On stability of small solitons of the 1–D NLS with a trapping delta potential

Scipio Cuccagna, Masaya Maeda

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Abstract

We consider a Nonlinear Schrödinger Equation with a very general non linear term and with a trapping δ potential on the line. We then discuss the asymptotic behavior of all its small solutions, generalizing a recent result by Masaki *et al.* [30]. We give also a result of dispersion in the case of defocusing equations with a non-trapping delta potential.

1 Introduction

In this paper we consider the Nonlinear Schrödinger Equation (NLS)

$$i\dot{u} = H_1 u + g(|u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \text{ with } u(0) = u_0 \in H^1(\mathbb{R}, \mathbb{C}), \tag{1.1}$$

with the Schrödinger operator (here $\delta(x)$ is the Dirac δ centered in 0)

$$H_q = -\partial_x^2 - q\delta(x) \text{ for } q \in \mathbb{R} \setminus \{0\}$$
 (1.2)

defined by $H_q := -\partial_x^2$ with domain

$$D(H_a) = \{ u \in H^1(\mathbb{R}, \mathbb{C}) \cap H^2(\mathbb{R} \setminus \{0\}, \mathbb{C}) \mid \partial_x u(0^+) - \partial_x u(0^-) = -qu(0) \}.$$
 (1.3)

For the nonlinearity, we assume $g \in C([0,\infty),\mathbb{R}) \cap C^3((0,\infty),\mathbb{R})$ and that there exist p > 0 and C > 0 s.t. for k = 0, 1, 2, 3 we have

$$|g^{(k)}(s)| \le C|s|^{p-k} \text{ for all } s \in (0,1].$$
 (1.4)

In particular, we have g(0) = 0 and the primitive G of g defined by

$$G'(s) = g(s) \text{ and } G(0) = 0.$$
 (1.5)

satisfies $|G(s)| \lesssim |s|^{p+1}$ for all $s \in (0,1)$.

Remark 1.1. A typical example we have in mind is $g(s) = \lambda s^p$ with p > 0 and $\lambda \in \{\pm 1\}$. In this case, our NLS can be written taking q = 1 in the form

$$i\dot{u} = H_a u + \lambda |u|^{2p} u,\tag{1.6}$$

which was considered by Masaki *et al.* [30] for the case $p \geq 2$. They also considered the cubic NLS for the cubic NLS, p = 1, with dispersive potential q < 0 in [31], where they proved dispersion, that is $||u(t)||_{L^{\infty}(\mathbb{R})} \lesssim t^{-\frac{1}{2}}$ as $t \to +\infty$, for appropriate very small solutions.

We recall, see [30], that the operator in (1.2) for q > 0 satisfies

$$\sigma_d(H_q) = \{-q^2/4\} \text{ with } \ker\left(H_q + q^2/4\right) = \operatorname{Sp}(\varphi_q) \text{ where } \varphi_q := \sqrt{q/2}e^{-\frac{q}{2}|x|}, \tag{1.7}$$

with $\operatorname{Sp}(\varphi_q) := \mathbb{C}\varphi_q$. Furthermore the point 0 is neither an eigenvalue nor a resonance for H_q , that is to say, the only $u_0 \in L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$ s.t. $H_q u_0 = 0$ is $u_0 = 0$.

We also have a spectral (orthogonal) decomposition

$$L^{2}(\mathbb{R}) = \operatorname{Sp}(\varphi_{q}) \oplus L_{c}^{2}(H_{q}) \tag{1.8}$$

with $L_c^2(H_q)$ the continuous spectrum component associated to H_q . We will consider the case q=1 and denote

$$L_c^2 := L_c^2(H_1)$$
 and $\varphi := \varphi_1$.

We will denote by P_c the projection onto L_c^2 . In particular,

$$P_c u := u - \left(\int_{\mathbb{R}} u \varphi \, dx \right) \varphi = u - \langle u, \varphi \rangle \, \varphi - \langle u, i\varphi \rangle \, i\varphi,$$

where

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}} f(x)\overline{g}(x)dx \text{ for } f, g : \mathbb{R} \to \mathbb{C} .$$
 (1.9)

We will also use the following notation.

- Given a Banach space $X, v \in X$ and $\delta > 0$ we set $D_X(v, \delta) := \{x \in X \mid ||v x||_X < \delta\}.$
- For $\gamma \in \mathbb{R}$ we set

$$L_{\gamma}^{2} := \{ u \in \mathcal{S}'(\mathbb{R}; \mathbb{C}) \mid \|u\|_{L_{\gamma}^{2}} := \|e^{\gamma|x|}u\|_{L^{2}} < \infty \}, \tag{1.10}$$

$$H_{\gamma}^{1} := \{ u \in \mathcal{S}'(\mathbb{R}; \mathbb{C}) \mid ||u||_{H_{\gamma}^{1}} := ||e^{\gamma|x|}u||_{H^{1}} < \infty \}.$$
 (1.11)

• For $f: \mathbb{C} \to X$ for some Banach space X, we set $D_1 f = \partial_{\operatorname{Re} z} f$ and $D_2 f = \partial_{\operatorname{Im} z} f$.

The eigenvalue of H_1 yields by bifurcation a family of standing waves solutions.

As in [15, 6, 30], we have the following, which we prove in the appendix.

Proposition 1.2 (Bound states). Let p > 0. Then there exists $\gamma_0 > 0$, $\alpha_0 > 0$ and C > 0 s.t. there exists a unique $Q \in C^1(D_{\mathbb{C}}(0, a_0); H^1_{\gamma_0})$ satisfying the gauge property

$$Q[e^{i\theta}z] = e^{i\theta}Q[z], \tag{1.12}$$

s.t. there exists $E \in C([0, a_0^2), \mathbb{R})$ s.t.

$$H_1Q[z] + g(|Q[z]|^2)Q[z] = E(|z|^2)Q[z], \tag{1.13}$$

and

$$||Q[z] - z\varphi||_{H^{1}_{\gamma_{0}}} \le C|z|^{2p+1}, \ ||D_{j}Q[z] - i^{j-1}\varphi||_{H^{1}_{\gamma}} \le C|z|^{2p}, \ \left|E(|z|^{2}) + \frac{1}{4}\right| \le C|z|^{2p}.$$
 (1.14)

Moreover, if 2p > 1 we have

$$Q[z] \in C^2\left(D_{\mathbb{C}}(0, a_0), H^1_{\gamma_0}\right),$$
(1.15)

and

$$||D_i D_k Q[z]||_{H^1} \le C|z|^{2p-1}, \quad j, k = 1, 2.$$
 (1.16)

Remark 1.3. In the case of power type nonlinearities $g(s) = s^p$, there is an explicit formula for Q[z]. See [12, 30].

Our first result is the following, related to [15, 33, 30], see [6] for more references.

Theorem 1.4. Assume p > 0 in (1.4). Then there exist $\epsilon_0 > 0$, $\gamma > 0$ and C > 0 such that for $\epsilon := \|u(0)\|_{H^1} < \epsilon_0$ the solution u(t) of (1.1) can be written uniquely for all times as

$$u(t) = Q[z(t)] + \xi(t) \text{ with } \xi(t) \in P_c H^1,$$
 (1.17)

s.t. we have

$$|z(t)| + \|\xi(t)\|_{H^1} \le C\epsilon \text{ for all } t \in [0, \infty),$$
 (1.18)

$$\int_0^\infty \|\xi\|_{H^1_{-\gamma}}^2 dt \le C\epsilon. \tag{1.19}$$

Theorem 1.4 claims that solutions with sufficiently small H^1 norm converge asymptotically to the set formed by the Q[z]. Indeed formula (1.19) is stating that, in an averaged sense, $\xi \xrightarrow{t\to\infty} 0$ locally in space. In Theorem 1.4 there is no proof of selection of ground state: we do not prove that up to a phase, z(t) has a limit as $t\to +\infty$. However, if we strengthen the hypotheses of the nonlinearity g(s), we obtain also the selection of ground states. This will be our second result. It requires a more subtle representation of u(t) than the one in (1.17), due to Gustafson et al. [15].

Definition 1.5. Consider the $a_0 > 0$ in Proposition 1.2.

$$\mathcal{H}_c[z] := \left\{ \eta \in L^2(\mathbb{R}) : \langle i \eta, D_1 Q \rangle = \langle i \eta, D_2 Q \rangle = 0 \right\}. \tag{1.20}$$

It is immediate that $\mathcal{H}_c[0] = L_c^2$. Our second result is the following.

Theorem 1.6. Let p > 1/2 in in (1.4). Then there exist $\epsilon_0 > 0$, $\gamma > 0$ and C > 0 such that for $\epsilon := \|u(0)\|_{H^1} < \epsilon_0$ the solution u(t) of (1.1) can be written uniquely for all times as

$$u(t) = Q[z(t)] + \eta(t) \text{ with } \eta(t) \in \mathcal{H}_c[z(t)], \tag{1.21}$$

s.t. we have

$$|z(t)| + ||\eta(t)||_{H^1} \le C\epsilon \text{ for all } t \in [0, \infty),$$
 (1.22)

$$\int_0^\infty \|\eta\|_{H^1_{-\gamma}}^2 dt \le C\epsilon. \tag{1.23}$$

and there exists a $z_+ \in \mathbb{C}$ such that

$$\lim_{t \to +\infty} z(t)e^{i\int_0^t E[z(s)]ds} = z_+. \tag{1.24}$$

We don't know if the last statement, with the limit (1.24), is true for $p \le 1/2$. We will prove Theorem 1.4 in Sect. 2 and Theorem 1.6 in Sect. 3.

Equations like (1.1) and its particular case (1.6) represent an interesting special type of the NLS in 1–D. Related models, obtained eliminating the linear $\delta(x)$ potential and replacing $g(|u|^2)u$ with $\sum_{j=1}^{n} \delta(x-x_j)g(|u|^2)u$, in some cases have been shown to satisfy very satisfactory characterizations of the global time behavior for all their finite energy solutions; see [23]–[26], which solved the *Soliton Risolution Conjecture* in these cases.

Returning to equation (1.1), Goodman et. al. [14] and Holmer et al. [16]–[19] have shown in the cubic case interesting patterns involving solitons, usually for finite intervals of time or numerically.

Some of these results have been proved for global times by Deift and Park [7], using the Inverse Scattering Transform. Masaki et al. [30] is a transposition to (1.6) of a result similar to Theorem 1.6, but for more regular potentials, by Mizumachi [33]. Similarly, the result in Masaki et al. [31] we described under (1.6) transposes to the case of δ potentials work on dispersion of very small solutions for NLS's with a non-trapping and quite regular potential in [9, 13, 35]. Even though they are usually motivated by the problem of stability of solitons, currently it is not so clear how to get from them results of the type $\|\xi(t)\|_{L^{\infty}(\mathbb{R})} \lesssim t^{-\frac{1}{2}}$ as $t \to +\infty$ for the error term $\xi(t)$ in (1.17) or for the $\eta(t)$ in (1.21). Such kind of transformation of results around 0 into results around a soliton exists in the context of the theory of Integrable Systems, where there are appropriate coordinate changes named Bäcklund and Darboux transforms, see for instance Deift and Park [7]. However for non integrable perturbations of the cubic NLS such coordinate changes represent an open question, c.f.r. the discussion in Mizumachi and Pelinovsky [34].

The main motivation for this paper is then to show the promise of an alternative method, involving positive commutators, which is classical in Quantum Mechanics, see for example Reed and Simon [38, pp. 157–163] and [2, 10]. In the nonlinear setting the method is also classical and has been extensively used to prove dispersion, like for example Morawetz estimates, see [5], or in the analysis of blow up, see for example [36, 32]. The method represents the tool of choice to complete the last step of the proof in the theory of stabilization developed by Kenig and Merle [20] (a possible alternative, the energy channel method of Duyckaerts et al. [11], has not been adapted yet to NLS's). In this paper we are inspired by the stability of various patterns studied for wave like equations by Kowalczyk et. al [27, 28]. The main point here, is that this method can be applied rather simply in the proof of Theorems 1.4 and 1.6.

In the nonlinear setting, an insidious problem arises from the fact that the commutators often have some negative eigenvalues. An important part of the proofs in papers such as [27] consists in showing that analogues of the ξ in (1.17) or of the η in (1.21), live where the commutator is positive. If in (1.1) we replace the δ -potential with a more regular one, proving such positivity, or circumventing possible negativity, appears to be mostly an open problem. See [43] for related problems. In the case of a δ -potential in 1-D, we show that this is an easy problem (see also Banica and Visciglia [3], Ikeda and Inui [29] and Richard [39]). This allows us to cover with a rather simple proof cases outside the reach of the theory in Masaki et al. [30], where the use of Strichartz estimates restricts consideration to $p \geq 2$. In this sense we go beyond the results for more regular potentials considered by Mizumachi [33], in turn related to [40, 41, 37, 15]. In some of these older papers there is a clear interest at obtaining the largest possible set of of values for the exponent p. However they are severely restricted by their dependence on dispersive and/or Strichartz estimates, not always sufficiently robust in nonlinear settings, see [42]. The commutator method can be more robust, as results such as [27, 28] show. In the literature some partial stability results have been obtained for subcritical nonlinearities in 2-D and 3-D by Kirr et al. [21, 22] using dispersive estimates. But exactly like for the dispersion results on small solutions of cubic NLS's with potentials in [9, 35, 13, 31] or for the theory initiated by Deift and Zhou [8] on the Scattering Transform in non-Integrable Systems, the ultimate test will be how pliable, widely utilizable and not too technically complicated will they be. Our point here is that the theory in [27, 28] seems the most promising.

Just as a final remark, to emphasize once more the effectiveness of the commutator method for equations like (1.1), in Sect. 4 we will also give a very simple proof of the following result for defocusing equations (1.1) with non–trapping δ potential, which to our knowledge is not in the literature.

Theorem 1.7. Consider equation (1.1) with q < 0, $g \ge 0$ everywhere and $sg(s) - G(s) \ge 0$ for any $s \ge 0$ for G defined in (1.5). Then for any $\gamma > 0$ there exists a $C_{\gamma} > 0$ s.t. for any $u_0 \in H^1(\mathbb{R}, \mathbb{C})$

the corresponding strong solution u(t) satisfies

$$\int_0^\infty \|u\|_{H^1_{-\gamma}}^2 dt < C_\gamma \left((E(u_0)Q(u_0))^{\frac{1}{2}} + Q(u_0) \right). \tag{1.25}$$

2 Proof of Theorem 1.4

2.1 Notation and coordinates

We have the following ansatz, which is an elementary consequence of the Implicit Function Theorem.

Lemma 2.1. There exist $c_0 > 0$ and C > 0 s.t. for all $u \in H^1$ with $||u||_{H^1} < c_0$, there exists a unique pair $(z, \xi) \in \mathbb{C} \times P_c H^1$ s.t.

$$u = Q[z] + \xi \text{ with } |z| + \|\xi\|_{H^1} \le C\|u\|_{H^1}.$$
 (2.1)

The map $u \to (z, \xi)$ is in $C^1(D_{H^1}(0, c_0), \mathbb{C} \times H^1)$.

Proof. Set

$$F(z,u) := \begin{pmatrix} \langle u - Q[z], \varphi \rangle \\ \langle u - Q[z], i\varphi \rangle \end{pmatrix}.$$

Then, by Proposition 1.2, we see that $F \in C^1(D_{\mathbb{C}}(0, a_0) \times H^1; \mathbb{R}^2)$ and moreover

$$\left. \frac{\partial F}{\partial (z_R, z_I)} \right|_{(z,u)=(0,0)} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Therefore, by implicit function theorem, we have the conclusion.

For $z, w \in \mathbb{C}$, we will use the notation

$$DQ[z]w := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Q[z+\varepsilon w].$$

Notice that by $Q[e^{i\theta}z] = e^{i\theta}Q[z]$, we have

$$iQ[z] = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} e^{i\varepsilon}Q[z] = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Q[e^{i\varepsilon}z] = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} Q[z+\varepsilon iz] = DQ[z]iz.$$

Further, for $w \in \mathbb{C}$, we set

$$f(w) := q(|w|^2)w.$$

The well posedness of problem (1.1) in $H^1(\mathbb{R})$ is considered by Goodman *et al.* [14], Fukuizumi *et al.* [12] and [1]. The energy and mass conservation imply the global well posedness of our small solutions, with representation (1.17) valid for all times along with the bound (1.18). So we can write the equation (1.1) in terms of the ansatz (1.17), or (3.1), and obtain the system

$$\langle iDQ(\dot{z} + iEz), i^{j-1}\varphi \rangle = \langle f(\xi) + \tilde{f}(z,\xi), i^{j-1}\varphi \rangle \text{ for } j = 1, 2,$$
 (2.3)

$$i\dot{\xi} = H_1 \xi + f(\xi) + \tilde{f}(z,\xi) - iDQ(\dot{z} + iEz), \tag{2.4}$$

where

$$\tilde{f}(z,\xi) := f(Q[z] + \xi) - f(\xi) - f(Q[z]) \tag{2.5}$$

In order to prove the last two points of Theorem 1.4, that is estimate (1.18) and the limit (1.24), we will use the method considered by Kowalczyk *et al.* in [27] and in their very recent paper [28].

2.2 The commutator method

Following [28], we introduce an even smooth function $\chi : \mathbb{R} \to [-1, 1]$ s.t.

$$\chi = 1 \text{ in } [-1, 1], \ \chi = 0 \text{ in } \mathbb{R} \setminus [-2, 2], \chi' \le 0 \text{ in } \mathbb{R}_+.$$
 (2.6)

For $A \gg 1$ large enough which will be fixed later, we set

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1 - \chi(x))\right) \text{ and } \psi_A(x) = \int_0^x \zeta_A^2(t)dt.$$
 (2.7)

One can easily verify

$$e^{-\frac{|x|}{A}} \le \zeta_A(x) \le 2e^{-\frac{|x|}{A}} \text{ for } A \ge 4.$$
 (2.8)

To each function $\xi \in H^1(\mathbb{R})$ we can associate

$$w := \zeta_A \xi. \tag{2.9}$$

Notice that there exist fixed constants C and A_0 s.t.

$$\langle (-\partial_x^2 + \delta)w, w \rangle \le C \|\xi\|_{H^1}^2 \text{ for all } A \ge A_0.$$
(2.10)

Also, we have

$$\|w'\|_{L^{2}}^{2} + \|\langle x \rangle^{-2} w\|_{L^{2}}^{2} \le C \langle (-\partial_{x}^{2} + \delta)w, w \rangle,$$
 (2.11)

where $\langle x \rangle := (1+|x|^2)^{1/2}$ and C=12. Indeed, it suffices to bound the second term. Since

$$w(x) = w(0) + \int_0^x w'(s) ds,$$

from Hölder inequality we have

$$|w(x)| \le |w(0)| + |x|^{1/2} ||w'||_{L^2} \le 2 \langle x \rangle^{1/2} \langle (-\partial_x^2 + \delta) w, w \rangle^{\frac{1}{2}}.$$
 (2.12)

Thus, we have

$$\|\langle x\rangle^{-2} w(x)\|_{L^{2}}^{2} \leq 4 \int_{\mathbb{R}} \langle x\rangle^{-3} dx \langle (-\partial_{x}^{2} + \delta)w, w\rangle \leq 12 \langle (-\partial_{x}^{2} + \delta)w, w\rangle.$$

We consider now the quadratic form

$$\mathcal{J}(\xi) := 2^{-1} \left\langle i\xi, \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi \right\rangle. \tag{2.13}$$

By the well posedness of (1.1) we can consider

$$\xi \in C^0(\mathbb{R}, D(H_1)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{C})) \subset C^0(\mathbb{R}, H^1(\mathbb{R}, \mathbb{C})) \cap C^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{C})) =: Y.$$

We claim that $\mathcal{J}(\xi) \in C^1(\mathbb{R}, \mathbb{R})$ with

$$\frac{d}{dt}\mathcal{J}(\xi) = \left\langle i\dot{\xi}, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \xi \right\rangle. \tag{2.14}$$

Indeed we can consider a sequence $\{\xi_n\}$ in $C^0(\mathbb{R}, H^2(\mathbb{R}, \mathbb{C})) \cap C^1(\mathbb{R}, H^2(\mathbb{R}, \mathbb{C}))$ converging to ξ in Y uniformly for t on compact sets. The functions $\mathcal{J}(\xi_n)$ belong to $C^1(\mathbb{R}, \mathbb{R})$ and furthermore their derivatives satisfy (2.14) with ξ replaced by ξ_n . From this formula we derive that the sequence $\{\frac{d}{dt}\mathcal{J}(\xi_n)\}$ converges uniformly on compact sets to the r.h.s. of (2.14). Since $\mathcal{J}(\xi_n) \xrightarrow{n \to \infty} \mathcal{J}(\xi)$ uniformly on compact sets, we conclude that $\mathcal{J}(\xi) \in C^1(\mathbb{R}, \mathbb{R})$ and that formula (2.14) is correct. From (2.14) we obtain

$$\frac{d}{dt}\mathcal{J}(\xi) + \left\langle iDQ(\dot{z} + iEz), \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \xi \right\rangle$$

$$= \left\langle H_1 \xi, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \xi \right\rangle + \left\langle f(\xi), \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \xi \right\rangle + \left\langle \tilde{f}(z, \xi), \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \xi \right\rangle.$$
(2.15)

The main result of this section is the following.

Proposition 2.2. There exist values $1 \gg a_0 > 0$ and $A \gg 1$ s.t. for $\xi \in P_cH^1$ and $|z| + \|\xi\|_{H^1} < a_0$ we have

r.h.s. of
$$(2.15) \ge \frac{1}{12} \left(\|w'\|_{L^2} + \|\langle x \rangle^{-2} w\|_{L^2}^2 \right)$$
 for $w = \zeta_A \xi$. (2.16)

The rest of this section is devoted to the proof of Proposition 2.2.

The key and single most important term is the quadratic form singled out in the following lemma, see [27, 28].

Lemma 2.3. For $w = \zeta_A \xi$ we have the equality

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = \left\langle H_{\frac{1}{2}} w, w \right\rangle + \frac{1}{2A} \left\langle V w, w \right\rangle \quad with \ V(x) := \chi''(x) \ |x| + 2\chi''(x) \ \frac{x}{|x|}. \tag{2.17}$$

Proof. Integrating by parts, see Corollary 8.10 [4], we obtain

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = -\frac{1}{2} \operatorname{Re} \int_{\mathbb{R}} \psi_A' \xi \overline{\xi}'' dx - \frac{1}{2} \int_{\mathbb{R}} \psi_A (|\xi'|^2)' dx$$

$$= \int_{\mathbb{R}} \psi_A' |\xi'|^2 dx + \frac{1}{4} \int_{\mathbb{R}} \psi_A'' (|\xi'|^2)' dx + \frac{1}{2} \operatorname{Re} \left(\psi_A'(0) \xi(0) \left(\overline{\xi}'(0^+) - \overline{\xi}'(0^-) \right) \right)$$

$$= \langle \psi_A' \xi', \xi' \rangle - \frac{1}{4} \langle \psi_A''' \xi, \xi \rangle - \frac{1}{2} \langle \delta \xi, \xi \rangle, \qquad (2.18)$$

where we used $\xi'(0^+) - \xi'(0^-) = -\xi(0)$, $\psi_A(0) = \psi''_A(0) = 0$ and $\psi'_A(0) = 1$. For the first term in the r.h.s. of (2.18), we have

$$\langle \psi'_{A} \xi', \xi' \rangle = \left\langle \zeta_{A}^{2} \left(\frac{w}{\zeta_{A}} \right)', \left(\frac{w}{\zeta_{A}} \right)' \right\rangle = \left\langle w' - \frac{\zeta'_{A}}{\zeta_{A}} w, w' - \frac{\zeta'_{A}}{\zeta_{A}} w \right\rangle$$

$$= \left\langle w', w' \right\rangle + \left\langle \left(\frac{\zeta'_{A}}{\zeta_{A}} \right)^{2} w, w \right\rangle - 2 \left\langle w', \frac{\zeta'_{A}}{\zeta_{A}} w \right\rangle$$

$$= \left\langle w', w' \right\rangle + \left\langle \left(\left(\frac{\zeta'_{A}}{\zeta_{A}} \right)' + \left(\frac{\zeta'_{A}}{\zeta_{A}} \right)^{2} \right) w, w \right\rangle = \left\langle w', w' \right\rangle + \left\langle \frac{\zeta''_{A}}{\zeta_{A}} w, w \right\rangle,$$

and for the second term we have

$$-\frac{1}{4}\left\langle \psi_A^{\prime\prime\prime}\xi,\xi\right\rangle = -\frac{1}{4}\left\langle \frac{(\zeta_A^2)^{\prime\prime}}{\zeta_A^2}w,w\right\rangle = -\frac{1}{2}\left\langle \left(\frac{\zeta_A^{\prime\prime}}{\zeta_A} + \left(\frac{\zeta_A^{\prime}}{\zeta_A}\right)^2\right)w,w\right\rangle.$$

Summing up we obtain

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = \left\langle H_{\frac{1}{2}} w, w \right\rangle + \frac{1}{2} \left\langle \left(\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right) w, w \right\rangle. \tag{2.19}$$

Finally, from

$$\zeta_A' = \frac{1}{A} \left(\chi'(x)|x| + (\chi(x) - 1) \frac{x}{|x|} \right) \zeta_A \text{ and}$$

$$\zeta_A'' = \frac{1}{A^2} \left(\chi'(x)|x| + (\chi(x) - 1) \frac{x}{|x|} \right)^2 \zeta_A + \frac{1}{A} \left(\chi''(x)|x| + 2\chi'(x) \frac{x}{|x|} \right) \zeta_A,$$

we conclude

$$A\left(\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A}\right)^2\right) = \chi''(x)|x| + 2\chi'(x)\frac{x}{|x|} = V(x). \tag{2.20}$$

Substituting (2.20) into (2.19) we obtain (2.17).

The main step in the proof of Proposition 2.2 is the following lemma.

Lemma 2.4. There exist a fixed constant $A_0 > 1$ s.t. for $A \ge A_0$ we have

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle \ge \frac{1}{5} \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \text{ for } w = \zeta_A \xi \text{ and all } \xi \in \mathcal{H}_c[0].$$
 (2.21)

Proof. We use formula (2.17) singling out the 1st term in the r.h.s. and writing it as

$$\left\langle H_{\frac{1}{2}}w,w\right\rangle =\frac{1}{4}\left\langle \left(-\partial_{x}^{2}+\delta\right)w,w\right\rangle +\frac{3}{4}\left\langle H_{1}w,w\right\rangle .$$

We now make the following claims.

Claim 2.5. There exist $A_0 > 0$ and $C_0 > 0$ s.t. for $A \ge A_0$ we have

$$\frac{1}{2A} \| \langle Vw, w \rangle \| \le \frac{C_0}{A} \left\langle (-\partial_x^2 + \delta)w, w \right\rangle \text{ for all } w \in H^1.$$
 (2.22)

Claim 2.6. There exist $A_0 > 1$ and $C_0 > 0$ s.t. for $A \ge A_0$ we have

$$\langle H_1 w, w \rangle \ge -\frac{C_0}{A^2} \langle (-\partial_x^2 + \delta) w, w \rangle \text{ for all } w = \zeta_A \xi \text{ with } \xi \in L_c^2.$$
 (2.23)

Let us assume Claims 2.5 and 2.6. Then we conclude

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = \left\langle H_{\frac{1}{2}} w, w \right\rangle + \frac{1}{2A} \left\langle V w, w \right\rangle$$

$$= \frac{1}{4} \left\langle \left(-\partial_x^2 + \delta \right) w, w \right\rangle + \frac{3}{4} \left\langle H_1 w, w \right\rangle + \frac{1}{2A} \left\langle V w, w \right\rangle$$

$$\geq \left(\frac{1}{4} - C_0 \left(\frac{3}{4} A^{-2} + A^{-1} \right) \right) \left\langle \left(-\partial_x^2 + \delta \right) w, w \right\rangle$$

which yields immediately (2.21). This, up to the proof of Claims 2.5 and 2.6, completes the proof of Lemma 2.4. \Box

Proof of Claim 2.5. By integrating (2.12),

$$\frac{1}{2A} \| \langle Vw, w \rangle \| \le \frac{2}{A} \int_{\mathbb{D}} |V(x)| \langle x \rangle \, dx \, \langle (-\partial_x^2 + \delta)w, w \rangle \text{ for all } w \in H^1.$$

Proof of Claim 2.6. Since $w = (\int_{\mathbb{R}} w \varphi_1 dx) \varphi_1 + P_c w$ and $\langle H_1 P_c w, P_c w \rangle \geq 0$, we have

$$\langle H_1 w, w \rangle = -\frac{1}{4} \left| \int_{\mathbb{D}} w \varphi_1 dx \right|^2 + \langle H_1 P_c w, P_c w \rangle \ge -\frac{1}{4} \left| \int_{\mathbb{D}} w \varphi_1 dx \right|^2 \text{ for all } w \in H^1.$$

On the other hand, for w as in (2.23) we have

$$\int_{\mathbb{R}} w\varphi_1 dx = \int_{\mathbb{R}} \xi \zeta_A \varphi_1 dx = \int_{\mathbb{R}} \xi \left(\zeta_A \varphi_1 - \varphi_1 \right) dx = -\int_{\mathbb{R}} w\varphi_1 \left(\frac{1}{\zeta_A} - 1 \right) dx.$$

Since $e^{-\frac{|x|}{A}} \le \zeta_A(x) \le 1$ and $e^{|x|} - 1 \le |x|e^{|x|}$ for all $x \in \mathbb{R}$, for $A \ge 4$

$$\left| \int_{\mathbb{R}} w \varphi_1 dx \right| \leq \int_{\mathbb{R}} |w| \varphi_1 \left(\frac{1}{\zeta_A} - 1 \right) dx \leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{2}} \left(e^{\frac{|x|}{A}} - 1 \right) dx$$
$$\leq \frac{1}{\sqrt{2}A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{2} + \frac{|x|}{A}} |x| dx \leq \frac{1}{\sqrt{2}A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{4}} |x| dx.$$

Furthermore, by (2.12)

$$\frac{1}{\sqrt{2}A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{4}} |x| dx \le \frac{\sqrt{2}}{A} \int_{\mathbb{R}} \langle x \rangle^{3/2} e^{-\frac{|x|}{4}} dx \left\langle (-\partial_x^2 + \delta) w, w \right\rangle^{1/2}.$$

This immediately leads to the lower bound (2.23).

By Lemma 2.4 we have found a lower bound on the 1st term in the r.h.s. of (2.15). We now examine the contribution to (2.16) of the term with $f(\xi) = g(|\xi|^2)\xi$.

Lemma 2.7. For any $\delta_0 > 0$ and $A \gg 1$ there exists $a_0 > 0$ s.t. for $\|\xi\|_{H^1} \leq a_0$ we have

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, f(\xi) \right\rangle \right| \le \delta_0 \|w'\|_{L^2}^2 \text{ for } w = \zeta_A \xi . \tag{2.24}$$

Proof. We follow [27, 28]. Recall that $f(\xi) = g(|\xi|^2)\xi$. Consider the G in (1.5). Then, we have

$$\left\langle \psi_A \partial_x \xi, g(|\xi|^2) \xi \right\rangle = \operatorname{Re} \int_{\mathbb{R}} \psi_A g(|\xi|^2) \overline{\xi} \partial_x \xi \, dx = \frac{1}{2} \int_{\mathbb{R}} \psi_A \partial_x G(|\xi|^2) \, dx = -\frac{1}{2} \int_{\mathbb{R}} \psi_A' G(|\xi|^2) \, dx.$$

Thus, by $\psi'_A = \zeta_A^2$, we have

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, f(\xi) \right\rangle \right| \le \int_{\mathbb{R}} \zeta_A^2 \left(|g(|\xi|^2)|\xi|^2 | + 2^{-1} |G(|\xi|^2)| \right) \, dx \le C \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)} \, dx.$$

Let $q = \frac{2p}{3} > 0$. Then, by the embedding $H^1(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)} dx \lesssim \|\xi\|_{H^1}^q \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} dx.$$

Therefore, it suffices to prove

$$\int_{\mathbb{D}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx \lesssim \|w'\|_{L^2}^2. \tag{2.25}$$

Following p. 793 [27], since 4(p-q) + 2 = 2(p+1) - q,

$$\begin{split} \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} dx &= \int_{\mathbb{R}} \zeta_A^{-(2p-q)} |w|^{2(p+1)-q} dx \\ &\lesssim \int_0^\infty e^{\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx + \int_{-\infty}^0 e^{-\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx \\ &\leq -\frac{2A}{2p-q} |w(0)|^{2p-q} + \frac{A}{2p-q} \int_{\mathbb{R}} e^{\frac{2p-q}{A}|x|} \left| (|w|^{2(p+1)-q})' \right| \, dx \\ &\lesssim A \int_{\mathbb{R}} \zeta_A^{-2p+q} |w|^{2p-q+1} |w'| \, dx = A \int_{\mathbb{R}} \zeta_A |\xi|^{2p-q+1} |w'| \, dx \\ &\leq A \|\xi\|_{H^1}^q \int_{\mathbb{R}} \zeta_A |\xi|^{2(p-q)+1} |w'| \, dx \\ &\leq A \|\xi\|_{H^1}^q \left(\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx \right)^{\frac{1}{2}} \|w'\|_{L^2} \\ &\leq \|w'\|_{L^2}^2 + \frac{1}{4} A^2 \|\xi\|_{H^1}^{2q} \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx. \end{split}$$

Thus, taking $\frac{1}{4}A^2a_0^{\frac{4}{3}p} \ll 1$, we have (2.25).

We now examine the contribution to (2.16) of the term with $\hat{f}(z,\xi)$.

Lemma 2.8. There exist $C_0 > 0$ and r > 0 and a neighborhood \mathcal{U} of the origin in \mathbb{C} s.t. for any pair $z, \xi \in \mathcal{U}$ we have

$$|\tilde{f}(z,\xi)| \le C_0 |Q[z]|^r |\xi|.$$
 (2.26)

Proof. It is enough to set $\zeta = Q[z]$ and then to prove

$$|f(\zeta + \xi) - f(\xi) - f(\zeta)| < C_0|\zeta|^r |\xi|. \tag{2.27}$$

We will prove (2.27) with r=1 if $2p \ge 1$ and with r=2p if $2p \le 1$. With these values of r, then $|\zeta| \ge |\xi|$ implies $|\zeta|^r |\xi| \le |\xi|^r |\zeta|$. This means that it is enough to consider the case $|\zeta| \ge |\xi|$. If $|\zeta| \le 2|\xi|$ we have $|\zeta| \sim |\xi|$ and it is elementary to conclude that each of the 3 terms in the l.h.s. of (2.27) is $\lesssim |\zeta|^r |\xi|$. Hence we are left with case $|\zeta| \ge 2|\xi|$. Notice that $|f(\xi)| \lesssim |\xi|^{2p+1} \le |\zeta|^r |\xi|$. So it is enough to prove that for a fixed $C_r > 0$ we have

$$|f(\zeta + \xi) - f(\zeta)| < C_r |\zeta|^r |w|. \tag{2.28}$$

By (1.4) we obtain the following, that implies (2.28) and completes the proof of the lemma:

$$|f(\zeta + \xi) - f(\zeta)| \le \int_0^1 \left| \frac{d}{dt} f(\zeta + t\xi) \right| dt$$

$$= \int_0^1 \left| g(|\zeta + t\xi|^2) \xi + 2g'(|\zeta + t\xi|^2) (\zeta + tw) \left(\text{Re} \left(\zeta \overline{w} \right) + t|w|^2 \right) \right| dt$$

$$\le C|\zeta|^{2p} |\xi| + C \left(|\zeta|^{2p} |\xi| + |\zeta|^{2p-1} |\zeta|^2 \right) \le 3C|\zeta|^{2p} |\xi|.$$

Lemma 2.9. For any $\delta_0 > 0$ and $A \gg 1$ there exists $a_0 > 0$ s.t. for $\|(z, \xi)\|_{\mathbb{C} \times H^1} \leq a_0$ we have

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, \tilde{f}(z, \xi) \right) \right\rangle \right| \leq \delta_0 \left(\|w'\|_{L^2}^2 + \|\langle x \rangle^{-2} w\|_{L^2}^2 \right) \text{ for } w = \zeta_A \xi .$$

Proof. First, we have

$$\left| \left\langle \frac{\psi_A'}{2} \xi, \tilde{f}(z, \xi) \right\rangle \right| \lesssim \int_{\mathbb{R}} |Q[z]| |w|^2 \, dx \lesssim a_0 \| \left\langle x \right\rangle^{-2} w \|_{L^2}^2.$$

Next, since $|\partial_x \xi| \lesssim e^{2\frac{|x|}{A}} (|w'| + |w|)$, we have

$$\left|\left\langle \psi_A \partial_x \xi, \tilde{f}(z,\xi) \right\rangle\right| \lesssim A \int_{\mathbb{R}} |Q[z]| e^{\frac{3}{A}|x|} \left(|w'| + |w|\right) |w| \, dx \lesssim a_0 A \left(\|w'\|_{L^2}^2 + \|\langle x \rangle^{-2} \, w\|_{L^2}^2\right).$$

Therefore, taking $a_0 A \ll 1$, we have the conclusion.

2.3 Closure of the estimates and completion of the proof of Theorem 1.4

We set

$$||w||_X^2 := ||w'||_{L^2}^2 + ||\langle x \rangle^{-2} w||_{L^2}^2. \tag{2.29}$$

In view of (2.15)–(2.16), for any T > 0, we have

$$\int_0^T \|w(t)\|_X^2 dt \lesssim \epsilon^2 + \left(\int_0^T \|w\|_X^2 dt\right)^{1/2} \|\dot{z} + iEz\|_{L^2((0,T))}. \tag{2.30}$$

From (2.3) we have

$$|\dot{z} + iEz| \lesssim ||\xi||_{L^{\infty}}^{2p} ||w||_X + |z|^{\min(2p,1)} ||w||_X.$$

Thus,

$$\|\dot{z} + iEz\|_{L^2((0,T))} \lesssim \epsilon^{\min(2p,1)} \left(\int_0^T \|w(t)\|_X^2 dt \right)^{1/2}.$$

Entering this in (2.30) we obtain

$$||w||_{L^2([0,T],X)}^2 \lesssim \epsilon^2 + \epsilon^{\min(2p,1)} ||w||_{L^2([0,T],X)}^2$$

Since $\min(2p,1) > 0$, we obtain $||w||_{L^2([0,T],X)} \le C_0\epsilon$ for a fixed $C_0 > 0$ and any T, and so

$$||w||_{L^2(\mathbb{R}_+,X)} \le C_0 \epsilon, \tag{2.31}$$

which for A sufficiently large implies the estimate (1.19) and ends the proof of Theorem 1.4.

3 Proof of Theorem 1.6

We have the following ansatz.

Lemma 3.1. There exists $c_0 > 0$ s.t. there exists a C > 0 s.t. for all $u \in H^1$ with $||u||_{H^1} < c_0$, there exists a unique pair $(z, \eta) \in \mathbb{C} \times (H^1 \cap \mathcal{H}_c[z])$ s.t.

$$u = Q[z] + \eta \text{ with } |z| + ||\eta||_{H^1} < C||u||_{H^1}.$$
(3.1)

The map $u \to (z, \eta)$ is in $C^1(D_{H^1}(0, c_0), \mathbb{C} \times H^1)$.

Proof. Set

$$F(z,u) := \begin{pmatrix} \langle u - Q[z], \mathrm{i} D_1 Q[z] \rangle \\ \langle u - Q[z], \mathrm{i} D_2 Q[z] \rangle \end{pmatrix}.$$

Then, since here 2p > 1, we have $F(z, u) \in C^1$ with formula (2.2) true for this function. We conclude by Implicit Function Theorem.

The following operator was introduced by Gustafson et al. [15].

Lemma 3.2. There exists $a_0 > 0$ such that for any $z \in \mathbb{C}$ with $|z| < a_0$ there exist $\alpha_j(z)$ (j = 1, 2) s.t. $||e^{\gamma|z|}\alpha_j||_{H^1} \lesssim |z|^{p-1}$ such that the \mathbb{R} -linear operator R[z] defined by

$$R[z]\xi := \xi + \langle i\xi, \alpha_1(z) \rangle \varphi + \langle i\xi, \alpha_2(z) \rangle i\varphi, \tag{3.2}$$

satisfies $R[z]: \mathcal{H}[0] \to \mathcal{H}_c[z]$ and $P_c|_{\mathcal{H}_c[z]} = R[z]^{-1}$.

Proof. We search for $\beta_j = \beta_j(z, \xi) \in \mathbb{R}$ (j = 1, 2) which satisfy

$$\langle i(\xi + (\beta_1 + i\beta_2)\varphi), D_j Q[z] \rangle = 0 \text{ for } j = 1, 2.$$
 (3.3)

Since

$$D(z) := \langle \varphi, D_1 Q[z] \rangle \langle i\varphi, D_2 Q[z] \rangle - \langle i\varphi D_1 Q[z] \rangle \langle \varphi, D_2 Q[z] \rangle = 1 + o(1),$$

where $o(1) \to 0$ as $|z| \to 0$, the system (3.3) have a unique solution

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = D(z)^{-1} \begin{pmatrix} \langle \varphi, D_2 Q[z] \rangle & -\langle \varphi, D_1 Q[z] \rangle \\ \langle i\varphi, D_2 Q[z] \rangle & -\langle i\varphi, D_1 Q[z] \rangle \end{pmatrix} \begin{pmatrix} \langle i\xi, D_1 Q[z] \rangle \\ \langle i\xi, D_2 Q[z] \rangle \end{pmatrix}.$$
(3.4)

Therefore, setting

$$\alpha_1(z) := D(z)^{-1} \left(\langle \varphi, D_2 Q[z] \rangle D_1 Q[z] - \langle \varphi, D_1 Q[z] \rangle D_2 Q[z] \right), \alpha_2(z) := D(z)^{-1} \left(\langle i\varphi, D_2 Q[z] \rangle D_1 Q[z] - \langle i\varphi, D_1 Q[z] \rangle D_2 Q[z] \right),$$

$$(3.5)$$

we see that R[z] defined by (3.2) with α_j given by (3.5) satisfies $R[z]:\mathcal{H}[0]\to\mathcal{H}[z]$. It is obvious that we have $P_cR[z]\xi=\xi$ for all $\xi\in\mathcal{H}[0]$. Finally, we have $R[z]P_c\xi=\xi$, for all $\xi\in\mathcal{H}[z]$. Indeed, since $R[z]P_c\xi$ is the unique element of $\mathcal{H}[z]$ of the form $P_c\xi+\beta\varphi$, we have the conclusion from the spectral decomposition $\xi=P_c\xi+\tilde{\beta}\varphi$ where $\tilde{\beta}=\int \xi\varphi\,dx$.

Thanks to Lemmas 3.1 and 3.2 we conclude the following.

Lemma 3.3. Picking $a_0 > 0$ small enough, we have

$$u = Q[z] + R[z]\xi \text{ for } (z,\xi) \in D_{\mathbb{C}}(0,a_1) \times (H^1 \cap L_c^2)$$
 (3.6)

such that $(z,\xi) \to u$ is C^1 and

$$|z| + ||\xi||_{H^1} \sim ||u||_{H^1}. \tag{3.7}$$

In terms decomposition (1.21)–(1.22) equation (1.1) can be expressed as follows:

$$\langle iDQ(\dot{z} + iEz), D_j Q \rangle - \langle i\eta, D_j DQ(\dot{z} + iEz) \rangle = \langle f(\eta) + \tilde{f}(z, \eta), D_j Q \rangle \text{ for } j = 1, 2$$
(3.8)

$$i\dot{\eta} = H\eta + f(\eta) + \tilde{f}(z,\eta) - iDQ(\dot{z} + iEz), \tag{3.9}$$

where

$$H\eta = H[z]\eta = H_1\eta + V[z]\eta \text{ with } V[z]\eta = -|Q[z]|^2\eta - 2Q[z]\operatorname{Re}(\eta\overline{Q[z]})$$
(3.10)

and where $f(\eta)$ and $\tilde{f}(z,\eta)$ are defined like in Sect. 2. Like above, we consider

$$\mathcal{J}(\eta) = 2^{-1} \left\langle i \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \eta, \eta \right\rangle. \tag{3.11}$$

Proceeding like in Sect. 2.2 we obtain

$$\frac{d}{dt}\mathcal{J}(\eta) - \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \eta, iDQ(\dot{z} + iEz) \right\rangle
= \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \eta, H_1 \eta \right\rangle + \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \eta, V[z] \eta \right\rangle
+ \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \eta, \tilde{f}(z, \eta) \right\rangle + \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) \eta, f(\eta) \right\rangle.$$
(3.12)

Then, like in Sect. 2.2, we have the following result.

Proposition 3.4. There exist values $1 \gg a_0 > 0$ and $A \gg 1$ s.t. for $\xi \in P_cH^1$ and $|z| + ||\eta||_{H^1} < a_0$ we have

r.h.s. of
$$(3.12) \ge \frac{1}{12} \left(\|w'\|_{L^2} + \|\langle x \rangle^{-2} w\|_{L^2}^2 \right)$$
 for $w = \zeta_A \eta$. (3.13)

Proof. The proof of Proposition 3.4 is exactly like the proof of Proposition 2.2, except that in (3.12) there is an additional term (the 2nd in the 2nd line) dealt with in Lemma 3.5 below, and that the analogue Lemma (2.21) continues to be true with an $\eta \in \mathcal{H}[z]$ instead of $\xi \in \mathcal{H}[0]$. We skip the elementary proof of the last point.

Lemma 3.5. There exist $a_0 > 0$ and $C_0 > 0$ s.t. for $|z| \le a_0$ we have

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \eta, V[z] \eta \right\rangle \right| \leq C_0 |z|^2 \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \text{ for } w = \zeta_A \eta \text{ and all } \eta \in H^1.$$

Proof. In view of the fact that V[z] is symmetric for our inner product, we have the following:

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \eta, V[z] \eta \right\rangle = 2^{-1} \left\langle \eta, (|Q[z]|^2)' \eta + 2(Q[z])' \operatorname{Re}(\eta \overline{Q[z]}) + 2Q[z] \operatorname{Re}(\eta \overline{Q[z]})' \right\rangle$$

$$\lesssim |z|^2 \left\langle (-\partial_x^2 + \delta) w, w \right\rangle.$$

End of the proof of Theorem 1.6. From (3.13) we have like in

$$||w||_{L^{2}((0,t),X)}^{2} \lesssim 2\epsilon^{2} + ||w||_{L^{2}((0,t),X)} ||\dot{z} + iEz||_{L^{2}((0,t))}.$$
(3.14)

From (3.9) we have

$$\begin{aligned} \|\dot{z} + iEz\|_{L^{2}((0,t))} &\lesssim \|\dot{z} + iEz\|_{L^{2}((0,t))} \|\langle x \rangle^{-2} w\|_{L^{\infty}((0,t),L^{2})} \\ &+ \left(\|z\|_{L^{\infty}((0,t))} + \|\langle x \rangle^{-2} w\|_{L^{\infty}((0,t),L^{2})} \right) \|w\|_{L^{2}((0,t),X)}^{2} \\ &\lesssim \epsilon \|\dot{z} + iEz\|_{L^{2}((0,t))} + \epsilon^{2} \|w\|_{L^{2}((0,t),X)}, \end{aligned}$$

and hence

$$\|\dot{z} + iEz\|_{L^2((0,t))} \lesssim \epsilon^2 \|w\|_{L^2((0,t),L^2)}.$$

Entering this in (3.14) we conclude, for a fixed $C_0 > 0$,

$$||w||_{L^2((0,t),X)} \le C_0 \epsilon.$$

We set now $\rho(t) := z(t)e^{i\int_0^t E[z(s)]ds}$. Then we have

$$\|\dot{\rho}\|_{L^{1}((0,t))} = \|\dot{z} + iEz\|_{L^{1}((0,t))}$$

$$\lesssim \|\dot{\rho}\|_{L^{1}((0,t))} \|\langle x\rangle^{-2} w\|_{L^{\infty}((0,t),L^{2})} + \left(\|z\|_{L^{\infty}((0,t))} + \|\langle x\rangle^{-2} w\|_{L^{\infty}((0,t),L^{2})}\right) \|w\|_{L^{2}((0,t),X)}^{2}.$$

From this we derive

$$\|\dot{\rho}\|_{L^1(\mathbb{R}_+)} \lesssim \epsilon \|w\|_{L^2(\mathbb{R}_+,X)}^2 \lesssim \epsilon^3.$$

The existence of ρ_+ and of the limit (1.24) follow. This ends the proof of (1.23)– (1.24).

4 Proof of Theorem 1.7

We know that there exists a unique global strong solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}, \mathbb{C}))$, and furthermore that energy and mass are constant

$$E(u(t)) = \frac{1}{2} \|\partial_x u(t)\|_{L^2(\mathbb{R})}^2 + \frac{|q|}{2} |u(t,0)|^2 + \frac{1}{2} \int_{\mathbb{R}} G(|u(t)|^2) dx = E(u_0),$$

$$Q(u(t)) = \frac{1}{2} \|u(t)\|_{L^2(\mathbb{R})}^2 = Q(u_0).$$

By well posedness and a density argument, it is enough to focus on the case $u_0 \in D(H_q)$, so that

$$u \in C^0(\mathbb{R}, D(H_1)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{C})).$$

Then we consider $\mathcal{J}(u)$, defined like in (2.13), and by the same argument of Sect. 2.2 we have

$$\frac{d}{dt}\mathcal{J}(u) = \left\langle i\dot{u}, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle = \left\langle i\dot{u}, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle$$
$$= \left\langle (-\partial_x^2 + |q|\delta(x))u + g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right)u \right\rangle.$$

By computations similar to Lemma 2.3, for $w = \zeta_A \xi$ and for the V(x) in (2.17), we have

$$\left\langle (-\partial_x^2 + |q|\delta(x))u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right)u \right\rangle = \left\langle \left(-\partial_x^2 + \frac{|q|}{2}\delta(x)\right)w, w \right\rangle + \frac{1}{2A} \left\langle Vw, w \right\rangle$$

$$\geq \frac{1}{2} \left\langle \left(-\partial_x^2 + \frac{|q|}{2}\delta(x)\right)w, w \right\rangle$$

for $A \ge A_0$ with A_0 a fixed sufficiently large constant.

On the other hand, by $\psi'_A > 0$ and the argument in the first few lines of Lemma 2.7,

$$\left\langle g(|u|^2)u, \frac{\psi_A'}{2}u \right\rangle + \left\langle g(|u|^2)u, \psi_A \partial_x u \right\rangle = \frac{1}{2} \left\langle g(|u|^2)|u|^2 - G(|u|^2), \psi_A' \right\rangle \ge 0.$$

Hence, for fixed constants

$$\int_{0}^{T} \|w(t)\|_{X}^{2} dt \lesssim \mathcal{J}(u(T)) - \mathcal{J}(u_{0}) \lesssim \sqrt{E(u_{0})Q(u_{0})} + Q(u_{0}),$$

which yields Theorem 1.7.

A Appendix.

We prove Proposition 1.2.

Lemma A.1. Set $R := \left(\left(H_1 + \frac{1}{4}\right)\big|_{P_cL^2}\right)^{-1}$. Then, for sufficiently small $\gamma > 0$, R is a bounded operator from L^2_{γ} to H^1_{γ} .

Proof. For case $\gamma = 0$ see Lemma 2.12 of [30]. For the case $\gamma > 0$, set $\chi_A(x) := \chi(x/A)$ where χ is given in (2.6). Set $\mu_{\gamma,A}(x) := e^{\gamma \sqrt{1+|x|^2}} \chi_A(x)$. Then, multiplying $H_1 R u = u$ by μ_A , we obtain

$$H_1\mu_{\gamma,A}Ru = [H_1, \mu_{\gamma,A}]Ru + \mu_{\gamma,A}u.$$

Notice that there exists a C > 0 s.t.

$$||[H_1, \mu_{\gamma, A}]u|| \le C \ \gamma \ ||\mu_{\gamma, A}u||_{H^1} \text{ for all } \gamma \in [0, 1] \text{ and } A \in [1, \infty).$$

This implies that for sufficiently small $\gamma > 0$,

$$\|\mu_{\gamma,A}Ru\|_{H^1} \lesssim \|\mu_{\gamma,A}u\|_{L^2} \lesssim \|u\|_{L^2}$$
.

Thus, taking $A \to \infty$, we have $||Ru||_{H^1_\alpha} \lesssim ||u||_{L^2_\alpha}$.

We consider $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ defined by

$$\tilde{h}(\rho,\mu) := g(\rho\mu^2)\mu. \tag{A.1}$$

For $\gamma < \frac{1}{2}$, we set $h: [0,1] \times H^1_{\gamma}(\mathbb{R},\mathbb{R}) \to L^2_{\gamma}(\mathbb{R},\mathbb{R})$ by

$$h(\rho, q)(x) := \tilde{h}(\rho, q(x)) = g(\rho q(x)^2)q(x).$$
 (A.2)

Notice that q in (A.1) is a number but q in (A.2) is a function.

Lemma A.2. We have $\tilde{h} \in C([0,1] \times \mathbb{R}, \mathbb{R}) \cap C^1((0,1] \times \mathbb{R}, \mathbb{R})$ and the estimates

$$|\tilde{h}(\rho,\mu)| \lesssim \rho^p |\mu|^{2p+1} \tag{A.3}$$

and

$$|\partial_{\rho}\tilde{h}(\rho,\mu)| \lesssim \rho^{p-1}|\mu|^{2p+1}, \quad |\partial_{\mu}\tilde{h}(\rho,\mu)| \lesssim \rho^{p}|\mu|^{2p}.$$
 (A.4)

Furthermore, for $\rho\mu \neq 0$, h is three times differentiable and we have

$$|\partial_{\rho}^{2}\tilde{h}(\rho,\mu)| \lesssim \rho^{p-2}|\mu|^{2p+1}, \quad |\partial_{\rho}\partial_{\mu}\tilde{h}(\rho,\mu)| \lesssim \rho^{p-1}|\mu|^{2p}, \quad |\partial_{\mu}^{2}\tilde{h}(\rho,\mu)| \lesssim \rho^{p}|\mu|^{2p-1} \tag{A.5}$$

and

$$|\partial_{\rho}^{3}\tilde{h}(\rho,\mu)| \lesssim \rho^{p-3}|\mu|^{2p+1}, \quad |\partial_{\mu}\partial_{\rho}^{3}h(\rho,\mu)| \lesssim \rho^{p-1}\mu^{2p}. \tag{A.6}$$

If $p > \frac{1}{2}$, we have $\tilde{h} \in C^2((0,1] \times \mathbb{R}, \mathbb{R})$.

Proof. By the definition of \tilde{h} , we have $C([0,1] \times \mathbb{R}, \mathbb{R}) \cap C^3((0,1] \times (\mathbb{R} \setminus \{0\}), \mathbb{R})$. Also, (A.3) is immediate from (1.4) and (A.1). At $(\rho, q) = (\rho, 0)$ with $\rho > 0$, \tilde{h} is differentiable w.r.t. ρ and μ having $\partial_{\rho}\tilde{h} = \partial_{q}\tilde{h} = 0$. One can see this easily from $\tilde{h}(\rho, 0) = 0$ and

$$\tilde{h}(\rho + \epsilon, 0) = 0, \ |\tilde{h}(\rho, \epsilon)| \lesssim \epsilon^{2p+1}.$$

Further, since for $\mu \neq 0$,

$$\partial_{\rho}\tilde{h}(\rho,q) = q'(\rho\mu^{2})\mu^{3}, \quad \partial_{\mu}\tilde{h}(\rho,\mu) = q(\rho\mu^{2}) + 2\rho q'(\rho\mu^{2})\mu^{2},$$
 (A.7)

we have (A.4) from (1.4), which imply that $\partial_{\rho}\tilde{h}$ and $\partial_{\mu}\tilde{h}$ are continuous at $(\rho,0)$. Differentiating (A.7) for $\rho, \mu \neq 0$, we have

$$\partial_{\rho}^{2}\tilde{h}(\rho,q) = g''(\rho\mu^{2})\mu^{5}, \quad \partial_{\rho}\partial_{\mu}\tilde{h}(\rho,\mu) = 3g'(\rho\mu^{2})\mu^{2} + 2\rho g''(\rho\mu^{2})\mu^{4},$$
$$\partial_{\mu}^{2}\tilde{h}(\rho,\mu) = 6\rho g'(\rho\mu^{2})\mu + 4\rho^{2}g''(\rho\mu^{2})\mu^{3}.$$

and

$$\partial_\rho^3 \tilde{h}(\rho,q) = g^{\prime\prime\prime}(\rho\mu^2)\mu^7, \quad \partial_\mu \partial_\rho^2 \tilde{h}(\rho,q) = 2\rho g^{\prime\prime\prime}(\rho\mu^2)\mu^6.$$

This implies that $\rho, \mu \neq 0$, we have (A.5) and (A.6). By (A.5), for the case p > 1/2, we see that h is twice continuously differentiable at $(\rho, 0)$ $(\rho \neq 0)$ with

$$\partial_{\rho}^{2}\tilde{h}(\rho,0) = \partial_{\rho}\partial_{\mu}\tilde{h}(\rho,0) = \partial_{\mu}^{2}\tilde{h}(\rho,0) = 0.$$

Therefore, we have the conclusion.

Lemma A.3. Let $\gamma \geq 0$. Let \tilde{h} , h be the functions given in (A.1) and (A.2). Then,

$$h \in C([0,1] \times H^1_{\gamma}, L^2_{\gamma}) \cap C^1((0,1] \times H^1_{\gamma}, L^2_{\gamma})$$
 (A.8)

and

$$\partial_{\rho}h(\rho,q)(x) = \partial_{\rho}\tilde{h}(\rho,q(x)), \ (\partial_{q}h(\rho,q)v)(x) = \partial_{\mu}\tilde{h}(\rho,q(x))v(x). \tag{A.9}$$

Proof. First of all, for $q \in H^1_{\gamma}$, we have $h(\rho, q) \in L^2_{\gamma}$. Indeed, from (A.3),

$$||h(\rho,q)||_{L^{2}_{\gamma}} = ||\tilde{h}(\rho,q(\cdot))||_{L^{2}_{\gamma}} \lesssim \rho^{p} ||q||_{L^{\infty}}^{2p} ||q||_{L^{2}_{\gamma}} \lesssim \rho^{p} ||q||_{H^{1}_{z}}^{2p+1}, \tag{A.10}$$

which implies also that h is continuous at $\{0\} \times H^1$.

Next, we show (A.9). For $(\rho, q) \in (0, 1] \times H^1_{\gamma}$, and $|\epsilon| < \rho$,

$$\begin{split} |\epsilon|^{-1} \|h(\rho + \epsilon, q) - h(\rho, q) - \epsilon \partial_{\rho} \tilde{h}(\rho, q)\|_{L_{\gamma}^{2}} &= |\epsilon|^{-1} \|\tilde{h}(\rho + \epsilon, q) - \tilde{h}(\rho, q) - \epsilon \partial_{\rho} \tilde{h}(\rho, q)\|_{L_{\gamma}^{2}} \\ &= \int_{0}^{1} \|\partial_{\rho} \tilde{h}(\rho + \tau_{1} \epsilon, q) \, d\tau - \partial_{\rho} \tilde{h}(\rho, q)\|_{L_{\gamma}^{2}} \, d\tau_{1} \\ &\lesssim \int_{0}^{1} \int_{0}^{1} \tau_{1} |\epsilon| \|\partial_{\rho}^{2} \tilde{h}(\rho + \tau_{1} \tau_{2} \epsilon, q)\|_{L_{\gamma}^{2}(\{x \in \mathbb{R} | q(x) \neq 0\})} \, d\tau_{1} d\tau_{2} \\ &\lesssim |\epsilon| \int_{0}^{1} \int_{0}^{1} \tau_{1} (\rho + \tau_{1} \tau_{2} \epsilon)^{p-2} \, d\tau_{1} d\tau_{2} \|q\|_{H_{\gamma}^{1}}^{2p+1} \to 0 \text{ as } \epsilon \to 0. \end{split}$$

Similarly,

$$||v||_{H_{\gamma}^{-1}}^{-1}||h(\rho, q + v) - h(\rho, q) - \partial_{\mu}\tilde{h}(\rho, q)v||_{L_{\gamma}^{2}} = ||v||_{H_{\gamma}^{-1}}^{-1}||\int_{0}^{1} \left(\partial_{\mu}\tilde{h}(\rho, q + \tau v) - \partial_{\mu}\tilde{h}(\rho, q)\right)v||_{L_{\gamma}^{2}}$$

$$\leq \sup_{\tau \in [0, 1]} ||\partial_{\mu}\tilde{h}(\rho, q + \tau v) - \partial_{\mu}\tilde{h}(\rho, q)||_{L^{\infty}} \to 0 \text{ as } ||v||_{H_{\gamma}^{1}} \to 0.$$
(A.11)

Here we have used the fact that $\partial_{\mu}\tilde{h}$ is uniformly continuous in $\left[\frac{\rho}{2},1\right] \times \left[-\|q\|_{L^{\infty}}-1,\|q\|_{L^{\infty}}+1\right]$ and $\|v\|_{L^{\infty}} \to 0$ if $\|v\|_{H^{1}_{\gamma}} \to 0$. By similar estimate, we see that $\partial_{\rho}h$ and $\partial_{q}h$ are continuous in $(0,1] \times H^{1}_{\gamma}$. From this, we have (A.8).

Lemma A.4. Let p > 1/2. Let \tilde{h} , h be the functions given in (A.1) and (A.2). Then,

$$h \in C([0,1] \times H^1_{\gamma}, L^2_{\gamma}) \cap C^2((0,1] \times H^1_{\gamma}, L^2_{\gamma}), \tag{A.12}$$

and

$$\partial_{\rho}^{2}h(\rho,q)(x) = \partial_{\rho}^{2}\tilde{h}(\rho,q(x)), \quad (\partial_{\rho}\partial_{q}h(\rho,q)v)(x) = \partial_{\rho}\partial_{\mu}\tilde{h}(\rho,q(x))v(x),
\partial_{q}^{2}h(\rho,q)(v,w)(x) = \partial_{\mu}^{2}\tilde{h}(\rho,q(x))v(x)w(x).$$
(A.13)

Proof. Since the argument is similar to the proof of Lemma A.3 we omit it. \Box

Lemma A.5. Let $\gamma \in [0, \frac{1}{2})$ and set

$$\mathfrak{e}(\rho,q) := \langle h(\rho,\varphi+q), \varphi \rangle$$
.

Then, $\mathfrak{e} \in C^1((0,1) \times H^1_{\gamma}(\mathbb{R},\mathbb{R}),\mathbb{R})$. Moreover, if p > 1/2, we have $\mathfrak{e} \in C^2((0,1) \times H^1_{\gamma}(\mathbb{R},\mathbb{R}),\mathbb{R})$.

Proof. Set $F \in C^{\infty}(L^2_{\gamma}; \mathbb{R})$ by $F(h) := \langle h, \varphi \rangle$. Since $\mathfrak{c}(\rho, q) = F \circ h(\rho + \varphi, q)$, we immediately have the conclusion from Lemmas A.3 and A.4.

Lemma A.6. Let $\gamma \in [0, \frac{1}{2})$. Set

$$\Phi(\rho, q) := \mathfrak{e}(\rho, q)(\varphi + q) - h(\rho, \varphi + q). \tag{A.14}$$

Then, $\Phi \in C^1((0,1] \times P_c H^1_{\gamma}(\mathbb{R},\mathbb{R}); P_c L^2_{\gamma}(\mathbb{R},\mathbb{R}))$. Moreover, if p > 1/2, we have $\Phi \in C^2((0,1] \times P_c H^1_{\gamma}(\mathbb{R},\mathbb{R}); P_c L^2_{\gamma}(\mathbb{R},\mathbb{R}))$

Proof. From Lemmas A.3, A.4 and A.5, it suffices to show $\Phi(\rho, q) \in P_c L_{\gamma}^2$. However, from the definition of \mathfrak{e} we obtain the following, which yields the conclusion:

$$\langle \Phi(\rho, q), \varphi \rangle = \mathfrak{e}(\rho, q) - \langle h(\rho, \varphi + q), \varphi \rangle = 0.$$

Lemma A.7. Take $\gamma_0 \in (0, 1/2)$ such that the conclusion of Lemma A.1 holds. Then there exists $\rho_0 > 0$ s.t. there exists a unique $q \in C^k((0, \rho_0), H^1_{\gamma})$ with k = 1 and k = 2 if $p > \frac{1}{2}$ satisfying

$$q(\rho) = R\Phi(\rho, q(\rho)), \tag{A.15}$$

and

$$\|q(\rho)\|_{H^1_{\gamma}} \lesssim \rho^p, \ \|\partial_{\rho}q(\rho)\|_{H^1_{\gamma}} \lesssim \rho^{p-1} \ and \ |\mathfrak{e}(\rho, q(\rho))| \lesssim \rho^p.$$
 (A.16)

Moreover, if p > 1/2 we have

$$\|\partial_{\rho}^{2}q(\rho)\|_{H_{\alpha}^{1}} \lesssim \rho^{p-2}.\tag{A.17}$$

Proof. By Lemma A.1 and Lemma A.3, we have

$$||R\Phi(\rho, q_1) - R\Phi(\rho, q_2)||_{H^1_{\gamma}} \lesssim ||\Phi(\rho, q_1) - \Phi(\rho, q_2)||_{L^2_{\gamma}} \lesssim ||h(\rho, \varphi + q_1) - h(\rho, \varphi + q_2)||_{L^2_{\gamma}}$$

$$\leq \int_0^1 ||\partial_q h(\rho, \varphi + q_2 + \tau(q_1 - q_2))||_{L^{\infty}} d\tau ||q_1 - q_2||_{L^2_{\gamma}} \lesssim \rho^p ||q_1 - q_2||_{H^1_{\gamma}},$$

for $q_1, q_2 \in \overline{D_{H^1_{\gamma}}(0,1)}$. Therefore, there exists $\rho_0 > 0$ s.t.

$$||R\Phi(\rho, q_1) - R\Phi(\rho, q_2)||_{H^1_{\gamma}(\mathbb{R}, \mathbb{R})} \le \frac{1}{2}||q_1 - q_2||_{H^1_{\gamma}},$$

for all $\rho \in (0, \rho_0)$ and $q_1, q_2 \in \overline{D_{H^1_{\gamma}(\mathbb{R},\mathbb{R})}(0, 1)}$. Thus, by contraction mapping principle, there exists a unique $q \in \overline{D_{H^1_{\gamma}(\mathbb{R},\mathbb{R})}(0, 1)}$ satisfying (A.15). We call $q(\rho)$ the fixed point of $R\Phi(\rho, \cdot)$ and set

$$F(\rho, q) := q - R\Phi(\rho, q).$$

Since one can show $\partial_q F|_{(\rho,q)=(\rho,q(q))}$ is invertible by using the estimate we have prepared, by the Implicit Function Theorem and by Lemma A.6 we have $q \in C^k((0,\rho_0),H^1_\gamma)$ with k=1 and k=2 if $p>\frac{1}{2}$.

We now prove (A.16). First, by the fact that $q(\rho)$ is the fixed point of $R\Phi(\rho,\cdot)$, Lemma (A.1) and (A.3) with the definition of h, Φ , we have

$$||q(\rho)||_{H^1_{\gamma}} = ||R\Phi(\rho)||_{H^1_{\gamma}} \lesssim ||\Phi(\rho)||_{L^2_{\gamma}} \lesssim ||h(\rho, \varphi + q(\rho))||_{L^2_{\gamma}} \lesssim \rho^p.$$

Next, by the definition of \mathfrak{e} , we have

$$|\mathfrak{e}(\rho, q(\rho))| \le ||h(\rho, \varphi + q(\rho))||_{L^2} \lesssim \rho^p.$$

Finally, since

$$\partial_{\rho}q = R\partial_{q}\Phi\partial_{\rho}q + R\partial_{\rho}\Phi,$$

and by the above argument, for sufficietly small $\rho > 0$, we have $\|(\mathrm{Id} - R\partial_q \Phi)^{-1}\|_{H^1_{\gamma} \to H^1_{\gamma}} \le 2$, we have the 2nd estimate of (A.16) by

$$\|\partial_{\rho}q\|_{H^{1}} = \|(\operatorname{Id} - R\partial_{q}\Phi)^{-1}R\partial_{q}\Phi\|_{H^{1}} \lesssim \|\partial_{q}\Phi\|_{L^{2}} \lesssim \|\partial_{q}h\|_{L^{2}} \lesssim \rho^{p-1}.$$

The estimate (A.17) can be proved similarly.

Proof of Proposition 1.2. Set $a_0 = \rho_0^{\frac{1}{2}}$ where $\rho_0 > 0$ is given in Lemma A.7. Set

$$Q[z] := z(\varphi + q(|z|^2))$$
 and $E(|z|^2) := -\frac{1}{4} + \mathfrak{e}(|z|^2, q(|z|^2)),$

where $q \in C^1((0, a_0^2), H^1_{\gamma}(\mathbb{R}, \mathbb{R}))$ is given in Lemma A.7. Then, (1.12) and (1.13) are immediate from the definition of Q, q and \mathfrak{e} . The first and third inequality of (1.14) follow from (A.16).

By Lemma A.7, we have $Q \in C(D_{\mathbb{C}}(0, a_0), H^1_{\gamma}) \cap C^{k}(D_{\mathbb{C}}(0, a_0) \setminus \{0\}, H^1_{\gamma})$ for k = 1 and k = 2 for $p > \frac{1}{2}$. However, since

$$||D_j Q[z] - i^{j-1} \varphi||_{H^1_+} = ||i^{j-1} q(|z|^2) + 2q'(|z|^2) z z_j||_{H^1_+} \lesssim |z|^{2p}, \ j = 1, 2,$$

we see that Q[z] is also continuously differentiable at z=0. Here, we have set $z=z_1+iz_2$ for $z_1, z_2 \in \mathbb{R}$. Similarly, if p>1/2, we see that Q[z] is twice continuously differentiable at the origin and satisfying the estimate (1.16). This finishes the proof.

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Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy. *E-mail Address*: scuccagna@units.it

Department of Mathematics and Informatics, Faculty of Science, Chiba University, Chiba 263-8522, Japan. E-mail Address: maeda@math.s.chiba-u.ac.jp