LONG TIME EXISTENCE FOR A STRONGLY DISPERSIVE BOUSSINESQ SYSTEM

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Abstract.

This paper is concerned with the one-dimensional version of a specific member of the (abcd) family of Boussinesq systems having the higher possible dispersion. We will establish two different long time existence results for the solutions of the Cauchy problem. The first result concerns the system (1.4) without a small parameter. If the initial data is of order $O(\varepsilon)$, we prove that the existence time scale is of $1/\varepsilon^{\frac{4}{3}}$ which improves the result $1/\varepsilon$ that could be obtained by a "dispersive" method. The second result is about the system (1.6) which involves a small parameter ϵ in front of the dispersive and nonlinear terms and which is the form obtained when the system is derived from the water wave system in the KdV/Boussinesq regime. If the initial data is of order $O(1)$, we obtain the existence time scale $1/\epsilon^{\frac{2}{3}}$ which improves the result $1/\sqrt{\epsilon}$ obtained by a dispersive method. These results were not included in the previous papers dealing with similar issues because of the presence of zeroes in the phases. The proof involves normal form transformations suitably modified away from the zero set of the phases.

Keywords : Boussinesq systems. Long time existence. Normal forms.

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1. Introduction

1.1. **The general setting.** The four-parameter (abcd) Boussinesq systems for *long wavelength, small amplitude* gravity-capillary surface water waves introduced in [6, 7] couples the elevation of the wave $\zeta = \zeta(x,t)$ to a measure of the horizontal velocity $\boldsymbol{v} = \boldsymbol{v}(x,t), x \in \mathbb{R}^N, N = 1,2, t \in \mathbb{R}$ and read as follows:

$$\begin{cases} \partial_t \zeta + \nabla \cdot \boldsymbol{v} + \epsilon \nabla \cdot (\zeta \boldsymbol{v}) + \epsilon\big(a \nabla \cdot \Delta \boldsymbol{v} - b \Delta \partial_t \zeta\big) = 0, \\ \partial_t \boldsymbol{v} + \nabla \zeta + \dfrac{\epsilon}{2} \nabla (|\boldsymbol{v}|^2) + \epsilon\big(c \nabla \Delta \zeta - d \Delta \partial_t \boldsymbol{v}\big) = \mathbf{0}. \end{cases} \tag{1.1}$$

Here a, b, c, d are modeling parameters which satisfy the constraint $a + b + c + d = \frac{1}{3} - \tau$ where $\tau \geq 0$ is a measure of surface tension effects, $\tau = 0$ for pure gravity waves.

In (1.1), the small parameter ϵ is defined by

$$\epsilon = a/h \sim (h/\lambda)^2,$$

where h denotes the mean depth of the fluid, a a typical amplitude of the wave and λ a typical horizontal wavelength.

It was established in [6] that, in suitable Sobolev classes, the error with solutions of the full water waves system and the approximation given by (1.1) is of order $O(\epsilon^2 t)$. This result is of course useful if one knows that the corresponding solutions of the water wave system in this regime and of the Boussinesq systems exist on time scales of at least $O(1/\epsilon)$. This has been proven in [3], see also [18], for the water wave systems and in [10, 11, 20, 22, 23] for all the locally-well posed Boussinesq systems except the case $b = d = 0, a = c > 0$ which is in some sense special since the "generic" case $b = d = 0, a, c > 0, a \neq c$ is linearly ill-posed.

Remark 1.1. *The global well-posedness of Boussinesq systems has been only established in a few one-dimensional cases, including the case $a = c = b = 0, d > 0$ that can be viewed as a dispersive perturbation of the hyperbolic Saint-Venant (shallow water) system, see* [4, 24]*, and the Hamiltonian cases $b = d > 0, a \leq 0, c < 0$, see* [8]. *We also refer to* [16, 17] *for scattering results in the energy space for those Hamiltonian cases when $b = d > 0$.*

Recall that the linearization of (1.1) around the null solution is well-posed (see [7]) provided that

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0, \tag{1.2}$$

$$\text{or} \quad a = c > 0, \quad b \geq 0, \quad d \geq 0. \tag{1.3}$$

Actually the linear well-posedness occurs when the non zero eigenvalues of the linearization of (1.1) at $(0,0)$

$$\lambda_{\pm}(\xi) = \pm i |\xi| \left(\frac{(1 - \epsilon a |\xi|^2)(1 - \epsilon c |\xi|^2)}{(1 + \epsilon d |\xi|^2)(1 + \epsilon b |\xi|^2)} \right)^{\frac{1}{2}}.$$

are purely imaginary.

This paper will focus on the exceptional case (1.3) with $b = d = 0$, $a = c = 1$ which is the only linearly well-posed case with eigenvalues having non trivial zeroes. Moreover we will restrict to the one-dimensional case, $N = 1$.

If (ζ, v) is a solution of (1.1), then by the scaling

$$\tilde{\zeta}(t,x) = \epsilon \zeta(\epsilon^{\frac{1}{2}} t, \epsilon^{\frac{1}{2}} x), \quad \tilde{\boldsymbol{v}}(t,x) = \epsilon \boldsymbol{v}(\epsilon^{\frac{1}{2}} t, \epsilon^{\frac{1}{2}} x),$$

$(\tilde{\zeta}, \tilde{\boldsymbol{v}})$ satisfies (1.1) with $\epsilon = 1$ (see also [7]).

In this article, we first establish the long time existence theory for the following strongly dispersive (1D) Boussinesq system

$$\begin{cases} \partial_t \zeta + (1 + \partial_x^2) \partial_x v + \partial_x (\zeta v) = 0, \\ \partial_t v + (1 + \partial_x^2) \partial_x \zeta + \dfrac{1}{2} \partial_x (v^2) = 0, \end{cases} \tag{1.4}$$

with initial data

$$\zeta|_{t=0} = \zeta_0, \quad v|_{t=0} = v_0 \tag{1.5}$$

which are of order $O(\varepsilon)$ in a suitable Sobolev class on time scales of order $O(1/\varepsilon^{\frac{4}{3}})$. A similar issue was discussed in [12] for multi-dimensional periodic water waves.

As a consequence, we will prove the long time existence of solutions to (1.1) with $b = d = 0, a = c = 1$ in the one-dimensional case, that is

$$\begin{cases} \partial_t \zeta + (1 + \epsilon\partial_x^2)\partial_x v + \epsilon\partial_x(\zeta v) = 0, \\ \partial_t v + (1 + \epsilon\partial_x^2)\partial_x \zeta + \dfrac{\epsilon}{2}\partial_x(v^2) = 0, \end{cases} \tag{1.6}$$

with initial data

$$\zeta|_{t=0} = \zeta_0, \quad v|_{t=0} = v_0 \tag{1.7}$$

which are of order $O(1)$, on time scales of order $O(1/\epsilon^{2/3})$.

Contrary to [22, 23] where only symmetrization techniques were used to establish the well-posedness of Boussinesq systems on time scales of order $O(1/\epsilon)$, we will use normal form transformations suitably modified to avoid the zero set of the phases. Normal form techniques were used to obtain global or long time existence results of small solutions to the full water wave system, see *e.g.,* [1, 12, 25].

We recall that the local well-posedness of (1.4) and (1.6) can be established by reducing to known results for the KdV equation.

Actually, as noticed in [8], the change of variable $\zeta = u + w,\ v = u - w$ reduces (1.6) to the following system:

$$\begin{cases} u_t + u_x + \epsilon u_{xxx} + \frac{\epsilon}{2}[3uu_x - ww_x - (uw)_x] = 0 \\ w_t - w_x - \epsilon w_{xxx} + \frac{\epsilon}{2}[uu_x - 3ww_x + (uw)_x] = 0 \end{cases}, \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1.8}$$

which is a system of KdV type with uncoupled (diagonal) linear part. Thus (see [8]) the Cauchy problem is easily seen to be locally well-posed for initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > \frac{3}{4}$ by the results in [13], [14].

On the other hand, as noticed in [21] Appendix A in a slightly different context, a minor modification of Bourgain's method as used in [15] allows to solve the Cauchy problem for (1.8) for data in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > -\frac{3}{4}$. We refer to [9] for details. It is worth noticing that in [9] the question of the dependence of the existence time with respect to ϵ is not considered but one can check that it is of order $O(1/\sqrt{\epsilon})$.

By using dispersive properties it has been moreover established in [19] that the two-dimensional version of (1.6) is well-posed in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), s > \frac{3}{2}$ on time scales of order $O(1/\sqrt{\epsilon})$. Note that neglecting the dispersive terms in (1.6) one gets by a standard symmetrization method the existence on time scales of order $O(1/\epsilon)$ but in the "hyperbolic" space $H^s(\mathbb{R}^2), s > 2$.

We also recall (see [8]) that (1.6) and (1.4) have an Hamiltonian structure given (for (1.6)) by

$$\partial_t \begin{pmatrix} \zeta \\ v \end{pmatrix} = J\text{grad } H \begin{pmatrix} \zeta \\ v \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}$$

and

$$H(\zeta, v) = \frac{1}{2}\int_{-\infty}^{\infty} (\epsilon\zeta_x^2 + \epsilon v_x^2 - \zeta^2 - v^2 - \epsilon v^2\zeta)dx.$$

Unfortunately, contrary to the case $b = d > 0, a \leq 0, c < 0$ mentioned above, it does not seem possible to use uniquely this structure to prove the global existence of small solutions.

The paper will be organized as follows. The Introduction will continue by some heuristics and the statements of the main results. Section 2 is devoted to some preliminary results. A symmetrization of the strongly dispersive system is given in Section 3 while Sections 4 and 5 are devoted to the proof of the main results, Theorem 1.1 and Theorem 1.2 respectively.

1.2. **Heuristics analysis of the system** (1.4)**.** In order to diagonalize the linear part of (1.4), we define

$$V = \zeta + i\frac{\partial_x}{|\partial_x|}v \quad \text{and} \quad \Lambda = (1 + \partial_x^2)|\partial_x|.$$

Then (1.4) is rewritten as

$$\partial_t V - i\Lambda V = \sum_{\mu,\nu \in \{+,-\}} Q_{\mu,\nu}(V^\mu, V^\nu), \tag{1.9}$$

where $V^+ = V$, $V^- = \overline{V}$ and $Q_{\mu,\nu}(V^\mu, V^\nu)$ are quadratic terms in V^μ and V^ν with symbol $q_{\mu,\nu}(\cdot,\cdot)$, i.e.,

$$\mathcal{F}\Big(Q_{\mu,\nu}(V^\mu, V^\nu)\Big)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} q_{\mu,\nu}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)d\eta. \tag{1.10}$$

One could check that $|q_{\mu,\nu}(\xi,\eta)| \sim |\xi|$. Since we aim to prove long time existence results for solutions of (1.4), we hope that the quadratic terms could be killed. To do so, we use normal form transformation techniques.

Defining the profile of V as follows

$$f(t,x) = e^{-it\Lambda}V(t,x), \quad \text{i.e.,} \quad \widehat{f}(t,\xi) = e^{-it\Lambda(\xi)}\widehat{V}(t,\xi),$$

we have

$$\partial_t \widehat{f} = \sum_{\mu,\nu\in\{+,-\}} \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Phi_{\mu,\nu}(\xi,\eta)} q_{\mu,\nu}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{f^\nu}(\eta)d\eta, \tag{1.11}$$

where the phase $\Phi_{\mu,\nu}(\xi,\eta)$ is defined by

$$\Phi_{\mu,\nu}(\xi,\eta) = -\Lambda(\xi) + \mu\Lambda(\xi-\eta) + \nu\Lambda(\eta).$$

To remove the quadratic terms in the right hand side of (1.11), we introduce the following normal forms transformation

$$g = f + \sum_{\mu,\nu\in\{+,-\}} A_{\mu,\nu}(f^\mu, f^\nu), \tag{1.12}$$

where

$$\mathcal{F}\Big(A_{\mu,\nu}(f^\mu, f^\nu)\Big)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Phi_{\mu,\nu}(\xi,\eta)} a_{\mu,\nu}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{f^\nu}(\eta)d\eta$$

with the symbol

$$a_{\mu,\nu}(\xi,\eta) = -\frac{q_{\mu,\nu}(\xi,\eta)}{i\Phi_{\mu,\nu}(\xi,\eta)}. \tag{1.13}$$

Thus, we have

$$\partial_t \widehat{g} = \frac{1}{2\pi}\sum_{\mu,\nu\in\{+,-\}} \int_{\mathbb{R}} e^{it\Phi_{\mu,\nu}(\xi,\eta)} a_{\mu,\nu}(\xi,\eta)\partial_t\Big(\widehat{f^\mu}(\xi-\eta)\widehat{f^\nu}(\eta)\Big)d\eta. \tag{1.14}$$

By virtue of (1.11), we see that the r.h.s. of (1.14) includes the cubic terms in (f^μ, f^ν, f^γ). Therefore, if the symbols of quadratic terms have "good" properties, for data of small size ε, the time scale $\frac{1}{\varepsilon^2}$ is much likely expected.

We will use the normal form techniques in another way, that is, integrating by parts with respect to time in the energy estimate. More precisely, energy estimate gives rise to

$$\frac{1}{2}\frac{d}{dt}\|V\|_{H^N}^2 = \sum_{\mu,\nu\in\{+,-\}} \big(Q_{\mu,\nu}(V^\mu, V^\nu)\,|\,V^+\big)_{H^N},$$

which implies that

$$\|V(t)\|_{H^N}^2 \lesssim \|V(0)\|_{H^N}^2 + \sum_{\mu,\nu\in\{+,-\}} \underbrace{\int_0^t \big(Q_{\mu,\nu}(V^\mu, V^\nu)\,|\,V^+\big)_{H^N} d\tau}_{I_{\mu,\nu}}. \tag{1.15}$$

For $I_{\mu,\nu}$, using (1.10) and the profiles, we have

$$\begin{aligned} I_{\mu,\nu} &= \frac{1}{(2\pi)^2}\int_0^t\int_{\mathbb{R}\times\mathbb{R}} \langle\xi\rangle^{2N} q_{\mu,\nu}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi d\tau \\ &= \frac{1}{(2\pi)^2}\int_0^t\int_{\mathbb{R}\times\mathbb{R}} e^{i\tau\Phi_{\mu,\nu}(\xi,\eta)}\langle\xi\rangle^{2N} q_{\mu,\nu}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{f^\nu}(\eta)\overline{\widehat{f^+}(\xi)}d\eta d\xi d\tau. \end{aligned}$$

Since

$$e^{i\tau\Phi_{\mu,\nu}(\xi,\eta)} = \frac{1}{i\Phi_{\mu,\nu}(\xi,\eta)}\frac{d}{d\tau}e^{i\tau\Phi_{\mu,\nu}(\xi,\eta)},$$

integrating by parts with respect to τ, we have

$$
\begin{aligned}
I_{\mu,\nu} =&\frac{1}{(2\pi)^2}\int_{\mathbb{R}\times\mathbb{R}} e^{i\tau\Phi_{\mu,\nu}(\xi,\eta)}\frac{\langle\xi\rangle^{2N}q_{\mu,\nu}(\xi,\eta)}{i\Phi_{\mu,\nu}(\xi,\eta)}\cdot\widehat{f^{\mu}}(\xi-\eta)\widehat{f^{\nu}}(\eta)\overline{\widehat{f^{+}}(\xi)}d\eta d\xi\big|_{\tau=0}^{t}\\
&-\frac{1}{(2\pi)^2}\int_0^t\int_{\mathbb{R}\times\mathbb{R}} e^{i\tau\Phi_{\mu,\nu}(\xi,\eta)}\frac{\langle\xi\rangle^{2N}q_{\mu,\nu}(\xi,\eta)}{i\Phi_{\mu,\nu}(\xi,\eta)}\cdot\partial_\tau\big(\widehat{f^{\mu}}(\xi-\eta)\widehat{f^{\nu}}(\eta)\overline{\widehat{f^{+}}(\xi)}\big)d\eta d\xi d\tau.
\end{aligned}
\tag{1.16}
$$

By virtue of (1.11), we see that the second term in the r.h.s of (1.16) includes inner product between the cubic terms of (f^μ, f^ν, f^γ) and f^+. The first term in the r.h.s of (1.16) may be controlled by the initial energy. If the symbols of quadratic terms have "good" properties, one may derive an energy estimate from (1.15) so that the time scale $\frac{1}{\varepsilon^2}$ is much likely expected, provided that the data is of small size ε.

However, the phase $\Phi_{\mu,\nu}(\xi,\eta)$ may equal 0 for some ξ and η. The symbol $a_{\mu,\nu}(\xi-\eta,\eta)$ in (1.13) is not well-defined for all $(\xi,\eta)\in\mathbb{R}^2$. While the integration by parts with respect to τ in (1.16) could not work for all $(\xi,\eta)\in\mathbb{R}^2$. We have to modify the normal forms transformation only on the "good frequencies set" that is far away from the zeroes of the phase $\Phi_{\mu,\nu}(\xi,\eta)$. Then the existence time scale may be enlarged. Although we could not obtain the time scale $\frac{1}{\varepsilon^2}$, we may get the existence time scale $\frac{1}{\varepsilon^{1+\delta}}$ (for some $\delta\in(0,1)$). It extends the local existence time scale $\frac{1}{\varepsilon}$ that can be obtained by a purely dispersive method as in [19].

In the present paper, we thus use normal form techniques after integration by parts with respect to time as in (1.16).

1.3. **The main results.** We now state the main results of this paper. The first one concerns the system (1.4) without the small parameter ϵ but with "small" initial data.

Theorem 1.1. *Assume that $(\zeta_0, v_0)\in H^{N_0}(\mathbb{R})$ for some $N_0\geq 4$ satisfying $\widehat{\zeta_0}(0)=\widehat{v_0}(0)=0$ and*

$$
\|\zeta_0\|_{H^{N_0}}^2+\|v_0\|_{H^{N_0}}^2=\varepsilon^2. \tag{1.17}
$$

There exists a small $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, there exists $T_\varepsilon=c_0\varepsilon^{-\frac{4}{3}}$ for some $c_0>0$ and a unique solution $(\zeta,v)\in C(0,T_\varepsilon;H^{N_0}(\mathbb{R}))$ of system (1.4)-(1.5) such that

$$
\sup_{(0,T_\varepsilon)}\big(\|\zeta(t)\|_{H^{N_0}}+\|v(t)\|_{H^{N_0}}\big)\leq C\big(\|\zeta_0\|_{H^{N_0}}+\|v_0\|_{H^{N_0}}\big), \tag{1.18}
$$

where $C>0$ is a universal constant.

Remark 1.2. *If $\widehat{\zeta_0}(0)=\widehat{v_0}(0)=0$, (1.4) shows that $\widehat{\zeta}(t,0)=\widehat{v}(t,0)=0$ holds for all time $t>0$. Therefore, throughout the whole paper, we shall use the condition $\widehat{\zeta}(t,0)=\widehat{v}(t,0)=0$.*

As a consequence of Theorem 1.1, we get the long time existence of solutions to system (1.6) :

Theorem 1.2. *Assume that $(\zeta_0, v_0)\in H^{N_0}(\mathbb{R})$ with $N_0\geq 4$ satisfying $\widehat{\zeta_0}(0)=\widehat{v_0}(0)=0$. There exist a small $\epsilon_0>0$ and a constant $T_0=T_0(\|(\zeta_0,v_0)\|_{H^{N_0}})$ such that for any $\epsilon\in(0,\epsilon_0]$, there exists a unique solution $(\zeta,v)\in C(0,T_0\epsilon^{-\frac{2}{3}};H^{N_0}(\mathbb{R}))$ of system (1.6)-(1.7) such that*

$$
\sup_{(0,T_0\epsilon^{-\frac{2}{3}})}\big(\|\zeta(t)\|_{H^{N_0}}+\|v(t)\|_{H^{N_0}}\big)\leq C\big(\|\zeta_0\|_{H^{N_0}}+\|v_0\|_{H^{N_0}}\big). \tag{1.19}
$$

Here $T_0=T_0(\|(\zeta_0,v_0)\|_{H^{N_0}})$ is a constant depending on $\|(\zeta_0,v_0)\|_{H^{N_0}}$.

Remark 1.3. *Contrary to the previous known results on long time existence of other (abcd) Boussinesq systems obtained in [10, 11, 20, 22, 23], we do not reach in Theorem 1.2 the expected time scales $O(1/\epsilon)$. Recall however that Theorem 1.2 improves the $O(1/\sqrt{\epsilon})$ result obtained by purely dispersive methods, see [19].*

1.4. **Comments on the proofs of Theorems 1.1 and 1.2.** We shall prove two different long time existence results in Theorems 1.1 and 1.2. The proofs of the theorems share some common features. To avoid losing derivative, we introduce the good unknowns (in the sense of Alinhac [2]) $(\zeta,\,u)$ via nonlinear and nonlocal transformation. Then the principal paralinearization parts for the new system of $V=\zeta+i\frac{\partial_x}{|\partial_x|}u$ (or $(\zeta,\,u)$) are symmetric (see (3.6) and (5.3)).

However, to enlarge the scale of the existence time, the difficulties of system (3.6) and (5.3) are different. For system (3.6), we want to prove an existence time of scale $O(1/\varepsilon^{4/3})$ when the data are of order $O(\varepsilon)$. The main difficulty arises from all the quadratic terms so that we have to deal with all the quadratic terms by the normal form transformation techniques which sketched in subsection 1.2. Whereas for system (5.3), we want to prove an existence time of scale $O(1/\epsilon^{2/3})$ when the data are of order $O(1)$ with small parameter ϵ. The key difficulty stems from the quadratic term that is of order $O(\sqrt{\epsilon})$ involving the low frequencies. We only apply the normal form transformation techniques to such $O(\sqrt{\epsilon})$ term. One could check that the normal form transformation could not improve the estimates involving other quadratic terms which are of order $O(\epsilon)$.

2. Preliminary

2.1. **Definitions and notations.** The notation $f \sim g$ means that there exists a constant C such that $\frac{1}{C} f \le g \le Cf$. $f \lesssim g$ means that there exists a constant C such that $f \le Cg$. We shall use C to denote a universal constant which may changes from line to line. For any $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the classical L^2 based Sobolev spaces with the norm $\|\cdot\|_{H^s}$. The notation $\|\cdot\|_{L^p}$ stands for the $L^p(\mathbb{R})$ norm for $1 \le p \le \infty$. For any $k \in \mathbb{N}$, we denote by

$$\|f\|_{W^{k,\infty}} = \sum_{j=0}^{k} \|\partial_x^j f\|_{L^\infty}.$$

The $L^2(\mathbb{R})$ scalar product is denoted by $(u\,|\,v)_2 \overset{\text{def}}{=} \int_{\mathbb{R}} u\bar{v}dx$.

If A, B are two operators, $[A, B] = AB - BA$ denotes their commutator.

The Fourier transform of a tempered distribution $u \in \mathcal{S}'$ is denoted by $\widehat{u}$, which is defined as follows

$$\widehat{u}(\xi) \overset{\text{def}}{=} \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(x)dx.$$

We use $\mathcal{F}^{-1}(f)$ to denote the inverse Fourier transform of $f(\xi)$.

If f and u are two functions defined on $\mathbb{R}$, the Fourier multiplier $f(D)u$ is defined in term of Fourier transforms, i.e.,

$$\widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

We shall use notations

$$\langle\xi\rangle = \big(1 + |\xi|^2\big)^{\frac{1}{2}}, \quad \langle\partial_x\rangle = \big(1 + |\partial_x|^2\big)^{\frac{1}{2}}.$$

For two well-defined functions $f(x)$, $g(x)$ and their bilinear form $Q(f, g)$, we use the convection that the symbol $q(\xi, \eta)$ of $Q(f, g)$ is defined in the following sense

$$\mathcal{F}\big(Q(f, g)\big)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} q(\xi, \eta)\hat{f}(\xi - \eta)\hat{g}(\xi)d\eta.$$

2.2. **Para-differential decomposition theory.** Our proof of the main results relies on suitable energy estimates for the solutions of (1.4) and (1.6). To do so, we introduce para-differential formulations (see *e.g.,* [5]) to symmetrize the systems (1.4) and (1.6).

We fix an even smooth function $\varphi : \mathbb{R} \to [0, 1]$ supported in $[-\frac{3}{2}, \frac{3}{2}]$ and equals to 1 in $[-\frac{5}{4}, \frac{5}{4}]$. For any $k \in \mathbb{Z}$, we define

$$\varphi_k(x) \overset{\text{def}}{=} \varphi(\frac{x}{2^k}) - \varphi(\frac{x}{2^{k-1}}), \quad \varphi_{\le k}(x) \overset{\text{def}}{=} \varphi(\frac{x}{2^k}) = \sum_{l \le k} \varphi_l(x). \quad \varphi_{\ge k}(x) \overset{\text{def}}{=} 1 - \varphi_{\le k-1}(x).$$

While for any interval I of $\mathbb{R}$, we define

$$\varphi_I(x) \overset{\text{def}}{=} \sum_{k \in I} \varphi_k(x) = \sum_{k \in I \cap \mathbb{Z}} \varphi_k(x).$$

Then for any $x \in \mathbb{R}$,

$$\sum_{k \in \mathbb{Z}} \varphi_k(x) = 1 \quad \text{and} \quad supp\, \varphi_k(\cdot) \in \{x \in \mathbb{R} \mid |x| \in [\frac{5}{8} 2^k, \frac{3}{2} 2^k]\}. \tag{2.1}$$

We use P_k, $P_{\le k}$, $P_{\ge k}$ and P_I to denote the Littlewood-Paley projection operators of the Fourier multiplier φ_k, $\varphi_{\le k}$, $\varphi_{\ge k}$ and φ_I, respectively.

We shall use the following para-differential decomposition: for any functions $f, g \in \mathcal{S}'(\mathbb{R})$,

$$fg = T_f g + T_g f + R(f, g), \tag{2.2}$$

with the para-differential operators being defined as follows

$$T_f g = \sum_{j\in\mathbb{Z}} P_{\le j-7} f \cdot P_j g, \quad R(f,g) = \sum_{j\in\mathbb{Z}} P_j f \cdot P_{[j-6,j+6]} g.$$

2.3. **Analysis of the phases.** In this subsection, we shall discuss the quadratic phase function $\Phi_{\mu,\nu}(\xi,\eta)$ which is defined as follows:

$$\Phi_{\mu,\nu}(\xi,\eta) = -\Lambda(\xi) + \mu\Lambda(\xi-\eta) + \nu\Lambda(\eta), \quad \mu,\nu\in\{+,-\}, \tag{2.3}$$

where $\Lambda(\xi)$ is defined by

$$\Lambda(\xi) = (1-|\xi|^2)|\xi| = |\xi| - |\xi|^3.$$

We first rewrite the explicit expressions of the phases.

Lemma 2.1. *For any $(\xi,\eta)\in\mathbb{R}^2$ with $\xi\neq\eta$, $\xi\neq 0$, $\eta\neq 0$, we have*

$$\Phi_{+,+}(\xi,\eta) = \begin{cases} 3|\xi||\xi-\eta||\eta|, \quad \textit{if} \quad (\xi-\eta)\cdot\eta > 0, \\ -\dfrac{1}{2}\min\{|\xi-\eta|,|\eta|\}\big(3|\xi|^2 + 3\max\{|\xi-\eta|^2,|\eta|^2\} + \min\{|\xi-\eta|^2,|\eta|^2\}\big) - 4\Big), \\ \qquad\qquad \textit{if} \quad (\xi-\eta)\cdot\eta < 0; \end{cases}$$

$$\Phi_{-,-}(\xi,\eta) = \begin{cases} \dfrac{1}{2}|\xi|\big(|\xi|^2 + 3|\xi-\eta|^2 + 3|\eta|^2 - 4\big), \quad \textit{if} \quad (\xi-\eta)\cdot\eta > 0, \\ \dfrac{1}{2}\max\{|\xi-\eta|,|\eta|\}\big(3|\xi|^2 + 3\min\{|\xi-\eta|^2,|\eta|^2\} + \max\{|\xi-\eta|^2,|\eta|^2\} - 4\big), \\ \qquad\qquad \textit{if} \quad (\xi-\eta)\cdot\eta < 0; \end{cases}$$

and

$$\Phi_{-,+}(\xi,\eta) = -\Phi_{+,+}(\eta,\xi), \quad \Phi_{+,-}(\xi,\eta) = -\Phi_{+,+}(\eta-\xi,\eta).$$

Proof. We derive the expressions of phases one by one.

(1) *For $\Phi_{+,+}$,* by the definition, we have

$$\begin{aligned} \Phi_{+,+}(\xi,\eta) &= -|\xi| + |\xi-\eta| + |\eta| + |\xi|^3 - |\xi-\eta|^3 - |\eta|^3 \\ &= (|\xi| - |\xi-\eta| - |\eta|)\Big(2|\xi|^2 - \big(1 + 2\mathrm{sign}\big((\xi-\eta)\cdot\eta\big)\big)|\xi-\eta||\eta| \\ &\qquad + |\xi|(|\xi-\eta| + |\eta|) - 1\Big) + 3|\xi||\xi-\eta||\eta|. \end{aligned}$$

If $(\xi-\eta)\cdot\eta > 0$, we have

$$|\xi| - |\xi-\eta| - |\eta| = 0,$$

which gives rise to

$$\Phi_{+,+}(\xi,\eta) = 3|\xi||\xi-\eta||\eta|.$$

If $(\xi-\eta)\cdot\eta < 0$, we have

$$\begin{aligned} &|\xi| = \big||\xi-\eta| - |\eta|\big| = \max\{|\xi-\eta|,|\eta|\} - \min\{|\xi-\eta|,|\eta|\}, \\ &\text{and} \quad |\xi| - |\xi-\eta| - |\eta| = -2\min\{|\xi-\eta|,|\eta|\}, \end{aligned}$$

which yields

$$\begin{aligned} \Phi_{+,+}(\xi,\eta) &= -2\min\{|\xi-\eta|,|\eta|\}\Big(2|\xi|^2 + |\xi-\eta||\eta| + |\xi|(|\xi-\eta|+|\eta|) - \frac{3}{2}|\xi|\max\{|\xi-\eta|,|\eta|\} - 1\Big) \\ &= -2\min\{|\xi-\eta|,|\eta|\}\big(\frac{3}{2}\max\{|\xi-\eta|^2,|\eta|^2\} + \min\{|\xi-\eta|^2,|\eta|^2\} - \frac{3}{2}|\xi-\eta||\eta| - 1\big) \\ &= -2\min\{|\xi-\eta|,|\eta|\}\Big(\frac{3}{4}\max\{|\xi-\eta|^2,|\eta|^2\} + \frac{1}{4}\min\{|\xi-\eta|^2,|\eta|^2\} + \frac{3}{4}(|\xi-\eta|-|\eta|)^2 - 1\Big) \\ &= -\frac{1}{2}\min\{|\xi-\eta|,|\eta|\}\big(3|\xi|^2 + 3\max\{|\xi-\eta|^2,|\eta|^2\} + \min\{|\xi-\eta|^2,|\eta|^2\}\big) - 4\big). \end{aligned}$$

(2) *For* $\Phi_{-,-}$, by the definition, we have

$$\begin{aligned}\Phi_{-,-}(\xi,\eta) &= -\big(|\xi|+|\xi-\eta|+|\eta|\big)+\big(|\xi|^3+|\xi-\eta|^3+|\eta|^3\big)\\ &= (|\xi|+|\xi-\eta|+|\eta|)\Big(|\xi|^2-|\xi|(|\xi-\eta|+|\eta|)+(|\xi-\eta|+|\eta|)^2-3|\xi-\eta||\eta|-1\Big)\\ &\quad +3|\xi||\xi-\eta||\eta|.\end{aligned}$$

If $(\xi-\eta)\cdot\eta>0$, we have

$$|\xi|=|\xi-\eta|+|\eta|,$$

and

$$\begin{aligned}\Phi_{-,-}(\xi,\eta) &= 2|\xi|\big(|\xi-\eta|^2+|\eta|^2+\frac{1}{2}|\xi-\eta||\eta|-1\big)\\ &= \frac{1}{2}|\xi|\big(|\xi|^2+3|\xi-\eta|^2+3|\eta|^2-4\big).\end{aligned}$$

If $(\xi-\eta)\cdot\eta<0$, we have

$$\begin{aligned}&|\xi|=\big||\xi-\eta|-|\eta|\big|=\max\{|\xi-\eta|,|\eta|\}-\min\{|\xi-\eta|,|\eta|\},\\ &\text{and}\quad |\xi|+|\xi-\eta|+|\eta|=2\max\{|\xi-\eta|,|\eta|\},\end{aligned}$$

which implies

$$\begin{aligned}\Phi_{-,-}(\xi,\eta) &= 2\max\{|\xi-\eta|,|\eta|\}\big(\max\{|\xi-\eta|^2,|\eta|^2\}+\frac{3}{2}\min\{|\xi-\eta|^2,|\eta|^2\}-\frac{3}{2}|\xi-\eta||\eta|-1\big)\\ &= \frac{1}{2}\max\{|\xi-\eta|,|\eta|\}\big(3|\xi|^2+3\min\{|\xi-\eta|^2,|\eta|^2\}+\max\{|\xi-\eta|^2,|\eta|^2\}-4\big).\end{aligned}$$

(3) *For* $\Phi_{+,-}$ *and* $\Phi_{-,+}$, by the definition, we have

$$\Phi_{-,+}(\xi,\eta)=-\Phi_{+,+}(\eta,\xi),\quad \Phi_{+,-}(\xi,\eta)=-\Phi_{+,+}(\eta-\xi,\eta).$$

The lemma is proved. □

As a consequence, defining

$$\Lambda_\epsilon(\xi)=(1-\epsilon|\xi|^2)|\xi|=|\xi|-\epsilon|\xi|^3,$$

and

$$\Phi^\epsilon_{\mu,\nu}(\xi,\eta)=-\Lambda_\epsilon(\xi)+\mu\Lambda_\epsilon(\xi-\eta)+\nu\Lambda_\epsilon(\eta),\quad \mu,\nu\in\{+,-\},\tag{2.4}$$

we obtain explicit expressions of the phases $\Phi^\epsilon_{\mu,\nu}(\xi,\eta)$ which involve the operator Λ_ϵ.

Lemma 2.2. *For any* $(\xi,\eta)\in\mathbb{R}^2$ *with* $\xi\neq\eta$, $\xi\neq0$, $\eta\neq0$, *we have*

$$\Phi_{+,-}(\xi,\eta)=\begin{cases}-3\epsilon|\xi||\xi-\eta||\eta|, & \textit{if}\quad \xi\cdot\eta<0,\\ \frac{1}{2}\min\{|\xi|,|\eta|\}\big(3\epsilon|\xi-\eta|^2+3\epsilon\max\{|\xi|^2,|\eta|^2\}+\epsilon\min\{|\xi|^2,|\eta|^2\}-4\big), & \\ \qquad\qquad \textit{if}\quad \xi\cdot\eta>0;\end{cases}$$

$$\Phi^\epsilon_{-,-}(\xi,\eta)=\begin{cases}\frac{1}{2}|\xi|\big(\epsilon|\xi|^2+3\epsilon|\xi-\eta|^2+3\epsilon|\eta|^2-4\big), & \textit{if}\quad (\xi-\eta)\cdot\eta>0,\\ \frac{1}{2}\max\{|\xi-\eta|,|\eta|\}\big(3\epsilon|\xi|^2+3\epsilon\min\{|\xi-\eta|^2,|\eta|^2\}+\epsilon\max\{|\xi-\eta|^2,|\eta|^2\}-4\big), & \\ \qquad\qquad \textit{if}\quad (\xi-\eta)\cdot\eta<0;\end{cases}$$

and

$$\Phi^\epsilon_{+,+}(\xi,\eta)=-\Phi^\epsilon_{+,-}(\eta-\xi,\eta),\quad \Phi^\epsilon_{-,+}(\xi,\eta)=-\Phi^\epsilon_{+,+}(\eta,\xi)=\Phi^\epsilon_{+,-}(\xi-\eta,\xi),\quad .$$

2.4. **Technical lemmas.**

Lemma 2.3. *Let f, g be smooth enough functions. Then,*

$$[\frac{\partial_x}{|\partial_x|}, T_f]g = 0. \tag{2.5}$$

Proof. By the definitions of commutator and para-differential operators, we have

$$\mathcal{F}\Big([\frac{\partial_x}{|\partial_x|}, T_f]g\Big)(\xi) = i\int_{\mathbb{R}}\big(\mathrm{sign}(\xi) - \mathrm{sign}(\eta)\big)\sum_{j\in\mathbb{Z}}\varphi_{\le j-7}(|\xi-\eta|)\varphi_j(|\eta|)\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta.$$

For fixed $\xi \in \mathbb{R}$, when $|\eta| \in (2^k, 2^{k+1}]$ with $k \in \mathbb{Z}$, we have

$$\sum_{j\in\mathbb{Z}}\varphi_{\le j-7}(|\xi-\eta|)\varphi_j(|\eta|) = \sum_{j=k}^{k+1}\varphi_{\le j-7}(|\xi-\eta|)\varphi_j(|\eta|) \le \varphi(\frac{|\xi-\eta|}{|\eta|}\cdot\frac{|\eta|}{2^{k-6}}) \le \varphi_{\le -6}(\frac{|\xi-\eta|}{|\eta|}). \tag{2.6}$$

Then we get $|\xi - \eta| \le 2^{-5}|\eta|$, which yields

$$\xi\cdot\eta > 0.$$

Otherwise,

$$|\xi-\eta| = |\xi| + |\eta| \ge |\eta|.$$

Therefore, we have $\xi \cdot \eta > 0$ and

$$\mathcal{F}\Big([\frac{\partial_x}{|\partial_x|}, T_f]g\Big)(\xi) = 0,$$

which implies (2.5). The lemma is proved. □

3. Symmetrization of the system (1.4)

In this section, we will symmetrize the system (1.4) by introducing good unknowns.

3.1. **Symmetrization of the system** (1.4)**.** By virtue of the para-differential decomposition, we rewrite (1.4) to

$$\begin{cases} \partial_t\zeta + (1+\partial_x^2)\partial_x v + \partial_x(T_v\zeta) + \partial_x(T_\zeta v) + \partial_x\big(R(\zeta, v)\big) = 0, \\ \partial_t v + (1+\partial_x^2)\partial_x\zeta + \partial_x(T_v v) + \frac{1}{2}\partial_x\big(R(v,v)\big) = 0. \end{cases} \tag{3.1}$$

We introduce good unknowns (ζ, u) with

$$u = v + B(\zeta, v), \tag{3.2}$$

where $B(\cdot,\cdot)$ is a bilinear operator defined as

$$B(f,g) = \frac{1}{2}T_f\big((1+\partial_x^2)^{-1}P_{\ge 6}g\big).$$

Without confusion, we sometimes use B to denote the bilinear term $B(\zeta, v)$.

Thanks to (3.1) and (3.2), we have

$$\partial_t\zeta + (1+\partial_x^2)\partial_x u + \partial_x(T_v\zeta) + \partial_x(T_\zeta u) = (1+\partial_x^2)\partial_x B - \partial_x\big(R(\zeta,v)\big) + \partial_x(T_\zeta B).$$

Since

$$(1+\partial_x^2)\partial_x B = \frac{1}{2}\partial_x\big(T_\zeta P_{\ge 6}u\big) - \frac{1}{2}\partial_x\big(T_\zeta P_{\ge 6}B\big) + \frac{1}{2}\partial_x\Big([\partial_x^2, T_\zeta](1+\partial_x^2)^{-1}P_{\ge 6}v\Big),$$

we have

$$\partial_t\zeta + (1+\partial_x^2)\partial_x u + \partial_x(T_v\zeta) + \frac{1}{2}\partial_x(T_\zeta u) = N_\zeta, \tag{3.3}$$

where

$$N_\zeta = -\frac{1}{2}\partial_x\big(T_\zeta P_{\le 5}u\big) - \frac{1}{2}\partial_x\big(T_\zeta P_{\ge 6}B\big) + \partial_x(T_\zeta B) + \frac{1}{2}\partial_x\Big([\partial_x^2, T_\zeta](1+\partial_x^2)^{-1}P_{\ge 6}v\Big) - \partial_x\big(R(\zeta,v)\big).$$

Using (1.4), (3.1) and (3.2), we also have

$$\begin{aligned}\partial_t u &= \partial_t v + B(\partial_t\zeta, v) + B(\zeta, \partial_t v)\\ &= -(1+\partial_x^2)\partial_x\zeta - \partial_x(T_v v) - \frac{1}{2}\partial_x\big(R(v,v)\big) + B(\partial_t\zeta, v) - B(\zeta, (1+\partial_x^2)\partial_x\zeta) - \frac{1}{2}B\big(\zeta, \partial_x(|v|^2)\big).\end{aligned}$$

Noticing that

$$B(\zeta,(1+\partial_x^2)\partial_x\zeta)=\frac{1}{2}T_\zeta\partial_x P_{\geq 6}\zeta,\quad \partial_x(T_v v)=\partial_x(T_v u)-\partial_x(T_v B),$$

we have

$$\partial_t u+(1+\partial_x^2)\partial_x\zeta+\partial_x(T_v u)+\frac{1}{2}\partial_x\big(T_\zeta\zeta\big)=N_u, \tag{3.4}$$

where

$$N_u=\frac{1}{2}\partial_x\big(T_\zeta P_{\leq 5}\zeta\big)+\frac{1}{2}T_{\partial_x\zeta}P_{\geq 6}\zeta+\partial_x(T_v B)-\frac{1}{2}\partial_x\big(R(v,v)\big)+B(\partial_t\zeta,v)-\frac{1}{2}B\big(\zeta,\partial_x(|v|^2)\big).$$

Now, we define

$$V=\zeta+i\frac{\partial_x}{|\partial_x|}u. \tag{3.5}$$

Thanks to (3.3) and (3.4), using (2.5), we have

$$\partial_t V-i\Lambda V+\partial_x(T_v V)-\frac{1}{2}i|\partial_x|(T_\zeta V)=N_\zeta+i\frac{\partial_x}{|\partial_x|}N_u, \tag{3.6}$$

where $\Lambda=|\partial_x|(1-|\partial_x|^2)$. The l.h.s of (3.6) is the quasi-linear part of system (1.4).

Denoting by

$$V^+=V,\quad V^-=\overline{V},$$

we shall rewrite the quadratic terms of (3.6) in terms of V^+ and V^-. Whereas we keep the cubic and quartic terms in terms of ζ and v.

Before ending this subsection, we provide a lemma involving the bilinear operator $B(\cdot,\cdot)$.

Lemma 3.1. *Assume that the real-valued functions $f\in L^\infty(\mathbb{R})$, $g\in H^s(\mathbb{R})$ for $s\geq -2$. There hold*

$$\mathcal{F}\big(B(f,g)\big)(\xi)=\overline{\mathcal{F}\big(B(f,g)\big)(-\xi)} \tag{3.7}$$

and

$$\|B(f,g)\|_{H^{s+2}}\leq C_B\|f\|_{L^\infty}\|g\|_{H^s}, \tag{3.8}$$

where $C_B>0$ is a universal constant.

Proof. By the definition of $B(\cdot,\cdot)$, we have

$$\mathcal{F}\big(B(f,g)\big)(\xi)=\frac{1}{4\pi}\int_{\mathbb{R}}\widehat{f}(\xi-\eta)\widehat{g}(\eta)(1-|\eta|^2)^{-1}\varphi_{\geq 6}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)d\eta, \tag{3.9}$$

and

$$\begin{aligned}\overline{\mathcal{F}\big(B(f,g)\big)(-\xi)}&=\frac{1}{4\pi}\int_{\mathbb{R}}\overline{\widehat{f}(-\xi-\eta)}\,\overline{\widehat{g}(\eta)}(1-|\eta|^2)^{-1}\varphi_{\geq 6}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|-\xi-\eta|)\varphi_j(|\eta|)d\eta\\&=\frac{1}{4\pi}\int_{\mathbb{R}}\overline{\widehat{f}(-\xi+\eta)}\,\overline{\widehat{g}(-\eta)}(1-|\eta|^2)^{-1}\varphi_{\geq 6}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)d\eta.\end{aligned}$$

Since f,g are real-valued functions, we have

$$\overline{\widehat{f}(-\xi+\eta)}=\widehat{f}(\xi-\eta),\quad \overline{\widehat{g}(-\eta)}=\widehat{g}(\eta),$$

which gives rise to

$$\overline{\mathcal{F}\big(B(f,g)\big)(-\xi)}=\frac{1}{4\pi}\int_{\mathbb{R}}\widehat{f}(\xi-\eta)\widehat{g}(\eta)(1-|\eta|^2)^{-1}\varphi_{\geq 6}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)d\eta.$$

Then we have

$$\mathcal{F}\big(B(f,g)\big)(\xi)=\overline{\mathcal{F}\big(B(f,g)\big)(-\xi)}.$$

Estimate (3.8) follows from the standard estimate on $T_f g$ and the definition of $B(f,g)$. This completes the proof of the lemma. $\square$

3.2. **Main proposition for the symmetric system** (3.6)**.** For (3.6), we state the following proposition.

Proposition 3.2. *Assume that* $(\zeta, v) \in H^{N_0}(\mathbb{R})$ *with* $N_0 \geq 4$ *solves* (1.4)*. Then* V *defined in* (3.5) *satisfies the following system*

$$\partial_t V - i\Lambda V = \mathcal{S}_V + \mathcal{Q}_V + \mathcal{R}_V + \mathcal{M}_V + \mathcal{L}_V + \mathcal{C}_V + \mathcal{N}_V, \tag{3.10}$$

where

- *The quadratic term* $\mathcal{S}_V$ *is of the form*
$$\mathcal{S}_V = S_{+,+}(V^+, V^+) + S_{-,+}(V^-, V^+).$$
And the symbol $s_{\mu,+}(\xi,\eta)$ *of* $S_{\mu,+}$ *(for* $\mu = +,-$*) satisfies*
$$\overline{s_{\mu,+}(\xi,\eta)} = -s_{\mu,+}(\xi,\eta), \tag{3.11}$$
$$|\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s_{\mu,+}(\xi,\eta) - \langle\eta\rangle^{2N_0}s_{-\mu,+}(\eta,\xi)\big)| \lesssim |\xi-\eta|\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{\max\{|\xi|,|\eta|\}}\Big). \tag{3.12}$$
- *The quadratic term* $\mathcal{Q}_V$ *is of the form*
$$\mathcal{Q}_V = Q_{+,-}(V^+, V^-) + Q_{-,-}(V^-, V^-).$$
And the symbol $q_{\mu,-}(\xi,\eta)$ *of* $Q_{\mu,-}$ *satisfies*
$$|q_{\mu,-}(\xi,\eta)| \lesssim |\xi|\cdot\varphi_{\leq 5}\big(|\eta|\big)\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big). \tag{3.13}$$
- *The quadratic term* $\mathcal{R}_V$ *is of the form*
$$\mathcal{R}_V = \sum_{\mu,\nu\in\{+,-\}} R_{\mu,\nu}(V^\mu, V^\nu).$$
And the symbol $r_{\mu,\nu}(\xi,\eta)$ *of* $R_{\mu,\nu}$ *satisfies*
$$|r_{\mu,\nu}(\xi,\eta)| \lesssim |\xi-\eta|\cdot\varphi_{\geq 6}\big(|\eta|\big)\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big). \tag{3.14}$$
- *The quadratic term* $\mathcal{M}_V$ *is of the form*
$$\mathcal{M}_V = \sum_{\mu,\nu\in\{+,-\}} M_{\mu,\nu}(V^\mu, V^\nu).$$
And the symbol $m_{\mu,\nu}(\xi,\eta)$ *of* $M_{\mu,\nu}$ *satisfies*
$$|m_{\mu,\nu}(\xi,\eta)| \lesssim |\xi|\cdot\varphi_{[-6,7]}\Big(\frac{|\xi-\eta|}{|\eta|}\Big). \tag{3.15}$$
- *The cubic term* $\mathcal{L}_V = \partial_x(T_B V)$ *satisfies*
$$\big|Re\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2\big\}\big| \lesssim \|\zeta\|_{L^\infty}\|v\|_{L^2}\|V\|^2_{H^{N_0}}. \tag{3.16}$$
- *The cubic term* $\mathcal{C}_V$ *satisfies*
$$\|\mathcal{C}_V\|_{H^{N_0}} \lesssim \|\zeta\|_{W^{1,\infty}}\big(\|\zeta\|^2_{H^{N_0}} + \|v\|^2_{H^{N_0}}\big). \tag{3.17}$$
- *The quartic term* $\mathcal{N}_V$ *satisfies*
$$\|\mathcal{N}_V\|_{H^{N_0}} \lesssim \|\zeta\|^2_{L^\infty}\|v\|^2_{H^{N_0}}. \tag{3.18}$$

Remark 3.3. *Proposition 3.2 shows that there is no loss of derivative for the nonlinear terms of* (3.6)*. Indeed, in the energy estimates, we shall use the symmetric structure of the quadratic terms* $\mathcal{S}_V$ *to avoid losing derivative(see* (3.12)*). We also use the symmetric structure of* $\mathcal{L}_V$ *to avoid losing derivative(see the proof of* (3.16)*).*

Remark 3.4. *For the symmetric system* (3.6) *or* (3.3)*-*(3.4)*, the standard energy estimates will provide the local existence on time of scale* $O(\frac{1}{\varepsilon})$*, for the initial data of size* ε*. To enlarge the existence time of the system, we shall use the new formulation* (3.10) *and the normal form transformations. Thanks to Proposition 3.2, if the quadratic terms equal zero, the estimates of the cubic terms and the quartic terms guarantee the existence time of scale* $\frac{1}{\varepsilon^2}$*. For the non-trivial quadratic terms in* (3.10)*, we shall apply normal forms transformation in the "good frequencies set"(far away from zeroes of the phases) to kill the quadratic terms to the cubic and quartic order terms, while for the quadratic terms in the "bad frequencies set" (in a small neighborhood of zeroes of phases), we will use the smallness size of the* *frequencies set. Combining the two estimates on both "good" and "bad" sets, the optimal "cut-off" of the frequencies spaces will determine an existence time of order* $\frac{1}{\varepsilon^{4/3}}$ *for* ε *sufficiently small.*

3.3. **Proof of Proposition 3.2.** In this subsection, we present the proof of Proposition 3.2.

Proof of Proposition 3.2. The nonlinear terms in the r.h.s. of (3.10) come from the nonlinear terms in (3.6). We rewrite (3.6) to be (3.10) with the nonlinear terms in the following forms

$$\begin{aligned}
&\mathcal{S}_V = -\partial_x(T_u V) + \frac{1}{2} i|\partial_x|(T_\zeta V),\\
&\mathcal{Q}_V = -\frac{1}{2}\partial_x\big(T_\zeta P_{\leq 5}u\big) - \frac{i}{2}|\partial_x|\big(T_\zeta P_{\leq 5}\zeta\big),\\
&\mathcal{R}_V = \frac{1}{2}\partial_x\Big([\partial_x^2, T_\zeta](1+\partial_x^2)^{-1}P_{\geq 6}u\Big) + \frac{i}{2}\frac{\partial_x}{|\partial_x|}\big(T_{\partial_x\zeta}P_{\geq 6}\zeta\big) - \frac{i\partial_x}{|\partial_x|}B((1+\partial_x^2)\partial_x u, u),\\
&\mathcal{M}_V = -\partial_x\big(R(\zeta,u)\big) + \frac{i}{2}|\partial_x|\big(R(u,u)\big),\\
&\mathcal{L}_V = \partial_x(T_B V),\\
&\mathcal{C}_V = \partial_x(T_\zeta B) - \frac{1}{2}\partial_x\big(T_\zeta P_{\geq 6}B\big) - \frac{1}{2}\partial_x\big([\partial_x^2, T_\zeta](1+\partial_x^2)^{-1}P_{\geq 6}B\big) - i|\partial_x|(T_v B)\\
&\qquad + \frac{i\partial_x}{|\partial_x|}B\big((1+\partial_x^2)\partial_x v, B\big) + \frac{i\partial_x}{|\partial_x|}B\big((1+\partial_x^2)\partial_x B, v\big) - \frac{i\partial_x}{|\partial_x|}B\big(\partial_x(\zeta v), v\big) + \partial_x\big(R(\zeta,B)\big)\\
&\qquad - \frac{i}{2}|\partial_x|\big(R(v,B)\big) - \frac{i}{2}|\partial_x|\big(R(B,v)\big) - \frac{i}{2}\frac{\partial_x}{|\partial_x|}B\big(\zeta, \partial_x(|v|^2)\big),\\
&\mathcal{N}_V = \frac{i\partial_x}{|\partial_x|}B\big((1+\partial_x^2)\partial_x B, B\big) - \frac{i}{2}|\partial_x|\big(R(B,B)\big).
\end{aligned}$$

Here we used the first equation of (1.4) and (3.2). Thanks to (3.5), we have

$$\zeta = \frac{1}{2}(V^+ + V^-) = \frac{1}{2}\sum_{\mu\in\{+,-\}} V^\mu, \quad u = \frac{i}{2}\frac{\partial_x}{|\partial_x|}(V^+ - V^-) = \frac{i}{2}\sum_{\mu\in\{+,-\}}\mu\frac{\partial_x}{|\partial_x|}V^\mu. \tag{3.19}$$

(1) *For the quadratic term* $\mathcal{S}_V$*,* by virtue of (3.19), we rewrite it in terms of V^+ and V^- as

$$\mathcal{S}_V = S_{+,+}(V^+, V^+) + S_{-,+}(V^-, V^+),$$

with

$$S_{\mu,+}(V^\mu, V^+) = -\mu\frac{i}{2}\partial_x(T_{\frac{\partial_x}{|\partial_x|}V^\mu}V^+) + \frac{1}{4}i|\partial_x|(T_{V^\mu}V^+).$$

By the definition of the para-differential operator, we have

$$\mathcal{F}\Big(S_{\mu,+}(V^\mu, V^+)\Big)(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} i\Big(\mu\frac{1}{2}\xi\,\mathrm{sign}(\xi-\eta) + \frac{1}{4}|\xi|\Big)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\widehat{V^\mu}(\xi-\eta)\widehat{V^+}(\eta)d\eta,$$

with the symbol

$$s_{\mu,+}(\xi,\eta) = i\Big(\mu\frac{1}{2}\xi\,\mathrm{sign}(\xi-\eta) + \frac{1}{4}|\xi|\Big)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|). \tag{3.20}$$

Then (3.20) yields $\overline{s_{\mu,+}(\xi,\eta)} = -s_{\mu,+}(\xi,\eta)$ which is exactly (3.11). Thanks to (2.6), we have

$$\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|) \lesssim \varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big),$$

which implies

$$\xi\cdot\eta > 0 \quad \text{and} \quad \frac{31}{32}|\eta| \leq |\xi| \leq \frac{33}{32}|\eta|. \tag{3.21}$$

Since

$$
\begin{aligned}
&\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s_{\mu,+}(\xi,\eta)-\langle\eta\rangle^{2N_0}s_{-\mu,+}(\eta,\xi)\big)\\
&=i\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\Big\{\langle\xi\rangle^{2N_0}\Big(\mu\frac{1}{2}\xi\,\mathrm{sign}(\xi-\eta)+\frac{1}{4}|\xi|\Big)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\\
&\quad-\langle\eta\rangle^{2N_0}\Big(\mu\frac{1}{2}\eta\,\mathrm{sign}(\xi-\eta)+\frac{1}{4}|\eta|\Big)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\xi|)\Big\},
\end{aligned}
$$

and

$$
|\varphi_j(|\xi|)-\varphi_j(|\eta|)|\lesssim\frac{1}{\min\{|\xi|,|\eta|\}}|\xi-\eta|,
$$

using (3.21), we obtain

$$
|\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s_{\mu,+}(\xi,\eta)-\langle\eta\rangle^{2N_0}s_{-\mu,+}(\eta,\xi)\big)|\lesssim|\xi-\eta|\cdot\varphi_{\leq-6}\Big(\frac{|\xi-\eta|}{\max\{|\xi|,|\eta|\}}\Big).
$$

This is exactly (3.12).

(2) *For the quadratic term* $\mathcal{Q}_V$, by virtue of (3.19), we rewrite it in terms of V^+ and V^- as

$$
\mathcal{Q}_V=\sum_{\mu,\nu\in\{+,-\}}Q_{\mu,\nu}(V^\mu,V^\nu),
$$

where

$$
Q_{\mu,\nu}(V^\mu,V^\nu)=-\nu\frac{i}{8}\partial_x\big(T_{V^\mu}\frac{\partial_x}{|\partial_x|}P_{\leq5}V^\nu\big)-\frac{i}{8}|\partial_x|\big(T_{V^\mu}P_{\leq5}V^\nu\big).
$$

Applying Fourier transformation to $Q_{\mu,\nu}(V^\mu,V^\nu)$, we have

$$
\mathcal{F}\Big(Q_{\mu,\nu}(V^\mu,V^\nu)\Big)(\xi)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{i}{8}\big(\nu\xi\mathrm{sign}(\eta)-|\xi|\big)\varphi_{\leq5}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)d\eta.
$$

Using (3.21), we have the symbol of $Q_{\mu,\nu}(V^\mu,V^\nu)$ as follows

$$
q_{\mu,\nu}(\xi,\eta)=\frac{i}{8}\big(\nu|\xi|-|\xi|\big)\varphi_{\leq5}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|) \tag{3.22}
$$

Thanks to (2.6), we obtain

$$
q_{\mu,+}(\xi,\eta)=0\quad\text{and}\quad|q_{\mu,-}(\xi,\eta)|\lesssim|\xi|\varphi_{\leq5}(|\eta|)\varphi_{\leq-6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big),
$$

which implies $Q_{\mu,+}=0$ and (3.13).

(3) *For the quadratic term* $\mathcal{R}_V$, by virtue of (3.19), we rewrite it in terms of V^+ and V^- as

$$
\mathcal{R}_V=\sum_{\mu,\nu\in\{+,-\}}R_{\mu,\nu}(V^\mu,V^\nu),
$$

where

$$
\begin{aligned}
R_{\mu,\nu}(V^\mu,V^\nu)&=\nu\frac{i}{8}\partial_x\Big([\partial_x^2,T_{V^\mu}](1+\partial_x^2)^{-1}\frac{\partial_x}{|\partial_x|}P_{\geq6}V^\nu\Big)\\
&\quad+\frac{i}{8}\frac{\partial_x}{|\partial_x|}\big(T_{\partial_xV^\mu}P_{\geq6}V^\nu\big)-\mu\nu\frac{i}{8}\frac{\partial_x}{|\partial_x|}\big(T_{(1+\partial_x^2)|\partial_x|V^\mu}(1+\partial_x^2)^{-1}\frac{\partial_x}{|\partial_x|}P_{\geq6}V^\nu\big).
\end{aligned}
$$

Here we used the definition of $B(\cdot,\cdot)$. Applying Fourier transformation to $R_{\mu,\nu}(V^\mu,V^\nu)$, we have

$$
\begin{aligned}
\mathcal{F}\Big(R_{\mu,\nu}(V^\mu,V^\nu)\Big)(\xi)&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{i}{8}\Big(\nu\big(|\xi|^2-|\eta|^2\big)(1-|\eta|^2)^{-1}\xi\mathrm{sign}(\eta)\\
&\quad-(\xi-\eta)\mathrm{sign}(\xi)+\mu\nu(1-|\xi-\eta|^2)|\xi-\eta|(1-|\eta|^2)^{-1}\mathrm{sign}(\xi)\mathrm{sign}(\eta)\Big)\\
&\quad\cdot\varphi_{\geq6}(|\eta|)\cdot\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)d\eta,
\end{aligned}
$$

which implies that the symbol of $R_{\mu,\nu}(\cdot,\cdot)$ is

$$\begin{aligned}r_{\mu,\nu}(\xi,\eta)=\frac{i}{8}\Big(&\nu\big(|\xi|^2-|\eta|^2\big)(1-|\eta|^2)^{-1}|\xi|-(\xi-\eta)\mathrm{sign}(\xi)\\&+\mu\nu(1-|\xi-\eta|^2)|\xi-\eta|(1-|\eta|^2)^{-1}\Big)\varphi_{\geq 6}(|\eta|)\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|),\end{aligned}$$

where we used the fact $\xi\cdot\eta>0$ (in (3.21)). Using (2.6), we obtain (3.14).

(3) *For the quadratic term $\mathcal{M}_V$*, by virtue of (3.19), we rewrite it in terms of V^+ and V^- as

$$\mathcal{M}_V=\sum_{\mu,\nu\in\{+,-\}}M_{\mu,\nu}(V^\mu,V^\nu),$$

where

$$M_{\mu,\nu}(V^\mu,V^\nu)=-\nu\frac{i}{4}\partial_x\big(R(V^\mu,\frac{\partial_x}{|\partial_x|}V^\nu)\big)-\mu\nu\frac{i}{8}|\partial_x|\big(R(\frac{\partial_x}{|\partial_x|}V^\mu,\frac{\partial_x}{|\partial_x|}V^\nu)\big).$$

Then we have

$$\begin{aligned}\mathcal{F}\Big(M_{\mu,\nu}(V^\mu,V^\nu)\Big)(\xi)=&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{i}{8}\Big(2\nu\xi\,\mathrm{sign}(\eta)+\mu\nu|\xi|\mathrm{sign}(\xi-\eta)\mathrm{sign}(\eta)\Big)\\&\times\sum_{j\in\mathbb{Z}}\varphi_j(|\xi-\eta|)\varphi_{[j-6,j+6]}(|\eta|)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)d\eta.\end{aligned}$$

and we obtain the symbol of $M_{\mu,\nu}(\cdot,\cdot)$ as follows

$$m_{\mu,\nu}(\xi,\eta)=\frac{i}{8}\Big(2\nu\xi\mathrm{sign}(\eta)+\mu\nu|\xi|\mathrm{sign}(\xi-\eta)\mathrm{sign}(\eta)\Big)\sum_{j\in\mathbb{Z}}\varphi_j(|\xi-\eta|)\varphi_{[j-6,j+6]}(|\eta|).$$

For fixed ξ, when $|\eta|\in(2^k,2^{k+1}]$ with $k\in\mathbb{Z}$, we have

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\varphi_j(|\xi-\eta|)\varphi_{[j-6,j+6]}(|\eta|)=\sum_{j=k-6}^{k+7}\varphi_j(|\xi-\eta|)\varphi_{[j-6,j+6]\cap[k,k+1]}(|\eta|)\\&\leq\varphi_{[k-6,k+7]}(|\xi-\eta|)\leq\varphi_{[-6,7]}\Big(\frac{|\xi-\eta|}{|\eta|}\Big),\end{aligned}$$

which implies (3.15).

(4) *For the cubic term $\mathcal{L}_V$*, we first have

$$\widehat{\mathcal{L}_V}(\xi)=\frac{i}{2\pi}\int_{\mathbb{R}}\xi\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\widehat{B}(\xi-\eta)\widehat{V}(\eta)d\eta.$$

Then there holds

$$\begin{aligned}&\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V\mid\langle\partial_x\rangle^{N_0}V\big)_2=\frac{1}{2\pi}\int_{\mathbb{R}}\langle\xi\rangle^{2N_0}\widehat{\mathcal{L}_V}(\xi)\overline{\widehat{V}(\xi)}d\xi\\&=\frac{i}{(2\pi)^2}\int_{\mathbb{R}^2}\xi\langle\xi\rangle^{2N_0}\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\widehat{B}(\xi-\eta)\widehat{V}(\eta)\overline{\widehat{V}(\xi)}d\eta d\xi\end{aligned}$$

and

$$\begin{aligned}&\overline{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V\mid\langle\partial_x\rangle^{N_0}V\big)_2}\\&=-\frac{i}{(2\pi)^2}\int_{\mathbb{R}^2}\xi\langle\xi\rangle^{2N_0}\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\eta|)\overline{\widehat{B}(\xi-\eta)}\widehat{V}(\xi)\overline{\widehat{V}(\eta)}d\eta d\xi\\&=-\frac{i}{(2\pi)^2}\int_{\mathbb{R}^2}\eta\langle\eta\rangle^{2N_0}\sum_{j\in\mathbb{Z}}\varphi_{\leq j-7}(|\xi-\eta|)\varphi_j(|\xi|)\overline{\widehat{B}(\eta-\xi)}\widehat{V}(\eta)\overline{\widehat{V}(\xi)}d\eta d\xi.\end{aligned}$$

Thanks to (3.7), we have

$$\overline{\widehat{B}(\eta-\xi)}=\widehat{B}(\xi-\eta)$$

which leads to

$$\begin{aligned}
&\mathrm{Re}\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2\big\} = \frac{1}{2}\Big(\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \overline{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2}\Big)\\
&= \frac{i}{8\pi^2}\int_{\mathbb{R}^2} l(\xi,\eta)\widehat{B}(\xi-\eta)\cdot\langle\eta\rangle^{N_0}\widehat{V}(\eta)\cdot\langle\xi\rangle^{N_0}\overline{\widehat{V}(\xi)}d\eta d\xi,
\end{aligned}$$

where

$$l(\xi,\eta) = \sum_{j\in\mathbb{Z}}\varphi_{\le j-7}(|\xi-\eta|)\big(\xi\langle\xi\rangle^{2N_0}\varphi_j(|\eta|) - \eta\langle\eta\rangle^{2N_0}\varphi_j(|\xi|)\big)\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}.$$

Using (2.6) for $l(\xi,\eta)$, there holds

$$|\xi|\sim|\eta|, \quad \xi\cdot\eta>0.$$

For fixed ξ,η, we have

$$|\varphi_j(|\xi|)-\varphi_j(|\eta|)| \lesssim \frac{1}{\min\{|\xi,|\eta||\}}|\xi-\eta| \lesssim \frac{1}{|\xi|}|\xi-\eta|.$$

Noticing that for fixed ξ,η, the summation in $l(\xi,\eta)$ is finite, we get

$$|l(\xi,\eta)|\lesssim|\xi-\eta|.$$

Then we obtain

$$\begin{aligned}
&\big|\mathrm{Re}\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2\big\}\big| \lesssim \int_{\mathbb{R}^2}|\xi-\eta||\widehat{B}(\xi-\eta)|\cdot\langle\eta\rangle^{N_0}|\widehat{V}(\eta)|\cdot\langle\xi\rangle^{N_0}|\widehat{V}(\xi)|d\eta d\xi\\
&\lesssim \|\xi\widehat{B}(\xi)\|_{L^1}\|\langle\xi\rangle^{N_0}\widehat{V}(\xi)\|_{L^2}^2 \lesssim \|\langle\xi\rangle^2\widehat{B}(\xi)\|_{L^2}\|V\|_{H^{N_0}}^2 \lesssim \|B\|_{H^2}\|V\|_{H^{N_0}}^2.
\end{aligned}$$

Thanks to (3.8), we have

$$\|B\|_{H^2} = \|B(\zeta,v)\|_{H^2} \lesssim \|\zeta\|_{L^\infty}\|v\|_{L^2}.$$

Thus we obtain

$$\big|\mathrm{Re}\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2\big\}\big| \lesssim \|\zeta\|_{L^\infty}\|v\|_{L^2}\|V\|_{H^{N_0}}^2.$$

This is (3.16).

(5) *For the cubic term $\mathcal{C}_V$*, we first have

$$\begin{aligned}
\|\mathcal{C}_V\|_{H^{N_0}} &\lesssim \big(\|\zeta\|_{W^{1,\infty}}+\|v\|_{W^{1,\infty}}\big)\|B\|_{H^{N_0+1}} + \|\partial_x\big([\partial_x^2,T_\zeta](1+\partial_x^2)^{-1}P_{\ge6}B\big)\|_{H^{N_0}}\\
&\quad+\|B\big((1+\partial_x^2)\partial_x v,B\big)\|_{H^{N_0}} + \|B\big((1+\partial_x^2)\partial_x B,v\big)\|_{H^{N_0}}\\
&\quad+\|B\big(\partial_x(\zeta v),v\big)\|_{H^{N_0}} + \|B\big(\zeta,\partial_x(|v|^2)\big)\|_{H^{N_0}}.
\end{aligned}\tag{3.23}$$

Since

$$[\partial_x^2,T_f]g = 2T_{\partial_x f}\partial_x g + T_{\partial_x^2 f}g,$$

we have

$$\begin{aligned}
\|\partial_x\big([\partial_x^2,T_\zeta](1+\partial_x^2)^{-1}P_{\ge6}B\big)\|_{H^{N_0}} &\lesssim \|T_{\partial_x\zeta}(1+\partial_x^2)^{-1}\partial_x^2P_{\ge6}B\|_{H^{N_0}}\\
&\quad+\|T_{\partial_x^2\zeta}(1+\partial_x^2)^{-1}\partial_xP_{\ge6}B\|_{H^{N_0}} + \|T_{\partial_x^3\zeta}(1+\partial_x^2)^{-1}P_{\ge6}B\|_{H^{N_0}}\\
&\lesssim \|\zeta\|_{W^{3,\infty}}\|B\|_{H^{N_0}}.
\end{aligned}$$

Thanks to (3.8), we can bound the last four terms of (3.23) by

$$\begin{aligned}
&\lesssim \|(1+\partial_x^2)\partial_x v\|_{L^\infty}\|B\|_{H^{N_0-2}} + \|(1+\partial_x^2)\partial_x B\|_{L^\infty}\|v\|_{H^{N_0-2}} + \|\partial_x(\zeta v)\|_{L^\infty}\|v\|_{H^{N_0-2}}\\
&\quad+\|\zeta\|_{L^\infty}\|\partial_x(|v|^2)\|_{H^{N_0-2}}\\
&\lesssim \|B\|_{H^{N_0}}\|v\|_{H^{N_0}} + \|\zeta\|_{W^{1,\infty}}\|v\|_{H^{N_0}}^2,
\end{aligned}$$

where we used the Sobolev inequality and the assumption $N_0\ge4$. Then we obtain

$$\|\mathcal{C}_V\|_{H^{N_0}} \lesssim \big(\|\zeta\|_{H^{N_0}}+\|v\|_{H^{N_0}}\big)\|B\|_{H^{N_0+1}} + \|\zeta\|_{W^{1,\infty}}\|v\|_{H^{N_0}}^2$$

Using (3.8) again, we have

$$\|B\|_{H^{N_0+1}} = \|B(\zeta,v)\|_{H^{N_0+1}} \lesssim \|\zeta\|_{L^\infty}\|v\|_{H^{N_0-1}}.$$

Thus, we obtain

$$\|\mathcal{C}_V\|_{H^{N_0}} \lesssim \|\zeta\|_{W^{1,\infty}}\big(\|\zeta\|_{H^{N_0}}^2+\|v\|_{H^{N_0}}^2\big).$$

This is (3.17).

(6) *For the quartic term* $\mathcal{N}_V$, using (3.8), we have

$$\|\mathcal{N}_V\|_{H^{N_0}} \lesssim \|(1+\partial_x^2)\partial_x B\|_{L^\infty}\|B\|_{H^{N_0}} + \|\partial_x B\|_{L^\infty}\|B\|_{H^{N_0}} \lesssim \|B\|_{H^{N_0}}^2.$$

Using (3.8) again for $B = B(\zeta, v)$, we obtain

$$\|\mathcal{N}_V\|_{H^{N_0}} \lesssim \|\zeta\|_{L^\infty}^2\|v\|_{H^{N_0}}^2.$$

This is (3.18). The proposition is proved. □

4. The proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. The proof relies on the continuity argument and the a priori estimates which are presented in the following subsections.

4.1. **Ansatz for the continuity argument.** Our first ansatz for the continuity argument is about the amplitude of ζ. We assume that

$$\|\zeta(t)\|_{L^\infty} \leq \frac{1}{2C_B}, \quad \text{for} \quad t \in [0, T_\varepsilon], \tag{4.1}$$

where C_B is a constant determined in Lemma 3.1. We define the energy functional

$$\mathcal{E}_{N_0}(t) = \|\zeta(t)\|_{H^{N_0}}^2 + \|v(t)\|_{H^{N_0}}^2.$$

Our second ansatz for the continuity argument is about the energy. We assume that

$$\mathcal{E}_{N_0}(t) \leq 2C_0\varepsilon^2, \quad \text{for} \quad t \in [0, T_\varepsilon], \tag{4.2}$$

where $C_0 > 1$ is a universal constant that will be determined in the end of the proof. We take

$$T_\varepsilon = \frac{C_1}{C_2}\varepsilon^{-\frac{4}{3}}, \quad C_0 = 2C_1,$$

where C_1, C_2 are constants stated in the following Proposition 4.1.

We use the standard continuity argument: since for small ε,

$$\mathcal{E}_{N_0}(0) = \varepsilon^2 < 2C_0\varepsilon^2, \quad \|\zeta(0)\|_{L^\infty} \leq \frac{1}{4C_B} < \frac{1}{2C_B},$$

the ansatz (4.1) and (4.2) hold on a short time interval $[0, t^*)$, where t^* is the maximal possible time on which (4.1) and (4.2) are correct. Without loss of generality, we assume that $T_\varepsilon = t^*$. To close the continuity argument, we need the following two steps:

Step 1. There exists a small constant $\varepsilon_0 > 0$, such that for all $\varepsilon < \varepsilon_0$, we can improve the ansatz (4.1) to

$$\|\zeta(t)\|_{L^\infty} \leq \frac{1}{4C_B}, \quad \text{for} \quad t \in [0, T_\varepsilon]. \tag{4.3}$$

Step 2. There exists a small constant $\varepsilon_0 > 0$, such that for all $\varepsilon < \varepsilon_0$, we can improve the ansatz (4.2) to

$$\mathcal{E}_{N_0}(t) \leq C_0\varepsilon^2, \quad \text{for} \quad t \in [0, T_\varepsilon]. \tag{4.4}$$

Theorem 1.1 follows from the above two steps and the local regularity theorem. To complete the above two steps, we need Proposition 4.1 in the following subsection. Thus, the rest of this section is concerned with the proof of Proposition 4.1.

4.2. **The a priori energy estimates.** The main result of this section is about the a priori estimates of (1.4)-(1.5) which is stated in the following proposition.

Proposition 4.1. *Under the ansatz* (4.1) *and* (4.2), *the solution* (ζ, v) *of* (1.4)-(1.5) *satisfies*

$$\mathcal{E}_{N_0}(t) \leq C_1\varepsilon^2 + C_2 t\varepsilon^{\frac{4}{3}} \cdot \varepsilon^2, \quad \textit{for any } t \in (0, T_\varepsilon], \tag{4.5}$$

where C_1 *and* C_2 *are two universal constants, and* $T_\varepsilon = \frac{C_1}{C_2}\varepsilon^{-\frac{4}{3}}$.

Proof. We shall divide the proof into several steps.

Step 1. The a priori energy estimate. Thanks to (3.8) and (4.1), we have

$$\|B(\zeta,\, v)\|_{H^{N_0}} \leq \frac{1}{2}\|v\|_{H^{N_0}},$$

which along with (3.2), (3.5) implies that

$$\mathcal{E}_{N_0}(t) \sim \|\zeta(t)\|^2_{H^{N_0}} + \|u(t)\|^2_{H^{N_0}} \sim \|V(t)\|^2_{H^{N_0}}, \quad \text{for} \quad t \in [0, T_\varepsilon]. \tag{4.6}$$

By virtue of (4.6), we perform the energy estimate of (3.10). First, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V(t)\|^2_{H^{N_0}} = \mathrm{Re}\big\{&\big(\langle\partial_x\rangle^{N_0}\mathcal{S}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}\mathcal{Q}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}\mathcal{R}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2\\
&+ \big(\langle\partial_x\rangle^{N_0}\mathcal{M}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}(\mathcal{C}_V + \mathcal{N}_V) \mid \langle\partial_x\rangle^{N_0}V\big)_2\big\}.\end{aligned}$$

Thanks to the estimates (3.16), (3.17) and (3.18) in Proposition 3.2, using (4.2) and (4.6), we obtain

$$\mathcal{E}_{N_0}(t) \lesssim \varepsilon^2 + |\mathrm{Re}(I + II + III + IV)| + t\varepsilon^4, \tag{4.7}$$

where

$$\begin{aligned}
I &\stackrel{\text{def}}{=} \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} s_{\mu,+}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^+}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi dt,\\
II &\stackrel{\text{def}}{=} \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} q_{\mu,-}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^-}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi dt,\\
III &\stackrel{\text{def}}{=} \sum_{\mu,\nu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} r_{\mu,\nu}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi dt,\\
IV &\stackrel{\text{def}}{=} \sum_{\mu,\nu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} m_{\mu,\nu}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi dt.
\end{aligned} \tag{4.8}$$

Step 2. The evolution equation and estimates of the profile. To estimate the quadratic terms in (4.8), we introduce the profiles f and g of V and $\langle\partial_x\rangle^{N_0}V$ as follows

$$f = e^{-it\Lambda}V \quad \text{and} \quad g = \langle\partial_x\rangle^{N_0}f.$$

Thanks to (4.6), we have

$$\mathcal{E}_{N_0}(t) \sim \|V(t)\|^2_{H^{N_0}} \sim \|f(t)\|^2_{H^{N_0}} = \|g(t)\|^2_{L^2}. \tag{4.9}$$

By virtue of the definition of f and the equation (3.6), we have

$$\partial_t f = e^{-it\Lambda}\Big(-\partial_x(T_v V) + \frac{i}{2}|\partial_x|(T_\zeta V) + N_\zeta + i\frac{\partial_x}{|\partial_x|}N_u\Big). \tag{4.10}$$

Notice that the r.h.s of (4.10) consists in quadratic terms and higher order terms.

To bound $\partial_t f$, we have to investigate the expressions of N_ζ and N_u. Thanks to (3.9) and (2.6), there holds

$$\mathrm{supp}\,\widehat{B(\cdot,\cdot)}(\xi) \subset \{\xi \in \mathbb{R} \mid |\xi| \geq 2^5\},$$

which along with the expressions of N_ζ and N_u shows that

$$P_{\leq 0}N_\zeta = -\partial_x P_{\leq 0}\big(\frac{1}{2}T_\zeta P_{\leq 5}v + R(\zeta, v)\big), \quad P_{\leq 0}N_u = \frac{1}{2}\partial_x P_{\leq 0}\big(T_\zeta P_{\leq 5}\zeta - R(v, v)\big).$$

Then we have

$$\|\frac{1}{|\partial_x|}P_{\leq 0}\partial_t f\|_{L^2} \lesssim \big(\|v\|_{L^\infty} + \|\zeta\|_{L^\infty}\big)\big(\|v\|_{L^2} + \|\zeta\|_{L^2} + \|V\|_{L^2}\big).$$

Whereas the expressions of N_ζ and N_u give rise to

$$\begin{aligned}\|\frac{1}{|\partial_x|}P_{\geq 1}\partial_t f\|_{H^{N_0}} \lesssim \big(\|v\|_{L^\infty} + \|\zeta\|_{W^{3,\infty}}\big)\big(&\|V\|_{H^{N_0}} + \|v\|_{H^{N_0}} + \|\zeta\|_{H^{N_0}} + \|u\|_{H^{N_0}}\\
&+ \|B(\zeta, v)\|_{H^{N_0}}\big) + \|B(\partial_t\zeta, v)\|_{H^{N_0-1}} + \|B\big(\zeta, \partial_x(|v|^2)\big)\|_{H^{N_0-1}}.\end{aligned}$$

The first equation of (1.4) shows that

$$\partial_t\zeta = -(1 + \partial_x^2)\partial_x v - \partial_x(\zeta v).$$

Thanks to (3.8) and (4.6), we obtain

$$\|\frac{1}{|\partial_x|}\partial_t f\|_{H^{N_0}} \lesssim \big(\|v\|_{W^{3,\infty}} + \|\zeta\|_{W^{3,\infty}}\big)\big(\|V\|_{H^{N_0}} + \|V\|_{H^{N_0}}^2\big) \lesssim \varepsilon^2 + \varepsilon^3 \lesssim \varepsilon^2. \tag{4.11}$$

Step 3. Estimate for $\mathrm{Re}(I)$. In this step, we shall prove

$$|\mathrm{Re}(I)| \lesssim \varepsilon^2 + t\varepsilon^{\frac{4}{3}} \cdot \varepsilon^2. \tag{4.12}$$

By the expression of I, using (3.11), we have

$$\bar{I} = -\sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} s_{\mu,+}(\xi,\eta)\overline{\widehat{V^\mu}(\xi-\eta)}\,\overline{\widehat{V^+}(\eta)}\widehat{V^+}(\xi)d\eta d\xi dt$$

Noticing that

$$\overline{\widehat{V^\mu}(\xi)} = \widehat{V^{-\mu}}(-\xi),$$

we have

$$\begin{aligned}\bar{I} &= -\sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} s_{\mu,+}(\xi,\eta)\widehat{V^{-\mu}}(\eta-\xi)\,\overline{\widehat{V^+}(\eta)}\widehat{V^+}(\xi)d\eta d\xi dt\\ &= -\sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\eta\rangle^{2N_0} s_{\mu,+}(\eta,\xi)\widehat{V^{-\mu}}(\xi-\eta)\widehat{V^+}(\eta)\overline{\widehat{V^+}(\xi)}d\eta d\xi dt.\end{aligned}$$

Since $\mathrm{Re}(I) = \frac{1}{2}(I+\bar{I})$, we obtain

$$\mathrm{Re}(I) = \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \tilde{s}_{\mu,+}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\cdot\langle\eta\rangle^{N_0}\widehat{V^+}(\eta)\cdot\langle\xi\rangle^{N_0}\widehat{V^-}(-\xi)d\eta d\xi dt,$$

where

$$\tilde{s}_{\mu,+}(\xi,\eta) = \langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s_{\mu,+}(\xi,\eta) - \langle\eta\rangle^{2N_0}s_{-\mu,+}(\eta,\xi)\big).$$

Thanks to (3.12), we have

$$\mathrm{supp}\,\tilde{s}_{\mu,+} \subset \mathbb{S} \stackrel{\mathrm{def}}{=} \{(\xi,\eta)\in\mathbb{R}^2 \mid |\xi-\eta| \le 2^{-5}\max\{|\xi|,|\eta|\}\}. \tag{4.13}$$

and

$$|\tilde{s}_{\mu,+}(\xi,\eta)| \lesssim |\xi-\eta|\cdot 1_{\mathbb{S}}(\xi,\eta). \tag{4.14}$$

For simplicity, we denote

$$\mathfrak{S}_\mu(\xi,\eta) \stackrel{\mathrm{def}}{=} \tilde{s}_{\mu,+}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\cdot\langle\eta\rangle^{N_0}\widehat{V^+}(\eta)\cdot\langle\xi\rangle^{N_0}\widehat{V^-}(-\xi).$$

To estimate $\mathrm{Re}(I)$, we rewrite $\mathfrak{S}_\mu(\xi,\eta)$ in terms of profiles $f^\pm, g^\pm$ as follows

$$\mathfrak{S}_\mu(\xi,\eta) = e^{it\Phi_{\mu,+}(\xi,\eta)}\tilde{s}_{\mu,+}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\cdot\widehat{g^+}(\eta)\cdot\widehat{g^-}(-\xi). \tag{4.15}$$

Thanks to Lemma 2.1, we have

$$\Phi_{-,+}(\xi,\eta) = -\Phi_{+,+}(\eta,\xi).$$

Then the estimate of $\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_-(\xi,\eta)d\eta d\xi dt$ is similar to $\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+(\xi,\eta)d\eta d\xi dt$. We only derive the estimate for $\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+(\xi,\eta)d\eta d\xi dt$.

By the expression of $\Phi_{+,+}(\xi,\eta)$, we divide $\mathfrak{S}_+(\xi,\eta)$ into two parts as follows

$$\mathfrak{S}_+(\xi,\eta) = \underbrace{\mathfrak{S}_+(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta>0}}_{\mathfrak{S}_+^{>0}(\xi,\eta)} + \underbrace{\mathfrak{S}_+(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}}_{\mathfrak{S}_+^{<0}(\xi,\eta)}.$$

Step 3.1. Estimate for the integral of $\mathfrak{S}_+^{>0}(\xi,\eta)$. For $(\xi-\eta)\cdot\eta>0$, Lemma 2.1 shows that

$$\Phi_{+,+}(\xi,\eta) = 3|\xi||\xi-\eta||\eta|.$$

Now we split the integral of $\mathfrak{S}_+^{>0}(\xi,\eta)$ into two parts which correspond to high and low frequencies respectively, i.e.,

$$|\xi| \ge 2^{-D-1} \quad \text{and} \quad |\xi| \le 2^{-D},$$

where $D\in\mathbb{N}$ is a large number that will be determined later on.

(1). For $|\xi|\geq 2^{-D-1}$, using (3.21) and (4.14), we have

$$\Big|\frac{\tilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}\Big|\lesssim\frac{1}{|\xi||\eta|}\sim\frac{1}{|\xi|^2}. \tag{4.16}$$

Using (4.15), we have

$$\mathfrak{S}^{>0}_{+}(\xi,\eta)=\frac{\tilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}\frac{d}{dt}e^{it\Phi_{\mu,+}(\xi,\eta)}\widehat{f^{+}}(\xi-\eta)\cdot\widehat{g^{+}}(\eta)\cdot\widehat{g^{-}}(-\xi)\cdot 1_{(\xi-\eta)\cdot\eta>0}.$$

Integrating by parts with respect to t, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}^{>0}_{+}(\xi,\eta)\varphi_{\geq -D}(|\xi|)d\eta d\xi dt\\ &=\underbrace{\int_{\mathbb{R}^2}\frac{\tilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}e^{i\tau\Phi_{\mu,+}(\xi,\eta)}\widehat{f^{+}}(\tau,\xi-\eta)\cdot\widehat{g^{+}}(\tau,\eta)\cdot\widehat{g^{-}}(\tau,-\xi)\cdot\varphi_{\geq -D}(|\xi|)\cdot 1_{(\xi-\eta)\cdot\eta>0}d\eta d\xi\Big|_{\tau=0}^{t}}_{A_1}\\ &\quad\underbrace{-\int_0^t\int_{\mathbb{R}^2}\frac{\tilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}e^{i\tau\Phi_{\mu,+}(\xi,\eta)}\partial_t\Big(\widehat{f^{+}}(\xi-\eta)\cdot\widehat{g^{+}}(\eta)\cdot\widehat{g^{-}}(-\xi)\Big)\cdot\varphi_{\geq -D}(|\xi|)1_{(\xi-\eta)\cdot\eta>0}d\eta d\xi d\tau}_{A_2}.\end{aligned}$$

Thanks to (3.21) and (4.16), we have

$$\begin{aligned}|A_1|&\lesssim\sum_{\tau\in\{0,t\}}\int_{\mathbb{S}}\frac{1}{|\eta|^2}|\widehat{f}(\tau,\xi-\eta)|\cdot|\widehat{g}(\tau,\eta)|\cdot|\widehat{g}(\tau,-\xi)|\varphi_{\geq -D}(|\xi|)d\eta d\xi\\ &\lesssim\sum_{\tau\in\{0,t\}}\|\widehat{f}(\tau,\xi)\|_{L^2}\|\frac{1}{|\xi|^2}\widehat{g}(\tau,\xi)\|_{L^1(|\xi|\geq 2^{-D-2})}\|\widehat{g}(\tau,\xi)\|_{L^2}\\ &\lesssim 2^{\frac{3}{2}D}\big(\|f(0)\|_{L^2}\|g(0)\|_{L^2}^2+\|f(t)\|_{L^2}\|g(t)\|_{L^2}^2\big),\end{aligned}$$

where we used the following formula in the last inequality

$$\|\frac{1}{|\xi|^r}\widehat{g}(\tau,\xi)\|_{L^1(|\xi|\geq 2^{-D-2})}\lesssim 2^{(r-\frac{1}{2})D}\|\widehat{g}(\tau,\xi)\|_{L^2},\quad\text{for any } r>\frac{1}{2}. \tag{4.17}$$

Whereas using (3.21) and (4.16), we have

$$\begin{aligned}|A_2|&\lesssim t\sup_{(0,t)}\int_{\mathbb{S}}\frac{1}{|\xi|^2}\Big(|\partial_t\widehat{f^{+}}(\xi-\eta)|\cdot|\widehat{g^{+}}(\eta)|\cdot|\widehat{g^{-}}(-\xi)|+|\widehat{f^{+}}(\xi-\eta)|\cdot|\partial_t\big(\widehat{g^{+}}(\eta)\widehat{g^{-}}(-\xi)\big)|\Big)\varphi_{\geq -D}(|\xi|)d\eta d\xi\\ &\lesssim t\sup_{(0,t)}\big(\|\frac{1}{|\xi|}\partial_t\widehat{f}(\xi)\|_{L^2}\|\frac{1}{|\xi|}\widehat{g}(\xi)\|_{L^1(|\xi|\geq 2^{-D-2})}\|\widehat{g}\|_{L^2}+\|\widehat{f}\|_{L^2}\|\frac{1}{|\xi|}\partial_t\widehat{g}(\xi)\|_{L^2}\|\frac{1}{|\xi|}\widehat{g}\|_{L^1(|\xi|\geq 2^{-D-2})}\big)\\ &\lesssim 2^{\frac{1}{2}D}t\sup_{(0,t)}\big(\|\frac{1}{|\partial_x|}\partial_t f\|_{L^2}\|g\|_{L^2}^2+\|f\|_{L^2}\|\frac{1}{|\partial_x|}\partial_t g\|_{L^2}\|g\|_{L^2}\big),\end{aligned}$$

where we used (4.17) in the last inequality.

Thanks to (4.2), (4.9) and (4.11), noticing that $g=\langle\partial_x\rangle^{N_0}f$, we have

$$\Big|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}^{>0}_{+}(\xi,\eta)\varphi_{\geq -D}(|\xi|)d\eta d\xi dt\Big|\lesssim 2^{\frac{3}{2}D}\varepsilon^3+2^{\frac{1}{2}D}t\varepsilon^4. \tag{4.18}$$

(2). For $|\xi|<2^{-D}$, we have $|\eta|<2^{-D+1}$ for any $(\xi,\eta)\in\mathbb{S}$. Using (4.14) and (4.15), we have

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}^{>0}_{+}(\xi,\eta)\varphi_{\leq -D-1}(|\xi|)d\eta d\xi dt\Big|\\ &\lesssim t\sup_{(0,t)}\int_{-2^{-D}}^{2^{-D}}\int_{-2^{-D+1}}^{2^{-D+1}}|\xi-\eta||\widehat{f}(\xi-\eta)|\cdot|\widehat{g}(\eta)|\cdot|\widehat{g}(\xi)|d\eta d\xi\\ &\lesssim t\sup_{(0,t)}\|\xi\widehat{f}(\xi)\|_{L^2}\|\widehat{g}(\xi)\|_{L^1(|\xi|<2^{-D+1})}\|\widehat{g}\|_{L^2}\lesssim 2^{-\frac{1}{2}D}t\sup_{(0,t)}\|\partial_x f\|_{L^2}\|g\|_{L^2}^2,\end{aligned}$$

which along with (4.2) and (4.9) implies that

$$\Big|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}^{>0}_{+}(\xi,\eta)\varphi_{\leq -D-1}(|\xi|)d\eta d\xi dt\Big|\lesssim 2^{-\frac{1}{2}D}t\varepsilon^3. \tag{4.19}$$

Taking $D=[\log_2\varepsilon^{-\frac{2}{3}}]$ (i.e, $2^D\sim\varepsilon^{-\frac{2}{3}}$) in (4.18) and (4.19), we have

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+^{>0}(\xi,\eta)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2. \tag{4.20}$$

Here the notation $[x]$ means the largest integer that does not exceed x.

Step 3.2. Estimate for the integral of $\mathfrak{S}_+^{<0}(\xi,\eta)$. For $(\xi,\eta)\in\mathbb{S}$ with $(\xi-\eta)\cdot\eta<0$, there holds

$$\xi\cdot\eta>0,\quad \frac{31}{32}|\eta|\le|\xi|<|\eta|. \tag{4.21}$$

Lemma 2.1 yields

$$\Phi_{+,+}(\xi,\eta)=-\frac{1}{2}|\xi-\eta|\big(3|\xi|^2+3|\eta|^2+|\xi-\eta|^2-4\big)=-2|\xi-\eta|\phi_{+,+}(\xi,\eta), \tag{4.22}$$

where

$$\phi_{+,+}(\xi,\eta)=\xi^2+\eta^2-\frac{1}{2}\xi\cdot\eta-1.$$

Now, we split the frequencies space into three parts as follows:

(1). For high frequencies $|\eta|\ge 2^5$ and low frequencies $|\eta|\le\frac{1}{2}$, using (3.12), we have

$$|\Phi_{+,+}(\xi,\eta)|\sim|\xi-\eta||\eta|^2\quad\text{and}\quad|\frac{\widetilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}|\lesssim\frac{1}{|\eta|^2},\quad\text{for}\quad|\eta|\ge 2^5,$$
$$|\Phi_{+,+}(\xi,\eta)|\sim|\xi-\eta|\quad\text{and}\quad|\frac{\widetilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}|\lesssim 1,\quad\text{for}\quad|\eta|\le\frac{1}{2}.$$

Similarly as in the derivation of (4.18), integrating by parts with respect to t, we have

$$|\int_0^t\int_{\mathbb{S}}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\big(\varphi_{\le -2}(|\eta|)+\varphi_{\ge 6}(|\eta|)\big)d\eta d\xi dt|\lesssim\varepsilon^3+t\varepsilon^4. \tag{4.23}$$

(2). For moderate frequencies with large modulation of $\phi_{+,+}(\xi,\eta)$, i.e.,

$$|\eta|\in[\frac{1}{4},\,2^6]\quad\text{and}\quad|\phi_{+,+}(\xi,\eta)|\ge 2^{-D-1},$$

using (4.14) and (4.22), we have

$$|\frac{\widetilde{s}_{+,+}(\xi,\eta)}{i\Phi_{+,+}(\xi,\eta)}|\lesssim\frac{1}{|\phi_{+,+}(\xi,\eta)|}\lesssim 2^D.$$

Following a similar argument as for (4.18), integrating by parts with respect to t, noticing that the integral set is bounded, we have

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)\varphi_{\ge -D}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^D\varepsilon^3+2^Dt\varepsilon^4. \tag{4.24}$$

(3). For moderate frequencies with small modulation of $\phi_{+,+}(\xi,\eta)$, i.e.,

$$|\eta|\in[\frac{1}{4},\,2^6]\quad\text{and}\quad|\phi_{+,+}(\xi,\eta)|\le 2^{-D},$$

using (4.21), we only consider the integral over the set

$$\mathbb{S}_+=\{(\xi,\eta)\in\mathbb{R}^2\mid\eta\in[\frac{1}{4},2^6],\,\frac{31}{32}\eta\le\xi<\eta\},$$

since the integral over the set

$$\mathbb{S}_-=\{(\xi,\eta)\in\mathbb{R}^2\mid\eta\in[-2^6,-\frac{1}{4}],\,\eta<\xi\le\frac{31}{32}\eta\},$$

could be estimated in a similar way.

Introducing the coordinate transformation on $\mathbb{S}_+$ as follows

$$\Psi:\mathbb{S}_+\to\widetilde{\mathbb{S}}_+\subset\mathbb{R}^2,$$
$$(\xi,\eta)\mapsto(\tilde{\xi},\eta)=(\phi_{+,+}(\xi,\eta),\eta),$$

we have

$$\det\big(\frac{\partial\Psi(\xi,\eta)}{\partial(\xi,\eta)}\big)=\frac{\partial\phi_{+,+}(\xi,\eta)}{\partial\xi}=2\xi-\frac{1}{2}\eta\sim\eta\sim 1, \tag{4.25}$$

which implies that Ψ is invertible and we denote by

$$(\xi,\eta)=\Psi^{-1}(\tilde{\xi},\eta).$$

Changing variables (ξ,η) to $(\tilde{\xi},\eta)$, using (4.25), we have

$$\begin{aligned}
&|\int_0^t\int_{\mathbb{S}_+}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\\
&\lesssim t\sup_{(0,t)}\int_{\frac14}^{2^6}\int_{-2^{-D}}^{2^{-D}}\big(|\widehat{f}(\xi-\eta)|\cdot|\widehat{g}(\xi)|\cdot 1_{\mathbb{S}_+}(\xi,\eta)\big)|_{(\xi,\eta)=\Psi^{-1}(\tilde{\xi},\eta)}\cdot|\widehat{g}(\eta)|d\tilde{\xi}d\eta\\
&\lesssim 2^{-\frac12 D}t\sup_{(0,t)}\|\widehat{g}(\eta)\|_{L^2}\Big(\int_{\frac14}^{2^6}\int_{-2^{-D}}^{2^{-D}}\big(|\widehat{f}(\xi-\eta)|^2\cdot|\widehat{g}(\xi)|^2\cdot 1_{\mathbb{S}_+}(\xi,\eta)\big)|_{(\xi,\eta)=\Psi^{-1}(\tilde{\xi},\eta)}d\tilde{\xi}d\eta\Big)^{\frac12}.
\end{aligned}$$

Then changing variables $(\tilde{\xi},\eta)$ to (ξ,η), using (4.25), we obtain

$$|\int_0^t\int_{\mathbb{S}_+}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^{-\frac12 D}t\sup_{(0,t)}\|f\|_{L^2}\|g\|_{L^2}^2,$$

which along with (4.2) and (4.9) implies that

$$|\int_0^t\int_{\mathbb{S}_+}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^{-\frac12 D}t\varepsilon^3. \tag{4.26}$$

The same estimate holds for $|\int_0^t\int_{\mathbb{S}_-}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|$.

Taking $D=[\log_2\varepsilon^{-\frac23}]$ (i.e., $2^D\sim\varepsilon^{-\frac23}$) in (4.24) and (4.26), we obtain

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+^{<0}(\xi,\eta)\cdot\varphi_{[-1,5]}(|\eta|)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac43}\cdot\varepsilon^2. \tag{4.27}$$

Thanks to (4.23) and (4.27), we obtain

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+^{<0}(\xi,\eta)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac43}\cdot\varepsilon^2. \tag{4.28}$$

Step 3.3. Estimate for Re(I). Combining (4.20) and (4.28), we get

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+(\xi,\eta)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac43}\cdot\varepsilon^2.$$

The same estimate holds for $\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_-(\xi,\eta)d\eta d\xi dt$. Then we obtain

$$|\mathrm{Re}(I)|\leq|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_+(\xi,\eta)d\eta d\xi dt|+|\int_0^t\int_{\mathbb{R}^2}\mathfrak{S}_-(\xi,\eta)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac43}\cdot\varepsilon^2,$$

This is exactly (4.12).

Step 4. Estimate for Re(II). In this step, we will prove

$$|\mathrm{Re}(II)|\lesssim\varepsilon^2+t\varepsilon^{\frac43}\cdot\varepsilon^2, \tag{4.29}$$

By the expression of II, denoting by

$$\mathfrak{Q}_{\mu,-}(\xi,\eta)\stackrel{\text{def}}{=}\tilde{q}_{\mu,-}(\xi,\eta)\cdot\widehat{V^\mu}(\xi-\eta)\cdot\langle\eta\rangle^{N_0}\widehat{V^-}(\eta)\cdot\langle\xi\rangle^{N_0}\overline{\widehat{V^+}(\xi)},$$

with

$$\tilde{q}_{\mu,-}(\xi,\eta)=\langle\eta\rangle^{-N_0}\langle\xi\rangle^{N_0}q_{\mu,-}(\xi,\eta),$$

we have

$$II=\sum_{\mu\in\{+,-\}}\int_0^t\int_{\mathbb{R}^2}\mathfrak{Q}_{\mu,-}(\xi,\eta)d\eta d\xi dt.$$

Step 4.1. Estimate for $\sum_{\mu\in\{+,-\}}\int_0^t\int_{\mathbb{R}^2}\mathfrak{Q}_{\mu,-}(\xi,\eta)d\eta d\xi dt$. Now, we rewrite $\mathfrak{Q}_{\mu,-}(\xi,\eta)$ in terms of profiles as follows

$$\mathfrak{Q}_{\mu,-}(\xi,\eta)=e^{it\Phi_{\mu,-}(\xi,\eta)}\tilde{q}_{\mu,-}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{g^-}(\eta)\widehat{g^-}(-\xi).$$

Thanks to (3.13), we have

$$|\widetilde{q}_{\mu,-}(\xi,\eta)| \lesssim |\xi| \cdot \varphi_{\leq 5}(|\eta|) \cdot \varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big). \tag{4.30}$$

Lemma 2.1 and the fact $\xi \cdot \eta > 0$ (in (3.21)) yield

$$\Phi_{+,-}(\xi,\eta) = \begin{cases} \frac{1}{2}|\eta|\big(3|\xi-\eta|^2 + 3|\xi|^2 + |\eta|^2 - 4\big) & \text{if} \quad |\xi| > |\eta|, \\ \frac{1}{2}|\xi|\big(3|\xi-\eta|^2 + 3|\eta|^2 + |\xi|^2 - 4\big), & \text{if} \quad |\xi| < |\eta|. \end{cases}$$

Then there hold

$$\Phi_{-,-}(\xi,\eta) = \Phi_{+,-}(\eta,\xi), \quad \text{and} \quad \Phi_{+,-}(\xi,\eta) = \Phi_{+,-}(\eta,\xi). \tag{4.31}$$

Due to (4.31), we only need to estimate the integral of $\mathfrak{Q}_{+,-}(\xi,\eta)$ over the set with restriction $|\xi| > |\eta|$.

For $|\xi| > |\eta|$, we have

$$\Phi_{+,-}(\xi,\eta) = 2|\eta|\phi_{+,-}(\xi,\eta) \quad \text{with} \quad \phi_{+,-}(\xi,\eta) = \eta^2 - \frac{3}{2}\xi\eta + \frac{3}{2}\xi^2 - 1.$$

A similar argument as in Step 3.2 leads to

$$|\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{+,-}(\xi,\eta) 1_{|\xi|>|\eta|} \varphi_{\leq -2}(|\eta|) d\eta d\xi dt| \lesssim \varepsilon^3 + t\varepsilon^4, \tag{4.32}$$

$$|\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{+,-}(\xi,\eta) 1_{|\xi|>|\eta|} \varphi_{[-1,5]}(|\eta|) \varphi_{\geq -D}(\phi_{+,-}(\xi,\eta)) d\eta d\xi dt| \lesssim 2^D\varepsilon^3 + 2^D t\varepsilon^4, \tag{4.33}$$

$$|\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{+,-}(\xi,\eta) 1_{|\xi|>|\eta|} \varphi_{[-1,5]}(|\eta|) \varphi_{\leq -D-1}(\phi_{+,-}(\xi,\eta)) d\eta d\xi dt| \lesssim 2^{-\frac{1}{2}D} t\varepsilon^3, \tag{4.34}$$

where $D \in \mathbb{N}$ need to be determined later on. Here we only verify (4.34). Indeed, since $|\xi-\eta| \leq 2^{-5}|\eta|$, we only consider the integral over set

$$\mathbb{S}_> = \{(\xi,\eta) \in \mathbb{R}^2 \mid \eta \in [\frac{1}{4}, 2^6],\, \eta < \xi \leq \frac{33}{32}\eta\},$$

since the same estimate will also hold for the integral over set

$$\mathbb{S}_< = \{(\xi,\eta) \in \mathbb{R}^2 \mid \eta \in [-2^6, -\frac{1}{4}],\, \frac{33}{32}\eta \leq \xi < \eta\}.$$

Introducing the coordinates transformation on $\mathbb{S}_>$ as follows

$$\begin{aligned} \Psi_> : \mathbb{S}_> &\to \widetilde{\mathbb{S}}_> \subset \mathbb{R}^2, \\ (\xi,\eta) &\mapsto (\xi,\tilde{\eta}) = (\xi, \phi_{+,-}(\xi,\eta)), \end{aligned}$$

we have

$$\det\big(\frac{\partial \Psi_>(\xi,\eta)}{\partial(\xi,\eta)}\big) = \frac{\partial \phi_{+,-}(\eta)}{\partial \eta} = 2\eta - \frac{3}{2}\xi \sim \eta \sim 1, \tag{4.35}$$

which implies that $\Psi_>$ is invertible. With (4.35), following the similar derivation of (4.26), we obtain (4.34).

Taking $D = [\log_2 \varepsilon^{-\frac{2}{3}}]$ (i.e., $2^D \sim \varepsilon^{-\frac{2}{3}}$) in (4.33) and (4.34), using (4.32), (4.33) and (4.34), we get

$$|\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{+,-}(\xi,\eta) \cdot 1_{(|\xi|>|\eta|)} d\eta d\xi dt| \lesssim \varepsilon^2 + t\varepsilon^{\frac{4}{3}} \cdot \varepsilon^2.$$

The same estimate hold for $\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{+,-}(\xi,\eta) \cdot 1_{(|\xi|<|\eta|)} d\eta d\xi dt$ and $\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{-,-}(\xi,\eta) d\eta d\xi dt$. We finally obtain

$$|\sum_{\mu \in \{+,-\}} \int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}_{\mu,-}(\xi,\eta) d\eta d\xi dt| \lesssim \varepsilon^2 + t\varepsilon^{\frac{4}{3}} \cdot \varepsilon^2.$$

This is (4.29).

Step 5. Estimate for Re(III). Firstly, we rewrite III in terms of the profiles as follows

$$III = \sum_{\mu,\nu \in \{+,-\}} \int_0^t \int_{\mathbb{R}^2} e^{it\Phi_{\mu,\nu}(\xi,\eta)} \widetilde{r}_{\mu,\nu}(\xi,\eta) \widehat{f^\mu}(\xi-\eta) \widehat{g^\nu}(\eta) \widehat{g^-}(-\xi) d\eta d\xi dt,$$

where

$$\widetilde{r}_{\mu,\nu}(\xi,\eta)=\langle\xi\rangle^{N_0}\langle\eta\rangle^{-N_0}r_{\mu,\nu}(\xi,\eta).$$

Thanks to (3.14), we have

$$|\widetilde{r}_{\mu,\nu}(\xi,\eta)|\lesssim|\xi-\eta|\cdot\varphi_{\geq 6}(|\eta|)\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big),$$

where we used $|\xi|\sim|\eta|$ which is stated in (3.21). After similar derivations as in Step 3 and Step 4, we obtain

$$\big|\mathrm{Re}(III)\big|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2. \tag{4.36}$$

Step 6. Estimate for Re(IV). In this step, we shall prove

$$|\mathrm{Re}(IV)|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2. \tag{4.37}$$

First, denoting by

$$\mathfrak{M}_{\mu,\nu}(\xi,\eta)\overset{\mathrm{def}}{=}\langle\xi\rangle^{2N_0}m_{\mu,\nu}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^\nu}(\eta)\overline{\widehat{V^+}(\xi)},$$

we have

$$IV=\sum_{\mu,\nu\in\{+,-\}}\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{\mu,\nu}(\xi,\eta)d\eta d\xi dt.$$

Thanks to (3.15), we have

$$\mathrm{supp}\,\mathfrak{M}_{\mu,\nu}\subset\{(\xi,\eta)\in\mathbb{R}^2\ |\,2^{-7}|\eta|\leq|\xi-\eta|\leq 2^8|\eta|\},$$

i.e., for any $(\xi,\eta)\in\mathrm{supp}\,\mathfrak{M}_{\mu,\nu}$,

$$|\xi-\eta|\sim|\eta|,\quad |\xi|\lesssim|\eta|. \tag{4.38}$$

By the definitions of the profiles, we rewrite $\mathfrak{M}_{\mu,\nu}(\xi,\eta)$ to

$$\mathfrak{M}_{\mu,\nu}(\xi,\eta)=e^{it\Phi_{\mu,\nu}(\xi,\eta)}\widetilde{m}_{\mu,\nu}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\widehat{g^\nu}(\eta)\overline{\widehat{g^+}(\xi)},$$

where

$$\widetilde{m}_{\mu,\nu}(\xi,\eta)=\langle\xi\rangle^{N_0}\langle\eta\rangle^{-N_0}m_{\mu,\nu}(\xi,\eta).$$

Due to (3.15) and (4.38), we have

$$|\widetilde{m}_{\mu,\nu}(\xi,\eta)|\lesssim|\xi|. \tag{4.39}$$

Thanks to Lemma 2.1, we shall only derive the estimates for the integral of $\mathfrak{M}_{+,+}(\xi,\eta)$.

Step 6.1. The integral over the set with $(\xi-\eta)\cdot\eta>0$. For $(\xi-\eta)\cdot\eta>0$, Lemma 2.1 yields

$$\Phi_{+,+}(\xi,\eta)=3|\xi||\xi-\eta||\eta|,$$

which along with (4.39) shows

$$\Big|\frac{\widetilde{m}_{+,+}(\xi,\eta)}{\Phi_{+,+}(\xi,\eta)}\Big|\lesssim\frac{1}{|\xi-\eta||\eta|}.$$

Similarly as the derivation of (4.20), using (4.38), we have

$$\Big|\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta>0}d\eta d\xi\Big|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2. \tag{4.40}$$

Step 6.2. The integral over the set with $(\xi-\eta)\cdot\eta<0$. For $(\xi-\eta)\cdot\eta<0$, Lemma 2.1 shows

$$\begin{aligned}&\Phi_{+,+}(\xi,\eta)=-\frac{1}{2}\min\{|\xi-\eta|,|\eta|\}\phi_{+,+}(\xi,\eta)\\ \text{with}\quad&\phi_{+,+}(\xi,\eta)=3|\xi|^2+3\max\{|\xi-\eta|^2,|\eta|^2\}+\min\{|\xi-\eta|^2,|\eta|^2\}-4.\end{aligned} \tag{4.41}$$

Now we split the frequency space into three parts as follows:

(1). For high frequencies $|\eta|>4$ *and low frequency* $|\eta|<2^{-9}$, using the fact that $|\xi-\eta|\in[2^{-7}|\eta|,\,2^8|\eta|]$ and (4.39), we have

$$|\Phi_{+,+}(\xi,\eta)|\sim|\eta|^3\quad\text{and}\quad\Big|\frac{\widetilde{m}_{+,+}(\xi,\eta)}{\Phi_{+,+}(\xi,\eta)}\Big|\lesssim\frac{1}{|\eta|^2},\quad\text{if}\quad|\eta|>4,$$

$$|\Phi_{+,+}(\xi,\eta)|\sim|\eta|\quad\text{and}\quad\Big|\frac{\widetilde{m}_{+,+}(\xi,\eta)}{\Phi_{+,+}(\xi,\eta)}\Big|\lesssim 1,\quad\text{if}\quad|\eta|<2^{-9}.$$

Following similar derivation as (4.18), using (4.38), we have

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}\cdot\big(\varphi_{\leq -10}(|\eta|)+\varphi_{\geq 3}(|\eta|)\big)d\eta d\xi dt|\lesssim\varepsilon^3+t\varepsilon^4. \tag{4.42}$$

(2). For moderate frequencies with large modulation of $\phi_{+,+}(\xi,\eta)$, i.e.,

$$|\eta|\in[2^{-10},8]\quad\text{and}\quad|\phi_{+,+}(\xi,\eta)|\geq 2^{-D-1},$$

following similar derivation as (4.24), we have

$$|\int_0^t\int_{\mathbb{S}_1'}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}\cdot\varphi_{[-9,2]}(|\eta|)\cdot\varphi_{\geq -D}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^D\varepsilon^3+2^D t\varepsilon^4. \tag{4.43}$$

(3). For moderate frequencies with small modulation of $\phi_{+,+}(\xi,\eta)$, i.e.,

$$|\eta|\in[2^{-10},8]\quad\text{and}\quad|\phi_{+,+}(\xi,\eta)|\leq 2^{-D},$$

we divide the integral set

$$\mathbb{S}'\overset{\text{def}}{=}\{(\xi,\eta)\in\mathbb{R}^2\mid(\xi-\eta)\cdot\eta<0,\,|\eta|\in[2^{-10},8],\ |\xi-\eta|\in[2^{-7}|\eta|,2^8|\eta|]\}$$

into two sets as follows

$$\mathbb{S}'=\underbrace{\{(\xi,\eta)\in\mathbb{S}'\mid\xi\cdot\eta>0,\ |\eta|>|\xi|\}}_{\mathbb{S}_1'}\cup\underbrace{\{(\xi,\eta)\in\mathbb{S}'\mid\xi\cdot\eta<0\}}_{\mathbb{S}_2'}.$$

(i). When $(\xi,\eta)\in\mathbb{S}_1'$, we have $|\xi-\eta|=|\eta|-|\xi|$. Due to (4.41), there holds

$$\phi_{+,+}(\xi,\eta)=4\xi^2+4\eta^2-2\xi\cdot\eta-4.$$

Following similar derivation as (4.26), we have

$$|\int_0^t\int_{\mathbb{S}_1'}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}\cdot\varphi_{[-9,2]}(|\eta|)\cdot\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^{-\frac12 D}t\varepsilon^3, \tag{4.44}$$

Here we only need to verify (4.44) on set

$$\mathbb{S}_{1+}'=\{(\xi,\eta)\in\mathbb{S}_1'\mid\eta>\xi>0,\ \eta\in[2^{-10},8]\}.$$

According to the proof of (4.26), it is reduced to check that there exists an invertible coordinates transformation on $\mathbb{S}_{1+}'$. Indeed, introducing the coordinates transformation on $\mathbb{S}_{1+}'$ as follows

$$\begin{aligned}\Psi_{1+}:\ &\mathbb{S}_{1+}'\to\widetilde{\mathbb{S}}'_{1+}\subset\mathbb{R}^2,\\ &(\xi,\eta)\mapsto(\xi,\tilde\eta)=(\xi,\phi_{+,+}(\xi,\eta)),\end{aligned}$$

we have

$$\det\Big(\frac{\partial\Psi_{1+}(\xi,\eta)}{\partial(\xi,\eta)}\Big)=\frac{\partial\phi_{+,+}(\xi,\eta)}{\partial\eta}=8\eta-2\xi\sim\eta\sim 1,$$

which implies that Ψ_{1+} is invertible.

(ii). When $(\xi,\eta)\in\mathbb{S}_2'$, we have $|\xi-\eta|=|\xi|+|\eta|$. Due to (4.41), there holds

$$\phi_{+,+}(\xi,\eta)=6\xi^2+4\eta^2-6\xi\cdot\eta-4.$$

Similarly as (4.44), we have

$$|\int_0^t\int_{\mathbb{S}_2'}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}\cdot\varphi_{[-9,2]}(|\eta|)\cdot\varphi_{\leq -D-1}(\phi_{+,+}(\xi,\eta))d\eta d\xi dt|\lesssim 2^{-\frac12 D}t\varepsilon^3. \tag{4.45}$$

Here we only need to check that there exists invertible coordinates transformation on $\mathbb{S}_2'$. Since $\xi\cdot\eta<0$ for any $(\xi,\eta)\in\mathbb{S}_2'$, we only consider the set

$$\mathbb{S}_{2>}'=\{(\xi,\eta)\in\mathbb{S}_2'\mid\xi<0,\,\eta\in[2^{-10},8]\}$$

Introducing coordinates transformation on $\mathbb{S}_{2>}'$ as follows

$$\begin{aligned}\Psi_{2>}:\ &\mathbb{S}_{2>}'\to\widetilde{\mathbb{S}}'_{2>}\subset\mathbb{R}^2,\\ &(\xi,\eta)\mapsto(\xi,\tilde\eta)=(\xi,\phi_{+,+}(\xi,\eta)),\end{aligned}$$

we have

$$\det\big(\frac{\partial\Psi_{2>}(\xi,\eta)}{\partial(\xi,\eta)}\big)=\frac{\partial\phi_{+,+}(\xi,\eta)}{\partial\eta}=8\eta-6\xi.$$

Since $|\xi-\eta|=|\xi|+|\eta|\in[2^{-7}|\eta|,2^{8}|\eta|]$, we have

$$\xi\in[-(2^{8}-1)\eta,0),$$

which along with the fact $\eta\in[2^{-10},8]$ implies

$$\det\big(\frac{\partial\Psi_{2>}(\xi,\eta)}{\partial(\xi,\eta)}\big)\sim\eta\sim1.$$

Then $\Psi_{2>}$ is invertible.

Taking $D=[\log_2\varepsilon^{-\frac{2}{3}}]$ (i.e., $2^D\sim\varepsilon^{-\frac{2}{3}}$) in (4.43), (4.44) and (4.45), we obtain

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{+,+}(\xi,\eta)\cdot 1_{(\xi-\eta)\cdot\eta<0}\cdot\varphi_{[-9,2]}(|\eta|)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2, \tag{4.46}$$

Thanks to (4.40), (4.42) and (4.46), we obtain

$$|\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{+,+}(\xi,\eta)d\eta d\xi dt|\lesssim\varepsilon^2+t\varepsilon^{\frac{4}{3}}\cdot\varepsilon^2. \tag{4.47}$$

The same estimate holds for $\int_0^t\int_{\mathbb{R}^2}\mathfrak{M}_{\mu,\nu}(\xi,\eta)d\eta d\xi dt$. Then we obtain (4.37).

Combining (4.7), (4.12), (4.29), (4.36) and (4.37), we finally obtain (4.5). The Proposition is proved. □

5. The proof of Theorem 1.2

In this section, we shall sketch the proof of Theorem 1.2. Since the small parameter ϵ is considered in (1.6), we have to modify the proof of Theorem 1.1 slightly.

5.1. **Symmetrization of** (1.6)**.** Similarly as the derivation of (3.6), we firstly introduce good unknowns (ζ,u) with

$$u=v+\epsilon B^\epsilon(\zeta,v), \tag{5.1}$$

where $B^\epsilon(\cdot,\cdot)$ is a bilinear operator defined as

$$B^\epsilon(f,g)=\frac{1}{2}T_f\big((1+\epsilon\partial_x^2)^{-1}\varphi_{\geq6}(\sqrt{\epsilon}|\partial_x|)g\big).$$

Without confusion, we sometimes use B^ϵ to denote the bilinear term $B^\epsilon(\zeta,v)$. Defining

$$V=\zeta+i\frac{\partial_x}{|\partial_x|}u, \tag{5.2}$$

we get

$$\partial_tV-i\Lambda_\epsilon V+\epsilon\partial_x(T_vV)-\frac{i}{2}\epsilon|\partial_x|(T_\zeta V)=N_\zeta^\epsilon+i\frac{\partial_x}{|\partial_x|}N_u^\epsilon, \tag{5.3}$$

where $\Lambda_\epsilon=|\partial_x|(1-\epsilon|\partial_x|^2)$ and

$$\begin{aligned}N_\zeta^\epsilon=&-\frac{\epsilon}{2}\partial_x\big(T_\zeta\varphi_{\leq5}(\sqrt{\epsilon}|\partial_x|)u\big)-\frac{\epsilon^2}{2}\partial_x\big(T_\zeta\varphi_{\geq6}(\sqrt{\epsilon}|\partial_x|)B^\epsilon\big)+\epsilon^2\partial_x(T_\zeta B^\epsilon)\\&+\frac{\epsilon^2}{2}\partial_x\Big([\partial_x^2,T_\zeta](1+\epsilon\partial_x^2)^{-1}\varphi_{\geq6}(\sqrt{\epsilon}|\partial_x|)v\Big)-\epsilon\partial_x\big(R(\zeta,v)\big),\\N_u^\epsilon=&\frac{\epsilon}{2}\partial_x\big(T_\zeta\varphi_{\leq5}(\sqrt{\epsilon}|\partial_x|)\zeta\big)+\frac{\epsilon}{2}T_{\partial_x\zeta}\varphi_{\geq6}(\sqrt{\epsilon}|\partial_x|)\zeta+\epsilon^2\partial_x(T_vB^\epsilon)-\frac{\epsilon}{2}\partial_x\big(R(v,v)\big)\\&+\epsilon B^\epsilon(\partial_t\zeta,v)-\frac{\epsilon^2}{2}B^\epsilon\big(\zeta,\partial_x(|v|^2)\big),\end{aligned}$$

where we used (2.5) and the definition of $B^\epsilon(\cdot,\cdot)$. Here we used the Fourier multipliers $\varphi_{\leq k}(\cdot)$, $\varphi_{\geq k}(\cdot)$ and $\varphi_k(\cdot)$, instead of their Littlewood-Paley projection operators $P_{\leq k}$, $P_{\geq k}$ and P_k, respectively (see subsection 2.2.).

Following the proof of Lemma 3.1, for any $f\in L^\infty(\mathbb{R})$ and $g\in H^s(\mathbb{R})$ with $s\geq-2$, we have

$$\mathcal{F}\big(B^\epsilon(f,g)\big)(\xi)=\overline{\mathcal{F}\big(B^\epsilon(f,g)\big)(-\xi)} \tag{5.4}$$

and

$$\|B^\epsilon(f,g)\|_{H^{s+k}} \le C_{B^\epsilon}\epsilon^{-\frac{k}{2}}\|f\|_{L^\infty}\|g\|_{H^s}, \quad \text{for} \quad k=0,1,2, \tag{5.5}$$

where $C_{B^\epsilon}>0$ is a universal constant.

5.2. **Main proposition on the symmetric system** (5.3)**.** For (5.3), arranging the quadratic terms in terms of V^+ and V^-, we have a proposition similar to Proposition 3.2.

Proposition 5.1. *Assume that* $(\zeta,v)\in H^{N_0}(\mathbb{R})$ *with* $N_0\ge 4$ *solves* (1.6)*. Then* V *defined in* (5.2) *satisfies the following system*

$$\partial_t V - i\Lambda_\epsilon V = \mathcal{S}_V^\epsilon + \mathcal{Q}_V^\epsilon + \mathcal{L}_V^\epsilon + \mathcal{N}_V^\epsilon, \tag{5.6}$$

where

- *The quadratic term* $\mathcal{S}_V^\epsilon$ *is of the form*

$$\mathcal{S}_V^\epsilon = S_{+,+}^\epsilon(V^+,V^+) + S_{-,+}^\epsilon(V^-,V^+).$$

  And the symbol $s_{\mu,+}^\epsilon(\xi,\eta)$ *of* $S_{\mu,+}^\epsilon$ *(for* $\mu=+,-$*) satisfies*

$$\overline{s_{\mu,+}^\epsilon(\xi,\eta)} = -s_{\mu,+}^\epsilon(\xi,\eta), \tag{5.7}$$

$$|\langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s_{\mu,+}^\epsilon(\xi,\eta) - \langle\eta\rangle^{2N_0}s_{-\mu,+}^\epsilon(\eta,\xi)\big)| \lesssim \epsilon|\xi-\eta|\cdot\varphi_{\le -6}\Big(\frac{|\xi-\eta|}{\max\{|\xi|,|\eta|\}}\Big). \tag{5.8}$$

- *The quadratic term* $\mathcal{Q}_V^\epsilon$ *is of the form*

$$\mathcal{Q}_V^\epsilon = Q_{+,-}^\epsilon(V^+,V^-) + Q_{-,-}^\epsilon(V^-,V^-).$$

  And the symbol $q_{\mu,-}^\epsilon(\xi,\eta)$ *of* $Q_{\mu,-}^\epsilon$ *satisfies*

$$|q_{\mu,-}^\epsilon(\xi,\eta)| \lesssim \epsilon|\xi|\cdot\varphi_{\le 5}\big(\sqrt{\epsilon}|\eta|\big)\cdot\varphi_{\le -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big). \tag{5.9}$$

- *The cubic term* $\mathcal{L}_V^\epsilon = \epsilon^2\partial_x(T_{B^\epsilon}V)$ *satisfies*

$$\big|Re\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V^\epsilon \,|\, \langle\partial_x\rangle^{N_0}V\big)_2\big\}\big| \lesssim \epsilon^2\|\zeta\|_{L^\infty}\|v\|_{H^2}\|V\|_{H^{N_0}}^2. \tag{5.10}$$

- *The remaining nonlinear term* $\mathcal{N}_V^\epsilon$ *satisfies*

$$\|\mathcal{N}_V^\epsilon\|_{H^{N_0}} \lesssim \epsilon\big(\|\zeta\|_{W^{3,\infty}} + \|v\|_{W^{3,\infty}}\big)\big(1+\|\zeta\|_{H^{N_0}}+\|v\|_{H^{N_0}}\big)^2\big(\|\zeta\|_{H^{N_0}}+\|v\|_{H^{N_0}}\big). \tag{5.11}$$

Remark 5.2. *The terms* $\mathcal{S}_V^\epsilon$*,* $\mathcal{Q}_V^\epsilon$ *and* $\mathcal{L}_V^\epsilon$ *in* (5.6) *correspond to* $\mathcal{S}_V$*,* $\mathcal{Q}_V$ *and* $\mathcal{L}_V$ *in* (3.10) *respectively. Whereas* $\mathcal{N}_V^\epsilon$ *in* (5.6) *is corresponding to the sum* $\mathcal{R}_V+\mathcal{M}_V+\mathcal{C}_V+\mathcal{N}_V$ *in* (3.10)*.*

Remark 5.3. *Proposition 5.1 reveals that the worst term is* $\mathcal{Q}_V^\epsilon$*. Indeed,* (5.9) *hints that term* $\mathcal{Q}_V^\epsilon$ *is of order* $O(\sqrt{\epsilon})$ *if there is no loss of derivative.*

Proof of Proposition 5.1. Thanks to (5.3), rewriting (5.3) to (5.6), we have

$$\begin{aligned}
&\mathcal{S}_V^\epsilon = -\epsilon\partial_x(T_vV) + \frac{i}{2}\epsilon|\partial_x|(T_\zeta V),\\
&\mathcal{Q}_V^\epsilon = -\frac{\epsilon}{2}\partial_x\big(T_\zeta\varphi_{\le 5}(\sqrt{\epsilon}|\partial_x|)u\big) - \epsilon\frac{i}{2}|\partial_x|\big(T_\zeta\varphi_{\le 5}(\sqrt{\epsilon}|\partial_x|)\zeta\big),\\
&\mathcal{L}_V^\epsilon = \epsilon^2\partial_x(T_{B^\epsilon}V),\\
&\mathcal{N}_V^\epsilon = \big(N_\zeta^\epsilon + \frac{\epsilon}{2}\partial_x\big(T_\zeta\varphi_{\le 5}(\sqrt{\epsilon}|\partial_x|)u\big)\big) + i\frac{\partial_x}{|\partial_x|}\big(N_u^\epsilon - \frac{\epsilon}{2}\partial_x\big(T_\zeta\varphi_{\le 5}(\sqrt{\epsilon}|\partial_x|)\zeta\big)\big).
\end{aligned}$$

Thanks to (5.2), we have

$$\zeta = \frac{1}{2}(V^+ + V^-) = \frac{1}{2}\sum_{\mu\in\{+,-\}}V^\mu, \quad u = \frac{i}{2}\frac{\partial_x}{|\partial_x|}(V^+ - V^-) = \frac{i}{2}\sum_{\mu\in\{+,-\}}\mu\frac{\partial_x}{|\partial_x|}V^\mu. \tag{5.12}$$

Using (5.12), we could rewrite $\mathcal{S}_V^\epsilon$ and $\mathcal{Q}_V^\epsilon$ in terms of V^+ and V^-. They would have similar expression as $\mathcal{S}_V$ and $\mathcal{Q}_V$ in the proof of Proposition 3.2. It is easy to check that there hold (5.8) and (5.9).

Similarly as in the derivation of (3.16), using the symmetric structure of $\mathcal{L}_V^\epsilon$ and (5.4), we have

$$\big|\mathrm{Re}\big\{\big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V^\epsilon \,|\, \langle\partial_x\rangle^{N_0}V\big)_2\big\}\big| \lesssim \epsilon^2\|B^\epsilon\|_{H^2}\|V\|_{H^{N_0}}^2,$$

which along with (5.5) implies the estimate (5.10).

For the remained nonlinear term $\mathcal{N}_V^\epsilon$, similarly as in the derivation of the estimates involving $\mathcal{C}_V$ and $\mathcal{N}_V$ in the proof of Proposition 3.2, using product estimates and (5.5), we obtain (5.11). The proposition is proved. $\square$

5.3. **Main a priori estimates for** (1.6)**.** Similarly as the proof of Theorem 1.1, the proof of Theorem 1.2 also relies on the continuity argument and the a priori energy estimates. Before stating the main a priori energy estimates of (1.6), we present the ansatz for the continuity arguments.

The first ansatz is involving the amplitude of ζ as follows

$$\epsilon\|\zeta(t)\|_{L^\infty} \leq \frac{1}{2C_{B^\epsilon}}, \quad \text{for} \quad t \in [0, T_0\epsilon^{-\frac{2}{3}}]. \tag{5.13}$$

We define the energy functional for (1.6) as

$$\mathcal{E}_{N_0}(t) = \|\zeta(t)\|_{H^{N_0}}^2 + \|v(t)\|_{H^{N_0}}^2.$$

For simplicity of the proof and without loss of generality, we assume

$$\|\zeta_0\|_{H^{N_0}}^2 + \|v_0\|_{H^{N_0}}^2 = 1. \tag{5.14}$$

Our second ansatz is about the energy and reads

$$\mathcal{E}_{N_0}(t) \leq 2C_0', \quad \text{for} \quad t \in [0, T_0\epsilon^{-\frac{2}{3}}], \tag{5.15}$$

where $C_0' > 1$ is an universal constant that will be determined in the end of the proof. We take

$$T_0 = \frac{C_1'}{C_2'}, \quad C_0' = 2C_1',$$

where C_1', C_2' are constants stated in the following Proposition 5.4. Thanks to Proposition 5.4, we could improve the ansatz (5.13) and (5.15). Precisely, there exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, we improve the ansatz (5.13) and (5.15) to

$$\epsilon\|\zeta(t)\|_{L^\infty} \leq \frac{1}{4C_{B^\epsilon}}, \quad \text{for} \quad t \in [0, T_0\epsilon^{-\frac{2}{3}}]$$
$$\text{and} \quad \mathcal{E}_{N_0}(t) \leq C_0', \quad \text{for} \quad t \in [0, T_0\epsilon^{-\frac{2}{3}}].$$

Then Theorem 1.2 follows from the above argument and the local regularity theorem.

Now, we focus on the a priori energy estimate which is established in the following proposition.

Proposition 5.4. *Assume that* $0 < \epsilon < 1$ *and there holds* (5.14). *Under the ansatz* (5.13) *and* (5.15), *the solution* (ζ, v) *of* (1.6)-(1.7) *satisfies*

$$\mathcal{E}_{N_0}(t) \leq C_1' + C_2' t\epsilon^{\frac{2}{3}}, \quad \textit{for any } t \in (0, T_0\epsilon^{-\frac{2}{3}}], \tag{5.16}$$

where C_1' *and* C_2' *are two universal constants, and* $T_0 = \frac{C_1'}{C_2'}$.

Proof. We shall use the formulation (5.6) to derive the energy estimates for the Boussinesq system (1.6). Due to Proposition 5.1, standard energy estimates will give rise to a local existence theorem with time scale of $O(1/\sqrt{\epsilon})$. To enlarge the existence time, we will apply the normal forms transformation to the worst term $\mathcal{Q}_V^\epsilon$. Now we sketch the proof.

Step 1. The a priori energy estimate. Thanks to (5.5) and (5.13), we have

$$\epsilon\|B^\epsilon(\zeta, v)\|_{H^{N_0}} \leq \frac{1}{2}\|v\|_{H^{N_0}},$$

which along with (5.1) and (5.2) implies

$$\mathcal{E}_{N_0}(t) \sim \|\zeta(t)\|_{H^{N_0}}^2 + \|u(t)\|_{H^{N_0}}^2 \sim \|V(t)\|_{H^{N_0}}^2, \quad \text{for} \quad t \in [0, T_0\epsilon^{-\frac{2}{3}}] \tag{5.17}$$

By virtue of (5.17), we start the energy estimate of (5.6) as follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V(t)\|_{H^{N_0}}^2 = \mathrm{Re}\{&\big(\langle\partial_x\rangle^{N_0}\mathcal{S}_V^\epsilon \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}\mathcal{Q}_V^\epsilon \mid \langle\partial_x\rangle^{N_0}V\big)_2\\ &+ \big(\langle\partial_x\rangle^{N_0}\mathcal{L}_V^\epsilon \mid \langle\partial_x\rangle^{N_0}V\big)_2 + \big(\langle\partial_x\rangle^{N_0}\mathcal{N}_V^\epsilon \mid \langle\partial_x\rangle^{N_0}V\big)_2\}.\end{aligned}$$

Thanks to the estimates (5.10) and (5.11) in Proposition 5.1, using (5.14), (5.15) and (5.17), we obtain

$$\mathcal{E}_{N_0}(t) \lesssim 1 + |\mathrm{Re}(I)| + |\mathrm{Re}(II)| + t\varepsilon, \tag{5.18}$$

where

$$
\begin{aligned}
I &\overset{\text{def}}{=} \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} s^{\epsilon}_{\mu,+}(\xi,\eta)\widehat{V^{\mu}}(\xi-\eta)\widehat{V^{+}}(\eta)\overline{\widehat{V^{+}}(\xi)}d\eta d\xi dt,\\
II &\overset{\text{def}}{=} \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \langle\xi\rangle^{2N_0} q^{\epsilon}_{\mu,-}(\xi,\eta)\widehat{V^{\mu}}(\xi-\eta)\widehat{V^{-}}(\eta)\overline{\widehat{V^{+}}(\xi)}d\eta d\xi dt.
\end{aligned}
\tag{5.19}
$$

Step 2. Estimate for Re(I). Similarly as Step 3 in the proof of Proposition 3.2, using symmetric structure of $\mathcal{S}^{\epsilon}_V$, we have

$$
\mathrm{Re}(I) = \frac{1}{2}(I+\bar{I}) = \sum_{\mu\in\{+,-\}} \int_0^t \int_{\mathbb{R}^2} \tilde{s}^{\epsilon}_{\mu,+}(\xi,\eta)\widehat{V^{\mu}}(\xi-\eta)\cdot\langle\eta\rangle^{N_0}\widehat{V^{+}}(\eta)\cdot\langle\xi\rangle^{N_0}\widehat{V^{-}}(-\xi)d\eta d\xi dt,
$$

where

$$
\tilde{s}^{\epsilon}_{\mu,+}(\xi,\eta) = \langle\xi\rangle^{-N_0}\langle\eta\rangle^{-N_0}\big(\langle\xi\rangle^{2N_0}s^{\epsilon}_{\mu,+}(\xi,\eta) - \langle\eta\rangle^{2N_0}s^{\epsilon}_{-\mu,+}(\eta,\xi)\big).
$$

Thanks to (5.8), we have

$$
|\tilde{s}^{\epsilon}_{\mu,+}(\xi,\eta)| \lesssim \epsilon|\xi-\eta|\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{\max\{|\xi|,|\eta|\}}\Big).
$$

Then we obtain

$$
\begin{aligned}
|\mathrm{Re}(I)| &\lesssim \epsilon\int_0^t\int_{\mathbb{R}^2} |\xi-\eta||\widehat{V}(\xi-\eta)|\cdot\langle\eta\rangle^{N_0}|\widehat{V}(\eta)|\cdot\langle\xi\rangle^{N_0}|\widehat{V}(\xi)|d\eta d\xi dt\\
&\lesssim \epsilon t\sup_{(0,t)}\|\xi\widehat{V}(\xi)\|_{L^1}\cdot\|\langle\xi\rangle^{N_0}\widehat{V}(\xi)\|^2_{L^2} \lesssim \epsilon t\sup_{(0,t)}\|V\|_{H^2}\|V\|^2_{H^{N_0}}
\end{aligned}
$$

which along with (5.14), (5.15) and (5.17) implies

$$
|\mathrm{Re}(I)| \lesssim \epsilon t. \tag{5.20}
$$

Step 3. Estimate for Re(II). Due to (5.9), the direct estimate for II will lead to one derivative loss or $\sqrt{\epsilon}$ loss. To improve the estimate, we shall apply the normal form transformation to this term.

Step 4.1. The evolution equation and estimates of the profile. Firstly, we introduce the profiles f, g of V and $\langle\partial_x\rangle^{N_0}V$ as follows

$$
f = e^{-it\Lambda_\epsilon}V \quad \text{and} \quad g = \langle\partial_x\rangle^{N_0}f.
$$

Thanks to (5.17), we have

$$
\mathcal{E}_{N_0}(t) \sim \|V\|^2_{H^{N_0}} \sim \|f\|^2_{H^{N_0}} = \|g\|^2_{L^2}. \tag{5.21}
$$

Due to the equation (5.3), we have

$$
\partial_t f = e^{-it\Lambda_\epsilon}\Big(-\epsilon\partial_x(T_v V) + \frac{i}{2}\epsilon|\partial_x|(T_\zeta V) + N^{\epsilon}_{\zeta} + i\frac{\partial_x}{|\partial_x|}N^{\epsilon}_u\Big). \tag{5.22}
$$

By virtue of definition of $B^{\epsilon}(\cdot,\cdot)$, we have

$$
\mathrm{supp}\,\widehat{B^{\epsilon}(\cdot,\cdot)}(\xi) \subset \{\xi\in\mathbb{R} \mid \sqrt{\epsilon}|\xi| \geq 2^5\},
$$

which along with the expressions of N^{ϵ}_{ζ} and N^{ϵ}_u implies

$$
\begin{aligned}
\varphi_{\leq 0}(\sqrt{\epsilon}|\partial_x|)N^{\epsilon}_{\zeta} &= -\epsilon\partial_x\varphi_{\leq 0}(\sqrt{\epsilon}|\partial_x|)\Big(\frac{1}{2}T_\zeta\varphi_{\leq 5}(\sqrt{\epsilon}|\partial_x|)v + R(\zeta,v)\Big),\\
\varphi_{\leq 0}(\sqrt{\epsilon}|\partial_x|)N^{\epsilon}_u &= \frac{\epsilon}{2}\partial_x\varphi_{\leq 0}(\sqrt{\epsilon}|\partial_x|)\big(T_\zeta\varphi_{\leq 5}(\sqrt{\epsilon}|\partial_x|)\zeta - R(v,v)\big).
\end{aligned}
$$

Then we have

$$
\|\frac{1}{|\partial_x|}\varphi_{\leq 0}(\sqrt{\epsilon}|\partial_x|)\partial_t f\|_{H^{N_0}} \lesssim \epsilon\big(\|\zeta\|_{L^\infty}+\|v\|_{L^\infty}\big)\big(\|\zeta\|_{H^{N_0}}+\|v\|_{H^{N_0}}+\|V\|_{H^{N_0}}\big).
$$

Due to the expressions of N^{ϵ}_{ζ} and N^{ϵ}_u, using (5.5), we also have

$$
\begin{aligned}
\|\frac{1}{|\partial_x|}\varphi_{\geq 1}(\sqrt{\epsilon}|\partial_x|)\partial_t f\|_{H^{N_0}} &\lesssim \epsilon\big(\|v\|_{W^{3,\infty}}+\|\zeta\|_{W^{3,\infty}}\big)\big(1+\|v\|_{W^{3,\infty}}+\|\zeta\|_{W^{3,\infty}}\big)\\
&\quad\times\big(\|V\|_{H^{N_0}}+\|v\|_{H^{N_0}}+\|\zeta\|_{H^{N_0}}+\|u\|_{H^{N_0}}\big)
\end{aligned}
$$

Thanks to (5.14), (5.15) and (5.17), we obtain

$$\|\frac{1}{|\partial_x|}\partial_t f\|_{H^{N_0}} \lesssim \epsilon. \tag{5.23}$$

Step 4.2. The profiles version for II. Denoting by

$$\mathfrak{Q}^\epsilon_\mu(\xi,\eta) = \langle\xi\rangle^{2N_0} q^\epsilon_{\mu,-}(\xi,\eta)\widehat{V^\mu}(\xi-\eta)\widehat{V^-}(\eta)\overline{\widehat{V^+}(\xi)},$$

we have

$$II = \int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}^\epsilon_+(\xi,\eta)d\eta d\xi dt + \int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}^\epsilon_-(\xi,\eta)d\eta d\xi dt.$$

Now we rewrite $\mathfrak{Q}^\epsilon_\mu(\xi,\eta)$ in terms of the profiles f and g as follows

$$\mathfrak{Q}^\epsilon_\mu(\xi,\eta) = e^{it\Phi^\epsilon_{\mu,-}(\xi,\eta)}\tilde{q}^\epsilon_{\mu,-}(\xi,\eta)\widehat{f^\mu}(\xi-\eta)\cdot\widehat{g^-}(\eta)\cdot\widehat{g^-}(-\xi),$$

where

$$\begin{aligned}
&\Phi^\epsilon_{\mu,-}(\xi,\eta) = -\Lambda_\epsilon(\xi) + \mu\Lambda_\epsilon(\xi-\eta) - \Lambda_\epsilon(\eta),\\
&\tilde{q}^\epsilon_{\mu,-}(\xi,\eta) = \langle\eta\rangle^{-N_0}\langle\xi\rangle^{N_0} q^\epsilon_{\mu,-}(\xi,\eta).
\end{aligned}$$

Thanks to (5.9), we have

$$\begin{aligned}
&|\tilde{q}^\epsilon_{\mu,-}(\xi,\eta)| \lesssim \epsilon|\xi|\cdot\varphi_{\leq 5}\big(\sqrt{\epsilon}|\eta|\big)\cdot\varphi_{\leq -6}\Big(\frac{|\xi-\eta|}{|\eta|}\Big),\\
&\operatorname{supp}\tilde{q}^\epsilon_{\mu,-} \subset \mathbb{S}^\epsilon \overset{\text{def}}{=} \{(\xi,\eta)\in\mathbb{R}^2 \mid \xi\cdot\eta > 0,\ \frac{31}{32}|\eta| \leq |\xi| \leq \frac{33}{32}|\eta|,\ \sqrt{\epsilon}|\eta| \leq 2^6\}.
\end{aligned} \tag{5.24}$$

Lemma 2.2 and the fact $\xi\cdot\eta > 0$ (in (5.24)) yield

$$\Phi^\epsilon_{+,-}(\xi,\eta) = \frac{1}{2}\min\{|\xi|,|\eta|\}\phi^\epsilon_{+,-}(\xi,\eta),$$

with

$$\phi^\epsilon_{+,-}(\xi,\eta) = \begin{cases} 6\epsilon\xi^2 - 6\epsilon\xi\cdot\eta + 4\epsilon\eta^2 - 4, & \text{if} \quad |\xi| > |\eta|,\\ 6\epsilon\eta^2 - 6\epsilon\xi\cdot\eta + 4\epsilon\xi^2 - 4, & \text{if} \quad |\xi| < |\eta|. \end{cases}$$

Then there hold

$$\Phi^\epsilon_{-,-}(\xi,\eta) = \Phi^\epsilon_{+,-}(\eta,\xi) \quad \text{and} \quad \Phi^\epsilon_{+,-}(\xi,\eta) = \Phi^\epsilon_{+,-}(\eta,\xi). \tag{5.25}$$

With (5.25), we only derive the estimate for the integral of $\mathfrak{Q}^\epsilon_+(\xi,\eta)$ over set $\mathbb{S}^\epsilon_>$ with

$$\mathbb{S}^\epsilon_> = \{(\xi,\eta)\in\mathbb{S}^\epsilon \,|\, |\xi| > |\eta|\}.$$

Step 4.3. Estimate for $\int_0^t\int_{\mathbb{S}^\epsilon_>}\mathfrak{Q}^\epsilon_+(\xi,\eta)d\eta d\xi dt$. We divide $\mathfrak{Q}^\epsilon_+(\xi,\eta)$ into three parts as follows:
(1). For low frequency $\sqrt{\epsilon}|\eta| \leq \frac{1}{2}$, using (5.24), we have

$$|\phi^\epsilon_{+,-}(\xi,\eta)| \sim 1 \quad \text{and} \quad |\frac{\tilde{q}^\epsilon_{+,-}(\xi,\eta)}{i\Phi^\epsilon_{+,-}(\xi,\eta)}| \lesssim \frac{\epsilon}{|\phi^\epsilon_{+,-}(\xi,\eta)|} \lesssim \epsilon. \tag{5.26}$$

Integrating by parts w.r.t t, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{S}^\epsilon_>}\mathfrak{Q}^\epsilon_+(\xi,\eta)\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)d\eta d\xi dt\\
&= \underbrace{\int_{\mathbb{S}^\epsilon_>}\frac{\tilde{q}^\epsilon_{+,-}(\xi,\eta)}{i\Phi^\epsilon_{+,-}(\xi,\eta)}e^{it\Phi^\epsilon_{+,-}(\xi,\eta)}\widehat{f^+}(\tau,\xi-\eta)\cdot\widehat{g^-}(\tau,\eta)\cdot\widehat{g^-}(\tau,-\xi)\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)d\eta d\xi}_{A^\epsilon_1}\Big|_{\tau=0}^t\\
&\quad -\underbrace{\int_0^t\int_{\mathbb{S}^\epsilon_>}\frac{\tilde{q}^\epsilon_{+,-}(\xi,\eta)}{i\Phi^\epsilon_{+,-}(\xi,\eta)}e^{it\Phi^\epsilon_{+,-}(\xi,\eta)}\partial_t\big(\widehat{f^+}(\xi-\eta)\cdot\widehat{g^-}(\eta)\cdot\widehat{g^-}(-\xi)\big)\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)d\eta d\xi dt}_{A^\epsilon_2}
\end{aligned}$$

Similarly as the derivation of (4.18) in Step 3.1 of proof to Proposition 3.2, using (5.24) and (5.26), we have

$$|A_1^\epsilon| \lesssim \epsilon \int_{\mathbb{S}_>^\epsilon} |\widehat{f}(\xi-\eta)|\cdot|\widehat{g}(\eta)|\cdot|\widehat{g}(-\xi)|d\eta d\xi \lesssim \epsilon\|\widehat{f}(\xi)\|_{L^1}\|\widehat{g}(\xi)\|_{L^2}^2 \lesssim \epsilon\|f\|_{H^1}\|g\|_{L^2}^2,$$

$$|A_2^\epsilon| \lesssim \epsilon t \sup_{(0,t)} \int_{\mathbb{S}_>^\epsilon} \big(|\partial_t\widehat{f}(\xi-\eta)|\cdot|\widehat{g}(\eta)|\cdot|\widehat{g}(-\xi)| + |\widehat{f}(\xi-\eta)|\cdot|\partial_t\big(\widehat{g^-}(\eta)\cdot\widehat{g^-}(-\xi)\big)|\big)\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)d\eta d\xi$$

$$\lesssim \epsilon t \sup_{(0,t)}\big(\|\partial_t f\|_{H^1}\|g\|_{L^2}^2 + \frac{1}{\sqrt{\epsilon}}\|f\|_{H^1}\|\frac{1}{|\partial_x|}\partial_t g\|_{L^2}\|g\|_{L^2}\big),$$

where we used the fact that $|\xi|\sim|\eta|$ and the following inequality in the last inequality

$$\sqrt{\epsilon}|\eta|\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)\lesssim 1.$$

Thanks to (5.14), (5.15), (5.21) and (5.23), we obtain

$$|\int_0^t\int_{\mathbb{S}_>^\epsilon}\mathfrak{Q}_+^\epsilon(\xi,\eta)\varphi_{\leq -2}(\sqrt{\epsilon}|\eta|)d\eta d\xi dt| \lesssim \epsilon + \epsilon^{\frac{3}{2}}t. \tag{5.27}$$

(2). For moderate frequencies with large modulation of phase, i.e., for

$$\frac{1}{4}\leq\sqrt{\epsilon}|\eta|\leq 2^6 \quad\text{and}\quad |\phi_{+,-}^\epsilon(\xi,\eta)|\geq 2^{-D-1},$$

we have

$$|\frac{\tilde{q}_{+,-}^\epsilon(\xi,\eta)}{i\Phi_{+,-}^\epsilon(\xi,\eta)}| \lesssim \frac{\epsilon}{|\phi_{+,-}^\epsilon(\xi,\eta)|}\lesssim 2^D\epsilon.$$

Following similar arguments as (5.27), integrating by parts with respect to t, we get

$$|\int_0^t\int_{\mathbb{S}_>^\epsilon}\mathfrak{Q}_+^\epsilon(\xi,\eta)\varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|)\varphi_{\geq -D}(\phi_{+,-}(\xi,\eta))d\eta d\xi dt| \lesssim 2^D\epsilon + 2^D\epsilon^{\frac{3}{2}}t. \tag{5.28}$$

(3). For moderate frequencies with small modulation of phase, i.e., for

$$\frac{1}{4}\leq\sqrt{\epsilon}|\eta|\leq 2^6 \quad\text{and}\quad |\phi_{+,-}^\epsilon(\xi,\eta)|\leq 2^{-D},$$

we divide the integral set into the following two parts

$$\underbrace{\{(\xi,\eta)\in\mathbb{S}^\epsilon\,|\,0<\eta<\xi\leq\frac{33}{32}\eta,\ \frac{1}{4}\leq\sqrt{\epsilon}\eta\leq 2^6\}}_{\mathbb{S}_{>,+}^\epsilon}\cup\underbrace{\{(\xi,\eta)\in\mathbb{S}^\epsilon\,|\,0>\eta>\xi\geq\frac{33}{32}\eta,\ -\frac{1}{4}\geq\sqrt{\epsilon}\eta\geq -2^6\}}_{\mathbb{S}_{>,-}^\epsilon}.$$

We only derive the estimate for the integral over the set $\mathbb{S}_{>,+}^\epsilon$. Now, introducing the coordinates transformation on $\mathbb{S}_{>,+}^\epsilon$ as follows:

$$\Psi_\epsilon : \mathbb{S}_{>,+}^\epsilon \to \widetilde{\mathbb{S}^\epsilon}_{>,+}\subset\mathbb{R}^2,$$
$$(\xi,\eta)\mapsto(\tilde{\xi},\eta) = (\phi_{+,-}^\epsilon(\xi,\eta),\eta),$$

we have

$$\det\Big(\frac{\partial\Psi_\epsilon(\xi,\eta)}{\partial(\xi,\eta)}\Big) = \frac{\partial\phi_{+,-}^\epsilon(\xi,\eta)}{\partial\xi} = \epsilon(12\xi-6\eta)\sim\epsilon\eta\sim\sqrt{\epsilon}. \tag{5.29}$$

Then Ψ_ϵ is invertible and we denote by

$$(\xi,\eta) = \Psi_\epsilon^{-1}(\tilde{\xi},\eta).$$

Changing the variables (ξ,η) to $(\tilde{\xi},\eta)$, using (5.24) and (5.29), we have

$$|\int_0^t\int_{\mathbb{S}_{>,+}^\epsilon}\mathfrak{Q}_+^\epsilon(\xi,\eta)\varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|)\varphi_{\leq -D-1}(\phi_{+,-}(\xi,\eta))d\eta d\xi dt|$$

$$\lesssim t\sup_{(0,t)}\int_{\frac{1}{4\sqrt{\epsilon}}}^{\frac{32}{\sqrt{\epsilon}}}\int_{-2^{-D}}^{2^{-D}}\big(\sqrt{\epsilon}|\xi||\widehat{f}(\xi-\eta)|\cdot|\widehat{g}(\eta)|\cdot|\widehat{g}(\xi)|1_{\mathbb{S}_{>,+}^\epsilon}\big)|_{(\xi,\eta)=\Psi_\epsilon^{-1}(\tilde{\xi},\eta)}d\tilde{\xi}d\eta$$

$$\lesssim t2^{-\frac{D}{2}}\sup_{(0,t)}\|g\|_{L^2}\Big(\int_{\frac{1}{4\sqrt{\epsilon}}}^{\frac{32}{\sqrt{\epsilon}}}\int_{-2^{-D}}^{2^{-D}}\big(|\widehat{f}(\xi-\eta)|^2\cdot|\widehat{g}(\xi)|^2 1_{\mathbb{S}_{>,+}^\epsilon}\big)|_{(\xi,\eta)=\Psi_\epsilon^{-1}(\tilde{\xi},\eta)}d\tilde{\xi}d\eta\Big)^{\frac{1}{2}},$$

where we used the fact that $\sqrt{\epsilon}|\xi| \sim \sqrt{\epsilon}|\eta| \sim 1$ in the last inequality. Then changing variables $(\tilde{\xi}, \eta)$ to (ξ, η), using (5.29), we have

$$|\int_0^t \int_{\mathbb{S}^\epsilon_{>,+}} \mathfrak{Q}^\epsilon_+(\xi,\eta)\varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|)\varphi_{\leq -D-1}(\phi_{+,-}(\xi,\eta))d\eta d\xi dt| \lesssim 2^{-\frac{D}{2}}\epsilon^{\frac{1}{4}} t \sup_{(0,t)} \|f\|_{L^2}\|g\|^2_{L^2},$$

which along with (5.14), (5.15) and (5.21) implies

$$|\int_0^t \int_{\mathbb{S}^\epsilon_{>,+}} \mathfrak{Q}^\epsilon_+(\xi,\eta)\varphi_{[-1,5]}(\sqrt{\epsilon}|\eta|)\varphi_{\leq -D-1}(\phi_{+,-}(\xi,\eta))d\eta d\xi dt| \lesssim 2^{-\frac{D}{2}}\epsilon^{\frac{1}{4}} t. \tag{5.30}$$

The same estimate holds for the integral over set $\mathbb{S}^\epsilon_{>,-}$.

Taking $D = [\log_2 \epsilon^{-\frac{5}{6}}]$ (i.e., $2^D \sim \epsilon^{-\frac{5}{6}}$) in (5.28) and (5.30), together with (5.27), we obtain that

$$|\int_0^t \int_{\mathbb{S}^\epsilon_{>}} \mathfrak{Q}^\epsilon_+(\xi,\eta)d\eta d\xi dt| \lesssim 1 + \epsilon^{\frac{2}{3}} t. \tag{5.31}$$

The same estimates hold for $\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}^\epsilon_+(\xi,\eta)d\eta d\xi dt$ and $\int_0^t \int_{\mathbb{R}^2} \mathfrak{Q}^\epsilon_-(\xi,\eta)d\eta d\xi dt$. Then we obtain

$$|\mathrm{Re}(II)| \lesssim 1 + \epsilon^{\frac{2}{3}} t. \tag{5.32}$$

Step 5. Final energy estimates. Combining (5.18), (5.20) and (5.32), we finally obtain

$$\mathcal{E}_{N_0}(t) \lesssim 1 + \epsilon^{\frac{2}{3}} t.$$

This is exactly (5.16). This completes the proof of the proposition. □

6. Final comments

1. It would be interesting to extend the results of the present paper to the two-dimensional version of (1.4) or (1.6).

2. As for other Boussinesq systems except those described in Remark 1.1, the global well-posedness (or finite time blow-up) of (1.6) is an open question.

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