# Finding Submodularity Hidden in Symmetric Difference 

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#### Abstract

A set function $f$ on a finite set $V$ is submodular if $f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y)$ for any pair $X, Y \subseteq V$. The symmetric difference transformation (SD-transformation) of $f$ by a canonical set $S \subseteq V$ is a set function $g$ given by $g(X)=f(X \Delta S)$ for $X \subseteq V$, where $X \Delta S=(X \backslash S) \cup$ ( $S \backslash X$ ) denotes the symmetric difference between $X$ and $S$. Submodularity and SD-transformations are regarded as the counterparts of convexity and affine transformations in a discrete space, respectively. However, submodularity is not preserved under SD-transformations, in contrast to the fact that convexity is invariant under affine transformations. This paper presents a characterization of SD-transformations preserving submodularity. Then, we are concerned with the problem of discovering a canonical set $S$, given the SD-transformation $g$ of a submodular function $f$ by $S$, provided that $g(X)$ is given by a function value oracle. A submodular function $f$ on $V$ is said to be strict if $f(X)+f(Y)>f(X \cup Y)+$ $f(X \cap Y)$ holds whenever both $X \backslash Y$ and $Y \backslash X$ are nonempty. We show that the problem is solved by using $\mathrm{O}(|V|)$ oracle calls when $f$ is strictly submodular, although it requires exponentially many oracle calls in general.


Keywords: Submodular functions, symmetric difference

## 1 Introduction

### 1.1 Submodular function and convexity

Submodular function on a finite set. For a set function $f: 2^{V} \rightarrow \mathbb{R}$ on a finite set $V$, we define

$$
\begin{equation*}
\Phi_{f}(X, Y) \stackrel{\text { def. }}{=} f(X)+f(Y)-f(X \cup Y)-f(X \cap Y) \tag{1}
\end{equation*}
$$

for any $X, Y \subseteq V$, for convenience of the arguments of the paper. A set function $f$ is submodular if $\Phi_{f}(X, Y) \geq 0$ hold $\int^{1}$ for any pair $X, Y \in 2^{V}$. In this paper, we do not assume $f(\emptyset)=0$ for a submodular function $f$, which is often assumed in the literature, but this is not essential to the arguments of the paper. A submodular function is strictly submodular if $\Phi_{f}(X, Y)>0$ holds whenever both $X \backslash Y$ and $Y \backslash X$ are nonempty. In contrast, a set function is modular if $\Phi_{f}(X, Y)=0$ holds for any pair $X, Y \in 2^{V}$.

Submodular function is an important concept particularly in the context of combinatorial optimization, and has many applications in economics, machine learning, etc. It is well-known that minimizing a submodular function given as its function value oracle is solved efficiently, by calling the value oracle (strongly) polynomial times [18, 9, 10, 12]. In contrast, maximizing submodular function, e.g., max cut, is NP-hard, and approximation algorithms have been developed e.g., [15, 4].

[^0]A celebrated characterization of a submodular function is described by the Lovász extension (see e.g., [1, 5, 14]). For a set function $f: 2^{V} \rightarrow \mathbb{R}$, the Lovász extension $\widehat{f}: \mathbb{R}^{V} \rightarrow \mathbb{R}$ is defined for $\boldsymbol{x}=(x(v)) \in \mathbb{R}^{V}$ which satisfies $x\left(v_{1}\right) \geq x\left(v_{2}\right) \geq \cdots \geq x\left(v_{|V|}\right)$ by $\widehat{f}(\boldsymbol{x}) \stackrel{\text { def. }}{=} \sum_{i=1}^{|V|} x\left(v_{i}\right)\left(f\left(\left\{v_{j} \mid j \leq i\right\}\right)-f\left(\left\{v_{j} \mid j \leq\right.\right.\right.$ $i-1\}))+f(\emptyset)$. Lovász [13] showed that a set function $f$ is submodular if and only if $\widehat{f}$ is convex. There are many other arguments to regard submodular functions as a discrete analogy of convex functions see e.g., [13, 14].

Convex function in continuous space. A function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ in a continuous space is convex if $\lambda f(\boldsymbol{x})+$ $(1-\lambda) f(\boldsymbol{y}) \geq f(\lambda \boldsymbol{x}+(1-\lambda) \boldsymbol{y})$ holds for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}$ and $\lambda \in[0,1]$ (see e.g., [17, 14]). An important property of a convex function (even on a convex set) is that local minimality guarantees the global minimality, and convexity is regarded as a tractable and useful class in the context of optimization. As another property, convexity is invariant under an affine map; Let $h: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ be an affine map given by $h(\boldsymbol{x}) \stackrel{\text { def. }}{=} A \boldsymbol{x}+\boldsymbol{b}$ with some $A \in \mathbb{R}^{n \times n}$ and $\boldsymbol{b} \in \mathbb{R}^{n}$ and let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a convex function. Then, the composition $g \stackrel{\text { def. }}{=} f \circ h$, i.e., $g(\boldsymbol{x})=f(A \boldsymbol{x}+\boldsymbol{b})$, is again a convex function.

Change-of-variables for submodular function. A change-of-variables is a fundamental technique for a function. For instance, it may not be trivial whether a continuous function $f(x, y)=8 x^{2}+10 x y+3 y^{2}$ is convex or not. Let $x=s-2 t$ and $y=-s+3 t$, then we get another function $g(s, t)=f(s-2 t,-s+3 t)=$ $s^{2}-t^{2}$. It is relatively easy to see that $g$ is not convex; that is confirmed by $\frac{1}{2} g(0,1)+\frac{1}{2} g(0,-1)<g(0,0)$ where $g(0,1)=-1, g(0,-1)=-1$ and $g=(0,0)=0$. Since convexity is invariant under an affine map, we see that $f$ is not convex.

This paper is motivated by a "change-of-variables" for submodular functions, as a discrete analogy. As the counter pert of affine transformations of convex functions, we will investigate symmetric difference transformations (SD-transformations) of submodular functions, which we will describe just below.

### 1.2 SD-transformation of a submodular function

Let $\sigma_{S}: 2^{V} \rightarrow 2^{V}$ denote the symmetric difference map (SD-map) by a set $S \subseteq V$, which is given by

$$
\begin{equation*}
\sigma_{S}(X) \stackrel{\text { def. }}{=} X \Delta S \tag{2}
\end{equation*}
$$

for any $X \subseteq V$, where $X \triangle S=(X \backslash Y) \cup(Y \backslash X)$ is the symmetric difference between $X$ and $S$. For a set function $f: 2^{V} \rightarrow \mathbb{R}$ and a set $S \subseteq V$, we say $g=f \circ \sigma_{S}$ is a symmetric difference transformation (SD-transformation) of $f$ by $S$, i.e., the SD-transformation is the set function $g: 2^{V} \rightarrow \mathbb{R}$ given by $g(X)=$ $f(X \triangle S)$ for any $X \subseteq V$.

It is not difficult to see that any bijective map on $2^{V}$ preserving the 1 -skeleton of the hypercube ("topology") is given by a combination of an SD-map ("origin-shift") and renaming elements of $V$. Obviously, submodularity is invariant under renaming elements of $V$. Thus, the SD-transformation is essential in a change-of-variables for submodular functions.

However, an SD-transformation of a submodular function is not submodular in general, in contrast to the fact that convexity is invariant under affine maps. Figure 1 shows an example. The left figure shows a submodular function $f: 2^{\{1,2,3\}} \rightarrow \mathbb{R}$, and the right figure shows its SD-transformation $g$ by the set $\{1,2\}$. We can check exhaustively that $f$ is submodular, while $g$ is not submodular since $\Phi_{g}(\{1\},\{3\})=$ $g(\{1\})+g(\{3\})-g(\{1,3\})-g(\emptyset)<0$.


Figure 1: (Left) A submodular function $f$ on $V=\{1,2,3\}$. (Right) $g=f \circ \sigma_{\{1,2\}}$.

### 1.3 Contribution

This paper characterizes SD-maps preserving the submodularity, i.e., given a submodular function $f$, we characterize $S \in 2^{V}$ for which $f \circ \sigma_{S}$ is again submodular. In Section 3, we present a characterization described by a Boolean system (Theorem 3.3), and rephrase it using a graph defined from $f$ (Theorem 3.7). By a similar and much simpler argument, we also remark that the modularity is invariant under SD-transformations (Proposition 3.8.

Then, we are concerned with the following problem.
Problem 1. Let $g: 2^{V} \rightarrow \mathbb{R}$ be an SD-transformation of a submodular function. Provided that $g$ is given by its function value oracle, the goal is to find a subset $T \subseteq V$ such that $h=g \circ \sigma_{T}$ is submodular.

We call a solution $T$ to Problem 1 a canonical set of $g$. Notice that a canonical set is not unique. In fact, we will show that if $T$ is a canonical set then $V \backslash T$ is also a canonical set (see Proposition 3.10 in Section 3.4.2). Once we find a canonical set $T$, we can apply many algorithms for submodular functions, such as minimization or maximization, to $g \circ \sigma_{T}$.

Unfortunately, Problem 1 requires exponentially many oracle calls, in the worst case. An easy example is given as follows (see e.g., [7]). Let $U \subseteq V$, then we define a set function $g: 2^{V} \rightarrow \mathbb{R}$ by $g(U)=-1$ and $g(X)=0$ for any other subset $X \subseteq V$. Then, the canonical sets are only $U$ and $V \backslash U$. Thus, it is not difficult (intuitively) to see that we need $2^{|V|}-2$ oracle calls in the worst case to solve Problem 1 (see also the proof of Proposition 4.3, for a detailed argument).

In Section 4 , we present a complete characterization of canonical sets (Theorem4.1). As an interesting consequence, we show that Problem 1 is solved by calling the function value oracle $\mathrm{O}(|V|)$ times if $f$ is strictly submodular (Theorem4.2). Once we find a canonical set of an SD-transformation $g$ of a submodular function, minimization of $g$ is easy using submodular function minimization, as we stated above. However, the converse is not true; we give an example in which Problem 1 requires exponentially many function value oracle calls even if we have all minimizers (or maximizers) of $g$ (Section 4.3).

### 1.4 Related works

Recognizing submodularity. It takes exponential time to check naively if a set function given by its function value oracle is submodular, in general. To be precise, the submodularity is confirmed in $2^{|V|}$. poly $(|V|)$ time, instead of checking $\Phi_{f}(X, Y) \geq 0$ for all $\binom{2^{|V|}}{2} \simeq\left(2^{|V|}\right)^{2}$ pairs $X, Y \in 2^{|V|}$ (see Section 22.

Goemans et al. [8] is concerned with approximating a submodular function with polynomially many oracle calls. For nonnegative monotone submodular functions $f$, they showed that an approximate function
$\widetilde{f}$ is constructed by calling poly $(|V|)$ times the function value oracle of $f$, such that $\widetilde{f}(X) \leq f(X) \leq$ $\alpha \widetilde{f}(X)$ for any $X \in 2^{V}$ with an approximation factor $\alpha=\mathrm{O}(\sqrt{|V|} \log |V|)$. Notice that $\widetilde{f}$ may not be submodular. They also gave a lower bound $\Omega(\sqrt{|V|} / \log |V|)$ of the approximation ratio with polynomially many oracle calls.

SD-transformation of a submodular function. Gillenwater et al. [7] are concerned with submodular Hamming distance $d_{f}(A, B)=f(A \Delta B)$ for $A, B \subseteq V$ given by a positive polymatroid function $f$, that is a monotone nondecreasing positive submodular function $f$ satisfying $f(\emptyset)=0$. Giving some applications in machine learning, such as clustering, structured prediction, and diverse $k$-best, they investigated the hardness and approximations of problems SH-min: $\min _{A \in \mathcal{C}} \sum_{i=1}^{m} f_{i}\left(A \triangle B_{i}\right)$ and SH-max: $\max _{A \in \mathcal{C}} \sum_{i=1}^{m} f_{i}(A \triangle$ $B_{i}$ ), where $f_{i}$ is a positive polymatroid, $B_{i} \subseteq V$, and $\mathcal{C}$ denotes a combinatorial constraint.

### 1.5 Organization

This paper is organized as follows. As a preliminary step, Section 2 is concerned with the 2 -faces of 0-1 hypercube. More precisely, Section 2.1 mentions the known fact that the submodularity is confirmed only by checking the submodularity on all 2-faces. Section 2.2 explicitly writes some basic facts of SD-map $\sigma_{S}$ on 2 -faces in concrete terms, to avoid a confusion in the following arguments.

Section 3 provides characterizations of SD-maps preserving the submodularity. Prior to the main theorems, Section 3.1 proves a key lemma using the argument in Section 2.2 . Sections 3.2 and 3.3 respectively show the main theorems. Section 3.4 make some remarks on Section 3 ,

Then, Section 4 is concerned with Problem 1. Section 4.1 characterizes canonical sets using a Boolean system. Section 4.2 presents a linear-time algorithm for Problem 1 provided that $f$ is strictly submodular. Section 4.3 gives some bad examples in general case. Finally, Section 5 concludes this paper.

## 2 Preliminary: 2-faces of a 0-1 hypercube

### 2.1 Submodularity is determined on 2-faces

Let $X \in 2^{V}$, and let $u, v \in V$ be a distinct pair. For convenience, let $X^{\prime}=X \backslash\{u, v\}$ then the four distinct subsets $X^{\prime}, X^{\prime} \cup\{u\}, X^{\prime} \cup\{v\}$ and $X^{\prime} \cup\{u, v\}$ of $V$ form a 2-face (a.k.a. polygonal face) of the $n$-dimensional 0-1 hypercube of the vertex set $2^{V}$. Let

$$
\begin{equation*}
\mathcal{P} \stackrel{\text { def. }}{=}\left\{(X,\{u, v\}) \mid X \subseteq V,\{u, v\} \in\binom{V \backslash X}{2}\right\}, \tag{3}
\end{equation*}
$$

representing the whole set of 2-faces of $n$-dimensional hypercube where $(X,\{u, v\})$ corresponds to the 2-face consisting of $X, X \cup\{u\}, X \cup\{v\}, X \cup\{u, v\}$. Notice that $|\mathcal{P}|=2^{n-2}\binom{n}{2}$ holds (cf., [3]).

For convenience, let $\breve{\Phi}_{f}: \mathcal{P} \rightarrow \mathbb{R}$ be defined by

$$
\begin{align*}
\check{\Phi}_{f}(X,\{u, v\}) & \stackrel{\text { def. }}{=} \Phi_{f}(X \cup\{u\}, X \cup\{v\})  \tag{4}\\
& =f(X \cup\{u\})+f(X \cup\{u\})-f(X \cup\{u, v\})-f(X)
\end{align*}
$$

for any $\left.(X,\{u, v\}) \in \mathcal{P}\right|^{2}$ The following characterization of submodular functions is known.
Theorem $2.1\left([\overline{19]})\right.$. A set function $f: 2^{V} \rightarrow \mathbb{R}$ is submodular if and only if $\check{\Phi}_{f}(p) \geq 0$ holds for any $p \in \mathcal{P}$.

[^1]
### 2.2 SD-map on 2-faces

In this section, we are concerned with the map over $\mathcal{P}$ provided by an SD-map $\sigma_{S}$ for a subset $S \subseteq V$, as a preliminary step of the arguments in the following sections. It is not be difficult to see that

$$
\left\{\sigma_{S}(X), \sigma_{S}(X \cup\{u\}), \sigma_{S}(X \cup\{v\}), \sigma_{S}(X \cup\{u, v\})\right\}
$$

again forms a 2 -face of a $0-1$ hypercube. To be precise, we can show the following proposition.
Proposition 2.2. For each $(X,\{u, v\}) \in \mathcal{P}$,

$$
\begin{aligned}
& \left\{\sigma_{S}(X), \sigma_{S}(X \cup\{u\}), \sigma_{S}(X \cup\{v\}), \sigma_{S}(X \cup\{u, v\})\right\} \\
& =\{Y, Y \cup\{u\}, Y \cup\{v\}, Y \cup\{u, v\}\}
\end{aligned}
$$

holds where $Y=(X \triangle S) \backslash\{u, v\}$.
Proof. We are concerned with three cases depending on whether $|\{u, v\} \cap S|=0$, 1, or 2 .
Case i) Suppose that $|\{u, v\} \cap S|=0$, i.e., $u \notin S$ and $v \notin S$. Notice that $u \notin X \triangle S$ and $v \notin X \triangle S$. Thus,

$$
\begin{array}{lll}
Y & & =X \Delta S \\
Y \cup\{u\} & =(X \Delta S) \cup\{u\} & =(X \cup\{u\}) \Delta S \\
Y \cup\{v\} & =(X \Delta S) \cup\{v\} & =(X \cup\{v\}) \Delta S \\
Y \cup\{u, v\} & =(X \Delta S) \cup\{u, v\} & =(X \cup\{u, v\}) \Delta S
\end{array}
$$

hold, where the right hand sides are respectively $\sigma_{S}(X), \sigma_{S}(X \cup\{u\}), \sigma_{S}(X \cup\{v\})$ and $\sigma_{S}(X \cup\{u, v\})$. Thus, we obtain the claim in this case.

Case ii) Suppose that $|\{u, v\} \cap S|=1$. Without loss of generality, we may assume that $u \in S$ and $v \notin S$. Then,

$$
\begin{array}{lll}
Y & =(X \Delta S) \backslash\{u\} & =(X \cup\{u\}) \Delta S \\
Y \cup\{u\} & & =X \Delta S \\
Y \cup\{v\} & =((X \Delta S) \backslash\{u\}) \cup\{v\} & =(X \cup\{u, v\}) \Delta S \\
Y \cup\{u, v\} & =(X \Delta S) \cup\{v\} & =(X \cup\{v\}) \Delta S
\end{array}
$$

hold, where the right hand sides are respectively $\sigma_{S}(X \cup\{u\}), \sigma_{S}(X), \sigma_{S}(X \cup\{u, v\})$ and $\sigma_{S}(X \cup\{v\})$. Thus, we obtain the claim in this case.

Case iii) Suppose that $|\{u, v\} \cap S|=2$, i.e., $u \in S$ and $v \in S$. Then,

$$
\begin{array}{lll}
Y & =(X \Delta S) \backslash\{u, v\} & =(X \cup\{u, v\}) \Delta S \\
Y \cup\{u\} & =(X \Delta S) \backslash\{v\} & =(X \cup\{v\}) \Delta S \\
Y \cup\{v\} & =(X \Delta S) \backslash\{u\} & =(X \cup\{u\}) \Delta S \\
Y \cup\{u, v\} & & =X \Delta S \tag{16}
\end{array}
$$

hold, where the right hand sides are respectively $\sigma_{S}(X \cup\{u, v\}), \sigma_{S}(X \cup\{v\}), \sigma_{S}(X \cup\{u\})$ and $\sigma_{S}(X)$. Thus, we obtain the claim.

Let $\check{\sigma}_{S}: \mathcal{P} \rightarrow \mathcal{P}$ for $S \subseteq V$ be defined by

$$
\begin{equation*}
\check{\sigma}_{S}(X,\{u, v\}) \stackrel{\text { def. }}{=}(Y,\{u, v\}) \tag{17}
\end{equation*}
$$

for any $(X,\{u, v\}) \in \mathcal{P}$ where $Y=(X \Delta S) \backslash\{u, v\}\}^{3}$ Then, $\check{\sigma}_{S}$ is the map on $\mathcal{P}$ provided by $\sigma_{S}$ by Proposition 2.2. Since $\sigma_{S}$ is bijective on $2^{V}$, Proposition 2.2 also implies the following.
Corollary 2.3. $\check{\sigma}_{S}$ is bijective.

## 3 SD-maps preserving submodulraity

This section characterizes $S \subseteq V$ for which $f \circ \sigma_{S}$ is submodular. Theorem 3.3 describes it using a Boolean system, and Theorem 3.7 rephrases it using a graph. As a preliminary argument, we give a key lemma in Section 3.1

### 3.1 Key lemma

This section mainly proves a technical lemma (Lemma 3.1), which intuitively characterizes SD-maps $\sigma_{S}$ preserving submodularity on a 2-face. We also remark that any SD-map $\sigma_{S}$ preserves modularlity on a 2 -face by a similar argument, at Lemma 3.2.
Lemma 3.1. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a submodular function. Suppose that a 2-face $p=(X,\{u, v\}) \in \mathcal{P}$ satisfies that

$$
\breve{\Phi}_{f}(p)>0 .
$$

Then for any subset $S \subseteq V$,

$$
\begin{equation*}
\check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(p)\right)>0 \tag{18}
\end{equation*}
$$

holds if and only if $|S \cap\{u, v\}| \equiv 0(\bmod 2)$.
Proof. By Proposition 2.2 and the definition (4) of $\check{\Phi}_{f}$, the condition (18) holds if and only if

$$
f \circ \sigma_{S}(Y \cup\{u\})+f \circ \sigma_{S}(Y \cup\{v\})>f \circ \sigma_{S}(Y \cup\{u, v\})+f \circ \sigma_{S}(Y)
$$

holds on the 2-face $(Y,\{u, v\})=\check{\sigma}(X,\{u, v\})$, where $Y=(X \triangle S) \backslash\{u, v\}$.
$(\Leftarrow)$ We show that (18) holds if $|S \cap\{u, v\}|=0$ or 2. If $|\{u, v\} \cap S|=0$, then

$$
\begin{aligned}
& \check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(X,\{u, v\})\right) \\
& \quad=f \circ \sigma_{S}(Y \cup\{u\})+f \circ \sigma_{S}(Y \cup\{v\})-f \circ \sigma_{S}(Y \cup\{u, v\})-f \circ \sigma_{S}(Y) \\
& \quad=f((Y \cup\{u\}) \Delta S)+f((Y \cup\{v\}) \Delta S)-f((Y \cup\{u, v\}) \Delta S)-f(Y \Delta S) \\
& \quad=f(X \cup\{u\})+f(X \cup\{v\})-f(X \cup\{u, v\})-f(X) \quad \text { (by (5)-(8) } \\
& \quad=\check{\Phi}_{f}(X,\{u, v\}) \\
& \quad>0
\end{aligned}
$$

hold, where the last inequality follows from the hypothesis that $\check{\Phi}_{f}(p)>0$.
If $|\{u, v\} \cap S|=2$, i.e., $u \in S$ and $v \in S$, then

$$
\begin{aligned}
& \check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(X,\{u, v\})\right) \\
& \quad=f \circ \sigma_{S}(Y \cup\{u\})+f \circ \sigma_{S}(Y \cup\{v\})-f \circ \sigma_{S}(Y \cup\{u, v\})-f \circ \sigma_{S}(Y) \\
& \quad=f((Y \cup\{u\}) \Delta S)+f((Y \cup\{v\}) \Delta S)-f((Y \cup\{u, v\}) \Delta S)-f(Y \Delta S) \\
& \quad=f(X \cup\{v\})+f(X \cup\{u\})-f(X)-f(X \cup\{u, v\}) \quad \text { (by (13)-16) } \\
& \quad=\breve{\Phi}_{f}(X,\{u, v\}) \\
& \quad>0
\end{aligned}
$$

[^2]\[

M_{f}=\left($$
\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{array}
$$\right) \underset{\leftarrow}{\leftarrow} \quad \leftarrow\left($$
\begin{array}{ll}
\leftarrow & \leftarrow(\emptyset,\{1,2\}) \text {-th } \\
& \leftarrow(\{1\},\{2,3\}) \text {-th } \\
\leftarrow & (\{2\}, 3\}) \text { th } \\
\leftarrow & (\{3\},\{1,3\}) \text {-th } \\
& (1,2\}) \text {-th }
\end{array}
$$\right.
\]

Figure 2: The matrix $M_{f}$ of the submodular function $f$ given in Figure 1 .
hold, where the last inequality follows from the hypothesis that $\breve{\Phi}_{f}(p)>0$. We obtain the claim.
$(\Rightarrow)$ We prove the contrapositive: if $|S \cap\{u, v\}|=1$ then (18) does not hold. Without loss of generality we may assume that $u \in S$ and $v \notin S$. Then

$$
\begin{aligned}
& \check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(X,\{u, v\})\right) \\
& \quad=f \circ \sigma_{S}(Y \cup\{u\})+f \circ \sigma_{S}(Y \cup\{v\})-f \circ \sigma_{S}(Y \cup\{u, v\})-f \circ \sigma_{S}(Y) \\
& \quad=f(Y \cup\{u\}) \Delta S)+f((Y \cup\{v\}) \Delta S)-f((Y \cup\{u, v\}) \Delta S)-f(Y \Delta S) \\
& \quad=f(X)+f(X \cup\{u, v\})-f(X \cup\{u\})-f(X \cup\{v\}) \quad \text { (by (9)-(12)) } \\
& \quad=-\check{\Phi}_{f}(X,\{u, v\}) \\
& \quad<0
\end{aligned}
$$

hold, where the last inequality follows from the hypothesis that $\check{\Phi}_{f}(p)>0$. Now, we obtain the claim.
The proof of the following lemma is similar to and much easier than the proof of Lemma 3.1, so it is omitted.

Lemma 3.2. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a submodular function. Suppose that a 2 -face $p \in \mathcal{P}$ satisfies that

$$
\check{\Phi}_{f}(p)=0 .
$$

Then for any subset $S \subseteq V$,

$$
\check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(p)\right)=0
$$

holds.

### 3.2 A characterization by a Boolean system

This section presents a characterization of SD-maps preserving submodularity, by using Lemmas 3.1 and 3.2 For any set function $f: 2^{V} \rightarrow \mathbb{R}$, let $M_{f} \in 2^{\mathcal{P} \times V}$ be a matrix whose ( $X,\{u, v\}$ )-th row vector is given by

$$
M_{f}[(X,\{u, v\}), \cdot]= \begin{cases}\boldsymbol{\chi}_{\{u, v\}}^{\top} & \text { if } \check{\Phi}_{f}(X,\{u, v\}) \neq 0,  \tag{19}\\ \mathbf{0}^{\top} & \text { otherwise },\end{cases}
$$

for each $(X,\{u, v\}) \in \mathcal{P}$, where $\chi_{S} \in 2^{V}$ denotes the characteristic column-vector of $S \subseteq V$, i.e., $\chi_{S}(w)=1$ if $w \in S$; otherwise $\chi_{S}(w)=0$. Figure 2 shows the matrix $M_{f}$ of the submodular function $f$ given in Figure 1. Then an SD-map $\sigma_{S}$ that preserves the submodularity of a submodular function $f$ is characterized by the next theorem.

$$
\bar{M}_{f}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{array}\right) \begin{gathered}
\leftarrow\{1,2\} \text {-th } \\
\leftarrow\{1,3\} \text {-th } \\
\leftarrow\{2,3\} \text {-th }
\end{gathered}
$$


(2)

Figure 3: The reduced matrix $\bar{M}_{f}$ (left) and the inequality graph $G_{f}$ (right) of the submodular function $f$ given in Figure 1.

Theorem 3.3. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a submodular function. For any $S \subseteq V, f \circ \sigma_{S}$ is submodular if and only if $M_{f} \chi_{S} \equiv \mathbf{0}(\bmod 2)$ holds.

To prove Theorem 3.3, we show the following lemma.
Lemma 3.4. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a submodular function. For any $S \subseteq V$ and for any $p=(X,\{u, v\}) \in \mathcal{P}$, $\breve{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(p)\right) \geq 0$ if and only if

$$
M_{f}[p, \cdot] \chi_{S} \equiv 0 \quad(\bmod 2)
$$

holds.
Proof. The proof is by a case analysis of $(X,\{u, v\}) \in \mathcal{P}$. If $\check{\Phi}_{f}(X,\{u, v\})=0$ holds for $(X,\{u, v\}) \in \mathcal{P}$, then $M_{f}[(X,\{u, v\}), \cdot]=\mathbf{0}$ holds by the definition of $M_{f}$. Using Lemma 3.2, the claim is easy in this case.

Suppose that $\check{\Phi}_{f}(X,\{u, v\}) \neq 0$ holds for $(X,\{u, v\}) \in \mathcal{P}$. Then,

$$
\begin{aligned}
M_{f}[(X,\{u, v\}), \cdot] \chi_{S} & =\chi_{S}(u)+\chi_{S}(v) \\
& =|\{u, v\} \cap S|
\end{aligned}
$$

by the definition of $M_{f}$. Now the claim follows from Lemma 3.1.
Proof of Theorem 3.3 Since $\check{\sigma}_{S}$ is bijective on $\mathcal{P}$ by Corollary $2.3 . \check{\Phi}_{f \circ \sigma_{S}}\left(\check{\sigma}_{S}(p)\right) \geq 0$ holds for any $p \in \mathcal{P}$ implies that $\check{\Phi}_{f \circ \sigma_{S}}(p) \geq 0$ holds for any $p \in \mathcal{P}$. Now, Theorem 3.3 is immediate from Lemma 3.4, using Theorem 2.1

### 3.3 An interpretation of Theorem 3.3 by a graph

This section interprets the characterization in Theorem 3.3 in terms of a graph defined from $f$. To begin with, we remark that the Boolean system $M_{f} \chi_{S} \equiv \mathbf{0}(\bmod 2)$ considered in Theorem 3.3 contains many redundant constraints since the rank of the matrix $M_{f} \in 2^{\mathcal{P} \times V}$ is at most $|V|$. Then, we define the reduced matrix ${ }^{4} \bar{M}_{f} \in 2^{\binom{V}{2} \times V}$ of $M_{f}$ where its $\{u, v\}$-th row vector $\left(\{u, v\} \in\binom{V}{2}\right)$ is given by

$$
\bar{M}_{f}[\{u, v\}, \cdot]= \begin{cases}\boldsymbol{\chi}_{\{u, v\}}^{\top} & \text { if } \exists(X,\{u, v\}) \in \mathcal{P}, \check{\Phi}_{f}(X,\{u, v\}) \neq 0,  \tag{20}\\ 0 & \text { otherwise },\end{cases}
$$

for each $\{u, v\} \in\binom{V}{2}$. We now make an observation, where the "only-if" part is trivial, and "if" part is not difficult by the definitions of $\bar{M}_{f}$ (see (20)).

[^3]Observation 3.5. For any $\boldsymbol{\chi} \in 2^{V}, M_{f} \boldsymbol{\chi} \equiv \mathbf{0}(\bmod 2)$ if and only if $\bar{M}_{f} \boldsymbol{\chi} \equiv \mathbf{0}(\bmod 2)$.
We can regard $\bar{M}_{f}$ as the (redundant) incidence matrix of what is called the inequality graph $G_{f}$ of $f$. Precisely, for any set function $f: 2^{V} \rightarrow \mathbb{R}$, let $G_{f}=\left(V, E_{f}\right)$ be an undirected graph with the edge set

$$
\begin{equation*}
E_{f} \stackrel{\text { def. }}{=}\left\{\left.\{u, v\} \in\binom{V}{2} \right\rvert\, \exists(X,\{u, v\}) \in \mathcal{P}, \check{\Phi}_{f}(X,\{u, v\}) \neq 0\right\} . \tag{21}
\end{equation*}
$$

Figure 3 shows the reduced matrix $\bar{M}_{f}$ and the inequality graph $G_{f}$ of the submodular function $f$ given in Figure 1

The following observation is trivial, too (see also the arguments on the graphic matroid [5]).
Observation 3.6. For any $S \subseteq V, \bar{M}_{f} \chi_{S} \equiv \mathbf{0}(\bmod 2)$ holds if and only if $\chi_{S}(u)=\chi_{S}(v)$ holds for any $\{u, v\} \in E_{f}$.

Observation 3.6 implies that $S$ is a canonical set if and only if every connected component of $G_{f}$ is included in or completely excluded from $S$. To be precise, let $U_{i} \subseteq V(i=1, \ldots, k)$ be the connected components of $G_{f}$ where $k$ is the number of connected components of $G_{f}$. Let $\mathcal{U}(f)$ denote the whole set family of unions of $U_{i}(i=1, \ldots, k)$, i.e.,

$$
\begin{equation*}
\mathcal{U}(f)=\left\{\bigcup_{i \in I} U_{i} \mid I \subseteq\{1,2, \ldots, k\}\right\} \tag{22}
\end{equation*}
$$

Then, we can conclude the following theorem as an easy consequence of Theorem 3.3 and Observations 3.5 and 3.6
Theorem 3.7. For any submodular function $f: 2^{V} \rightarrow \mathbb{R}, f \circ \sigma_{S}$ is submodular if and only if $S \in \mathcal{U}(f)$.

### 3.4 Remarks on Results in Section 3

This section makes some remarks concerning the arguments in Section 3. Sections 3.4.1 and 3.4.2 are easy implications of Lemmas 3.1 and 3.2. Sections 3.4.3, 3.4.4 and 3.4.5 are remarks on Theorem 3.7.

### 3.4.1 SD-transformation of a modular function

Proposition 3.8. If a set function $f: 2^{V} \rightarrow \mathbb{R}$ is modular then $f \circ \sigma_{S}$ is modular for any $S \subseteq V$.
Proof. The claim is immediate from Lemma 3.2 .
We will use Proposition 3.8 in Section 4.3 .

### 3.4.2 Complement of a canonical set

Proposition 3.9. If a set function $f: 2^{V} \rightarrow \mathbb{R}$ is submodular then $f \circ \sigma_{V}$ is submodular.
Proof. Notice that $M_{f} \chi_{V}=\mathbf{0}$ holds by the definition (19) of $M_{f}$. The claim is immediate from Theorem 3.3.

The following proposition for Problem 1 immediately follows from Proposition 3.9 .
Proposition 3.10. Let $g: 2^{V} \rightarrow \mathbb{R}$ be an SD-transformation of a submodular function. If $T \subseteq V$ is a canonical set of $g$, so is $V \backslash T$.
Proof. By the hypothesis, $h=g \circ \sigma_{T}$ is submodular. Proposition 3.9 implies that $h^{\prime}=h \circ \sigma_{V}$ is submodular. Since the symmetric difference is commutative and associative, we see that $h^{\prime}=h \circ \sigma_{V}=$ $g \circ \sigma_{T} \circ \sigma_{V}=g \circ \sigma_{V \backslash T}$, and we obtain the claim.


Figure 4: (Left) Submodular function $f$ given by (24) on $V=\{1,2,3\}$. (Right) The inequality graph $G_{f}$.

### 3.4.3 Nontrivial example of many canonical sets

Using Theorem 3.7, we give a nontrivial example of submodular functions which have many SD-transformations that are submodular. For an arbitrary finite set $V$ and an arbitrary partition $U_{1}, U_{2}, \ldots, U_{k}$ of $V$, let $f: 2^{V} \rightarrow \mathbb{R}$ be a set function defined by

$$
\begin{equation*}
f(X)=\min _{W \in \mathcal{U}}|X \Delta W| \tag{23}
\end{equation*}
$$

for any $X \subseteq V$, where $\mathcal{U}=\left\{\bigcup_{i \in I} U_{i} \mid I \subseteq\{1,2, \ldots, k\}\right\}$. This function represents the edit distance from the nearest point in the Boolean sublattice $\mathcal{U}$ of $2^{V}$.

Proposition 3.11. The set function $f$ given by (23) is submodular.
Proposition 3.12. For the submodular function $f$ given by (23), $f \circ \sigma_{S}$ is submodular if and only if $S \in \mathcal{U}$.
See Appendix A for the proofs of Propositions 3.11 and 3.12

### 3.4.4 A connected component of an inequality graph is not a clique, in general

As for the inequality graph of a submodular function defined by (21), it would be natural to ask if $U_{i}$ is a clique, considering the transitivity of $=$. However, it is not true in general. Let $V=\{1,2,3\}$, and let $f: 2^{V} \rightarrow \mathbb{R}$ be a set function given by

$$
f(X)= \begin{cases}0 & \text { if } X=\emptyset  \tag{24}\\ 2 & \text { if } X=\{1,3\} \\ 1 & \text { otherwise }\end{cases}
$$

for $X \in 2^{V}$ (see Figure 4 left). We can check that $f$ is submodular by Theorem 2.1. Then, its inequality graph is given by $G_{f}=(V,\{\{1,2\},\{2,3\}\})$, since $\Phi_{f}(\emptyset,\{1,3\})=\Phi_{f}(\{2\},\{1,3\})=0, \breve{\Phi}_{f}(\emptyset,\{x, y\})=$ $1 \neq 0, \check{\Phi}_{f}(\emptyset,\{y, z\})=1 \neq 0$ (see Figure 4 right). Clearly, the unique connected component of $G_{f}$ is not a clique.

### 3.4.5 Connection to the inseparable decomposition

In fact, SD-transformations preserving the submodularity are closely related to the inseparable decomposition of a submodular function. Let $\rho: 2^{V} \rightarrow \mathbb{R}$ be a submodular function satisfying $\rho(\emptyset)=0{ }_{[ }^{5}$ A nonempty subset $U \subseteq V$ is separable if there exists $X \subset U(X \neq \emptyset)$ such that $\rho(U)=\rho(X)+\rho(U \backslash X)$ holds; otherwise $U$ is inseparable [2, 16, 1, 11, 5, 6].

Theorem 3.13 (see e.g., [11, 6]). For a submodular function $f: 2^{V} \rightarrow \mathbb{R}, V$ is uniquely partitioned into inseparable subsets $U_{1}, \ldots, U_{k}$ with an appropriate $k$. For this partition,

$$
\begin{equation*}
\rho(X)=\sum_{i=1}^{k} \rho\left(X \cap U_{i}\right) \tag{25}
\end{equation*}
$$

holds for any $X \in 2^{V}$. Moreover, this partition is constructible in polynomial time ${ }^{6}$
We can show that $U_{1}, \ldots, U_{k}$ form an inseparable decomposition of $\rho$ if and only if each $U_{i}$ is the vertex set of a connected component of the inequality graph $G_{\rho}$. See Appendix B for more details.

Thus, the following theorem is an easy consequence of Theorems 3.7 and 3.13 .
Theorem 3.14. Given a subdmodular function $f: 2^{V} \rightarrow \mathbb{R}$ by its function value oracle, and given $S \subseteq V$, the question of $f \circ \sigma_{S}$ is subdmodular is decidable in polynomial time.

Proof. The inseparable decomposition of $f$ can be found in polynomial time by Theorem 3.13, where the decomposition $U_{1}, \ldots, U_{k}$ corresponds to the connected components of the inequality graph $G_{f}$ (see Proposition B.1). By Theorem 3.7, $f \circ \sigma_{S}$ is subdmodular if and only if $S \in \mathcal{U}(f)$. The latter condition is checkable in linear time.

We emphasize that Theorem 3.14 does not imply that Problem 1, which is to find unknown $S$, is solvable in polynomial time. The next section is concerned with Problem1, using the characterizations given in this section.

## 4 Finding A Canonical Set

### 4.1 A characterization of canonical sets

We are now concerned with Problem 1. Section 4.1 presents a characterization of canonical sets of an SD-transformation of a submodular function. For any set function $g: 2^{V} \rightarrow \mathbb{R}$, let $\boldsymbol{b}_{g} \in 2^{\mathcal{P}}$ be defined by

$$
\boldsymbol{b}_{g}[(Z,\{u, v\})]= \begin{cases}0 & \text { if } \check{\Phi}_{g}(Z,\{u, v\}) \geq 0  \tag{26}\\ 1 & \text { otherwise }\end{cases}
$$

for any $(Z,\{u, v\}) \in \mathcal{P}$.
Theorem 4.1. Let $g: 2^{V} \rightarrow \mathbb{R}$ be an SD-transformation of a submodular function. Then, $h=g \circ \sigma_{T}$ for $T \subseteq V$ is submodular if and only if

$$
M_{g} \boldsymbol{\chi}_{T} \equiv \boldsymbol{b}_{g} \quad(\bmod 2)
$$

holds where $\chi_{T}$ is the characteristic vector of $T$.

[^4]Proof. Suppose that $g$ is given by $g=f \circ \sigma_{S}$ for a submodular function $f$ and $S \subseteq V$. Firstly, we claim that

$$
\begin{equation*}
M_{g} \boldsymbol{\chi}_{S} \equiv \boldsymbol{b}_{g} \quad(\bmod 2) \tag{27}
\end{equation*}
$$

holds. By Lemma 3.4, $\check{\Phi}_{g}\left(\check{\sigma}_{S}(X,\{u, v\})\right) \geq 0$ if and only if $M_{f}[(X,\{u, v\}), \cdot] \boldsymbol{\chi}_{S} \equiv 0(\bmod 2)$. By the definition 19 ) of $M_{f}$, and Lemmas 3.1 and 3.2.

$$
\begin{equation*}
M_{g}\left[\check{\sigma}_{S}(X,\{u, v\}), \cdot\right]=M_{f}[(X,\{u, v\}), \cdot] \tag{28}
\end{equation*}
$$

holds for any $(X,\{u, v\}) \in \mathcal{P}$. Thus, $\check{\Phi}_{g}\left(\check{\sigma}_{S}(X,\{u, v\})\right) \geq 0$ if and only if $M_{g}\left[\check{\sigma}_{S}(X,\{u, v\}), \cdot\right] \chi_{S} \equiv 0$ $(\bmod 2)$. This implies 27), since $\check{\sigma}_{S}$ is bijective on $\mathcal{P}$ by Corollary 2.3.

Then, (27) and the hypothesis that $M_{g} \chi_{T}=\boldsymbol{b}_{g}$ imply that

$$
\begin{equation*}
M_{g} \boldsymbol{\chi}_{S}+M_{g} \boldsymbol{\chi}_{T} \equiv \boldsymbol{b}_{g}+\boldsymbol{b}_{g} \equiv \mathbf{0} \quad(\bmod 2) \tag{29}
\end{equation*}
$$

holds. Meanwhile,

$$
\begin{equation*}
M_{g} \chi_{T}+M_{g} \chi_{S}=M_{g}\left(\chi_{T}+\chi_{S}\right) \tag{30}
\end{equation*}
$$

holds. Notice that $\chi_{S \Delta T} \equiv \chi_{S}+\chi_{T}(\bmod 2)$ holds. Thus, 29) and (30) imply that

$$
\begin{equation*}
M_{g} \chi_{S \Delta T} \equiv \mathbf{0} \quad(\bmod 2) \tag{31}
\end{equation*}
$$

holds. By 28, (31) also implies

$$
\begin{equation*}
M_{f} \chi_{S \Delta T} \equiv \mathbf{0} \quad(\bmod 2) \tag{32}
\end{equation*}
$$

holds. Then, $f \circ \sigma_{S \Delta T}$ is submodular by Theorem 3.3. It is easy to observe that

$$
g \circ \sigma_{T}=f \circ \sigma_{S} \circ \sigma_{T}=f \circ \sigma_{S \Delta T}
$$

holds, meaning that $T$ is a canonical set. We obtain the claim.

### 4.2 Linear-time algorithm for strictly submodular function

Interestingly, Problem 1 is solvable in linear time for strictly submodular function. Precisely, it is described as follows.

Theorem 4.2. Problem 1 is solved in $2|V| \cdot \mathbf{E O}+\mathrm{O}(|V|)$ time if the set function $f$ is strictly submodular, where $\mathbf{E O}$ denotes the time complexity of an oracle call to know the value of $g(X)$ for a set $X \subseteq V$.

Proof. Since $f$ is strictly submodular, $G_{f}$ is connected. In particular, let $u^{*} \in V$ be arbitrary. Then $\check{\Phi}_{g}\left(\emptyset,\left\{u^{*}, v\right\}\right) \neq 0$ holds for any $v \in V \backslash\left\{u^{*}\right\}$. Thus, we can obtain a canonical set $T \subseteq V$ by solving the Boolean system $M_{g}\left[\left(\emptyset,\left\{u^{*}, v\right\}\right), \cdot\right] \boldsymbol{\chi}_{T} \equiv \boldsymbol{b}_{g}\left[\left(\emptyset,\left\{u^{*}, v\right\}\right)\right](\bmod 2)$ for $v \in V \backslash\left\{u^{*}\right\}$. (Recall (19) and (26) for the definitions of $M_{g}$ and $\boldsymbol{b}_{g}$.) It is not difficult to see that the solution of the Boolean system is a solution of Problem 1 by Theorems 4.1 and 3.7 .

In fact, the solution of the Boolean system is simply given as follows: Set $T:=\emptyset$ for initialization. For each $v \in V \backslash\left\{u^{*}\right\}$, set $T:=T \cup\{v\}$ if $\check{\Phi}_{g}\left(\emptyset,\left\{u^{*}, v\right\}\right)<0$. See Algorithm 1 for a formal description. It is not difficult to observe that the obtained $T$ provides a solution of the Boolean system by Observation 3.6, Computing $\check{\Phi}_{g}\left(\emptyset,\left\{u^{*}, v\right\}\right)$ requires the values of $g(\emptyset), g\left(\left\{u^{*}\right\}\right), g(\{v\})$ and $g\left(\left\{u^{*}, v\right\}\right)$ for $v \in V \backslash\left\{u^{*}\right\}$. Now the time complexity is easy.

## Algorithm 1.

Given a function value oracle of $g: 2^{V} \rightarrow \mathbb{R}$.
Set $T:=\emptyset$. Choose $u^{*} \in V$ arbitrarily.
Get the values $g(\emptyset)$ and $g\left(\left\{u^{*}\right\}\right)$.
For each $v \in V \backslash\left\{u^{*}\right\}$,
Get the values $g(\{v\})$ and $g\left(\left\{u^{*}, v\right\}\right)$.
If $\check{\Phi}_{g}\left(\emptyset,\left\{u^{*}, v\right\}\right)<0$, then set $T:=T \cup\{v\}$.
Output $T$.

### 4.3 Minimizer/Maximize is helpless for finding a canonical set

Once we obtain a canonical set $T$ for an SD-transformation $g$ of a submodular function, we can find the minimum value of $g$ using a submodular function minimization algorithm. However the opposite is not always true; finding a canonical set is sometimes hard even if all minimizers of $g$ are given.
Proposition 4.3. Problem 1 requires $2^{|V|}-2$ function value oracle calls in the worst case, even if all minimizers of $g$ are given.

Proof. We give an instance of Problem 1 with a unique minimizer, for which any algorithm needs to call the function value oracle at least $2^{|V|}-2$ times to solve Problem 1 . For any $U \subseteq V$ such that $U \neq \emptyset$, let $g_{U}: 2^{V} \rightarrow \mathbb{R}$ be a set function defined by

$$
g_{U}(X)= \begin{cases}|X|-\frac{1}{2} & (\text { if } X=U)  \tag{33}\\ |X| & \text { (otherwise) }\end{cases}
$$

for $X \in 2^{V}$. Observe that $g_{U}(X)>0$ for any $X \neq \emptyset$, meaning that $\emptyset$ is the unique minimizer of $g_{U}$ with the minimum value $g_{U}(\emptyset)=0$. We claim that exactly $U$ and $V \backslash U$ are the canonical sets of $g_{U}$. Let

$$
r_{U}(X)= \begin{cases}-\frac{1}{2} & (\text { if } X=U)  \tag{34}\\ 0 & (\text { otherwise })\end{cases}
$$

for $X \in 2^{V}$, and let $d(X) \stackrel{\text { def. }}{=}|X|$ for $X \in 2^{V}$. Then,

$$
g_{U}(X)=r_{U}(X)+d(X)
$$

holds. Clearly $r_{U} \circ \sigma_{U}$ is submodular. Since $d$ is a modular function, $d \circ \sigma_{U}$ is again modular by Proposition 3.8, Notice that

$$
g_{U} \circ \sigma_{U}=\left(r_{U}+d\right) \circ \sigma_{U}=r_{U} \circ \sigma_{U}+d \circ \sigma_{U}
$$

holds. Since the sum of submodular functions is submodular [5], $g_{U} \circ \sigma_{U}$ is submodular, meaning that $U$ is a canonical set of $g$. It is easy to observe $G_{r_{U}}$ is connected, and hence $G_{g_{U}}=G_{f}$ is connected since $d$ is modular. By Theorem 3.7, we see that only $U$ and $V \backslash U$ are canonical sets of $g$.

To prove that no algorithm finds a canonical set of $g_{U}$ with at most $2^{|V|}-3$ function value oracle calls, we show the existence of an adversarial oracle. Suppose that an arbitrary algorithm calls the value oracle of $g_{U}$ $2^{|V|}-3$ times. For the $2^{|V|}-3$ queries, our adversarial oracle answers their cardinalities. Let $X, Y, Z \in 2^{V}$ be the remaining sets. Without loss of generality, we may assume that both $X \neq V \backslash Y$ and $X \neq V \backslash Z$ hold. (Note that $Z=V \backslash Y$ may hold.) Since only $U$ and $V \backslash U$ are the canonical sets of $g_{U}$, both $X$ and $Y$ cannot be canonical sets at the same time. This implies that the algorithm cannot determine $X, Y$ or $Z$; if the algorithm answers $X$ then our oracle can set $Y=U$, meaning that $X$ is a wrong answer, and if the algorithm answers $Y$ or $Z$ then our oracle can set $X=U$.

In contrast to minimization, maximization of a submodular function, e.g., max cut, is NP-hard. Even if all maximizers are given, finding a canonical set is hard, too. The SD-transformation $g_{U}$ given in the proof of Proposition 4.3 also witnesses it.

Corollary 4.4. Problem 1 requires $2^{|V|}-2$ function value oracle calls in the worst case, even if all maximizer of $g$ are given.

Proof. Let $U \subseteq V$ satisfy $U \neq V$ and let $g_{U}$ be a set function defined by (33). Clearly, $V$ is the unique maximizer of $g$ with the maximum value $g(V)=|V|$. Finding canonical set of $g$ requires $2^{|V|}-2$ oracle calls, by the same argument as Proposition 4.3.

### 4.4 A Remark on Theorem 4.1

In fact, the hypothesis "Let $g: 2^{V} \rightarrow \mathbb{R}$ be an SD-transformation of a submodular function." in Theorem 4.5 is redundant. An enhanced theorem is described as follows.

Theorem 4.5. Let $g: 2^{V} \rightarrow \mathbb{R}$ be an arbitrary set function. Then, $h=g \circ \sigma_{T}$ for $T \subseteq V$ is submodular if and only if

$$
M_{g} \boldsymbol{\chi}_{T} \equiv \boldsymbol{b}_{g} \quad(\bmod 2)
$$

holds where $\chi_{T}$ is the characteristic vector of $T$.
Theorem 4.5 states that if $M_{g} \boldsymbol{\chi} \equiv \boldsymbol{b}_{g}(\bmod 2)$ does not have a solution then any SD-transformation of $g$ is NOT submodular. The "only-if" part is immediate from Theorem4.1. The "if" part is also not difficult by Theorem 3.3 (and Theorem 2.1), and obtained in a similar way as the proof of Theorem 4.1. We here omit the detailed proof.

## 5 Concluding Remark

This paper has been concerned with SD-transformations of submodular functions. We gave characterizations of SD-transformations preserving the submodularity in Section 3. We also showed that canonical sets are found in linear time for SD-transformations of a strictly submodular functions in Section 4. It is a natural question whether there is another interesting class of submodular functions for which a canonical set is found efficiently. A related question is whether there is a nontrivial class of transformations (maps) preserving submodularity.

We remark that it is not difficult to extend the results to submodular functions on distributive lattices, instead of Boolean lattices. Extensions to submodular functions on a general lattice, i.e., containing $M_{3}$ or $N_{5}, L$-convex functions and $M$-convex functions [14] on integer lattice, or $k$-submodular functions are interesting problems.

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## A Supplemental Proofs in Section 3.4.3

This section proves Propositions 3.11 and 3.12 . The set function which we are concerned with here is given by

$$
f(X)=\min _{W \in \mathcal{U}}|X \Delta W| \quad \text { (recall (23)) }
$$

for $X \in 2^{V}$, where $\mathcal{U}=\left\{\bigcup_{i \in I} U_{i} \mid I \subseteq\{1,2, \ldots, k\}\right\}$ for a partition $U_{1}, \ldots, U_{k}$ of $V$.

## A. 1 Proof of Proposition 3.11

Proposition A. 1 (Proposition 3.11). The set function $f$ given by 23) is submodular.
Proof. Since $U_{1}, \ldots, U_{k}$ is a partition of $V$,

$$
\begin{align*}
|X \Delta W| & =\sum_{i=1}^{m}\left|(X \triangle W) \cap U_{i}\right| \\
& =\sum_{i=1}^{m}\left|\left(X \cap U_{i}\right) \Delta\left(W \cap U_{i}\right)\right| \tag{35}
\end{align*}
$$

holds for any $X \in 2^{V}$ and $W \in \mathcal{U}$. Notice that

$$
\left|\left(X \cap U_{i}\right) \Delta\left(W \cap U_{i}\right)\right|= \begin{cases}\left|U_{i} \backslash X\right| & \text { if } U_{i} \subseteq W  \tag{36}\\ \left|X \cap U_{i}\right| & \text { otherwise, (i.e., } U_{i} \cap W=\emptyset \text { since } W \in \mathcal{U}, \text { ) }\end{cases}
$$

hold. Then,

$$
\begin{align*}
f(X) & =\min _{W \in \mathcal{U}}|X \Delta W| \\
& =\min _{W \in \mathcal{U}} \sum_{i=1}^{m}\left|\left(X \cap U_{i}\right) \Delta\left(W \cap U_{i}\right)\right|  \tag{35}\\
& =\sum_{i=1}^{m} \min \left\{\left|U_{i} \backslash X\right|,\left|X \cap U_{i}\right|\right\} \tag{36}
\end{align*}
$$

holds for any $X \in 2^{V}$. For convenience, we define $h_{U}: 2^{V} \rightarrow \mathbb{R}$ for $U \subseteq V$ by

$$
\begin{equation*}
h_{U}(X) \stackrel{\text { def. }}{=} \min \{|X \cap U|,|U \backslash X|\} \tag{37}
\end{equation*}
$$

for $X \in 2^{V}$. Then, $f(X)=\sum_{i=1}^{m} h_{U_{i}}(X)$ holds for any $X \in 2^{V}$. We will prove that $h_{U}(X)$ is submodular in the following Lemma A. 2 . Since the sum of submodular functions is again submodular (see e.g., [5]), and we obtain the claim.

Lemma A.2. The set function $h_{U}$ defined by (37) is submodular.
Proof. For convenience, let $X^{\prime}=X \cap U$ and $Y^{\prime}=Y \cap U$, where we may assume that $\left|X^{\prime}\right|=|X \cap U| \leq$ $\left|Y^{\prime}\right|=|Y \cap U|$ holds, without loss of generality. Then,

$$
h_{U}(X)= \begin{cases}|X \cap U|=\left|X^{\prime}\right| & \text { if }\left|X^{\prime}\right| \leq|U| / 2  \tag{38}\\ |U \backslash X|=\left|U \backslash X^{\prime}\right|=|U|-\left|X^{\prime}\right| & \text { otherwise }\end{cases}
$$

hold for any $X \in 2^{V}$. We consider the following three cases.
Case i) Suppose that $\left|X^{\prime}\right| \leq|U| / 2$ and $\left|Y^{\prime}\right| \leq|U| / 2$ hold. Then, $h_{U}(X)=\left|X^{\prime}\right|$ and $h_{U}(Y)=\left|Y^{\prime}\right|$ hold. Since $\left|X^{\prime} \cap Y^{\prime}\right| \leq\left|X^{\prime}\right| \leq|U| / 2$,

$$
h_{U}(X \cap Y)=\min \{|(X \cap Y) \cap U|,|U \backslash(X \cap Y)|\}=\left|X^{\prime} \cap Y^{\prime}\right|
$$

hold. Observe that

$$
h_{U}(X \cup Y)=\min \{|(X \cup Y) \cap U|,|U \backslash(X \cup Y)|\} \leq|(X \cup Y) \cap U|=\left|X^{\prime} \cup Y^{\prime}\right|
$$

always hold. Thus,

$$
\begin{aligned}
\Phi_{h_{U}}(X, Y) & =h_{U}(X)+h_{U}(Y)-h_{U}(X \cup Y)-h_{U}(X \cap Y) \\
& \geq\left|X^{\prime}\right|+\left|Y^{\prime}\right|-\left|X^{\prime} \cup Y^{\prime}\right|-\left|X^{\prime} \cap Y^{\prime}\right|=0
\end{aligned}
$$

hold where the last equality follows from that the cardinality function is modular. We obtain the claim in the case.

Case ii) Suppose that $\left|X^{\prime}\right|>|S| / 2$ and $\left|Y^{\prime}\right|>|S| / 2$ hold. Then, $h_{U}(X)=|U|-\left|X^{\prime}\right|$ and $h_{U}(Y)=$ $|U|-\left|Y^{\prime}\right|$ hold. Since $\left|X^{\prime} \cup Y^{\prime}\right| \geq\left|Y^{\prime}\right|>|U| / 2$,

$$
h_{U}(X \cup Y)=\min \{|(X \cup Y) \cap U|,|U \backslash(X \cup Y)|\}=|U \backslash(X \cup Y)|=|U|-\left|X^{\prime} \cup Y^{\prime}\right|
$$

hold. Observe that

$$
h_{U}(X \cap Y)=\min \{|(X \cap Y) \cap U|,|U \backslash(X \cap Y)|\} \leq|U \backslash(X \cap Y)|=|U|-\left|X^{\prime} \cap Y^{\prime}\right|
$$

always holds. Thus,

$$
\begin{aligned}
\Phi_{h_{U}}(X, Y) & =h_{U}(X)+h_{U}(Y)-h_{U}(X \cup Y)-h_{U}(X \cap Y) \\
& \geq\left(|U|-\left|X^{\prime}\right|\right)+\left(|U|-\left|Y^{\prime}\right|\right)-\left(|U|-\left|X^{\prime} \cup Y^{\prime}\right|\right)-\left(|U|-\left|X^{\prime} \cap Y^{\prime}\right|\right)=0
\end{aligned}
$$

hold where the last equality follows that the cardinality function is modular. We obtain the claim in the case.
Case iii) Suppose that $\left|X^{\prime}\right| \leq|S| / 2$ and $\left|Y^{\prime}\right|>|S| / 2$ hold. Then, $h_{U}(X)=\left|X^{\prime}\right|$ and $h_{U}(Y)=$ $|U|-\left|Y^{\prime}\right|$ hold. Since $\left|X^{\prime} \cup Y^{\prime}\right| \geq\left|Y^{\prime}\right|>|U| / 2$,

$$
h_{U}(X \cup Y)=\min \{|(X \cup Y) \cap U|,|U \backslash(X \cup Y)|\}=|U|-\left|X^{\prime} \cup Y^{\prime}\right|
$$

holds. Similarly, since $\left|X^{\prime} \cap Y^{\prime}\right| \leq\left|Y^{\prime}\right| \leq|U| / 2$,

$$
h_{U}(X \cap Y)=\min \{|(X \cap Y) \cap U|,|U \backslash(X \cap Y)|\}=\left|X^{\prime} \cap Y^{\prime}\right|
$$

holds. Thus,

$$
\begin{aligned}
\Phi_{h_{U}}(X, Y) & =h_{U}(X)+h_{U}(Y)-h_{U}(X \cup Y)-h_{U}(X \cap Y) \\
& =\left|X^{\prime}\right|+\left(|U|-\left|Y^{\prime}\right|\right)-\left(|U|-\left|X^{\prime} \cup Y^{\prime}\right|\right)-\left(\left|X^{\prime} \cap Y^{\prime}\right|\right) \\
& =2|U|+\left(\left|X^{\prime}\right|-\left|X^{\prime} \cap Y^{\prime}\right|\right)+\left(\left|X^{\prime} \cup Y^{\prime}\right|-\left|Y^{\prime}\right|\right) \\
& \geq 0
\end{aligned}
$$

holds. We obtain the claim.

## A. 2 Proof of Proposition 3.12

Proposition A. 3 (Proposition 3.12). For the submodular function $f$ given by (23), $f \circ \sigma_{S}$ is submodular if and only if $S \in \mathcal{U}$.
Proof. $(\Leftarrow)$ We show that $S \in \mathcal{U}$ is a canonical set. Let $g=f \circ \sigma_{S}$. Then

$$
\begin{equation*}
g(X)=f \circ \sigma_{S}(X)=f(X \triangle S)=\min _{W \in \mathcal{U}}|(X \Delta S) \Delta W|=\min _{W \in \mathcal{U}}|X \Delta(S \Delta W)| \tag{39}
\end{equation*}
$$

holds for any $X \in 2^{V}$. Notice that $W^{\prime}=W \Delta S$ is in $\mathcal{U}$ for any $W \in \mathcal{U}$ and $S \in \mathcal{U}$. Thus,

$$
\text { (39) }=\min _{W^{\prime} \in \mathcal{U}}\left|X \Delta W^{\prime}\right|,
$$

which implies that $g=f$, and hence $g$ is subdmodular by Proposition 3.11.
$(\Rightarrow)$ We prove the contraposition: if $S \notin \mathcal{U}$ then $g=f \circ \sigma_{S}$ is not submodular. By the hypothesis that $S \notin \mathcal{U}$, there exists $U_{i}$ such that $S \cap U_{i} \neq \emptyset$ and $S \cap U_{i} \neq U_{i}$. Let $X=S \Delta U_{i}$, and we claim that $\Phi_{g}(X, S)<0$. First, remark that $g(X)=g\left(U_{i} \Delta S\right)=f\left(U_{i}\right)=0$ and $g(S)=f(\emptyset)=0$ hold. Next,

$$
g(X \cup S)=g\left(S \cup U_{i}\right)=f\left(\left(S \cup U_{i}\right) \Delta S\right)=f\left(U_{i} \backslash S\right)>0
$$

where the last inequality follows from the assumption $U_{i} \cap T \neq U_{i}$ and the fact that $f(X)>0$ unless $X \in \mathcal{U}$ by the definition of $f$ (recall (23))). Similarly,

$$
\begin{aligned}
g(X \cap S) & =g\left(\left(S \triangle U_{i}\right) \cap T\right)=g\left(S \backslash U_{i}\right) \\
& =f\left(\left(S \backslash U_{i}\right) \Delta S\right)=f\left(S \backslash\left(S \backslash U_{i}\right)\right)=f\left(S \cap U_{i}\right) \\
& >0
\end{aligned}
$$

hold where the last inequality follows from the assumption $S \cap U_{i} \neq \emptyset$. Thus,

$$
\begin{aligned}
\Phi_{g}(X, S) & =g(X)+g(S)-g(X \cup S)-g(X \cap S) \\
& <0
\end{aligned}
$$

hold, and we obtain the claim.

## B Supplement to Section 3.4.5

This section shows the connection between the connected components of the inequality graph $G_{f}$ given in Section 3.3 and the inseparable decomposition (cf. [2, 16, 1, 11, 5, 6]) for submodular functions. Precisely, we show the following.
Proposition B.1. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a submodular function. For any set $U \subseteq V, \Phi_{f}(U, \bar{U})=0$ holds if and only if $U$ and $\bar{U}$ are disconnected in the inequality graph $G_{f}$, where $\bar{U}=V \backslash U$.

Notice that Proposition B. 1 implies that $U_{1}, \ldots, U_{k}$ are inseparable decomposition of $f$ if and only if each $U_{i}$ is a connected component of $G_{f}$. To prove Proposition B.1, we will use the following Corollary B. 3 of Theorem B.2. In fact, the following Theorem B.2 is a part of Theorem 3.13. Here we will give a simpler proof in a naive way without using the arguments on a base polytope.
Theorem B. 2 (cf. [2, 16, 1, 11, 5, (6]). Let $\rho: 2^{V} \rightarrow \mathbb{R}$ be a subdmodular function satisfying that $\rho(\emptyset)=0$. Suppose for $U \subset V(U \neq \emptyset)$ that $\rho(V)=\rho(U)+\rho(\bar{U})$ holds where $\bar{U}=V \backslash U$. Then,

$$
\begin{equation*}
\rho(X)=\rho(X \cap U)+\rho(X \cap \bar{U}) \tag{40}
\end{equation*}
$$

holds for any $X \in 2^{V}$.

Proof. To begin with, we remark that (40) is trivial for $X$ satisfying $X \subseteq U$ or $X \subseteq \bar{U}$. Thus, we prove (40) for $X$ satisfying both $X \cap U \neq \emptyset$ and $X \cap \bar{U} \neq \emptyset$. Since $\rho$ is submodular and $\rho(\emptyset)=0$,

$$
\begin{align*}
\rho(X) & \leq \rho(X \cap U)+\rho(X \cap \bar{U})  \tag{41}\\
\rho(X \cup U)+\rho(X \cap U) & \leq \rho(X)+\rho(U)  \tag{42}\\
\rho(X \cup \bar{U})+\rho(X \cap \bar{U}) & \leq \rho(X)+\rho(\bar{U})  \tag{43}\\
\rho(V)+\rho(X) & \leq \rho(X \cup U)+\rho(X \cup \bar{U}) \tag{44}
\end{align*}
$$

hold, respectively. By summing up (42), (43) and (44), we obtain that

$$
\begin{equation*}
\rho(X \cap U)+\rho(X \cap \bar{U}) \leq \rho(X) \tag{45}
\end{equation*}
$$

holds, where we used the hypothesis that $\rho(V)=\rho(U)+\rho(\bar{U})$. Now, (41) and (45) imply (40).
As a corollary of Theorem B.2, we obtain the following.
Corollary B.3. Let $f: 2^{V} \rightarrow \mathbb{R}$ be a subdmodular function (and $f(\emptyset)=0$ may not hold). Suppose for $U \subset V(U \neq \emptyset)$ that $f(V)+f(\emptyset)=f(U)+f(\bar{U})$ holds where $\bar{U}=V \backslash U$. Then,

$$
\begin{equation*}
f(X)+f(\emptyset)=f(X \cap U)+f(X \cap \bar{U}) \tag{46}
\end{equation*}
$$

holds for any $X \in 2^{V}$.
Proof. Let $\rho(X)=f(X)-f(\emptyset)$ for any $X \in 2^{V}$, then $\rho$ satisfies the hypothesis of Theorem B.2. Notice that $f(X)=\rho(X)+f(\emptyset)$ holds for any $X \in 2^{V}$. Thus, (40) implies (46).

Now we prove Proposition B.1.
Proof of Proposition B.1] $(\Leftarrow)$ Suppose that $U$ and $\bar{U}$ are disconnected in $G_{f}$. Then, we prove

$$
\begin{equation*}
f(X)+f(\emptyset)=f(X \cap U)+f(X \cap \bar{U}) \tag{47}
\end{equation*}
$$

holds for any $X \in 2^{V}$, by an induction of the size $|X|$. Notice that (47) is trivial for $|X|=0$ and $|X|=1$. Inductively assuming that (47) holds for any $Y \in 2^{V}$ satisfying $|Y| \leq k$, we prove 47) for any $X \in 2^{V}$ satisfying $|X|=k+1$. Notice that (47) is trivial if $X \subseteq U$ or $X \subseteq \bar{U}$, and we assume that $X \cap U \neq \emptyset$ and $X \cap \bar{U} \neq \emptyset$. Let $u \in X \cap U, v \in X \cap \bar{U}$, and $X^{\prime}=X \backslash\{u, v\}$. Since $\{u, v\} \in E_{f}$,

$$
\begin{equation*}
f\left(X^{\prime} \cup\{u\} \cup\{v\}\right)+f\left(X^{\prime}\right)=f\left(X^{\prime} \cup\{u\}\right)+f\left(X^{\prime} \cup\{v\}\right) \tag{48}
\end{equation*}
$$

holds. By the induction hypothesis, we obtain that

$$
\begin{align*}
& f\left(X^{\prime} \cup\{u\}\right)+f(\emptyset)=f\left(\left(X^{\prime} \cup\{u\}\right) \cap U\right)+f\left(X^{\prime} \cap \bar{U}\right)  \tag{49}\\
& f\left(X^{\prime} \cup\{v\}\right)+f(\emptyset)=f\left(X^{\prime} \cap U\right)+f\left(\left(X^{\prime} \cup\{v\}\right) \cap \bar{U}\right) \tag{50}
\end{align*}
$$

hold, as well as

$$
\begin{equation*}
f\left(X^{\prime} \cap U\right)+f\left(X^{\prime} \cap \bar{U}\right)=f\left(X^{\prime}\right)+f(\emptyset) \tag{51}
\end{equation*}
$$

holds. By summing up (48)-(51), we obtain

$$
f\left(\left(X^{\prime} \cup\{u\}\right) \cup\{v\}\right)+f(\emptyset)=f\left(\left(X^{\prime} \cup\{u\}\right) \cap U\right)+f\left(\left(X^{\prime} \cup\{v\}\right) \cap \bar{U}\right)
$$

which implies 477 holds for the $X$. Let $X=V$, then we obtain the claim.


Figure 5: For Proof of Proposition B.1 $(\Rightarrow)$
$\Leftrightarrow$ Suppose $f(U)+f(\bar{U})=f(V)+f(\emptyset)$ holds. Assume for a contradiction that there is a pair $u \in U$ and $v \in \bar{U}$ such that $\{u, v\} \in E_{f}$. This means that there is $X \subseteq V \backslash\{u, v\}$ such that $\check{\Phi}(X,\{u, v\}) \neq 0$ holds, by the definition (21) of $E_{f}$. For convenience, let $A=X \cap U$ and let $B=X \cap \bar{U}$ (see Figure 5). By Corollary B. 3 .

$$
\begin{align*}
f(X \cup\{u\})+f(\emptyset) & =f(A \cup\{u\})+f(B)  \tag{52}\\
f(X \cup\{v\})+f(\emptyset) & =f(A)+f(B \cup\{v\})  \tag{53}\\
f(X \cup\{u, v\})+f(\emptyset) & =f(A \cup\{u\})+f(B \cup\{v\})  \tag{54}\\
f(X)+f(\emptyset) & =f(A)+f(B) \tag{55}
\end{align*}
$$

hold, respectively. Thus,

$$
\begin{aligned}
\check{\Phi}_{f}(X,\{u, v\})= & f(X \cup\{u\})+f(X \cup\{v\})-f(X \cup\{u, v\})-f(X) \\
= & f(A \cup\{u\})+f(B)+f(A)+f(B \cup\{v\}) \\
& \quad-(f(A \cup\{u\})+f(B \cup\{v\}))-(f(A)+f(B)) \\
= & 0
\end{aligned}
$$

holds, which contradicts to the assumption that $(X,\{u, v\}) \in \mathcal{P}$ satisfies $\check{\Phi}_{f}(X,\{u, v\}) \neq 0$.


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    ${ }^{1}$ Clearly, the condition $\Phi_{f}(X, Y) \geq 0$ is equivalent to $f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y)$, which is often used.

[^1]:    ${ }^{2}$ For the simplicity of the notation, we use the notation $\check{\Phi}(X,\{u, v\})$ instead of $\check{\Phi}((X,\{u, v\}))$. At the same time, we also use the notation $\check{\Phi}(p)$ for $p=(X,\{u, v\}) \in \mathcal{P}$.

[^2]:    ${ }^{3}$ For the simplicity of the notation we use $\check{\sigma}_{S}(X,\{u, v\})$, instead of $\check{\sigma}_{S}((X,\{u, v\}))$. At the same time, we also use the notation $\check{\sigma}(p)$ for $p=(X,\{u, v\}) \in \mathcal{P}$.

[^3]:    ${ }^{4}$ The reduced matrix $\bar{M}_{f}$ itself does not imply any "improvement of computational complexity" to $M_{f}$ : to construct $\bar{M}_{f}$ we have to check (almost) all $(X,\{u, v\}) \in \mathcal{P}$ in the worst case, to confirm if $\exists X \subseteq V \backslash\{u, v\}$ satisfying the condition. See also Proposition 4.3 .

[^4]:    ${ }^{5}$ For any submodular function $f$, let $\rho(X)=f(X)-f(\emptyset)$ for $X \in 2^{V}$, then $\rho$ is submodular satisfying $\rho(\emptyset)=0$.
    ${ }^{6}$ It takes $\mathrm{O}\left(n^{2}\right)$ time if we have a base of $f$ (see [6, 5]).

