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LINEAR POSITIVE SYSTEMS MAY HAVE A REACHABLE SUBSET FROM THE ORIGIN THAT IS EITHER POLYHEDRAL OR NONPOLYHEDRAL*

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Abstract. Positive systems with positive inputs and positive outputs are used in several branches of engineering, biochemistry, and economics. Both control theory and system theory require the concept of reachability of a time-invariant discrete-time linear positive system. The subset of the state set that is reachable from the origin is therefore of interest. The reachable subset is in general a cone in the positive vector space of the positive real numbers. It is established in this paper that the reachable subset can be either a polyhedral or a nonpolyhedral cone. For a single-input case, a characterization is provided of when the infinite-time and the finite-time reachable subsets are polyhedral. An example is provided for which the reachable subset is nonpolyhedral. Finally, for the case of polyhedral reachable subset(s), a method is provided to verify if a target set can be reached from the origin using positive inputs.

Key words. linear positive systems, reachable subset, polyhedral cone, positive recursion

AMS subject classifications. 93C15, 93B05

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1. Introduction.

Motivation and scope. In this paper, the focus is on the reachable subset from the origin of a single-input time-invariant discrete-time linear positive system. It will be proven that such a reachable subset can be either a polyhedral or a nonpolyhedral cone. A characterization is provided of when this reachable subset is polyhedral.

A positive system may arise in many areas of science and of engineering, such as econometrics [42], biochemical reactors [31], compartmental systems [26, 32], and transportation systems [47, 53], to name a few. The variables in such systems represent growth rates, concentration levels, mass accumulation, flows, etc. Obviously, variables of this nature can only assume values that are zero or strictly positive.

For problems of control and system theory with positive systems, a solid body of concepts, theorems, and algorithms has been developed. Of particular interest is the theory of linear positive systems [5], which is based on the theory of positive matrices and their geometric equivalent, polyhedral cones, [6, 14, 21, 33].

While the theory of linear positive systems overlaps with the theory of linear systems, there are distinct differences between the two. Therefore, several concepts of linear systems cannot be directly generalized to linear positive systems without reformulation. One such property is the notion of reachability and controllability of a linear positive system.

The motivation of the investigation of reachability and controllability of a linear positive system is in (1) their use in control theory as an equivalent condition for the existence of a control law for particular control objectives; and (2) in the theory

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of realization and of system identification. In a positive system, as it arises in the research areas mentioned above, one may want to know whether from a specified initial state a particular terminal state can be reached by application of a positive input to the system. The state to be reached can be a set of concentrations of chemical substances in a bioreactor or a concentration in a compartment which is, e.g., a model of tissue in a human being. More generally, one may want to characterize all states of a linear positive system that can be reached from the zero initial state using positive inputs, which is also the object of interest for realization theory of linear positive systems. The choice for the reachable subset from the origin is essential for realization theory. Observability of a linear positive system is then of interest only for states in the reachable set. A characterization of that view of observability does not currently exist in the literature. The condition formulated in the paper [27] is too strong because it is based on the assumption that the reachable set from the origin is the entire positive vector space $\mathbb{R}_+^n$. Therefore, characterizing all states of a linear positive system that can be reached from the zero initial state using positive inputs is the problem to be investigated in this paper. More details on the problem formulation may be found in section 3.

Previous work. Below, the vector space of tuples of the positive real numbers will be referred to as the *positive vector space*; it is formally defined in section 2.

Controllability and reachability of a discrete-time linear positive system has been widely studied and there is a considerable literature. This literature is briefly surveyed below. In most of the literature it is emphasized that the characterization of controllability of a discrete-time linear positive system takes a very different form than that of its counterpart for discrete-time linear systems [8, 12, 20]. In addition, while reachability of a linear system may be achieved in a number of steps equal to the state-space dimension, [43], for discrete-time linear positive systems this does not hold. For a linear positive system the number of steps required to reach a certain point in the positive orthant can be larger than the dimension of the system, as noted in [12], where this is illustrated using the model of a pharmacokinetic system.

The concept of reachability used in the literature of discrete-time linear positive systems is whether every state of the positive vector space can be reached from the origin either in finite time or in infinite time. The result is then a characterization of this considered concept of reachability. Publications that are based on that approach include [3, 8, 19, 24, 50].

Reachability of a discrete-time linear positive system is characterized using a graph-theoretic approach, and canonical reachable or canonical controllable forms are derived as well in [8, 50]. The authors of [24] have established a link between positive state controllability and positive input controllability of a related system, which is then used to obtain a controllability criterion. A survey of results on controllability and reachability of positive systems is provided in [10, 35]. Controllability results for special classes of 1- and 2-dimensional positive systems are provided in [34].

It is worth mentioning that the constrained reachability problem for a discrete-time linear system in the presence of a disturbance with respect to a target tube or a target set has been widely discussed in the literature [7, 22, 25, 39]. Among others [22] investigates this problem by constructing a sequence of target sets. The reachability problem is then transformed into a certain inclusion check on the last target set of this sequence. The authors of [22] also provide an approximate bounding ellipsoid algorithm to calculate the sequence of target sets and the associated input sequence. In [7], constrained reachability with respect to a target set is studied as a special case of reachability with respect to a target tube, and the authors provide an algorithm to construct the sequence of modified target sets when these sets are known to be polyhedral. In the above-mentioned literature, checking reachability or controllability of a target set requires one to directly or indirectly construct certain modified target sets in an iterative manner. In addition, it is not known in advance whether a target set can be reached in finite time.

Contribution of this work. The contribution of this paper to control and system theory is described next. Attention is restricted to a time-invariant discrete-time linear positive system. The problem for a continuous-time linear positive system is different. The results are mostly for a single-input system. The object of interest is the reachable subset from the origin state in either finite time or in infinite time. The problem is to characterize this reachable subset, in particular to determine whether the reachable subset is either a polyhedral cone or a nonpolyhedral cone. This problem is of interest to both control theory and to realization theory.

The problem considered in this paper differs from the reachability or controllability problems treated in the literature. In the literature, the problem whether any state of the positive vector space can be reached by use of a positive input from the zero initial state has been investigated and a corresponding characterization of this concept has been provided. In this paper the focus is on the characterization of the reachability subset which will often be a strict subset of the positive vector space. Moreover, whether the reachable subset is a polyhedral cone or a nonpolyhedral cone will be investigated. In the existing literature the reachable subset has to equal the positive vector space which is a polyhedral cone. Surprisingly, as presented in this paper, there exists an example of a linear positive system of which the reachable subset from the origin is a nonpolyhedral cone in the positive vector space. A consequence of this is that the reachable subset has to be investigated for the following cases: for a prespecified finite time, for an arbitrary finite time, and for infinite time. It will also be shown that the reachable subset can in general not be determined in a number of steps that equals the dimension of the state set but that the number of steps can be strictly larger than the dimension of the state set.

The specific contributions of the paper are then as follows. A characterization of when the infinite-time reachable subset is a polyhedral cone, is provided in Theorem 4.5. A characterization of when the finite-time reachable subset from the origin is a polyhedral cone, is provided in Theorem 5.2. An example of linear positive system for which the reachable subset is nonpolyhedral is provided in Example 4.8. Results for the problem of when the reachable set contains a particular cone of terminal states are summarized in Propositions 6.2 and 6.4.

The structure of the paper is described next. Section 2 presents necessary background knowledge on positive matrices and positive systems. It also reports key terminology of controllability and reachability and links this to linear positive systems while highlighting the existing view of the characterization of controllability and reachability of linear positive systems in the literature. Section 3 presents the approach of this paper and the problem formulation. The characterization of the infinite-time reachable set as a polyhedral cone is provided in section 4. The characterization of the finite-time reachable set as a polyhedral cone is provided in section 5. Numerical verifiable conditions for the polyhedrality of the reach set in terms of the spectrum of the system matrix are provided also in those sections. Section 6 provides results on how to determine reachability for a specified control objective in the form of a subset of the positive vector space of a linear positive system.

2. Preliminaries.

Positive real numbers, positive matrices, and cones. The reader is assumed to be familiar with the integers, the real numbers, and vector spaces. Denote the set of integers by $\mathbb{Z}$, the set of strictly positive integers by $\mathbb{Z}_+ = \{1, 2, \dots\}$, and the set of the natural numbers by $\mathbb{N} = \{0, 1, 2, \dots\}$. For $n \in \mathbb{Z}_+$ denote $\mathbb{Z}_n = \{1, 2, \dots, n\}$.

The *real numbers* are denoted by $\mathbb{R}$, the set of the *positive real numbers* or the *positive numbers* by $\mathbb{R}_+ = [0, \infty)$, and the set of the *strictly positive real numbers* by $\mathbb{R}_{s+} = (0, \infty) \in \mathbb{R}_+$. The term *positive real numbers* is preferred by the authors over the term *nonnegative real numbers* which occurs in the literature. The term *positive real numbers* is used in the book [15, p. 19].

Define the *positive vector space* of tuples of the positive real numbers as the tuple $(\mathbb{R}_+, \mathbb{R}_+^n)$ with the algebraic operations described next. The set of the positive real numbers is closed with respect to addition and to multiplication. There does not exist an additive inverse while in the subset $(0, \infty)$ there always exists a multiplicative inverse. The set of positive vectors $\mathbb{R}_+^n$ is closed with respect to addition but there does not exist an additive inverse in this set. The vector of all-ones in $\mathbb{R}^n$ is denoted by $\mathbb{1}_n$. When used without a subscript $\mathbb{1}$ is a vector of appropriate dimension of which all elements are equal to one.

For an integer $m \in \mathbb{Z}_+$ and a set of positive vectors $\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \dots,\ \boldsymbol{a}_m$, $\boldsymbol{a}_i \in \mathbb{R}_+^n$, define in the positive vector space the set

$$\text{(1)}\quad \text{conv}([\boldsymbol{a}_1\ \dots\ \boldsymbol{a}_m]) = \left\{ \boldsymbol{x} \in \mathbb{R}_+^n \mid \boldsymbol{x} = \sum_{i=1}^{m} \lambda_i \boldsymbol{a}_i,\ \lambda_i \geq 0,\ i = 1, \dots, m,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

as the *convex polytope* generated by $\boldsymbol{a}_i$, $i = 1, \dots, m$.

Define in the vector space of the real numbers $\mathbb{R}^n$ the *open ball* with center $x \in \mathbb{R}^n$ and with radius $r \in (0, \infty)$ as the set

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\|_2 < r\}. \tag{2}$$

The norm on $\mathbb{R}^n$ is the Euclidean norm, $\|\boldsymbol{x}\|_2 = (\sum_{i=1}^n\ x_i^2)^{1/2}$. This norm is also used on $\mathbb{R}_+^n$. An open ball in the positive vector space $\mathbb{R}_+^n$ is defined in a similar manner with $\boldsymbol{y} \in \mathbb{R}$ replaced by $\boldsymbol{y} \in \mathbb{R}_+^n$ in (2).

A *positive matrix* $\boldsymbol{A}$ of size $n \times m$ for $n,\ m \in \mathbb{Z}_+$ is a matrix of which each element $A_{i,j} = A_{ij}$ belongs to the positive real numbers $\mathbb{R}_+$. The set of such matrices is denoted by $\mathbb{R}_+^{n \times m}$.

The geometric viewpoint of positive vectors is formulated in terms of rays and of cones as defined next. A *ray* is a half-line $Y \subset \mathbb{R}_+^n$ for $n \in \mathbb{Z}_+$ described by a direction vector $\boldsymbol{x} \in \mathbb{R}_+^n \backslash \{0\}$ such that for all $c \in \mathbb{R}_+$, Y contains all elements of the form $c \cdot \boldsymbol{x} \in Y$. Equivalently,

$$\exists\ n \in \mathbb{Z}_+,\ \exists\ \boldsymbol{x} \in \mathbb{R}_+^n \backslash \{0\},\quad C(x) = \{c \cdot \boldsymbol{x} \in \mathbb{R}_+^n \mid \forall\ c \in \mathbb{R}_+\}.$$

Below $c \cdot \boldsymbol{x}$ will be denoted by $c\ \boldsymbol{x}$.

A *cone* is a nonempty subset $C \subseteq \mathbb{R}_+^n$ such that (1) if $\boldsymbol{x} \in C$ and $c \in \mathbb{R}_+$ then $c\ \boldsymbol{x} \in C$; and (2) if $\boldsymbol{x},\ \boldsymbol{y} \in C$ then $\boldsymbol{x} + \boldsymbol{y} \in C$. It follows that $0 \in C$ for any cone C. By definition, a cone always includes the zero element of the positive vector space. That zero element is called the *apex* of the cone. Cones with an apex not at zero of the positive vector space are not used in this paper.

A cone C is called a *polyhedral cone* if there exists an integer $m \in \mathbb{Z}_+$ and a set of positive vectors $a_1,\ a_2,\ \dots,\ a_m \in C \subseteq \mathbb{R}_+^n$ such that, for any $\boldsymbol{x} \in C$ there exist positive real numbers $y_i \in \mathbb{R}_+$ for $i = 1, \ldots, m$, $\boldsymbol{x} = \sum_{i=1}^{m} y_i \, \boldsymbol{a}_i$. Equivalently, C is a polyhedral cone if

$$\begin{gathered} \exists\ m \in \mathbb{Z}_+,\ \exists\ \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in C, \text{ such that} \\ C = \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \mid \exists\ \boldsymbol{y} \in \mathbb{R}^m_+ \text{ such that } \boldsymbol{x} = \boldsymbol{A}\boldsymbol{y} \right\}, \text{ where} \\ \boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{bmatrix} \in \mathbb{R}^{n \times m}_+, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}. \end{gathered}$$

In the representation used above, the cone will also be denoted by

$$C = \text{cone}([\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_m]) = \text{cone}(\boldsymbol{A}) \tag{3}$$

for the positive matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}_+$ with the understanding that the cone is generated by the columns of the matrix $\boldsymbol{A}$. Moreover, with little abuse of the notation for $\boldsymbol{A} \in \mathbb{R}^{n \times m}_+$ and $\boldsymbol{X} \in \mathbb{R}^{n \times p}_+$, the cone generated by stacking up the $m + p$ columns of the matrices $\boldsymbol{A}$ and $\boldsymbol{X}$ will be denoted by $C = \text{cone}([\boldsymbol{A}\ \boldsymbol{X}])$.

A cone is called a *nonpolyhedral cone* if it is not polyhedral. This implies that there does not exist a finite number $m \in \mathbb{Z}_+$ as in the above definition. The term *round cone* could also be used in this case. An example of a round cone is the well-known ice cream cone which may be found in [6, Ex. 1.2.2].

An example of a polyhedral cone is given by

$$\begin{gathered} C = \{\boldsymbol{x} \in \mathbb{R}^4_+ \mid \exists\ \boldsymbol{y} \in \mathbb{R}^4_+ \text{ such that } \boldsymbol{x} = \boldsymbol{A}\boldsymbol{y}\} \\ \text{with } \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \end{gathered}$$

A *boundary ray* of a cone C is a ray of the cone that lies on the boundary of the cone. A ray lies on the boundary of a cone if for every $\epsilon \in (0, 1)$ sufficiently small and for every element $\boldsymbol{x}$ of the ray, the ball $B(\boldsymbol{x}, \epsilon)$ includes an element outside the cone.

It is called an *extreme (boundary) ray* of the cone if it cannot be written as the strict convex combination of two different rays. Thus $\boldsymbol{x} \in C$ is an extreme ray if there do not exist vectors $\boldsymbol{y},\ \boldsymbol{z} \in C$ that are boundary rays and a scalar $c \in (0, 1)$ such that $\boldsymbol{x} = c\,\boldsymbol{y} + (1 - c)\,\boldsymbol{z}$. In the above example, each of the columns of the matrix $\boldsymbol{A}$ is an extremal ray of cone cone$(\boldsymbol{A})$.

More technical concepts and results regarding positive matrices may be found in Appendix A because these are well known and not a contribution of this paper.

The reader may find additional information on positive real numbers, positive matrices, and cones in the books [6, 9, 44].

Linear positive systems.

DEFINITION 2.1. *Define a* discrete-time linear positive system *with* system matrix $\boldsymbol{A}$ *and* input matrix $\boldsymbol{B}$ *by the representation*

$$\begin{aligned} &\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\ t_0 \in \mathbb{N},\ \boldsymbol{x(t_0)} = \boldsymbol{x}_0, \\ &\boldsymbol{A} \in \mathbb{R}^{n \times n}_+,\ \boldsymbol{B} \in \mathbb{R}^{n \times m}_+,\ t \in T = \{t_0,\ t_0 + 1,\ t_0 + 2, \ldots\}, \\ &\boldsymbol{x}_0 \in \mathbb{R}^n_+,\ \boldsymbol{u} : T \to \mathbb{R}^m_+,\ \boldsymbol{x} : T \to \mathbb{R}^n_+. \end{aligned} \tag{4}$$

An explicit expression for the state function of a discrete-time linear positive system is well known and provided by the formula

$$\boldsymbol{x}(t) = \boldsymbol{A}^{t-t_0}\boldsymbol{x}_0 + \sum_{s=t_0}^{t-t_0-1} \boldsymbol{A}^s \ \boldsymbol{B}\boldsymbol{u}(t-1-s), \ \forall \ t \in T, \tag{5}$$

$$(t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t-1)}{\mapsto} (t, \boldsymbol{x}(t)), \text{ where} \tag{6}$$
$$\boldsymbol{u}(t_0 : t-1) = (\boldsymbol{u}(t_0), \ \boldsymbol{u}(t_0+1), \ldots, \boldsymbol{u}(t-1)).$$

For a *time-invariant discrete-time linear positive system* we may assume $t_0 = 0$ in Definition 2.1 and in the explicit solution (5) as the time axis can be shifted to the zero time without affecting the trajectories.

Definition 2.1 requires that the mathematical objects of the definition exist. An alternative definition, which may be found in the literature, defines a linear positive system as a linear system with a state space $X = \mathbb{R}^n$ and requires that for any initial state $\boldsymbol{x}_0 \in \mathbb{R}_+^n$ and any positive input function $\boldsymbol{u} : T \to \mathbb{R}_+^m$, the resulting state function $\boldsymbol{x}$ is such that for all $t \in T$, $\boldsymbol{x}(t) \in \mathbb{R}_+^n$. It can then be proven that this alternative definition leads to the condition that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive matrices. Thus the alternative definition leads back to the form of Definition 2.1.

Books on positive systems or books with chapters on positive systems include [5, 20, 34].

Terminology of controllability and reachability. The literature of control and system theory is not standardized in regard to the terms controllability and reachability. The authors have chosen to use in this paper the terms as introduced by Kalman in Chapter 2 of the book [38, Def. 2.13, Def. 2.14, p. 32]. Almost the same definitions may be found in [49, Def. 3.1.1]. Related papers of Kalman on controllability are [36, 37].

Consider the discrete-time linear system with the representation (4) and the corresponding solution (5). Associate with this system the *initial tuple* $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ consisting of the initial time t_0 and the initial state $\boldsymbol{x}_0$, where t_0 will often be taken to be zero, $t_0 = 0$, and the *terminal tuple* $(t_1, \boldsymbol{x}_1)$ consisting of the terminal time t_1 and the terminal state x_1, where $t_1 \in T$ and $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$. The solution displayed above is then denoted as the *transition*

$$(t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}(t_1)).$$

In systems theory one distinguishes between reachability and controllability: for reachability one considers an initial tuple consisting of an initial time and an initial state as fixed and one has to determine which tuples of a terminal time and a terminal state can be reached by the use of a positive input; for controllability one considers a terminal time and terminal state as fixed and one has to determine from which tuples of an initial time and an initial state one can reach the selected terminal state at the terminal time by the use of a positive input.

In the case of a time-invariant system the concepts of reachability and of controllability do not depend on the initial time because the time axis can be shifted to the zero time without affecting the trajectories.

DEFINITION 2.2. *Consider a linear positive system as defined in Definition* 2.1.

(a) *Fix an initial tuple* $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$. *The terminal tuple* $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$ *is called* reachable *from the initial tuple (i.e., can be* reached *from the initial*

tuple), if there exists a positive input $\boldsymbol{u} : \{t_0, t_0+1, \ldots, t_1-1\} \to \mathbb{R}_+^m$ *such that the transition* $(t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}(t_1)) = (t_1, \boldsymbol{x}_1)$ *exists for this system. (Kalman states this for* $\boldsymbol{x}_0 = 0$*.) The terminal tuple is called* reachable from the origin *if it is reachable from the initial tuple* $(t_0, 0) \in T \times \mathbb{R}_+^n$.
Define the reachable set *from* $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ *as*

$$\mathrm{Reachset}(t_0, \boldsymbol{x}_0) = \left\{ \begin{array}{l} \boldsymbol{x}_1 \in \mathbb{R}_+^n | \ \exists\, t_1 \in T, \ \exists\, \boldsymbol{u} : \{t_0, \ldots, t_1-1\} \to \mathbb{R}_+^m, \\ \textit{such that } (t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, x_1) \end{array} \right\}.$$

(b) *Fix a terminal tuple* $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$*. The initial tuple* $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ *is called* controllable *to the terminal tuple (i.e., can be* controlled *to the terminal tuple) if there exists an input* $\boldsymbol{u} : \{t_0, t_0+1, \ldots, t_1-1\} \to \mathbb{R}_+^m$ *such that the transition* $(t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}_1)$ *exists for this system. (Kalman requires that the terminal state* $\boldsymbol{x}_1 = 0$*.) The initial tuple is called* controllable to the origin *if it is controllable to the terminal tuple* $(t_1, 0) \in T \times \mathbb{R}_+^n$.
Define the controllable set *to the terminal tuple* $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$ *as*

$$\mathrm{Conset}(t_1, \boldsymbol{x}_1) = \left\{ \begin{array}{l} \boldsymbol{x}_0 \in \mathbb{R}_+^n | \ \exists\, t_0 \in T, \ \exists\, \boldsymbol{u} : \{t_0, \ldots, t_1-1\} \to \mathbb{R}_+^m, \\ \textit{such that } (t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, x_1) \end{array} \right\}.$$

For linear systems, not necessarily a linear positive system, the following result holds.

LEMMA 2.3 (see [49, Lem. 3.1.5]). *Consider a time-invariant discrete-time linear system (not necessarily a linear positive system). The system is a reachable system on the interval* $\{t_0, \ldots, t_1\}$*, if and only if it is reachable from the origin on the same interval.*

The above result does not hold for linear positive systems as the following example shows.

Example 2.4. Consider the time-invariant linear positive system

$$\boldsymbol{x}(t+1) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

Then the reachable set from the origin is the full positive vector space $\mathbb{R}_+^2$. If $\boldsymbol{x}_0 = (1,1)^T$ then the reachable set from that initial state equals

$$X(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}_+^2 | \ x_1 \geq 0.5 x_{0,1} = 0.5, \ x_2 \geq x_{0,2} = 1\}.$$

Hence the state $\boldsymbol{x}_1 = (0.4, 0.4)^T$ can never be reached from $\boldsymbol{x}_0$ using positive inputs. Thus reachability from the origin and from an arbitary initial state of the positive vector space are different concepts for linear positive systems.

From the above example it is clear that the reachable set from the origin and the reachable set from an arbitrary initial state are different objects. In this paper attention is restricted to the reachable set from the origin.

Existing results on reachability and controllability of linear positive systems. The existing view of the characterization of controllability and reachability as known in the literature, is discussed below. In most papers of the literature, the characterization of controllability or of reachability of a linear positive system is based on the following definition.

DEFINITION 2.5 (see [20, p. 74, Def. 7]). *A linear positive system is said to be completely reachable if all states $\boldsymbol{x} \geq 0$ are reachable in finite time from the origin, that is, if $X_{\mathrm{r}} = \mathbb{R}_+^n$, where X_{r} denotes the cone of all reachable states in finite time using a positive input.*

The underlying idea behind Definition 2.5 probably originates from making an analogy to reachability of linear systems. This definition is based on the assumption that the state space equals $X = \mathbb{R}^n$. Note that in Definition 2.5 $X_{\mathrm{r}} \subseteq \mathbb{R}_+^n$ by definition, hence the equality $X_{\mathrm{r}} = \mathbb{R}_+^n$ holds if in addition $\mathbb{R}_+^n \subseteq X_{\mathrm{r}}$. The following theorem states a necessary and sufficient condition for reachability with respect to Definition 2.5 for the single-input case.

THEOREM 2.6 (see [20, Thm. 27]). *A discrete-time linear positive system with a single-input is completely reachable if it is possible to reorder its state variable in such a way that the input $\boldsymbol{u}$ directly influences only $\boldsymbol{x}_1$, and $\boldsymbol{x}_i$ directly influences $\boldsymbol{x}_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

Additional results may be found in [20, Chap. 8].

The criterion for complete reachability of a linear positive system with multiple inputs based on Definition 2.5 is more involved, but it is required that the controllability matrix of the corresponding linear system, $[\boldsymbol{B}\ \boldsymbol{AB}\ \ldots\ \boldsymbol{A}^k\boldsymbol{B}]$, includes a monomial submatrix of dimension n for some $k \in \mathbb{N}_+$ [8, 10, 12, 19, 50]. Such conditions are often too strong to be satisfied by most practical linear positive systems.

For several examples of linear positive systems, complete reachability as in Definition 2.5 is not required. For example in economic systems, one would be interested to know whether a certain growth rate can be achieved, which corresponds to checking whether a certain extremal ray of a cone inside the positive vector space is reachable. In biochemical reactors, it may be of interest to know whether a set of desired mass concentrations can be reached by applying a particular input (for example, a flow of materials).

An example follows that illustrates the concept of reachability stated above.

Example 2.7. Consider the discrete-time time-invariant linear positive system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\boldsymbol{u}(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

with

$$\boldsymbol{A} = \begin{bmatrix} 4 & 4 \\ 11 & 2 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \boldsymbol{x}_0 = 0.$$

It is of interest to determine whether the states in the cone $K \subset \mathbb{R}_+^2$, defined by (7) and illustrated by Figure 1, can be reached in finite time:

$$K : \begin{cases} 3x_1 - 2x_2 \geq 0, \\ 3x_2 - 2x_1 \geq 0, \\ x_1 \geq 0,\ x_2 \geq 0. \end{cases} \tag{7}$$

Since $K \subset \mathbb{R}_+^2$, in order to answer this question using the classical approach, one needs to check the reachability of $\mathbb{R}_+^2$, which is very conservative considering the fact that K occupies only a small portion of $\mathbb{R}_+^2$. It can be verified that

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} & \ldots & \boldsymbol{A}^k\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 12 & \cdots \\ 1 & 24 & \cdots \end{bmatrix}$$

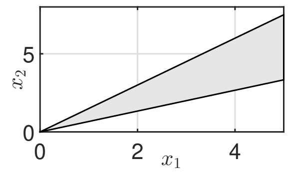


FIG. 1. *Example 2.7. The shaded area, associated with K, represents the region of interest for which controllability needs to be checked.*

does not include a monomial submatrix of dimension 2 for any $k \in \mathbb{N}_+$. Therefore, the conditions of Theorem 2.6 do not hold and we cannot deduce anything about the reachability of K. Nevertheless, invoking Theorem 5.2 and using the results of section 6, it turns out that K is reachable from the origin in a finite number of steps.

3. Approach of this paper. This paper changes the focus of reachability of a linear positive system. In the classical literature the system is reachable from the origin if the reach set from the origin equals the entire positive vector space $\mathbb{R}^n_+$.

In this paper, the approach is to determine the reachable set from the origin, in either finite time or in infinite time, as defined below. The reachable set is then the main object of study. In this paper, there is no requirement that the reachable set from the origin equals the positive vector space $\mathbb{R}^n_+$.

In the late 1960s and the 1970s the geometric viewpoint gained momentum in control and system theory. This viewpoint was developed by Wonham [52] for time-invariant linear control systems using the concept of a linear subspace of a vector space. The geometric approach to control of nonlinear control systems was described in the book [41]. Later this led to the development of control theory in differential-geometric structures [29, 30], and in algebraic-geometric structures such as rings [48].

In the geometric approach to control systems the main concept is the reachable set from the origin. In the context of observability, it is the kernel of the output map, but that will not be treated in this paper. For linear positive systems, the main geometric concept is a cone in the positive vector space $\mathbb{R}^n_+$. This geometric object allows the use of abstract algebra for theory and algorithms. Therefore, in this paper the geometric approach to linear positive systems is used.

Based on this new viewpoint, the system theoretic problem under study is characterize the reachable set from the origin of a linear positive system. The reachable set from the origin is by definition a cone in the positive vector space. A question is then, is the reachable set from the origin a polyhedral cone or a nonpolyhedral cone?

Remark 3.1. The above formulation has been for decades the approach to reachability in systems theory. The reachable set from the origin is defined as stated above. The reachable set in general may be a strict subset of the ambient space in which it is situated. The reader may want to look at the definitions of the reachable subset for discrete-time polynomial systems [48], for continuous-time polynomial systems [2], rational systems [45], and infinite-dimensional linear systems [13].

Concepts. The reachable set and its role in the problem of reachability and of controllability of linear positive systems have been already discussed in the literature [8, 10, 12, 19, 50]. Below the concept inspired by [12] is used. Recall that only reachability from the origin, the zero initial state, is considered and that the system is restricted to have an input with only one component. Recall the formula of the state transition of a time-invariant discrete-time linear positive system as

$$\begin{aligned} \boldsymbol{x}(t) &= \sum_{s=0}^{t-1} \boldsymbol{A}^s \, \boldsymbol{b} \, u(t-1-s) \\ &= \left[\begin{array}{cccc} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \ldots & \boldsymbol{A}^{t-1}\boldsymbol{b} \end{array} \right] \left[\begin{array}{c} \boldsymbol{u}(t-1) \\ \boldsymbol{u}(t-2) \\ \vdots \\ \boldsymbol{u}(0) \end{array} \right] \end{aligned} \tag{8}$$

with $\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b}) = \left[\begin{array}{cccc} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \ldots & \boldsymbol{A}^{k-1}\boldsymbol{b} \end{array} \right]$ being the *controllability matrix of index k*.

It is useful to have notation for the infinite reachable set and to contrast that with the finite reachable set, which is the purpose of the following definition.

DEFINITION 3.2. *Consider a single-input time-invariant discrete-time linear positive system with representation*

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\boldsymbol{u}(t), \;\; \boldsymbol{x}(0) = 0. \tag{9}$$

Define the following subsets of the state space: the k-step reachable subset from the origin*, the* finite-time reachable subset from the origin*, and the* infinite-time reachable subset from the origin*, respectively, as the sets,*

$$\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}) = \left\{ \boldsymbol{x} \in \mathbb{R}_+^n \,|\, \exists \boldsymbol{u} : \mathbb{N}^{k-1} \to \mathbb{R}_+, \; (0,0) \overset{\boldsymbol{u}}{\mapsto} (k, \boldsymbol{x}) \right\} \quad \forall\, k \in \mathbb{Z}_+, \tag{10}$$

$$\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b}) = \cup_{k=0}^{\infty} \mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}), \tag{11}$$

$$\mathrm{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b}) = \overline{\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})}. \tag{12}$$

Here, the notation $\overline{S}$ denotes the closure of the set S with respect to the Euclidean topology.

The reachable subsets defined above are subsets of the state set. To simplify the terminology, in the remainder of the paper these sets are referred to as the *reachable set from the origin* or as the *reachable set*, without the use of the term *subset*.

Once a reachable set has been defined, there is no need for the concept of complete reachability.

PROPOSITION 3.3. *The k-step reachable subset, the finite-time reachable subset, and the infinite-time reachable subset of Definition* 3.2*, each from the zero initial state, equal, respectively, the expressions*

$$\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}) = \mathrm{cone}\big(\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big), \tag{13}$$

$$\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b}) = \mathrm{cone}\big([\boldsymbol{b} \;\, \boldsymbol{A}\boldsymbol{b} \;\, \boldsymbol{A}^{\boldsymbol{2}}\boldsymbol{b} \;\, \ldots]\big), \tag{14}$$

$$\mathrm{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b}) = \overline{\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})}, \;\; \textit{where} \tag{15}$$

$$\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{b} \;\, \boldsymbol{A}\boldsymbol{b} \;\, \boldsymbol{A}^{\boldsymbol{2}}\boldsymbol{b} \;\, \ldots \;\, \boldsymbol{A}^{\boldsymbol{k-1}}\boldsymbol{b}]. \tag{16}$$

Proof. The proof is skipped as it can be derived in a straightforward manner. The reader is referred to [10, 11] for similar proofs. The proof could also be deduced from the corresponding definition in [12]. ∎

Problem formulation. Having characterized the infinite-time and the finite-time reachable sets from the origin, the main questions of this paper are discussed next.

PROBLEM 3.4. *For a single-input time-invariant linear positive system, the problems to be addressed in this paper are the following:*

(a) *Is the finite-time reachable set from the origin* $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ *a polyhedral cone or a nonpolyhedral cone?*

(b) *Is the infinite-time reachable set from the origin* $\mathrm{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$ *a polyhedral cone or a nonpolyhedral cone?*

(c) *If the control objective is specified as a cone in the positive vector space or as a subset of that space, is that control objective subset then contained in the reachable set from the origin?*

Note that the k-time reachable set is by definition always a polyhedral set.

4. When is the infinite-time reachable set a polyhedral set? In this section, we investigate the polyhedrality of $\mathrm{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$, and characterize this in terms of necessary and sufficient conditions on the system matrix $\boldsymbol{A}$.

The reader is expected to have knowledge of concepts and of results of positive linear algebra as summarized in Appendix A. The notations used below may be found in Appendix A.

As summarized in Appendix A, a positive matrix which is nonzero and of dimension $n \geq 2$ is either irreducible or can be fully reduced. The analysis of the matrix $\boldsymbol{A}^k$ for $k \in \mathbb{Z}_+$ or for its limit, $\lim_{k\to\infty} \boldsymbol{A}^k$, can then be carried out (1) for irreducible positive matrices and, (2) for fully reduced matrices. Below, the case of an irreducible system matrix $\boldsymbol{A}$ is carried out. The case of a fully reduced positive matrix is then relatively simple based on the results for the irreducible case [6].

For the remainder of this section, the reader should keep in mind the restriction to an irreducible positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$.

PROPOSITION 4.1. *Consider the linear positive system given in* (4). *Assume that* $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ *is irreducible with cyclicity index* $1 \leq h \leq n$ *and* $\boldsymbol{b} \in \mathbb{R}_+^n$. *Then, the infinite-time reachable set from the origin,* $\mathrm{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$, *is polyhedral if and only if there exists a* $k^* \in \mathbb{Z}_+$ *such that*

$$\boldsymbol{A}^{k^*}\boldsymbol{b} \in \mathrm{cone}\left([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \ \boldsymbol{A}_{\mathrm{f},0}b\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}]\right), \tag{17}$$

where matrices $\boldsymbol{A}_{\mathrm{f},i}$ *are introduced in Definition* A.5.

Proof. The result is almost obvious by geometric considerations except for the presence of the set of vectors $\{\boldsymbol{A}_{\mathrm{f},0}b, \ldots, \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}\}$.

Sufficiency: We will show that

$$C = \mathrm{cone}\left([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}]\right)$$

is $\boldsymbol{A}$-invariant. Let $\boldsymbol{x} = \sum_{i=0}^{k^*-1} c_i \boldsymbol{A}^i \boldsymbol{b} + \sum_{i=0}^{h-1} c_{\mathrm{f},i}\boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}$ for arbitrary positive coefficients $\boldsymbol{c} \in \mathbb{R}_+^{k^*}$ and $\boldsymbol{c}_{\mathrm{f}} \in \mathbb{R}_+^h$. We then have

$$\boldsymbol{A}\boldsymbol{x} = \sum_{i=0}^{k^*-1} c_i \boldsymbol{A}^{i+1}\boldsymbol{b} + \sum_{i=0}^{h-1} c_{\mathrm{f},i}\boldsymbol{A}\boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}. \tag{18}$$

Using (17), and noting that (see Definition A.5)

$$\boldsymbol{A}\boldsymbol{A}_{\mathrm{f},i} = \boldsymbol{A}_{\mathrm{f},i+1}, \quad i = 0, \dots, h-2, \qquad (19)$$
$$\boldsymbol{A}\boldsymbol{A}_{\mathrm{f},h-1} = \big(\rho(\boldsymbol{A})\big)^h \boldsymbol{A}_{\mathrm{f},0},$$

(18) can be expressed as $\boldsymbol{A}\boldsymbol{x} = \sum_{i=0}^{k^*-1} c'_i \boldsymbol{A}^i \boldsymbol{b} + \sum_{i=0}^{h-1} c'_{\mathrm{f},i} \boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}$ for some $\boldsymbol{c}' \in \mathbb{R}_+^{k^*}$ and some $\boldsymbol{c}'_{\mathrm{f},i} \in \mathbb{R}_+^h$. This proves that $\boldsymbol{A}\boldsymbol{x} \in C$ for any $\boldsymbol{x} \in C$. Hence, the system trajectory (8) remains in C and $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = C$ is polyhedral.

Necessity: Let $\boldsymbol{x}_\infty = \lim_{k\to\infty} \frac{\boldsymbol{A}^k \boldsymbol{b}}{(\rho(\boldsymbol{A}))^k}$. Note that $\boldsymbol{x}_\infty$ is characterized by the set of h vectors $\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}, \dots, \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}$ [12, Thm. 2](also see proof of Lemma A.6). In fact, Lemma A.6 states that $\boldsymbol{x}_\infty \in \mathrm{cone}([\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. By the definition of $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ as the closure of $\mathrm{Reachset}_\mathrm{f}(\boldsymbol{A}, \boldsymbol{b})$, and by the above explanation of $\boldsymbol{x}_\infty$, the extremal rays of the polyhedral $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ belong to the sequence $\{\boldsymbol{A}^k\boldsymbol{b} \in \mathbb{R}_+^n, \ k \in \mathbb{N}\}$ or are extremal rays of the cone $\mathrm{cone}([\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. Again, by the assumption that $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral, there exists a finite $k^* \in \mathbb{Z}_+$ such that $\boldsymbol{A}^{k^*}\boldsymbol{b} \in \mathrm{cone}([\boldsymbol{b} \ \dots \ \boldsymbol{A}^{k^*-1}\boldsymbol{b} \ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. ☐

It is clear that if (17) is established for an integer $k^* \in \mathbb{Z}_+$, it will hold for any $k \geq k^*$. The smallest integer $k^* \in \mathbb{Z}_+$ satisfying (17) is called the *vertex number* and denoted by k_{vert}^∞ of the reachable set $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$. Following the steps of the proof of Proposition 4.1, we can put forward the following corollary.

COROLLARY 4.2. *Given* $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ *irreducible with cyclicity index* $h \in \{1, \dots, n\}$ *and* $\boldsymbol{b} \in \mathbb{R}_+^n$*, the following statements are equivalent:*

(a) $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ *is polyhedral.*

(b) *There exists an integer* $k_{\mathrm{vert}}^\infty \in \mathbb{Z}_+$ *such that* $\mathrm{cone}\big([\boldsymbol{b} \ \boldsymbol{A}\boldsymbol{b} \ \dots \ \boldsymbol{A}^{k-1}\boldsymbol{b} \ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}]\big)$ *is* $\boldsymbol{A}$*-invariant for* $k \geq k_{\mathrm{vert}}^\infty$.

(c) *There exists an integer* $k_{\mathrm{vert}}^\infty \in \mathbb{Z}_+$ *such that for all* $k \geq k_{\mathrm{vert}}^\infty$*, the matrix equation*

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{X},$$

has a solution $\boldsymbol{X} \in \mathbb{R}_+^{(k+h)\times(k+h)}$*, where,*

$$\boldsymbol{M} = \left[\begin{array}{cccccccc} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \dots & \boldsymbol{A}^{k-1}\boldsymbol{b} & \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b} & \dots & \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b} \end{array}\right].$$

DEFINITION 4.3. *A square positive matrix* $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ *is said to have a* positive recursion *if the following holds:*

$$\exists m \in \mathbb{N}, \ \exists c_i \in \mathbb{R}_+ \text{ for } i = 0, \dots, m-1 \text{ such that} \qquad (20)$$
$$\boldsymbol{A}^m = \sum_{i=0}^{m-1} c_i \boldsymbol{A}^i$$

or, equivalently, if

$$g(\lambda) = \lambda^m - \sum_{i=0}^{m-1} c_i \lambda^i = 0 \quad \forall \lambda \in \mathrm{spec}(\boldsymbol{A}).$$

In terms of the characteristic polynomial of $\boldsymbol{A}$, $p_{\boldsymbol{A}}$, the existence of a positive recursion implies that $g = p_{\boldsymbol{A}}\, Q$, where Q is a polynomial of degree q with $0 \leq q \leq m$.

It is then immediate that

$$m = n + q \geq n. \tag{21}$$

Before presenting our main results on polyhedrality of reachable subsets, we report a key theorem ([46, Thm. 5]). In the following Q denotes the set of all real polynomials of the form $c_n x^n - \sum_{i=0}^{n-1} c_i x^i$, where $n \geq 1$, $c_n > 0$, and $c_i \geq 0$ for all i.

THEOREM 4.4 (see [46, Thm. 5]). *Let $\{a_1, \dots, a_k\}$ be given complex numbers, and let $P(x)$ be the polynomial $x^k - a_1 x^{k-l} - \cdots - a_k$. Then conditions* (A), (B)*, and* (C) *below are equivalent:*

(A) *Any infinite sequence $(u_n)_{n\geq 0}$ of complex numbers which satisfies the recursion $u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_k u_n$ for $n \geq 0$, also satisfies a recursion with positive coefficients.*

(B) *The polynomial $P(x)$ divides a polynomial in Q.*

(C) *In the case the polynomial $P(x)$ has a positive root r, then all conditions* (1)–(4) *below are satisfied:*

  (C1) *$r \geq |\alpha|$ for any root α of $P(x)$;*

  (C2) *if $\alpha = r$ for some root α of $P(x)$, then α/r is a root of unity;*

  (C3) *all roots $P(x)$ with absolute value r are simple;*

  (C4) *if $P(r) = P(r\epsilon) = 0$, where $\epsilon^k = 1$ with $k \geq 1$ minimal, then $P(x)$ has no roots of the form $s\omega$, where $0 < s < r$ and $\omega^k = 1$.*

We are now in the position to state a characterization of Proposition 4.1 in terms of spec($\boldsymbol{A}$), hence, providing numerically verifiable conditions as to when (17) holds.

THEOREM 4.5 (polyhedrality of $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$). *Given an irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}_+^n$, the following statements are equivalent:*

(a) *The infinite-time reachable subset is polyhedral, hence there exists an integer $k^* \in \mathbb{Z}_+$ such that*

$$\begin{aligned} &\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) \\ &= \mathrm{cone}\big([\mathrm{conmat}_{k^*}(\boldsymbol{A}, \boldsymbol{b}) \ \ \boldsymbol{A}_{f,0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\big). \end{aligned}$$

Denote the lowest integer for which the above equality holds by $k_{vert}^\infty \in \mathbb{Z}_+$.

(b) *The matrix $\boldsymbol{A}_2$ defined in Definition* A.3*, satisfies a positive recursion.*

(c) *If there exists a positive $\lambda_r \in \mathrm{spec}(\boldsymbol{A}_2)$, then the following conditions all hold:*

  (c1) *$\lambda_r = \rho(\boldsymbol{A}_2)$;*

  (c2) *for any $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$, $\lambda = \rho(\boldsymbol{A}_2)\exp\big(\phi_\lambda 2\pi i\big)$, where $\phi_\lambda \in \mathbb{Q}$ is a rational number;*

  (c3) *$\sigma^\rho(\boldsymbol{A}_2)$, defined in Definition* A.3*, includes only simple eigenvalues;*

  (c4) *given $M \in \mathbb{Z}_+$ by Lemma* A.4*, no $\lambda^- \in \sigma^-(\boldsymbol{A}_2)$ has a polar angle which is an integer multiple of $2\pi/Mh$.*

Note that the condition of Theorem 4.5(a) involves the determination of the integer k^*, which is in principle a test with an infinite number of steps. Similarly, condition (b) of Theorem 4.5 is a test with an infinite number of steps. However, condition (c) of the theorem is a finite test though it requires the exact eigenvalues.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Since $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral, according to Corollary 4.2, there is a sufficiently large $k \geq n - h$ such that the equation

$$\boldsymbol{A}[\boldsymbol{b} \ \boldsymbol{A}\boldsymbol{b} \ \dots \ \boldsymbol{A}^{k-1}\boldsymbol{b} \ \boldsymbol{A}_{f,0} \ \dots \ \boldsymbol{A}_{f,h-1}] = [\boldsymbol{b} \ \boldsymbol{A}\boldsymbol{b} \ \dots \ \boldsymbol{A}^{k-1}\boldsymbol{b} \ \boldsymbol{A}_{f,0}\boldsymbol{b} \ \dots \ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\boldsymbol{X}$$

has a solution $\boldsymbol{X} \geq 0$. It can be easily verified using (17)–(19) that

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{0} \\ \boldsymbol{X}_3 & \boldsymbol{X}_2 \end{bmatrix}, \ \boldsymbol{X_1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & & 0 & \alpha_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{k-1} \end{bmatrix}, \tag{22}$$

$$\boldsymbol{X_2} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \rho^h(\boldsymbol{A}) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \ \boldsymbol{X_3} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_0 \\ 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 0 & & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & \beta_{h-1} \end{bmatrix}, \tag{23}$$

constitutes a solution, where $\boldsymbol{X}_1 \in \mathbb{R}_+^{k\times k}$, $\boldsymbol{X}_2 \in \mathbb{R}_+^{h\times h}$, and $\boldsymbol{X}_3 \in \mathbb{R}_+^{h\times k}$. Let $p_{\boldsymbol{X}_1}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{X}_1)$ and $p_{\boldsymbol{X}_2}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{X}_2)$. Since by assumption, $k \geq n-h$ and $\operatorname{rank}\big(\operatorname{conmat}_n(\boldsymbol{A}, \boldsymbol{b})\big) = n$, due to [4, Lem. 3.10], $p_{\boldsymbol{A}}(\lambda)$ divides $p_{\boldsymbol{X}}(\lambda) = p_{\boldsymbol{X}_1}(\lambda)p_{\boldsymbol{X}_2}(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))(\lambda^k - \alpha_{k-1}\lambda^{k-1} - \cdots - \alpha_0)$. Since $\boldsymbol{A}$ is irreducible with cyclicity index h, $p_{\boldsymbol{A}}(\lambda)$ can be expressed as $p_{\boldsymbol{A}}(\lambda) = p_{\boldsymbol{A}_1}(\lambda)p_{\boldsymbol{A}_2}(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))p_{\boldsymbol{A}_2}(\lambda)$. Therefore, $p_{\boldsymbol{A}_2}(\lambda)$ divides $p_{\boldsymbol{X}_2}(\lambda)$, which, due to statements (A) and (B) of Theorem 4.4, proves that $\boldsymbol{A}_2$ has a positive recursion of the form $\boldsymbol{A}_2^{k^*} - \gamma_{k^*-1}\boldsymbol{A}_2^{k^*-1} - \cdots - \gamma_0 \boldsymbol{I} = 0$ for some $n-h \leq k^* \leq k$ and for some $\boldsymbol{\gamma} \in \mathbb{R}_+^{k^*}$. Assume $\boldsymbol{A}_2$ has a positive eigenvalue. Since $\boldsymbol{A}_2$ satisfies a positive recursion, the statements ((C1)–(C4)) in (C) of Theorem 4.4 hold for $p_{\boldsymbol{A}_2}(\lambda)$. It is straightforward to check that this implies that (c1)–(c4) holds.[1]

(c) $\Rightarrow$ (b) $\Rightarrow$ (a): Assume $\boldsymbol{A}_2$ has a positive eigenvalue. We need to prove that statements (c1)–(c4) imply a positive recursion for $\boldsymbol{A}_2$ of the form $\boldsymbol{A}_2^{k^*} - \alpha_{k^*-1}\boldsymbol{A}_2^{k^*-1} - \cdots - \alpha_0 \boldsymbol{I} = 0$ for $k^* \geq n-h$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^{k^*}$ and, that in turn, implies polyhedrality of the infinite-time reachable subset.

First we show that the statements (c1)–(c4) imply the statements (C1)–(C4) of Theorem 4.4. The statement $\lambda_r \in \sigma^\rho(\boldsymbol{A}_2)$ implies (C1) of Theorem 4.4. The requirement of all $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$ having a rational polar phase implies (C2). The requirement of all $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$ being simple implies (C3), and (C4) is implied from $\sigma^-(\boldsymbol{A}_2)$ including no eigenvalue with polar phase $2\pi m/Mh$ for any $m \in \mathbb{Z}$ [6, Thm. 2.2.20]. Next, invoking the equivalence between (C) and (B) of Theorem 4.4 for $p_{\boldsymbol{A_2}}(\lambda)$, one can observe that there is a polynomial $Q(\lambda)$ of positive degree such that

$$g(\lambda) = p_{\boldsymbol{A_2}}(\lambda)Q(\lambda) = \lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_0 = 0 \tag{24}$$

for $k^* \geq n-h$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^{k^*}$. It follows from (20) that $\boldsymbol{A}_2$ has a positive recursion, which results in (b).

Given (b), there exists a polynomial $g(\lambda)$ of degree $k^* \geq n-h$ satisfying (24) from which one concludes that $p_{\boldsymbol{A}}(\lambda) = p_{\boldsymbol{A}_1}(\lambda)p_{\boldsymbol{A}_2}(\lambda)$ divides $h(\lambda) = p_{\boldsymbol{A}_1}(\lambda)g(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))(\lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_0)$. Now consider the equation $\boldsymbol{AM} = \boldsymbol{MX}$ with $\boldsymbol{M} = [\boldsymbol{b}\ \boldsymbol{Ab}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \boldsymbol{A}_{f,0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]$, where $\boldsymbol{X} \in \mathbb{R}^{(n+k^*)\times(n+k^*)}$ is an unknown matrix. Since $\operatorname{conmat}_{k^*}(\boldsymbol{A}, \boldsymbol{b})$ is full rank by assumption and $k^* \geq n-h$, $\boldsymbol{M}$

[1]Condition $\lambda_r \in \sigma^\rho(\boldsymbol{A}_2)$ follows from (C1) of Theorem 4.4, and conditions (c2) and (c3) are, respectively, a direct result of (C2) and (C3). Finally, (c4) is implied from (C4) using Lemma A.4.

is of full rank as well. Then, it is known from [4, Lem. 10] that $p_{\boldsymbol{A}}(\lambda)$ divides $p_{\boldsymbol{X}}(\lambda)$. Hence, we can choose $\boldsymbol{X}$ such that $p_{\boldsymbol{X}}(\lambda) = h(\lambda)$. A possible choice of $\boldsymbol{X}$, having substituted k^* for k, is then given by (22)–(23). It is clear from (22)–(23) that $\boldsymbol{X}$ admits a positive solution. Based on Corollary 4.2, this implies that $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral. ∎

Remark 4.6. For a polyhedral $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ the following can be observed:

(a) Due to (21) and from the second part of the proof of Theorem 4.5 the vertex number of $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$, k^∞_{vert}, is at least $n - h$, which implies that $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ has at least n generators. It has exactly n generators (i.e., it is simplicial) if and only if the characteristic polynomial $p_{\boldsymbol{A}_2}$ of $\boldsymbol{A}_2$ has nonpositive coefficients.

(b) In the view of Lemma A.6, $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ can be expressed as $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k-1}\boldsymbol{b}\ \boldsymbol{v}_{f,0}\ \ldots\ \boldsymbol{v}_{f,h-1}]\big)$, where $\boldsymbol{v}_{f,0}, \ldots, \boldsymbol{v}_{f,h-1}$ are the h distinct positive eigenvectors of $\boldsymbol{A}^h$ associated with the eigenvalue $\rho^h(\boldsymbol{A})$.

Example 4.7 (polyhedral Reachset$\infty(\boldsymbol{A}, \boldsymbol{b})$). Consider the discrete-time linear time-invariant positive system of Definition 2.1 with system matrices

$$\boldsymbol{A} = \begin{bmatrix} 0.9727 & 0 & 0.0263 \\ 0.0388 & 0.1273 & 0.2156 \\ 0 & 3.4497 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

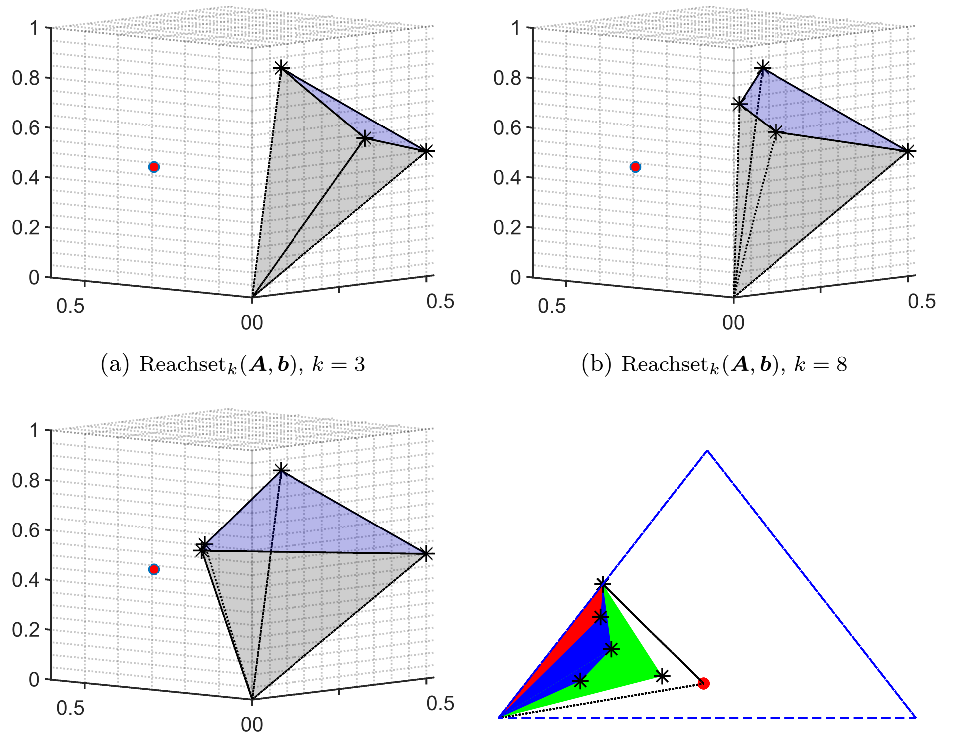
where $\boldsymbol{A}$ is primitive, i.e., is irreducible with cyclicity index $h = 1$. We have $\text{spec}(\boldsymbol{A}) = \{1, 0.9, -0.8\}$. We can assume $\boldsymbol{A}_1 = 1$, and $\boldsymbol{A}_2 = \text{diag}(0.9, -0.8)$. Using Theorem 4.5, it is immediate that conditions (c1) and (c2) hold as $\lambda = 0.9$ is a simple eigenvalue of $\boldsymbol{A}_2$, which equals the spectral radius of $\boldsymbol{A}_2$. Condition (c1) holds as well since the polar angle of $\lambda = -0.8$ is not an integer multiple of the polar angle of $\lambda = 0.9$. Hence, it can be concluded that the infinite-time reachable subset $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral. We can also conclude that $\boldsymbol{A}_2$ has a positive recursion, which is readily verified as $p_{\boldsymbol{A}_2}(\lambda) = \lambda^2 - 0.1\lambda - 0.72$. Figure 2 illustrates the growth of $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$. It can be observed that $\text{Reachset}_\text{f}(\boldsymbol{A}, \boldsymbol{b})$ is not polyhedral since the cone keeps growing for increasing values of k. Its closure is, however, polyhedral as shown in Figure 2(d).

Example 4.8 (nonpolyhedral $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$). Consider the time-invariant discrete-time linear positive system of Definition 2.1 with system matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0.5 \\ 0 & 0.4 & 1 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $\boldsymbol{A}$ has cyclicity index $h = 1$ with $\text{spec}(\boldsymbol{A}) = \{-1.05, 0.7116, 1.3383\}$. One can assume $\boldsymbol{A}_1 = 1.3383$ and $\boldsymbol{A}_2 = \text{diag}(-1.05, 0.7116)$. It is immediate that condition (c1) of Theorem 4.5 is not satisfied as $0.7116 \neq \rho(\boldsymbol{A}_2)$. Therefore, based on this theorem, $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is not polyhedral. This is illustrated by Figure 3(d), from which it is clear that $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is approaching a round cone as introduced in section 2.

5. When are the finite-time reachable subsets a polyhedral set? The polyhedrality of the finite-time reachability set from the origin, $\text{Reachset}_\text{f}(\boldsymbol{A}, \boldsymbol{b})$, will be proven to be a special case of polyhedrality of $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ but with stricter requirements.



(a) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 3$

(b) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 8$

(c) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 19$

(d) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 3$ (red), 8 (blue and red), 19 (green, blue and red) and $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$(the triangle with the red vertex)

FIG. 2. (a, b, c): *The growth of the reachability cone* $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$ *of Example* 4.7 *for different values of* k*, where generators of the cone are marked by asterisks, and the Frobenius eigenvector is marked by a red dot.* (d): *The growth of the reachable cone mapped on the* 3*-dimensional simplex* $S = \{\boldsymbol{x} \in \mathbb{R}^3_+ | \mathbb{1}^{\mathrm{T}} \boldsymbol{x} = 1\}$.

In this section we investigate the polyhedrality of the finite-time reachable set from the origin, $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$. Consider a linear positive system with an irreducible system matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$ with the cyclicity index $h \in \{1, \ldots, n\}$. It follows from Proposition 4.1 that the finite-time reachable set from the origin $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral if and only if there exists a positive integer $k^* \in \mathbb{Z}_+$ such that

$$\mathrm{Reachset}_{k^*+1}(\boldsymbol{A}, \boldsymbol{b}) \subseteq \mathrm{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b}) \tag{25}$$

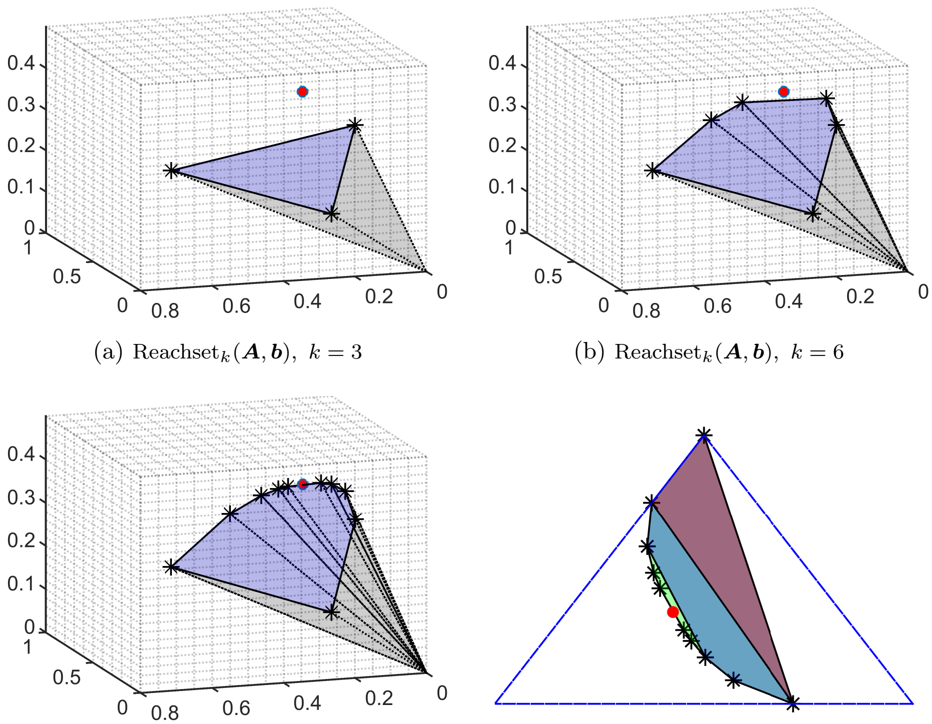
$$\Leftrightarrow \boldsymbol{A}^{k^*}\boldsymbol{b} \in \mathrm{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b}). \tag{26}$$

The smallest k^* for which (26) holds is referred to as the *vertex number*, k_{vert}, of $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$. Note that (26) also implies that

$$\mathrm{cone}\big([\boldsymbol{A}_{f,0}\boldsymbol{b} \ \ldots \ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\big) \subseteq \mathrm{Reachset}_{k_{\mathrm{vert}}}(\boldsymbol{A}, \boldsymbol{b}), \tag{27}$$

which is clearly a restriction on (17).

COROLLARY 5.1. *For an irreducible* $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$ *with cyclicity index* $1 \leq h \leq n$ *and for* $b \in \mathbb{R}_+$*, equivalence of the following statements follows directly from the above argument:*



(a) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 3$

(b) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 6$

(c) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 10$

(d) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 3$ (red region), $k = 6$ (red and blue regions), $k = 10$ (red, blue and green regions). $\mathrm{Reachset}\infty(\boldsymbol{A}, \boldsymbol{b})$ approaches a "round cone."

FIG. 3. (a, b, c): *The growth of the reachability cone* $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$ *of Example* 4.8 *for different values of k, where generators of the cone are marked by asterisks, and the Frobenius eigenvector is marked by a red dot.* (d): *The growth of the reachable cone mapped on the* 3*-dimensional simplex* $S = \{\boldsymbol{x} \in \mathbb{R}^3_+ | \mathbb{1}^{\mathrm{T}}\boldsymbol{x} = 1\}$.

(a) $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ *is polyhedral.*

(b) *There exists an integer* $k_{\mathrm{vert}} \in \mathbb{Z}_+$ *such that* $\mathrm{cone}\big([\boldsymbol{b}\ \ \boldsymbol{A}\boldsymbol{b}\ \ \ldots\ \ \boldsymbol{A}^k\boldsymbol{b}]\big)$ *is* $\boldsymbol{A}$*-invariant for any* $k \geq k_{\mathrm{vert}}$.

(c) *There exists an integer* $k_{\mathrm{vert}} \in \mathbb{Z}_+$ *such that for the matrix equation*

$$\boldsymbol{A}[\boldsymbol{b}\ \ \boldsymbol{A}\boldsymbol{b}\ \ \ldots\ \ \boldsymbol{A}^{k-1}\boldsymbol{b}] = [\boldsymbol{b}\ \ \boldsymbol{A}\boldsymbol{b}\ \ \ldots\ \ \boldsymbol{A}^{k-1}\boldsymbol{b}]\boldsymbol{X},$$

there exists a solution $\boldsymbol{X} \in \mathbb{R}_+^{(k)\times(k)}$ *with* $k \geq k_{\mathrm{vert}}$.

(d) *Based on* (27) *and Lemma* A.6*, there exists an integer* $k_{\mathrm{vert}} \in \mathbb{Z}_+$ *such that for any* $k \geq k_{\mathrm{vert}}$, $\mathrm{cone}\big([\boldsymbol{v}_{f,0}\ \ \ldots\ \ \boldsymbol{v}_{f,h-1}]\big) \subseteq \mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$.

The following theorem provides necessary and sufficient conditions on $\mathrm{spec}(\boldsymbol{A})$ for polyhedrality of $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$. These conditions turn out to be a conservative version of those of Theorem 4.5.

THEOREM 5.2 (polyhedrality of $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$). *Let* $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$ *be irreducible with index of cyclicity* $h \in \{1, \ldots, n\}$ *and* $\boldsymbol{b} \in \mathbb{R}^n_+$*. Then the following statements are equivalent:*

(a) *The finite-time controllable subset is polyhedral and hence there exists an integer* $k^* \in \mathbb{Z}_+$, $k^* \geq k_{\mathrm{vert}}$, *such that* $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b}) = \mathrm{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b})$.
(b) $\boldsymbol{A}$ *has a positive recursion.*
(c) *The matrix* $\boldsymbol{A}_2$, *defined in Definition* A.3, *does not have any positive eigenvalue.*

Proof. (a) ⇒ (b) ⇒ (c): Based on 4.2 with $k \geq n$ we obtain

$$\boldsymbol{A}\big(\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big) = \big(\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big)\boldsymbol{X},$$

where $\boldsymbol{X} \in \mathbb{R}_+^{k \times k}$ is given by

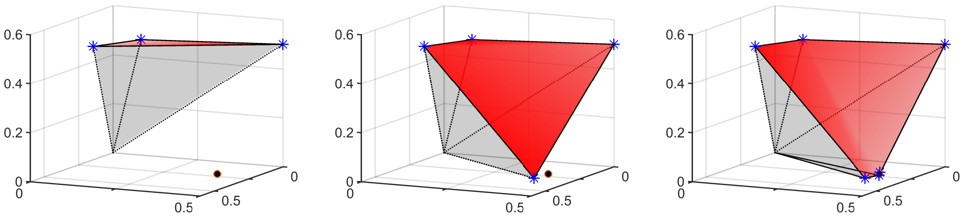
$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & & 0 & \alpha_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{k-1} \end{bmatrix}.$$

Since, by assumption, $\mathrm{conmat}_n(\boldsymbol{A}, \boldsymbol{b})$ is of full rank and $k \geq n$, there exists [4, Lem. 3.10] a polynomial $Q(\lambda)$ of positive degree such that $p_{\boldsymbol{A}}(\lambda)Q(\lambda) = p_{\boldsymbol{X}}(\lambda) = \lambda^k - \alpha_{k-1}\lambda^{k-1} - \cdots - \alpha_1\lambda - \alpha_0$, which, in the view of Definition 4.3, proves that $\boldsymbol{A}$ has a positive recursion. Noting that (b) is equivalent to condition (B) of Theorem 4.4 [46, Thm. 5], all conditions (C1)–(C4) are then fulfilled. In particular, (C4) holds as conditions (C1)–(C3) are already satisfied for a positive irreducible matrix due to the Perron–Frobenius theorem [6, Thms. 2.1.4, 2.2.20]. Condition (C4) requires that no eigenvalue $\lambda^- \in \sigma^-(\boldsymbol{A})$ has a polar angle of $2\pi k/h$ for $k = 0, \ldots, h-1$. Since $\mathrm{spec}(\boldsymbol{A})$ is invariant under a polar rotation of $2\pi m/h$ for any $m \in \mathbb{Z}$, no $\lambda^- \in \sigma^-(\boldsymbol{A})$ is then positive. Noting that for an irreducible matrix, $\big(\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\}\big) \cap \mathbb{R}_{\mathrm{s}+} = \emptyset$ and that $\mathrm{spec}(\boldsymbol{A}_2) = (\sigma^-(\boldsymbol{A}) \cup \sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\})$, one concludes that $\boldsymbol{A}_2$ has no positive eigenvalue.

(c) ⇒ (b) ⇒ (a): Given (c), we have $\mathrm{spec}(\boldsymbol{A}_2) \cap \mathbb{R}_{\mathrm{s}+} = \emptyset$. For an irreducible matrix it holds that $\big(\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\}\big) \cap \mathbb{R}_{\mathrm{s}+} = \emptyset$. Since $\mathrm{spec}(\boldsymbol{A}_2) = \sigma^-(\boldsymbol{A}) \cup (\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\})$, it follows that $\sigma^-(\boldsymbol{A}) \cap \mathbb{R}_{\mathrm{s}+} = \emptyset$, from which it can be immediately concluded that $\nexists \lambda \in \sigma^-(\boldsymbol{A}),\ \lambda = |\lambda| \exp(i2\pi m/h)$ for any $m \in \mathbb{Z}$. Hence, we establised that (C4) of Theorem 4.4 [46, Thm. 5] holds for $p_{\boldsymbol{A}}(\lambda)$. Moreover, statements (C1)–(C3) hold as well for $p_{\boldsymbol{A}}$ as $\boldsymbol{A}$ is irreducible. Therefore, due to (B) of Theorem 4.4, there exists a polynomial Q of positive degree, such that $p_{\boldsymbol{A}}(\lambda)Q(\lambda) = \lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_1\lambda - \alpha_0$, where $k^* \geq n$ and $\alpha_i \geq 0$, $i = 0, 1, \ldots, k^* - 1$. This proves that $\boldsymbol{A}$ has a positive recursion based on Definition 4.3. Then, (a) immediately follows as $\boldsymbol{A}^{k^*}\boldsymbol{b} = \sum_{i=0}^{k^*-1} \alpha_i \boldsymbol{A}^i \boldsymbol{b}$. ∎

Remark 5.3. Note that since $\deg\big(Q(\lambda)\big) \geq 0$, k_{vert} of $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ is at least n, and it equals n if and only if $p_{\boldsymbol{A}}(\lambda) = \lambda^n - \alpha_{n-1}\lambda^{n-1} - \cdots - \alpha_1\lambda - \alpha_0$ with $\alpha_i \geq 0$, $i = 0, \ldots, n-1$. Hence $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ is a simplicial cone (i.e., has n generators) if and only if the characteristic polynomial of $\boldsymbol{A}$ has nonpositive coefficients. One such matrix is a cyclic matrix with cyclicity index $h = n$ as $p_{\boldsymbol{A}}(\lambda) = \lambda^n - \rho^n(\boldsymbol{A})$.

Comparing Theorem 4.5 to Theorem 5.2 reveals that the latter is a restricted version of the former. For example, Theorem 4.5(b) requires a part of $\boldsymbol{A}$ (i.e., $\boldsymbol{A}_2$) to have a positive recursion while Theorem 5.2(b) requires the entire $\boldsymbol{A}$ to have a positive recursion.



(a) $\mathrm{Reachset}_k(\boldsymbol{A},\boldsymbol{b}),\ k=3$ (b) $\mathrm{Reachset}_k(\boldsymbol{A},\boldsymbol{b}),\ k=4$ (c) $\mathrm{Reachset}_k(\boldsymbol{A},\boldsymbol{b}),\ k=6$

FIG. 4. *Example* 5.4*: growth of the reachability cone mapped on the* 3*-dimensional simplex* $S=\{\boldsymbol{x}\in\mathbb{R}^3_+|\mathbb{1}^{\mathrm{T}}\boldsymbol{x}=1\}$*; the generators of the cone and the Frobenius eigenvector are, respectively, marked by asterisks and a dot.*

Example 5.4 (polyhedral $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$). Consider the time-invariant discrete-time linear positive system of Definition 2.1 with system matrices

$$\boldsymbol{A}=\begin{bmatrix}0 & 1.6333 & 1.1049 & 0\\ 23.5667 & 6.0944 & 0 & 0\\ 0 & 0 & 1.1225 & 1.0672\\ 0 & 1.6611 & 0 & 0.7830\end{bmatrix},\quad \boldsymbol{b}=\begin{bmatrix}0\\0\\1\\1\end{bmatrix},$$

where $\boldsymbol{A}$ is irreducible with cyclicity index $h=1$. It can be verified that $\mathrm{spec}(\boldsymbol{A})=\{10,-4,1+1i,1-1i\}$. One can recognize that no eigenvalue of $\boldsymbol{A}_2=\mathrm{diag}(-4,1+i,1-i)$ is positive. Therefore, condition (c3) of Theorem 5.2 holds and it follows that $\boldsymbol{A}$ has a positive recursion. In fact, it can be verified that in this case it holds that $\boldsymbol{A}^6=166.7569\boldsymbol{I}_4+16.1434\boldsymbol{A}+39.7036\boldsymbol{A}^4+6.0262\boldsymbol{A}^5$, where $\boldsymbol{I}_4$ denotes the identity matrix of dimension 4×4. In addition, we can conclude that $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$ is polyhedral with $k_{\mathrm{vert}}=6$. This is illustrated by Figure 4, where it is observed that $\mathrm{Reachset}_k(\boldsymbol{A},\boldsymbol{b})$ stops growing for $k\geq 6$, i.e., $\mathrm{Reachset}_k(\boldsymbol{A},\boldsymbol{b})=\mathrm{Reachset}_6(\boldsymbol{A},\boldsymbol{b})$ for any $k\geq 6$. One can also notice from Figure 4(c) that $C_{\mathrm{lim}}\subset\mathrm{Reachset}_{k_{\mathrm{vert}}}(\boldsymbol{A},\boldsymbol{b})$ with C_{lim} introduced in Definition A.5. Note that in this particular example, since $h=1$, we have $C_{\mathrm{lim}}=\mathrm{cone}(\boldsymbol{A}_{f,0}\boldsymbol{b})=\{c\boldsymbol{v}_f|c\in\mathbb{R}_{\mathrm{s}+}\}$, where $\boldsymbol{v}_f$ is the Frobenius eigenvector of $\boldsymbol{A}^h$.

Remark 5.5 (concluding remark on Theorems 4.5 and 5.2). Theorems 4.5 and 5.2 emphasize the equivalence between the three statements; but this does not imply that all cases are directly verifiable. In fact, it is very difficult to verify statement (b) directly especially since $k^{\infty}_{\mathrm{vert}}$ and k_{vert} are not known a priori. In practice, statement (a) is practically what one is interested in, and (c) provides numerically verifiable conditions. Statement (b) serves the dual purpose of facilitating the proof and providing insight into otherwise-very-abstract statement (a) and statement (c) by relating them to the matrix having a (partial) positive recursion. Moreover, the characterization (b) will be useful for a different algebraic characterization which is to be developed.

Special case. So far it has been assumed that $\mathrm{rank}(\mathrm{conmat}_n(\boldsymbol{A},\boldsymbol{b}))=n$. Based on this assumption, the polyhedrality of the finite-time reachable set only depends on the spectrum of $\boldsymbol{A}$. In addition, $k_{\mathrm{vert}}\geq n$ for $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$. We now point out that in the absence of such an assumption, $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$ can depend on the structure of $\boldsymbol{b}$ and that the vertex number can be less than n. In particular, it will be shown that $k_{\mathrm{vert}}=h$ if $\boldsymbol{b}\in\mathbb{R}^n_+$ is of a particular structure.

THEOREM 5.6. *Let* $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ *be irreducible with cyclicity index* h *with* $0 \le h \le n-1$. *Then,* $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b}) = \mathrm{cone}\big(\mathrm{conmat}_h(\boldsymbol{A},\boldsymbol{b})\big)$ *if* $\boldsymbol{b} \in \mathrm{cone}\big([\boldsymbol{v}_{f,0} \ \ldots \ \boldsymbol{v}_{f,h-1}]\big)$, *where* $\boldsymbol{v}_{f,i}$, $i = 0,\ldots,h-1$ *are the* h *positive eigenvectors of* $\boldsymbol{A}^h$.

Proof. Assume $\boldsymbol{b} = \sum_{i=0}^{h-1} c_i \boldsymbol{v}_{f,i}$ for some $\boldsymbol{c} \in \mathbb{R}_+^h$. Then, since

$$\boldsymbol{A}^h\boldsymbol{b} = \sum_{i=0}^{h-1} c_i \rho^h(\boldsymbol{A})\boldsymbol{v}_{f,i} = \rho^h(\boldsymbol{A})\boldsymbol{b},$$

it is immediate to see that $\boldsymbol{A}\big(\mathrm{conmat}_h(\boldsymbol{A},\boldsymbol{b})\big) = \big(\mathrm{conmat}_h(\boldsymbol{A},\boldsymbol{b})\big)\boldsymbol{X}$ has a positive solution

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \rho^h(\boldsymbol{A}) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

which, in the view of Corollary 5.1, completes the proof. ∎

For $\boldsymbol{A}$ primitive (i.e., $h = 1$), this results in the obvious case of $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$ being a ray along the Frobenius eigenvector $\boldsymbol{v}_f$ of $\boldsymbol{A}$ when $\boldsymbol{b} = c\boldsymbol{v}_f$ for any $c \ge 0$.

6. Does the reachable set contain a prespecified set? A direct consequence of the polyhedrality of the infinite- or finite-time reachable subset discussed in sections 4 and 5 is that it enables us to determine whether a given subset of the positive vector space is reachable from the origin. Given a cone $C_{\mathrm{obj}} \subseteq \mathbb{R}_+^n$ of control objectives or a subset of $\mathbb{R}_+^n$, the problem considered here is to investigate whether C_{obj} is contained in $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b})$ or in $\mathrm{Reachset}_\infty(\boldsymbol{A},\boldsymbol{b})$. Of particular interest is when $C_{\mathrm{obj}} \subset \mathbb{R}_+^n$ is a polyhedral cone or a polytope. Note that if the control objective cone C_{obj} is not polyhedral then one can outer approximate it by a polyhedral cone $C_{\mathrm{out}} \subseteq \mathbb{R}_+^n$ such that $C_{\mathrm{obj}} \subset C_{\mathrm{out}}$.

Here, it is assumed that the reachability cone or its closure is polyhedral and that its corresponding vertex number or an upper bound of it is known. Note that the authors are not aware of any method to directly compute an upper bound for k_{vert} or for k_{vert}^∞. Nonetheless, such an upper bound could be imposed by the length of the control sequence that can be practically applied. Let $N \in \mathbb{Z}_+$ denote an upper bound to k_{vert}^∞ or, where applicable, an upper bound to k_{vert}. Hence $\mathrm{Reachset}_\infty(\boldsymbol{A},\boldsymbol{b}) = \mathrm{cone}\big([\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b} \ \boldsymbol{v}_{f,0} \ \ldots \ \boldsymbol{v}_{f,h-1}]\big)$ and/or $\mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b}) = \mathrm{cone}\big([\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b}]\big)$.

PROPOSITION 6.1. *Let* $C_{\mathrm{obj}} = \mathrm{cone}([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m])$ *or* $C_{\mathrm{obj}} = \mathrm{conv}([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m])$, *where* $\boldsymbol{p}_i \in \mathbb{R}_+^n$, $i = 1,\ldots,m$. *Then*

(a) C_{obj} *is reachable in finite time if and only if*

$$\forall\ \boldsymbol{p} \in \{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_m\}, \quad \boldsymbol{p} \in \mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b}).$$

(b) C_{obj} *is reachable in infinite time (to be called* almost reachable*) if and only if*

$$\forall \boldsymbol{p} \in \{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_m\},\ \boldsymbol{p} \in \mathrm{Reachset}_\infty(\boldsymbol{A},\boldsymbol{b}),\ \textit{and}$$
$$\exists\ \boldsymbol{p}' \in \{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_m\} \textit{ such that } \boldsymbol{p}' \notin \mathrm{Reachset_f}(\boldsymbol{A},\boldsymbol{b}).$$

Proof. The proof is obvious from Proposition 3.3 and considering the fact that a cone can be expressed as a positive combination of its generators. ∎

It is obvious from Proposition 6.1 that checking for reachability from the origin involves checking the following condition for each $i \in \{1, \ldots, m\}$:

$$\exists \boldsymbol{x}_i \in \{\boldsymbol{z} | \boldsymbol{M}\boldsymbol{z} = \boldsymbol{p}_i, \boldsymbol{z} \in \mathbb{R}_+^N\}, \tag{28}$$

where $\boldsymbol{M} \in \mathbb{R}_+^{n \times N}$. Depending on the problem being investigated, either $\boldsymbol{M} = [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b} \ \boldsymbol{v}_{f,0} \ \ldots \ \boldsymbol{v}_{f,h-1}]$ or $\boldsymbol{M} = [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b}]$.

In general, since $N \geq n$ (see Remarks 4.6 and 5.3), (28) defines an underdetermined system of equations. It is known that the positive solution of (28) is not unique in general [18, 51], and that uniqueness is guaranteed when the solution is sufficiently sparse [18]. The author of [16] characterizes necessary and sufficient conditions on the polytope $P = \mathrm{conv}(\boldsymbol{M})$ for uniqueness of the solution, and he proves that a unique solution exists if and only if P is k-neighborly.[2] In [17, 51], an equivalent condition is presented in terms of the null space of $\boldsymbol{M}$. In this regard, this problem relates to the *sparse measurement* problem, where the aim is to reconstruct a positive sparse vector from lower-dimensional linear measurements [40]. The results in this field do not directly apply here as the necessary sparsity condition is usually not met. In addition, we are not interested in finding the sparsest solution of (28), which is normally an NP-hard problem [18].

Consider for $n \in \mathbb{Z}_+$ the positive matrix $A \in \mathbb{R}_+^{n\times n}$. Let $N \in \mathbb{Z}_+$ with $N > n$ be an upper bound of k_{vert} or an upper bound of k_{vert}^{∞}. Denote by $C(N, n)$ the size of the set of all n-subsets of $\mathbb{Z}_N = \{1, \ldots, N\}$. Let the index set $\mathcal{I}_j$ be an n-subset (i.e., $|\mathcal{I}_j| = n$) of $\mathbb{Z}_N$ for $j = 1, 2, \ldots, C(N, n)$ such that $\cup_{j=1}^{C(N,n)} \mathcal{I}_j = \mathbb{Z}_N$ and $\mathcal{I}_j \neq \mathcal{I}_k$, $j, k = 1, 2, \ldots, C(N, n)$, $j \neq k$.

Let $\boldsymbol{I}_{\mathcal{I}_j}$ denote the matrix with n columns, where the columns are chosen from columns of $\boldsymbol{I}_N$ (i.e., the identity matrix of dimension N) according to the index set $\mathcal{I}_j$ and let $C_{\text{obj}} = \mathrm{cone}([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m])$.

Proposition 6.2. *Consider the above defined objects. Then, for any $i \in \{1, \ldots, m\}$, (28) has a solution $\boldsymbol{x}_i$ if and only if*

$$\boldsymbol{X}^i = \left\{ \boldsymbol{x}_j^i | \boldsymbol{x}_j^i = \boldsymbol{I}_{\mathcal{I}_j} (\boldsymbol{M}\boldsymbol{I}_{\mathcal{I}_j})^{-1} \boldsymbol{p}_i, \ \boldsymbol{x}_j^i \in \mathbb{R}_+^N, \ j = 1, \ldots, C(N, n) \right\} \tag{29}$$

is a nonempty set.

Proof. From our assumption we have $\boldsymbol{p}_i \in \mathrm{cone}(\boldsymbol{M})$. Since $N > n$, due to the Carathéodory theorem [1], $\boldsymbol{p}_i$ also lies in at least one simplicial cone generated by n columns of $\boldsymbol{M}$. Let $\mathcal{J}^i \subset \{1, \ldots, N\}$ with $|\mathcal{J}^i| = n$ be an index set composed of the indices of the columns generating this simplicial cone, and let $\boldsymbol{M}_{\mathcal{J}^i}$ denote the columns of $\boldsymbol{M}$ corresponding to $\mathcal{J}^i$. We can then write $\boldsymbol{p}_i \in \mathrm{cone}(\boldsymbol{M}_{\mathcal{J}^i})$, which can be expressed as $\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i}\boldsymbol{z}^i = \boldsymbol{p}_i$ having a solution $\boldsymbol{z}^i \in \mathbb{R}_+^n$. Since $\boldsymbol{M}$ has full row rank and $\boldsymbol{I}_{\mathcal{J}^i}$ is of full column rank, one obtains $\boldsymbol{z}^i = (\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i})^{-1}\boldsymbol{p}_i$. Finally, we obtain a solution $\boldsymbol{x}_j^i \in \mathbb{R}_+^N$, where $\boldsymbol{x}_j^i = \boldsymbol{I}_{\mathcal{J}^i}\boldsymbol{z}^i = \boldsymbol{I}_{\mathcal{J}^i}(\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i})^{-1}\boldsymbol{p}_i$.

The converse is proved in a straightforward manner by noticing that every $\boldsymbol{z} \in \boldsymbol{X}^i$ satisfies (28). ∎

[2] A k-neighborly polytope is a convex polytope in which every set of k or fewer vertices forms a face [54, 23].

Remark 6.3. Let $\boldsymbol{X}^i = \{\boldsymbol{x}_1^i, \ldots, \boldsymbol{x}_{q_i}^i\}$ for some $q_i \in \mathbb{Z}_+$. It is then clear from the proof of Proposition 6.2 that the set of solutions of (28) is the convex hull of $\boldsymbol{X}^i$, i.e., we have for (28) that $\boldsymbol{x}_i \in \mathrm{conv}(\boldsymbol{X}^i)$.

Note that even though Proposition 6.2 provides a method to determine whether $C_{\mathrm{obj}} \subseteq \mathrm{cone}(\boldsymbol{M})$ by checking the inclusion of C_{obj} in any simplicial subcone of $\mathrm{cone}(\boldsymbol{M})$, the computational complexity of this method can be prohibitive as the check must be conducted for all $C(N, n)$ simplicial subcones in the worst case. A more practical approach is then presented by the following proposition.

PROPOSITION 6.4. *Let*

$$\begin{aligned} \boldsymbol{M}_{\mathrm{f}} &= [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1} b], \\ \boldsymbol{M}_{\infty} &= [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1} b \ \boldsymbol{v}_{\mathrm{f},0} \ \ldots \ \boldsymbol{v}_{\mathrm{f},h-1}], \\ C_{\mathrm{obj}} &= \mathrm{cone}([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m]). \end{aligned}$$

Define the following optimization problem for each $i \in \{1, \ldots, m\}$:

$$\begin{aligned} &\min_{\boldsymbol{x}_i} \ \boldsymbol{x}_i^{\mathrm{T}} \mathbb{1} \\ &\quad \textit{subject to } \boldsymbol{M}\boldsymbol{x}_i = \boldsymbol{p}_i, \quad \textit{and } \boldsymbol{x}_i \geq 0. \end{aligned} \tag{30}$$

We then have the following:

(a) *The optimization problem* (30) *with $\boldsymbol{M} = \boldsymbol{M}_{\infty}$ has an optimal solution $\boldsymbol{x}_i^* \in \mathbb{R}_+^N$ if and only if* (28) *has a solution with $\boldsymbol{M} = \boldsymbol{M}_{\infty}$.*

(b) *The optimization problem* (30) *with $\boldsymbol{M} = \boldsymbol{M}_{\mathrm{f}}$ has an optimal solution $\boldsymbol{x}_i^* \in \mathbb{R}_+^N$ if and only if* (28) *has a solution with $\boldsymbol{M} = \boldsymbol{M}_{\mathrm{f}}$.*

Proof. If (28) has a solution, the set $\boldsymbol{X}^i$ in (29) is nonempty. As mentioned in Remark 6.3, the feasible set of (30) is $\mathrm{conv}(\boldsymbol{X}^i)$. Therefore, the convex optimization problem with linear penalty function converges to the minimum 1-norm solution in the feasible set. The converse is obvious. ∎

Example 6.5. We conclude this section with an example illustrating the application of Proposition 6.4. Consider the system matrices of Example 5.4. Let C_{obj} be the polytope given by

$$C_{\mathrm{obj}} = \Big\{ \boldsymbol{p} \in \mathbb{R}_+^4 \Big| \boldsymbol{p} = \sum_{i=1}^{4} \lambda_i \boldsymbol{p}_i, \lambda_i \geq 0, \sum_{i=1}^{4} \lambda_i = 1 \Big\},$$

where

$$\begin{aligned} \boldsymbol{p}_1 &= [1, \ 3, \ 1, \ 1]^{\mathrm{T}}, \ \boldsymbol{p}_2 = [1, \ 3, \ 4, \ 3]^{\mathrm{T}}, \\ \boldsymbol{p}_3 &= [1, \ 2, \ 2, \ 1]^{\mathrm{T}}, \ \boldsymbol{p}_4 = [1, \ 1, \ 2, \ 1]^{\mathrm{T}}. \end{aligned}$$

We will now check whether the system initially at rest can be steered to any point in C_{obj} in finite time. From Example 5.4, it is known that $k_{\mathrm{vert}} = 6$. Thus taking $\boldsymbol{M} = [\boldsymbol{b} \ \boldsymbol{A}\boldsymbol{b} \ \ldots \ \boldsymbol{A}^5\boldsymbol{b}]$, we solve the linear programming problem (30) using the dual-simplex algorithm implemented in the MATLAB Optimization Toolbox. The optimal solutions are obtained as

$$\begin{aligned} \boldsymbol{x}_1^* &= [0.1209, \ 0.3735, \ 0, \ 0.0078, \ 0, \ 0.0001]^{\mathrm{T}}, \\ \boldsymbol{x}_2^* &= [2.3460, \ 0.6165, \ 0.0876, \ 0, \ 0.0003, \ 0]^{\mathrm{T}}, \\ \boldsymbol{x}_3^* &= [0.2989, \ 0.6982, \ 0.0473, \ 0, \ 0.0003, \ 0]^{\mathrm{T}}, \\ \boldsymbol{x}_4^* &= [0.2517, \ 0.7798, \ 0.0071, \ 0, \ 0.0003, 0]^{\mathrm{T}}. \end{aligned}$$

Hence, the vertices of C_{obj} can be reached from the origin in a finite number of steps using positive inputs, which are determined by the solution vectors $\boldsymbol{x}_i^*$. Moreover, since $k_{\text{vert}} = 6$, every vertex of C_{obj} can be reached in at most 6 steps from the origin. Since C_{obj} is the convex hull of its vertices, we can conclude that any point $\boldsymbol{p} = \sum_{i=1}^{4} \lambda_i \boldsymbol{p}_i \in C_{\text{obj}}$ can be reached from the origin in at most 6 steps using the input sequence $\boldsymbol{u}^* = \sum_{i=1}^{4} \lambda_i \boldsymbol{x}_i^*$.

7. Conclusions and future work. The main contribution of the paper is the result that the reachable set from the origin of a linear positive system can be either a polyhedral cone or a nonpolyhedral cone depending on the system matrices. Among other applications, this has direct consequences for the realization problem, where the choice for the reachable subset from the origin is essential as observability of a linear positive system is then of interest only for states in the reachable set.

For a single-input case, necessary and sufficient conditions for polyhedrality of the reachable set from the origin and its closure are provided. These conditions are expressed in terms of characteristics of eigenvalues of the system. Finally, the paper presents a method to determine for a positive linear system whether a given target set in the positive orthant can be reached from the origin.

There are several technical issues to be studied. Is it possible to determine in a finite number of steps for a positive matrix whether there exists a positive recursion for it?

In this paper, we have focused on the single input case, where $\boldsymbol{b} \in \mathbb{R}_+^n$. The problem of characterizing the reachability set from the origin for the multi-input case is an interesting problem because the results developed here are not directly applicable. The main issue, as noted in [4], is that the direct sum of two nonpolyhedral cones may still result in a polyhedral cone. Therefore, one cannot apply the results of this paper to a set of systems $(\boldsymbol{A}, \boldsymbol{b}_i)$ separately, with $\boldsymbol{b}_i$ being a column of $\boldsymbol{B}$.

Finally, it is also of interest to investigate the geometry of the reachable set when the controllability matrix is not of full rank. As far as the authors of this paper know, this is still an open issue.

Appendix A. Positive matrices. The reader finds in this appendix a summary of the theory of positive matrices including concepts and decompositions as far as is necessary for the understanding of this paper. This theory is well known and therefore not stated in the body of the paper.

Decompositions of positive matrices. As is well known in the theory of positive matrices, such matrices can be either reducible or irreducible as defined next. See the books [6, 28] for the definitions.

Definition A.1. *Consider a positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ for $n \in \mathbb{Z}_+$. Call this matrix* reducible *if*

$$\begin{aligned}
&\exists\ \boldsymbol{P} \in \mathbb{R}_+^{n\times n}, \ \textit{a permutation matrix,} \\
&\exists\ n_1, n_2 \in \mathbb{Z}_+, \quad \exists\ A_{11} \in \mathbb{R}^{n_1 \times n_1}, \ A_{12} \in \mathbb{R}^{n_1 \times n_2}, \ A_{22} \in \mathbb{R}^{n_2 \times n_2}, \\
&\textit{such that } n = n_1 + n_2 \textit{ and}
\end{aligned}$$

$$A = P \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} P^T. \tag{31}$$

Call the matrix A irreducible *if* (1) $A \neq 0$ *and* (2) *A is not reducible.*

Call the matrix A fully reduced *if either $n = 1$ or there exists a transformation by a permutation matrix P so that PAP^T has a decomposition in upper-block-diagonal* *form with only irreducible submatrices on the block-diagonal. Thus the lower-block-diagonal matrices are all zero. The particular form of a fully reduced positive matrix is thus*

$$A = P \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1,n-1} & A_{1,n} \\ 0 & A_{22} & A_{23} & \dots & A_{2,n-1} & A_{2,n} \\ 0 & 0 & A_{33} & \dots & A_{3,n-1} & A_{3,n} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & \dots & A_{n-1,n-1} & A_{n-1,n} \\ 0 & 0 & & \dots & 0 & A_{n,n} \end{pmatrix} P^T, \tag{32}$$

where $P \in \mathbb{R}_+^{n\times n}$ is a permutation matrix and the matrices on the block-diagonal of (32) *are all irreducible positive matrices.*

Decompositions of positive matrices based on eigenvalues. Recall that for a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ the *spectrum* is defined as the set of its eigenvalues and the *spectral radius* is defined as $\rho(\boldsymbol{A}) = \max_{\lambda\in\mathrm{spec}(\boldsymbol{A})} |\lambda|$. It follows from [6, Thm. 1.3.2] that every positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ has at least one eigenvalue which equals its spectral radius.

Definition A.2 (see [6, Def. 2.2.26]). *Define for an integer $n \in \mathbb{Z}_+$ and an irreducible positive matrix $A \in \mathbb{R}_+^{n\times n}$, the* index of cyclicity *of A as the number $h \in \mathbb{Z}_+$ such that h equals the maximum number of distinct eigenvalues of A which are in modulus equal to the spectral radius $\rho(A)$. In mathematical notation,*

$$h = \max\{k \in \mathbb{Z}_n |\ \forall\ i \in \mathbb{Z}_k,\ |\lambda_i(A)| = \rho(A)\}. \tag{33}$$

It follows from the comment above the previous definition that $h \geq 1$. If $h \geq 2$ then one says that the matrix A is cyclic of index h.

Definition A.3. *Consider an integer $n \in \mathbb{Z}_+$ and an irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$. Partition the set of eigenvalues into the following two subsets: $\sigma^\rho(\boldsymbol{A})$, which is the spectrum of $\boldsymbol{A}$ on the circle centered at the origin with radius $\rho(\boldsymbol{A})$, and $\sigma^-(\boldsymbol{A})$, which is the spectrum of $\boldsymbol{A}$ strictly inside the disc centered at the origin with radius $\rho(\boldsymbol{A})$. Hence,*

$$\begin{aligned} \sigma^\rho(\boldsymbol{A}) &= \{\lambda \in \mathrm{spec}(\boldsymbol{A})|\ |\lambda(\boldsymbol{A})| = \rho(\boldsymbol{A})\}, \\ \sigma^-(\boldsymbol{A}) &= \{\lambda \in \mathrm{spec}(\boldsymbol{A})|\ |\lambda(\boldsymbol{A})| < \rho(\boldsymbol{A})\} \end{aligned} \tag{34}$$

with

$$\begin{aligned} \mathrm{spec}(\boldsymbol{A}) = \sigma^\rho(\boldsymbol{A}) \cup \sigma^-(\boldsymbol{A}),\ \sigma^\rho(\boldsymbol{A}) \cap \sigma^-(\boldsymbol{A}) = \emptyset, \\ n_1 = |\sigma^\rho(\boldsymbol{A})| = h,\ n_2 = |\sigma^-(\boldsymbol{A})| = n - n_1 = n - h. \end{aligned}$$

In addition, there exists a nonsingular matrix $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ such that the matrix $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is block diagonal with

$$\begin{aligned} &\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \text{Block-diag}(\boldsymbol{A}_1, \boldsymbol{A}_2), \\ &\quad \boldsymbol{A}_1 \in \mathbb{R}^{n_1\times n_1},\ \boldsymbol{A}_2 \in \mathbb{R}^{n_2\times n_2},\ \mathrm{spec}(\boldsymbol{A}_1) = \sigma^\rho(\boldsymbol{A}),\ \mathrm{spec}(\boldsymbol{A}_2) = \sigma^-(\boldsymbol{A}). \end{aligned} \tag{35}$$

Finally, define the sets $\sigma^\rho(\boldsymbol{A}_2)$ and $\sigma^-(\boldsymbol{A}_2)$ in a similar manner with $\boldsymbol{A}$ being replaced by $\boldsymbol{A}_2$ in (34) *and define the set $\sigma^0(\boldsymbol{A}) \subseteq \sigma^\rho(\boldsymbol{A}_2)$ as the set of all eigenvalues of $\boldsymbol{A}_2$ whose polar angle is a rational multiple of 2π.*

The notation $|\sigma^\rho(A)|$ denotes the number of elements of the indicated set. That the decomposition (35) is indeed a partition follows from the Perron–Frobenius theorem [6, Thms. 2.1.4, 2.2.20] and from the concept of spectral radius as the maximal value of the absolute values of all eigenvalues. In general, the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ depend on $\boldsymbol{S}$. However, the relations (35) hold for any such $\boldsymbol{S}$. When the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are used in the body of the paper, then these are characterized by their spectra. Also note that an contrary to $\sigma^\rho(\boldsymbol{A})$, $\sigma^\rho(\boldsymbol{A}_2)$ can be an empty set.

Next, we present the following lemma about the existence of a subset of eigenvalues that are among the $(M\ h)$th root of unity for some $M \in \mathbb{Z}_+$. This lemma is used in the following for deriving the conditions on $\operatorname{spec}(\boldsymbol{A})$ for $\boldsymbol{A}$ to have a positive recursion.

Lemma A.4. *Consider the objects of Definition* A.3.
Then, there exists a minimal integer $M \in \mathbb{Z}_+$ such that

$$\sigma_0(\boldsymbol{A}) \subseteq \Big\{\lambda \in \operatorname{spec}(\boldsymbol{A}_2)\Big|\ \lambda = \rho(\boldsymbol{A}_2)\exp(\frac{2\pi k}{Mh}i), k = 0,\dots,Mh-1\Big\} \tag{36}$$

or, equivalently, there exists a minimal integer $M \in \mathbb{Z}_+$ such that the eigenvalues of $\boldsymbol{A}_2/\rho(\boldsymbol{A}_2)$ with unit modulus whose arguments are a rational multiple of 2π are among the $(M\ h)$th roots of unity.

Proof of Lemma A.4. Let δ^0 be a set of $n_{\delta^0} \in \mathbb{Z}_+$ members of σ^0 with the property that the difference between the polar angle of any two members of δ^0 is not an integer multiple of $2\pi/h$ or, formally, we define $\delta^0 = \{\lambda_1,\dots,\lambda_{n_{\delta^0}} \in \sigma^0 | \arg(\lambda_i) - \arg(\lambda_j) \neq 2z\pi/h, i \neq j, z \in \mathbb{Z}\}$. For $\lambda_j \in \delta^0$, $j = 1,\dots,n_{\delta^0}$, let $\arg(\lambda_j) = \frac{2\pi p_j}{q_j}$. Define the sets $\sigma_j^0 \subset \sigma^0$ for $j = 1,\dots,n_{\delta^0}$ as

$$\sigma_j^0 = \Big\{\lambda \in \operatorname{spec}(\boldsymbol{A}_2)\Big|\lambda = \rho(\boldsymbol{A}_2)\exp\big((k/h + p_j/q_j)2\pi i\big),\ k = 0,\dots,h-1\Big\}$$

or, equivalently, using the notation $s_{j,k} \equiv kq_j + hp_j (\mathrm{mod}\ hq_j)$,

$$\sigma_j^0 = \Big\{\lambda \in \operatorname{spec}(\boldsymbol{A}_2)\Big|\lambda = \rho(\boldsymbol{A}_2)\exp\big(\frac{s_{j,k}}{hq_j}2\pi i\big),\ k = 0,\dots,h-1\Big\}.$$

It is clear that $\sigma_1^0,\dots,\sigma_{n_{\delta^0}}^0$ are mutually disjoint. In addition, since the eigenvalues of $\boldsymbol{A}$ are invariant under polar rotation of $2k\pi/h$ for any $k \in \mathbb{Z}$, we have $\sigma^0 = \cup_{j=1}^{n_{\delta^0}}\sigma_j^0$. Noting that $0 \le s_{j,k} \le hq_j - 1$ for $k = 0,\dots,h-1$ and for $j = 1,\dots,n_{\delta^0}$, one observes that σ_0 has the form proposed in (36) by choosing $M = \operatorname{lcm}(q_1,\dots,q_{n_{\delta^0}})$. ◻

It follows from [6, Thm. 2.2.20] that if the matrix $A \in \mathbb{R}_+^{n\times n}$ is irreducible and if A is of index of cyclicity $h \ge 2$ then there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ and matrices $\{A_{i,i+1} \in \mathbb{R}_+^{n_i\times n_{i+1}},\ i = 0,1,\dots,h-1\ (\mathrm{mod}\ h)\}$ such that,

$$\sum_{i=0}^{h-1} n_i = n,$$

$$\boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} 0 & \boldsymbol{A}_{1,2} & 0 & \dots & 0 & 0 \\ 0 & 0 & \boldsymbol{A}_{2,3} & \dots & 0 & 0 \\ \vdots & & 0 & \ddots & \boldsymbol{A}_{h-2,h-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boldsymbol{A}_{h-1,h} \\ \boldsymbol{A}_{h,1} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}\boldsymbol{P}^T \tag{37}$$

with square diagonal blocks.

One then says that the positive matrix $\boldsymbol{A}$ is *cogredient* to the block matrix of (37) [6, Def. 2.1.2].

The irreducible positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ is called *primitive* if its trace is strictly positive; see [6, Def. 2.1.8, Cor. 2.2.28].

It follows from the proof of [6, Thm. 2.2.30] that if the matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ is irreducible and if A is of index of cyclicity $h \geq 2$ then there exists a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ such that

$$\boldsymbol{A}^h = \boldsymbol{P} \begin{pmatrix} \boldsymbol{C}_{1,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \boldsymbol{C}_{2,2} & 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{C}_{h-1,h-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boldsymbol{C}_{h,h} \end{pmatrix} \boldsymbol{P}^T,$$

$$\forall\ i \in \mathbb{Z}_h,\ \boldsymbol{C}_{i,i} \in \mathbb{R}_+^{n_i \times n_i} \text{ are primitive matrices with } \rho(\boldsymbol{C}_{i,i}) = \rho^h(\boldsymbol{A}),$$

$$\sum_{i=1}^{h} n_i = n.$$

Sources for the above theory are not only [6] but also the book [9, Chap. 3].

Limits of powers of positive matrices. It follows from Theorem [6, Thm. 2.4.1] that for a primitive irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$, the following limit exists:

$$\lim_{k \to \infty} \left(\frac{\boldsymbol{A}}{\rho(\boldsymbol{A})} \right)^k \in \mathbb{R}_+^{n \times n}.$$

Next the above results can be combined. Consider an irreducible matrix $A \in \mathbb{R}_+^{n \times n}$. Assume that the index of cyclity of A is such that $h \geq 2$. It then follows from the above that A^h is cogredient to a block diagonal matrix with primitive irreducible matrices on the diagonal. From the above existence of the limit then it follows that

$$\boldsymbol{A}^h = \boldsymbol{P} \begin{pmatrix} \boldsymbol{C}_{1,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \boldsymbol{C}_{2,2} & 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{C}_{h-1,h-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boldsymbol{C}_{h,h} \end{pmatrix} \boldsymbol{P}^T,$$

$$\begin{aligned} \lim_{k \to \infty} \left(\frac{\boldsymbol{A}^h}{\rho^h(\boldsymbol{A})} \right)^k &= \boldsymbol{P} \text{ Block-diag}\Big(\lim_{k \to \infty} (\boldsymbol{C}_{1,1}/\rho^k(\boldsymbol{C}_{1,1}), \dots, \lim_{k \to \infty} (\boldsymbol{C}_{h,h}/\rho^k(\boldsymbol{C}_{h,h}) \Big)\ \boldsymbol{P}^T \\ &= \boldsymbol{P} \text{ Block-diag}(\boldsymbol{C}_{\infty,1,1}, \dots, \boldsymbol{C}_{\infty,h,h})\ \boldsymbol{P}^T \in \mathbb{R}_+^{n \times n}. \end{aligned}$$

Next, we introduce the following lemma that characterizes the limit behavior of $\text{conmat}_k(\boldsymbol{A}, \boldsymbol{b})$ as $k \to \infty$, and that is used for characterizing the infinite-time reachable subset $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$.

DEFINITION A.5. *Let the positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ be irreducible with index of cyclicity h with $1 \leq h \leq n$ and let $\boldsymbol{b} \in \mathbb{R}_+^m$. Define the matrices and the* limit cone *according to*

$$\forall\ i \in \{0, \dots, h-1\}, \quad \boldsymbol{A}_{f,i} = \lim_{k \to \infty} \left(\left(\frac{\boldsymbol{A}}{\rho(\boldsymbol{A})} \right)^h \right)^k \boldsymbol{A}^i,$$

$$C_{\lim} = \text{cone}([\boldsymbol{A}_{f,0}\boldsymbol{b}\ \dots\ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]).$$

Define for $i = 0, \dots, h-1$ *the positive eigenvectors* $\boldsymbol{v}_{f,i} \in \mathbb{R}_+^n$ *of the* h *distinct eigenvalues of the matrix* $\boldsymbol{A}^h$ *associated with the Perron root of* $\rho^h(\boldsymbol{A})$*; thus,*

$$\boldsymbol{A}^h \ v_{f,i} = \rho^h(\boldsymbol{A}) \ v_{f,i}.$$

LEMMA A.6. *Consider the objects of Definition* A.5. *Then the limit cone satisfies*

$$C_{\lim} \subseteq \operatorname{cone}([\boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}]).$$

Proof of Lemma A.6. Since $\boldsymbol{A}$ is irreducible, there exists a monomial matrix $\boldsymbol{S} \in \mathbb{R}_+^{n\times n}$ [6] such that

$$\hat{\boldsymbol{A}} = \boldsymbol{S}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{S} = \begin{bmatrix} 0_{n_1} & A_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0_{n_{h-1}} & A_{h-1} \\ A_h & 0 & \dots & \dots & 0_{n_h} \end{bmatrix}, \ \hat{\boldsymbol{b}} = \boldsymbol{S}^{\mathrm{T}} b,$$

where $0_{n_i} \in \mathbb{R}^{n_i \times n_i}$, $i \in \mathbb{N}$, are square blocks with $\sum_{i=1}^h n_i = n$, and where A_i has no zero rows or columns with $L_1 = \prod_{i=1}^h A_i$ being an irreducible matrix. Then we have $\hat{\boldsymbol{A}}^h = \operatorname{diag}(L_1, \dots, L_h)$, where $L_k = \prod_{i=k}^h A_i \prod_{j=1}^{\operatorname{mod}(h+k-1,h)} A_j$ is a primitive matrix of dimension $n_k \times n_k$ with Perron root $\rho^h(\boldsymbol{A})$. Define the matrix $\hat{\boldsymbol{A}}_{f,i} = \lim_{p\to\infty} \frac{\hat{\boldsymbol{A}}^{ph}}{\rho^{ph}} \hat{\boldsymbol{A}}^i$ for $i = 0, \dots, h-1$. Since L_i, $i = 1, \dots, h$ is primitive, it follows from [6] that

$$\hat{\boldsymbol{A}}_{f,0} = \begin{bmatrix} \boldsymbol{x}_1^1 & \dots & \boldsymbol{x}_1^{n_1} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \boldsymbol{x}_2^1 & \dots & \boldsymbol{x}_2^{n_2} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x}_h^1 & \dots & \boldsymbol{x}_h^{n_h} \end{bmatrix},$$

where $\boldsymbol{x}_i^k = c_i^k \boldsymbol{x}_i$ with c_i^k, $k = 1, \dots, n_i$, being some positive scalars and with $\boldsymbol{x}_i \in \mathbb{R}_{\mathrm{s}+}^{n_i \times n_i}$ being the Frobenius eigenvector of L_i. Note that due to the block structure of $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{A}}_{f,i}$ retains the same structure as $\hat{\boldsymbol{A}}_{f,0}$ up to a scaled permutation of its columns for $i = 1, \dots, h-1$. Hence, we have $\hat{\boldsymbol{A}}_{f,i} \hat{\boldsymbol{b}} \in \operatorname{cone}(\boldsymbol{C})$, where

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{x}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{x}_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \boldsymbol{x}_h \end{bmatrix}.$$

In the original coordinates, we have $\boldsymbol{A}_{f,i} \boldsymbol{b} \in \operatorname{cone}(\boldsymbol{SC})$. Clearly, since the columns of $\boldsymbol{C}$ are the positive eigenvectors of $\hat{\boldsymbol{A}}^h$ and since $\boldsymbol{S}$ is monomial, we have $\boldsymbol{SC} = [\boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}]$, where $\boldsymbol{v}_{f,i} \in \mathbb{R}_+^{n\times n}$ is the $(i+1)$th positive eigenvector of $\boldsymbol{A}^h$ for $i = 0, \dots, h-1$. This proves $\operatorname{cone}([\boldsymbol{A}_{f,0} b \ \dots \ \boldsymbol{A}_{f,h-1} b]) \subseteq \operatorname{cone}([\boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}])$. ∎

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