

ON THE MICROSCOPIC MODELING OF VEHICULAR TRAFFIC ON GENERAL NETWORKS

RINALDO M. COLOMBO*, HELGE HOLDEN†, AND FRANCESCA MARCELLINI‡

Abstract. We introduce a formalism to deal with the microscopic modeling of vehicular traffic on a road network. Traffic on each road is uni-directional, and the dynamics of each vehicle is described by a Follow-the-Leader model. From a mathematical point of view, this amounts to define a system of ordinary differential equations on an arbitrary network. A general existence and uniqueness result is provided, while priorities at junctions are shown to hinder the stability of solutions. We investigate the occurrence of the Braess paradox in a time-dependent setting within this model. The emergence of Nash equilibria in a non-stationary situation results in the appearance of Braess type paradoxes, and this is supported by numerical simulations.

Key words. Vehicular traffic, Networks, Follow-the-Leader model, Braess paradox, Nash equilibria.

AMS subject classifications. 90B20, 91B74, 91D10.

1. Introduction. The literature on the modeling of vehicular traffic has been growing very quickly in recent years. A variety of approaches coexists, typically they can be characterized as either macroscopic or microscopic.

The former ones are usually based on partial differential equations, their prototype being the Lighthill–Whitham [27] and Richards [32] model. Deep criticisms [16] led to the formulation of entirely new continuum models, such as [2], or multiphase models [4, 8, 13, 20, 29] and models on networks, starting from [21] up to the recent monograph [18].

Microscopic models also have a long tradition, see [19]. They are usually denoted as Follow-the-Leader models, the dynamics being governed by the interaction between a vehicle and the vehicle immediately in front of it. More precisely, we have

$$\dot{x}_\alpha = v \left(\frac{\ell}{x_{\alpha-1} - x_\alpha} \right),$$

where $x_\alpha < x_{\alpha-1}$ denotes the position of two consecutive vehicles, each of length ℓ , driving with a velocity function v .

Various connections between the two scalings are found in the literature, referring to limiting procedures yielding the macroscopic models as limit of the microscopic ones, as in [1, 17, 22, 23], or mixing the two scales [11, 12, 26]. Note however that most macroscopic models prescribe traffic rules at junctions that also require some sort of flow maximization, see [18] for more details. In the construction below, no such maximization is used, and this will make a continuum limit more complicated. However, the chosen priority rules are sufficient to single out a unique evolution. Other approaches have been studied in the literature.

Apart from models based on differential equations, many other mathematical tools are used in the literature to describe traffic on networks and, where possible, to account for Braess paradox. For instance, a stochastic approach can be found in [3], an evolutionary variational inequality model is studied in [31], while queue theory is

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applied in [28]. The assessment of the network performance due to selfish routing can be found in [33]. In contrast to these approaches, here the dynamics is fully described by ODEs, with simple priority rules at junctions.

Modern vehicular traffic offers a plethora of modeling challenges – complicated network geometries, roundabouts, traffic lights, traffic obstructions, a combination of various agents (pedestrians, bicyclists, a wide range of different vehicles), noise, pollution, etc. We here focus on a general network with only one type of vehicles, but we provide a consistent and rigorous model for behavior at junctions based on a Follow-the-Leader model. See also [14, 15] for related work.

As far as we are aware of, the microscopic modeling of traffic on a network has not been formalized systematically before.

Our approach yields a model that comprises a system of (discontinuous) ordinary differential equations (ODEs) on a network with a concrete behavior at junctions. Moreover, the present model comprises the presence of different priorities between roads. Below, we present a framework where rigorous statements about the microscopic modeling of vehicle dynamics, complying with priority rules, can be formalized, proved, and numerically computed.

Within this structure, we formalize an ODE-based model and provide an existence and uniqueness result for the corresponding evolution, see Theorem 2.1. By means of an example, we show that the usual well-posedness estimates may not hold. Indeed, and consistently with everyday experience, small changes in the departure time of a single vehicle may lead to large changes in the arrival time of that vehicle, due for instance to arriving slightly earlier or later at junctions where priority has to be yielded, see Remark 2.2.

A main aim for us has been to investigate the ubiquitous Braess paradox in a time-dependent setting through deterministic differential equations. As far as we know, in this context, the Braess phenomenon has so far only been analyzed mathematically in the stationary case. Recall first the simplest example of Braess paradox. We have a

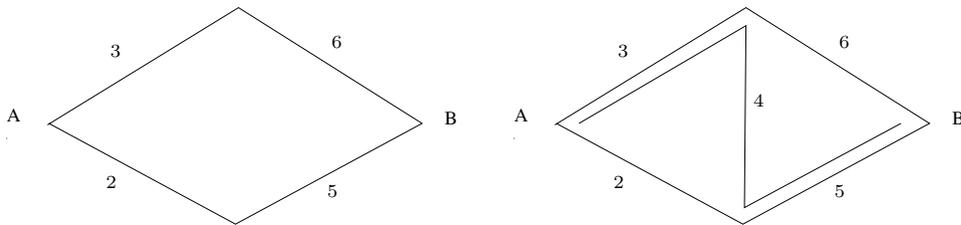


FIG. 1. *Left: network consisting of two routes connecting A to B. The first route consists of the roads 3 and 6; the second route consists of the roads 2 and 5. Right: network consisting of three routes connecting A to B. The first route consists of the roads 3 and 6; the second route consists of the roads 2 and 5. The third route consists of the roads 3, 4, and 5.*

network consisting of two routes connecting A to B , where the first route consists of the roads 3 and 6, while the second route consists of the roads 2 and 5, see Figure 1 (left). Traffic is unidirectional in the direction from A to B . The roads 2 and 6 are equal, with unlimited capacity, and the travel time is 45 minutes independently of the number of vehicles. The roads 3 and 5 are also equal and the travel time is $N/100$, where N is the number of vehicle traveling on the road. We suppose that 4000 vehicles move from A to B . Each driver chooses the fastest route and the resulting Nash equilibrium amounts to 2000 drivers traveling along each road. Correspondingly,

we find a travel time of 65 minutes for each driver.

Then, we add a new road, say number 4, as in Figure 1 (right), characterized by a negligible travel time. Drivers start using the new road choosing the route consisting of roads 3, 4, and 5, reducing their travel time. However, since the new route [3, 4, 5] is more convenient than both [3, 6] and [2, 5], more and more drivers choose this new route. As a result, the travel time increases to 80 minutes for everyone. This is the paradox: contrary to common sense, adding a new road to a network may make travel times worse for everyone.

This paradox was introduced by Braess in 1968 [5] with a different example, see also [30], and it has been observed in real situations. In 1968, for instance, a highway segment was closed in Stuttgart and traffic improved, see [24]. In 1990, in New York the 42nd street was closed for one day and, again unexpectedly, traffic improved, see [25].

This paradox appears in other situations as well, not only modeling vehicular traffic. In crowd dynamics, the well-known phenomenon of reducing the evacuation time from a closed space by suitably positioning obstacles near exits that direct the crowd movement (and closing a number of paths) is described through a partial differential equation model in [9].

Our aim is to capture the Braess paradox in a non-stationary setting in the present Follow-the-Leader model. For simplicity we study the case of the network depicted in Figure 1. The present framework allows us to show the dynamic emergence of a Braess-like situation in a fully non-stationary setting. In contrast to the examples typically found in the literature [5, 10, 30], in the examples below we start from an *empty* network. As vehicles enter it, the measured travel times show the rise of Braess paradox, as shown by numerical computations.

A key role is here played by our postulating the behavior of drivers as described by a Nash equilibrium. Indeed, we view drivers as players competing in a non-cooperative way to reduce their travel times, see also [6, 7, 10]. In particular situations, the solution of the Follow-the-Leader model at Nash equilibrium leads to the emergence of non-stationary Braess-like situations as supported numerically.

The next section is devoted to the definition of the microscopic model on a network. Section 3 is devoted to the emergence of Braess paradox, obtained as Nash equilibrium within the framework of the model here introduced. The last section collects the analytic proofs.

2. Formal Framework. The standard first-order *Follow-the-Leader* model is based on the following Cauchy problem for a system of ordinary differential equations:

$$(2.1) \quad \begin{cases} \dot{x}_1 = V_{\max} \\ \dot{x}_\alpha = v\left(\frac{\ell}{x_{\alpha-1} - x_\alpha}\right) & \alpha \in \{2, \dots, n\}, \\ x_\alpha(0) = x_\alpha^o & \alpha \in \{1, \dots, n\}. \end{cases}$$

Here, n drivers labeled by their positions x_1, \dots, x_n drive at speed $v(\ell/(x_{\alpha-1} - x_\alpha))$, where ℓ is the length of each vehicle and the speed v satisfies the condition:

(SpeedLaw) v is a Lipschitz continuous function and attains values in $[0, V_{\max}]$, i.e., $v \in \mathbf{W}^{1,\infty}(\mathbb{R}^+; [0, V_{\max}])$, and it is a (weakly) decreasing function such that $v(\rho) = 0$ for all $\rho \geq 1$.

The constant V_{\max} is an upper bound for the speed of all vehicles. The drivers' initial positions are x_0^o, \dots, x_n^o . It is well-known that the assumption $x_\alpha^o - x_{\alpha-1}^o \geq \ell$ for $\alpha \in \{1, \dots, n\}$ ensures that the solutions to (2.1) keep satisfying the same bound, i.e., $x_\alpha(t) - x_{\alpha-1}(t) \geq \ell$ for all α and for all $t \geq 0$, meaning that no collision ever occurs.

We now introduce a formalism to deal with the extension of (2.1) to a general network.

Network Structure. The network is a collection of m real intervals: each of them representing a road. Roads are of three types:

- **Entry Roads:** they are copies of the (open) half-line $]-\infty, 0[$;
- **Middle Roads:** they are bounded intervals of the type $[0, L_j[$, where $L_j > 0$ is the road length;
- **Exit Roads:** they are copies of the half-line $[0, +\infty[$.

Entry Roads and Exit Roads have infinite length. We assume throughout that the vehicle length ℓ is negligible with respect to the (finite) length of each Middle Road: $\ell \ll L_j$ for all j indexing a Middle Road.

To simplify various expressions, it is convenient to assign $L_j = 0$ for all j indexing an Entry Road. It can also be of use to set $L_j = +\infty$ for each Exit Road. This convention allows us to introduce the following terminology, of use below: for each Middle Road or Entry Road j , the *end of the road* is the real interval $]L_j - \ell, L_j[$. Here, to define the end of the road we use the vehicle length ℓ but choosing a different length ℓ' , with $\ell' > \ell$, is also possible.

Road indices are assigned so that whenever two or more roads enter the same junction, drivers on roads with lower indices have priority. Throughout, we assume

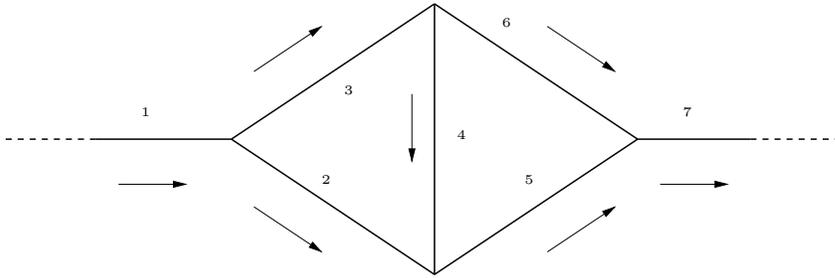


FIG. 2. The network notation in the case of Braess network, see [5]. Note that roads are numbered so that at each junction, roads coming from the right have the priority.

that junctions either have a single incoming road, or have a single outgoing road. The case of general junctions with several incoming and outgoing roads can be treated by the same methods described below, at the cost of a more intricate formalism.

Drivers' Route Choices. The n drivers are indexed by α , running between 1 and n . Each driver's route is identified by the sequence of the indices of the roads that constitute the route. We denote by \mathcal{R}_α the route followed by driver α . For instance, with reference to the Braess network in Figure 2, the route followed by the driver $\alpha = 1$ choosing the "lower" route is identified by $\mathcal{R}_1 = [1, 2, 5, 7]$. If the driver $\alpha = 2$ follows the route passing through the road 4, then $\mathcal{R}_2 = [1, 3, 4, 5, 7]$.

Throughout, $r_\alpha(t)$ stands for the index of the road along which the α th driver is traveling at time t . We also write $j' = \mathcal{N}_\alpha(j)$ meaning that the α th driver at the end of the j th road enters the j' th road. For instance, with reference to Figure 2, if the route of the driver $\alpha = 1$ is $\mathcal{R}_1 = [1, 3, 6, 7]$, then we have $\mathcal{N}_1(1) = 3$, $\mathcal{N}_1(3) = 6$, and $\mathcal{N}_1(6) = 7$.

Along each road, we identify the α th driver's position through the time dependent variable x_α ranging in $]-\infty, 0[$ along Entry Roads, in $[0, L_j[$ along Middle Roads and in $[0, +\infty[$ along Exit Roads.

A key assumption in the construction below amounts to require that no *loop* is possible for any driver:

(NoLoop): No route can contain the same road twice.

Note that the network itself may well contain loops, but condition **(NoLoop)** requires that none of them can be part of a route.

Of use below is also the following, quite natural, requirement:

(NoDeadEnd) The last road in each route is an Exit Road.

Drivers' Speed. We now specify the speed chosen by the α th driver, depending on the position and on that of the vehicles preceding the driver. We consider several special cases.

Far from Junctions. At time t the driver is positioned at $x_\alpha(t)$ driving along road $j = r_\alpha(t)$. As long as the α th driver is not at the end of the road indexed by $r_\alpha(t)$, i.e., $x_\alpha(t) < L_{r_\alpha(t)} - \ell$, the speed only depends on the free space ahead, similarly to what happens in (2.1):

(2.2)

$$\dot{x}_\alpha = \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} \alpha \text{ is not at the end of the road;} \\ \text{no one is on the same road in front of } \alpha. \end{cases} \\ v_{r_\alpha(t)}\left(\frac{\ell}{p-x_\alpha}\right) & \text{if } \begin{cases} \text{someone is on the same road in front of } \alpha; \\ p \text{ is the position of the nearest vehicle in front of} \\ \alpha \text{ on the same road.} \end{cases} \end{cases}$$

$$= \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} x_\alpha(t) < L_{r_\alpha(t)} - \ell; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset. \end{cases} \\ v_{r_\alpha(t)}\left(\frac{\ell}{p-x_\alpha}\right) & \text{if } \begin{cases} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} \neq \emptyset; \\ p = \min \{x_{\alpha'} \in [0, L_{r_\alpha(t)}] : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\}. \end{cases} \end{cases}$$

Indeed, the set $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\}$ identifies the (indices α' of) drivers preceding α along the road $r_\alpha(t)$ where α is driving at time t . If no such driver exists, α drives at the maximal speed $V_{r_\alpha(t)}$ possible along the road $r_\alpha(t)$. On the other hand, if $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} \neq \emptyset$, then the speed $\dot{x}_\alpha(t)$ of the α th driver is adjusted to the distance between α and the driver at position p , who is the one immediately in front of α , as usual in a *Follow-the-Leader* model.

Note that if $r_\alpha(t)$ is an Exit Road, then we understand that the condition $x_\alpha < L_{r_\alpha(t)} - \ell$ is true for all x_α .

A Fork in the Road. Consider a junction with one road (either an Entry or a Middle Road) entering it and any number of roads exiting it. At time t driver α is close to the end of the Entry Road or the Middle Road $r_\alpha(t)$, in the sense that $x_\alpha(t) \in [L_{r_\alpha(t)} - \ell, L_{r_\alpha(t)}]$. Driver α chooses the speed $\dot{x}_\alpha(t)$ taking into consideration only those drivers preceding him/her along the road $r_\alpha(t)$ or present in the next road $\mathcal{N}_\alpha(r_\alpha(t))$ he/she is going to take, see Figure 3.

We then set

$$(2.3) \quad \dot{x}_\alpha = \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} \alpha \text{ is at the end of the road;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{no one is on the road where } \alpha \text{ is going.} \end{cases} \\ v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) & \text{if } \begin{cases} \alpha \text{ is at the end of the road;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{someone is on the road where } \alpha \text{ is going;} \\ p \text{ is the position of the nearest vehicle in front of} \\ \alpha \text{ on the same road.} \end{cases} \end{cases}$$

$$= \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} = \emptyset. \end{cases} \\ v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) & \text{if } \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} \neq \emptyset; \\ p = \min \{x_{\alpha'} \in [0, L_{\mathcal{N}_\alpha(r_\alpha(t))}] : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}. \end{cases} \end{cases}$$

Indeed, when $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}$ is empty, no one is preceding the α th driver along his/her route and the α th driver proceeds at full speed. On the other hand, if $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} \neq \emptyset$, then the driver immediately preceding α is at position p , as defined in (2.3). The resulting speed $\dot{x}_\alpha(t)$ of the α th

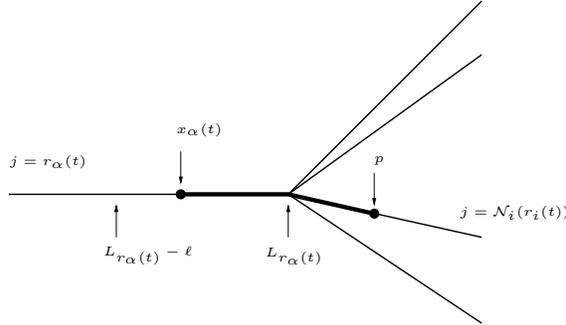


FIG. 3. Notation used in (2.3). The α th driver, located at $x_\alpha(t)$ is approaching the end of the $r_\alpha(t)$ road, in the sense that $x_\alpha(t) \in [L_{r_\alpha(t)} - \ell, L_{r_\alpha(t)}]$, and its predecessor is at p along the road $\mathcal{N}_\alpha(r_\alpha(t))$.

driver is then chosen according to the usual *Follow-the-Leader* rule, with $p + L_{r_\alpha(t)} - x_\alpha(t)$ being the physical distance measured along the road between the α th driver and his/her predecessor, see Figure 3.

Roads Merging. Consider now a junction with several roads entering a single road. We assume that the roads' indexing respects the roads' priorities, in the sense that if the roads j and j' enter the same junction and $j < j'$, then the drivers on the road j have priority over those on road j' . Call J the set of indices of the roads entering the junction under consideration.

First, we deal with the case of a driver coming from the road that has the priority over all the other incoming roads. In this case, we have $r_\alpha(t) = \min J$ by assumption.

We then set

$$(2.4) \quad \dot{x}_\alpha = \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} \alpha \text{ is at the end of the road;} \\ \alpha\text{'s road has the priority;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{no one is on the road where } \alpha \text{ is going.} \end{cases} \\ v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) & \text{if } \begin{cases} \alpha \text{ is at the end of the road;} \\ \alpha\text{'s road has the priority;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{someone is on the road where } \alpha \text{ is going;} \\ p \text{ is the position of the nearest vehicle in front of} \\ \alpha \text{ on the same road.} \end{cases} \end{cases}$$

$$= \begin{cases} V_{r_\alpha(t)} & \text{if } \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ r_\alpha(t) = \min J; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} = \emptyset. \end{cases} \\ v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) & \text{if } \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ r_\alpha(t) = \min J; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} \neq \emptyset; \\ p = \min \{x_{\alpha'} \in [0, L_{r_\alpha(t)}] : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}. \end{cases} \end{cases}$$

Similarly to the previous case of the fork in the road, i.e., equation (2.3), $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}$ is empty whenever the α th driver has free road ahead. When $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}$ is nonempty, p as defined in (2.4) is the position of the first driver in front of α , and $p + L_{r_\alpha(t)} - x_\alpha(t)$ is the length of the free road in front of the driver α , see Figure 4 (right).

Let now the α th driver approach the junction along the road $r_\alpha(t)$ which yields to other roads, so that $r_\alpha(t) > \min J$. Assume that at the end of road j entering the junction (i.e., $j \in J$) there is no one that has the priority over the road $r_\alpha(t)$ (i.e., $j < r_\alpha(t)$), i.e., $\bigcup_{j \in J: j < r_\alpha(t)} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = j \text{ and } x_j(t) > L_j - \ell\} = \emptyset$, and there is no one in the road where α is entering (i.e., $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} = \emptyset$). Then, α drives at full speed $V_{r_\alpha(t)}$:

$$(2.5) \quad \dot{x}_\alpha = V_{r_\alpha(t)} \quad \text{if } \begin{cases} \alpha \text{ is at the end of the road;} \\ \alpha\text{'s road does not have the priority;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{no one is on the road where } \alpha \text{ is going;} \\ \text{no one is at the end of roads having priority over } \alpha. \end{cases}$$

$$= V_{r_\alpha(t)} \quad \text{if } \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ r_\alpha(t) > \min J; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} = \emptyset; \\ \bigcup_{j \in J: j < r_\alpha(t)} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = j \text{ and } x_j(t) > L_j - \ell\} = \emptyset. \end{cases}$$

As soon as another driver, say α' , is present near to the end of road $j' = r_{\alpha'}(t)$ (i.e., $x_{\alpha'}(t) \in [L_{j'} - \ell, L_{j'}]$) entering the junction (i.e., $j' \in J$) and having priority

over the $r_\alpha(t)$ road (i.e., $j' = r_{\alpha'}(t) < r_\alpha(t)$), the α th driver has to yield to α' and stop, see (2.5) and Figure 4 (left).

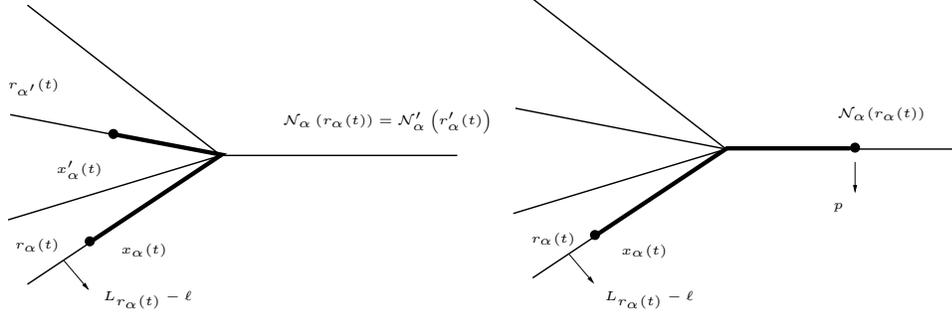


FIG. 4. Left: notation used in (2.6): the α th driver is near the end of the $r_\alpha(t)$ road and gives way to the α' th driver, since $r_{\alpha'}(t) < r_\alpha(t)$. Right: notation used in (2.7): the α th driver adjusts his/her speed to the position p of the nearest driver in front of him/her on road $\mathcal{N}_\alpha(r_\alpha(t))$.

$$(2.6) \quad \dot{x}_\alpha = 0 \quad \text{if} \quad \begin{cases} \alpha \text{ is at the end of the road;} \\ \alpha\text{'s road does not have the priority;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{someone is at the end of roads having priority over } \alpha. \end{cases}$$

$$= 0 \quad \text{if} \quad \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ r_\alpha(t) \neq \min J; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \bigcup_{j \in J : j < r_\alpha(t)} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = j \text{ and } x_j(t) > L_j - \ell\} \neq \emptyset. \end{cases}$$

Finally, consider the case when no one is present on the road having priority over the road, indexed by $r_\alpha(t)$, where the α th driver is moving (i.e., $\bigcup_{j \in J : j < r_\alpha(t)} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = j \text{ and } x_j(t) > L_j - \ell\} = \emptyset$), but other vehicles are present on the $\mathcal{N}_\alpha(r_\alpha(t))$ road where α is heading (i.e., $\{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} \neq \emptyset$), see Figure 4 (right). Then, the α th driver adapts his/her speed to the vehicle in front of him/her:

$$(2.7) \quad \dot{x}_\alpha = v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) \quad \text{if} \quad \begin{cases} \alpha \text{ is at the end of the road;} \\ \alpha\text{'s road does not have the priority;} \\ \text{no one is on the same road in front of } \alpha; \\ \text{no one is at the end of roads having priority over } \alpha; \\ \text{someone is on the road where } \alpha \text{ is going;} \\ p \text{ is the position of the nearest vehicle in front of } \\ \alpha \text{ on the same road.} \end{cases}$$

$$= v_{r_\alpha(t)} \left(\frac{\ell}{p + L_{r_\alpha(t)} - x_\alpha} \right) \quad \text{if} \quad \begin{cases} x_\alpha(t) > L_{r_\alpha(t)} - \ell; \\ r_\alpha(t) > \min J; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = r_\alpha(t) \text{ and } x_{\alpha'}(t) > x_\alpha(t)\} = \emptyset; \\ \bigcup_{j \in J : j < r_\alpha(t)} \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = j, x_j(t) > L_j - \ell\} = \emptyset; \\ \{\alpha' \in \{1, \dots, n\} : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\} \neq \emptyset; \\ p = \min \{x_{\alpha'} \in [0, L_{r_\alpha(t)}] : r_{\alpha'}(t) = \mathcal{N}_\alpha(r_\alpha(t))\}. \end{cases}$$

Existence and Uniqueness of Solutions. Summarizing, the above formulas (2.2)–(2.7) define a system of n ordinary differential equations, which we write

$$(2.8) \quad \dot{x}_\alpha = \mathcal{V}_\alpha(t, x)$$

for short. The definitions above ensure that $\mathcal{V}_\alpha(t, x) \in [0, V_{\max}]$ for all $i = 1, \dots, n$, $t \in [0, T]$ and $x \in \mathbb{R}^n$.

We now introduce a condition that states the absence of collisions among drivers. Recall that at time t driver α is located at $x_\alpha(t)$ on road $r_\alpha(t)$.

(NoCollision) For all $\alpha', \alpha'' \in \{1, \dots, n\}$, if $r_{\alpha'} = r_{\alpha''}$, then $|x_{\alpha'} - x_{\alpha''}| \geq \ell$.

Observe that the above condition does *not* rule out the following situation. Driver α is located at x_α on road $j = r_\alpha(t)$ and, say, very near the junction located at the end of road j , so that $x_\alpha \in]L_j - \ell, L_j[$. Driver α' moves along road $j' = r_{\alpha'}(t)$, also entering the same junction and has the priority over road j , so that $j' < j$. When α' passes the junction, α is stopped and there may well be a time at which the distance between α and α' is smaller than ℓ , but with α and α' being on different roads, so that no actual collision takes place.

THEOREM 2.1. *Consider a network of m interconnected roads containing at least one Entry Road and one Exit Road. For $j = 1, \dots, m$, on road j a speed law v_j satisfying **(SpeedLaw)** is given. Assign to n drivers routes $\mathcal{R}_1, \dots, \mathcal{R}_n$ satisfying the **(NoLoop)** and **(NoDeadEnd)** conditions. Each driver α is assigned an initial position x_α^0 in the first road of α 's route \mathcal{R}_α and these initial positions satisfy condition **(NoCollision)**.*

*Then, the system of differential equations (2.8) admits a unique solution on the time interval $[0, +\infty[$. Moreover, at any positive t , the positions $x_\alpha(t)$ of the drivers at time t along roads $r_\alpha(t)$, keep satisfying condition **(NoCollision)**.*

REMARK 2.2. System (2.8) may not have good stability properties concerning the dependence of solutions on the initial data, which is consistent with the common driving experience.

Indeed, consider the case in Figure 5. The Entry Roads 1 and 2 end in the same junction, where the Exit Road 3 begins. Road 1 yields priority to road 2. For simplicity, choose the same speed law, say $v(\rho) = 1 - \rho$, along all roads.

Fix a sufficiently small $\varepsilon > 0$. At time $t = 0$, driver 1 is at $x_1^0 = -\ell - \varepsilon$, while driver 2 is at $x_2^0 = -\varepsilon^2$, see Figure 5 (left). Then, the solution to (2.8) consists in driver 1 passing through the junction and with driver 2 following. On the other hand,

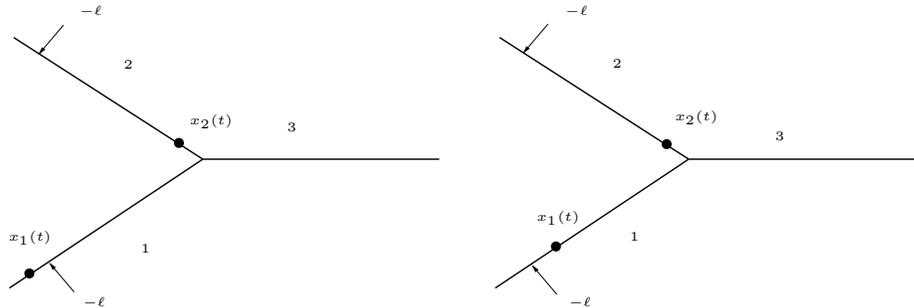


FIG. 5. These initial data are arbitrarily near but lead to solutions to (2.8) that are uniformly distant, due to the presence of priority rules.

if driver 1 starts from $x_1^0 = -\ell + \varepsilon$ with driver 2 always at $x_2^0 = -\varepsilon^2$, see Figure 5

(right), then driver 2 stops owing priority to driver 1. The two resulting solutions are uniformly different as $\varepsilon \rightarrow 0$.

3. Emergence of Braess Paradox. In this section we show the emergence of Braess paradox in a non-stationary setting, obtained within the framework of the system of differential equations (2.8) on the network depicted in Figure 2.

The seven roads are numbered as in Figure 2 and the Middle Roads are assigned the lengths $L_2 = L_3 = L_5 = L_6 = \sqrt{2}$ and $L_4 = 2$. We consider the routes

$$(3.1) \quad \mathcal{R}_0 = [1, 3, 6, 7], \quad \mathcal{R}_1 = [1, 2, 5, 7], \quad \text{and} \quad \mathcal{R}_2 = [1, 3, 4, 5, 7],$$

using the following priorities:

$$(3.2) \quad \begin{array}{l} \text{road 4 has the priority over road 2,} \\ \text{road 5 has the priority over road 6.} \end{array}$$

This means that the route \mathcal{R}_2 with the Middle Road 4, has priority over the other routes. Along road j we use the speed law v_j , for $j = 1, \dots, 7$, where

$$(3.3) \quad \begin{array}{l} v_1(\rho) = 0.9(1 - \rho), \quad v_2(\rho) = 0.6\sqrt{1 - \rho}, \quad v_3(\rho) = (1 - \rho)^{10}, \\ v_4(\rho) = 8.0(1 - \rho), \quad v_5(\rho) = 1.2(1 - \rho)^6, \quad v_6(\rho) = \sqrt{1 - \rho}, \quad v_7(\rho) = 1 - \rho. \end{array}$$

The vehicles' length is $\ell = 0.1$. We consider $n = 180$ drivers leaving at time $t = 0$ from positions x_1^0, \dots, x_{180}^0 evenly spaced in the interval $[-36, -0.1]$. Through a random number generator, we randomly assign the route to each driver according to the proportions $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_k$ being the percentage of driver following the route \mathcal{R}_k . Thus $\vartheta_k \in [0, 1]$ with $\sum_k \vartheta_k = 1$.

By means of Euler polygonals, with time step $h = 0.01$, we compute (approximate) solutions to (2.8). Each integration is repeated 20 times with different route assignments to the drivers, but assigning the same frequencies ϑ_0, ϑ_1 , and ϑ_2 . For each driver α , we compute the travel time as the first time step when α is on road 7. Then, all travel times are averaged over the drivers following the same route, and the results are displayed in Table 1.

Assigned Distrib.			Effective Distribution			Travel Time			Mean
ϑ_0	ϑ_1	ϑ_2	Θ_0	Θ_1	Θ_2	T_0	T_1	T_2	$\sum_i \Theta_i T_i$
0.00	0.00	1.00	0.0000	0.0000	1.0000	//	//	105.4	105.4
0.05	0.05	0.90	0.05222	0.05028	0.8975	105.6	106.7	100.1	100.7
0.06	0.06	0.88	0.05833	0.05861	0.8831	102.7	100.3	99.33	99.58
0.07	0.07	0.86	0.07056	0.07028	0.8592	100.0	101.6	98.21	98.58
0.06	0.04	0.90	0.05917	0.03444	0.9064	101.7	106.4	101.9	102.1
0.04	0.06	0.90	0.04333	0.06167	0.8950	100.0	95.95	99.10	98.95
0.30	0.30	0.40	0.3083	0.2761	0.4156	76.99	77.55	79.84	78.33
0.45	0.45	0.10	0.4467	0.4486	0.1047	63.06	65.45	60.79	63.89
0.47	0.47	0.06	0.4761	0.4633	0.06056	61.93	62.63	60.02	62.14
0.50	0.50	0.00	0.4983	0.5017	0.0000	58.45	60.00	//	59.23

TABLE 1

Sample results obtained from integrating (2.8). The travel times T_k , for $k = 0, 1, 2$, are the averages of the times at which drivers following route k enter road 7. The mean travel time is $\sum_{k=0}^2 \Theta_k T_k$, where Θ_k is the actual portion of drivers following route k .

Here, the *travel time* T_k is the average time that drivers following route k need to reach road 7.

The bold travel times in Table 1 display situations fully coherent with Braess paradox and with $(\vartheta_0, \vartheta_1, \vartheta_2) = (0, 0, 1)$ being a Nash equilibrium for the travel times. Note also that all displayed integrations are consistent with a *weak*, but still surprising, form of the Braess paradox, in the sense that the overall mean travel times with the new road 4 being present are all clearly larger than the mean travel time without road 4.

Figure 6 displays a sample integration of the model described by (2.8) with speed laws (3.3), where we can see the effect of the priority of route $\mathcal{R}_2 = [1, 3, 4, 5, 7]$,

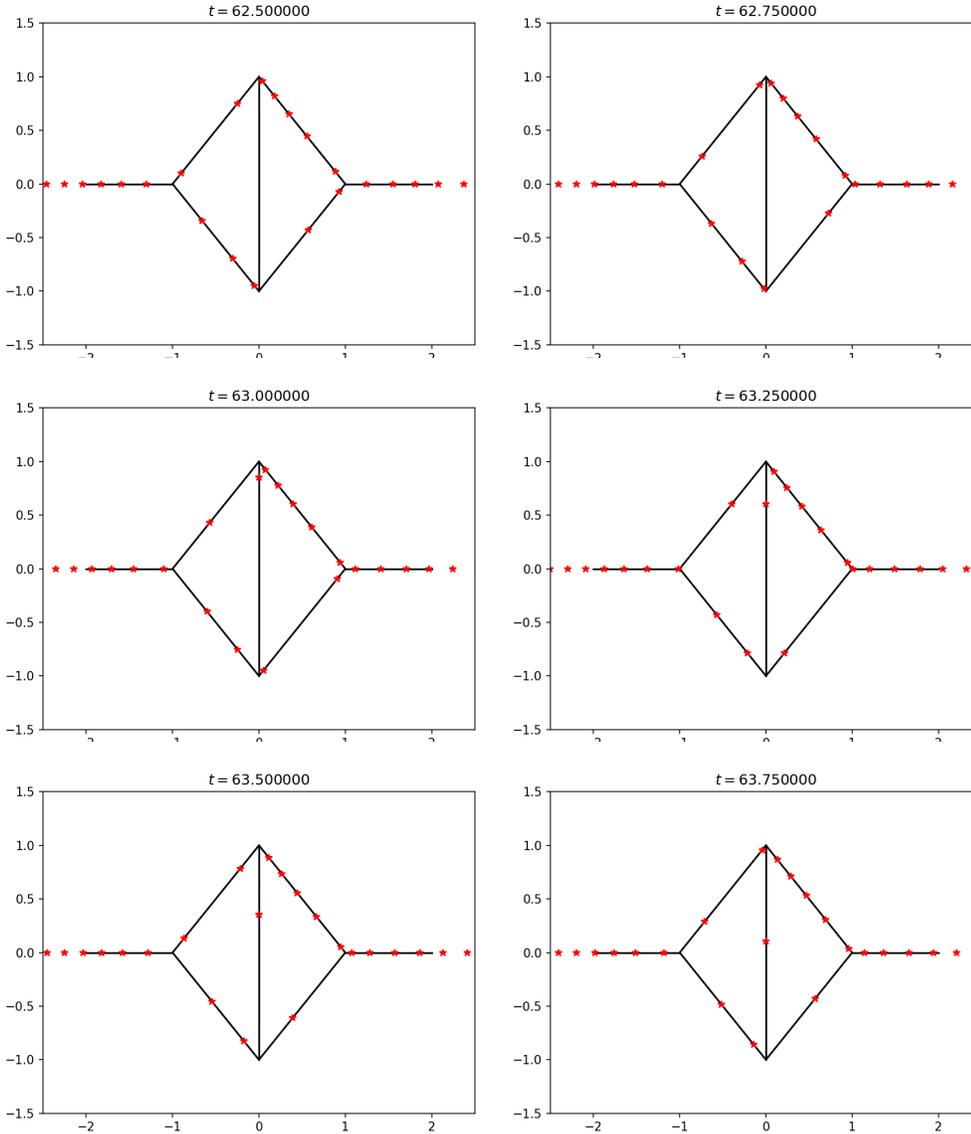


FIG. 6. Sample integration of the model described by (2.8). Note that the top right road yields priority to the bottom right one. As a consequence, all during the (short) time interval displayed, the 5 vehicles in the top right road line up and give priority to the vehicles coming from their right.

4. Analytic Proofs. The following lemma tackles the basic local existence part of Theorem 2.1.

LEMMA 4.1. *With the assumptions and notations of Theorem 2.1, assume that at time $\bar{t} \geq 0$ the drivers are distributed along the network at positions $\bar{x}_\alpha(\bar{t})$ satisfying condition **(NoCollision)**. Then, there exists a positive ε and uniquely determined functions $x_\alpha: [\bar{t}, \bar{t} + \varepsilon] \rightarrow \mathbb{R}$ solving (2.8). Moreover, for all $t \in [\bar{t}, \bar{t} + \varepsilon]$, condition **(NoCollision)** holds.*

Proof. The proof is divided in three steps.

1. For all α such that $r_\alpha(\bar{t})$ is an Exit Road, the function x_α can be uniquely defined on $[\bar{t}, +\infty[$ solving a standard Follow-the-Leader ODE system. Note that, by the standard properties of this model, these x_α satisfy the **(NoCollision)** condition.

2. If Middle Roads and Entry Roads are empty, the proof is finished.

Otherwise, introduce the set of Entry and Middle Roads where there is at least one driver at time \bar{t} whose next road is an Exit Road:

$$J_1 = \{j: \exists \alpha \in \{1, \dots, n\} \text{ with } r_\alpha(\bar{t}) = j \text{ and } \mathcal{N}_\alpha(r_\alpha(\bar{t})) \text{ is an Exit Road}\}.$$

Each road $j \in J_1$ ends at a junction where an Exit Road begins. Consider one of these junctions, say C and call j_1, \dots, j_k the roads in J_1 entering C . We may assume that $j_1 < j_2 < \dots < j_k$, so that road j_1 has the priority. The drivers along road j_1 are at positions $x_{\alpha_1}(\bar{t}), x_{\alpha_2}(\bar{t}), \dots, x_{\alpha_\nu}(\bar{t})$, with $L_{j_1} > x_{\alpha_1}(\bar{t}) > x_{\alpha_2}(\bar{t}) > \dots > x_{\alpha_\nu}(\bar{t}) \geq 0$. The trajectory of driver α_1 is uniquely determined, since all trajectories along Exit Roads are known. Therefore, along road j_1 , the usual Cauchy theorem for ODEs ensures the existence and uniqueness of a solution to (2.8) at least on the time interval $[\bar{t}, \bar{t} + \varepsilon]$, where $\varepsilon = (L_{j_1} - x_{\alpha_1}(\bar{t})) / V_{\max}$ and, by construction, condition **(NoCollision)** holds.

Assume now that all drivers' trajectories along roads j_1, \dots, j_{h-1} are uniquely defined on the time interval $[\bar{t}, \bar{t} + \varepsilon]$, for a positive ε . Denote by $\alpha_1, \dots, \alpha_\nu$ the drivers on road j_h , with $L_{j_h} > x_{\alpha_1}(\bar{t}) > x_{\alpha_2}(\bar{t}) > \dots > x_{\alpha_\nu}(\bar{t}) \geq 0$. The speed of α_1 is a unique non-negative \mathbf{L}^1 function defined at least on the time interval $[\bar{t}, \bar{t} + \varepsilon']$, where $\varepsilon' = \min\{\varepsilon, (L_{j_h} - x_{\alpha_1}(\bar{t})) / V_{\max}\}$, so that the trajectory of α_1 solves (2.8) and is uniquely defined. Iteratively, the same holds first for the trajectories of $\alpha_2, \dots, \alpha_\nu$ and then along all other roads j_{h+1}, \dots, j_k , always complying with condition **(NoCollision)**.

3. By condition **(NoLoop)**, the above procedure can be iterated, covering the whole network and without considering the same interval twice. Indeed, consider the set of roads entering J_i :

$$J_{i+1} = \{j: \exists \alpha \in \{1, \dots, n\} \text{ with } r_\alpha(\bar{t}) = j \text{ and } \mathcal{N}_\alpha(r_\alpha(\bar{t})) \in J_i\}$$

and proceed exactly as in the step **2** above.

Here, a unique solution to (2.8) was constructed on the time interval $[\bar{t}, \bar{t} + \varepsilon_*]$, complying with condition **(NoCollision)**, where ε_* is the minimum of a finite quantity of positive numbers. The proof is completed. \square

Below, for each driver, we also use the time-dependent coordinate $y_\alpha(t)$, which quantifies the total distance driven by the driver α at time t . For instance, with

reference to Figure 2, if the driver $\alpha = 2$ follows the route $\mathcal{R}_2 = [0, 2, 3, 4, 6]$, starting from $x_2^o \in]-\infty, 0[$ in the Entry Road $j = 0$ at time 0 and at time t is moving along road 4, then $r_2(t) = 4$ and $y_2(t) = |x_2^o| + L_2 + L_3 + x_2(t)$, with $x_2(t) \in [0, L_4[$.

Given the route $\mathcal{R}_\alpha = [j_0, j_1, \dots, j_k]$ (with road lengths L_{j_0}, L_{j_1}, \dots) for driver α , the initial position x_α^o and $x_\alpha(t)$, the length $y_\alpha(t)$ covered by the α th driver is uniquely determined. Indeed, if at time t the α driver is along road $j_{i_*} = r_\alpha(t)$, we have

$$(4.1) \quad y_\alpha(t) = \begin{cases} |x_\alpha^o| + \sum_{i < i_*} L_{j_i} + x_\alpha(t) & \alpha \text{ starts from an Entry Road,} \\ L_{i_1} - x_\alpha^o + \sum_{i < i_*} L_{j_i} + x_\alpha(t) & \alpha \text{ starts from a Middle Road,} \\ x_\alpha(t) - x_\alpha^o & \alpha \text{ starts from an Exit Road.} \end{cases}$$

The inverse correspondence is straightforward.

Proof of Theorem 2.1. By Lemma 4.1, for given initial data, problem (2.8) admits a unique solution on the interval $[0, \varepsilon_0]$, for a positive ε_0 .

Prolong, in a unique way, the solution to (2.8) on the time interval $[\varepsilon_0, \varepsilon_0 + \varepsilon_1]$ applying Lemma 4.1.

We claim that a solution to (2.8) can be uniquely constructed on all $[0, +\infty[$. Indeed, assume (by contradiction) that the above procedure yields a solution x_α , for $\alpha \in \{1, \dots, n\}$ defined on the maximal time interval $[0, T[$, for a positive T . For all $\alpha \in \{1, \dots, n\}$, the corresponding function $t \mapsto y_\alpha(t)$ is defined on $[0, T[$ and it is Lipschitz continuous, hence it is uniformly continuous and can be uniquely extended by continuity to the time interval $[0, T]$. As a consequence, also x_α can be uniquely extended to the whole interval $[0, T]$. At time T , we thus apply again Lemma 4.1, obtaining a solution defined on $[0, T + \varepsilon_*[$, for a positive ε_* . This contradicts the maximality of the above choice of T . \square

5. Conclusions. This paper provides the analytic framework to use microscopic traffic model on road networks. Traffic at junctions is ruled by fixed priority rules, so that queues may form and disappear, depending on the overall traffic distribution. Existence and uniqueness of solutions is proved, while continuous dependence may fail, which is consistent with everyday experience. Moreover, vehicles may not collide, once the initial datum assigned is reasonable.

This framework is then used to describe a non-stationary instance of Braess paradox. Adding a very fast road to an existing network may increase the travel times. Here, a game theoretic approach was used, each driver being a player aiming at minimizing his/her travel time.

On the basis of the present results, further questions arise and can be tackled. A very appealing research direction concerns the *control* of network traffic. For instance, following [6, 7], can the introduction of a suitable toll avoid the insurgence of Braess paradox? Once Theorem 2.1 is extended to time dependent priority rules (i.e., traffic lights), which seems a merely technical issue, is it possible to find optimal timings at the junctions that minimize travel times?

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