

A continuation principle for periodic BV-continuous state-dependent sweeping processes

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Abstract

We consider a Caratheodory differential equation with a state-dependent convex constraint that changes BV-continuously in time (a perturbed BV-continuous state-dependent sweeping processes). By setting up an appropriate catching-up algorithm we prove solvability of the initial value problem. Then, for sweeping processes with T -periodic right-hand-sides, we prove the existence of at least one T -periodic solution. Finally, we further consider a T -periodic sweeping process which is close to an autonomous sweeping process with a constant constraint and prove the existence of a T -periodic solution specifically located near the boundary switched equilibrium of the autonomous sweeping process.

Keywords: Sweeping process, perturbation theory, continuation principle, periodic solution, BV continuous state-dependent convex constraint

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1. Introduction

A variety of applications in elastoplasticity, economy, electrical circuits (see Adly et al [1] and references therein) lead to a constraint differential equation

$$-\dot{x}(t) \in N_{A(t)}(x(t)) + f(t, x(t)), \quad x \in E, \quad (1)$$

with a convex moving set $t \mapsto A(t)$ of just bounded variation (with respect to the Hausdorff metric). Here E is a finite-dimensional vector space and

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$N_A(x)$ is a so-called normal cone defined for closed convex $A \subset E$ as

$$N_A(x) = \begin{cases} \{\xi \in E : \langle \xi, c - x \rangle \leq 0, \text{ for any } c \in A\}, & \text{if } x \in A, \\ \emptyset, & \text{if } x \notin A. \end{cases} \quad (2)$$

Whereas the case of Lipschitz $t \mapsto A(t)$ always leads (Edmond-Thibault [10]) to the existence and uniqueness of an absolutely continuous solution $x(t)$ for any initial condition (under natural assumptions on f), the case where $t \mapsto A(t)$ is a convex-valued function of bounded variation doesn't ensure solvability of (1) in the class of absolutely continuous functions. That is why an extended concept of the derivative (called Radon-Nikodym concept) is required in (1) when the map $t \mapsto A(t)$ is a function of bounded variation, in which case equation (1) is usually formulated in terms of differential measure dx of BV continuous function x and Lebesgue measure dt as

$$-dx \in N_{A(t)}(x) + f(t, x)dt, \quad x \in E. \quad (3)$$

Existence and uniqueness of solutions to (3) and solvability of a periodic problem has been established by Castaing and Monteiro Marques in [5]. The problem of existence and uniqueness of solutions in the unperturbed case ($f \equiv 0$) was addressed in Moreau [26], Monteiro Marques [22], Valadier [29]. Further state-independent extensions of (3) were considered in Adly et al [1], Edmond-Thibault [10], Colombo and Monteiro Marques [6].

Motivated by applications to elastoplastic models with hardening and softening (see e.g. Gilles-Ulisse [12], Kunze [18]), the present paper investigates the initial-value and periodic problems to the following state-dependent version of (3):

$$-dx \in N_{A+a(t)+c(x)}(x) + f(t, x)dt, \quad x \in E, \quad (4)$$

where a is a BV-continuous function and $c : E \mapsto E$ is a Lipschitz function.

To prove the existence of solutions to (4) we introduce a new catching-up algorithm which allows to construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ of approximations of the solution x of (4). The existence of T -periodic solutions to (4) is proved by establishing that

$$d(I - P^n, Q) \neq 0 \quad (5)$$

for the Poincare maps P^n of the n -th approximation of the catching-up scheme and suitable $Q \in E$. Here $d(I - P^n, Q)$ is the topological degree of the map P^n with respect to an open bounded set Q (see Krasnoselskii-Zabreiko [16]). After we get the existence of a fixed point for P^n we pass

to the limit as $n \rightarrow \infty$ on the respective T -periodic solutions of sweeping process (4) and get the existence of a T -periodic solution to (4) even though we don't know whether $\lim_{n \rightarrow \infty} P^n(x)$ exists or not.

The paper offers global and local sufficient conditions to ensure (5). The global sufficient condition is based on construction of such a convex set Q which contains all possible values of the set $A + a(t) + c(x(t))$ for all possible solutions of (4). In this way, we can show that $P^n(Q) \subset \overline{Q}$ for sufficiently large $n \in \mathbb{N}$, which ensures (5).

To design sufficient conditions that ensure the validity of (5) in a desired region Q (local sufficient conditions), we are no longer allowed to enlarge Q as much as we want, so we have to seek for alternative deformations of (5) that stick to the given region Q . We go here by a continuation approach and replace (4) by a parameter dependent sweeping process

$$-dx \in N_{A+a(t,\lambda)+c(x,\lambda)}(x) + f(t, x, \lambda)dt, \quad x \in E, \quad \lambda \in \mathbb{R}. \quad (6)$$

Accordingly, the relation (5) gets replaced by

$$d(I - P^{\lambda,n}, Q) \neq 0. \quad (7)$$

We, therefore, assume that (5) corresponds to (7) with some $\lambda = \lambda_1$ and prove the validity of (7) for $\lambda = \lambda_1$ building upon some good properties of $P^{\lambda,n}$ for $\lambda = 0$ combined with nondegenerate homotopy between $P^{\lambda_1,n}$ and $P^{0,n}$.

As for possible good properties of $P^{0,n}$ we offer both topological (Theorem 3) and algebraic (Theorem 16) conditions. The topological condition simply assumes that (7) holds for $\lambda = 0$, that leads to an analogue of standard continuation principles available for ordinary differential equations, see e.g. Capietto et al [4] and Kamenskii et al [14].

To obtain easily verifiable algebraic conditions that ensure the validity of (7) for $\lambda = 0$, this paper takes a straightforward route and simply offers sufficient conditions for asymptotic stability of a point x_0 of the target set Q . Such an approach is based on the fact that the topological degree of a Poincare map in the neighborhood of an asymptotically stable fixed point equals 1. However, just assuming that x_0 is an asymptotically stable equilibrium of (6) with $\lambda = 0$ is not of interest because it leads to periodic solutions that don't interact with the boundary of the constraint of (6) when $\lambda > 0$. Such

periodic solutions will simply be solutions of the differential equation

$$-dx = f(t, x, \lambda)dt, \quad x \in E, \quad \lambda \in \mathbb{R}. \quad (8)$$

That is why a non-equilibrium concept of an asymptotically stable point x_0 is required to design periodic solutions of (6) which are intrinsically sweeping (i.e. interact with the boundary of the constraint of (6)).

The required concept of asymptotically stable point x_0 has been recently developed in Kamenskii-Makarenkov [13] based on the notion of switched boundary equilibrium well known in control theory (see e.g. Bolzern-Spinelli [3]). To introduce the concept of switched boundary equilibrium for sweeping process (6) at $\lambda = 0$, we will assume that, at $\lambda = 0$, sweeping process (6) takes the form

$$-\dot{x} \in N_A(x) + f_0(x), \quad x \in E, \quad (9)$$

where A is just a constant convex closed bounded set and f_0 is Lipschitz continuous. Following Kamenskii-Makarenkov [13], x_0 is a switched boundary equilibrium of (9), if $f_0(x_0)$ is normal to the boundary ∂A and if $f_0(x_0)$ points inwards A . To prove the validity of (7) for $\lambda = 0$ this paper, therefore, offers sufficient conditions for asymptotic stability of switched boundary equilibrium x_0 of (9).

The paper is organized as follows. In the next section of the paper we introduce a formal definition of sweeping process (6) following Castaing and Monteiro Marques [5]. The fundamental result of the paper (Theorem 1) on solvability of the initial-value problem for (4) (and, therefore, for (6) too) is formulated in Section 3. In the same section we introduce our concept of *generalized initial condition* that we repeatedly use in the paper later and which allows us to consider solutions of (4) with initial conditions outside $A + a(t) + c(x)$. We simply say that $x(t)$ is a solution of (4) with initial condition $q \in E$, if $x(0)$ is the solution of $x(0) = \text{proj}(q, A + a(0) + c(x(0)))$ (which has a unique solution $x(0) = V^0(q)$ according to Lemma 6).

Sections 4 and 5 contain formulations of our results on the existence of T -periodic solutions to (4). Section 4 offers a theorem (Theorem 2) saying that any T -periodic state-dependent sweeping process (4) always admits at least one T -periodic solution, if the right-hand-sides of (4) are T -periodic. Remarkably, the theorem doesn't assume uniqueness or continuous dependence of solutions of (4) on initial conditions.

Abstract results on continuation of T -periodic solutions to (6) are presented in Section 5. We assume that for $\lambda = 0$ the sweeping process (6) admits a Poincare map P^0 (over time T) and formulate (Theorem 3) a standard continuation principle: *if the topological degree $d(I - P^0 \circ V^0, Q) \neq 0$ for some open bounded set $Q \subset E$ and if none points of the boundary of Q are initial conditions of T -periodic solutions of sweeping process (6) for any $\lambda \in [0, \lambda_1]$ (non-degenerate deformation), then, for any $\lambda \in [0, \lambda_1]$, sweeping process (6) admits a T -periodic solution x .* A result on the existence of $\lambda_1 > 0$ such that the non-degenerate deformation assumption of Theorem 3 holds is also presented (Theorem 4) in Section 5.

Section 6 contains proofs of Theorems 1-4. The proof of the existence of solutions is based on introducing (section 6.2) a new implicit catching-up scheme (16)-(19), which in turn relies on the following two ideas: **(i)** Castaing and Monteiro Marques change of the variables [5, Theorem 4.1] that converts (section 6.1) the perturbed sweeping process (6) with differential measure dx into a non-perturbed sweeping process (14) for the derivative $\frac{du}{|du|}$ with respect to the variation measure $|du|$ of du ; **(ii)** Kunze and Monteiro Marques lemma ([18, Lemma 7]) to resolve (Lemma 6) the implicit catching-up scheme (16)-(19) with respect to the implicit variable. Furthermore, our Lemma 6 extends [18, Lemma 7] by proving continuous dependence of scheme (16)-(19) on initial condition, that gave us continuity of Poincare maps $P^{\lambda,n}$ (section 6.3). The convergence of the scheme (16)-(19) is established in section 6.4 where we prove (Lemma 8) convergence of the approximations $\{u_n\}_{n \in \mathbb{N}}$ of solution u of (14) and then prove (Lemma 9) convergence of the respective approximations $\{x_n\}_{n \in \mathbb{N}}$ of solution x of sweeping process (6). In other words, Lemma 9 states that the change of the variables of Castaing and Monteiro Marques [5, Theorem 4.1] is continuous with respect to time-discretization. Finally, a result by Monteiro Marques [22, p. 15-16] (which is also Proposition 6 in Valadier [29]) is used to prove (Theorem 10 of section 6.5) that the limit of catching-up scheme (16)-(19) is a solution of (6).

Section 8 is devoted to establishing conditions for the validity of (7) at $\lambda = 0$ in a neighborhood Q of a switched boundary equilibrium x_0 . Specifically, as mentioned earlier, we assume that, for $\lambda = 0$ sweeping process (6) takes the form (9) and discover conditions for asymptotic stability of $x_0 \in \partial A$. In particular, in section 8 we extend the two-dimensional approach of Makarenkov and Niwanthi Wadippuli [21] and derive a differential equation of sliding motion along ∂A , for which x_0 is a regular equilibrium whose stability can be

investigated (Theorem 15) over the eigenvalues of the respective linearization. Assuming that the real parts of these eigenvalues are negative we conclude that $d(I - P^0 \circ V_0, Q) = 1$ and establish (Theorem 16) the existence of T -periodic solutions near x_0 for all BV-continuous state-dependent sweeping processes (6) that approaches (9) when $\lambda \rightarrow 0$.

Conclusions and Acknowledgments sections conclude the paper.

2. Definition of solution

In what follows, $\mathcal{B}([0, T])$ is the family of Borel subsets of $[0, T]$. A *Borel vector measure* on $[0, T]$ is a map $\mu : \mathcal{B}([0, T]) \rightarrow E$ such that $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ for any sequence $\{B_n\}_{n=1}^{\infty}$ of mutually disjoint elements of $\mathcal{B}([0, T])$, see Recupero [27, §2.4] or Dinculeanu [9, Definition 1, §III.14.4, p. 297].

According to Dinculeanu [9, Theorem 1, §III.17.2, p. 358] (see also Recupero [27]), any BV-continuous function $x : [0, T] \rightarrow E$ admits a unique vector measure of bounded variation $dx : \mathcal{B}([0, T]) \rightarrow E$ (called *Stieltjes measure* in [9]) such that for every $0 < t_1 < t_2 < T$ we have

$$\begin{aligned} dx((t_1, t_2)) &= x(t_2^-) - x(t_1^+), & dx([t_1, t_2]) &= x(t_2^+) - x(t_1^-), \\ dx([t_1, t_2)) &= x(t_2^-) - x(t_1^+), & dx((t_1, t_2]) &= x(t_2^+) - x(t_1^+), \end{aligned}$$

where

$$x(t^-) = \lim_{\tau \rightarrow t^-} x(\tau), \quad x(t^+) = \lim_{\tau \rightarrow t^+} x(\tau), \quad 0 < t < T.$$

A vector Borel measure $d\mu$ is called continuous with respect to a scalar Borel measure $d\nu$ (or simply $d\nu$ -continuous), if $\lim_{\nu(D) \rightarrow 0} \mu(D) = 0$, see Diestel-Uhl [8, p. 11]. If a vector measure $d\mu$ is $d\nu$ -continuous then, according to Radon-Nikodym Theorem [8, p. 59] there is a $d\nu$ -integrable function $g : [0, T] \mapsto E$ such that

$$d\mu(D) = \int_D g d\nu, \quad \text{for all } D \in \mathcal{B}([0, T]).$$

In this case, the function g is called Radon-Nikodym derivative of $d\mu$ with respect to $d\nu$ (or density) and is denoted by $\frac{d\mu}{d\nu}$. Furthermore, according to Moreau-Valadier [25, Proposition 1] (see also Valadier [29, Theorem 3]), the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$ can be computed as

$$\frac{d\mu}{d\nu}(t) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \frac{d\mu([t, t + \varepsilon])}{d\nu([t, t + \varepsilon])}, \quad d\nu - a.e. \text{ on } [0, T].$$

We will use the following definition of the solution of (4) (Castaing and Monteiro Marques [5, §1]).

Definition 1. *A BV continuous function x is called a solution of (4), if there exists a finite measure $d\nu$ for which both differential measure dx and Lebesgue measure dt are $d\nu$ -continuous, and such that*

$$-\frac{dx}{d\nu}(t) \in N_{A+a(t)+c(x(t))}(x(t)) + f(t, x(t))\frac{dt}{d\nu}(t), \quad d\nu - a.e. \text{ on } [0, T].$$

3. Existence of solutions

It is customary (see [19, Theorem 6]) to assume that the initial condition q of sweeping process (4) satisfies

$$q \in A + a(0) + c(q). \quad (10)$$

However, it will be convenient for our analysis to define solutions of (4) for any initial condition $q \in E$, that we will term a *generalized initial condition*. We take advantage of the fact, that for contracting map c , the equation

$$v = \text{proj}(q, A + a(0) + c(v))$$

always has a solution $v = V(q)$ (see Lemma 6) and $V \in C^0(E, E)$.

The core of this paper is the following Theorem 1 on the existence of solutions to (4). As itself, the theorem won't loss anything by dropping the generalized initial condition concept. However, considering the generalized initial conditions will be convenient for applications of Theorem 1 to the problem of the occurrence of periodic solutions from a boundary equilibrium, that we consider in this paper later (Theorem 16).

Theorem 1. *Assume that $A \subset E$ is a nonempty closed convex bounded set, $a : [0, T] \rightarrow E$ is BV-continuous on $[0, T]$, $x \mapsto c(x)$ is globally Lipschitz with Lipschitz constant $0 < L_2 < 1$, and $(t, x) \mapsto f(t, x)$ is Caratheodory in (t, x) with respect to Lebesgue measure and globally Lipschitz in x . Then, for any generalized initial condition $q \in E$, the sweeping process (4) admits a solution, defined on $[0, T]$, with the initial condition $x(0) = V(q)$. In particular, sweeping process (4) admits a solution on $[0, T]$, for any initial condition $x(0) = q$, where q satisfies (10).*

4. Global existence of periodic solutions

In this section we offer a result saying that, under the conditions of Theorem 1, sweeping process (4) always has a periodic solution, if the right-hand-sides are T -periodic.

Theorem 2. *Assume that conditions of theorem 1 hold and let $L_2 \in (0, 1)$ be the Lipschitz constant of c as introduced in theorem 1. Denoting by $\xi \in E$ the unique solution of $c(\xi) = \xi$, consider the set*

$$\Omega = \bigcup_{t \in [0, T]} \Omega_t, \quad \Omega_t = \bigcup_{b \in A(t)} \left\{ x : \|x - \xi\| < \frac{\|b\|}{1 - L} \right\}.$$

Then sweeping process (4) admits a solution $t \mapsto x(t)$ such that

$$x(T) = x(0) \in \overline{\Omega}. \quad (11)$$

In particular, $t \mapsto x(t)$ is a T -periodic solution of (4), if both $t \mapsto a(t)$ and $t \mapsto f(t, x)$ are T -periodic.

Remark 1. Throughout the paper we prefer to work with functions defined on $[0, T]$ only. When saying $t \mapsto x(t)$ is a T -periodic solution of (4), we mean that $t \mapsto x(t)$ becomes a T -periodic solution after all functions are extended to $\mathbb{R}$ by T -periodicity.

5. Continuation of periodic solutions

This section considers a λ -dependent sweeping process (6) for measures dx and dt , and discovers how the existence of periodic solutions for $\lambda > 0$ can be concluded from an appropriate knowledge about (6) at $\lambda = 0$.

We will assume that BV-continuity of a of Theorem 1 holds uniformly with respect to λ , i.e.

$$\begin{aligned} \text{var}(a(\cdot, \lambda), [s, t]) &\leq \text{var}(\bar{a}, [s, t]), \quad \lambda \in [0, 1], \\ \text{where } \bar{a} : [0, T] &\rightarrow \mathbb{R} \text{ is a BV continuous function.} \end{aligned} \quad (12)$$

The map V^λ for (6) now depends on the parameter λ and is defined as the unique solution (according to Lemma 6) of the equation

$$v = \text{proj}(q, A + a(0, \lambda) + c(v, \lambda)).$$

We will call sweeping process (6) T -periodic, if

$$a(t + T, \lambda) \equiv a(t, \lambda), \quad f(t + T, x, \lambda) \equiv f(t, x, \lambda).$$

In what follows, $d(I - \bar{P}, Q)$ is the topological degree of the vector field $I - \bar{P}$ on an open bounded set $Q \subset E$, see e.g. Krasnoselskii-Zabreiko [16].

Theorem 3. *Assume that T -periodic sweeping process (6) possesses the following regularity:*

- I) *The set $A \subset E$ is nonempty, convex, closed, and bounded. The function a satisfies (12). The function $x \mapsto c(x, \lambda)$ is globally Lipschitz with Lipschitz constant $0 < L_2 < 1$. The function $(t, x) \mapsto f(t, x, \lambda)$ is Caratheodory in (t, x) with respect to Lebesgue measure and globally Lipschitz in x , and both the Lipschitz constants are independent of $\lambda \in [0, 1]$. Furthermore, a , c , and f are continuous in $\lambda \in [0, 1]$ uniformly with respect to $t \in [0, T]$ and $x \in E$.*

Assume, that the existence of a T -periodic solution for $\lambda = 0$ is given in the following extended way:

- II) *There exists an open bounded $Q \subset E$ such that, when $\lambda = 0$, the solution of (6) is unique for any initial condition $x(0) \in V^0(\bar{Q})$, none of the elements of $V^0(\partial Q)$ are initial conditions of T -periodic solutions of (6) with $\lambda = 0$, and for the Poincare map P^0 of (6) with $\lambda = 0$ one has*

$$d(I - P^0 \circ V^0, Q) \neq 0.$$

Finally, assume the following homotopy through $\lambda \in [0, \lambda_1]$:

- III) *There exists $\lambda_1 \in (0, 1]$ such that sweeping process (6) doesn't have periodic solutions x with initial condition $x(0) \in V^\lambda(\partial Q)$, $\lambda \in [0, \lambda_1]$.*

Then, for any $\lambda \in [0, \lambda_1]$, sweeping process (6) admits a T -periodic solution x with the initial condition $x(0) \in V^\lambda(Q)$.

Note, for $\lambda > 0$, we don't know whether or not the solutions of sweeping process (6) are uniquely defined by the initial condition or depend continuously on λ . That is why the statement of the theorem is not a direct consequence of II) as it usually happens in topological degree based existence results. In particular, we cannot establish any type of continuity of solutions as $\lambda \rightarrow 0$. That is why the next theorem is not a direct consequence of Theorem 3.

Theorem 4. *Assume that sweeping process (6) is T -periodic. Assume that conditions I) and II) of Theorem 3 hold. Then, there exists $\lambda_1 > 0$ such that condition III) of Theorem 3 holds, and, therefore, for any $\lambda \in [0, \lambda_1]$, sweeping process (6) admits a T -periodic solution x with the initial condition $x(0) \in V^\lambda(Q)$.*

6. The catching-up algorithm and proofs of the abstract existence results

6.1. An equivalent non-perturbed formulation of the initial perturbed sweeping process

Recall, that for a BV-continuous function $u : [0, T] \rightarrow E$, the *variation measure* $|du|$ (also called *modulus measure*) is defined, for any $D \in \mathcal{B}([0, T])$, as (see Diestel-Uhl [8, Definition 4, p. 2], Recupero [27, §2.4])

$$\begin{aligned} |du|(D) &= \\ &= \sup \left\{ \sum_{n=1}^{\infty} \|u(D_n)\| : D = \bigcup_{n=1}^{\infty} D_n, D_n \in \mathcal{B}([0, T]), D_i \cap D_j = \emptyset \text{ if } i \neq j \right\}. \end{aligned}$$

For a BV-continuous function $u : [0, T] \rightarrow \mathbb{R}$, the differential measure du is always $|du|$ -continuous (it follows e.g. from Diestel-Uhl [8, Theorem 1, p. 10]), i.e. a $|du|$ -integrable density $\frac{du}{|du|}$ is well defined. Moreover, according to Castaing and Monteiro Marques [5, Theorem 4.1], if x is a solution of the perturbed sweeping process (6), then the BV continuous function u defined by

$$u(t) = x(t) + \int_0^t f(\tau, x(\tau)) d\tau \quad (13)$$

is a solution to the non-perturbed sweeping process

$$-\frac{du}{|du|}(t) \in N_{A+a(t,\lambda)+c(x(t),\lambda)+\int_0^t f(\tau,x(\tau),\lambda)d\tau}(u(t)), \quad |du| - a.e. \text{ on } [0, T]. \quad (14)$$

Lemma 5. *Assume that $(t, x, \lambda) \mapsto f(t, x, \lambda)$ is Caratheodory in (t, x) with respect to Lebesgue measure and is globally Lipschitz in x with Lipschitz constant independent of $t \in [0, T]$ and $\lambda \in [0, 1]$. Then, for any BV continuous $u : [0, T] \rightarrow E$, the equation (13) admits a unique BV continuous solution $x : [0, T] \rightarrow E$.*

Lemma 5 is a direct consequence of Lemma 9 that we prove below.

Combining [5, Theorem 4.1] and Lemma 5, we can formulate the following equivalent definition of the solution of (6).

Definition 2. *A BV continuous function x is called a solution of perturbed sweeping process (6), if the function u given by (13) is a solution of the non-perturbed sweeping process (14).*

6.2. The catching-up algorithm

For each fixed $n \in \mathbb{N}$, we partition $[0, T]$ into smaller intervals by the points $\{t_0, t_1, \dots, t_n\} \subset [0, T]$ defined by

$$t_0 = 0, \quad t_n = T, \quad t_{i+1} - t_i = \frac{T}{n}, \quad i \in \overline{1, n}.$$

In what follows, we fix some initial condition

$$x(0) = u(0) = q,$$

where q satisfies

$$q \in A + a(0, \lambda) + c(q, \lambda), \tag{15}$$

and use the ideas of Definition 2 in order to construct piecewise-linear functions u_n and x_n (linear on each $[t_i, t_{i+1}]$) that serve as approximations of the solutions u and x of Definition 2. The construction will be implemented iteratively through the intervals $[t_i, t_{i+1}]$ starting from $i = 0$, and moving towards $i = n - 1$.

Denoting

$$u_n(0) = q, \quad x_n(0) = q, \quad u_i^n = u_n(t_i), \quad x_i^n = x_n(t_i), \quad i \in \overline{0, n},$$

we apply the implicit iterative scheme

$$u_{i+1}^n = \text{proj} \left[u_i^n, A + a(t_{i+1}, \lambda) + c \left(u_{i+1}^n - \int_0^{t_i} f(\tau, x_n(\tau), \lambda) d\tau, \lambda \right) + \int_0^{t_i} f(\tau, x_n(\tau), \lambda) d\tau \right], \quad (16)$$

$$x_{i+1}^n = u_{i+1}^n - \int_0^{t_i} f(\tau, x_n(\tau), \lambda) d\tau, \quad (17)$$

$$u_n(t) = u_i^n + \frac{t - t_i}{t_{i+1} - t_i} (u_{i+1}^n - u_i^n), \quad t \in [t_i, t_{i+1}], \quad (18)$$

$$x_n(t) = x_i^n + \frac{t - t_i}{t_{i+1} - t_i} (x_{i+1}^n - x_i^n), \quad t \in [t_i, t_{i+1}], \quad (19)$$

successively from $i = 0$ to $i = n - 1$. Next lemma uses the idea of the implicit scheme of Kunze and Monteiro Marques ([19, Lemma 7]) and it proves that for each $i \in \overline{0, n-1}$ we can extend the definition of u_n and x_n from $[0, t_i]$ to $[0, t_{i+1}]$ according to (16)-(19).

Lemma 6. *Consider a set-valued function*

$$C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi), \quad s_1, s_2 \in [0, T], \quad u \in E, \quad \xi \in W,$$

where $A \subset E$ is a nonempty closed convex bounded set, $\tilde{a} : \mathbb{R} \times W \rightarrow E$, $\tilde{c} : \mathbb{R} \times E \times W \rightarrow E$, and W is a finite dimensional Euclidean space. Assume that

$$\text{var}(\tilde{a}(\cdot, \xi), [s, t]) \leq \text{var}(\bar{a}, [s, t]), \quad \xi \in W,$$

where $\bar{a} : [0, T] \rightarrow \mathbb{R}$ is a BV continuous function,

and $(s, \xi) \rightarrow \tilde{a}(s, \xi)$ is continuous in $\xi \in W$ uniformly in $s \in [0, T]$. Assume that $(s, u, \xi) \mapsto \tilde{c}(s, u, \xi)$ is continuous in $\xi \in W$ uniformly in $(s, u) \in [0, T] \times E$ and satisfies the Lipschitz condition

$$\|\tilde{c}(s, u, \xi) - \tilde{c}(t, v, \xi)\| \leq L_1 |s - t| + L_2 \|u - v\|, \\ \text{for any } s, t \in [0, T], \quad u, v \in E, \quad \xi \in W,$$

with $L_1 > 0$ and $L_2 \in (0, 1)$. Then, for any $\tau_1, \tau_2, s_1, s_2 \in [0, T]$ and any $u \in E$ there exists an unique $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ such that

$$v \in C(\tau_1, \tau_2, v, \xi) \quad \text{and} \quad v = \text{proj}(u, C(\tau_1, \tau_2, v, \xi)). \quad (20)$$

Moreover, $v \in C^0([0, T] \times [0, T] \times [0, T] \times [0, T] \times E \times W, E)$. If, in addition,

$$u \in C(s_1, s_2, u, \xi),$$

then

$$\|v - u\| \leq \frac{\text{var}(\bar{a}, [s_1, \tau_1]) + L_1|\tau_2 - s_2|}{1 - L_2}. \quad (21)$$

The following key estimate is required for the proof of Lemma 6.

Lemma 7. *Let C be a convex set of E . Then, for any vectors $u, c \in E$,*

$$\|\text{proj}(u, C) - \text{proj}(u, C + c)\| \leq \|c\|.$$

Proof. From the definition of projections $v_1 = \text{proj}(u, C)$ and $v_2 = \text{proj}(u, C + c)$ we have (see e.g. Kunze and Monteiro Marques [19, §2])

$$u - v_1 \in N_C(v_1) \quad \text{and} \quad u - v_2 \in N_{C+c}(v_2). \quad (22)$$

Since $v_2 - c \in C$ and $v_1 + c \in C + c$, we conclude from (22) that

$$\langle u - v_1, v_2 - c - v_1 \rangle \leq 0 \quad \text{and} \quad \langle u - v_2, v_1 + c - v_2 \rangle \leq 0,$$

or, rearranging the terms,

$$\langle v_1 - u, v_1 - v_2 \rangle \leq \langle u - v_1, c \rangle \quad \text{and} \quad \langle u - v_2, v_1 - v_2 \rangle \leq \langle v_2 - u, c \rangle.$$

Finally, we add both inequalities together and get

$$\langle v_1 - v_2, v_1 - v_2 \rangle \leq \langle v_2 - v_1, c \rangle \leq \|v_1 - v_2\| \cdot \|c\|,$$

which implies the statement. $\square$

Proof of Lemma 6. Step 1. *The existence of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$.* Define $F \in C^0(E, E)$ as $F(v) = \text{proj}(u, C(\tau_1, \tau_2, v, \xi))$. Using Lemma 7, we have

$$\begin{aligned} & \|F(v_1) - F(v_2)\| = \\ & = \|\text{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v_1, \xi)) - \text{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v_2, \xi))\| \leq \\ & \leq \|\tilde{c}(\tau_2, v_1, \xi) - \tilde{c}(\tau_2, v_2, \xi)\| \leq L_2\|v_1 - v_2\|, \end{aligned} \quad (23)$$

so the existence of $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ with the required property (20) follows by applying the contraction mapping theorem (see e.g. Rudin [28, Theorem 9.23]).

Step 2. *Continuity of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$.* To prove the continuity of v , let $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ and $\bar{v} = v(\bar{\tau}_1, \bar{\tau}_2, \bar{s}_1, \bar{s}_2, \bar{u}, \bar{\xi})$ where $s_1, s_2, \bar{s}_1, \bar{s}_2 \in [0, T]$, $\tau_1, \tau_2, \bar{\tau}_1, \bar{\tau}_2 \in [0, T]$, $\xi, \bar{\xi} \in W$ and $u, \bar{u} \in E$.

First observe that

$$\begin{aligned} \|\bar{v} - v\| &= \\ &= \|\text{proj}(\bar{u}, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \text{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi))\| \\ &\leq \|\text{proj}(\bar{u}, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \text{proj}(u, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi}))\| \\ &\quad + \|\text{proj}(u, A + \tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi})) - \text{proj}(u, A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi))\|. \end{aligned}$$

Since for any nonempty, closed, convex set $C \subset E$ and any vectors $\bar{u}, u \in E$, we have (see e.g. Mordukhovich-Nam [24, Proposition 1.79])

$$\|\text{proj}(\bar{u}, C) - \text{proj}(u, C)\| \leq \|\bar{u} - u\|, \quad (24)$$

then, using also Lemma 7, we conclude that

$$\begin{aligned} \|\bar{v} - v\| &\leq \|\bar{u} - u\| + \|\tilde{a}(\bar{\tau}_1, \bar{\xi}) + \tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi}) - \tilde{a}(\tau_1, \xi) - \tilde{c}(\tau_2, v, \xi)\| \leq \\ &\leq \|\bar{u} - u\| + \|\tilde{a}(\bar{\tau}_1, \bar{\xi}) - \tilde{a}(\bar{\tau}_1, \xi)\| + \text{var}(\bar{a}, [\tau_1, \bar{\tau}_1]) + \\ &\quad + \|\tilde{c}(\bar{\tau}_2, \bar{v}, \bar{\xi}) - \tilde{c}(\tau_2, v, \xi)\| + L_1|\bar{\tau}_2 - \tau_2| + L_2\|\bar{v} - v\|, \end{aligned} \quad (25)$$

so that the required continuity of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ follows from $0 \leq L_2 < 1$.

Step 3. *Proof of the estimate (21).* Assuming that $u \in C(s_1, s_2, u, \xi)$, we have follow the lines of (25) to get

$$\|v - u\| = \|\text{proj}(u, C(\tau_1, \tau_2, v, \xi)) - u\| = \min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\|.$$

But $C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi)$ and $C(\tau_1, \tau_2, v, \xi) = A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)$. Therefore,

$$\begin{aligned} \min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| &\leq \|\tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi) - \tilde{a}(\tau_1, \xi) - \tilde{c}(\tau_2, v, \xi)\| \leq \\ &\leq \text{var}(\bar{a}, [s_1, \tau_1]) + L_1|\tau_2 - s_2| + L_2\|u - v\|, \end{aligned} \quad (26)$$

which implies (21).

The proof of the lemma is complete. $\square$

Remark 2. *On the validity of Lemma 6 when $A + c(t, \xi)$ is replaced by a more a more general term $A(t, \xi)$.*

One can observe that estimate (23) holds also in the case where $A + \tilde{a}(t, \xi)$ takes a more general form $A(t, \xi)$. Furthermore, if $d_H(A_1, A_2)$ is the Hausdorff distance between nonempty closed sets $A_1, A_2 \subset E$ and $A(t, \xi)$ satisfies

$$d_H(A(s), A(t)) \leq \text{var}(\bar{a}, [s, t]), \quad (27)$$

then (26) holds as well since

$$\begin{aligned} \min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| &\leq d_H(A(s_1, \xi) + \\ &+ \tilde{c}(s_2, u, \xi), A(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)). \end{aligned}$$

To summarize, the existence of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ (Step 1) and the estimate (21) (Step 3) still hold, if $A + \tilde{a}(t, \xi)$ is replaced by $A(t, \xi)$ satisfying (27).

At the same time, the continuity of the function $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ can no longer be established when $A + \tilde{a}(t, \xi)$ is replaced by $A(t, \xi)$. Indeed, the core of estimate (25) is Lemma 7 which doesn't allow a generalization when $C + c$ is replaced by an arbitrary set D . One might be tempted to believe that the conclusion of Lemma 7 can be replaced by

$$\|\text{proj}(u, C) - \text{proj}(u, D)\| \leq d_H(C, D), \quad (28)$$

when $C + c$ is replaced by just D , but formula (28) appears to be wrong as our Fig. 1 illustrates.

On the other hand Monteiro Marques [23, Proposition 4.7, p. 26] implies that

$$\begin{aligned} \|\text{proj}(u, C) - \text{proj}(u, D)\| &\leq \\ &\leq \sqrt{2(\text{dist}(u, C) + \text{dist}(u, D))} \cdot \\ &\cdot \sqrt{d_H(C, D)}, \end{aligned} \quad (29)$$

which could potentially help to obtain other versions of Lemma 6, that we don't pursue in this paper.

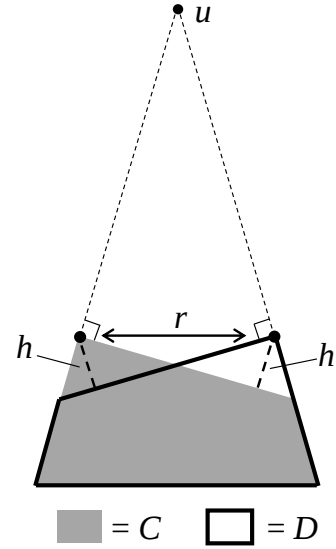


Figure 1: Illustration of the incorrectness of formula (28). Here $h = d_H(C, D)$, and $r = \|\text{proj}(u, C) - \text{proj}(u, D)\|$.

Corollary 1. *Assume that condition I) of Theorem 3 holds. Then, for any (q, λ) satisfying (15) the implicit scheme (16)-(19) is solvable iteratively from $i = 0$ to $i = n-1$ and the respective iterations $x_i^n = x_i^n(q, \lambda)$ and $u_i^n = u_i^n(q, \lambda)$ are continuous in (q, λ) on $E \times [0, 1]$. Moreover,*

$$\|u_{i+1}^n(q, \lambda) - u_i^n(q, \lambda)\| \leq \frac{\text{var}(\bar{a}, [t_i, t_{i+1}]) + L_1 T/n}{1 - L_2}, \quad i \in \overline{0, n-1},$$

where $L_1 > 0$ and $L_2 \in (0, 1)$.

Proof. Let $\xi = ((\xi_1, \xi_2, \dots, \xi_{n+1}), \xi_{n+2}) \in E^{n+1} \times \mathbb{R}$ be defined as

$$\xi_i = x_{i-1}^n, \quad i \in \overline{1, n+1}, \quad \xi_{n+2} = \lambda.$$

Therefore, the rule (19) defines a function $\Psi : E^{n+1} \times \mathbb{R} \rightarrow C^0([0, T], E)$ that relates $\xi \in E^{n+1} \times \mathbb{R}$ to a piecewise linear function $x_n(t)$ defined on $[0, T]$. The statement of the Corollary 1 now follows by applying Lemma 6 with

$$\begin{aligned} \tilde{c}(s, u, \xi) &= \left(u - \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau, \xi_{n+2} \right) + \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau, \\ \tilde{a}(s, \xi) &= a(s, \xi_{n+2}). \end{aligned}$$

The proof of the corollary is complete. $\square$

6.3. The Poincare map associated to the catching-up algorithm

Even though we cannot ensure the existence of a Poincare map for sweeping process (6), we can associate the following Poincare map

$$P^{\lambda, n}(q) = x_n(T)$$

to the approximations x_n of the catching-up algorithm (16)-(19). Corollary 1 allows to formulate the following property of the map $P^{\lambda, n}$.

Corollary 2. *Assume that condition I) of Theorem 3 holds. Consider an open bounded set $Q \subset E$. Then, for each fixed $\lambda \in [0, 1]$ and $n \in \mathbb{N}$, the Poincare map $q \mapsto P^{\lambda, n}(q)$ is continuous on $\overline{Q}$.*

6.4. Convergence of the catching-up algorithm

Let $(u_n(t, q, \lambda), x_n(t, q, \lambda))$ be the solution $(u_n(t), x_n(t))$ of the catching-up algorithm (16)-(19) with the parameter $\lambda \in [0, 1]$ and the initial condition $u_n(0) = x_n(0) = q$.

Lemma 8. *Assume that condition I) of Theorem 3 holds. Consider a sequence $(\lambda_n, q_n) \rightarrow (\lambda_0, q_0)$ as $n \rightarrow \infty$ of $[0, 1] \times E$ satisfying (15) for each $n \in \mathbb{N}$. Then, there exists a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$ converges as $k \rightarrow \infty$ uniformly in $t \in [0, T]$.*

Proof. Step 1. *Boundedness of $\{u_n(t, q_n, \lambda_n)\}_{n \in \mathbb{N}}$.* Let $u_i^n, i \in \overline{0, n}$, be the approximations given by (16)-(19) with $q = q_n$ and $\lambda = \lambda_n$. By Corollary 1,

$$\|u_n(t, q_n, \lambda_n)\| \leq \|q_n\| + \frac{1}{(1 - L_2)} (\text{var}(\bar{a}, [0, T]) + L_1 T),$$

so the sequence $\{u_n(t, q_n, \lambda_n)\}_{n \in \mathbb{N}}$ is bounded uniformly on $[0, T]$.

Step 2. *Equicontinuity of $\{u_n(t, q_n, \lambda_n)\}_{n \in \mathbb{N}}$.* Fix $\varepsilon > 0$. Since $\text{var}(\bar{a}, [s, t]) \rightarrow 0$ as $|s - t| \rightarrow 0$ (see e.g. Lojasiewicz [20, Theorem 1.3.4, p. 16]), we can choose $\delta_1 > 0$ such that

$$\frac{\text{var}(\bar{a}, [s, t]) + L_1(t - s)}{1 - L_2} < \frac{\varepsilon}{3}, \quad \text{for all } 0 \leq s \leq t \leq T \text{ with } t - s < \delta_1. \quad (30)$$

Fix some $0 \leq s \leq t \leq T$ satisfying $t - s < \delta_1$ and denote by $i_s, i_t \in \overline{0, n-1}$ such indexes that

$$s \in [t_{i_s}, t_{i_s+1}], \quad t \in [t_{i_t}, t_{i_t+1}].$$

Then we can estimate $\|u_n(t) - u_n(s)\|$ as follows:

$$\begin{aligned} \|u_n(t) - u_n(s)\| &\leq \\ &\leq \|u_n(s) - u_n(t_{i_s+1})\| + \|u_n(t_{i_s+1}) - u_n(t_{i_t})\| + \|u_n(t_{i_t}) - u_n(t)\| \leq \\ &\leq \text{var}(u_n, [t_{i_s}, t_{i_s+1}]) + \text{var}(u_n, [t_{i_s+1}, t_{i_t}]) + \text{var}(u_n, [t_{i_t}, t_{i_t+1}]). \end{aligned}$$

The second term is smaller than $\varepsilon/3$ by (30) right away. Assuming that $n \geq T/\delta_1$, the property (30) ensures that first and third terms are each smaller than $\varepsilon/3$ as well. So we proved that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \leq s \leq t \leq T \text{ with } t - s < \delta_1, \text{ and } n \geq T/\delta_1.$$

Since there is only a finite number of $n \in \mathbb{N}$ with $n < T/\delta_1$, we can find $\delta_2 > 0$ such that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \leq s \leq t \leq T \text{ with } t - s < \delta_2, \text{ and } n < T/\delta_1.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we finally obtain

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \leq s \leq t \leq T \text{ with } t - s < \delta, \text{ and } n \in \mathbb{N}.$$

The conclusion of the Lemma now follows by applying the Arzela-Ascoli theorem (see e.g. Rudin [28, Theorem 7.25]). $\square$

Remark 3. Establishing the existence of a converging subsequence $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$ needs more work compared to what we did in the proof of Lemma 8 because the direct corollary of (17)

$$x_{i+1}^n - x_i^n = u_{i+1}^n - u_i^n + \int_{t_{i-1}}^{t_i} f(\tau, x_n(\tau), \lambda) d\tau$$

doesn't imply uniform boundedness of $x_n(t, q_n, \lambda_n)$, $n \in \mathbb{N}$, directly.

To prove the convergence of $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$ we will now extend the discrete map (18) to such an operator $F_n : C([0, T], E) \rightarrow C([0, T], E)$ whose fixed point is exactly $t \mapsto x_n(t, q_n, \lambda_n)$. The convergence of x_{n_k} will then follow from the continuity of F_n in n at $n = \infty$.

Let us define $P_n : C([0, T], E) \rightarrow E^{n+1}$, $l^- : E^{n+1} \rightarrow E^{n+1}$ and $Q_n : E^{n+1} \rightarrow C([0, T], E^{n+1})$ as

$$P_n(x) = \left(x(0), x\left(\frac{T}{n}\right), \dots, x\left((n-1)\frac{T}{n}\right), x(T) \right), \quad x \in C([0, T], E),$$

$$[l^-(y)]_1 = 0, \quad [l^-(y)]_i = y_{i-1}, \quad i \in \overline{2, n+1}, \quad y \in E^{n+1},$$

$$Q_n(y)(t) = \frac{t - t_{i-1}}{1/n} y_{i+1} + \frac{t_i - t}{1/n} y_i, \quad y \in E^{n+1}, \quad t \in [t_{i-1}, t_i), \quad i \in \overline{1, n},$$

$$Q_n(y)(t_n) = y_{n+1}, \quad y \in E^{n+1}, \quad \text{since } t_n = T.$$

For a fixed $\lambda \in [0, 1]$ and a continuous function $u : [0, T] \rightarrow E$, we introduce a continuous extension of (18) as

$$\begin{aligned} (F_n x)(t) &= (Q_n P_n u)(t) - (Q_n l^- P_n J)(t), \quad t \in [0, T], \\ \text{where } J(t) &= \int_0^t f(\tau, x(\tau), \lambda) d\tau. \end{aligned} \quad (31)$$

Then, for $x \in C([0, T], E)$ satisfying $x = F_n x$, one has

$$\begin{aligned} x(0) &= (Q_n P_n u)(0) - (Q_n l^- P_n J)(0) = [P_n u]_1 - [l^- P_n J]_1 = u(0) - 0, \\ x(t_1) &= [P_n u]_2 - [l^- P_n J]_2 = u(t_1) - [P_n J]_1 = u(t_1) - J(0) = u(t_1), \\ x(t_2) &= u(t_2) - J(t_1), \\ &\dots \\ x(t_n) &= u(t_n) - J(t_{n-1}). \end{aligned}$$

Therefore, if u_n and x_n are given by (16)-(19), then, letting $u = u_n$ in (31), the fixed point x of F_n verifies $x(t_i) = x_n(t_i)$, $i \in \overline{0, n}$. And, since the function $t \mapsto (F_n x)(t)$ is linear on $[t_i, t_{i+1}]$, $i \in \overline{0, n-1}$, we conclude $x_n = x$. In other words, if u in (31) is given by $u = u_n$, then x_n is the unique fixed point of F_n .

Lemma 9. *Assume that the conditions of Lemma 5 hold. Then, there exists $\alpha > 0$ and $L \in (0, 1)$ such that*

$$\|F_n(x_1) - F_n(x_2)\|^* \leq L \|x_1 - x_2\|^*, \quad n \in \mathbb{N},$$

for any $x_1, x_2, u \in C([0, T], E)$, $\lambda \in [0, 1]$, and

$$\|x\|^* = \max_{t \in [0, T]} e^{-\alpha t} \|x(t)\|.$$

Moreover, for each $x, u \in C([0, T], E)$, and $\lambda \in [0, 1]$, one has

$$\lim_{n \rightarrow \infty} \|F_n(x) - F(x)\| = 0, \quad \text{where } F(x)(t) = u(t) - \int_0^t f(\tau, x(\tau), \lambda) d\tau,$$

where $\|\cdot\|$ is the max-norm on $[0, T]$ and F is a contraction in the norm $\|\cdot\|^*$.

Proof. Step 1. Using the definition of Q_n , l^- , and P_n , we have

$$(Q_n l^- P_n J)(t_{i-1}) = [l^- P_n J]_i = [P_n J]_{i-1} = J(t_{i-2}), \quad i \in \overline{2, n+1}.$$

So that

$$(F_n x)(t_i) = u(t_i) - J(t_{i-1}).$$

Fix $i \in \overline{1, n-1}$ and choose any $t \in [t_i, t_{i+1}]$. Then,

$$\begin{aligned} \|F_n(x_1)(t) - F_n(x_2)(t)\| &\leq \\ &\leq \max \{ \|F_n(x_1)(t_i) - F_n(x_2)(t_i)\|, \|F_n(x_1)(t_{i+1}) - F_n(x_2)(t_{i+1})\| \} = \\ &= \max \left\{ \left\| \int_0^{t_{i-1}} f(\tau, x_1(\tau), \lambda) d\tau - \int_0^{t_{i-1}} f(\tau, x_2(\tau), \lambda) d\tau \right\|, \right. \\ &\quad \left. \left\| \int_0^{t_i} f(\tau, x_1(\tau), \lambda) d\tau - \int_0^{t_i} f(\tau, x_2(\tau), \lambda) d\tau \right\| \right\} \leq \\ &\leq \bar{L} \int_0^{t_i} \|x_1(\tau) - x_2(\tau)\| d\tau \leq \bar{L} \int_0^{t_i} e^{\alpha\tau} \|x_1 - x_2\|^* d\tau, \end{aligned}$$

where $\bar{L} > 0$ is the global Lipschitz constant of $x \mapsto f(t, x, \lambda)$ and $\alpha > 0$ is an arbitrary constant. Therefore,

$$e^{-\alpha t} \|F_n(x_1)(t) - F_n(x_2)(t)\| \leq \frac{\bar{L}}{\alpha} (e^{\alpha(t_i-t)} - e^{-\alpha t}) \|x_1 - x_2\|^* \leq \frac{\bar{L}}{\alpha} \|x_1 - x_2\|^*,$$

which holds for any $t \in [0, T]$. The case $t \in [0, t_1]$ can be considered along the same lines. This proves the contraction part of the lemma.

Step 2. To prove the convergence part, fix $i \in \overline{1, n-1}$ again and consider $t \in [t_i, t_{i+1}]$. Since $(Q_n P_n u)(t_i) = u(t_i)$, we have

$$\begin{aligned} \|(Q_n P_n u)(t) - u(t)\| &\leq \|(Q_n P_n u)(t) - (Q_n P_n u)(t_i)\| + \|u(t) - u(t_i)\| \leq \\ &\leq \|u(t_{i+1}) - u(t_i)\| + \|u(t) - u(t_i)\|, \end{aligned}$$

so that the convergence of $(Q_n P_n u)(t)$ to $u(t)$ as $n \rightarrow \infty$ follows from continuity of u . The convergence of $(Q_n l^- P_n J)(t)$ follows same lines. Indeed, since $(Q_n l^- P_n J)(t_{i+1}) = J(t_i)$, one has

$$\begin{aligned} \|(Q_n l^- P_n J)(t) - J(t)\| &\leq \\ &\leq \|(Q_n l^- P_n J)(t) - (Q_n l^- P_n J)(t_{i+1})\| + \|J(t) - J(t_i)\| \leq \\ &\leq \|J(t_{i-1}) - J(t_i)\| + \|J(t) - J(t_i)\| \end{aligned}$$

and the convergence of $(Q_n l^- P_n J)(t)$ to $J(t)$ as $n \rightarrow \infty$ follows from continuity of $J(t)$.

The proof of the lemma is complete. $\square$

Corollary 3. *Assume that condition I) of Theorem 3 holds. Let $\{n_k\}_{k \in \mathbb{N}}$ be the subsequence given by Lemma 8 (which ensures the convergence of $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$). Consider the limit*

$$u(t) = \lim_{k \rightarrow \infty} u_{n_k}(t, q_{n_k}, \lambda_{n_k}).$$

Let $x(t)$ be the solution of the respective equation (13) (which exists according to Lemma 5). Then $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$ converges uniformly in $t \in [0, T]$, and

$$x(t) = \lim_{k \rightarrow \infty} x_{n_k}(t, q_{n_k}, \lambda_{n_k}). \quad (32)$$

Proof. The conclusion follows from the inequality

$$\begin{aligned} \|x - x_n\|^* &= \|F(x) - F_n(x_n)\|^* \leq \|F(x) - F_n(x)\|^* + \|F_n(x) - F_n(x_n)\|^* \leq \\ &\leq \|F(x) - F_n(x)\|^* + L\|x - x_n\|^*, \end{aligned}$$

where $L \in (0, 1)$ is given by Lemma 9. $\square$

6.5. Verifying that the limit of the catching-up algorithm is indeed a solution

Theorem 10. *Let the conditions of Corollary 3 hold and let $u(t)$ and $x(t)$ be as given by this corollary. Then, $u(t)$ is a solution of sweeping process (14) with the parameters $x(t)$, $\lambda = \lim_{k \rightarrow \infty} \lambda_{n_k}$, and the initial condition $u(0) = \lim_{k \rightarrow \infty} q_{n_k}$. Accordingly, by Definition 2, $x(t)$ is a solution of perturbed sweeping process (6).*

Proof. Let $\phi(t)$, $t \in [0, T]$, be an arbitrary continuous selector of the moving set of (14), i.e.

$$\phi(t) \in A + a(t, \lambda) + c(x(t), \lambda) + \int_0^t f(\tau, x(\tau), \lambda) d\tau, \quad t \in [0, T].$$

According to Monteiro Marques [22, p. 15-16] (see also Valadier [29, Proposition 6]) it is sufficient to prove that

$$\int_s^t \langle \phi(\tau), du(\tau) \rangle \geq \frac{1}{2} (\|u(t)\|^2 - \|u(s)\|^2), \quad 0 \leq s \leq t \leq T, \quad (33)$$

which we now establish using the ideas of Kunze and Monteiro Marques [19].

Without loss of generality we will assume that $\{n_k\}_{k \in \mathbb{N}} = \mathbb{N}$, and replace n_k , $k \in \mathbb{N}$, by n , $n \in \mathbb{N}$ in the formulation of the theorem. Fix $t > 0$ and select $i \in \overline{0, n-1}$ such that $t \in [t_i, t_{i+1}]$. Introduce $\hat{c}_n(t)$ as

$$\hat{c}_n(t) = \text{proj} \left(\phi(t), A + a(t_{i+1}, \lambda_n) + c(x_{i+1}^n, \lambda_n) + \int_0^{t_i} f(\tau, x_n(\tau), \lambda_n) d\tau \right).$$

Then, by (16) and by convexity of A , we have (see e.g. Kunze and Monteiro Marques [19, formula (4)])

$$\langle u_n(t_{i+1}) - u_n(t_i), u_n(t_{i+1}) - \hat{c}_n(t) \rangle \leq 0, \quad t \in [t_i, t_{i+1}],$$

from where

$$\begin{aligned} \langle u_n(t_{i+1}) - u_n(t_i), u_n(t) - \hat{c}_n(t) \rangle &\leq \\ &\leq \langle u_n(t_{i+1}) - u_n(t_i), u_n(t) - u_n(t_{i+1}) \rangle \leq \|u_n(t_{i+1}) - u_n(t)\|^2, \end{aligned}$$

or

$$\langle u_n(t_{i+1}) - u_n(t_i), \hat{c}_n(t) \rangle \geq -\|u_n(t_{i+1}) - u_n(t_i)\|^2 + \langle u_n(t_{i+1}) - u_n(t_i), u_n(t) \rangle,$$

for any $t \in [t_i, t_{i+1}]$. Using the linearity of u_n on $[t_i, t_{i+1}]$, we conclude

$$\begin{aligned} \langle \hat{c}_n(t), u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i) \rangle &\geq \\ &\geq \langle u_n(t), u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i) \rangle - \langle u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i), (u_n(t_{i+1}) - u_n(t_i)) \rangle, \end{aligned}$$

for any $t_i \leq \bar{t}_i \leq t \leq \bar{t}_{i+1} \leq t_{i+1}$. Therefore, denoting $\tau_{j,k} = \bar{t}_i + (j + \frac{1}{2}) \frac{\bar{t}_{i+1} - \bar{t}_i}{k}$ for $j \in \{0, 1, \dots, k-1\}$, one has (same approach is used e.g. in part (ii) of the proof of Monteiro Marques [23, Theorem 2.1, p. 12, second formula from below])

$$\begin{aligned} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \langle \hat{c}_n(\tau), du_n(d\tau) \rangle &= \\ &= \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \left\langle \hat{c}_n(\tau_{j,k}), u_n \left(\bar{t}_i + (j+1) \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right) - u_n \left(\bar{t}_i + j \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right) \right\rangle \geq \\ &\geq \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \left\langle u_n(\tau_{j,k}), u_n \left(\bar{t}_i + (j+1) \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right) - u_n \left(\bar{t}_i + j \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right) \right\rangle + R_n, \end{aligned}$$

where the reminder R_n is given by

$$\begin{aligned} R_n &= - \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \left\langle u_n \left(\bar{t}_i + (j+1) \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right) - \right. \\ &\quad \left. - u_n \left(\bar{t}_i + j \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right), u_n(t_{i+1}) - u_n(t_i) \right\rangle = \\ &= - \langle u_n(\bar{t}_{i+1}) - u_n(\bar{t}_i), u_n(t_{i+1}) - u_n(t_i) \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\bar{t}_i}^{\bar{t}_{i+1}} \langle \hat{c}_n(\tau), du_n(d\tau) \rangle = \\ &= \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \left\langle u_n(\tau_{j,k}), u'_n(\tau_{j,k}) \frac{\bar{t}_{i+1} - \bar{t}_i}{k} \right\rangle + R_n = \\ &= \frac{1}{2} \lim_{k \rightarrow \infty} \sum_{j=0}^{k-1} \left(\frac{d}{d\tau} \|u_n(\tau)\|^2 \right) \Big|_{\tau=\tau_{j,k}} \cdot \frac{\bar{t}_{i+1} - \bar{t}_i}{k} + R_n = \\ &= \frac{1}{2} \int_{\bar{t}_i}^{\bar{t}_{i+1}} \frac{d}{d\tau} \|u_n(\tau)\|^2 d\tau + R_n = \frac{1}{2} (\|u_n(\bar{t}_{i+1})\|^2 - \|u_n(\bar{t}_i)\|^2) + R_n. \end{aligned}$$

This result can now be used to estimate the required integral (33) as follows

$$\int_s^t \langle \hat{c}_n(\tau), du_n(d\tau) \rangle \geq \frac{1}{2} (\|u_n(t)\|^2 - \|u_n(s)\|^2) + R, \quad (34)$$

where

$$|R| \leq \text{var}(u_n, [s, t]) \cdot \max_{i \in \{0, n-1\}} \|u_n(t_{i+1}) - u_n(t_i)\|.$$

But according to Corollary 1,

$$\text{var}(u_n, [s, t]) \leq \frac{\text{var}(\bar{a}, [s, t])}{1 - L_2} + \frac{L_1 |t - s|}{1 - L_2}.$$

Therefore, the desired statement (33) follows from (34) by passing to the limit as $n \rightarrow \infty$ (the passage to the limit is valid e.g. by Monteiro Marques [23, Theorem 2.1(ii)-(iii)] combined with formula (26) of p. 7 of the same book).

The proof of the theorem is complete. $\square$

6.6. Proof of Theorem 1 (sweeping process without a parameter)

Theorem 1 is a direct consequence of Theorem 10. One just view sweeping process (4) as sweeping process (6) with $\lambda = 0$.

Remark 4. Using Remark 2, Theorem 1 can be directly extended to sweeping processes of the form

$$-dx \in N_{A(t,\lambda)+c(x,\lambda)}(x) + f(t, x, \lambda)dt, \quad x \in E, \quad \lambda \in \mathbb{R}, \quad (35)$$

where A is a set-valued function with nonempty closed convex bounded values that satisfies the property

$$\begin{aligned} d_H(A(s, \lambda), A(t, \lambda)) &\leq \text{var}(\bar{a}, [s, t]), \quad \lambda \in [0, 1], \\ \text{where } \bar{a} : [0, T] &\rightarrow \mathbb{R} \text{ is a BV continuous function.} \end{aligned} \quad (36)$$

6.7. Proofs of Theorems 3 and 4 (sweeping process with a parameter)

Proof of Theorem 3. Step 1. First we prove that there exists $N > 0$ such that $d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q)$ is defined for $n \geq N$ and $\lambda \in [0, \lambda_1]$. Assuming the contrary, we get a sequence $n_k \rightarrow \infty$, $\lambda_k \rightarrow \lambda_0 \in [0, \lambda_1]$, and a converging sequence $\{q_k\}_{k \in \mathbb{N}} \subset \partial Q$ such that

$$P^{\lambda_k, n_k} \circ V^{\lambda_k, n_k}(q_k) = q_k, \quad k \in \mathbb{N}. \quad (37)$$

Applying Lemma 8, Corollary 3, and Theorem 10 we conclude that $q_0 = \lim_{k \rightarrow \infty} q_k \in \partial Q$ is the initial condition of the T -periodic solution (32) of sweeping process (6) with $\lambda = \lambda_0$, which contradicts conditions III) of Theorem 3.

The conclusion of Step 1, in particular, implies that

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) = d(I - P^{0,n} \circ V^{0,n}, Q), \quad n \geq N, \quad \lambda \in [0, \lambda_1].$$

Step 2. Here we use assumption II (uniqueness) of Theorem 3 to conclude that

$$P^{0,n} \circ V^{0,n}(q) \rightarrow P^0 \circ V^0(q), \quad \text{as } n \rightarrow \infty,$$

uniformly with respect to $q \in \bar{Q}$. Thus, we can diminish $N > 0$ in such a way that $d(I - P^{0,n} \circ V^{0,n}, Q) = d(I - P^0 \circ V^0, Q)$, $n \geq N$, which gives

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) \neq 0, \quad n \geq N, \quad \lambda \in [0, \lambda_1].$$

Therefore, for each $\lambda \in [0, \lambda_1]$ there exists $q_n \in Q$ such that the approximations $\{x_n(\cdot, q_n, \lambda)\}_{n \geq N}$ are T -periodic, so this sequence has a convergent subsequence which converges to a T -periodic solution of (6) with initial condition $q = \lim_{n_k \rightarrow \infty} q_{n_k}$ as $n \rightarrow \infty$ according to Corollary 3.

The proof of the theorem is complete. $\square$

The proof of Theorem 4 follows the lines of the proof of Theorem 3. The only difference is in the beginning of Step 1, which now proves the existence of both $N > 0$ and $\lambda_1 \in (0, 1]$ such that $d(I - P^{\lambda, n} \circ V^{\lambda, n}, Q)$ is defined for $n \geq N$ and $\lambda \in [0, \lambda_1]$. Assuming the contrary, we get a sequence $n_k \rightarrow \infty$, $\lambda_k \rightarrow 0 \in [0, 1]$, and a converging sequence $\{q_k\}_{k \in \mathbb{N}} \subset \partial Q$ such that (37) holds, that leads to the existence of a T -periodic solution to sweeping process (6) with $\lambda = 0$, contradicting condition II) of Theorem 3. The rest of the proof of Theorem 4 follows the proof of Theorem 3 just literally.

7. Proof of the theorem on the global existence of periodic solutions

To prove Theorem 2 we will use the following well-known result (see e.g. Krasnoselskii-Zabreiko [16, Theorem 6.2]):

Theorem 11. *Let $\bar{P} : E \rightarrow E$ be a continuous map and let $Q \subset E$ be an open bounded convex set. If $\bar{P}(Q) \subset \bar{Q}$ and if $\bar{P}$ doesn't have fixed points on ∂Q , then*

$$d(I - \bar{P}, Q) = 1.$$

Proof of Theorem 2. Let Ω_1 be the 1-neighborhood of Ω . Since Ω is convex, then Ω_1 is convex as well. We will view sweeping process (4) as sweeping process (6) with $\lambda = 0$. So we consider the map

$$\bar{P}^{0, n}(x) = P^{0, n}(V(x)),$$

where $P^{0, n}$ is as introduced in Section 6.3 and V is as introduced in Section 3. We claim that

$$\bar{P}^{0, n}(\Omega_1) \subset \Omega, \quad \text{for all } n \in \mathbb{N}. \quad (38)$$

We have $V(x) \in \Omega$ by the definition of the map V . Then, according to the catching-up scheme (16)-(19), we have that

$$x_{i+1}^n \in A + a(t_{i+1}, 0) + c(x_{i+1}^n), \quad \text{i.e. } x_{i+1}^n \in \Omega_{t_{i+1}}, \quad i \in \overline{0, n-1},$$

and so $x_n(T) \in \Omega_T$, which implies (38).

Using the continuity of $P^{0,n}$ (Corollary 2) and V (Lemma 6) along with Theorem 11, we get the existence of $q_n \in \Omega$ such that $\bar{P}^{0,n}(q_n) = q_n$, which implies

$$P^{0,n}(q_n) = q_n, \quad n \in \mathbb{N},$$

because $V(q_n) \in \Omega$. In other words, we have $x_n(T, q_n, 0) = x_n(0, q_n, 0)$ for all $n \in \mathbb{N}$. Now, Theorem 10 applied with $\lambda_n = 0$, implies the existence of a convergent subsequence $\{x_{n_k}(t, q_{n_k}, 0)\}$ whose limit $x(t)$ is solution of (4) with the required T -periodicity property (11). The proof is complete. $\square$

8. Existence of periodic solutions in the neighborhood of a boundary equilibrium (the theorem and its proof)

This section uses the following extension of Theorem 11 (see e.g. Krasnoselskii-Zabreiko [16, Theorem 31.1]):

Theorem 12. *Let $\bar{P} : E \rightarrow E$ be a continuous map and let $Q \subset E$ be an open bounded set. If $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large, then*

$$d(I - \bar{P}, Q) = 1.$$

The main assumption of this section is that sweeping processes (6) reduces to

$$-\dot{x} \in N_A(x) + f_0(x), \quad x \in E, \quad (39)$$

when $\lambda = 0$ and that (39) posses a switched equilibrium on the boundary ∂A (as was earlier introduced in Kamenskii-Makarenkov [13] in 2d). To introduce the definition of a switched boundary equilibrium $x_0 \in \partial A$, we assume that in some neighborhood $Q \subset \mathbb{R}^n$ of x_0 the boundary ∂A is smooth and can be described as

$$\partial A \cap Q = \{x \in Q : H(x) = 0\}, \quad \text{where } H \in C^1(\mathbb{R}^n, \mathbb{R}).$$

Definition 3. *A point $x_0 \in \partial A$ is a switched boundary equilibrium of sweeping process (39), if*

$$H(x) > 0, \quad \text{for all } x \in Q \setminus A,$$

and

$$H'(x_0) = \alpha f(x_0) \quad \text{for some } \alpha < 0.$$

As the definition says, x_0 is not an equilibrium of f , however the next two lemmas imply that the solution of (39) with the initial condition at x_0 don't leave x_0 .

If x_0 is a switched equilibrium, then Q can be considered so small that

$$\langle f(x), H'(x) \rangle < 0, \quad \text{for all } x \in \partial A \cap Q. \quad (40)$$

The next lemma claims that $\partial A \cap Q$ is a sliding region for sweeping process (39).

Lemma 13. *Let $x_0 \in \partial A$ be a switched equilibrium of (39) and let $Q \subset E$ be such a neighborhood of x_0 that (40) holds. Consider a solution x of (39) with an initial condition $x_0 \in \partial A \cap Q$. Let $t_1 > 0$ be such that $x(t) \in Q$ for all $t \in [0, t_1]$. Then $x(t) \in \partial A$ for all $t \in [0, t_1]$.*

Proof. Let us assume, by contradiction, that there exists $t_{\text{escape}} \in [0, t_1]$ where $x(t)$ escapes from ∂A , i.e.

$$t_{\text{escape}} = \max\{t_0 \geq 0 : x(t) \in Q, H(x(t)) = 0, t \in [0, t_0]\} < t_1.$$

By the definition of t_{escape} , for any $\delta > 0$ there exist $t_\delta \in [t_{\text{escape}}, t_{\text{escape}} + \delta]$ such that $H(x(t)) < 0$ for each $t \in (t_{\text{escape}}, t_\delta]$. Since, the solution $x(t)$ satisfies $\dot{x}(t) = -f_0(x(t))$ on $(t_{\text{escape}}, t_\delta]$, by the Mean-Value Theorem

$$H(x(t_\delta)) - H(x(t_{\text{escape}})) = -H'(x(t_\delta^*))f_0(x(t_\delta^*))(t_\delta - t_{\text{escape}}),$$

for some $t_\delta^* \in (t_{\text{escape}}, t_\delta)$. This yields

$$H'(x(t_{\text{escape}}))f_0(x(t_{\text{escape}})) \geq 0,$$

as $\delta \rightarrow 0$, contradicting (40).

The proof of the lemma is complete. $\square$

As it happens in the theory of Filippov systems (see [11]), the dynamics of (39) in the sliding region is described by a smooth differential equation. Indeed, let us introduce the differential equation

$$-\dot{x} = \bar{f}(x), \quad \text{where } \bar{f}(x) = f_0(x) - \pi_{H'(x)}(f_0(x)) \text{ and } \pi_L(\xi) = \frac{1}{\|L\|^2} \langle \xi, L \rangle L. \quad (41)$$

Next lemma says that (41) is the equation of sliding motion for sweeping process (39) in the neighborhood of switched equilibrium $x_0 \in \partial A_0$.

Lemma 14. *Let the conditions of Lemma 13 hold and let $x(t)$ be the sliding solution $x(t)$, $t \in [0, t_1]$, of sweeping process (39) as introduced in Lemma 13. Then $x(t)$ is a solution of (41) on $[0, t_1]$.*

Proof. Fix $t \in [0, t_1]$ such that $\dot{x}(t)$ exists. Then, from (39),

$$-\dot{x}(t) = \alpha H'(x(t)) + f_0(x(t)), \quad \text{with some } \alpha > 0,$$

or

$$\alpha H'(x(t)) = -\pi_{H'(x(t))}(f_0(x(t))) + [-f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t)))] - \dot{x}(t). \quad (42)$$

From the definition of $\pi_L(\xi)$ we have

$$\langle -f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t))), H'(x(t)) \rangle = 0.$$

On the other hand, from Lemma 13,

$$\langle \dot{x}(t), H'(x(t)) \rangle = 0.$$

Therefore, taking the scalar product of (42) with $H'(x(t))$, we get

$$\alpha = -\frac{1}{\|H'(x(t))\|^2} \langle f_0(x(t)), H'(x(t)) \rangle,$$

which completes the proof. $\square$

Lemma 14 implies that the boundary ∂A is an invariant manifold for the differential equation (41). The definition (41) reduces the dimension of the image of f_0 by 1. Therefore, the image of the map $\bar{f}$ acts to a space of dimension $\dim E - 1$, which implies that one eigenvalue of the Jacobian $\bar{f}'(x_0)$ is always zero.

We now offer an asymptotic stability result which can be of independent interest in applications of perturbed sweeping processes.

Theorem 15. *Let $x_0 \in \partial A$ be a switched equilibrium of (39). If real parts of $\dim E - 1$ eigenvalues of the Jacobian $\bar{f}'(x_0)$ are negative, then x_0 is a uniformly asymptotically stable point of sweeping process (39).*

Proof. Step 1. *Convergence to ∂A .* Let $B_r(x_0)$ be a ball of radius r centered at x_0 . Let us show that there exists $r > 0$ such that for any $\xi \in B_r(x_0) \cap A$, the solution $t \mapsto X(t, \xi)$ of

$$\dot{x} = -f_0(x) \quad (43)$$

with the initial condition $X(0, \xi) = \xi$ reaches ∂A at time some time $\tau(\xi) > 0$. The proof will be through the Implicit Function Theorem applied to

$$F(t, x) = H(X(t, x)).$$

We have $F(0, x_0) = 0$ and $F_t(0, x_0) = -H'(x_0)f_0(x_0) \neq 0$ by the definition of switched equilibrium. Therefore, Implicit Function Theorem (see e.g. Rudin [28, Theorem 9.28]) ensures the existence of $\xi \rightarrow \tau(\xi)$ defined and continuous on a sufficiently small ball $B_r(x_0)$ and such that $\tau(x_0) = 0$.

It remains to show that $\tau(\xi) > 0$ for all $\xi \in B_r(x_0) \cap A$. Since, according to the definition of switched equilibrium, $H'(x_0)^T$ is a normal to A pointing outwards to A , it is sufficient to prove that $\tau(\xi) > 0$ for $\xi = x_0 - \lambda H'(x_0)^T$ with all $\lambda > 0$ sufficiently small. So we introduce a scalar function

$$G(\lambda) = \tau(x_0 - \lambda H'(x_0)^T)$$

and want to prove that $G'(0) > 0$. Using the formula for the derivative of the implicit function (see [28, Theorem 9.28])

$$\tau'(x_0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)$$

and so

$$G'(0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)(-H'(x_0)^T) = H'(x_0)f_0(x_0)\|H'(x_0)\|^2,$$

which is indeed positive according to Definition 3.

Finally, let us fix $\xi \in B_r(x_0) \cap A$ and let $x(t)$ be the solution of (39) with the initial condition $x(0) = \xi$. Since the conclusion of the Implicit Function Theorem comes with uniqueness, we have that $X(t, \xi) \notin \partial A$, $t \in [0, \tau(\xi))$. Therefore, $X(t, \xi) = x(t)$, for any $t \in [0, \tau(\xi))$, which implies that $\lim_{t \rightarrow \tau(\xi)} X(t, \xi) = \lim_{t \rightarrow \tau(\xi)} x(t)$ and so $x(\tau(\xi)) \in \partial A$.

Step 2. *Convergence along ∂A .* Lemmas 13 and 14 combined with the negativeness of real parts of $\dim E - 1$ eigenvalues of $\bar{f}'(x_0)$ imply that there exists an neighborhood $x_0 \in Q \subset E$ such that any solution of (39) with the

initial condition $x(0) \in Q \cap \partial A$ converges to x_0 along ∂A as $t \rightarrow \infty$ and the convergence is uniform with respect to the initial condition.

Making now $r > 0$ in Step 1 so small that $\cup_{\xi \in B_r(x_0)} X(\tau(\xi), \xi) \in Q$ (which is possible by continuity of $\xi \rightarrow \tau(\xi)$), we combine Step 1 and Step 2 to conclude that any solution of (39) with $x(0) \in B_r(x_0)$ approaches x_0 as $t \rightarrow \infty$.

The proof of the theorem is complete. $\square$

We are now in the position to combine theorems 4, 11, and 15 when the following condition holds for (6) at $\lambda = 0$:

$$a(t, 0) \equiv 0, \quad c(x, 0) \equiv 0, \quad f(t, x, 0) \equiv f_0(x) \text{ with } f_0 \in C^1(E, E). \quad (44)$$

Theorem 16. *Assume that condition I) of Theorem 3 holds. Assume, that for $\lambda = 0$ sweeping process (6) is smooth autonomous, i.e. satisfies (44). If real parts of $n - 1$ eigenvalues of $\bar{f}'(x_0)$ are negative for some switched equilibrium $x_0 \in \partial A$, then there exists $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1]$ sweeping process (6) admits a periodic solution $x_\lambda(t) \rightarrow x_0$ as $\lambda \rightarrow 0$.*

Proof. Let $\bar{P}(x) = P^0(V^0(x))$. By Theorem 15, there exists an open bounded set $x_0 \in Q \subset E$ such that $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large. Therefore, Theorem 12 ensures that condition II) of Theorem 3 holds, so Theorem 4 applies. $\square$

Similar to Theorem 16 results have been obtained for ordinary differential equations by Bernstein-Halanai [2] and Cronin [7].

9. Conclusions

By extending the implicit catching-up scheme of Kunze and Monteiro Marques [19] to perturbed sweeping processes, we proved solvability of BV-continuous state-dependent sweeping processes with Lipschitz dependence on the state. We further used topological degree arguments to establish the existence of periodic solutions to sweeping processes of this type. The analysis is carried out for the simplest possible moving set $C(t) = A + a(t) + c(x)$ throughout the entire paper, that allowed us to focus on the development of core mathematical ideas rather than on its possible generalizations. We explain in Remarks 2 and 4 how the existence result (Theorem 1) immediately extends to the moving set of the form $C(t) = A(t) + c(x)$. At the

same time, Remark 2 shows that our method of proof of continuity of approximations $x_n(t)$ on the initial condition fails for moving sets of the form $C(t) = A(t) + c(x)$. Since continuous dependence of $x_n(t)$ on the initial condition is the main ingredient in our proof of the existence of periodic solutions, the respective main theorems (Theorem 2 and Theorem 16) do not readily extend even to the moving set of the form $C(t) = A(t) + c(x)$. We don't know whether or not an alternative approach (e.g. formula (29) quoted from [23, Proposition 4.7, p. 26]) can deal with any more general state-dependent moving constraints.

The existence of a T -periodic solutions to a sweeping process with T -periodic right-hand-sides and convex moving set would be an immediate result when uniqueness and continuous dependence of solutions on initial conditions holds. The difficulty we overcame when proving the existence of periodic solutions comes from the fact that uniqueness and continuous dependence on initial conditions of solutions of BV-continuous state-dependent sweeping processes is still an open question even when the dependence on the state is Lipschitz continuous (for state-independent sweeping processes uniqueness and continuous dependence is established e.g in Castaing and Monteiro Marques [5] and Adly et al [1]).

The second part of the paper concerns sweeping processes with a parameter λ , for which we developed a topological degree based continuation principle. As an application of the continuation principle, we proved the occurrence of periodic solutions at a specific location being a neighborhood of a switched boundary equilibrium. Specifically, we assumed that for $\lambda = 0$, the sweeping process is autonomous and admits an asymptotically stable switched boundary equilibrium x_0 . We then proved the occurrence of T -periodic solutions from x_0 when the parameter λ increases and the sweeping process becomes nonautonomous (and T -periodic). The condition for asymptotic stability of x_0 can be replaced by assuming that the topological index of x_0 is different from 0. Such a condition can be also expressed in terms of the eigenvalues of the linearization $\bar{f}(x_0)$ of sliding differential equation (41), see e.g. Krasnoselskii-Zabreiko [16, Theorem 6.1] and [16, Theorem 7.4] (which will be required to account for the vector field outside of the boundary of the constraint).

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