# Parameter-robust convergence analysis of fixed-stress split iterative method for multiple-permeability poroelasticity systems 

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#### Abstract

We consider flux-based multiple-porosity/multiple-permeability poroelasticity systems describing mulitplenetwork flow and deformation in a poro-elastic medium, also referred to as MPET models. The focus of the paper is on the convergence analysis of the fixed-stress split iteration, a commonly used coupling technique for the flow and mechanics equations defining poromechanical systems. We formulate the fixed-stress split method in this context and prove its linear convergence. The contraction rate of this fixed-point iteration does not depend on any of the physical parameters appearing in the model. This is confirmed by numerical results which further demonstrate the advantage of the fixed-stress split scheme over a fully implicit method relying on norm-equivalent preconditioning.


Keywords: multiple-porosity/multiple-permeability poroelasiticity, MPET system, fixed-stress split iterative coupling, convergence analysis

## 1 Introduction

Double-porosity poroelasticity models have been used to describe the motion of liquids in fissured rocks as early as in 4]. As a generalization of Biot's theory of consolidation, [6, 7], they have been further extended in the framework of multiple-network poroelastic theory (MPET) where the deformable elastic matrix is permeated by more than two fluid networks with differing porosities and permeabilities. The latter find important applications in biophysics and medicine, see [27, 12, 15, 28.

In a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{d}, d=2,3$, the mathematical model is described by the MPET system:

$$
\begin{align*}
-\operatorname{div} \boldsymbol{\sigma}+\sum_{i=1}^{n} \alpha_{i} \nabla p_{i} & =\boldsymbol{f} \text { in } \Omega \times(0, T),  \tag{1a}\\
\boldsymbol{v}_{i} & =-K_{i} \nabla p_{i} \text { in } \Omega \times(0, T),  \tag{1b}\\
-\alpha_{i} \operatorname{div} \dot{\boldsymbol{u}}-\operatorname{div} \boldsymbol{v}_{i}-c_{p_{i}} \dot{p}_{i}-\sum_{\substack{j=1 \\
j \neq i}}^{n} \beta_{i j}\left(p_{i}-p_{j}\right) & =g_{i} \text { in } \Omega \times(0, T), \tag{1c}
\end{align*}
$$

where (1a) and (1b) are for $i=1, \ldots, n$. Here

$$
\begin{equation*}
\boldsymbol{\sigma}=2 \mu \boldsymbol{\epsilon}(\boldsymbol{u})+\lambda \operatorname{div}(\boldsymbol{u}) \boldsymbol{I} \quad \text { and } \quad \boldsymbol{\epsilon}(\boldsymbol{u})=\frac{1}{2}\left(\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^{T}\right), \tag{2}
\end{equation*}
$$

denote the effective stress and the strain tensor respectively and the Lamé parameters $\lambda$ and $\mu$ are expressed in terms of the modulus of elasticity $E$ and the Poisson ratio $\nu \in[0,1 / 2)$ by
$\lambda:=\frac{\nu E}{(1+\nu)(1-2 \nu)}, \mu:=\frac{E}{2(1+\nu)}$. The displacement $\boldsymbol{u}$, fluxes $\boldsymbol{v}_{i}$ and corresponding pressures $p_{i}$, $i=1, \ldots, n$, are the unknown physical quantities.

The constants $\alpha_{i}$ in (1a) are known as Biot-Willis parameters while $\boldsymbol{f}$ represents the body force density. The hydraulic conductivity tensors $K_{i}$ in (1b) are defined as the permeability divided by the viscosity of the $i$-the network. The constants $c_{p_{i}}$ in (1C) denote the constrained specific storage coefficients, see e.g. [25] and the references therein. The network transfer coefficients $\beta_{i j}$ couple the network pressures and hence $\beta_{i j}=\beta_{j i}$. Fluid extractions or injections enter the system via the source terms $g_{i}$ in (1C).

The system (1) is well posed under proper boundary and initial conditions. For stability reasons, this system is discretized in time by an implicit method. This creates a coupled static problem in each time step. The latter can be solved fully implicit, using a loose or explicit coupling, or an iterative coupling. In general, the loosely or explicitly coupled approach is less accurate than the fully implicit one which, however, is normally more computationally expensive. Iterative coupling is a commonly used alternative to avoid the disadvantages of the aforementioned approaches. The most popular procedures in this category are the undrained split, the fixed-stress split, the drained split and the fixed-strain split iterative methods. As shown in 20, in contrast to the drained split and the fixed-strain split methods, the undrained split and fixed-stress split methods are unconditionally stable.

Convergence estimates and the rate of convergence for latter methods have been derived in [23] for the quasi-static Biot system. The convergence and error analysis of an iterative coupling scheme for solving a fully discretized Biot system based on the fixed-stress split has been provided in [2]. Linear convergence in energy norms of a variant of the fixed-stress split iteration applied to heterogenous media has been shown in [9] for linearized Biot's equations.

Other variants of the fixed-stress split iterative scheme include a two-grid algorithm in which the flow subproblem of the Biot system is solved on a fine grid whereas the poromechanics subproblem is solved on a coarse grid, see [13], or the multi-rate fixed-stress split iterative scheme which exploits different time scales for the mechanics and flow problems by taking several finer time steps for flow within one coarse time step for the mechanics of the system, see [1].

The fixed-stress split scheme has also been successfully applied and proved convergent for spacetime finite element approximations of the quasi-static Biot system, cf. 5]. In the context of unsaturated materials, it can be used for linearization of non-linear poromechanics problems. When combined with Anderson acceleration, as shown in [10], this yields a highly efficient method. The optimization of the stabilization parameter that serves the acceleration of the fixed-stress iterative method is considered for the Biot problem in the two-field formulation in [26].

In this paper we propose a fixed-stress split method for the MPET system. We prove its linear convergence and, furthermore, show with a proper choice for the stabilization parameter that the rate of convergence is independent of the physical parameters in the model. These theoretical findings are also tested computationally. The obtained numerical results support the proven convergence rate estimate and demonstrate the precedence of the fixed-stress split iterative method over the MinRes algorithm with norm-equivalent preconditioning.

The remainder of the paper is structured as follows. In Section 2 we introduce notation and recall some important stability properties of the flux-based MPET system, see [18], also [17] for the special case of Biot's system, which are to be used later. Section 3 contains the main contribution of the paper. There, the fixed-stress algorithm for the MPET system is formulated and a parameter-robust convergence rate estimate proven. Numerical tests for the proposed fixedstress split iterative coupling scheme are presented in Section 6. Section 7 gives concluding remarks.

## 2 Properties of the flux-based MPET problem

Firstly, we present the operator form of the MPET equations (1). After imposing boundary and initial conditions to this system to obtain a well-posed problem, we use the backward Euler method for its time discretization. Subsequently, a static problem in each time step has to be solved which with rescaling and proper variable substitutions has the form:

$$
\begin{equation*}
\mathscr{A}\left[\boldsymbol{u}^{T}, \boldsymbol{v}_{1}^{T}, \ldots, \boldsymbol{v}_{n}^{T}, p_{1}, \ldots, p_{n}\right]^{T}=\left[\boldsymbol{f}^{T}, \mathbf{0}^{T}, \ldots, \mathbf{0}^{T}, g_{1}, \ldots, g_{n}\right]^{T} \tag{3}
\end{equation*}
$$

Here

$$
\mathscr{A}:=\left[\begin{array}{ccccccccc}
-\operatorname{div} \epsilon-\lambda \nabla \operatorname{div} & 0 & \ldots & \ldots & 0 & \nabla & \ldots & \ldots & \nabla  \tag{4}\\
0 & R_{1}^{-1} I & 0 & \ldots & 0 & \nabla & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\
0 & 0 & \ldots & 0 & R_{n}^{-1} I & 0 & \ldots & 0 & \nabla \\
-\operatorname{div} & -\operatorname{div} & 0 & \ldots & 0 & \tilde{\alpha}_{11} I & \alpha_{12} I & \ldots & \alpha_{1 n} I \\
\vdots & 0 & \ddots & & \vdots & \alpha_{21} I & \ddots & & \alpha_{2 n} I \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & \vdots \\
-\operatorname{div} & 0 & \ldots & 0 & -\operatorname{div} & \alpha_{n 1} I & \alpha_{n 2} I & \ldots & \tilde{\alpha}_{n n} I
\end{array}\right]
$$

is the rescaled operator, $\tau$ the time step size and

$$
R_{i}^{-1}:=\tau^{-1} K_{i}^{-1} \alpha_{i}^{2}, \quad \alpha_{p_{i}}:=\frac{c_{p_{i}}}{\alpha_{i}^{2}}, \quad \beta_{i i}:=\sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{i j}, \quad \alpha_{i j}:=\frac{\tau \beta_{i j}}{\alpha_{i} \alpha_{j}}, \quad \tilde{\alpha}_{i i}:=-\alpha_{p_{i}}-\alpha_{i i}
$$

for $i, j=1, \cdots, n$. General and plausible assumptions for the scaled parameters, namely,

$$
\begin{equation*}
\lambda>0, \quad R_{1}^{-1}, \ldots, R_{n}^{-1}>0, \quad \alpha_{p_{1}}, \ldots, \alpha_{p_{n}} \geq 0, \quad \alpha_{i j} \geq 0, \quad i, j=1, \ldots, n \tag{5}
\end{equation*}
$$

are made.

### 2.1 Preliminaries and notation

Denote $\boldsymbol{v}^{T}:=\left(\boldsymbol{v}_{1}^{T}, \ldots, \boldsymbol{v}_{n}^{T}\right), \boldsymbol{z}^{T}:=\left(\boldsymbol{z}_{1}^{T}, \ldots, \boldsymbol{z}_{n}^{T}\right), \boldsymbol{p}^{T}:=\left(p_{1}, \ldots, p_{n}\right), \boldsymbol{q}^{T}:=\left(q_{1}, \ldots, q_{n}\right)$ where $\boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{V}=\boldsymbol{V}_{1} \times \cdots \times \boldsymbol{V}_{n}, \boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{P}=P_{1} \times \cdots \times P_{n}$ and $\boldsymbol{U}=\left\{\boldsymbol{u} \in H^{1}(\Omega)^{d}: \boldsymbol{u}=\mathbf{0}\right.$ on $\left.\Gamma_{u}, D\right\}, \boldsymbol{V}_{i}=$ $\left\{\boldsymbol{v}_{i} \in H(\operatorname{div}, \Omega): \boldsymbol{v}_{i} \cdot \boldsymbol{n}=0\right.$ on $\left.\Gamma_{p_{i}, N}\right\}, P_{i}=L^{2}(\Omega)$, and $P_{i}=L_{0}^{2}(\Omega)$ if $\Gamma_{\boldsymbol{u}, D}=\Gamma=\partial \Omega$.

The weak formulation of system (3) reads as: Find $(\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$, such that for any $(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$ there hold

$$
\begin{align*}
(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w}))+\lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{w})-\sum_{i=1}^{n}\left(p_{i}, \operatorname{div} \boldsymbol{w}\right) & =(\boldsymbol{f}, \boldsymbol{w}),  \tag{6a}\\
\left(R_{i}^{-1} \boldsymbol{v}_{i}, \boldsymbol{z}_{i}\right)-\left(p_{i}, \operatorname{div} \boldsymbol{z}_{i}\right) & =0,  \tag{6b}\\
-\left(\operatorname{div} \boldsymbol{u}, q_{i}\right)-\left(\operatorname{div} \boldsymbol{v}_{i}, q_{i}\right)+\tilde{\alpha}_{i i}\left(p_{i}, q_{i}\right)+\sum_{\substack{j=1 \\
j \neq i}}^{n} \alpha_{i j}\left(p_{j}, q_{i}\right) & =\left(g_{i}, q_{i}\right),  \tag{6c}\\
& i=1, \ldots, n, n,
\end{align*}
$$

or, equivalentely, $\mathcal{A}((\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}),(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}))=F(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q})$ for $(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$, where $F(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q})=$ $(\boldsymbol{f}, \boldsymbol{w})+\sum_{i=1}^{n}\left(g_{i}, q_{i}\right)$ and

$$
\begin{aligned}
& \mathcal{A}((\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}),(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}))=(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{w}))+\lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{w})-\sum_{i=1}^{n}\left(p_{i}, \operatorname{div} \boldsymbol{w}\right)+\sum_{i=1}^{n}\left(R_{i}^{-1} \boldsymbol{v}_{i}, \boldsymbol{z}_{i}\right) \\
& -\sum_{i=1}^{n}\left(p_{i}, \operatorname{div} \boldsymbol{z}_{i}\right)-\sum_{i=1}^{n}\left(\operatorname{div} \boldsymbol{u}, q_{i}\right)-\sum_{i=1}^{n}\left(\operatorname{div} \boldsymbol{v}_{i}, q_{i}\right)-\sum_{i=1}^{n}\left(\alpha_{p_{i}}+\alpha_{i i}\right)\left(p_{i}, q_{i}\right)+\sum_{i=1}^{n} \sum_{\substack{j=1 \\
j \neq i}}^{n} \alpha_{j i}\left(p_{j}, q_{i}\right) \\
& =(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{w}))+\lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{w})-\sum_{i=1}^{n}\left(p_{i}, \operatorname{div} \boldsymbol{w}\right)+\sum_{i=1}^{n}\left(R_{i}^{-1} \boldsymbol{v}_{i}, \boldsymbol{z}_{i}\right)-\sum_{i=1}^{n}\left(p_{i}, \operatorname{div} \boldsymbol{z}_{i}\right) \\
& -\sum_{i=1}^{n}\left(\operatorname{div} \boldsymbol{u}, q_{i}\right)-(\operatorname{Div} \boldsymbol{v}, \boldsymbol{q})-\left(\left(\Lambda_{1}+\Lambda_{2}\right) \boldsymbol{p}, \boldsymbol{q}\right) .
\end{aligned}
$$

Here we have denoted $(\operatorname{Div} \boldsymbol{v})^{T}:=\left(\operatorname{div} \boldsymbol{v}_{1}, \ldots, \operatorname{div} \boldsymbol{v}_{n}\right)$ and

$$
\Lambda_{1}:=\left[\begin{array}{cccc}
\alpha_{11} & -\alpha_{12} & \ldots & -\alpha_{1 n} \\
-\alpha_{21} & \alpha_{22} & \ldots & -\alpha_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n 1} & -\alpha_{n 2} & \ldots & \alpha_{n n}
\end{array}\right], \quad \Lambda_{2}:=\left[\begin{array}{cccc}
\alpha_{p_{1}} & 0 & \ldots & 0 \\
0 & \alpha_{p_{2}} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \alpha_{p_{n}}
\end{array}\right] .
$$

Furthermore, define $R^{-1}:=\max \left\{R_{1}^{-1}, \ldots, R_{n}^{-1}\right\}, \lambda_{0}:=\max \{1, \lambda\}$ and the $n \times n$ matrices

$$
\Lambda_{3}:=\left[\begin{array}{cccc}
R & 0 & \ldots & 0 \\
0 & R & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & R
\end{array}\right], \quad \Lambda_{4}:=\left[\begin{array}{cccc}
\frac{1}{\lambda_{0}} & \ldots & \ldots & \frac{1}{\lambda_{0}} \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
\frac{1}{\lambda_{0}} & \cdots & \ldots & \frac{1}{\lambda_{0}}
\end{array}\right]
$$

that are used later in the convergence analysis of the fixed-stress coupling iteration. It is easy to show that $\Lambda_{i}$ are symmetric positive semidefinite (SPSD) for $i=1,2,4$ while $\Lambda_{3}$ is symmetric positive definite (SPD).

Moreover, we denote

$$
\Lambda:=\sum_{i=1}^{4} \Lambda_{i}
$$

which obviously is an SPD matrix and therefore, can be used to define the parameter-matrixdependent norms $\|\cdot\|_{\boldsymbol{U}},\|\cdot\|_{\boldsymbol{V}},\left\|_{\cdot}\right\|_{\boldsymbol{P}}$ induced by the inner products:

$$
\begin{align*}
(\boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{U}} & =(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w}))+\lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{w})  \tag{7a}\\
(\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{V}} & =\sum_{i=1}^{n}\left(R_{i}^{-1} \boldsymbol{v}_{i}, \boldsymbol{z}_{i}\right)+\left(\Lambda^{-1} \operatorname{Div} \boldsymbol{v}, \operatorname{Div} \boldsymbol{z}\right)  \tag{7b}\\
(\boldsymbol{p}, \boldsymbol{q})_{\boldsymbol{P}} & =(\Lambda \boldsymbol{p}, \boldsymbol{q}) \tag{7c}
\end{align*}
$$

As shown in [18], these norms are crucial to show the parameter-robust stability of the MPET system.

### 2.2 Stability properties

The following inf-sup conditions for the spaces $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{P}$ are assumed to be fulfilled in the analysis presented in this paper:

$$
\begin{array}{r}
\inf _{q \in P_{i}} \sup _{\boldsymbol{v} \in \boldsymbol{V}_{i}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\operatorname{div}}\|q\|} \geq \beta_{d}, \quad i=1, \ldots, n, \\
\inf _{\left(q_{1}, \cdots, q_{n}\right) \in P_{1} \times \cdots \times P_{n}} \sup _{\boldsymbol{u} \in \boldsymbol{U}} \frac{\left(\operatorname{div} \boldsymbol{u}, \sum_{i=1}^{n} q_{i}\right)}{\|\boldsymbol{u}\|_{1}\left\|\sum_{i=1}^{n} q_{i}\right\|} \geq \beta_{s} \tag{9}
\end{array}
$$

for some constants $\beta_{d}>0$ and $\beta_{s}>0$, see [11, 8]. Then from [18], we know that the MPET problem (6) is uniformly well-posed, namely the three assertions in Theorem 1 hold:

## Theorem 1.

(i) There exists a positive constant $C_{b}$ independent of the parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in$ $\{1, \ldots, n\}$ and the network scale $n$ such that the inequality

$$
|\mathcal{A}((\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}),(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}))| \leq C_{b}\left(\left\|\left.\boldsymbol{u}\right|_{\boldsymbol{U}}+\right\| \boldsymbol{v}\left\|_{\boldsymbol{V}}+\right\| \boldsymbol{p} \|_{\boldsymbol{P}}\right)\left(\|\boldsymbol{w}\|_{\boldsymbol{U}}+\|\boldsymbol{z}\|_{\boldsymbol{V}}+\|\boldsymbol{q}\|_{\boldsymbol{P}}\right)
$$

holds true for any $(\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P},(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$.
(ii) There is a constant $\omega>0$ independent of the parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in\{1, \ldots, n\}$ and the number of networks $n$ such that

$$
\begin{equation*}
\inf _{(\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}) \in \boldsymbol{X}} \sup _{(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}) \in \boldsymbol{X}} \frac{\mathcal{A}((\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}),(\boldsymbol{w} ; \boldsymbol{z} ; \boldsymbol{q}))}{\left(\|\boldsymbol{u}\|_{\boldsymbol{U}}+\|\boldsymbol{v}\|_{\boldsymbol{V}}+\|\boldsymbol{p}\|_{\boldsymbol{P}}\right)\left(\|\boldsymbol{w}\|_{\boldsymbol{U}}+\|\boldsymbol{z}\|_{\boldsymbol{V}}+\|\boldsymbol{q}\|_{\boldsymbol{P}}\right)} \geq \omega \tag{10}
\end{equation*}
$$

where $\boldsymbol{X}:=\boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$.
(iii) The MPET system (6) has a unique solution $(\boldsymbol{u} ; \boldsymbol{v} ; \boldsymbol{p}) \in \boldsymbol{U} \times \boldsymbol{V} \times \boldsymbol{P}$ and the following stability estimate holds:

$$
\begin{equation*}
\|\boldsymbol{u}\|_{\boldsymbol{U}}+\|\boldsymbol{v}\|_{\boldsymbol{V}}+\|\boldsymbol{p}\|_{P} \leq C_{1}\left(\|\boldsymbol{f}\|_{\boldsymbol{U}^{*}}+\|\boldsymbol{g}\|_{\boldsymbol{P}^{*}}\right) \tag{11}
\end{equation*}
$$

where $C_{1}$ is a positive constant independent of the parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in\{1, \ldots, n\}$ and the network scale $n$, and $\|\boldsymbol{f}\|_{\boldsymbol{U}^{*}}=\sup _{\boldsymbol{w} \in \boldsymbol{U}} \frac{(\boldsymbol{f} \boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{U}}}, \quad\|\boldsymbol{g}\|_{\boldsymbol{P}^{*}}=\sup _{\boldsymbol{q} \in \boldsymbol{P}} \frac{(\boldsymbol{g}, \boldsymbol{q})}{\|\boldsymbol{q}\|_{P}}=\left\|\Lambda^{-\frac{1}{2}} \boldsymbol{g}\right\|$.

### 2.3 A norm equivalent preconditioner

Consider the block-diagonal operator

$$
\begin{gathered}
\mathscr{B}:=\left[\begin{array}{ccc}
\mathscr{B}_{\boldsymbol{u}}^{-1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathscr{B}_{\boldsymbol{v}}^{-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathscr{B}_{\boldsymbol{p}}^{-1}
\end{array}\right], \text { where } \mathscr{B}_{\boldsymbol{u}}=-\operatorname{div} \boldsymbol{\epsilon}-\lambda \nabla \operatorname{div}, \\
\mathscr{B}_{\boldsymbol{v}}=\left[\begin{array}{cccc}
R_{1}^{-1} I & 0 & \ldots & 0 \\
0 & R_{2}^{-1} I & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & R_{n}^{-1} I
\end{array}\right]-\left[\begin{array}{cccc}
\tilde{\gamma}_{11} \nabla \operatorname{div} & \tilde{\gamma}_{12} \nabla \operatorname{div} & \ldots & \tilde{\gamma}_{1 n} \nabla \operatorname{div} \\
\tilde{\gamma}_{21} \nabla \operatorname{div} & \tilde{\gamma}_{22} \nabla \operatorname{div} & \ldots & \tilde{\gamma}_{2 n} \nabla \operatorname{div} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{\gamma}_{n 1} \nabla \operatorname{div} & \tilde{\gamma}_{n 2} \nabla \operatorname{div} & \ldots & \tilde{\gamma}_{n n} \nabla \operatorname{div}
\end{array}\right],
\end{gathered}
$$

and

$$
\mathscr{B}_{\boldsymbol{p}}=\left[\begin{array}{cccc}
\gamma_{11} I & \gamma_{12} I & \ldots & \gamma_{1 n} I \\
\gamma_{21} I & \gamma_{22} I & \ldots & \gamma_{2 n} I \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n 1} I & \gamma_{n 2} I & \ldots & \gamma_{n n} I
\end{array}\right]
$$

Here, $\gamma_{i j}, \tilde{\gamma}_{i j}, i, j=1, \ldots, n$ are the entries of $\Lambda$ and $\Lambda^{-1}$, respectively.
As substantiated in [18], the stability results for the operator $\mathscr{A}$ imply that the operator $\mathscr{B}$ is a uniform norm-equivalent (canonical) block-diagonal preconditioner that is robust with respect to all model and discretization parameters.

Note that the existence of this canonical uniform block-diagonal preconditioner can be transferred to the discrete level as long as discrete inf-sup conditions analogous to (8) and (9) are satisfied, cf. [18].

## 3 Fixed-stress method for MPET model

In the proposed fixed-stress split iterative coupling scheme for the MPET system, and as for Biot's equations, we first solve the flow and then the mechanics problem where, in order to avoid instabilities, a stabilization term is added to the flow equation. Note that generalizing the fixedstress iteration from the Biot to the (flux-based) MPET model is not straightforward due to the involvement of $n$ pressures $p_{i}$ and $n$ fluxes $\boldsymbol{v}_{i}$. Our formulation suggests a stabilization that employs the sum of the pressures which later shows itself to be vital in the convergence analysis of the scheme.

In order to elucidate our approach, we present the fixed-stress splitting scheme for the continuous problem first. Let $\boldsymbol{u}^{k}, \boldsymbol{v}_{i}^{k}$ and $p_{i}^{k}$ denote the $k$-th fixed-stress iterates for $\boldsymbol{u}, \boldsymbol{v}_{i}$ and $p_{i}$ respectively, $i=1, \ldots, n$. The single rate fixed-stress coupling iteration is given by the following algorithm:

$$
\begin{align*}
& \hline \text { Algorithm 1: Fixed-stress coupling iteration for the MPET system } \\
& \hline \text { Step a: Given } \boldsymbol{u}^{m} \text {, we solve for } \boldsymbol{v}_{i}^{m+1} \text { and } p_{i}^{m+1} \\
& \qquad \begin{array}{r}
\left(-\operatorname{div} \boldsymbol{v}_{i}^{m+1}, q_{i}\right)-\left(\left(\alpha_{p_{i}}+\alpha_{i i}\right) p_{i}^{m+1}, q_{i}\right)+\left(\sum_{\substack{j=1 \\
j \neq i}}^{n} \alpha_{i j} p_{j}^{m+1}, q_{i}\right)-L\left(\sum_{j=1}^{n} p_{j}^{m+1}, q_{i}\right) \\
\\
=\left(g_{i}, q_{i}\right)-L\left(\sum_{j=1}^{n} p_{j}^{m}, q_{i}\right)+\left(\operatorname{div} \boldsymbol{u}^{m}, q_{i}\right), \quad 1 \leq i \leq n,
\end{array} \tag{12}
\end{align*}
$$

and

$$
\begin{equation*}
\left(R_{i}^{-1} \boldsymbol{v}_{i}^{m+1}, \boldsymbol{z}_{i}\right)-\left(p_{i}^{m+1}, \operatorname{div} \boldsymbol{z}_{i}\right)=\mathbf{0}, \quad 1 \leq i \leq n . \tag{13}
\end{equation*}
$$

Step b: Given $\boldsymbol{v}_{i}^{m+1}$ and $p_{i}^{m+1}$, we solve for $\boldsymbol{u}^{m+1}$

$$
\begin{equation*}
\left(\boldsymbol{\epsilon}\left(\boldsymbol{u}^{m+1}\right), \boldsymbol{\epsilon}(\boldsymbol{w})\right)+\lambda\left(\operatorname{div} \boldsymbol{u}^{m+1}, \operatorname{div} \boldsymbol{w}\right)=(\boldsymbol{f}, \boldsymbol{w})+\sum_{i=1}^{n}\left(p_{i}^{m+1}, \operatorname{div} \boldsymbol{w}\right) . \tag{14}
\end{equation*}
$$

Our main result is formulated in terms of the following quantities:

$$
\begin{align*}
\boldsymbol{e}_{u}^{k} & =\boldsymbol{u}^{k}-\boldsymbol{u} \in \boldsymbol{U}  \tag{15a}\\
\boldsymbol{e}_{v_{i}}^{k} & =\boldsymbol{v}_{i}^{k}-\boldsymbol{v}_{i} \in \boldsymbol{V}_{i}, \quad i=1, \ldots, n  \tag{15b}\\
e_{p_{i}}^{k} & =p_{i}^{k}-p_{i} \in P_{i}, \quad i=1, \ldots, n \tag{15c}
\end{align*}
$$

denoting the errors of the $k$-th iterates $\boldsymbol{u}^{k}, \boldsymbol{v}_{i}^{k}, p_{i}^{k}, i=1, \ldots, n$ generated by Algrorithm (1. The error block-vectors $\boldsymbol{e}_{v}^{k}$ and $\boldsymbol{e}_{p}^{k}$ are defined by $\left(\boldsymbol{e}_{v}^{k}\right)^{T}=\left(\left(\boldsymbol{e}_{v_{1}}^{k}\right)^{T}, \ldots,\left(\boldsymbol{e}_{v_{n}}^{k}\right)\right)^{T},\left(\boldsymbol{e}_{p}^{k}\right)^{T}=\left(e_{p_{1}}^{k}, \ldots, e_{p_{n}}^{k}\right)$. Since $\boldsymbol{u}, \boldsymbol{v}_{i}, p_{i}, i=1, \ldots, n$ are the exact solutions of (6), the error equations

$$
\begin{align*}
&\left(-\operatorname{Div} \boldsymbol{e}_{v}^{m+1}, \boldsymbol{q}\right)-\left(\left(\Lambda_{1}+\Lambda_{2}\right) \boldsymbol{e}_{p}^{m+1}, \boldsymbol{q}\right)-L\left(\sum_{i=1}^{n} e_{p_{i}}^{m+1}, \sum_{i=1}^{n} q_{i}\right)=-L\left(\sum_{i=1}^{n} e_{p_{i}}^{m}, \sum_{i=1}^{n} q_{i}\right) \\
&+\left(\operatorname{div} \boldsymbol{e}_{u}^{m}, \sum_{i=1}^{n} q_{i}\right),  \tag{16a}\\
&\left(R^{-1} \boldsymbol{e}_{v}^{m+1}, \boldsymbol{z}\right)-\left(\boldsymbol{e}_{p}^{m+1}, \operatorname{Div} \boldsymbol{z}\right)=0  \tag{16b}\\
&\left(\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right), \boldsymbol{\epsilon}(\boldsymbol{w})\right)+\lambda\left(\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \operatorname{div} \boldsymbol{w}\right)=\left(\sum_{i=1}^{n} e_{p_{i}}^{m+1}, \operatorname{div} \boldsymbol{w}\right), \tag{16c}
\end{align*}
$$

hold, the latter of which playing a key role in the presented convergence analysis.
Note that in the following we do not make any further restrictive assumptions on the parameters in (6) but consider the general situation in which only (5) needs to be satisfied. Useful for deriving and defining the tuning parameter $L$ is the constant $c_{K}$ in the estimate

$$
\begin{equation*}
\|\boldsymbol{\epsilon}(\boldsymbol{w})\| \geq c_{K}\|\operatorname{div}(\boldsymbol{w})\| \quad \text { for all } \boldsymbol{w} \in \boldsymbol{U} \tag{17}
\end{equation*}
$$

which is used for $\boldsymbol{w}=\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}$ in the proof of the next Lemma. 1
We perform the convergence analysis in two steps. The first one is the proof of the following lemma.

Lemma 2. The errors $\boldsymbol{e}_{u}^{m+1}, \boldsymbol{e}_{v}^{m+1}$ and $\boldsymbol{e}_{p}^{m+1}$ of the ( $m+1$ )-st fixed-stress iterate generated by Algorithm 1 for $L \geq \frac{1}{\lambda+c_{K}^{2}}$ satisfy the estimate

$$
\begin{align*}
& \frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right)+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}, \quad m=0,1,2, \ldots \tag{18}
\end{align*}
$$

Proof. Setting $\boldsymbol{z}=\boldsymbol{e}_{v}^{m+1}, \boldsymbol{q}=-\boldsymbol{e}_{p}^{m+1}, \boldsymbol{w}=\boldsymbol{e}_{u}^{m+1}$ in (16a) $-(16 \mathrm{c})$, it follows that

$$
\begin{align*}
& \left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+L\left(\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right), \sum_{i=1}^{n} e_{p_{i}}^{m+1}\right)=\left(\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right), \sum_{i=1}^{n} e_{p_{i}}^{m+1}\right) . \tag{19}
\end{align*}
$$

[^0]Using the identity

$$
\left(\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right), \sum_{i=1}^{n} e_{p_{i}}^{m+1}\right)=\frac{1}{2}\left(\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}-\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}\right),
$$

equation (19) can be rewritten as

$$
\begin{align*}
& \left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}  \tag{20}\\
& =\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\left(\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right), \sum_{i=1}^{n} e_{p_{i}}^{m+1}\right) .
\end{align*}
$$

Now, taking $\boldsymbol{w}=\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}$ in (16c) we obtain

$$
\begin{equation*}
\left(\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right), \sum_{i=1}^{n} e_{p_{i}}^{m+1}\right)=\left(\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right), \boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right)+\lambda\left(\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right), \tag{21}
\end{equation*}
$$

and, substituting (21) in (20), conclude that

$$
\begin{aligned}
& \left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2} \\
& =\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\left(\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right), \boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right)+\lambda\left(\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right) \\
& \leq \\
& \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right) \\
& \quad+\frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}+\lambda\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}\right) .
\end{aligned}
$$

The latter inequality can be expressed equivalently in the form

$$
\begin{align*}
& \frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right)+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}  \tag{22}\\
& \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}+\lambda\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}\right) .
\end{align*}
$$

To estimate the last term in (22) consider (16c) again. Subtracting the $m$-th error from the $(m+1)$-st, choosing $\boldsymbol{w}=\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}$ and applying Cauchy's inequality yields

$$
\begin{equation*}
\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}+\lambda\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2} \leq\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|\left\|\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right)\right\| \tag{23}
\end{equation*}
$$

Next, from (17) we have that $\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\| \geq c_{K}\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|$, which implies

$$
\left(c_{K}^{2}+\lambda\right)\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\| \leq\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|,
$$

that is,

$$
\begin{equation*}
\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\| \leq \frac{1}{\lambda+c_{K}^{2}}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\| \tag{24}
\end{equation*}
$$

Hence

$$
\begin{align*}
\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2}+\lambda\left\|\operatorname{div}\left(\boldsymbol{e}_{u}^{m+1}-\boldsymbol{e}_{u}^{m}\right)\right\|^{2} & \leq \frac{1}{\lambda+c_{K}^{2}}\left\|\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right)\right\|^{2}  \tag{25}\\
& \leq L\left\|\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right)\right\|^{2}
\end{align*}
$$

Therefore, using (25) in (22), we obtain

$$
\begin{aligned}
& \frac{1}{2}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right)+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}-\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2} \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}+\frac{L}{2}\left\|\sum_{i=1}^{n}\left(e_{p_{i}}^{m+1}-e_{p_{i}}^{m}\right)\right\|^{2},
\end{aligned}
$$

which completes the proof.
Using (18), we can prove that $\sum_{i=1}^{n} e_{p_{i}}^{m} \xrightarrow{m \rightarrow \infty} 0$, which is stated in the following theorem.
Theorem 3. Let $c_{K}$ and $\beta_{s}$ denote the constants in (17) and (9), respectively. The single rate fixed-stress iterative method for the static MPET problem (6) defined in Algorithm $\mathbb{1}$ is a contraction that converges linearly for any $L \geq 1 /\left(\lambda+c_{K}^{2}\right)$ independent of the model parameters and the time step size $\tau$. The errors $\boldsymbol{e}_{p}^{m}$ in this case satisfy the inequality

$$
\begin{equation*}
\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq \operatorname{rate}^{2}(\lambda)\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2} \tag{26}
\end{equation*}
$$

with

$$
\begin{equation*}
\operatorname{rate}^{2}(\lambda) \leq \frac{1}{\frac{L^{-1}}{\beta_{s}^{-2}+\lambda}+1} \tag{27}
\end{equation*}
$$

For $L=\frac{1}{\lambda+c_{K}^{2}}$, the convergence factor in (26) can be estimated by

$$
\begin{equation*}
\operatorname{rate}^{2}(\lambda) \leq \frac{1}{\frac{\lambda+c_{K}^{2}}{\beta_{s}^{-2}+\lambda}+1} \leq \max \left\{\frac{\beta_{s}^{-2}}{c_{K}^{2}+\beta_{s}^{-2}}, \frac{1}{2}\right\} \tag{28}
\end{equation*}
$$

Proof. By the Stokes inf-sup condition, we have that for any $\sum_{i=1}^{n} e_{p_{i}}^{m+1}$ there exists $\boldsymbol{w}_{p} \in \boldsymbol{U}$ such that

$$
\begin{equation*}
\operatorname{div} \boldsymbol{w}_{p}=\sum_{i=1}^{n} e_{p_{i}}^{m+1} \quad \text { and } \quad\left\|\boldsymbol{\epsilon}\left(\boldsymbol{w}_{p}\right)\right\| \leq \beta_{s}^{-1}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|, \tag{29}
\end{equation*}
$$

where $\beta_{s}$ is the Stokes inf-sup constant in (9). Hence,

$$
\left\|\boldsymbol{\epsilon}\left(\boldsymbol{w}_{p}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{w}_{p}\right\|^{2} \leq\left(\beta_{s}^{-2}+\lambda\right)\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}
$$

Taking $\boldsymbol{w}=\boldsymbol{w}_{p}$ in (16c) and using (29) yields

$$
\begin{equation*}
\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}=\left(\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right), \boldsymbol{\epsilon}\left(\boldsymbol{w}_{p}\right)\right)+\lambda\left(\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \operatorname{div} \boldsymbol{w}_{p}\right) \tag{30}
\end{equation*}
$$

Now, applying Cauchy's inequality, we obtain

$$
\begin{align*}
& \left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right)^{\frac{1}{2}}\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{w}_{p}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{w}_{p}\right\|^{2}\right)^{\frac{1}{2}}  \tag{31}\\
& \quad \leq\left(\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2}\right)^{\frac{1}{2}}\left(\beta_{s}^{-2}+\lambda\right)^{\frac{1}{2}}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|
\end{align*}
$$

which implies

$$
\begin{equation*}
\left(\beta_{s}^{-2}+\lambda\right)^{-1}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2} . \tag{32}
\end{equation*}
$$

Given Lemma 2 and (32), we therefore obtain

$$
\begin{aligned}
& \frac{1}{2}\left(\beta_{s}^{-2}+\lambda\right)^{-1}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}+\left\|R^{-1 / 2} \boldsymbol{e}_{v}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}
\end{aligned}
$$

and hence

$$
\left(\frac{1}{2 \beta_{s}^{-2}+2 \lambda}+\frac{L}{2}\right)\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}
$$

or, equivalently,

$$
\left(\frac{L^{-1}}{\beta_{s}^{-2}+\lambda}+1\right)\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2} \leq\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|^{2}
$$

which proves (26)-(27). Finally, (28) follows from (27) by choosing $L=\frac{1}{\lambda+c_{K}^{2}}$ and noting that $\frac{1}{\frac{\lambda+c_{K}^{2}}{\beta_{s}^{-2}+\lambda}+1}$ is a monotone function for $\lambda>0$.

Note that $\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|$ only defines a seminorm of $\boldsymbol{e}_{p}^{m}$ and Theorem 3 indicates the convergence rate of $\boldsymbol{e}_{p}$ in this seminorm. It still remains at this point unclear whether $\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\| \rightarrow 0$ guarantees that $\boldsymbol{e}_{p}^{m}$ converges to $\mathbf{0}$.

Theorem 5, as stated later, clarifies this and demonstrates the uniform convergence of $\boldsymbol{e}_{u}^{m}, \boldsymbol{e}_{v}^{m}$ and $\boldsymbol{e}_{p}^{m}$ for the fixed-stress iterative method utilizing the uniform stability results from [18]. Before we present Theorem 5, we introduce the matrices:

$$
\Lambda_{L}:=\left[\begin{array}{cccc}
L & \ldots & \ldots & L \\
\vdots & & & \vdots \\
\vdots & & & \vdots \\
L & \ldots & \ldots & L
\end{array}\right]
$$

and

$$
\Lambda_{e}:=\Lambda+\Lambda_{L}
$$

Analogous to the assertion of Lemma 1 in [18], the properties of $\Lambda_{e}$ are as follows in Lemma [4] Lemma 4. Let $\tilde{\Lambda}=\Lambda_{3}+\Lambda_{4}+\Lambda_{L}, \tilde{\Lambda}^{-1}=\left(\tilde{b}_{i j}\right)_{n \times n}$, then $\tilde{\Lambda}$ is SPD and for any $n$-dimensional vector $\boldsymbol{x}$, we have

$$
\begin{align*}
& \left(\Lambda_{e} \boldsymbol{x}, \boldsymbol{x}\right) \geq(\tilde{\Lambda} \boldsymbol{x}, \boldsymbol{x}) \geq\left(\Lambda_{3} \boldsymbol{x}, \boldsymbol{x}\right)  \tag{33}\\
& \left(\Lambda_{e}^{-1} \boldsymbol{x}, \boldsymbol{x}\right) \leq\left(\tilde{\Lambda}^{-1} \boldsymbol{x}, \boldsymbol{x}\right) \leq\left(\Lambda_{3}^{-1} \boldsymbol{x}, \boldsymbol{x}\right)=R^{-1}(\boldsymbol{x}, \boldsymbol{x}) . \tag{34}
\end{align*}
$$

Also,

$$
\begin{equation*}
0<\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{b}_{i j} \leq\left(\frac{1}{\lambda_{0}}+L\right)^{-1} \tag{35}
\end{equation*}
$$

Subsequently, we can use $\Lambda_{e}$ to define the following parameter-dependent norms:

$$
\begin{align*}
(\boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{U}} & =(\epsilon(\boldsymbol{u}), \epsilon(\boldsymbol{w}))+\lambda(\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{w})  \tag{36a}\\
(\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{V}_{e}} & =\sum_{i=1}^{n}\left(R_{i}^{-1} \boldsymbol{v}_{i}, \boldsymbol{z}_{i}\right)+\left(\Lambda_{e}^{-1} \operatorname{Div} \boldsymbol{v}, \operatorname{Div} \boldsymbol{z}\right)  \tag{36b}\\
(\boldsymbol{p}, \boldsymbol{q})_{\boldsymbol{P}_{e}} & =\left(\Lambda_{e} \boldsymbol{p}, \boldsymbol{q}\right) \tag{36c}
\end{align*}
$$

As stated in the following theorem, the fixed-stress coupling iteration for the MPET system converges uniformly.

Theorem 5. Consider the fixed-stress coupling iteration according to Algorithm 1 and assume that $L \geq 1 /\left(\lambda+c_{K}^{2}\right)$. Then the errors $\boldsymbol{e}_{u}^{m}$, $\boldsymbol{e}_{v}^{m}$ and $\boldsymbol{e}_{p}^{m}$ defined in (15), measured in the norms induced by (36), satisfy the estimates

$$
\begin{gather*}
\left\|\boldsymbol{e}_{u}^{m}\right\|_{\boldsymbol{U}} \leq C_{u}[\operatorname{rate}(\lambda)]^{m}  \tag{37}\\
\left\|\boldsymbol{e}_{v}^{m}\right\|_{\boldsymbol{V}_{e}}+\left\|\boldsymbol{e}_{p}^{m}\right\|_{\boldsymbol{P}_{e}} \leq C_{v p}[\operatorname{rate}(\lambda)]^{m} \tag{38}
\end{gather*}
$$

where the constants $C_{u}$ and $C_{v p}$ are independent of the model parameters and the time step size $\tau$. Furthermore, the convergence rate rate $(\lambda)$ satisfies (27).

Proof. In the same manner as we derived (25) we find

$$
\left\|\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right)\right\|^{2}+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\|^{2} \leq\left(\frac{1}{c_{K}^{2}+\lambda}\right)\left\|\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|^{2}
$$

which shows (37). Moreover, rewriting the error equations (16a)-(16c) and using the definition of $\Lambda_{L}$ we deduce the variational problem

$$
\begin{gather*}
\left(\boldsymbol{\epsilon}\left(\boldsymbol{e}_{u}^{m+1}\right), \boldsymbol{\epsilon}(\boldsymbol{w})\right)+\lambda\left(\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \operatorname{div} \boldsymbol{w}\right)-\left(\sum_{i=1}^{n} e_{p_{i}}^{m+1}, \operatorname{div} \boldsymbol{w}\right)=0 \\
-\left(R^{-1} \boldsymbol{e}_{v}^{m+1}, \boldsymbol{z}\right)-\left(\boldsymbol{e}_{p}^{m+1}, \operatorname{Div} \boldsymbol{z}\right)=0 \\
=-L\left(\sum_{i=1}^{m+1}, \sum_{i=1}^{n} q_{i}^{m}\right)-\left(\operatorname{Div} \boldsymbol{e}_{v}^{m+1}, \boldsymbol{q}\right)-\left(\left(\Lambda_{1}+\Lambda_{2}+\Lambda_{L}\right) \boldsymbol{e}_{p}^{m+1}, \boldsymbol{q}\right)  \tag{39}\\
\left.q_{i}\right)+\left(\operatorname{div} \boldsymbol{e}_{u}^{m}-\operatorname{div} \boldsymbol{e}_{u}^{m+1}, \sum_{i=1}^{n} q_{i}\right)
\end{gather*}
$$

Denote $g_{e}=-L \sum_{i=1}^{n} e_{p_{i}}^{m}+\operatorname{div} \boldsymbol{e}_{u}^{m}-\operatorname{div} \boldsymbol{e}_{u}^{m+1}$, then by the triangle inequality, (24) and the contraction estimate (26), it follows that

$$
\begin{align*}
\left\|g_{e}\right\| & =\left\|-L \sum_{i=1}^{n} e_{p_{i}}^{m}+\operatorname{div} \boldsymbol{e}_{u}^{m}-\operatorname{div} \boldsymbol{e}_{u}^{m+1}\right\| \\
& \leq L\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|+\frac{1}{\lambda+c_{K}^{2}}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}-\sum_{i=1}^{n} e_{p_{i}}^{m+1}\right\|  \tag{40}\\
& \leq L\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\|+\frac{2}{\lambda+c_{K}^{2}}\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\| \\
& \leq 3 L\left\|\sum_{i=1}^{n} e_{p_{i}}^{m}\right\| \leq 3 L[\operatorname{rate}(\lambda)]^{m}\left\|\sum_{i=1}^{n} e_{p_{i}}^{0}\right\|
\end{align*}
$$

Next, by taking $\boldsymbol{f}=\mathbf{0}, \boldsymbol{g}=\left(g_{e}, g_{e}, \cdots, g_{e}\right)^{T}$ and replacing $\Lambda_{1}+\Lambda_{2}$ by $\Lambda_{1}+\Lambda_{2}+\Lambda_{L}$ in (6) and using the uniform stability estimate (11) with $\Lambda$ replaced by $\Lambda_{e}$, we obtain

$$
\begin{equation*}
\left\|\boldsymbol{e}_{u}^{m+1}\right\|_{\boldsymbol{U}}+\left\|\boldsymbol{e}_{v}^{m+1}\right\|_{\boldsymbol{V}_{e}}+\left\|\boldsymbol{e}_{p}^{m+1}\right\|_{\boldsymbol{P}_{e}} \leq C_{1}\|\boldsymbol{g}\|_{\boldsymbol{P}_{e}^{*}}=C_{1}\left\|\Lambda_{e}^{-\frac{1}{2}} \boldsymbol{g}\right\|=C_{1}\left(\Lambda_{e}^{-1} \boldsymbol{g}, \boldsymbol{g}\right)^{\frac{1}{2}} \tag{41}
\end{equation*}
$$

Further, by Lemma 4 and (40), we have

$$
\begin{align*}
\left(\Lambda_{e}^{-1} \boldsymbol{g}, \boldsymbol{g}\right) \leq\left(\tilde{\Lambda}^{-1} \boldsymbol{g}, \boldsymbol{g}\right) & =(\tilde{\Lambda}^{-1}(\underbrace{g_{e}, g_{e} \ldots, g_{e}}_{n})^{T},(\underbrace{g_{e}, g_{e} \ldots, g_{e}}_{n})^{T}) \\
& =\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{b}_{i j}\right)\left(g_{e}, g_{e}\right) \leq\left(\frac{1}{\lambda_{0}}+L\right)^{-1}\left(g_{e}, g_{e}\right) \\
& \leq 9\left(\frac{1}{\lambda_{0}}+L\right)^{-1} L^{2}[\operatorname{rate}(\lambda)]^{2 m}\left\|\sum_{i=1}^{n} e_{p_{i}}^{0}\right\|^{2}  \tag{42}\\
& \leq 9 L[\operatorname{rate}(\lambda)]^{2 m}\left\|\sum_{i=1}^{n} e_{p_{i}}^{0}\right\|^{2}
\end{align*}
$$

Combining (41) and (42) then implies (37) and (38).

## 4 Discrete MPET problem

In this section, mass conservative discretizations of the MPET model are discussed, see also [18, 17]. The analysis here can be similarly used for other stable discretizations of the MPET model.

### 4.1 Notation

We consider a shape-regular triangulation $\mathcal{T}_{h}$ of $\Omega$ into triangles/tetrahedrons. Here, the subscript $h$ indicates the mesh-size. The set of all interior edges/faces and the set of all boundary edges/faces of $\mathcal{T}_{h}$ are denoted by $\mathcal{E}_{h}^{I}$ and $\mathcal{E}_{h}^{B}$ respectively and their union by $\mathcal{E}_{h}$.

We define the broken Sobolev spaces

$$
H^{s}\left(\mathcal{T}_{h}\right)=\left\{\phi \in L^{2}(\Omega), \text { such that }\left.\phi\right|_{T} \in H^{s}(T) \text { for all } T \in \mathcal{T}_{h}\right\}
$$

for $s \geq 1$.
We next introduce the notion of jumps [•] and averages $\{\cdot\}$. Let $T_{1}$ and $T_{2}$ be two elements from the triangulation sharing an edge or face $e$ and $\boldsymbol{n}_{1}$ and $\boldsymbol{n}_{2}$ be the corresponding unit normal vectors to $e$ pointing to the exterior of $T_{1}$ and $T_{2}$. Then for $q \in H^{1}\left(\mathcal{T}_{h}\right), \boldsymbol{v} \in H^{1}\left(\mathcal{T}_{h}\right)^{d}$ and $\boldsymbol{\tau} \in H^{1}\left(\mathcal{T}_{h}\right)^{d \times d}$ and any $e \in \mathcal{E}_{h}^{I}$ we define

$$
[q]=\left.q\right|_{\partial T_{1} \cap e}-\left.q\right|_{\partial T_{2} \cap e}, \quad[\boldsymbol{v}]=\left.\boldsymbol{v}\right|_{\partial T_{1} \cap e}-\left.\boldsymbol{v}\right|_{\partial T_{2} \cap e}
$$

and

$$
\{\boldsymbol{v}\}=\frac{1}{2}\left(\left.\boldsymbol{v}\right|_{\partial T_{1} \cap e} \cdot \boldsymbol{n}_{1}-\left.\boldsymbol{v}\right|_{\partial T_{2} \cap e} \cdot \boldsymbol{n}_{2}\right), \quad\{\boldsymbol{\tau}\}=\frac{1}{2}\left(\left.\boldsymbol{\tau}\right|_{\partial T_{1} \cap e} \boldsymbol{n}_{1}-\left.\boldsymbol{\tau}\right|_{\partial T_{2} \cap e} \boldsymbol{n}_{2}\right),
$$

while for $e \in \mathcal{E}_{h}^{B}$,

$$
[q]=\left.q\right|_{e}, \quad[\boldsymbol{v}]=\left.\boldsymbol{v}\right|_{e}, \quad\{\boldsymbol{v}\}=\left.\boldsymbol{v}\right|_{e} \cdot \boldsymbol{n}, \quad\{\boldsymbol{\tau}\}=\left.\boldsymbol{\tau}\right|_{e} \boldsymbol{n} .
$$

### 4.2 Mixed finite element spaces and discrete formulation

In order to discretize the flow equations, we use a mixed finite element method to approximate the fluxes and pressures whereas for the mechanics problem we apply a discontinuous Galerkin method to approximate the displacement. The considered finite element spaces are denoted by:

$$
\begin{aligned}
\boldsymbol{U}_{h} & =\left\{\boldsymbol{u} \in H(\operatorname{div} ; \Omega):\left.\boldsymbol{u}\right|_{T} \in \boldsymbol{U}(T), T \in \mathcal{T}_{h} ; \boldsymbol{u} \cdot \boldsymbol{n}=0 \text { on } \partial \Omega\right\}, \\
\boldsymbol{V}_{i, h} & =\left\{\boldsymbol{v} \in H(\operatorname{div} ; \Omega):\left.\boldsymbol{v}\right|_{T} \in \boldsymbol{V}_{i}(T), T \in \mathcal{T}_{h} ; \boldsymbol{v} \cdot \boldsymbol{n}=0 \text { on } \partial \Omega\right\}, \quad i=1, \ldots, n, \\
P_{i, h} & =\left\{q \in L^{2}(\Omega):\left.q\right|_{T} \in Q_{i}(T), T \in \mathcal{T}_{h} ; \int_{\Omega} q d x=0\right\}, \quad i=1, \ldots, n,
\end{aligned}
$$

where $\boldsymbol{V}_{i}(T) / Q_{i}(T)=\mathrm{RT}_{l-1}(T) / \mathrm{P}_{l-1}(T)$ and $\boldsymbol{U}(T)=\mathrm{BDM}_{l}(T)$ or $\boldsymbol{U}(T)=\mathrm{BDFM}_{l}(T)$ for $l \geq 1$. For each of these choices, we would like to point out that $\operatorname{div} \boldsymbol{U}(T)=\operatorname{div} \boldsymbol{V}_{i}(T)=Q_{i}(T)$ is satisfied.

As commented also in [17, 18], for all $e \in \mathcal{E}_{h}$ and for all $\boldsymbol{\tau} \in H^{1}\left(\mathcal{T}_{h}\right)^{d}, \boldsymbol{u} \in \boldsymbol{U}_{h}$ it holds that

$$
\begin{equation*}
\int_{e}\left[\boldsymbol{u}_{n}\right] \cdot \boldsymbol{\tau} d s=0, \quad \text { from which it follows } \quad \int_{e}[\boldsymbol{u}] \cdot \boldsymbol{\tau} d s=\int_{e}\left[\boldsymbol{u}_{t}\right] \cdot \boldsymbol{\tau} d s \tag{43}
\end{equation*}
$$

where $\boldsymbol{u}_{n}$ and $\boldsymbol{u}_{t}$ denote the normal and tangential component of $\boldsymbol{u}$ respectively.
Using the notation

$$
\boldsymbol{v}_{h}^{T}=\left(\boldsymbol{v}_{1, h}^{T}, \cdots \boldsymbol{v}_{n, h}^{T}\right), \quad \boldsymbol{p}_{h}^{T}=\left(p_{1, h}, \cdots, p_{n, h}\right), \quad \boldsymbol{z}_{h}^{T}=\left(\boldsymbol{z}_{1, h}^{T}, \cdots \boldsymbol{z}_{n, h}^{T}\right), \quad \boldsymbol{q}_{h}^{T}=\left(q_{1, h}, \cdots, q_{n, h}\right),
$$

$$
\boldsymbol{V}_{h}=\boldsymbol{V}_{1, h} \times \cdots \times \boldsymbol{V}_{n, h}, \quad \boldsymbol{P}_{h}=P_{1, h} \times \cdots \times P_{n, h}, \quad \boldsymbol{X}_{h}=\boldsymbol{U}_{h} \times \boldsymbol{V}_{h} \times \boldsymbol{P}_{h}
$$

the discretization of the variational problem (6) can be expressed as: Find $\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h},\right) \in \boldsymbol{X}_{h}$, such that for any $\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right) \in \boldsymbol{X}_{h}$ and $i=1, \ldots, n$

$$
\begin{align*}
a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{w}_{h}\right)+\lambda\left(\operatorname{div} \boldsymbol{u}_{h}, \operatorname{div} \boldsymbol{w}_{h}\right)-\sum_{i=1}^{n}\left(p_{i, h}, \operatorname{div} \boldsymbol{w}_{h}\right) & =\left(\boldsymbol{f}, \boldsymbol{w}_{h}\right),  \tag{44a}\\
\left(R_{i}^{-1} \boldsymbol{v}_{i, h}, \boldsymbol{z}_{i, h}\right)-\left(p_{i, h}, \operatorname{div} \boldsymbol{z}_{i, h}\right) & =0,  \tag{44b}\\
-\left(\operatorname{div} \boldsymbol{u}_{h}, q_{i, h}\right)-\left(\operatorname{div} \boldsymbol{v}_{i, h}, q_{i . h}\right)+\tilde{\alpha}_{i i}\left(p_{i, h}, q_{i, h}\right)+\sum_{\substack{j=1 \\
j \neq i}}^{n} \alpha_{i j}\left(p_{j, h}, q_{i, h}\right) & =\left(g_{i}, q_{i, h}\right) \tag{44c}
\end{align*}
$$

where

$$
\begin{align*}
a_{h}(\boldsymbol{u}, \boldsymbol{w})= & \sum_{T \in \mathcal{T}_{h}} \int_{T} \boldsymbol{\epsilon}(\boldsymbol{u}): \boldsymbol{\epsilon}(\boldsymbol{w}) d x-\sum_{e \in \mathcal{E}_{h}} \int_{e}\{\boldsymbol{\epsilon}(\boldsymbol{u})\} \cdot\left[\boldsymbol{w}_{t}\right] d s  \tag{45}\\
& -\sum_{e \in \mathcal{E}_{h}} \int_{e}\{\boldsymbol{\epsilon}(\boldsymbol{w})\} \cdot\left[\boldsymbol{u}_{t}\right] d s+\sum_{e \in \mathcal{E}_{h}} \int_{e} \eta h_{e}^{-1}\left[\boldsymbol{u}_{t}\right] \cdot\left[\boldsymbol{w}_{t}\right] d s
\end{align*}
$$

$\tilde{\alpha}_{i i}=-\alpha_{p_{i}}-\alpha_{i i}$, and $\eta$ is a stabilization parameter independent of the parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}$, $\alpha_{i j}$, where $i, j \in\{1, \ldots, n\}$, the network scale $n$ and the mesh size $h$.

The discrete variational problem (44) corresponds to the weak formulation (6) with homogeneous boundary conditions. The DG discretizations for general rescaled boundary conditions can be found in [18, (17].

### 4.3 Stability properties

Let $\boldsymbol{u}$ be a function from $\boldsymbol{U}_{h}$ and consider the mesh dependent norms

$$
\begin{gather*}
\|\boldsymbol{u}\|_{h}^{2}=\sum_{K \in \mathcal{T}_{h}}\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{0, K}^{2}+\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\left\|\left[\boldsymbol{u}_{t}\right]\right\|_{0, e}^{2}, \\
\|\boldsymbol{u}\|_{1, h}^{2}=\sum_{K \in \mathcal{T}_{h}}\|\nabla \boldsymbol{u}\|_{0, K}^{2}+\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\left\|\left[\boldsymbol{u}_{t}\right]\right\|_{0, e}^{2}, \\
\|\boldsymbol{u}\|_{D G}^{2}=\sum_{K \in \mathcal{T}_{h}}\|\nabla \boldsymbol{u}\|_{0, K}^{2}+\sum_{e \in \mathcal{E}_{h}} h_{e}^{-1}\left\|\left[\boldsymbol{u}_{t}\right]\right\|_{0, e}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K}^{2}|\boldsymbol{u}|_{2, K}^{2} \tag{46}
\end{gather*}
$$

and

$$
\begin{equation*}
\|\boldsymbol{u}\|_{\boldsymbol{U}_{h}}^{2}=\|\boldsymbol{u}\|_{D G}^{2}+\lambda\|\operatorname{div} \boldsymbol{u}\|^{2} . \tag{47}
\end{equation*}
$$

The well-posedness and approximation properties of the DG formulation are detailed in [19, 16]. Here we briefly present some important results:

- $\|\cdot\|_{D G},\|\cdot\|_{h}$, and $\|\cdot\|_{1, h}$ are equivalent on $\boldsymbol{U}_{h}$; that is

$$
\|\boldsymbol{u}\|_{D G} \bar{\sim}\|\boldsymbol{u}\|_{h} \bar{\sim}\|\boldsymbol{u}\|_{1, h}, \text { for all } \boldsymbol{u} \in \boldsymbol{U}_{h}
$$

- $a_{h}(\cdot, \cdot)$ from (45) is continuous and it holds true that

$$
\begin{equation*}
\left|a_{h}(\boldsymbol{u}, \boldsymbol{w})\right| \lesssim\|\boldsymbol{u}\|_{D G}\|\boldsymbol{w}\|_{D G}, \quad \text { for all } \quad \boldsymbol{u}, \boldsymbol{w} \in H^{2}\left(\mathcal{T}_{h}\right)^{d} . \tag{48}
\end{equation*}
$$

- The inf-sup conditions

$$
\begin{align*}
& \inf _{\left(q_{1, h}, \cdots, q_{n, h}\right) \in P_{1, h} \times \cdots \times P_{n, h}} \sup _{\boldsymbol{u}_{h} \in \boldsymbol{U}_{h}} \frac{\left(\operatorname{div} \boldsymbol{u}_{h}, \sum_{i=1}^{n} q_{i, h}\right)}{\left\|\boldsymbol{u}_{h}\right\|_{1, h}\left\|\sum_{i=1}^{n} q_{i, h}\right\|} \geq \beta_{s d},  \tag{49}\\
& \inf _{q_{i, h} \in P_{i, h}} \sup _{\boldsymbol{v}_{i, h} \in \boldsymbol{V}_{i, h}} \frac{\left(\operatorname{div} \boldsymbol{v}_{i, h}, q_{i, h}\right)}{\left\|\boldsymbol{v}_{i, h}\right\|_{\operatorname{div}}\left\|q_{i, h}\right\|} \geq \beta_{d d}, \quad i=1, \ldots, n,
\end{align*}
$$

are valid for our choice of $\boldsymbol{U}_{h}, \boldsymbol{V}_{h}$ and $\boldsymbol{P}_{h}$, see [24], and the positive constants $\beta_{s d}$ and $\beta_{d d}$ are independent of $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}$ for $i, j \in\{1, \ldots, n\}$, the network scale $n$ and the mesh size $h$. - $a_{h}(\cdot, \cdot)$ is coercive, namely

$$
\begin{equation*}
a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{u}_{h}\right) \geq \alpha_{a}\left\|\boldsymbol{u}_{h}\right\|_{h}^{2}, \quad \text { for all } \quad \boldsymbol{u}_{h} \in \boldsymbol{U}_{h}, \tag{50}
\end{equation*}
$$

where $\alpha_{a}>0$ is a constant independent of the model and discretization parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}$, $\alpha_{i j}, i, j \in\{1, \ldots, n\}, n$ and $h$.
Using the definition of the matrices $\Lambda_{1}$ and $\Lambda_{2}$, we define the bilinear form

$$
\begin{align*}
& \mathcal{A}_{h}\left(\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right),\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right)\right)=a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{w}_{h}\right)+\lambda\left(\operatorname{div} \boldsymbol{u}_{h}, \operatorname{div} \boldsymbol{w}_{h}\right)-\sum_{i=1}^{n}\left(p_{i, h}, \operatorname{div} \boldsymbol{w}_{h}\right) \\
& +\sum_{i=1}^{n}\left(R_{i}^{-1} \boldsymbol{v}_{i, h}, \boldsymbol{z}_{i, h}\right)-\left(\boldsymbol{p}_{h}, \operatorname{Div} \boldsymbol{z}_{h}\right)-\left(\operatorname{div} \boldsymbol{u}_{h}, \sum_{i=1}^{n} q_{i, h}\right)-\left(\operatorname{Div} \boldsymbol{v}_{h}, \boldsymbol{q}_{h}\right)-\left(\left(\Lambda_{1}+\Lambda_{2}\right) \boldsymbol{p}_{h}, \boldsymbol{q}_{h}\right) \tag{51}
\end{align*}
$$

related to problem (44a)-(44c).
Similar to Theorem 1, the following uniform stability results can be found in [18].

## Theorem 6.

(i) For any $\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right) \in \boldsymbol{X}_{h},\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right) \in \boldsymbol{X}_{h}$ there exists a positive constant $C_{b d}$ independent of the parameters $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in\{1, \ldots, n\}$, the network scale $n$ and the mesh size $h$ such that the inequality

$$
\left|\mathcal{A}_{h}\left(\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right),\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right)\right)\right| \leq C_{b d}\left(\left\|\boldsymbol{u}_{h}\right\|_{\boldsymbol{U}_{h}}+\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{p}_{h}\right\|_{\boldsymbol{P}}\right)\left(\left\|\boldsymbol{w}_{h}\right\|_{\boldsymbol{U}_{h}}+\left\|\boldsymbol{z}_{h}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{q}_{h}\right\|_{\boldsymbol{P}}\right)
$$

holds true.
(ii) There exists a constant $\beta_{0}>0$ independent of the model and discretization parameters $\lambda$, $R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in\{1, \ldots, n\}, n$ and $h$, such that

$$
\begin{equation*}
\inf _{\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right) \in \boldsymbol{X}_{h}} \sup _{\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right) \in \boldsymbol{X}_{h}} \frac{\mathcal{A}_{h}\left(\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right),\left(\boldsymbol{w}_{h} ; \boldsymbol{z}_{h} ; \boldsymbol{q}_{h}\right)\right)}{\left(\left\|\boldsymbol{u}_{h}\right\|_{\boldsymbol{U}_{h}}+\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{p}_{h}\right\|_{\boldsymbol{P}}\right)\left(\left\|\boldsymbol{w}_{h}\right\|_{\boldsymbol{U}_{h}}+\left\|\boldsymbol{z}_{h}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{q}_{h}\right\|_{\boldsymbol{P}}\right)} \geq \beta_{0} \tag{52}
\end{equation*}
$$

(iii) Let $\left(\boldsymbol{u}_{h} ; \boldsymbol{v}_{h} ; \boldsymbol{p}_{h}\right) \in \boldsymbol{X}_{h}$ solve (44a) -(44c) and

$$
\|\boldsymbol{f}\|_{\boldsymbol{U}_{h}^{*}}=\sup _{\boldsymbol{w}_{h} \in \boldsymbol{U}_{h}} \frac{\left(\boldsymbol{f}, \boldsymbol{w}_{h}\right)}{\left\|\boldsymbol{w}_{h}\right\|_{\boldsymbol{U}_{h}}}, \quad\|\boldsymbol{g}\|_{\boldsymbol{P}^{*}}=\sup _{\boldsymbol{q}_{h} \in \boldsymbol{P}_{h}} \frac{\left(\boldsymbol{g}, \boldsymbol{q}_{h}\right)}{\left\|\boldsymbol{q}_{h}\right\|_{\boldsymbol{P}}}
$$

Then the estimate

$$
\begin{equation*}
\left\|\boldsymbol{u}_{h}\right\|_{\boldsymbol{U}_{h}}+\left\|\boldsymbol{v}_{h}\right\|_{\boldsymbol{V}}+\left\|\boldsymbol{p}_{h}\right\|_{\boldsymbol{P}} \leq C_{2}\left(\|\boldsymbol{f}\|_{\boldsymbol{U}_{h}^{*}}+\|\boldsymbol{g}\|_{\boldsymbol{P}^{*}}\right) \tag{53}
\end{equation*}
$$

holds with a constant $C_{2}$ independent of $\lambda, R_{i}^{-1}, \alpha_{p_{i}}, \alpha_{i j}, i, j \in\{1, \ldots, n\}$, the network scale $n$ and the mesh size $h$.

## 5 Fixed-stress method for the discrete MPET model

In the manner of Algorithm 1, we formulate the fixed-stress method for the mixed continuousdiscontinuous Galerkin finite element method (44):

Algorithm 2 : Fixed-stress method for the discrete MPET problem
Step a: Given $\boldsymbol{u}_{h}^{m}$, we solve for $\boldsymbol{v}_{i, h}^{m+1}$ and $p_{i, h}^{m+1}$

$$
\begin{gather*}
\left(-\operatorname{div} \boldsymbol{v}_{i, h}^{m+1}, q_{i, h}\right)-\left(\left(\alpha_{p_{i, h}}+\alpha_{i i}\right) p_{i, h}^{m+1}, q_{i, h}\right)+\left(\sum_{\substack{j=1 \\
j \neq i}}^{n} \alpha_{i j} p_{j, h}^{m+1}, q_{i, h}\right)-L\left(\sum_{j=1}^{n} p_{j, h}^{m+1}, q_{i, h}\right)  \tag{54}\\
=\left(g_{i}, q_{i, h}\right)-L\left(\sum_{j=1}^{n} p_{j, h}^{m}, q_{i, h}\right)+\left(\operatorname{div} \boldsymbol{u}^{m}, q_{i, h}\right), \quad 1 \leq i \leq n,
\end{gather*}
$$

and

$$
\begin{equation*}
\left(R_{i}^{-1} \boldsymbol{v}_{i, h}^{m+1}, \boldsymbol{z}_{i, h}\right)-\left(p_{i, h}^{m+1}, \operatorname{div} \boldsymbol{z}_{i, h}\right)=0, \quad 1 \leq i \leq n . \tag{55}
\end{equation*}
$$

Step b: Given $\boldsymbol{v}_{i, h}^{m+1}$ and $p_{i, h}^{m+1}$, we solve for $\boldsymbol{u}_{h}^{m+1}$

$$
\begin{equation*}
a_{h}\left(\boldsymbol{u}_{h}^{m+1}, \boldsymbol{w}_{h}\right)+\lambda\left(\operatorname{div} \boldsymbol{u}_{h}^{m+1}, \operatorname{div} \boldsymbol{w}_{h}\right)=\left(\boldsymbol{f}, \boldsymbol{w}_{h}\right)+\sum_{i=1}^{n}\left(p_{i, h}^{m+1}, \operatorname{div} \boldsymbol{w}_{h}\right) . \tag{56}
\end{equation*}
$$

The main convergence result for Algorithm 2 is formulated in terms of the following quantities corresponding to the discrete case:

$$
\begin{align*}
\boldsymbol{e}_{u_{h}}^{k} & =\boldsymbol{u}_{h}^{k}-\boldsymbol{u}_{h} \in \boldsymbol{U}_{h},  \tag{57a}\\
\boldsymbol{e}_{v_{i, h}}^{k} & =\boldsymbol{v}_{i, h}^{k}-\boldsymbol{v}_{i, h} \in \boldsymbol{V}_{i, h}, \quad i=1, \ldots, n,  \tag{57b}\\
e_{p_{i, h}}^{k} & =p_{i, h}^{k}-p_{i, h} \in P_{i, h}, \quad i=1, \ldots, n, \tag{57c}
\end{align*}
$$

denoting the errors of the $k$-th iterates $\boldsymbol{u}_{h}^{k}, \boldsymbol{v}_{i, h}^{k}, p_{i, h}^{k}, i=1, \ldots, n$ generated by Algrorithm 2, In the discrete case, the useful constant for defining the tuning parameter $L$ is the constant $c_{K_{d}}$ from the estimate

$$
\begin{equation*}
a_{h}\left(\boldsymbol{w}_{h}, \boldsymbol{w}_{h}\right) \geq c_{K_{d}}^{2}\left\|\operatorname{div} \boldsymbol{w}_{h}\right\|^{2} \quad \text { for all } \boldsymbol{w}_{h} \in \boldsymbol{U}_{h} \tag{58}
\end{equation*}
$$

Note that $c_{K_{d}}$ is strictly positive and independent of the mesh size $h$.
Using the approach applied to proving Lemma 2, for the continuous MPET model we obtain the corresponding lemma for the discrete case as follows:
Lemma 7. The errors $\boldsymbol{e}_{u_{h}}^{m+1}$, $\boldsymbol{e}_{v_{h}}^{m+1}$ and $\boldsymbol{e}_{p_{h}}^{m+1}$ of the $(m+1)$-st fixed-stress iterate generated by Algorithm 图 for $L \geq \frac{1}{\lambda+c_{K_{d}}^{2}}$ satisfy the estimate

$$
\begin{align*}
& \frac{1}{2}\left(a_{h}\left(\boldsymbol{e}_{u_{h}}^{m+1}, \boldsymbol{e}_{u_{h}}^{m+1}\right)+\lambda\left\|\operatorname{div} \boldsymbol{e}_{u_{h}}^{m+1}\right\|^{2}\right)+\left\|R^{-1 / 2} \boldsymbol{e}_{v_{h}}^{m+1}\right\|^{2}+\left\|\left(\Lambda_{1}+\Lambda_{2}\right)^{1 / 2} \boldsymbol{e}_{p_{h}}^{m+1}\right\|^{2} \\
& \quad+\frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i, h}}^{m+1}\right\|^{2} \leq \frac{L}{2}\left\|\sum_{i=1}^{n} e_{p_{i, h}}^{m}\right\|^{2}, \quad m=0,1,2, \ldots \tag{59}
\end{align*}
$$

By Lemma 7, again following the proof of Theorem 3 for the continuous MPET model, we obtain the corresponding statements, Theorem 8, for the discrete case:

Theorem 8. Let $c_{K_{d}}$ and $\beta_{s d}$ denote the constants in (58) and (49) respectively. The single rate fixed-stress iterative method for the discrete static MPET problem (44) defined in Algorithm $\mathbf{Q}^{2}$ is a contraction that converges linearly for any $L \geq 1 /\left(\lambda+c_{K_{d}}^{2}\right)$ independent of the model parameters, the time step size $\tau$ and the mesh size $h$. The errors $\boldsymbol{e}_{p_{h}}^{m}$ in this case satisfy the inequality

$$
\begin{equation*}
\left\|\sum_{i=1}^{n} e_{p_{i, h}}^{m+1}\right\|^{2} \leq \operatorname{rate}_{d}^{2}(\lambda)\left\|\sum_{i=1}^{n} e_{p_{i, h}}^{m}\right\|^{2} \tag{60}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{rate}_{d}^{2}(\lambda) \leq \frac{1}{\frac{L^{-1}}{\beta_{s d}^{-2}+\lambda}+1} \tag{61}
\end{equation*}
$$

For $L=\frac{1}{\lambda+c_{K_{d}}^{2}}$, the convergence factor in (60) can be estimated by

$$
\begin{equation*}
\operatorname{rate}_{d}^{2}(\lambda) \leq \frac{1}{\frac{\lambda+c_{K_{d}}^{2}}{\beta_{s d}^{-2}+\lambda}+1} \leq \max \left\{\frac{\beta_{s d}^{-2}}{c_{K_{d}}^{2}+\beta_{s d}^{-2}}, \frac{1}{2}\right\} \tag{62}
\end{equation*}
$$

Note that Theorem 8 only gives the convergence rate of $\boldsymbol{e}_{p_{h}}^{m}$ in the semi-norm $\left\|\sum_{i=1}^{n} e_{p_{i, h}}^{m}\right\|$. However, we can combine the estimates in Theorem 8 with the uniform stability result presented in Theorem 6 and follow the proof of Theorem 5 to obtain the following convergence results for $\boldsymbol{e}_{u_{h}}^{m}, \boldsymbol{e}_{v_{h}}^{m}$ and $\boldsymbol{e}_{p_{h}}^{m}$ in their respective parameter-dependent full norms:
Theorem 9. The errors $\boldsymbol{e}_{u_{h}}^{m}, \boldsymbol{e}_{v_{h}}^{m}$ and $\boldsymbol{e}_{p_{h}}^{m}$ defined in (57) measured in the norms induced by (47) and (7) satisfy the estimates:

$$
\begin{gather*}
\left\|\boldsymbol{e}_{u_{h}}^{m}\right\|_{\boldsymbol{U}_{h}}^{2} \leq C_{u d}\left[\operatorname{rate}_{d}(\lambda)\right]^{2 m}  \tag{63}\\
\left\|\boldsymbol{e}_{v_{h}}^{m}\right\|_{\boldsymbol{V}_{e}}^{2}+\left\|\boldsymbol{e}_{p_{h}}^{m}\right\|_{\boldsymbol{P}_{e}}^{2} \leq C_{v p d}\left[\operatorname{rate}_{d}(\lambda)\right]^{2 m} \tag{64}
\end{gather*}
$$

where the constants $C_{u d}$ and $C_{v p d}$ are independent of the model parameters, the time step size and the mesh size.

## 6 Numerical results

In our numerical test setup, we assume that:

- $\Omega=[0,1]$ is partitioned into $2 N^{2}$ right-angled triangles with catheti of length $h=\frac{1}{N}$;
- Problem (6) is discretized by a strongly conservative discontinuous Galerkin method based on a mixed finite element space formed by the triplet of $\mathrm{BDM}_{1} / \mathrm{RT}_{0} / \mathrm{P}_{0}^{d c}$ elements;
- the constant for the fixed-stress splitting is $L=\frac{1}{1+\lambda}$;
- the iterative process is terminated when residual reduction by a factor $10^{8}$ in the combined norm induced by the inner products (7) (the norm induced by the inverse of the preconditioner) is reached.

Numerical experiments have been performed in FEniCS, [3, 22], and their aim was:
(i) to validate the theoretical estimates for the convergence of the fixed-stress splitting;
(ii) to compare the performance of the latter with the preconditioned MinRes algorithm using the norm-equivalent preconditioner proposed in [18].

### 6.1 The two-network model

The Biot-Barenblatt model involves two pressures and two fluxes. In our notation, it has the following formulation:

$$
\begin{align*}
-\operatorname{div}\left(\boldsymbol{\sigma}-p_{1} \boldsymbol{I}-p_{2} \boldsymbol{I}\right)=\boldsymbol{f}, &  \tag{65a}\\
R_{i}^{-1} \boldsymbol{v}_{i}+\nabla p_{i} & =0,  \tag{65b}\\
-\operatorname{div} \boldsymbol{u}-\operatorname{div} \boldsymbol{v}_{i}-\alpha_{p_{i}} p_{i}+\sum_{\substack{j=1 \\
j \neq i}}^{2} \alpha_{i j} p_{j} & =g_{i},  \tag{65c}\\
& i=1,2 .
\end{align*}
$$

Specifically, the subject of numerical study in this subsection is the cantilever bracket benchmark problem, see [14, for which $\boldsymbol{f}=\mathbf{0}, g_{1}=g_{2}=0$. The boundary $\Gamma$ of the domain $\Omega=[0,1]^{2}$ is split into bottom, right, top and left boundaries denoted by $\Gamma_{1}, \Gamma_{2}, \Gamma_{3}$ and $\Gamma_{4}$ respectively and

$$
\begin{aligned}
\left(\boldsymbol{\sigma}-p_{1} \boldsymbol{I}-p_{2} \boldsymbol{I}\right) \boldsymbol{n} & =(0,0)^{T} & & \text { on } \Gamma_{1} \cup \Gamma_{2}, \\
\left(\boldsymbol{\sigma}-p_{1} \boldsymbol{I}-p_{2} \boldsymbol{I}\right) \boldsymbol{n} & =(0,-1)^{T} & & \text { on } \Gamma_{3}, \\
\boldsymbol{u} & =\mathbf{0} & & \text { on } \Gamma_{4}, \\
p_{1} & =2 & & \text { on } \Gamma, \\
p_{2} & =20 & & \text { on } \Gamma .
\end{aligned}
$$

Table 1 gives the base values of the model parameters as taken from [21]. We have varied the parameter $K_{2}$ over a wider range than $K_{1}$ since, at least, for the MinRes iteration it happened to be the more interesting case. The results in Tables 2 4 show very clearly the robust behaviour of the fixed-stress iteration with respect to mesh refinements and variation of the hydraulic conductivities $K_{1}$ and $K_{2}$, and also $\lambda$. Furthermore, they demonstrate its advantage over the MinRes method in terms of rate of convergence.

### 6.2 The four-network model

This subsection is devoted to the four-network MPET model. As with the previous example, the boundary $\Gamma$ of $\Omega$ is split into bottom $\left(\Gamma_{1}\right)$, right $\left(\Gamma_{2}\right)$, top $\left(\Gamma_{3}\right)$, and left $\left(\Gamma_{4}\right)$ boundaries. The considered boundary conditions are chosen as:

$$
\begin{aligned}
\left(\boldsymbol{\sigma}-p_{1} \boldsymbol{I}-p_{2} \boldsymbol{I}-p_{3} \boldsymbol{I}-p_{4} \boldsymbol{I}\right) \boldsymbol{n} & =(0,0)^{T} & & \text { on } \Gamma_{1} \cup \Gamma_{2}, \\
\left(\boldsymbol{\sigma}-p_{1} \boldsymbol{I}-p_{2} \boldsymbol{I}-p_{3} \boldsymbol{I}-p_{4} \boldsymbol{I}\right) \boldsymbol{n} & =(0,-1)^{T} & & \text { on } \Gamma_{3}, \\
\boldsymbol{u} & =\mathbf{0} & & \text { on } \Gamma_{4}, \\
p_{1} & =2 & & \text { on } \Gamma, \\
p_{2} & =20 & & \text { on } \Gamma, \\
p_{3} & =30 & & \text { on } \Gamma, \\
p_{4} & =40 & & \text { on } \Gamma,
\end{aligned}
$$

Table 1: Base values of model parameters for a Barenblatt model.

| parameter | value | unit |
| :---: | :---: | :---: |
| $\lambda$ | 4.2 | MPa |
| $\mu$ | 2.4 | MPa |
| $c_{p_{1}}$ | 54 | $(\mathrm{GPa})^{-1}$ |
| $c_{p_{2}}$ | 14 | $(\mathrm{GPa})^{-1}$ |
| $\alpha_{1}$ | 0.95 |  |
| $\alpha_{2}$ | 0.12 |  |
| $\beta$ | 5 | $10^{-10} \mathrm{~kg} /(\mathrm{m} \cdot \mathrm{s})$ |
| $K_{1}$ | 100 | $10^{-10} \mathrm{~kg} /(\mathrm{m} \cdot \mathrm{s})$ |
| $K_{2}$ | 6.18 | $10^{-15} \mathrm{~m}^{2}$ |
|  | 27.2 | $10^{-15} \mathrm{~m}^{2}$ |

Table 2: Number of preconditioned MinRes and fixed-stress splitting iterations for residual reduction by a factor $10^{8}$ in the norm induced by the preconditioner when solving the Barenblatt problem.

| $h$ | $\beta$ |  | $K_{2}$ |  | $K_{2} \cdot 10^{2}$ |  | $K_{2} \cdot 10^{4}$ |  | $K_{2} \cdot 10^{6}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
|  |  | $K_{1}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
|  |  | $K_{1}$ | 16 | 8 | 21 | 8 | 37 | 8 | 29 | 8 |
| $\frac{1}{32}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1}$ | 16 | 8 | 26 | 8 | 38 | 8 | 27 | 8 |
| $\frac{1}{64}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1} \cdot 10^{-1}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |
|  |  | $K_{1}$ | 18 | 8 | 32 | 8 | 38 | 8 | 27 | 8 |

whereas the right hand sides are $\boldsymbol{f}=\mathbf{0}, g_{1}=g_{2}=g_{3}=g_{4}=0$.
Table 5)shows the base values of the parameters which have been taken from [28]. The presented numerical results in Table 6 demonstrate again the superiority of the fixed-stress splitting method over the preconditioned MinRes algorithm and its robustness with respect to large variations of the coefficients $\lambda, K_{3}$ and $K=K_{1}=K_{2}=K_{4}$.

## 7 Concluding remarks

To the best of our knowledge, this paper is the first example of a proposed and analyzed fixedstress splitting scheme for a three-field formulation of the MPET model. Fundamental to the linear convergence of the evolved algorithm is the incorporation of stabilization that employs the sum of all pressures. By applying the stability results proven in [18], we have demonstrated that

Table 3: Number of preconditioned MinRes and fixed-stress splitting iterations for residual reduction by a factor $10^{8}$ in the norm induced by the preconditioner when solving the Barenblatt problem where we have redefined $\lambda:=0.01 \cdot \lambda$.

| $h$ | $\beta$ |  | $K_{2}$ |  | $K_{2} \cdot 10^{2}$ |  | $K_{2} \cdot 10^{4}$ |  | $K_{2} \cdot 10^{6}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
|  |  | $K_{1} \cdot 10^{-1}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
|  |  | $K_{1}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
|  |  | $K_{1} \cdot 10^{-1}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
|  |  | $K_{1}$ | 24 | 11 | 38 | 11 | 71 | 11 | 42 | 11 |
| $\frac{1}{32}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1} \cdot 10^{-1}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1} \cdot 10^{-1}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1}$ | 25 | 10 | 45 | 10 | 66 | 10 | 38 | 10 |
| $\frac{1}{64}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 25 | 10 | 57 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1} \cdot 10^{-1}$ | 25 | 10 | 57 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1}$ | 25 | 10 | 57 | 10 | 66 | 10 | 38 | 10 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | 25 | 10 | 57 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1} \cdot 10^{-1}$ | 25 | 10 | 57 | 10 | 66 | 10 | 38 | 10 |
|  |  | $K_{1}$ | 25 | 10 | 57 | 16 | 66 | 10 | 38 | 10 |

Table 4: Number of preconditioned MinRes and fixed-stress splitting iterations for residual reduction by a factor $10^{8}$ in the norm induced by the preconditioner when solving the Barenblatt problem where we have redefined $\lambda:=100 \cdot \lambda$.

| $h$ | $\beta$ |  | $K_{2}$ | $K_{2} \cdot 10^{2}$ | $K_{2} \cdot 10^{4}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | 42 | 82 | 162 | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $4 \quad 2$ | $8 \quad 2$ | 162 | 14 | 2 |
|  |  | $K_{1}$ | $4 \quad 2$ | $8 \quad 2$ | $16 \quad 2$ | 14 | 2 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | $4 \quad 2$ | 82 | 162 | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $4 \quad 2$ | $8 \quad 2$ | 162 | 14 | 2 |
|  |  | $K_{1}$ | $4 \quad 2$ | $8 \quad 2$ | $16 \quad 2$ | 14 | 2 |
| $\frac{1}{32}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 2$ | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 2$ | 14 | 2 |
|  |  | $K_{1}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 3$ | 14 | 2 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 2$ | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 2$ | 14 | 2 |
|  |  | $K_{1}$ | $6 \quad 2$ | $12 \quad 2$ | $20 \quad 2$ | 14 | 2 |
| $\frac{1}{64}$ | 5E-10 | $K_{1} \cdot 10^{-2}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |
|  |  | $K_{1}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |
|  | 1E-8 | $K_{1} \cdot 10^{-2}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |
|  |  | $K_{1} \cdot 10^{-1}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |
|  |  | $K_{1}$ | $7 \quad 2$ | $16 \quad 2$ | $21 \quad 2$ | 14 | 2 |

Table 5: Base values of model parameters for a four-network MPET model.

| parameter | value | unit |
| :---: | :---: | :---: |
| $\lambda$ | 505 | $\mathrm{Nm}^{-2}$ |
| $\mu$ | 216 | $\mathrm{Nm}^{-2}$ |
| $c_{p_{1}}=c_{p_{2}}=c_{p_{3}}=c_{p_{4}}$ | $4.5 \cdot 10^{-10}$ | $\mathrm{~m}^{2} \mathrm{~N}^{-1}$ |
| $\alpha_{1}=\alpha_{2}=\alpha_{3}=\alpha_{4}$ | 0.99 |  |
| $\beta_{12}=\beta_{24}$ | $1.5 \cdot 10^{-19}$ | $\mathrm{~m}^{2} \mathrm{~N}^{-1} \mathrm{~s}^{-1}$ |
| $\beta_{23}$ | $2.0 \cdot 10^{-19}$ | $\mathrm{~m}^{2} \mathrm{~N}^{-1} \mathrm{~s}^{-1}$ |
| $\beta_{34}$ | $1.0 \cdot 10^{-13}$ | $\mathrm{~m}^{2} \mathrm{~N}^{-1} \mathrm{~s}^{-1}$ |
| $K_{1}=K_{2}=K_{4}=K$ | $\left(1.0 \cdot 10^{-10}\right) /\left(2.67 \cdot 10^{-3}\right)$ | $\mathrm{m}^{2} / \mathrm{Nsm}^{-2}$ |
| $K_{3}$ | $\left(1.4 \cdot 10^{-14}\right) /\left(8.9 \cdot 10^{-4}\right)$ | $\mathrm{m}^{2} / \mathrm{Nsm}^{-2}$ |

Table 6: Number of preconditioned MinRes and fixed-stress splitting iterations for residual reduction by a factor $10^{8}$ in the norm induced by the preconditioner when solving the four-network MPET problem.

| $h$ |  |  | $K_{3} \cdot 10^{-2}$ |  | $K_{3}$ |  | $K_{3} \cdot 10^{2}$ |  | $K_{3} \cdot 10^{4}$ |  | $K_{3} \cdot 10^{6}$ |  | $K_{3} \cdot 10^{10}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{16}$ | $\lambda$ | $K \cdot 10^{-2}$ | 34 | 10 | 34 | 10 | 26 | 10 | 23 | 10 | 21 | 10 | 21 | 10 |
|  |  | $K$ | 24 | 10 | 24 | 10 | 24 | 10 | 22 | 10 | 21 | 10 | 19 | 10 |
|  |  | $K \cdot 10^{2}$ | 21 | 10 | 21 | 10 | 23 | 10 | 23 | 10 | 31 | 10 | 30 | 10 |
|  | $\lambda \cdot 10^{4}$ | $K \cdot 10^{-2}$ | 18 | 2 | 23 | 2 | 24 | 2 | 34 | 2 | 34 | 2 | 34 | 2 |
|  |  | $K$ | 11 | 2 | 17 | 2 | 34 | 2 | 31 | 2 | 31 | 2 | 31 | 2 |
|  |  | $K \cdot 10^{2}$ | 9 | 2 | 14 | 2 | 32 | 2 | 21 | 2 | 14 | 2 | 14 | 2 |
|  | $\lambda \cdot 10^{8}$ | $K \cdot 10^{-2}$ | 14 | 2 | 14 | 2 | 12 | 2 | 12 | 2 | 12 | 2 | 12 | 2 |
|  |  | $K$ | 11 | 2 | 14 | 2 | 9 | 2 | 7 | 2 | 7 | 2 | 7 | 2 |
|  |  | $K \cdot 10^{2}$ | 9 | 2 | 14 | 2 | 9 | 2 | 5 | 2 | 5 | 2 | 5 | 2 |
| $\frac{1}{32}$ | $\lambda$ | $K \cdot 10^{-2}$ | 34 | 10 | 32 | 10 | 26 | 10 | 23 | 10 | 19 | 10 | 19 | 10 |
|  |  | $K$ | 24 | 10 | 24 | 10 | 24 | 10 | 22 | 10 | 21 | 10 | 20 | 10 |
|  |  | $K \cdot 10^{2}$ | 21 | 10 | 21 | 10 | 21 | 10 | 26 | 10 | 41 | 10 | 39 | 10 |
|  | $\lambda \cdot 10^{4}$ | $K \cdot 10^{-2}$ | 18 | 2 | 25 | 2 | 30 | 2 | 34 | 2 | 34 | 2 | 34 | 2 |
|  |  | $K$ | 12 | 2 | 20 | 2 | 35 | 2 | 31 | 2 | 31 | 2 | 31 | 2 |
|  |  | $K \cdot 10^{2}$ | 9 | 2 | 18 | 2 | 34 | 2 | 21 | 2 | 14 | 2 | 14 | 2 |
|  | $\lambda \cdot 10^{8}$ | $K \cdot 10^{-2}$ | 14 | 2 | 14 | 2 | 12 | 2 | 12 | 2 | 12 | 2 | 12 | 2 |
|  |  | K | 12 | 2 | 14 | 2 | 9 | 2 | 7 | 2 | 7 | 2 | 7 | 2 |
|  |  | $K \cdot 10^{2}$ | 11 | 2 | 14 | 2 | 9 | 2 | 6 | 2 | 5 | 2 | 5 | 2 |
| $\frac{1}{64}$ | $\lambda$ | $K \cdot 10^{-2}$ | 34 | 10 | 32 | 10 | 26 | 10 | 21 | 10 | 19 | 10 | 19 | 10 |
|  |  | K | 24 | 10 | 24 | 10 | 24 | 10 | 23 | 10 | 22 | 10 | 21 | 10 |
|  |  | $K \cdot 10^{2}$ | 21 | 10 | 21 | 10 | 21 | 10 | 36 | 10 | 45 | 10 | 45 | 10 |
|  | $\lambda \cdot 10^{4}$ | $K \cdot 10^{-2}$ | 20 | 2 | 28 | 2 | 34 | 2 | 34 | 2 | 34 | 2 | 34 | 2 |
|  |  | K | 13 | 2 | 25 | 2 | 36 | 2 | 31 | 2 | 31 | 2 | 31 | 2 |
|  |  | $K \cdot 10^{2}$ | 6 | 2 | 25 | 2 | 36 | 2 | 21 | 2 | 14 | 2 | 14 | 2 |
|  | $\lambda \cdot 10^{8}$ | $K \cdot 10^{-2}$ | 14 | 2 | 14 | 2 | 12 | 2 | 12 | 2 | 12 | 2 | 12 | 2 |
|  |  | $K$ | 12 | 2 | 14 | 2 | 9 | 2 | 7 | 2 | 7 | 2 | 7 | 2 |
|  |  | $K \cdot 10^{2}$ | 12 | 2 | 14 | 2 | 9 | 2 | 6 | 2 | 5 | 2 | 5 | 2 |

the contraction rate of this fixed-point iteration is independent of any model physical parameters. Furthermore, the performed numerical experiments have clearly demonstrated the efficiency of the presented fixed-stress scheme along with its superiority over a fully implicit scheme which utilizes a norm-equivalent preconditioner.

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[^0]:    ${ }^{1}$ The constant $c_{K}$ is related to Korn's inequality and, while in general not easy to bound tightly from below, can be estimated sufficiently in the discrete setting.

