theta_2\|\Lambda^3\|}{2}-\|\Lambda^5\|\|\varrho\|\ge\left(\Lambda^4-\frac{\theta_1\|\gamma\|}{2}-\frac{\theta_2\|\Lambda^3\|}{2}-\|\Lambda^5\|\|\varrho\|\right)_\star>0$. So we can choose δ small enough such that $\left(\Lambda^4-\frac{\theta_1\|\gamma\|}{2}-\frac{\theta_2\|\Lambda^3\|}{2}-\|\Lambda^5\|\|\varrho\|\right)_\star-2\delta>0$, which implies

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T(\widetilde{R}_t-\widetilde{R}'_t)^2\,dt\right]+\mathbb{E}\left[\int_0^T(\widetilde{Q}_t-\widetilde{Q}'_t)^2\,dt\right]&\le C\mathfrak{d}\mathbb{E}\left[\int_0^T(Q_t-Q'_t)^2\,dt\right]\\
&\quad+C\mathfrak{d}\mathbb{E}\left[\int_0^T(R_t-R'_t)^2\,dt\right].
\end{aligned}
$$

The preceding estimate and the dynamic for $\widetilde{Q}$ allow us to estimate $\widetilde{Q}$ in terms of $\widetilde{R}$ as follows:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|\widetilde{Q}_t-\widetilde{Q}'_t|^2\right]\\
&\le C\mathbb{E}\left[\int_0^T|\widetilde{R}_s-\widetilde{R}'_s|^2\,ds\right]+C\int_0^T\mathbb{E}\left[|\widetilde{Q}'_s-\widetilde{Q}_s|^2\right]\,ds \qquad (2.22)\\
&\le \mathfrak{d}C\mathbb{E}\left[\int_0^T(Q_t-Q'_t)^2\,dt\right]+\mathfrak{d}C\mathbb{E}\left[\int_0^T(R_t-R'_t)^2\,dt\right].
\end{aligned}$$

By (2.22), a similar argument as in (2.18) yields the existence of a uniform C such that for any $\epsilon>0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}\left|\widetilde{H}_t-\widetilde{H}'_t\right|^2\right]&\le C\mathbb{E}\left[\sup_{0\le s\le T}|\widetilde{Q}_s-\widetilde{Q}'_s|^2\right]+C\mathbb{E}\left[\int_0^T|\widetilde{R}_t-\widetilde{R}'_t|^2\,dt\right]\\
&\quad+C\mathfrak{d}\mathbb{E}\left[\sup_{0\le s\le T}|Q_s-Q'_s|^2\right]+C\mathfrak{d}\mathbb{E}\left[\int_0^T|R_t-R'_t|^2\,dt\right]. \qquad (2.23)
\end{aligned}$$

Now we return to the expression of $\widetilde{Q}$ in terms of $\widetilde{H}$, from which we have by (2.22), (2.23), and the same argument as in (2.11) that

$$(2.24)\qquad \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{\widetilde{Q}_t-\widetilde{Q}'_t}{(T-t)^\alpha}\right|^2\right]\le C\mathfrak{d}\|Q-Q'\|_\alpha^2+C\mathfrak{d}\mathbb{E}\left[\int_0^T|R_t-R'_t|^2\,dt\right].$$

We choose $\mathfrak{d}$ small enough such that Φ is a contraction. Thus, we have a fixed point which is a solution to (2.6) when $\mathfrak{p}$ is replaced by $\mathfrak{p}+\mathfrak{d}$.

Since $\mathfrak{d}$ does not depend on $\mathfrak{p}$ or f, iterating the above argument finitely often, we see that (2.6) admits a solution. Taking $\mathfrak{p}=1$ and $f=0$ then yields the existence of a solution to (2.1).

Step 2. *Uniqueness of solutions.* Let us assume to the contrary that there exist two solutions $(Q,H,R,Z^H,Z^R)\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}$ and $(Q',H',R',Z^{H'},Z^{R'})$ $\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}\times L^{2,-}_{\mathbb{F}}$ to (2.1). As in the proof of Step 1, integration by parts for $(Q-Q')(R-R')$ yields

$$(2.25)\qquad \mathbb{E}\left[\int_0^T(R_t-R'_t)^2+(Q_t-Q'_t)^2\,dt\right]=0.$$

Second, by the expression of $(Q-Q')$ in terms of $R-R'$, (2.25) yields that

$$(2.26)\qquad \mathbb{E}\left[\sup_{0\le t\le T}|Q_t-Q'_t|^2\right]=0.$$

Third, the expression for $(H-H')$, (2.25), and (2.26) yield that for any $\epsilon>0$

$$(2.27)\qquad \mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|H_t-H'_t|^2\right]=0.$$

Finally, by the expression for $(Q-Q')$ in terms of $(H-H')$, (2.25), (2.26), (2.27), and arbitrariness of ϵ yield that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{Q_t-Q'_t}{(T-t)^\alpha}\right|^2\right]=0. \tag{2.28}$$

By a standard estimate for linear BSDEs we have for each $\epsilon>0$,

$$\mathbb{E}\left[\int_0^{T-\epsilon}|Z_t^H-Z_t^{H'}|^2\,dt\right]=0,\quad \mathbb{E}\left[\int_0^{T-\epsilon}|Z_t^R-Z_t^{R'}|^2\,dt\right]=0. \qquad \square$$

Remark 2.5. From the proof of Lemma 2.3 and Theorem 2.4 (see, e.g., (2.8) and (2.10)), we see that for $\overline{f}\equiv 0$, the regularity of the solution can be increased to $(Q,H)\in\mathcal{H}_\beta\times\mathcal{H}_\varsigma$, where $\varsigma<\frac{1}{2}\wedge\beta$. This is the case in [25].

The following corollary is important for the analysis of our leader-follower game of optimal portfolio liquidation analyzed below. It implies that the follower's optimal response function is linear convex and hence that the leader's control problem is convex.

COROLLARY 2.6. *For each $(\overline{f},\overline{g})\in S^2_{\mathbb{F}}\times L^2_{\mathbb{F}}$, denote by $(Q,H,R)(\overline{f},\overline{g})$ the solution to* (2.1). *Then, the mapping $(\overline{f},\overline{g})\in S^2_{\mathbb{F}}\times L^2_{\mathbb{F}}\to(Q,H,R)(\overline{f},\overline{g})\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}$ is well defined and for $\rho\in[0,1]$,*

$$(Q,H,R)(\rho(\overline{f},\overline{g})+(1-\rho)(\overline{f}',\overline{g}'))=\rho(Q,H,R)(\overline{f},\overline{g})+(1-\rho)(Q,H,R)(\overline{f}',\overline{g}').$$

Proof. By Theorem 2.4, for each $(\overline{f},\overline{g})\in S^2_{\mathbb{F}}\times L^2_{\mathbb{F}}$, there exists a unique solution (Q,H,R) to (2.1). Thus, the mapping is well defined. Uniqueness of the solution and linearity of the system yield the desired equality. $\square$

Using the same arguments as in the proof of Theorem 2.4 we can also get existence of a unique solution to the "penalized version" of (2.1) where the terminal state constraint on the forward process is replaced by the terminal condition of the backward process $R_T=2nQ_T$. To this end, we introduce the BSDE,

$$-dA_t^n=\left(\Lambda_t^4-\Lambda_t^1(A_t^n)^2\right)\,dt-Z_t^{A^n}\,dW_t,\quad A_T^n=2n.$$

Existence and uniqueness of a solution to this equation has been established in [5]. Moreover, for each $t\in[0,T)$,

$$\lim_{n\to\infty}A_t^n=A_t,\ \text{a.s.}. \tag{2.29}$$

When the terminal state constraint is replaced by the penalty term introduced above, the system (2.1) translates into the following system:

$$\begin{cases} dQ_t^n=\left(-\Lambda_t^1R_t^n-\Lambda_t^2\mathbb{E}\left[\gamma_tQ_t^n|\mathcal{F}_t^0\right]+\overline{f}_t^n\right)\,dt,\\ -dH_t^n=\Big(-\Lambda_t^1A_t^n-\Lambda_t^2A_t^n\mathbb{E}[\gamma_tQ_t^n|\mathcal{F}_t^0]+A_t^n\overline{f}_t^n+\Lambda_t^3\mathbb{E}[\zeta_tR_t^n|\mathcal{F}_t^0]\\ \qquad\qquad+\Lambda_t^5\mathbb{E}[\varrho_tQ_t^n|\mathcal{F}_t^0]+\overline{g}_t^n\Big)\,dt-Z_t^{H^n}\,dW_t,\\ -dR_t^n=\left(\Lambda_t^4Q_t^n+\Lambda_t^3\mathbb{E}[\zeta_tR_t^n|\mathcal{F}_s^0]+\Lambda_t^5\mathbb{E}[\varrho_tQ_t^n|\mathcal{F}_t^0]+\overline{g}_t^n\right)\,dt-Z_t^{R^n}\,dW_t,\\ Q_0^n=\chi,\ H_T^n=0,\ R_T^n=2nQ_T^n. \end{cases} \tag{2.30}$$

COROLLARY 2.7. *Assume that for each fixed $n \in \mathbb{N}$, $(\overline{f}^n, g^n) \in S^2_{\mathbb{F}} \times L^2_{\mathbb{F}}$. Then, for each $n \in \mathbb{N}$ the FBSDE* (2.30) *admits a unique solution $(Q^n, H^n, R^n) \in \mathcal{H}_{\alpha,n} \times S^2_{\mathbb{F}} \times L^2_{\mathbb{F}}$, where*

$$\mathcal{H}_{\alpha,n} = \left\{ X \in S^2_{\mathbb{F}} : \mathbb{E}\left[\sup_{0 \le t \le T} \left| \frac{X_t}{(T - t + \frac{1}{n})^\alpha} \right|^2 \right] < \infty \right\}.$$

Remark 2.8. Note that in (2.30), the terminal condition for H^n is 0 so H^n is defined on $[0,T]$. In (2.1) the process H is only defined on $[0,T)$, due to the singularity of the process A at the terminal time.

2.2. Convergence. We now prove an approximation result for the system (2.1) in terms of the systems (2.30) as $n \to \infty$. The convergence result is established under the additional assumption that for any $0 \le t_1 < t_2 \le T$,

$$e^{-\int_{t_1}^{t_2} \Lambda^1_u A_u \, du} \le C \frac{T - t_2}{T - t_1} \quad \text{and} \quad e^{-\int_{t_1}^{t_2} \Lambda^1_u A^n_u \, du} \le C \frac{T - t_2 + \frac{1}{n}}{T - t_1 + \frac{1}{n}}. \tag{2.31}$$

We refer the reader to [25] for sufficient conditions on the model parameters under which this assumption is satisfied.

The proof of the following lemma can be found in [25, Lemma 4.4].

LEMMA 2.9. *Let $\overline{f}^n \in S^2_{\mathbb{F}}$ and $\overline{g}^n \in L^2_{\mathbb{F}}$ be two sequences of progressively measurable stochastic processes, and let (Q^n, H^n, R^n) be the solution to the system* (2.30). *If the sequences $\overline{f}^n$ and $\overline{g}^n$ are bounded in $S^2_{\mathbb{F}}$ and $L^2_{\mathbb{F}}$ uniformly in n, respectively, then*

$$\sup_n \|Q^n\|_{\alpha,n} + \sup_n \|H^n\|_{S^{2,-}} + \sup_n \|R^n\|_{L^2} \le C \left(\sup_n \|\overline{f}^n\|_{S^2} + \sup_n \|\overline{g}^n\|_{L^2} \right) < \infty.$$

LEMMA 2.10. *Let $\overline{f}^n$ and $\overline{g}^n$ be two sequences of stochastic processes satisfying the conditions in Lemma* 2.9. *Then there exists $\overline{f} \in L^2_{\mathbb{F}}$, $\overline{g} \in L^2_{\mathbb{F}}$ and a convex combination of a subsequence of $(\overline{f}^n, \overline{g}^n)$ converging to $(\overline{f}, \overline{g})$ in L^ν with $1 < \nu < 2$, i.e.,*

$$\lim_{N \to \infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{N} \sum_{k=1}^N (\overline{f}^{n_k}_t, \overline{g}^{n_k}_t) - (\overline{f}_t, \overline{g}_t) \right|^\nu dt \right] = 0. \tag{2.32}$$

Proof. Since the sequence $(\overline{f}^n, \overline{g}^n)$ is L^2 uniformly bounded, the proof of [8, Theorem 2.1] (see also [35]) tells us there exists a subsequence of $(\overline{f}^n, \overline{g}^n)$ and a progressively measurable stochastic processes $(\overline{f}, \overline{g})$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N (\overline{f}^{n_k}, \overline{g}^{n_k}) - (\overline{f}, \overline{g}) = 0 \quad \text{a.e. a.s. on } [0,T] \times \Omega.$$

Fatou's lemma implies that

$$\mathbb{E}\left[\int_0^T |(\overline{f}_t, \overline{g}_t)|^2 \, dt \right] \le \liminf_{N \to \infty} \frac{1}{N} \sum_{k=1}^N \mathbb{E}\left[\int_0^T |(\overline{f}^{n_k}_t, \overline{g}^{n_k}_t)|^2 \, dt \right] < \infty.$$

Thus, Vitali's convergence result implies (2.32). ∎

The following theorem proves a convergence result for the FBSDE systems associated with the unconstrained penalized control problems to the system associated with the constrained one. The result is key to our maximum principle for the leader-follower game introduced above.

THEOREM 2.11. *Let $(\overline{f}^n, \overline{g}^n)$ be a sequence satisfying the conditions in Lemma 2.10 and $(\overline{f}, \overline{g}) \in L^2_{\mathbb{F}} \times L^2_{\mathbb{F}}$ be the limit. Let (Q^n, H^n, R^n) and (Q, H, R) be the solution to* (2.30) *and* (2.1)*, respectively. We further assume the limit $\overline{f}$ belongs to $S^2_{\mathbb{F}}$. Then there exists a convex combination of a subsequence of $\left(\frac{1}{N}\sum_{k=1}^N Q^{n_k}, \frac{1}{N}\sum_{k=1}^N H^{n_k}, \frac{1}{N}\sum_{k=1}^N R^{n_k}\right)$ converging to (Q, H, R) in $S^\nu_{\mathbb{F}} \times L^1_{\mathbb{F}} \times L^\nu_{\mathbb{F}}$ with $1 < \nu < 2$, i.e.,*

$$\lim_{N' \to \infty} \mathbb{E}\left[\sup_{0 \le t \le T} \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} Q_t^{n_k} - Q_t \right|^\nu \right] = 0,$$

$$\lim_{N' \to \infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} H_t^{n_k} - H_t \right| dt \right] = 0,$$

$$\lim_{N' \to \infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} R_t^{n_k} - R_t \right|^\nu dt \right] = 0.$$

Proof. The uniform boundedness of $\overline{f}^n$ and $\overline{g}^n$ implies the uniform boundedness of R^n in L^2 (Lemma 2.9) and the uniform boundedness of $\frac{1}{N}\sum_{k=1}^N R^{n_k}$ in L^2. Thus, [8, Theorem 2.1] again yields the existence of a progressively measurable process $\overline{R} \in L^2_{\mathbb{F}}$ and a subsequence of $\frac{1}{N}\sum_{k=1}^N R^{n_k}$ such that

$$\lim_{N' \to \infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} R_t^{n_k} - \overline{R}_t \right|^\nu dt \right] = 0. \tag{2.33}$$

By (2.32), the convergence of the same convex combination holds for $(\overline{f}^n, \overline{g}^n)$:

$$\lim_{N' \to \infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} (\overline{f}_t^{n_k}, \overline{g}_t^{n_k}) - (\overline{f}_t, \overline{g}_t) \right|^\nu dt \right] = 0. \tag{2.34}$$

Define $\overline{Q}$ as the unique solution in $S^2_{\mathbb{F}}$ to the following mean field SDE in terms of the limits $\overline{f}$ and $\overline{R}$:

$$\overline{Q}_t = \chi + \int_0^t \left(-\Lambda^1_s \overline{R}_s - \Lambda^2_s \mathbb{E}[\gamma_s \overline{Q}_s | \mathcal{F}^0_s] + \overline{f}_s \right) ds. \tag{2.35}$$

Standard SDE estimates, (2.33), and (2.34) yield

$$\lim_{N' \to \infty} \mathbb{E}\left[\sup_{0 \le t \le T} \left| \frac{1}{N'} \sum_{j=1}^{N'} \frac{1}{N_j} \sum_{k=1}^{N_j} Q_t^{n_k} - \overline{Q}_t \right|^\nu \right] = 0. \tag{2.36}$$

Now define $\overline{H}$ in terms of the limits $\overline{f}$, $\overline{R}$, and $\overline{Q}$ as

$$\begin{aligned}\overline{H}_t =& \mathbb{E}\Big[\int_t^T e^{-\int_t^s \Lambda^1_u A_u \, du} \big(-\Lambda^2_s A_s \mathbb{E}[\gamma_s \overline{Q}_s | \mathcal{F}^0_s] + A_s \overline{f}_s + \Lambda^3_s \mathbb{E}[\zeta_s \overline{R}_s | \mathcal{F}^0_s] \\ & + \Lambda^5_s \mathbb{E}[\varrho_s \overline{Q}_s | \mathcal{F}^0_s] + \overline{g}_s \big) \, ds \Big| \mathcal{F}_t \Big].\end{aligned} \tag{2.37}$$

By (2.3), (2.31), and the Hölder inequality, we have for any $h \in L^{\nu}_{\mathbb{F}}$

$$
\begin{aligned}
\mathbb{E}\left[\int_t^T e^{-\int_t^s \Lambda_u^1 A_u\,du} A_s|h_s|\,ds\,\middle|\,\mathcal{F}_t\right] &\le \frac{1}{T-t}\mathbb{E}\left[\int_t^T |h_s|\,ds\,\middle|\,\mathcal{F}_t\right] \\
&\le \frac{1}{(T-t)^{\frac{1}{\nu}}}\mathbb{E}\left[\int_t^T |h_s|^{\nu}\,ds\,\middle|\,\mathcal{F}_t\right]^{\frac{1}{\nu}}.
\end{aligned}
\tag{2.38}
$$

Thus, by (2.3), (2.38), and the Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j} H_t^{n_k} - \overline{H}_t\right| \\
\le &\frac{C}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\left(\int_t^T \left|e^{-\int_t^s \Lambda_u^1 A_u^{n_k}\,du}A_s^{n_k} - e^{-\int_t^s \Lambda_u^1 A_u\,du}A_s\right|ds\right)^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&\times\left(\mathbb{E}\left[\sup_{0\le s\le T}|\mathbb{E}[Q_s^{n_k}|\mathcal{F}_s^0]|^2 + \sup_{0\le s\le T}(\overline{f}_s^{n_k})^2\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&+\frac{C}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\int_t^T\left|e^{-\int_t^s \Lambda_u^1 A_u^{n_k}\,du} - e^{-\int_t^s \Lambda_u^1 A_u\,du}\right|^2 ds\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&\cdot\left(\mathbb{E}\left[\int_t^T|\overline{g}_s^{n_k}|^2\,ds\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&+\frac{C}{T-t}\mathbb{E}\left[\int_t^T\mathbb{E}\left[\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}Q_s^{n_k} - \overline{Q}_s\right|\middle|\mathcal{F}_s^0\right] + \left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}f_s^{n_k} - \overline{f}_s\right|ds\middle|\mathcal{F}_t\right] \\
&+\frac{C}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\int_t^T\left|e^{-\int_t^s \Lambda_u^1 A_u^{n_k}} - e^{-\int_t^s \Lambda_u^1 A_u}\right|^2 ds\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&\cdot\left(\mathbb{E}\left[\int_t^T\mathbb{E}[(R_s^{n_k})^2 + (Q_s^{n_k})^2|\mathcal{F}_s^0]\,ds\middle|\mathcal{F}_t\right]\right)^{\frac{1}{2}} \\
&+C\mathbb{E}\left[\int_t^T\mathbb{E}\left[\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}R_s^{n_k} - \overline{R}_s\right| + \left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}Q_s^{n_k} - \overline{Q}_s\right|\middle|\mathcal{F}_s^0\right]ds\middle|\mathcal{F}_t\right] \\
&+C\mathbb{E}\left[\int_t^T\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\overline{g}_s^{n_k} - \overline{g}_s\right|ds\middle|\mathcal{F}_t\right]
\end{aligned}
$$

Applying the Hölder inequality again along with Doob's maximal inequality, the uniform boundedness of $(Q^n, R^n, \overline{f}^n, \overline{g}^n)$, the dominated convergence theorem, and the convergence (2.29) we get

$$
\mathbb{E}\left[\int_0^T\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}H_t^{n_k} - \overline{H}_t\right|dt\right]
$$

$$
\begin{aligned}
&\le C\sup_n\left(\mathbb{E}\left[\sup_{0\le s\le T}|Q^n_s|^2+\sup_{0\le s\le T}|\overline{f}^n_s|^2\right]\right)^{\frac12}\\
&\quad\times\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\int_0^T\left(\int_t^T\left|e^{-\int_t^s\Lambda^1_uA^{n_k}_u\,du}A^{n_k}_s-e^{-\int_t^s\Lambda^1_uA_u\,du}A_s\right|\,ds\right)^2\,dt\right]\right)^{\frac12}\\
&\quad+C\sup_n\left(\mathbb{E}\left[\int_0^T|\overline{g}^n_t|^2\,dt\right]\right)^{\frac12}\\
&\quad\cdot\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\int_0^T\int_t^T\left|e^{-\int_t^s\Lambda^1_uA^{n_k}_u\,du}-e^{-\int_t^s\Lambda^1_uA_u\,du}\right|^2\,ds\,dt\right]\right)^{\frac12}\\
&\quad+C\int_0^T\frac{1}{(T-t)^{\frac1\nu}}\,dt\\
&\quad\cdot\left(\mathbb{E}\left[\int_0^T\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}Q^{n_k}_s-\overline{Q}_s\right|^\nu+\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}f^{n_k}_s-\overline{f}_s\right|^\nu\,ds\right]\right)^{\frac1\nu}\\
&\quad+C\sup_n\left(\mathbb{E}\left[\int_0^T(R^n_s)^2+(Q^n_s)^2\,ds\right]\right)^{\frac12}\\
&\quad\cdot\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(\mathbb{E}\left[\int_0^T\int_t^T\left|e^{-\int_t^s\Lambda^1_uA^{n_k}_u\,du}-e^{-\int_t^s\Lambda^1_uA_u\,du}\right|^2\,ds\,dt\right]\right)^{\frac12}\\
&\quad+C\mathbb{E}\left[\int_0^T\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}R^{n_k}_s-\overline{R}_s\right|+\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}Q^{n_k}_s-\overline{Q}_s\right|\,ds\right]\\
&\quad+C\mathbb{E}\left[\int_0^T\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}\overline{g}^{n_k}_s-\overline{g}_s\right|\,ds\right]\\
&\to 0\qquad\text{as } N'\to\infty.
\end{aligned}
\tag{2.39}
$$

Let $\widehat{R}=A\overline{Q}+\overline{H}$. For any $\widetilde{T}<T$, by (2.36) and (2.39) we have

$$\lim_{N'\to\infty}\mathbb{E}\left[\int_0^{\widetilde{T}}\left|\frac{1}{N'}\sum_{j=1}^{N'}\frac{1}{N_j}\sum_{k=1}^{N_j}R^{n_k}_t-\widehat{R}_t\right|\,dt\right]=0.$$

Thus, (2.33) implies that for any $\widetilde{T}<T$,

$$\mathbb{E}\left[\int_0^{\widetilde{T}}|\widehat{R}_t-\overline{R}_t|\,dt\right]=0.$$

This proves that

$$\widehat{R}=\overline{R},\ \text{a.e. a.s. on } [0,T]\times\Omega.$$

Moreover,

$$(\overline{Q},\overline{H},\widehat{R})\in\mathcal{H}_\alpha\times S^{2,-}_{\mathbb{F}}\times L^2_{\mathbb{F}}.$$

Indeed, since $\overline{R} \in L^2_{\mathbb{F}}$ and $\overline{R} = \widehat{R}$ a.e. a.s. on $[0,T] \times \Omega$, we have that $\widehat{R} \in L^2_{\mathbb{F}}$. Moreover, (2.35) implies that $\overline{Q} \in S^2_{\mathbb{F}}$, from which (2.37) implies $H \in S^{2,-}_{\mathbb{F}}$, and taking $\widehat{R} = A\overline{Q} + \overline{H}$ into (2.35) yields $\overline{Q} \in \mathcal{H}_\alpha$. By (2.37), there exists $Z^{\overline{H}} \in L^{2,-}_{\mathbb{F}}$ such that $(\overline{Q}, \widehat{R}, \overline{H}, Z^{\overline{H}})$ satisfies the second equation in (2.1). Moreover, since $\widehat{R} = A\overline{Q} + \overline{H}$ the triple $(\overline{Q}, \widehat{R}, Z^{\widehat{R}})$ satisfies the third equation in (2.1), where $Z^{\widehat{R}} = \overline{Q}Z^A + Z^{\overline{H}}$. This implies for each $\epsilon > 0$

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon} |Z_t^{\widehat{R}}|^2\,dt\right] \le C\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}\left|\int_0^t Z_s^{\widehat{R}}\,dW_s\right|^2\right] \quad \text{(BDG inequality)}\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|A_t\overline{Q}_t + \overline{H}_t|^2\right] + C\mathbb{E}\left[\int_0^T |\overline{Q}_t|^2 + |\widehat{R}_t|^2 + |\overline{g}_t|^2\,dt\right] \ \text{(by the SDE for } \widehat{R}\text{)}\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T-\epsilon}|\overline{Q}_t|^2 + \sup_{0\le t\le T-\epsilon}|\overline{H}_t|^2\right] + C\mathbb{E}\left[\int_0^T |\overline{Q}_t|^2 + |\widehat{R}_t|^2 + |\overline{g}_t|^2\,dt\right]\\
&\qquad (A \text{ is bounded on } [0,T-\epsilon])\\
&< \infty.
\end{aligned}
$$

Hence, $(\overline{Q}, \overline{H}, \widehat{R}, Z^{\overline{H}}, Z^{\widehat{R}})$ satisfies the system (2.1). The uniqueness of solutions in $\mathcal{H}_\alpha \times S^{2,-}_{\mathbb{F}} \times L^2_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}} \times L^{2,-}_{\mathbb{F}}$ yields the desired convergence result. ∎

3. A MFC problem of optimal portfolio liquidation. In this section, we solve the single player portfolio liquidation model with expectations feedback introduced in section 1.2.1. Setting $\chi = x$, $\Lambda^1 = -\Lambda^2 = \zeta = \frac{1}{2\eta}$, $\gamma = \Lambda^3 = \varrho = \kappa$, $\Lambda^4 = 2\lambda$, $\Lambda^5 = -\kappa\mathbb{E}\big[\frac{1}{2\eta}\big|\mathcal{F}^0\big]$, $\overline{f} = 0$, and $\overline{g} = \widetilde{g}$, then FBSDE (1.2) reduces to FBSDE (1.7). We make the following assumption.

Assumption 3.1. *The process $\widetilde{g}$ belongs to $L^2_{\mathbb{F}}$. The progressively measurable stochastic processes η, κ, and λ are nonnegative and essentially bounded. Moreover, there exists some $\theta' > 0$ such that*

$$
\eta_\star - \frac{\|\kappa\|}{2\theta'} > 0, \qquad \lambda_\star - \|\kappa\|\theta' > 0. \tag{3.1}
$$

Assumption 3.1 implies Assumption 2.1. In particular, a direct computation shows that condition (3.1) implies Assumption 2.1(ii) with $\theta_1 = \theta_2 = \theta'$. The condition (3.1) means that the mean-field interaction is sufficiently weak. A similar condition has been made in the game-theoretic literature before; see, e.g., [28] and references therein.

The trader's objective is to minimize the cost function $J(\cdot)$ introduced in (1.4) over the set of admissible controls

$$
\mathcal{A}_{\mathbb{F}}(x) = \left\{\xi \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) : \int_0^T \xi_s\,ds = x\right\}.
$$

A standard stochastic maximum principle suggests the candidate optimal strategy is given by

$$
\xi_t^* = \frac{Y_t - \mathbb{E}[\kappa_t X_t|\mathcal{F}_t^0]}{2\eta_t}, \tag{3.2}
$$

where $(X,Y) \in \mathcal{H}_\alpha \times L^2_{\mathbb{F}}$ is the unique solution to the FBSDE system (1.7). Standard arguments show that $\xi^* \in \mathcal{A}_{\mathbb{F}}(x)$. To prove that ξ^* is indeed the unique optimal control, we establish an auxiliary result that substitutes for the lack of convexity of the Hamiltonian for our MFC problem.

LEMMA 3.2. *For every $t \in [0,T)$, we have*

(3.3)
$$\begin{aligned}&\mathbb{E}\left[\kappa_t X_t \mathbb{E}[\xi_t|\mathcal{F}_t^0] + \eta_t \xi_t^2 + \lambda_t X_t^2\right] - \mathbb{E}\left[\kappa_t X_t' \mathbb{E}[\xi_t'|\mathcal{F}_t^0] + \eta_t (\xi_t')^2 + \lambda_t (X_t')^2\right] \\ &\geq \mathbb{E}\left[\left(\mathbb{E}[\kappa_t X_t'|\mathcal{F}_t^0] + 2\eta_t \xi_t'\right)(\xi_t - \xi_t') + 2\lambda_t X_t'(X_t - X_t') + \kappa_t (X_t - X_t')\mathbb{E}[\xi_t'|\mathcal{F}_t^0]\right].\end{aligned}$$

Moreover, the above inequality becomes an equality if and only if $\xi_t = \xi_t'$ a.s..

Proof. To prove (3.2), it is equivalent to show for $t \in [0,T)$,

$$\mathbb{E}\left[\eta_t(\xi_t - \xi_t')^2 + \lambda_t (X_t - X_t')^2 + \mathbb{E}[(\xi_t - \xi_t')|\mathcal{F}_t^0]\mathbb{E}[\kappa_t (X_t - X_t')|\mathcal{F}_t^0]\right] \geq 0.$$

Note that using Young's inequality

$$\begin{aligned}&\left|\mathbb{E}\left[\mathbb{E}[(\xi_t - \xi_t')|\mathcal{F}_t^0]\mathbb{E}[\kappa_t (X_t - X_t')|\mathcal{F}_t^0]\right]\right| \\ &\leq \|\kappa\|\mathbb{E}\left[\mathbb{E}[|\xi_t - \xi_t'||\mathcal{F}_t^0]\mathbb{E}[|X_t - X_t'||\mathcal{F}_t^0]\right] \\ &\leq \frac{\|\kappa\|}{2\theta'}\mathbb{E}\left[\left(\mathbb{E}[|\xi_t - \xi_t'||\mathcal{F}_t^0]\right)^2\right] + \frac{\|\kappa\|\theta'}{2}\mathbb{E}\left[\left(\mathbb{E}[|X_t - X_t'||\mathcal{F}_t^0]\right)^2\right].\end{aligned}$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[\eta_t(\xi_t - \xi_t')^2 + \lambda_t (X_t - X_t')^2 + \mathbb{E}[(\xi_t - \xi_t')|\mathcal{F}_t^0]\mathbb{E}[\kappa_t (X_t - X_t')|\mathcal{F}_t^0]\right] \\ &\geq \mathbb{E}\Bigg[\left(\eta_\star - \frac{\|\kappa\|}{2\theta'}\right)(\xi_t - \xi_t')^2 + \left(\lambda_\star - \frac{\|\kappa\|\theta'}{2}\right)(X_t - X_t')^2 \\ &\quad - \|\kappa\|\mathbb{E}[|\xi_t - \xi_t'||\mathcal{F}_t^0]\mathbb{E}[|X_t - X_t'||\mathcal{F}_t^0]\Bigg] \\ &\quad + \frac{\|\kappa\|}{2\theta'}\mathbb{E}\left[(\xi_t - \xi_t')^2\right] + \frac{\|\kappa\|\theta'}{2}\mathbb{E}\left[(X_t - X_t')^2\right] \\ &\geq \mathbb{E}\Bigg[\left(\eta_\star - \frac{\|\kappa\|}{2\theta'}\right)(\xi_t - \xi_t')^2 + \left(\lambda_\star - \frac{\|\kappa\|\theta'}{2}\right)(X_t - X_t')^2 \\ &\quad - \|\kappa\|\mathbb{E}[|\xi_t - \xi_t'||\mathcal{F}_t^0]\mathbb{E}[|X_t - X_t'||\mathcal{F}_t^0]\Bigg] \\ &\quad + \frac{\|\kappa\|}{2\theta'}\mathbb{E}\left[(\mathbb{E}[|\xi_t - \xi_t'||\mathcal{F}_t^0])^2\right] + \frac{\|\kappa\|\theta'}{2}\mathbb{E}\left[(\mathbb{E}[|X_t - X_t'||\mathcal{F}_t^0])^2\right] \\ &\geq \mathbb{E}\left[\left(\eta_\star - \frac{\|\kappa\|}{2\theta'}\right)(\xi_t - \xi_t')^2 + \left(\lambda_\star - \frac{\|\kappa\|\theta'}{2}\right)(X_t - X_t')^2\right] \\ &\geq 0.\end{aligned}$$

The second claim is obvious from the above estimate. ◻

We are now ready to state and prove the main result of this section.

THEOREM 3.3. *Under Assumption* 3.1 *the process ξ^* defined in* (3.2) *is the unique optimal control to the MFC problem* (1.4)–(1.5).

Proof. To prove the optimality of the candidate strategy ξ^* we fix an arbitrary control $\xi \in \mathcal{A}_{\mathbb{F}}(x)$ and denote by X^* and X the corresponding state processes. For any $\epsilon > 0$, it follows from Lemma 3.2 that

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon} \kappa_t X_t \mathbb{E}[\xi_t|\mathcal{F}_t^0] + \widetilde{g}_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2 \, dt\right] \\
&\quad - \mathbb{E}\left[\int_0^{T-\epsilon} \kappa_t X_t^* \mathbb{E}[\xi_t^*|\mathcal{F}_t^0] + \widetilde{g}_t X_t^* + \eta_t (\xi_t^*)^2 + \lambda_t (X_t^*)^2 \, dt\right] \\
&\geq \mathbb{E}\left[\int_0^{T-\epsilon} \left(\mathbb{E}[\kappa_t X_t^*|\mathcal{F}_t^0] + 2\eta_t \xi_t^*\right)(\xi_t - \xi_t^*) + (2\lambda_t X_t^* + \kappa_t \mathbb{E}[\xi_t^*|\mathcal{F}_t^0] + \widetilde{g}_t)(X_t - X_t^*) \, dt\right].
\end{aligned} \tag{3.4}
$$

Integration by part yields

$$
\begin{aligned}
&\mathbb{E}\left[Y_{T-\epsilon}(X_{T-\epsilon} - X_{T-\epsilon}^*)\right] \\
&= -\mathbb{E}\left[\int_0^{T-\epsilon} Y_t(\xi_t - \xi_t^*) \, dt\right] - \mathbb{E}\left[\int_0^{T-\epsilon} (X_t - X_t^*) \left(\kappa_t \mathbb{E}\left[\left.\frac{Y_t^*}{2\eta_t}\right| \mathcal{F}_t^0\right]\right.\right. \\
&\qquad \left.\left. -\kappa_t \mathbb{E}\left[\left.\frac{1}{2\eta_t}\right| \mathcal{F}_t^0\right] \mathbb{E}[\kappa_t X_t^*|\mathcal{F}_t^0] + 2\lambda_t X_t^* + \widetilde{g}_t\right) \, dt\right] \\
&= -\mathbb{E}\left[\int_0^{T-\epsilon} Y_t(\xi_t - \xi_t^*) \, dt\right] - \mathbb{E}\left[\int_0^{T-\epsilon} (X_t - X_t^*) \left(\kappa_t \mathbb{E}\left[\xi_t^*|\mathcal{F}_t^0\right] + 2\lambda_t X_t^* + \widetilde{g}_t\right) \, dt\right].
\end{aligned} \tag{3.5}
$$

Putting (3.5) into (3.4), we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{T-\epsilon} \kappa_t X_t \mathbb{E}[\xi_t|\mathcal{F}_t^0] + \widetilde{g}_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2 \, dt\right] \\
&\quad - \mathbb{E}\left[\int_0^{T-\epsilon} \kappa_t X_t^* \mathbb{E}[\xi_t^*|\mathcal{F}_t^0] + \widetilde{g}_t X_t^* + \eta_t (\xi_t^*)^2 + \lambda_t (X_t^*)^2 \, dt\right] \\
&\quad + \mathbb{E}\left[Y_{T-\epsilon}(X_{T-\epsilon} - X_{T-\epsilon}^*)\right] \\
&\geq \mathbb{E}\left[\int_0^{T-\epsilon} \left(\mathbb{E}[\kappa_t X_t^*|\mathcal{F}_t^0] + 2\eta_t \xi_t^* - Y_t\right)(\xi_t - \xi_t^*) \, dt\right] = 0.
\end{aligned} \tag{3.6}
$$

Letting $\epsilon \to 0$, a similar argument as the proof of [25, Theorem 2.9] yields that

$$
\lim_{\epsilon \to 0} \mathbb{E}[Y_{T-\epsilon}(X_{T-\epsilon} - X_{T-\epsilon}^*)] = 0.
$$

Thus, (3.6) implies

$$
J(\xi) \geq J(\xi^*).
$$

In order to prove the uniqueness of optimal controls, let ξ' be another optimal control. Then, (3.6) yields

$$
\begin{aligned}
0 &= \mathbb{E}\left[\int_0^T \kappa_t X_t \mathbb{E}[\xi'_t|\mathcal{F}^0_t] + \widetilde{g}_t X'_t + \eta_t(\xi'_t)^2 + \lambda_t (X'_t)^2\,dt\right] \\
&\quad - \mathbb{E}\left[\int_0^T \kappa_t X^*_t \mathbb{E}[\xi^*_t|\mathcal{F}^0_t] + \widetilde{g}_t X^*_t + \eta_t(\xi^*_t)^2 + \lambda_t (X^*_t)^2\,dt\right] \\
&\geq \mathbb{E}\left[\int_0^T \left(\mathbb{E}[\kappa_t X^*_t|\mathcal{F}^0_t] + 2\eta_t \xi^*_t - Y_t\right)(\xi'_t - \xi^*_t)\,dt\right] = 0.
\end{aligned}
$$

Thus, (3.4) holds with an equality. The second claim in Lemma 3.2 implies the uniqueness. ∎

4. A Stackelberg game of optimal portfolio liquidation. In this section, we solve the Stackelberg game of optimal portfolio liquidation introduced in section 1.2.2 above. We make the following assumption which implies Assumption 2.1 and condition (2.31).

ASSUMPTION 4.1. (i) *The processes* $\widetilde{\kappa}^0$, κ, η, $1/\eta$, *and* λ *belong to* $L^\infty_{\mathbb{F}}([0,T]\times\Omega;[0,\infty))$.

(ii) *The processes* $\overline{\kappa}^0$, κ^0, η^0, $1/\eta^0$, *and* λ^0 *belong to* $L^\infty_{\mathbb{F}^0}([0,T]\times\Omega;[0,\infty))$.

(ii) *For some positive constants* θ', θ, *and* $\overline{\theta}$,

$$\eta_\star - \frac{\|\kappa\|}{2\theta'} > 0, \quad \lambda_\star - \|\kappa\|\theta' > 0.$$

and

$$\eta^0_\star - \frac{\|\kappa^0\|}{2\theta} > 0, \quad \lambda^0_\star - \frac{\|\kappa^0\|\theta}{2} - \frac{\|\overline{\kappa}^0\|\overline{\theta}}{2} > 0, \quad \overline{\lambda_\star} - \frac{\|\overline{\kappa}^0\|}{2\overline{\theta}} > 0.$$

(iv) *For any* $0 \leq s < t \leq T$,

$$e^{-\int_s^t \frac{A_u}{2\eta_u}\,du} \leq C\left(\frac{T-t}{T-s}\right)$$

and

$$e^{-\int_s^t \frac{A^n_u}{2\eta_u}\,du} \leq C\left(\frac{T-t+\frac{1}{n}}{T-s+\frac{1}{n}}\right).$$

The problem of the Stackelberg leader is to minimize the cost functional (1.9) over the set of admissible controls

$$\mathcal{A}_{\mathbb{F}^0}(x^0) = \left\{\xi^0 \in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R}) : \int_0^T \xi^0_s\,ds = x^0\right\}.$$

The follower's optimal response function is given by

$$\overline{\xi}_t := \overline{\xi}_t(\xi^0) := \frac{Y_t(\xi^0) - \mathbb{E}[\kappa_t X_t(\xi^0)|\mathcal{F}^0_t]}{2\eta_t}, \tag{4.1}$$

where (X,Y) is the solution to (1.7) with $\widetilde{g} = \widetilde{\kappa}^0\xi^0$. We will occasionally drop the dependence on ξ^0 if there is no confusion. Under Assumption 4.1 the solution (X,Y) enjoys better regularity properties than the process (Q,H) established in Theorem 2.4, due to Remark 2.5 and the estimate (2.3).

COROLLARY 4.2. *Under Assumption* 4.1*, the solution to* (1.7) *belongs to* $\mathcal{H}_1 \times S^2_{\mathbb{F}}$*. Moreover,* $Y = AX + B$ *with* $B \in \mathcal{H}_\varsigma$*.*

In the next section we first prove that the leader's problem has a unique solution if the terminal state constraints are replaced by finite penalty terms and establish a necessary maximum principle for the penalized problem. Subsequently, we prove the convergence of the state and adjoint equations of the penalized problems as the degree of penalization tends to infinity.

4.1. The penalized problem: Existence and maximum principle. The penalized optimization problem is obtained by replacing the terminal state constraint on the leader's and follower's state process by a finite penalty term. The leader's problem consists of minimizing the cost functional

$$
\begin{aligned}
J^{0,n}(\xi^0) = \mathbb{E}\bigg[\int_0^T & \overline{\kappa}^0_s \mathbb{E}[\overline{\xi}^n_s | \mathcal{F}^0_s] X^0_s + \kappa^0_s \xi^0_s X^0_s + \eta^0_s (\xi^0_s)^2 \\
& + \lambda^0_s (X^0_s)^2 + \overline{\lambda}_s (\mathbb{E}[\overline{\xi}^n_s | \mathcal{F}^0_s])^2 \, ds + n (X^0_T)^2 \bigg]
\end{aligned}
\tag{4.2}
$$

over all controls $\xi^0 \in L^2_{\mathbb{F}^0}$ subject to the state dynamics

$$
\begin{cases}
dX^0_t = -\xi^0_t \, dt, \\
dX_t = -\dfrac{Y_t - \mathbb{E}[\kappa_t X_t | \mathcal{F}^0_t]}{2\eta_t} \, dt, \\
-dY_t = \left(\kappa_t \mathbb{E}\left[\dfrac{Y_t}{2\eta_t} \middle| \mathcal{F}^0_t \right] - \kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t} \middle| \mathcal{F}^0_t \right] \mathbb{E}[\kappa_t X_t | \mathcal{F}_t] + 2\lambda_t X_t + \widetilde{\kappa}^0_t \xi^0_t \right) dt - Z_t \, dW_t, \\
X_0 = x, \; X^0_0 = x^0, \; Y_T = 2nX_T,
\end{cases}
\tag{4.3}
$$

where the optimal response for the penalized follower $\overline{\xi}^n$ is defined as follows in terms of (X, Y) in (4.3),

$$
\overline{\xi}^n = \frac{Y - \mathbb{E}[\kappa X | \mathcal{F}^0]}{2\eta}.
$$

We are now going to show that the penalized optimization problem has a unique solution. Similar arguments could be used to prove the existence of an optimal control for the original problem. They would not, however, give us an open-loop characterization of the optimal control.

THEOREM 4.3. *For each* $n \in \mathbb{N}$*, the penalized optimization problem* (4.2)–(4.3) *admits a unique optimal control in* $L^2_{\mathbb{F}^0}$*.*

Proof. In view of Corollary 2.7 the system (4.3) is well-posed for each fixed $\xi^0 \in L^2_{\mathbb{F}^0}$. The representation of the cost functional

$$
\begin{aligned}
& J^{0,n}(\xi^0) \\
& = \mathbb{E}\Bigg[\int_0^T \frac{\overline{\kappa}^0_t}{2} \left(\sqrt{\overline{\theta}} X^0_t + \frac{\mathbb{E}\left[\overline{\xi}^n_t \middle| \mathcal{F}^0_t \right]}{\sqrt{\overline{\theta}}} \right)^2 + \frac{\kappa^0_t}{2} \left(\sqrt{\theta} X^0_t + \frac{\xi^0_t}{\sqrt{\theta}} \right)^2 \\
& \quad + \left(\lambda^0_t - \frac{\overline{\kappa}^0_t \overline{\theta}}{2} - \frac{\kappa^0_t \theta}{2} \right) (X^0_t)^2 \\
& \quad + \left(\eta^0_t - \frac{\kappa^0_t}{2\theta} \right) (\xi^0_t)^2 + \left(\overline{\lambda}_t - \frac{\overline{\kappa}^0_t}{2\overline{\theta}} \right) \left(\mathbb{E}\left[\overline{\xi}^n_t \middle| \mathcal{F}^0_t \right] \right)^2 dt + n (X^0_T)^2 \Bigg]
\end{aligned}
\tag{4.4}
$$

along with Corollary 2.6 and Assumption 4.1 shows that $J^{0,n}$ is strictly convex. Uniqueness of the optimal strategy follows.

Let $J^* = \inf_{\xi^0 \in L^2_{\mathbb{F}^0}} J^{0,n}(\xi^0)$. Then $J^* < \infty$ because $J^{0,n}(x^0/T)$ is bounded. Let $\{\xi^{0,n,m}\} \subseteq L^2_{\mathbb{F}^0}$ be a sequence such that

$$\lim_{m\to\infty} J^{0,n}(\xi^{0,n,m}) = J^*.$$

By Assumption 4.1 this implies

$$\sup_m \mathbb{E}\left[\int_0^T (\xi_s^{0,n,m})^2\,ds\right] < C. \tag{4.5}$$

Thus, Lemma 2.10 implies the existence of some $\xi^{0,n,*} \in L^2_{\mathbb{F}^0}$ such that

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^T \left|\overline{\xi}_t^{0,n,N} - \xi_t^{0,n,*}\right|^\nu dt\right] = 0,\ 1<\nu<2, \tag{4.6}$$

where

$$\overline{\xi}^{0,n,N} = \frac{1}{N}\sum_{k=1}^N \xi^{0,n,m_k}.$$

Let $(X^{0,n,*}, X^{n,*}, Y^{n,*})$ be the solution to (4.3) associated with $\xi^{0,n,*}$. Then the same argument as in the proof of Theorem 2.11 implies

$$\lim_{\overline{N}\to\infty} \mathbb{E}\left[\int_0^T \left|\frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} \frac{1}{N_j}\sum_{k=1}^{N_j}(X_t^{n,m_k}, Y_t^{n,m_k}) - (X_t^{n,*}, Y_t^{n,*})\right|^\nu dt\right] = 0,\ 1<\nu<2.$$

Moreover, (4.6) yields

$$\lim_{\overline{N}\to\infty} \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} \frac{1}{N_j}\sum_{k=1}^{N_j} X_t^{0,n,m_k} - X_t^{0,n,*}\right|^\nu dt\right] = 0,\ 1<\nu<2.$$

The integrand of $J^{0,n}$ (4.4) is nonnegative by Assumption 4.1(i)–(iii). Thus, Fatou's lemma and the convexity of $J^{0,n}$ imply that

$$\begin{aligned} J^{0,n}(\xi^{0,n,*}) &\le \liminf_{\overline{N}\to\infty} J^{0,n}\left(\frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} \overline{\xi}^{0,n,N_j}\right) \\ &\le \liminf_{\overline{N}\to\infty} \frac{1}{\overline{N}}\sum_{j=1}^{\overline{N}} \frac{1}{N_j}\sum_{k=1}^{N_j} J^{0,n}\left(\xi^{0,n,m_k}\right) = J^*. \end{aligned}$$ ∎

From now on, we denote the unique optimal control for the penalized optimization (4.2)–(4.3) by $\xi^{0,n,*}$, whose existence has been established in Theorem 4.3. The following theorem provides a characterization of $\xi^{0,n,*}$.

THEOREM 4.4 (necessary maximum principle). *The optimal control $\xi^{0,n,*}$ admits the following representation:*

$$\xi_t^{0,n,*} = \frac{p_t^n + \mathbb{E}[\widetilde{\kappa}_t^0 q_t^n | \mathcal{F}_t^0] - \kappa_t^0 X_t^{0,n,*}}{2\eta_t^0}, \quad \textit{a.e. a.s. on } [0,T]\times\Omega, \tag{4.7}$$

where $X^{0,n,*}$, p^n, *and* q^n *satisfy the following FBSDE system:*

$$
\begin{cases}
dX_t^{0,n,*} = -\xi_t^{0,n,*}\,dt,\\
dX_t^{n,*} = -\xi_t^{n,*}\,dt,\\
-dY_t^{n,*} = \left(\kappa_t\mathbb{E}\left[\xi_t^{n,*}|\mathcal{F}_t^0\right] + 2\lambda_t X_t^{n,*} + \widetilde{\kappa}_t^0\xi_t^{0,n,*}\right)dt - Z_t\,dW_t,\\
-dp_t^n = \left(\overline{\kappa}_t^0\mathbb{E}\left[\xi_t^{n,*}|\mathcal{F}_t^0\right] + \kappa_t^0\xi_t^{0,n,*} + 2\lambda_t^0 X_t^{0,n,*}\right)dt - Z_t\,dW_t^0,\\
-dq_t^n = \left(-\frac{r_t^n}{2\eta_t} - \mathbb{E}\left[\kappa_t q_t^n|\mathcal{F}_t^0\right]\frac{1}{2\eta_t} + \overline{f}_t^n\right)dt,\\
-dr_t^n = \left(-2\lambda_t q_t^n + \kappa_t\mathbb{E}\left[\frac{r_t}{2\eta_t}\middle|\mathcal{F}_t^0\right] + \kappa_t\mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}[\kappa_t q_t^n|\mathcal{F}_t^0] + \overline{g}_t^n\right)\\
\qquad\cdot dt - Z_t\,dW_t,\\
X_0^0 = x^0,\ X_0 = x,\ Y_T^{n,*} = 2nX_T^{n,*},\ p_T^n = 2nX_T^{0,n,*},\ r_T^n = -2nq_T^n,\ q_0^n = 0,
\end{cases}
\tag{4.8}
$$

where

$$
\xi_t^{n,*} = \frac{Y_t^{n,*} - \mathbb{E}[\kappa_t X_t^{n,*}|\mathcal{F}_t^0]}{2\eta_t},
\tag{4.9}
$$

$$
\overline{f}_t^n = \frac{\overline{\kappa}_t^0 X_t^{0,n,*}}{2\eta_t} + \frac{\overline{\lambda}_t}{\eta_t}\mathbb{E}\left[\xi_t^{n,*}|\mathcal{F}_t^0\right],
\tag{4.10}
$$

and

$$
\overline{g}_t^n = -\kappa_t\mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\overline{\kappa}_t^0 X_t^{0,n,*} - 2\overline{\lambda}_t\kappa_t\mathbb{E}\left[\frac{1}{2\eta_t}\middle|\mathcal{F}_t^0\right]\mathbb{E}\left[\xi_t^{n,*}|\mathcal{F}_t^0\right].
\tag{4.11}
$$

Proof. A unique optimal control $\xi^{0,n,*}$ exists due to Theorem 4.3. It is to be viewed as an exogenous input to the FBSDE system (4.8). Thus, the system $(X^{n,*}, Y^{n,*})$ is a special case of (2.30) by taking (4.9) into account. Corollary 2.7 implies that the system is well-posed. Considering $\overline{f}^n$ and $\overline{g}^n$ as inputs, the system (q^n, r^n) is well-posed, again due to Corollary 2.7. The characterization (4.7) is then a direct result of stochastic maximum principle for control of FBSDE with partial information; cf [40]. ∎

The ansatz $p^n = \overline{A}^n X^{0,n,*} + \overline{p}^n$ shows that the equation for p^n could be dropped from the above system. It yields the following BSDEs for the processes $\overline{A}^n$ and $\overline{p}^n$ that will be used in the next subsection:

$$
\begin{cases}
-d\overline{A}_t^n = \left(-\frac{(\overline{A}_t^n)^2}{2\eta_t^0} + \frac{\kappa_t^0\overline{A}_t^n}{2\eta_t^0} + 2\lambda_t^0\right)dt - Z_t^{\overline{A}^n}\,dW_t^0,\\
\overline{A}_T^n = 2n
\end{cases}
\tag{4.12}
$$

and

$$
\begin{cases}
-d\overline{p}_t^n = \left(-\frac{\overline{A}_t^n\overline{p}_t^n}{2\eta_t^0} - \frac{\overline{A}_t^n\mathbb{E}[\widetilde{\kappa}_t^0 q_t^n|\mathcal{F}_t^0]}{2\eta_t^0} + \kappa_t^0\xi_t^{0,n,*} + \overline{\kappa}_t^0 E\left[\xi_t^{n,*}|\mathcal{F}_t^0\right]\right)dt - Z_t^{\overline{p}^n}\,dW_t^0,\\
\overline{p}_T^n = 0.
\end{cases}
\tag{4.13}
$$

4.2. The optimal solution to the Stackelberg game. Let us recall that $\xi^{0,n,*}$ denotes the leader's optimal control for penalized optimization with index $n \in \mathbb{N}$. The uniform boundedness of $J^{0,n}(x^0/T)$ in $n \in \mathbb{N}$ implies

$$\sup_n \mathbb{E}\left[\int_0^T \left|\xi_t^{0,n,*}\right|^2 dt + n(X_T^{0,n,*})^2\right] < \infty. \tag{4.14}$$

Thus, the same arguments as in the proof of Lemma 2.10 yield the existence of a progressively measurable process

$$\xi^{0,*} \in L^2_{\mathbb{F}^0}(\Omega \times [0,T];\mathbb{R}) \tag{4.15}$$

such that

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^T \left|\frac{1}{N}\sum_{k=1}^N \xi_t^{0,n_k,*} - \xi_t^{0,*}\right|^\nu dt\right] = 0,\ 1 < \nu < 2. \tag{4.16}$$

Our goal is to prove that $\xi^{0,*}$ is the leader's unique optimal strategy in the original state-constrained Stackelberg game. To this end, we first establish a representation of $\xi^{0,*}$ in terms of the solution to the systems (1.10), (1.12), and (1.13) by proving that the solutions to the system of state and adjoint equations (4.8) for the unconstrained penalized MFC problem Cesaro converge to the solutions to the systems (1.7), (1.10), (1.12), and (1.13). From this, we then deduce a sufficient maximum principle for the leader's MFC problem from which we conclude the optimality of the candidate strategy $\xi^{0,*}$.

4.2.1. Approximation. With the limit $\xi^{0,*}$ at hand, we can consider the FBSDE system (1.7), (1.10), (1.12), and (1.13) with ξ^0 replaced by $\xi^{0,*}$. The system (1.7) for (X^*, Y^*) is well-posed, due to Corollary 4.2. The system for (q, r) is well-posed, due to the following corollary.

COROLLARY 4.5. *If we take* $\chi = 0$, $\Lambda^1 = -\Lambda^2 = \zeta = 1/2\eta$, $\gamma = \Lambda^3 = \varrho = \kappa$, $\Lambda^4 = 2\lambda$, $\Lambda^5 = -\kappa\mathbb{E}\left[\frac{1}{2\eta}\middle|\mathcal{F}^0\right]$, $Q = -q$,

$$\overline{f} = \frac{\kappa^0 X^{0,*}}{2\eta} + \frac{\overline{\lambda}}{\eta}\mathbb{E}\left[\xi^*|\mathcal{F}^0\right] \tag{4.17}$$

and

$$\overline{g} = -\kappa\mathbb{E}\left[\frac{1}{2\eta}\middle|\mathcal{F}^0\right]\overline{\kappa}^0 X^{0,*} - 2\overline{\lambda}\kappa\mathbb{E}\left[\frac{1}{2\eta}\middle|\mathcal{F}^0\right]\mathbb{E}\left[\xi^*|\mathcal{F}^0\right], \tag{4.18}$$

where

$$\xi^* = \frac{Y^*}{2\eta} - \frac{1}{2\eta}\mathbb{E}[\kappa_t X^*|\mathcal{F}^0]. \tag{4.19}$$

Then the system (1.2) *reduces* (1.13). *Hence, existence and uniqueness of a solution holds for* (1.13). *Moreover,* $r = -Aq + D$ *with* $D \in S^{2,-}_{\mathbb{F}}$.

We now introduce two BSDEs that we expect to be the limits to (4.12) and (4.13):

$$\begin{cases} -d\overline{A}_t = \left(-\dfrac{\overline{A}_t^2}{2\eta_t^0} + \dfrac{\kappa_t^0\overline{A}_t}{2\eta_t^0} + 2\lambda_t^0\right)dt - Z_t\,dW_t^0, \\ \lim\limits_{t\nearrow T}\overline{A}_t = \infty \end{cases} \tag{4.20}$$

and

$$
\begin{cases} -d\overline{p}_t = \left(-\frac{\overline{A}_t \overline{p}_t}{2\eta_t^0} - \frac{\overline{A}_t \mathbb{E}[\widetilde{\kappa}_t^0 q_t | \mathcal{F}_t^0]}{2\eta_t^0} + \kappa_t^0 \xi_t^{0,*} + \overline{\kappa}_t^0 E\left[\xi_t^* | \mathcal{F}_t^0\right] \right) dt - Z_t^{\overline{p}}\, dW_t^0, \\ \overline{p}_T = 0. \end{cases} \tag{4.21}
$$

where ξ^* and $\xi^{0,*}$ are defined in (4.19) and (4.15), respectively. The following lemma confirms our guess. It shows that the solutions to the FBSDE system (4.8) converge to the solutions to the FBSDE systems (1.7), (1.10), (1.13), and (4.21) in the same sense as the optimal solutions to the unconstrained penalized problems converge to the candidate solution of the constrained problem.

LEMMA 4.6. *For $1 < \nu < 2$, it holds that*

$$
\lim_{N\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} \left| \frac{1}{N} \sum_{k=1}^{N} X_t^{0,n_k,*} - X_t^{0,*} \right|^{\nu} \right] = 0, \tag{4.22}
$$

$$
\lim_{\overline{N}\to\infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{\overline{N}} \sum_{j=1}^{\overline{N}} \frac{1}{N_j} \sum_{k=1}^{N_j} X_t^{n_k,*} - X_t^* \right|^{\nu} dt \right] = 0, \tag{4.23}
$$

$$
\lim_{\overline{N}\to\infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{\overline{N}} \sum_{j=1}^{\overline{N}} \frac{1}{N_j} \sum_{k=1}^{N_j} Y_t^{n_k,*} - Y_t^* \right|^{\nu} dt \right] = 0, \tag{4.24}
$$

$$
\lim_{\widetilde{N}\to\infty} \mathbb{E}\left[\sup_{0\le t\le T} \left| \frac{1}{\widetilde{N}} \sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i} \sum_{j=1}^{\overline{N}_i} \frac{1}{N_j} \sum_{k=1}^{N_j} q_t^{n_k} - q_t \right|^{\nu} \right] = 0, \tag{4.25}
$$

$$
\lim_{\widetilde{N}\to\infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{\widetilde{N}} \sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i} \sum_{j=1}^{\overline{N}_i} \frac{1}{N_j} \sum_{k=1}^{N_j} r_t^{n_k} - r_t \right|^{\nu} dt \right] = 0, \tag{4.26}
$$

$$
\lim_{\widetilde{N}\to\infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{\widetilde{N}} \sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i} \sum_{j=1}^{\overline{N}_i} \frac{1}{N_j} \sum_{k=1}^{N_j} \overline{p}_t^{n_k} - \overline{p}_t \right|^{\nu} dt \right] = 0. \tag{4.27}
$$

Proof. The convergence (4.22) follows from (4.16) and the definition of $X^{0,*}$. Taking $\chi = x$, $\zeta = \Lambda^1 = -\Lambda^2 = 1/2\eta$, $\gamma = \Lambda^3 = \varrho = \kappa$, $\Lambda^4 = 2\lambda$, $\Lambda^5 = -\kappa\mathbb{E}\left[\frac{1}{2\eta}\middle|\mathcal{F}^0\right]$, $\overline{f}^n = 0$ and $\overline{g}^n = \widetilde{\kappa}^0\xi^{0,n,*}$ in (2.30) the convergence (4.23), (4.24) follows from Theorem 2.11, due to the uniform L^2 boundedness of $\overline{g}^n$.

In (2.30), let $\chi = 0$, $\Lambda^1 = -\Lambda^2 = \zeta = 1/2\eta$, $\gamma = \Lambda^3 = \varrho = \kappa$, $\Lambda^4 = 2\lambda$, $\Lambda^5 = -\kappa\mathbb{E}\left[\frac{1}{2\eta}\middle|\mathcal{F}^0\right]$, $Q^n = -q^n$, and $(\overline{f}^n, \overline{g}^n)$ as in (4.10) and (4.11). It follows from (4.22)–(4.24) that

$$
\lim_{\overline{N}\to\infty} \mathbb{E}\left[\int_0^T \left| \frac{1}{\overline{N}} \sum_{j=1}^{\overline{N}} \frac{1}{N_j} \sum_{k=1}^{N_j} (\overline{f}_t^{n_k}, \overline{g}^{n_k}) - (\overline{f}_t, \overline{g}_t) \right|^{\nu} dt \right] = 0, \tag{4.28}
$$

where $\overline{f}$ and $\overline{g}$ are defined as in (4.17) and (4.18), respectively. By Corollary 4.2 and the estimate (2.3), we have $\overline{f} \in S^2_{\mathbb{F}}$ and $\overline{g} \in L^2_{\mathbb{F}}$. So (4.25) and (4.26) follow again from Theorem 2.11. By (4.16), (4.23), (4.24), and (4.25) we also have (4.27). ∎

The preceding lemma yields a representation on the candidate optimal strategy in terms of the candidate optimal state and adjoint processes akin to the maximum principle for the penalized problem.

THEOREM 4.7. *The limit $\xi^{0,*}$ in* (4.16) *admits the following representation:*

$$\xi^{0,*}_t = \frac{p_t + \mathbb{E}[\widetilde{\kappa}^0_t q_t|\mathcal{F}^0_t] - \kappa^0_t X^{0,*}_t}{2\eta^0_t}, \quad a.e.\ a.s.\ on\ [0,T]\times\Omega, \tag{4.29}$$

where $p = \overline{A}X^{0,*} + \overline{p}$. Moreover, $\xi^{0,*} \in \mathcal{A}_{\mathbb{F}}(x^0)$ and p satisfies the dynamic (1.12).

Proof. The characterization (4.29) follows from Theorem 4.4 and Lemma 4.6. It remains to verify the admissibility of $\xi^{0,*}$. The fact that $\xi^{0,*}$ belongs to $L^2_{\mathbb{F}^0}$ is due to (4.15). By (4.14),

$$\lim_{n\to\infty} \mathbb{E}[(X^{0,n,*}_T)^2] = 0.$$

By (4.22),

$$\lim_{\widetilde{N}\to\infty} \mathbb{E}\left[\left|\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j} X^{0,n_k,*}_T - X^{0,*}_T\right|^{\nu}\right] = 0.$$

Thus,

$$\begin{aligned}
&\mathbb{E}[|X^{0,*}_T|^{\nu}] \\
&\le 2\mathbb{E}\left[\left|\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j} X^{0,n_k,*}_T - X^{0,*}_T\right|^{\nu}\right] \\
&+ 2\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}} \frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j} \mathbb{E}\left[|X^{0,n_k,*}_T|^{\nu}\right] \to 0,
\end{aligned}$$

which implies $X^{0,*}_T = 0$ a.s.. Finally, starting from $p = \overline{A}X^{0,*} + \overline{p}$ by integration by parts and taking into account the characterization (4.29), we know p satisfies (1.12). ∎

4.2.2. Sufficient maximum principle. In this section, a sufficient maximum principle is established, from which we obtain the optimality of $\xi^{0,*}$ for the leader's MFC problem.

THEOREM 4.8 (sufficient maximum principle). *Under Assumption* 4.1*, $\xi^{0,*}$ given by Theorem* 4.7 *is the unique optimal strategy to the leader's optimization problem.*

Proof. We denote by $(X^{0,*}, X^*, Y^*)$ the states corresponding to $\xi^{0,*}$ and by (X^0, X, Y) the states corresponding to a generic strategy $\xi^0 \in L^2_{\mathbb{F}^0}$. The verification is split into three steps.

Step 1. By Corollary 2.6, X and Y are convex in ξ^0 in the sense that

$$(X(\rho\xi^0 + (1-\rho)\xi^{0'}), Y(\rho\xi^0 + (1-\rho)\xi^{0'})) = \rho(X(\xi^0), Y(\xi^0)) + (1-\rho)(X(\xi^{0'}), Y(\xi^{0'})).$$

Thus, J^0 is strictly convex in ξ^0. As a result, there is at most one optimal strategy.

Step 2. Integration by part for $(X^0 - X^{0,*})p$, $(X - X^*)r$, and $(Y - Y^*)q$ on $[0, \widetilde{T}]$ for $0 \le \widetilde{T} < T$ yields

$$
\begin{aligned}
&\mathbb{E}\left[(X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right] + \mathbb{E}\left[(X_{\widetilde{T}} - X^*_{\widetilde{T}})r_{\widetilde{T}}\right] + \mathbb{E}\left[(Y_{\widetilde{T}} - Y^*_{\widetilde{T}})q_{\widetilde{T}}\right] \\
= &-\mathbb{E}\left[\int_0^{\widetilde{T}} (X^0_t - X^{0,*}_t)\left(\overline{\kappa}^0_t \mathbb{E}\left[\xi^*_t|\mathcal{F}^0_t\right] + \kappa^0_t \xi^{0,*}_t + 2\lambda^0_t X^{0,*}_t\right) dt\right] \\
&-\mathbb{E}\bigg[\int_0^{T-\epsilon} \mathbb{E}[\kappa_t(X_t - X^*_t)|\mathcal{F}^0_t]\bigg(- \overline{\kappa}^0_t \mathbb{E}\bigg[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\bigg] X^{0,*}_t \\
&\quad - 2\overline{\lambda}_t \mathbb{E}\bigg[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\bigg]\mathbb{E}[\xi^*_t|\mathcal{F}^0_t]\bigg) dt\bigg] \\
&-\mathbb{E}\left[\int_0^{\widetilde{T}} \mathbb{E}\left[\frac{Y_t - Y^*_t}{2\eta_t}\bigg|\mathcal{F}^0_t\right]\left(\overline{\kappa}^0_t X^{0,*}_t + 2\overline{\lambda}_t \mathbb{E}\left[\xi^*_t|\mathcal{F}^0_t\right]\right) dt\right] \\
&-\mathbb{E}\left[\int_0^{\widetilde{T}} (p_t + \mathbb{E}[\widetilde{\kappa}^0_t q_t|\mathcal{F}^0_t])(\xi^0_t - \xi^{0,*}_t)\, dt\right],
\end{aligned}
$$

where we recall ξ^* is defined in (4.19).

Step 3. To prove the optimality of the strategy (4.29) we define, for any $\widetilde{T} < T$ the cost functional

$$
\begin{aligned}
\widetilde{J}^0(\xi^0) = \mathbb{E}\Bigg[\int_0^{\widetilde{T}} \overline{\kappa}^0_t \left(\mathbb{E}\left[\frac{Y_t}{2\eta_t}\bigg|\mathcal{F}^0_t\right] - \mathbb{E}\left[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\right]\mathbb{E}[\kappa_t X_t|\mathcal{F}^0_t]\right) X^0_t + \kappa^0_t \xi^0_t X^0_t + \eta^0_t (\xi^0_t)^2 \\
+\lambda^0_t (X^0_t)^2 + \overline{\lambda}_t \left|\mathbb{E}\left[\frac{Y_t}{2\eta_t}\bigg|\mathcal{F}^0_t\right] - \mathbb{E}\left[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\right]\mathbb{E}[\kappa_t X_t|\mathcal{F}^0_t]\right|^2 dt\Bigg].
\end{aligned}
$$

By direct calculation we have

$$
\begin{aligned}
&\widetilde{J}^0(\xi^0) - \widetilde{J}^0(\xi^{0,*}) \\
&\ge \mathbb{E}\left[\int_0^{T-\epsilon} (X^0_t - X^{0,*}_t)\left(\overline{\kappa}^0_t \mathbb{E}[\xi^*_t|\mathcal{F}^0_t] + \kappa^0_t \xi^0_t + 2\lambda^0_t \xi^{0,*}_t\right) dt\right] \\
&\quad + \mathbb{E}\left[\int_0^{T-\epsilon} \mathbb{E}\left[\frac{Y_t - Y^*_t}{2\eta_t}\bigg|\mathcal{F}^0_t\right]\left(\overline{\kappa}^0_t X^{0,*}_t + 2\overline{\lambda}_t \mathbb{E}[\xi^*_t|\mathcal{F}^0_t]\right) dt\right] \\
&\quad + \mathbb{E}\bigg[\int_0^{T-\epsilon} \mathbb{E}[\kappa_t(X_t - X^*_t)|\mathcal{F}^0_t]\bigg(- \overline{\kappa}^0_t \mathbb{E}\bigg[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\bigg] X^{0,*}_t \\
&\quad\quad - 2\overline{\lambda}_t \mathbb{E}\bigg[\frac{1}{2\eta_t}\bigg|\mathcal{F}^0_t\bigg]\mathbb{E}[\xi^*_t|\mathcal{F}^0_t]\bigg) dt\bigg] \\
&\quad + \mathbb{E}\left[\int_0^{T-\epsilon} (\xi^0_t - \xi^{0,*}_t)\left(\kappa^0_t X^{0,*}_t + 2\eta^0_t \xi^{0,*}_t\right) dt\right].
\end{aligned} \tag{4.30}
$$

Plugging the result into Step 2 into (4.30) and taking into account the characterization (4.29), we have

$$
\widetilde{J}^0(\xi^0) - \widetilde{J}^0(\xi^{0,*}) + \mathbb{E}\left[(X^0_{\widetilde{T}} - X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right] + \mathbb{E}\left[(X_{\widetilde{T}} - X^*_{\widetilde{T}})r_{\widetilde{T}}\right] + \mathbb{E}\left[(Y_{\widetilde{T}} - Y^*_{\widetilde{T}})q_{\widetilde{T}}\right] \ge 0.
$$

The same estimate as in the proof of [25, Theorem 2.9] yields that

$$\lim_{\widetilde{T}\nearrow T}\mathbb{E}\left|(X^0_{\widetilde{T}}-X^{0,*}_{\widetilde{T}})p_{\widetilde{T}}\right|=0.$$

Moreover, Corollaries 4.2 and 4.5 imply that

$$\begin{aligned}
&\mathbb{E}\left[(X_{\widetilde{T}}-X^*_{\widetilde{T}})r_{\widetilde{T}}\right]+\mathbb{E}\left[(Y_{\widetilde{T}}-Y^*_{\widetilde{T}})q_{\widetilde{T}}\right]\\
&=\mathbb{E}\left[(X_{\widetilde{T}}-X^*_{\widetilde{T}})(-A_{\widetilde{T}}q_{\widetilde{T}}+D_{\widetilde{T}})+\left(A_{\widetilde{T}}X_{\widetilde{T}}+B_{\widetilde{T}}-A_{\widetilde{T}}X^*_{\widetilde{T}}-B^*_{\widetilde{T}}\right)q_{\widetilde{T}}\right]\\
&=\mathbb{E}\left[(X_{\widetilde{T}}-X^*_{\widetilde{T}})D_{\widetilde{T}}+(B_{\widetilde{T}}-B^*_{\widetilde{T}})q_{\widetilde{T}}\right]\\
&\to 0 \qquad \text{as } \widetilde{T}\nearrow T.
\end{aligned}$$

This completes the proof. ∎

Corollary 4.9. *There exists a convex combination of the value functions converging to $J^0(\xi^{0,*})$, i.e.,*

$$\lim_{\widetilde{N}\to\infty}\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}J^{0,n_k}(\xi^{0,n_k,*})=J^0(\xi^{0,*}).$$

Proof. Recall that $X^{0,n_k,*}$ and $\xi^{n_k,*}$ are the optimal states of the leader and the optimal strategy of the follower corresponding to $\xi^{0,n_k,*}$, respectively. Due to the additional penalty term in the definition of J^{0,n_k} and because $\xi^{0,*}$ is an admissible strategy for the penalized problem,[3]

$$J^0(\xi^{0,n_k,*})\le J^{0,n_k}(\xi^{0,n_k,*})=\inf_{\xi\in L^2_{\mathbb{F}^0}([0,T]\times\Omega;\mathbb{R})}J^{0,n_k}(\xi)\le J^0(\xi^{0,*})$$

Denote by $K(\widetilde{N})$ the cost functional with (ξ^0,X^0,ξ) in J^0 replaced by

$$\left(\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}\xi^{0,n_k,*},\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}X^{0,n_k,*}\right.$$
$$\left.,\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}\xi^{n_k,*}\right).$$

By the convexity, we have

$$K(\widetilde{N})\le\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}J^0(\xi^{0,n_k,*})\le J^0(\xi^{0,*}).$$

The function J^0 admits a representation similar to (4.4). By Assumption 4.1(i–iii) the integrand of J^0 is nonnegative. By Lemma 4.6, (4.16), and Fatou's lemma

$$J^0(\xi^{0,*})\le\liminf_{\widetilde{N}\to\infty}K(\widetilde{N})\le\liminf_{\widetilde{N}\to\infty}\frac{1}{\widetilde{N}}\sum_{i=1}^{\widetilde{N}}\frac{1}{\overline{N}_i}\sum_{j=1}^{\overline{N}_i}\frac{1}{N_j}\sum_{k=1}^{N_j}J^0(\xi^{0,n_k,*})\le J^0(\xi^{0,*}).$$ ∎

[3]Notice that $J^0(\xi^{0,n_k,*})$ is well-defined even though $\xi^{0,n_k,*}$ may not be admissible for the constrained optimization problem.

4.3. Numerical simulations. We close this paper with a preliminary numerical analysis of the Stackelberg game previously analyzed. To the best of our knowledge, no numerical methods for simulating the mean-field FBSDEs arising in our game are yet available. We therefore simulate a deterministic benchmark model with constant coefficients. In this case, the conditional mean-field FBSDEs reduce to deterministic forward-backward ODEs that can be solved numerically using the MATLAB package `bvpsuite` [34]. Figure 1 (left) shows the optimal positions for the leader (solid) and follower (dashed) for the parameter values $\eta = 0.5, \kappa = 0.5, \lambda = 2, \kappa^0 = 0.5, \overline{\kappa}^0 = 0.5, \eta^0 = 0.5, \widetilde{\kappa}^0 = 1, \lambda^0 = 2, \overline{\lambda} = 1$, and $T = 1, x^0 = 8, x = 0$. In particular, we see that a beneficial round trip exists for the follower. The right plot shows the leader's cost as a function of the initial portfolio for the same parameter in a model with follower (solid) and a benchmark model without follower (dashed). For these choices of model parameters, the leader benefits from the presence of the follower. Figure 2 shows the same quantities as Figure 1, except that the impact of the leader on the follower is now much stronger: $\widetilde{\kappa}^0 = 10$. In this case, the leader suffers from the presence of the follower.

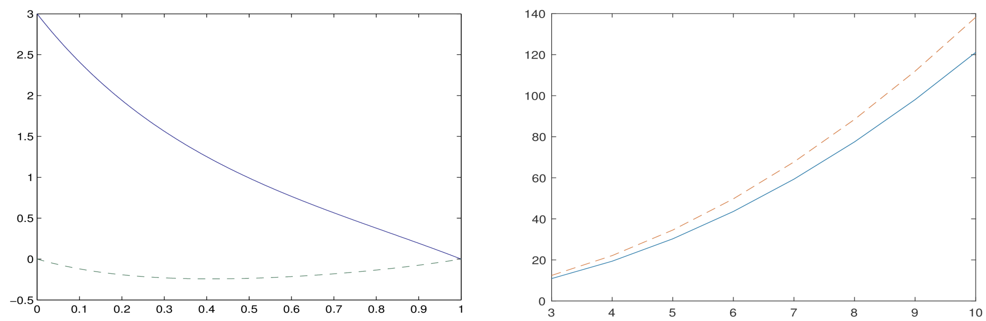

FIG. 1. *Left: optimal position for the leader (solid) and follower (dashed). Right: leader's cost function in a model with (solid) and without follower (dashed). Weak impact of leader on follower.*

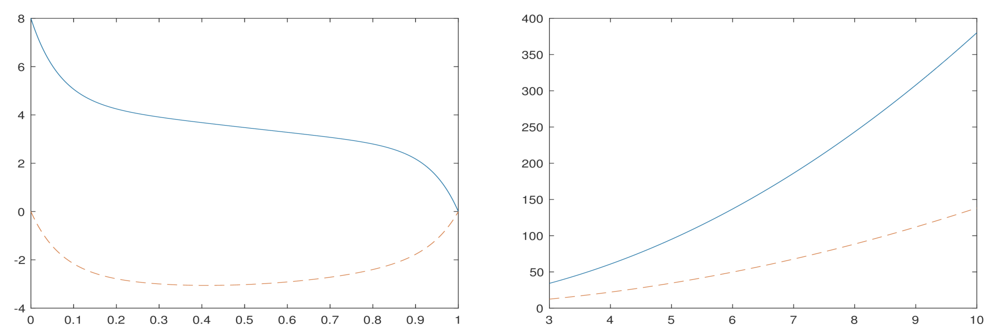

FIG. 2. *Left: optimal position for the leader (solid) and follower (dashed). Right: leader's cost function in a model with (solid) and without follower (dashed). Strong impact of leader on follower.*

5. Conclusion. We established existence and uniqueness of solutions results for linear McKean–Vlasov FBSDEs with a terminal state constraint on the forward process. The general results were used to solve novel MFC problems and mean-field leader-follower games of optimal portfolio liquidation. For the leader-follower game it could be viewed as an MFC problem where the state dynamics follow a controlled FBSDE. For such problems we proved a novel stochastic maximum principle. The proof was based on an approximation method. We proved that both the sequence of optimal solutions and the sequence of state and adjoint equations associated with a family of penalized problems Cesaro converge to a unique limit that yields the optimal solution, respectively, the state and adjoint equations to the original state-constrained problem.

REFERENCES

[1] B. Acciaio, J. Backhoff, and R. Carmona, *Extended Mean Field Control Problems: Stochastic Maximum Principle and Transport Perspective*, preprint, https://arxiv.org/abs/1802.05754, 2018.

[2] S. Ahuja, *Wellposedness of mean field games with common noise under a weak monotonicity condition*, SIAM J. Control Optim., 54 (2016), pp. 30–48, https://doi.org/10.1137/140974730.

[3] S. Ahuja, W. Ren, and T. Yang, *Forward-backward Stochastic Differential Equations with Monotone Functionals and Mean Field Games with Common Noise*, preprint, https://arxiv.org/abs/1611.04680, 2016.

[4] D. Andersson and B. Djehiche, *A maximum principle for SDEs of mean-field type*, Appl. Math. Optim., 63 (2011), pp. 341–356.

[5] S. Ankirchner, M. Jeanblanc, and T. Kruse, *BSDEs with singular terminal condition and a control problem with constraints*, SIAM J. Control Optim., 52 (2014), pp. 893–913, https://doi.org/10.1137/130923518.

[6] M. Basei and H. Pham, *Linear-quadratic McKean-Vlasov Stochastic Control Problems with Random Coefficients on Finite and Infinite Horizon, and Applications*, preprint, https://arxiv.org/abs/1711.09390, 2017.

[7] M. Bayraktar and A. Munk, *Mini-flash crashes, model risk, and optimal execution*, Market Microstructure and Liquidity, 4 (2018), 1850010.

[8] V. Benes, I. Karatzas, D. Ocone, and H. Wang, *Control with partial observations and an explicit solution of Mortensen's equation*, Appl. Math. Optim., 49 (2004), pp. 217–239.

[9] A. Bensoussan, M. Chau, Y. Lai, and P. Yam, *Linear-quadratic mean field Stackelberg games with state and control delays*, SIAM J. Control Optim., 55 (2017), pp. 2748–2781, https://doi.org/10.1137/15M1052937.

[10] A. Bensoussan, M. Chau, and P. Yam, *Mean field Stackelberg games: Aggregation of delayed instructions*, SIAM J. Control Optim., 53 (2015), pp. 2237–2266, https://doi.org/10.1137/140993399.

[11] A. Bensoussan, P. Yam, and Z. Zhang, *Well-posedness of mean-field type forward-backward stochastic differential equations*, Stochastic Process. Appl., 125 (2015), pp. 3327–3354.

[12] P. Cardaliaguet, M. Cirant, and A. Porretta, *Remarks on Nash Equilibria in Mean Field Game Models with a Major Player*, preprint, https://arxiv.org/abs/1811.02811, 2018.

[13] R. Carmona and F. Delarue, *Probabilistic analysis of mean-field games*, SIAM J. Control Optim., 51 (2013), pp. 2705–2734, https://doi.org/10.1137/120883499.

[14] R. Carmona and F. Delarue, *Forward-backward stochastic differential equations and controlled McKean-Vlasov dynamics*, Ann. Probab., 43 (2015), pp. 2647–2700, https://doi.org/10.1214/14-AOP946.

[15] R. Carmona and P. Wang, *An alternative approach to mean field game with major and minor players, and applications to herders impacts*, Appl. Math. Optim., 76 (2017), pp. 5–27.

[16] R. Carmona and X. Zhu, *A probabilistic approach to mean field games with major and minor players*, Ann. Appl. Probab., 26 (2016), pp. 1535–1580.

[17] P. Casgrain and S. Jaimungal, *Mean-field with Differing Beliefs for Algorithmic Trading*, preprint, https://arxiv.org/abs/1810.06101, 2018.

[18] P. Casgrain and S. Jaimungal, *Mean Field Games with Partial Information for Algorithmic Trading*, preprint, https://arxiv.org/abs/1803.04094, 2018.

[19] J. Cvitanić, D. Possamaï, and N. Touzi, *Moral hazard in dynamic risk management*, Manag. Sci., 63 (2017), pp. 3328–3346.

[20] J. Cvitanić, D. Possamaï, and N. Touzi, *Dynamic programming approach to principal-agent problems*, Finance Stoch., 22 (2018), pp. 1–37.

[21] J. Cvitanić, X. Wan, and J. Zhang, *Optimal compensation with hidden action and lump-sum payment in a continuous-time model*, Appl. Math. Optim., 59 (2009), pp. 99–146.

[22] J. Cvitanić and J. Zhang, *Contract Theory in Continuous Time Models*, Springer-Verlag, Berlin, 2012.

[23] R. Elie, T. Mastrolia, and D. Possamaï, *A tale of a principal and many many agents*, Math. Oper. Res., 44 (2019), pp. 440–467.

[24] D. Firoozi and P. Caines, *The execution problem in finance with major and minor traders: A mean field game formulation*, Advances in Dynamic and Mean Field Games, Ann. Internat. Soc. Dynam. Games 15, Birkhäuser/Springer, Cham, 2017, pp. 107–130.

[25] G. Fu, P. Graewe, U. Horst, and A. Popier, *A Mean Field Game of Optimal Portfolio Liquidation*, preprint, https://arxiv.org/abs/1804.04911, 2018.

[26] P. Graewe and U. Horst, *Optimal trade execution with instantaneous price impact and stochastic resilience*, SIAM J. Control Optim., 55 (2017), pp. 3707–3725, https://doi.org/10.1137/16M1105463.

[27] P. Graewe, U. Horst, and E. Séré, *Smooth solutions to portfolio liquidation problems under price-sensitive market impact*, Stochastic Process. Appl., 128 (2018), pp. 979–1006.

[28] U. Horst, *Stationary equilibria in discounted stochastic games with weakly interacting players*, Games Econom. Behav., 51 (2005), pp. 83–108.

[29] Y. Hu, J. Huang, and T. Nie, *Linear-quadratic-Gaussian mixed mean-field games with heterogeneous input constraints*, SIAM J. Control Optim., 56 (2018), pp. 2835–2877, https://doi.org/10.1137/17M1151420.

[30] M. Huang, *Large-population LQG games involving a major player: the Nash certainty equivalence principle*, SIAM J. Control Optim., 48 (2010), pp. 3318–3353, https://doi.org/10.1137/080735370.

[31] M. Huang, R. Malhamé, and P. Caines, *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–252.

[32] X. Huang, S. Jaimungal, and M. Nourian, *Mean-field game strategies for optimal execution*, Appl. Math. Finance, 26 (2019), pp. 153–185.

[33] S. Ji and X. Zhou, *A maximum principle for stochastic optimal control with terminal state constraints, and its applications*, Commun. Inf. Syst., 6 (2006), pp. 321–338.

[34] G. Kitzhofer, O. Koch, G. Pulverer, C. Simon, and E. Weinmüller, *The new matlab code bvpsuite for the solution of singular implicit BVPs*, J. Numer. Anal. Ind. Appl. Math., 10 (2010), pp. 113–134.

[35] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Hungar., 18 (1967), pp. 217–229.

[36] T. Kruse and A. Popier, *Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting*, Stochastic Process. Appl., 126 (2016), pp. 2554–2592, https://doi.org/10.1016/j.spa.2016.02.010.

[37] J.-M. Lasry and P.-L. Lions, *Mean field games*, Jpn. J. Math., 2 (2007), pp. 229–260, https://doi.org/10.1007/s11537-007-0657-8.

[38] J. Ma, Z. Wu, D. Zhang, and J. Zhang, *On well-posedness of forward-backward SDEs—a unified approach*, Ann. Appl. Probab., 25 (2015), pp. 2168–2214, https://doi.org/10.1214/14-AAP1046.

[39] M. Nourian and P. Caines, *ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents*, SIAM J. Control Optim., 51 (2013), pp. 3302–3331, https://doi.org/10.1137/120889496.

[40] B. Øksendal and A. Sulem, *Maximum principle for optimal control of forward-backward stochastic differential equations with jumps*, SIAM J. Control Optim., 48 (2009), pp. 2945–2976, https://doi.org/10.1137/080739781.

[41] A. Popier and C. Zhou, *Second order BSDE under monotonicity condition and liquidation problem under uncertainty*, Ann. Appl. Probab., 29 (2019), pp. 1685–1739.

[42] J. Yong, *Forward-backward stochastic differential equations with mixed initial-terminal conditions*, Trans. Amer. Math. Soc., 362 (2010), pp. 1047–1096.