A HIGHER-ORDER MAXIMUM PRINCIPLE FOR IMPULSIVE OPTIMAL CONTROL PROBLEMS

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Abstract. We consider a nonlinear system, affine with respect to an unbounded control u which is allowed to range in a closed cone. To this system we associate a Bolza type minimum problem, with a Lagrangian having sublinear growth with respect to u. This lack of coercivity gives the problem an *impulsive* character, meaning that minimizing sequences of trajectories happen to converge towards discontinuous paths. As is known, a distributional approach does not make sense in such a nonlinear setting, where, instead, a suitable embedding in the graph-space is needed. We provide higher order necessary optimality conditions for properly defined impulsive minima, in the form of equalities and inequalities involving iterated Lie brackets of the dynamical vector fields. These conditions are derived under very weak regularity assumptions and without any constant rank conditions.

1. Introduction

In this paper we establish necessary optimality conditions for the *space-time, impulsive extension* of the free end-time optimal control problem

$$\text{minimize } \Psi(T,x(T))+\int_0^T \ell(x(t),u(t),a(t))\,dt, \tag{1}$$

where the minimization is performed over the set of *processes* (T,u,a,x,v) verifying

$$\begin{cases} \dfrac{dx}{dt}(t)=f(x(t),a(t))+\displaystyle\sum_{i=1}^m g_i(x(t))u^i(t), \quad \text{a.e. } t\in[0,T],\\ \dfrac{dv}{dt}(t)=|u(t)|,\\ (x,v)(0)=(\check{x},0), \quad v(T)\le K, \quad (T,x(T))\in\mathfrak{T}. \end{cases} \tag{2}$$

Here, $0<K\le+\infty$, the *target* $\mathfrak{T}$ is a closed subset of $\mathbb{R}_+\times\mathbb{R}^n$, the control a ranges in a compact set $A\subset\mathbb{R}^q$, and standard regularity hypotheses are verified by the vector fields f, g_i and the cost functions ℓ, Ψ. Instead, less usual assumptions are made on the control maps u and on the u-growth of the Lagrangian ℓ. Precisely:
(i) the *unbounded* controls u, which take values in a closed cone $\mathcal{C}\subseteq\mathbb{R}^m$, are L^1 functions verifying $\|u\|_1:=\int_0^T|u(t)|dt=v(T)\le K$;
(ii) the Lagrangian $\ell:\mathbb{R}^n\times\mathcal{C}\times A\to\mathbb{R}$ has the form $\ell(x,u,a)=\ell_0(x,a)+\ell_1(x,u)$, with a continuous *recession function* $\hat{\ell}_1$ given by

$$\hat{\ell}_1(x,w^0,w):=\lim_{r\to w^0} r\ell_1\left(x,\frac{w}{r}\right), \quad \text{for } (x,w^0,w)\in\mathbb{R}^n\times\mathbb{R}_+\times\mathcal{C}.$$

In particular ℓ has sublinear growth in u.

On the one hand, optimal control problems with such a slow u-growth in the cost are motivated by several applications [13, 15, 22, 17, 27, 9, 34, 16]. For instance, a dynamics affine in the unbounded controls governs the motion of a mechanical system of mutually linked rigid bodies. In that case, the controlled parameters are the speeds of the *shape* coordinates.

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On the other hand, the lack of a sufficiently fast u-growth of the cost ℓ may cause minimizing sequences of trajectories to tend towards discontinuous paths.[1] Motivated by that, and following a nowadays standard approach [2] [42, 37, 14, 29, 28, 39, 23], one 'compactifies' the problem by embedding the original control system in the *extended, space-time,* system

$$
\begin{cases}
\dfrac{dy^0}{ds}(s) = w^0(s), \\
\dfrac{dy}{ds}(s) = f(y(s),\alpha(s))w^0(s) + \displaystyle\sum_{i=1}^{m} g_i(y(s))w^i(s), \\
\dfrac{d\beta}{ds}(s) = |w(s)|, \\
(y^0, y, \beta)(0) = (0, \check{x}, 0), \\
(y^0(S), y(S), \beta(S)) \in \mathfrak{T} \times [0, K],
\end{cases}
\qquad \text{a.e. } s \in [0, S], \tag{3}
$$

and considering the *extended* cost functional

$$
\Psi(y^0(S), y(S)) + \int_0^S \ell^e(y(s), w^0(s), w(s), \alpha(s))ds, \tag{4}
$$

where $\ell^e(x, w^0, w, a) := \ell_0(x, a)w^0 + \hat{\ell}_1(x, w^0, w)$. In particular, for problem (3)-(4) one can consider *bounded* controls verifying $w^0(s) + |w(s)| = 1$, $w(s) \in \mathcal{C}$, $w^0(s) \geq 0$, and $a(s) \in A$, for a.e. $s \in [0, S]$. As soon as $w^0 > 0$ a.e., (3)-(4) is nothing but a time reparameterization of the original control problem (1)-(2), and the unboundedness of u is reflected in the possibility of taking w^0 going to 0. By allowing processes $(S, w^0, w, \alpha, y^0, y, \beta)$ such that $w^0 = 0$ a.e. on non-degenerate subintervals $[s_1, s_2]$, we are embedding (1)-(2) in a more general problem. More precisely, the time-variable $t = y^0(s)$ has a constant value $\bar{t}$ on such an interval $[s_1, s_2]$, while the space trajectory $y(s)$ evolves according to the nonlinear dynamics $\frac{dy}{ds} = \sum_{i=1}^{m} g_i(y)w^i$. In particular, the jump $x(\bar{t}+) - x(\bar{t}-) = y(s_2) - y(s_1)$ depends on the restriction $w_{|[s_1,s_2]}$. Our necessary conditions concern a minimum for the extended, space-time, problem (3)-(4). We begin by exploiting the *rate-independence* of the extended problem in order to establish a First Order Maximum Principle, in Theorem 3.1. While the idea is nothing but new –see, for instance, [40, 32, 28, 7, 30]– here we stress a kind of competition among the Hamiltonian corresponding to the drift and the 'non-drift' Hamiltonian.

However, the main novelty of the present paper consists in a Maximum Principle containing *higher-order* necessary conditions which involve iterated Lie brackets (see Theorem 4.1). The crucial tool to prove our conditions consists in the construction of variations by coupling asymptotic formulas for Lie brackets and suitable insertions of impulsive intervals. In particular, in connection with sets of lines contained in the cone C (see Section 2), the dual product of the adjoint path with the corresponding iterated Lie brackets turns out to vanish.

Lie bracket-involving necessary conditions have been widely investigated within classical geometric control theory, a quite incomplete list of reference being [3, 25, 26, 41, 38, 10]. Instead, as far as impulsive control theory is concerned, the only results involving higher order –actually, second order– conditions deal, at our knowledge, with the so-called *commutative case*, where $[g_i, g_j] \equiv 0$ for all $i, j = 1, \dots, m$, (see e.g. [8, 6, 19]). Let us mention that our conditions might be regarded as a generalization to impulsive trajectories of [18], where one considers the (non extended) minimum time problem with L^∞, but not a priori

[1] For instance, if one considers the minimum time problem –i.e. $\ell \equiv 1$ and $\Psi \equiv 0$– with a target in $\mathbb{R}^n$ intersecting the orbit of the vector field g_1 issuing from the initial point $\check{x}$, the infimum value is zero and the 'extended' optimal trajectory should run through the mentioned orbit *with infinite speed*.

[2] Because of the nonlinearity of the dynamics, a *distributional* interpretation lacks essential prerequisites for robustness [24].

bounded, controls (see Remark 4.2). Let us stress that we assume neither the invariance of the dimension of the Lie algebra generated by $g_1, \dots, g_m$ nor the mere existence of such algebra. This is made possible by the fact that our Lie bracket-shaped variations are based on rather general asymptotic formulas, which demand very low regularity on the vector fields: in particular, most of these formulas assume *just the continuity* of the involved Lie brackets.

The article is organized as follows. In Subsection 1.1 we introduce some general notation and a set of definitions and technical results involving Lie brackets. The optimal control problem is described in detail in Section 2, together with its *space-time* extension. In Section 3, Theorem 3.1, we state a First Order Maximum Principle. In Section 4, Theorem 4.1, we establish our main result, namely the Higher Order Maximum Principle, whose proof is given in Section 5.

1.1. **Notations and preliminaries.**

1.1.1. *Some basic notation.* Let $N \geq 1$ be an integer. For any $i \in \{1, \dots, N\}$, we write $\mathbf{e}_i$ for the ith element of the canonical basis of $\mathbb{R}^N$. Given $\check{x} \in \mathbb{R}^N$, $\mathbb{B}_N(\check{x}) := \{x \in \mathbb{R}^N : |x - \check{x}| \leq 1\}$, $\mathbb{B}_N := \mathbb{B}_N(0)$, and $\partial\mathbb{B}_N := \{x \in \mathbb{R}^N : \ |x| = 1\}$. A subset $\mathcal{K} \subseteq \mathbb{R}^N$ is a *cone* if $\alpha x \in \mathcal{K}$ whenever $\alpha > 0$, $x \in \mathcal{K}$. Given a subset $X \subseteq \mathbb{R}^N$, we will use $X^\perp$ to denote the *polar* of X, i.e. $X^\perp \doteq \{p \in \mathbb{R}^N : \ p \cdot x \leq 0, \quad \text{for all} \quad x \in X\}$. Given an interval I and $X \subseteq \mathbb{R}^N$, we write $AC(I, X)$ for the space of absolutely continuous functions, $C^0(I, X)$ for the space of continuous functions, $L^1(I, X)$ for the Lebesgue space of L^1-functions, and $L^\infty(I, X)$ for the Lebesgue space of measurable, essentially bounded functions, respectively, defined on I and assuming values in X. As customary, we shall use $\|\cdot\|_{L^\infty(I,X)}$, and $\|\cdot\|_{L^1(I,X)}$ to denote the essential supremum norm and the L^1-norm, respectively. When no confusion may arise, we will simply write $\|\cdot\|_\infty$ and $\|\cdot\|_1$. We set $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0]$. Given an integer $k \geq 0$ and an open subset $\Theta \subseteq \mathbb{R}^N$, we say that a function $F\colon \Theta \to \mathbb{R}^N$ is *of class* C^k if it possesses continuous partial derivatives up to order k in Θ. Given a real-valued function $F : [a, b] \to \mathbb{R}$, we define the essential infimum of F as $\operatorname{ess\,inf} F := \sup\big\{r \in \mathbb{R} : \ \operatorname{meas}\{x \in [a, b] : \ F(x) < r\} = 0\big\}$, where meas denotes the Lebesgue measure. Finally, for all τ_1, $\tau_2 \in (0, +\infty)$ and for any pair $(z_1, z_2) \in C^0([0, \tau_1], \mathbb{R}^N) \times C^0([0, \tau_2], \mathbb{R}^N)$, let us define the distance

$$\mathrm{d}\big((\tau_1, z_1), (\tau_2, z_2)\big) := |\tau_1 - \tau_2| + \|\tilde{z}_1 - \tilde{z}_2\|_\infty, \tag{5}$$

where for any $z \in C^0([0, \tau], \mathbb{R}^N)$, $\tilde{z}$ denotes its continuous constant extension to $\mathbb{R}_+$.

1.1.2. *Boltyanski approximating cones.*

Definition 1.1. *Let Z be a subset of $\mathbb{R}^N$ for some integer $N \geq 1$. Fix $z \in Z$. We say that a convex cone $\mathcal{K} \subseteq \mathbb{R}^N$ is a* Boltyanski approximating cone for Z at z *if there exist a convex cone $C \subset \mathbb{R}^M$ for some integer $M \geq 0$, a neighborhood V of 0 in $\mathbb{R}^M$, and a continuous map $F : V \cap C \to Z$ such that: $F(0) = z$; there exists a linear map $L : \mathbb{R}^M \to \mathbb{R}^N$ verifying $F(v) = F(0) + Lv + o(|v|)$ for all $v \in V \cap C$; $LC = \mathcal{K}$.*

Definition 1.2. *Let us consider two subsets $\mathcal{A}_1, \mathcal{A}_2$ of a topological space $\mathcal{X}$. If $y \in \mathcal{A}_1 \cap \mathcal{A}_2$, we say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are* locally separated at y *provided there exists a neighborhood V of y such that $\mathcal{A}_1 \cap \mathcal{A}_2 \cap V = \{y\}$.*

The following open-mapping-based result characterizes set-separation in terms of linear separation of approximating cones (see e.g. [41]).

Theorem 1.1. *Let Z_1 and Z_2 be subsets of $\mathbb{R}^N$, $z \in Z_1 \cap Z_2$ and let $\mathcal{K}_1$, $\mathcal{K}_2 \subseteq \mathbb{R}^N$ be Boltyanski approximating cones for Z_1 and Z_2, respectively, at z. If $\mathcal{K}_1$ or $\mathcal{K}_2$ is not a subspace and Z_1, Z_2 are locally separated at z, then $\mathcal{K}_1$ and $\mathcal{K}_2$ are* linearly separated, *namely there exists a covector $\lambda \in \mathbb{R}^N$ such that $0 \neq \lambda \in \mathcal{K}_1^\perp \cap (-\mathcal{K}_2^\perp)$.*

1.1.3. *Lie brackets.* Given a fixed sequence $\mathbf{X} = (X_1, X_2, \ldots)$ of distinct objects called *variables*, we call *words* the finite ordered strings consisting of the variables X_i, the *left parenthesis* [and the *right parenthesis*], and the comma. We shall use $W(\mathbf{X})$ to denote the set of words. For instance, $X_2X_5X_4$ and $X_3, [X_{13}[,]]X_{61}[$ are words.
Given any word $W \in W(\mathbf{X})$, we use $\mathrm{Seq}(W)$ to denote the word obtained from W by deleting all left and right brackets and all commas. We call *length* of a word $W \in W(\mathbf{X})$, and write $\mathrm{Lgth}(W)$, the cardinality of $\mathrm{Seq}(W)$. For instance, if $W = [[[X_4, X_6], X_7], [X_8, X_9]]$, $\mathrm{Seq}(W) = X_4X_6X_7X_8X_9$ and $\mathrm{Lgth}(W) = 5$.

Definition 1.3. *We call* formal bracket of length 1 *any word of length 1 and we will say that the* bracket of two members W_1, W_2 *of* $W(\mathbf{X})$ *is the word* $[W_1, W_2]$.
We call formal iterated brackets *–or, simply,* brackets*– of* $\mathbf{X}$ *the elements of the smallest subset* $IB(\mathbf{X}) \subseteq W(\mathbf{X})$ *such that:* $IB(\mathbf{X})$ *contains the brackets of length* 1*; if* W_1, $W_2 \in IB(\mathbf{X})$*, then* $[W_1, W_2] \in IB(\mathbf{X})$*; for any* $b \in IB(\mathbf{X})$*,* $\mathrm{Seq}(b) = X_{\mu+1}, \ldots, X_{\mu+m}$ *for some* $\mu \geq 0$ *and* $m > 0$.

Notice that $\mathrm{Lgth}([b_1, b_2]) = \mathrm{Lgth}(b_1) + \mathrm{Lgth}(b_2)$, for every pair of brackets b_1, b_2. Let b be a bracket of length $m > 1$. Then there exists a unique pair (b_1, b_2) of brackets such that $b = [b_1, b_2]$. The pair (b_1, b_2) is the *factorization* of b, and b_1, b_2 are known, respectively, as the *left factor* and the *right factor* of b. Any substring of b which is itself an iterated bracket is called a *subbracket* of b.

Definition 1.4. *If* b *is a bracket and* S *is a subbracket of* b*, let us define* $\mathfrak{d}(S; b)$ *by a backward recursion on* S*:* $\mathfrak{d}(b; b) := 0$*,* $\mathfrak{d}(S_1; b) := \mathfrak{d}(S_2; b) := 1 + \mathfrak{d}([S_1, S_2]; b)$*. We shall refer to* $\mathfrak{d}(S; b)$ *as the* number of differentiations of S in b.

It is easy to prove that $\mathfrak{d}(S; b)$ is equal to the number of right brackets that occur in b to the right of S minus the number of left brackets that occur in b to the right of S. For example, if $b = b(X_3, X_4, X_5) := \big[X_3\,,\ [X_4, X_5]\big]$, then $\mathfrak{d}([X_4, X_5]; b) = 1$, $\mathfrak{d}(X_3; b) = 1$, $\mathfrak{d}(X_4; b) = 2$, $\mathfrak{d}(X_5; b) = 2$.

Definition 1.5 (Classes C^{b+k} and $C^{b+k-1,1}$)**.** *Let* b *be a bracket of degree* $m \geq 1$*, with* $\mathrm{Seq}(b) = X_{\mu+1} \ldots X_{\mu+m}$*,* $\mu \geq 0$*. Let* $\mathbf{f} = (f_1, \ldots, f_\nu)$ *be a finite sequence of vector fields, with* $\nu \geq \mu + m$*, and let* $k \geq 0$ *be an integer. We say that* $\mathbf{f}$ is of class C^{b+k} *if* f_j *is of class* $C^{\mathfrak{d}(X_j; b)+k}$ *for each* $j \in \{\mu + 1, \ldots, \mu + m\}$.

For example, if $b = \big[[X_3, X_4], [[X_5, X_6], X_7]\big]$ and $\mathbf{f} = (f_1, \ldots, f_8)$ (so $m = 5$, $\nu = 8$, $\mu = 2$), then $\mathbf{f} \in C^{b+3}$ if, and only if, $f_3, f_4, f_7 \in C^5$ and $f_5, f_6 \in C^6$. It is easy to verify the following result.

Proposition 1.2. *Let* b*,* k*, and* $\mathbf{f} = (f_1, \ldots, f_\nu)$ *be as in Def. 1.5, and let* (b_1, b_2) *be the factorization of* b*. Then* $\mathbf{f} \in C^{b+k}$ *if, and only if,* $\mathbf{f} \in C^{b_1+k+1}$ *and* $\mathbf{f} \in C^{b_2+k+1}$.

We are now ready to plug vector fields in place of indeterminates in a bracket.

Definition 1.6. *For integers* $\mu \geq 0$*,* $m, \nu \geq 1$*, such that* $\mu + m \leq \nu$*, let* b *be a formal bracket such that* $\mathrm{Seq}(b) = X_{\mu+1} \ldots X_{\mu+m}$ *and let* $\mathbf{f} = (f_1, \ldots, f_\nu)$ *be a* ν*-tuple of continuous vector fields. Let* S *be a subbracket of* b*. If* $\mathrm{Lgth}(S) = 1$*, i.e.* $S = X_j$ *for some* $j = \mu + 1, \ldots, \mu + m$*, we define the vector field* $S(\mathbf{f})$ *as* $S(\mathbf{f}) := X_j(\mathbf{f}) := f_j$*. If* $\mathrm{Lgth}(S) > 1$*,* $S = [S_1, S_2]$*, and either* $S \neq b$ *or, when* $S = b$*, one assumes* $\mathbf{f} \in C^b$*, we set* $S(\mathbf{f}) := [S_1(\mathbf{f}), S_2(\mathbf{f})]$*. We shall call* $S(\mathbf{f})$ *the Lie bracket corresponding to the formal bracket* S *and the sequence* $\mathbf{f}$ *of vector fields.*

We call *switch-number* of a formal bracket b the number r_b defined recursively as: $r_b := 1$, if $b = X_j$ for some j; $r_b := 2\big(r_{b_1} + r_{b_2}\big)$ if $\mathrm{Lgth}(b) \geq 2$ and $b = [b_1, b_2]$. For instance, the switch-numbers of $\big[[X_3, X_4], [[X_5, X_6], X_7]\big]$ and $[[X_5, X_6], X_7]$ are 28 and 10, respectively. We will call *length* and *switch-number of a Lie bracket* $B = b(f_{\mu+1}, \ldots f_{\mu+m})$ the length and the switch-number of the associated formal bracket b, respectively.

2. The optimization problems

In this section we introduce rigorously the optimization problem over L^1-controls and its embedding in an impulsive problem.

Throughout the paper we shall assume the following set of hypotheses:

(Hp) (i) *the target* $\mathfrak{T} \subset \mathbb{R}_+ \times \mathbb{R}^n$ *is a closed subset; the control set* $A \subset \mathbb{R}^q$ *is compact;*[3] *the* unbounded control set $\mathcal{C} \subseteq \mathbb{R}^m$ *is a closed cone of the form* $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$, *where* $m_1 + m_2 = m$, $\mathcal{C}_1 \subseteq \mathbb{R}^{m_1}$ *is a closed cone that contains the lines* $\{r\mathbf{e}_i : \ r \in \mathbb{R}\}$, *for* $i = 1, \dots, m_1$, *and* $\mathcal{C}_2 \subset \mathbb{R}^{m_2}$ *is a closed cone which does not contain straight lines;* [4]

(ii) *the* drift dynamics $f \colon \mathbb{R}^n \times A \to \mathbb{R}^n$ *is continuous and has continuous partial derivatives* $\dfrac{\partial f}{\partial x^1}, \dots, \dfrac{\partial f}{\partial x^n}$;

(iii) *the vector fields* $g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}^n$ *are continuously differentiable;*

(iv) *the* Lagrangian $\ell \colon \mathbb{R}^n \times \mathcal{C} \times A \to \mathbb{R}$ *can be written as* $\ell(x, u, a) = \ell_0(x, a) + \ell_1(x, u)$, *where* ℓ_0 *and the recession function*

$$\hat{\ell}_1(x, w^0, w) := \lim_{r \to w^0} r\ell_1\left(x, \frac{w}{r}\right), \quad \textit{for all } (x, w^0, w) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{C}$$

are continuous with continuous partial derivatives with respect to x.

(v) *the* final cost $\Psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable.*

Clearly, by standard cut-off methods one might assume the differentiability hypotheses in (ii)-(v) only on a neighborhood of the extended optimal trajectory considered in Thms. 3.1, 4.1.

2.1. **The original optimal control problem.**

We define the *set* $\mathcal{U}$ *of strict-sense controls* as $\mathcal{U} := \bigcup_{T>0} \{T\} \times L^1\big([0,T], \mathcal{C} \times A\big)$.

Definition 2.1. *For any strict-sense control* $(T, u, a) \in \mathcal{U}$, *we call* (T, u, a, x, v) *a* strict-sense process *if* (x, v) *is the (unique) Carathéodory solution to*

$$\begin{cases} \dfrac{dx}{dt}(t) = f(x(t), a(t)) + \displaystyle\sum_{i=1}^{m} g_i(x(t))u^i(t), \\ \dfrac{dv}{dt}(t) = |u(t)|, \\ (x, v)(0) = (\check{x}, 0). \end{cases} \qquad \text{a.e. } t \in [0, T], \tag{6}$$

Furthermore, we say that (T, u, a, x, v) *is* feasible *if* $(T, x(T), v(T)) \in \mathfrak{T} \times [0, K]$.

The original optimal control problem is defined as

$$\text{(P)} \qquad \begin{cases} \text{minimize } \Psi(T, x(T)) + \displaystyle\int_0^T \ell(x(t), u(t), a(t))\, dt \\ \text{over the set of feasible strict-sense processes } (T, u, a, x, v). \end{cases}$$

Definition 2.2. *We call a feasible strict-sense process* $(\bar{T}, \bar{u}, \bar{a}, \bar{x}, \bar{v})$ *a* local strict-sense minimizer *of* (P) *if there exists* $\delta > 0$ *such that*

$$\Psi(\bar{T}, \bar{x}(\bar{T})) + \int_0^{\bar{T}} \ell(\bar{x}(t), \bar{u}(t), \bar{a}(t))dt \ \le \ \Psi(T, x(T)) + \int_0^T \ell(x(t), u(t), a(t))dt \tag{7}$$

---

[3]Through minor changes one might generalize this hypothesis with the fact that, for every (x, u), the function $a \mapsto (f(x, a), l(x, u, a))$ is bounded.

[4]Hypothesis (i) on $\mathcal{C}$ is by no means restrictive, since it can be recovered by replacing the single vector fields g_i with suitable linear combinations of $\{g_1, \dots, g_m\}$.

for every feasible strict-sense process (T, u, a, x, v) *verifying* $\mathrm{d}\Big((T, x, v), (\bar{T}, \bar{x}, \bar{v})\Big) < \delta$, *where* d *is the distance defined in* (5)*. If relation* (7) *is satisfied for all feasible strict-sense processes, we say that* $(\bar{T}, \bar{u}, \bar{a}, \bar{x}, \bar{v})$ *is a* global strict-sense minimizer*.*

Remark 2.1. By adding the trivial equations $\dfrac{dx^0}{dt}(t) = 1$, $\dfrac{d\hat{x}}{dt}(t) = u(t)$, where $\hat{x} = (x^{n+1}, \dots, x^{n+m})$, we can allow ℓ, f, g_i, for $i = 1, \dots, m$, to depend on t and on the function $U(t) := \int_0^t u(\tau)\, d\tau$, while Ψ might depend on U as well.

2.2. **The space-time optimal control problem.** We refer to the set $\mathcal{W} := \bigcup_{S>0}\{S\} \times \Big\{(w^0, w, \alpha) \in L^\infty([0, S], \mathbb{R}_+ \times \mathcal{C} \times A) : \operatorname{ess\,inf}(w^0 + |w|) > 0\Big\}$ as the *set of space-time controls.*

Definition 2.3. *For any* $(S, w^0, w, \alpha) \in \mathcal{W}$*, we say that* $(S, w^0, w, \alpha, y^0, y, \beta)$ *is a* space-time process *if* (y^0, y, β) *is the unique Carathéodory solution of*

$$
\begin{cases}
\dfrac{dy^0}{ds}(s) = w^0(s), \\
\dfrac{dy}{ds}(s) = f(y(s), \alpha(s))w^0(s) + \displaystyle\sum_{i=1}^{m} g_i(y(s))w^i(s), & \text{a.e. } s \in [0, S], \\
\dfrac{d\beta}{ds}(s) = |w(s)|, \\
(y^0, y, \beta)(0) = (0, \check{x}, 0).
\end{cases}
\tag{8}
$$

We say that $(S, w^0, w, \alpha, y^0, y, \beta)$ *is* feasible *if* $(y^0(S), y(S), \beta(S))$ *belongs to* $\mathfrak{T} \times [0, K]$*.*

We define the *extended or space-time problem* as

$$
(\mathrm{P_{s\text{-}t}}) \qquad \begin{cases}
\text{minimize } \Psi(y^0(S), y(S)) + \displaystyle\int_0^S \ell^e\big((y, w^0, w, \alpha)(s)\big)\, ds \\
\text{over feasible space-time processes } (S, w^0, w, \alpha, y^0, y, \beta),
\end{cases}
$$

where the *extended Lagrangian* ℓ^e is given by

$$
\ell^e(x, w^0, w, a) := \ell_0(x, a)w^0 + \hat{\ell}_1(x, w^0, w), \quad \text{for } (x, w^0, w, a) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{C} \times A.
$$

Definition 2.4. *A feasible space-time process* $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \bar{\beta})$ *is said to be a* local minimizer for the space-time problem ($\mathrm{P_{s\text{-}t}}$) *if there exists* $\delta > 0$ *such that*

$$
\Psi((\bar{y}^0, \bar{y})(\bar{S})) + \int_0^{\bar{S}} \ell^e((\bar{y}, \bar{w}^0, \bar{w}, \bar{\alpha})(s))\, ds \le \Psi((y^0, y)(S)) + \int_0^S \ell^e((y, w^0, w, \alpha)(s))\, ds
\tag{9}
$$

for all feasible $(S, w^0, w, \alpha, y^0, y, \beta)$ *satisfying* $\mathrm{d}\Big((S, y^0, y, \beta), (\bar{S}, \bar{y}^0, \bar{y}, \bar{\beta})\Big) < \delta$*, where* d *is as in* (5)*. If* (9) *is satisfied for all feasible space-time processes, we call* $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \beta)$ *a* global space-time minimizer*.*

Observe that the space-time system (8) is *rate-independent.* Precisely, given a strictly increasing, surjective, and bi-Lipschitzian function $\sigma : [0, S] \to [0, \tilde{S}]$, $(\tilde{S}, \tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{y}^0, \tilde{y}, \tilde{\beta})$ is a space-time process if, and only if, $(S, w^0, w, \alpha, y^0, y, \beta)$ given by $(w^0, w) := \Big((\tilde{w}^0, \tilde{w}) \circ \sigma\Big)\dfrac{d\sigma}{ds}$, $(\alpha, y^0, y, \beta) := \Big(\tilde{\alpha}, \tilde{y}^0, \tilde{y}, \tilde{\beta}\Big) \circ \sigma$ [5] is a space-time process of (8) (see [29, Sect. 3]). In this case, $(\tilde{S}, \tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{y}^0, \tilde{y}, \tilde{\beta})$ is feasible if, and only if, $(S, w^0, w, \alpha, y^0, y, \beta)$ is feasible, and the associated costs coincide. Let us call *equivalent* any two space-time processes $(\tilde{S}, \tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{y}^0, \tilde{y}, \tilde{\beta})$, $(S, w^0, w, \alpha, y^0, y, \beta)$ as above. The following result is quite straightforward:

[5] Since every L^1-equivalence class contains Borel measurable representatives, we tacitly assume that all L^1-maps we are considering are Borel measurable when necessary.

Lemma 2.1. *A feasible space-time process* $(\bar S,\bar w^0,\bar w,\bar\alpha,\bar y^0,\bar y,\bar\beta)$ *is a local (resp., a global) minimizer for the space-time problem* ($\mathrm{P}_{\text{s-t}}$) *if, and only if, every equivalent space-time process is a local (resp., a global) minimizer, and the costs coincide.*

As a consequence, the extended problem can be regarded as a problem on the quotient space. Therefore, without loss of generality, one can replace a minimizer with its *canonical parameterization,* defined as follows:

Definition 2.5. *We say that* $(S_c,w_c^0,w_c,\alpha_c,y_c^0,y_c,\beta_c)$ *is the* canonical parameterization *of a space-time process* $(S,w^0,w,\alpha,y^0,y,\beta)$ *if*

$$(w_c^0,w_c):=\Big((w^0,w)\circ\sigma^{-1}\Big)\frac{d\sigma^{-1}}{ds},\quad (\alpha_c,y_c^0,y_c,\beta_c):=(\alpha,y^0,y,\beta)\circ\sigma^{-1},$$

where $\sigma(s):=\int_0^s\big(w^0(r)+|w(r)|\big)\,dr$, $s\in[0,S]$, $S_c:=\sigma(S)=y^0(S)+\beta(S)$.

Note that $w_c^0(s)+|w_c(s)|=1$ for a.e. $s\in[0,S_c]$. We introduce the subset of *canonical space-time controls*

$$\mathcal W_c:=\left\{(S,w^0,w,\alpha)\in\mathcal W:\ w^0(s)+|w(s)|=1\quad\text{a.e. } s\in[0,S]\right\},$$

and call *canonical* also the corresponding space-time processes. One can easily verify that a canonical space-time process coincides with its canonical parameterization.

2.3. **The space-time embedding.**

The original control system (6) can be embedded into the space-time system (8). Precisely, by the chain rule, given a strict-sense process (T,u,a,x,v), by setting

$$\sigma(t):=\int_0^t(1+|u(\tau)|)\,d\tau,\quad S:=\sigma(T),\quad y^0:=\sigma^{-1}:[0,S]\to[0,T], \tag{10}$$

one obtains that

$$(S,w^0,w,\alpha,y^0,y,\beta):=\left(S,\frac{dy^0}{ds},(u\circ y^0)\cdot\frac{dy^0}{ds},a\circ y^0,y^0,x\circ y^0,v\circ y^0\right) \tag{11}$$

is a (canonical) space-time process with $w^0>0$ a.e.. Conversely, given a space-time process $(S,w^0,w,\alpha,y^0,y,\beta)$ with $w^0>0$ a.e., the increasing surjective function $y^0:[0,S]\to[0,T]$, has an absolutely continuous inverse $\sigma:[0,T]\to[0,S]$ (see e.g. [20]), and $(T,u,a,x,v):=\big(T,(w\circ\sigma)\frac{d\sigma}{dt},\alpha\circ\sigma,y\circ\sigma,\beta\circ\sigma\big)$ is a strict-sense process. Hence, the family of strict-sense processes can be identified with the subfamily of space-time processes $(S,w^0,w,\alpha,y^0,y,\beta)$ having $w^0>0$ a.e..

The impulsive, space-time extension of the original optimal control problem consists in allowing the control w^0 to vanish in a set of positive measure. The s-intervals where w^0 vanishes represent the 'impulses', namely the s-intervals of instantaneous evolution of both the control and the state (see e.g. [14, 29]).[6]

The notions of strict-sense and space-time local minimizer are consistent, as stated in the following easy consequence of Lemma 2.1 above and [4, Prop. 2.7]:

Lemma 2.2. *A process* $(\bar T,\bar u,\bar a,\bar x,\bar v)$ *is a* strict-sense local minimizer *for problem* (P) *if, and only if,* $(\bar S,\bar w^0,\bar w,\bar\alpha,\bar y^0,\bar y,\bar\beta)$ *defined as in* (10)-(11) *is a space-time local minimizer for* ($\mathrm{P}_{\text{s-t}}$) *among feasible space-time processes with* $w^0>0$ *a.e..*

[6] Let us point out that one can equivalently give a t-based description of this extension using bounded variation trajectories as in [5, 31, 7].

3. A First Order Maximum Principle

Due to the rate-independence of the space-time control system discussed in Subsection 2.2, we can always assume that a local minimizer $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \bar{\beta})$ for ($\mathrm{P_{s\text{-}t}}$) is canonical.

Let us set

$$W := \{(w^0, w) \in \mathbb{R}_+ \times \mathcal{C} : \; w^0 + |w| = 1\}. \tag{12}$$

Let us consider the *unmaximized Hamiltonian* $H : \mathbb{R}^{n+1+n+1+1} \times \mathbb{R}_+ \times \mathcal{C} \times A \to \mathbb{R}$ and the *Hamiltonian* $\mathbf{H} : \mathbb{R}^{n+1+x+1+1} \to \mathbb{R}$, defined by setting

$$H(x, p_0, p, \pi, \lambda, w^0, w, a) := p_0 w^0 + p \cdot \Big(f(x,a) w^0 + \sum_{i=1}^{m} g_i(x) w^i \Big) + \pi |w| - \lambda \ell^e(x, w^0, w, a),$$

$$\mathbf{H}(x, p_0, p, \pi, \lambda) := \max_{(w^0, w, a) \in W \times A} H(x, p_0, p, \pi, \lambda, w^0, w, a).$$

Theorem 3.1 (**First Order Maximum Principle).** *Let $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \bar{\beta})$ be a canonical local minimizer for the space-time problem* ($\mathrm{P_{s\text{-}t}}$). *Then, for every Boltyanski approximating cone Γ of the target $\mathfrak{T}$ at $(\bar{y}^0, \bar{y})(\bar{S})$, there exists a multiplier $(p_0, p, \pi, \lambda) \in \mathbb{R} \times AC\big([0, \bar{S}], \mathbb{R}^n\big) \times \mathbb{R}_- \times \mathbb{R}_+$ verifying:*

(i) (non-triviality)

$$(p_0, p, \lambda) \neq (0,0,0)\,. \tag{13}$$

Furthermore, if $\bar{y}^0(\bar{S}) > 0$, then (13) *can be strengthened to*

$$(p, \lambda) \neq (0,0). \tag{14}$$

(ii) (non-tranversality)

$$(p_0, p(\bar{S}), \pi) \in \left[-\lambda \left(\frac{\partial \Psi}{\partial t}\big((\bar{y}^0, \bar{y})(\bar{S})\big) \,,\, \frac{\partial \Psi}{\partial x}\big((\bar{y}^0, \bar{y})(\bar{S})\big) \right) - \Gamma^{\perp} \right] \times J, \tag{15}$$

where $J := \{0\}$ if $\bar{\beta}(\bar{S}) < K$, and $J := (0, +\infty)$ if $\bar{\beta}(\bar{S}) = K$.[7] *In particular,*

$$\pi = 0 \quad \textit{provided} \quad \bar{\beta}(\bar{S}) < K. \tag{16}$$

(iii) (adjoint equation) *The path p solves, for a.e. $s \in [0, \bar{S}]$,*

$$\frac{dp}{ds}(s) \;=\; -\frac{\partial H}{\partial x}\left(\bar{y}(s), p(s), \pi, \lambda, \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\right). \tag{17}$$

(iv) (First order maximization) *For a.e. $s \in [0, \bar{S}]$,*

$$H\Big(\bar{y}(s), p_0, p(s), \pi, \lambda, \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\Big) = \mathbf{H}\Big(\bar{y}(s), p_0, p(s), \pi, \lambda\Big). \tag{18}$$

(v) (Vanishing of the Hamiltonian)

$$\mathbf{H}\Big(\bar{y}(s), p_0, p(s), \pi, \lambda\Big) = 0, \quad \textit{for all } s \in [0, \bar{S}]. \tag{19}$$

Proof. The Pontryagin Maximum Principle based on Boltyanski approximating cones (see e.g. [41, 38]) yields the existence of a multiplier $(p_0, p, \pi, \lambda) \in \mathbb{R} \times AC\big([0, \bar{S}], \mathbb{R}^n\big) \times \mathbb{R}_- \times \mathbb{R}_+$ verifying the non-transversality condition (15), the adjoint equation (17), the maximum relation (18), the conservation (19), and the non-triviality condition $(p_0, p, \pi, \lambda) \neq 0$. So, it remains to prove the strengthened non-triviality condition (13). This can be done by using the same elementary arguments as in the proof of [30, Theorem 3.1].[8] □

[7] It is tacitly meant that, as an approximating cone to the (T, x, v)-target $\mathfrak{T} \times [0, K]$ at $(\bar{y}^0, \bar{y}, \bar{\beta})(\bar{S})$, one chooses $\Gamma \times \mathbb{R}$ if $\bar{\beta}(\bar{S}) < K$ and $\Gamma \times (-\infty, 0]$ if $\bar{\beta}(\bar{S}) = K$. In particular, $(\Gamma \times \mathbb{R})^{\perp} = \Gamma^{\perp} \times \{0\}$ if $\bar{\beta}(\bar{S}) < K$ and $(\Gamma \times (-\infty, 0])^{\perp} = \Gamma^{\perp} \times \mathbb{R}_+$ when $\bar{\beta}(\bar{S}) = K$.

[8] The fact that in [30] one makes use of the limiting normal cone instead of the polar of the Boltyanski cone plays no role in the proof of this result.

Definition 3.1. *A process* $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \bar{\beta})$ *is called an* extremal *if it obeys the conditions in Theorem 3.1 for some multiplier* (p_0, p, π, λ). *If there is a choice of the multiplier with* $\lambda = 0$, *then the extremal* $(\bar{S}, \bar{y}^0, \bar{y}, \bar{\beta}, \bar{w}^0, \bar{w}, \bar{\alpha})$ *is called* abnormal, *otherwise it is called* normal. *Finally, the extremal is said to be* strictly abnormal *if every choice of the multiplier* (p_0, p, π, λ) *verifies* $\lambda = 0$.

When $\ell_1(x, \cdot)$ is positively 1-homogeneous, so that for any $(x, w^0, w, a) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{C} \times A$ one has $\ell^e(x, w^0, w, a) = \ell_0(x, a)w^0 + \ell_1(x, w)$, let us define the *drift Hamiltonian* $\mathbf{H}^{(\mathrm{dr})}$ and the *impulse Hamiltonian* $\mathbf{H}^{(\mathrm{imp})}$:

$$\mathbf{H}^{(\mathrm{dr})}\Big(x, p_0, p, \lambda\Big) := \max_{a\in A}\Big\{p_0 + p\cdot f(x,a) - \lambda\ell_0(x,a)\Big\},$$
$$\mathbf{H}^{(\mathrm{imp})}\Big(x, p, \pi, \lambda\Big) := \max_{w\in\mathcal{C}, |w|=1}\left\{p\cdot\sum_{i=1}^m g_i(x)w^i + \pi - \lambda\ell_1(x,w)\right\}.$$

Corollary 3.2. *Let* $\ell_1(x, \cdot)$ *be positively* 1*-homogeneous and let* $(\bar{S}, \bar{w}^0, \bar{w}, \bar{\alpha}, \bar{y}^0, \bar{y}, \bar{\beta})$ *be a canonical extremal obeying the conditions in Theorem 3.1 for some multiplier* (p_0, p, π, λ). *Then there exists a zero-measure subset* $\mathcal{N} \subset [0, \bar{S}]$ *such that, for every* $s \in [0, \bar{S}] \setminus \mathcal{N}$, *one has*

$$H\Big(\bar{y}(s), p_0, p(s), \pi, \lambda, \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\Big) = \mathbf{H}\Big(\bar{y}(s), p_0, p(s), \pi, \lambda\Big)$$
$$= \max\Big\{\mathbf{H}^{(\mathrm{dr})}(\bar{y}(s), p_0, p(s), \lambda)\,,\; \mathbf{H}^{(\mathrm{imp})}(\bar{y}(s), p(s), \pi, \lambda)\Big\} = 0, \tag{20}$$

$$\bar{w}^0(s)\Big[p_0 + p(s)\cdot f(\bar{y}(s), \bar{\alpha}(s)) - \lambda\ell_0(\bar{y}(s), \bar{\alpha}(s))\Big] = 0, \tag{21}$$

$$p(s)\cdot\sum_{i=1}^m g_i(\bar{y}(s))\bar{w}^i(s) + \pi|\bar{w}(s)| - \lambda\ell_1(\bar{y}(s), \bar{w}(s)) = 0. \tag{22}$$

In particular, if for some $s \in [0, \bar{S}] \setminus \mathcal{N}$ *one has*

i) $\mathbf{H}^{(\mathrm{dr})}(\bar{y}(s), p_0, p(s), \lambda) < 0$, *then* $\bar{w}^0(s) = 0$ *and*

$$p(s)\cdot\sum_{i=1}^m g_i(\bar{y}(s))\bar{w}^i(s) + \pi - \lambda\ell_1(\bar{y}(s), \bar{w}(s)) = \mathbf{H}^{(\mathrm{imp})}(\bar{y}(s), p_0, p(s), \pi, \lambda) = 0;$$

ii) $\mathbf{H}^{(\mathrm{imp})}(\bar{y}(s), p(s), \pi, \lambda) < 0$, *then* $\bar{w}(s) = 0$ *and*

$$p_0 + p(s)\cdot f(\bar{y}(s), \bar{\alpha}(s)) - \lambda\ell_0(\bar{y}(s), \bar{\alpha}(s)) = \mathbf{H}^{(\mathrm{dr})}(\bar{y}(s), p_0, p(s), \lambda) = 0.$$

Proof. By (19) it follows that for every $s \in [0, \bar{S}]$, one has

$$p_0w^0 + p(s)\cdot\Big(f(\bar{y}(s), a)w^0 + \sum_{i=1}^m g_i(\bar{y}(s))w^i\Big) + \pi|w| - \lambda\Big(\ell_0(\bar{y}(s), a)w^0 + \ell_1(\bar{y}(s), w)\Big) \le 0$$

for all $(w^0, w, a) \in W \times A$. Now, by choosing $w = 0$ one gets that $w^0 = 1$ and

$$p_0 + p(s)\cdot f(\bar{y}(s), a) - \lambda\ell_0(\bar{y}(s), a) \le 0, \quad \text{for all } a \in A,$$

while taking $w^0 = 0$ one obtains

$$p(s)\cdot\sum_{i=1}^m g_i(\bar{y}(s))w^i + \pi - \lambda\ell_1(\bar{y}(s), w) \le 0, \quad \text{for all } (w, a) \in \mathcal{C}\times A,\ |w| = 1.$$

Therefore, $\mathbf{H}^{(\mathrm{dr})}(\bar{y}(s), p(s), \pi, \lambda) \le 0$ and $\mathbf{H}^{(\mathrm{imp})}(\bar{y}(s), p(s), \pi, \lambda) \le 0$. In fact, it must be that $\max\big\{\mathbf{H}^{(\mathrm{dr})}(\bar{y}(s), p_0, p(s), \lambda)\,,\, \mathbf{H}^{(\mathrm{imp})}(\bar{y}(s), p(s), \pi, \lambda)\big\} = 0$, since, otherwise, both Hamiltonians would be negative, which contradicts (19). By taking $\mathcal{N} \subset [0, \bar{S}]$ to be the zero-measure subset such that the first order maximization (18) is verified in $[0,\bar S]\setminus\mathcal{N}$, we get (20). If $s\in[0,\bar S]\setminus\mathcal{N}$, by (18), (19) one has that

$$\bar w^0(s)\Big[p_0+p(s)\cdot f(\bar y(s),\bar\alpha(s))-\lambda\ell_0(\bar y(s),\bar\alpha(s))\Big] \\ +\Big[p(s)\cdot\sum_{i=1}^m g_i(\bar y(s))\bar w^i(s))+\pi|\bar w(s)|-\lambda\ell_1(\bar y(s),\bar w(s))\Big]=0.$$

Since the above argument implies that both terms in this equality are nonpositive, they necessarily vanish, namely (21) and (22) are verified.

To prove i), suppose $\mathbf{H}^{(\mathrm{dr})}(\bar y(s),p(s),\pi,\lambda)<0$. Then (21) implies $\bar w^0(s)=0$, so that $|\bar w(s)|=1$ and the thesis i) follows by (22). Finally, to prove ii) assume that $\mathbf{H}^{(\mathrm{imp})}(\bar y(s),p(s),\pi,\lambda)<0$, then $\bar w(s)=0$ due to (22) and in view of the positive 1-homogeneity of H w.r.t. (w^0,w). Hence $\bar w^0(s)=1$ and (21) yields ii). □

Remark 3.1. *Under the same hypotheses of Cor. 3.2,* $\mathbf{H}^{(\mathrm{dr})}(\bar y(s),p_0,p(s),\lambda)=0$ *for all* $s\in[s_1,s_2]$ *as soon as* $s_1,s_2\in[0,\bar S]$ *are such that* $\bar w^0(s)>0$ *for a.e.* $s\in[s_1,s_2]\subseteq[0,\bar S]$.

Corollary 3.3. *Let* $(\bar S,\bar w^0,\bar w,\bar\alpha,\bar y^0,\bar y,\bar\beta)$ *be a canonical extremal for the space-time problem* $(\mathrm{P}_{\text{s-t}})$ *and let* (p_0,p,π,λ) *be a corresponding multiplier. If*

$$\pi=0 \quad \textit{and} \quad \lambda\ell^e(\bar y(s),0,\pm\mathbf{e}_i,a)=0,\quad \textit{for all } s\in[0,\bar S],\ i=1,\dots,m_1,^{9} \tag{23}$$

then $p(s)\cdot g_i(\bar y(s))=0$ *for all* $s\in[0,\bar S]$, $i=1,\dots,m_1$.

Proof. By (19) it follows that for every $s\in[0,\bar S]$ and $(w^0,w,a)\in W\times A$, one has $p_0w^0+p(s)\cdot\Big(f(\bar y(s),a)w^0+\sum_{i=1}^m g_i(\bar y(s))w^i\Big)-\lambda\ell^e(\bar y(s),w^0,w,a)\le 0$. Therefore, choosing $w^0=0$ and $w=\pm\mathbf{e}_i$ for any $i=1,\dots,m_1$, one gets the thesis. □

Remark 3.2. From Theorem 3.1, one has $\pi=0$ as soon as $\bar\beta(\bar S)<K$. Moreover, the hypothesis $\lambda\ell^e(\bar y(s),0,\pm\mathbf{e}_i,a)=0$ is obviously satisfied when the extremal $(\bar S,\bar w^0,\bar w,\bar\alpha,\bar y^0,\bar y,\bar\beta)$ is abnormal and one chooses $\lambda=0$, or if $\hat\ell_1(x,0,w)=0$ for all $(x,w)\in\mathbb{R}^n\times(\mathbb{R}^{m_1}\times\{0\}^{m_2})$. (This includes, in particular, the case $\ell_1\equiv 0$, as in the minimum time problem, where $\ell_0\equiv 1$.)

4. A Higher Order Maximum Principle

Let us begin with a regularity notion for Lie brackets of the vector fields $g_1,\dots,g_{m_1}$.

Definition 4.1. *For every integer* $k\ge 0$*, we say that a vector field* B *is a* C^k-admissible Lie bracket *if* $B=b(F_1,\dots,F_q)$*, where* b *is a formal bracket and* $(F_1,\dots,F_q)$ *is a* q*-tuple of class* C^{b+k} *of vector fields in* $\{g_1,\dots,g_{m_1}\}$ *(see Def.1.5). We will use* $\mathfrak{B}^k$ *to denote the set of* C^k*-admissible Lie brackets of length* ≥ 2.

4.1. **Higher order conditions.**

Theorem 4.1 (**Higher Order Maximum Principle**). *Assume that hypothesis* **(Hp)** *is satisfied with* $\hat\ell_1(\cdot,0,\cdot)\equiv 0$*. Let* $(\bar S,\bar w^0,\bar w,\bar\alpha,\bar y^0,\bar y,\bar\beta)$ *be a canonical local minimizer for the space-time problem* $(\mathrm{P}_{\text{s-t}})$ *that verifies* $\bar\beta(\bar S)<K$*. Then, for every Boltyanski approximating cone* Γ *of the target* $\mathfrak{T}$ *at* $(\bar y^0,\bar y)(\bar S)$*, there exists a multiplier* $(p_0,p,\pi,\lambda)\in\mathbb{R}\times AC\big([0,\bar S],\mathbb{R}^n\big)\times\mathbb{R}_-\times\mathbb{R}_+$ *with* $\pi=0$ *that satisfies all the conditions of Theorem 3.1 and, moreover, verifies*

$$p(s)\cdot g_i(\bar y(s))=0,\qquad \textit{for all } s\in[0,\bar S],\ i=1,\dots,m_1, \tag{24}$$

$$p(s)\cdot B(\bar y(s))=0,\qquad \textit{for all } s\in[0,\bar S],\ B\in\mathfrak{B}^0. \tag{25}$$

The proof of this theorem is postponed to Section 5.

[9] By the definition of ℓ^e, it is clear that the quantities $\ell^e(\bar y(s),0,\pm\mathbf{e}_i,a)$ do not depend on a.

Remark 4.1. *Requiring the condition $\hat{\ell}_1(\cdot,0,\cdot)\equiv 0$ is crucial for the general validity of Theorem 4.1. Otherwise, the variations corresponding to brackets of length $h\geq 2$ would produce a perturbation of order $\varepsilon^{\frac{1}{h}}$ of the cost variable, so having infinite derivative w.r.t. ε. Since the same variation would produce a change of order ε in the dynamical variables, the separation Theorem 1.1 turns out to be not applicable. However, as soon as the minimizer is strictly abnormal, one might be able to deduce some results involving Lie brackets also for the case $\hat{\ell}_1(\cdot,0,\cdot)\neq 0$ as well, possibly via some higher-order open mapping argument. This would be similar to what happens in the case of sub-Riemannian geometry* [1]. *We leave this issue as an open question.*

Remark 4.2. *Since we obtained the higher order necessary conditions under the only prerequisite that the involved Lie brackets are continuous, one might wonder to which extent such a regularity hypothesis can be further weakened. For instance, one might prove an extension of Theorem 4 by means of* set-valued Lie brackets *of non smooth vector fields, as studied in* [35, 36, 21].

In the sequel we will use the notation $f_a(\cdot):=f(\cdot,a)$.

Corollary 4.2. *Assume that hypothesis* **(Hp)** *is satisfied with $\hat{\ell}_1(\cdot,0,\cdot)\equiv 0$, and let $(\bar{S},\bar{w}^0,\bar{w},\bar{\alpha},\bar{y}^0,\bar{y},\bar{\beta})$ be a canonical local minimizer of* ($\mathrm{P}_{\text{s-t}}$) *that verifies $\bar{\beta}(\bar{S})<K$. Given a Boltyanski approximating cone Γ of the target $\mathfrak{T}$ at $(\bar{y}^0,\bar{y})(\bar{S})$, let (p_0,p,λ) be a multiplier as in Theorem 4.1. Then, for any Lie bracket $B\in\mathfrak{B}^1\cup\{g_1,\dots,g_m\}$, one has* [10]

$$p(s)\cdot\Big(\big[f_{\bar{\alpha}(s)},B\big](\bar{y}(s))\bar{w}^0(s)+\sum_{j=m_1+1}^{m}\big[g_j,B\big](\bar{y}(s))\bar{w}^j(s)\Big)\\ =-\lambda\frac{\partial\ell^e}{\partial x}(\bar{y}(s),\bar{w}^0(s),\bar{w}(s),\bar{\alpha}(s))\cdot B(\bar{y}(s)), \tag{26}$$

for a.e. $s\in[0,S]$. In particular, if $m_1=m$ and the condition

$$\lambda\frac{\partial\ell^e}{\partial x}(\bar{y}(s),\bar{w}^0(s),\bar{w}(s),\bar{\alpha}(s))\cdot B(\bar{y}(s))=0\quad \textit{for a.e. } s\in[0,\bar{S}] \tag{27}$$

is satisfied, one obtains

$$p(s)\cdot\Big(\big[f_{\bar{\alpha}(s)},B\big](\bar{y}(s))\Big)\bar{w}^0(s)=0\quad\textit{for a.e. } s\in[0,\bar{S}]. \tag{28}$$

Proof. Condition (26) can be obtained by differentiating (24) or (25) and remembering that the derivative of p verifies the adjoint equation (17). □

Remark 4.3. Condition (27) is satisfied for all $s\in[0,\bar{S}]$ in at least two important situations, namely in the *abnormal case*, i.e. if $\lambda=0$, or when $\ell=\ell_0+\ell_1(u)$, with ℓ_0, ℓ_1 independent of x and $\hat{\ell}_1(0,w)\equiv 0$ (for instance, in the minimum time problem).

Remark 4.4 (Linear Systems)**.** Let us consider the linear system

$$\frac{dx}{dt}=Cx+Eu,\qquad u\in\mathbb{R}^m,$$

where C, E are $n\times n$ and $n\times m$ real matrices, respectively. For the vector fields $f(x)=:Cx$, and g_i, where $g_{ij}:=E_{ji}$ for each $i=1,\dots,m$, $j=1,\dots,n$, the conditions involving Lie brackets of the g_i become trivial, since $[g_i,g_j]=0$. However, because of the linearity of $f(x)=Cx$, further higher order conditions can be trivially deduced under assumption (27). Indeed, condition (24) reduces to

$$p(s)\cdot E=0,\quad \text{for all } s\in[0,\bar{S}], \tag{29}$$

while, due to (27), the adjoint equation now reads $\frac{dp}{dt}=-p\cdot C$. Therefore by differentiating (29) $n-1$ times, we get the *additional necessary conditions* $p\cdot[f,g_i]=p\cdot[f,[f,g_i]]=$

[10] *I.e.*, B is a C^1-admissible Lie bracket (possibly of length 1), see Definition 4.1.

$p \cdot [f, [f, [\dots, [f, g_i] \dots]]] = 0$, for all $i = 1, \dots, m$, which correspond to the $n-1$ matrix relations

$$p \cdot CE = 0, \quad p \cdot C^2E = 0, \; \dots, \; p \cdot C^{n-1}E = 0. \tag{30}$$

Remark 4.5. *As observed in the Introduction, some motivations for studying impulsive systems are to be found in Classical Mechanics. This is a reason why one might be interested in extending previous results to manifolds. Actually, such an extension does not present any special difficulty, in that the thesis of Theorem 4.1 has a* chart-independent character.

4.2. **Fully impulsive processes.** The necessary conditions established in Theorems 3.1 and 4.1 can be used to get information on the structure of optimal trajectories: for instance, one can wonder under which conditions an optimal trajectory is a finite concatenation of impulsive and non impulsive paths (as it occurs e.g. in the example in [4]). Though an accurate investigation in this direction goes beyond the objectives of this paper, let us highlight some rank conditions that happen to force an optimal process $(\bar S, \bar w^0, \bar w, \bar\alpha, \bar y^0, \bar y, \bar\beta)$ to be *fully impulsive*. By this we mean that it evolves in zero time, namely $\bar y^0(\bar S) = 0$, or, equivalently, $\bar w^0 = 0$ a.e. on $[0, \bar S]$.

To state our result, we introduce two rank-type assumptions:

(I) C^0-Pointwise Rank Conditions *at* $x \in \mathbb{R}^n$.

$(\mathbf{I}.1)_x$ *there exists an integer* $r \geq 0$ *and iterated Lie brackets* $B_1, \dots, B_r \in \mathfrak{B}^0$ *such that*

$$\operatorname{span}\{B_1, \dots, B_r, g_1, \dots, g_{m_1}\}(x) = \mathbb{R}^n; \;[11] \tag{31}$$

$(\mathbf{I}.2)_x$ *for every* $a \in A$, *there exist integers* $r \geq 0$, $k \geq 0$, *and iterated Lie brackets* $B_1, \dots, B_r \in \mathfrak{B}^0$, $\hat B_1, \dots, \hat B_k \in \mathfrak{B}^1$, *such that*

$$\operatorname{span}\big\{B_1, ..., B_r, [f_a, \hat B_1], ..., [f_a, \hat B_k], g_1, ..., g_{m_1}, [f_a, g_1], ..., [f_a, g_{m_1}]\big\}(x) = \mathbb{R}^n. \tag{32}$$

(II) Kalman Controllability Condition. *The system is linear and the Kalman Controllability Condition is verified, namely*

$$\frac{dx}{dt} = Cx + Eu, \quad \textit{and} \quad \operatorname{rank}(E \;\; CE \;\; C^2E \;\; \dots \;\; C^{n-1}E) = n,$$

where C, E *are* $n \times n$ *and* $n \times m$ *real matrices, respectively.*

We will consider the following assumption:

(Hp1) *Hypothesis* **(Hp)** *holds and, moreover,* (i) *the target is time-invariant, namely* $\mathfrak{T} = \mathbb{R} \times \hat{\mathfrak{T}}$, *with* $\hat{\mathfrak{T}} \subseteq \mathbb{R}^n$; (ii) *the final cost* Ψ *is time-independent;* (iii) *the Lagrangian* ℓ *is strictly positive and* $\hat\ell(\cdot, 0, \cdot) \equiv 0$.

Theorem 4.3. *Let us assume hypothesis* **(Hp1)**. *Let* $(\bar S, \bar w^0, \bar w, \bar\alpha, \bar y^0, \bar y, \bar\beta)$ *be a canonical local minimizer for* ($\mathrm{P_{s\text{-}t}}$) *such that* $\beta(\bar S) < K$, *and let* (p_0, p, λ) *be a multiplier as in Theorem 4.1. If* one *of the options* (a)–(c) *below is verified, then the process* $(\bar S, \bar w^0, \bar w, \bar\alpha, \bar y^0, \bar y, \bar\beta)$ *is fully impulsive.*

(a) *For every* $s \in [0, \bar S]$, *the* C^0*-Pointwise Rank Condition* $(\mathbf{I}.1)_{\bar y(s)}$ *is verified.*

(b) *For every* $s \in [0, \bar S]$, *the* C^0*-Pointwise Rank Condition* $(\mathbf{I}.2)_{\bar y(s)}$ *is verified, while* $J := \big\{s \in [0, \bar S] : (\mathbf{I}.1)_{\bar y(s)}$ *is not verified*$\big\} \neq \emptyset$. *Furthermore,* $m_1 = m$, *and* $\lambda \frac{\partial \ell^e}{\partial x}(\bar y(s), \bar w^0(s), \bar w(s), \bar\alpha(s)) = 0$ *for a.e.* $s \in J$.

(c) *The system is linear, the Kalman Controllability Condition* (**II**) *is verified, and* $\lambda \frac{\partial \ell^e}{\partial x}(\bar y(s), \bar w^0(s), \bar w(s), \bar\alpha(s)) = 0$ *for a.e.* $s \in [0, \bar S]$.

Preliminarily, let us prove the following result:

[11] We mean that $\{\zeta_1, \dots, \zeta_N\} = \emptyset$ as soon as $N = 0$.

Lemma 4.4. *Assume* (i) *and* (ii) *in hypothesis* **(Hp1)**, *and let* $\pi = 0$. *Then for any subset* $\mathcal{J} \subseteq [0, T]$ *of positive measure one has neither*

$$p(s) = 0 \quad \text{and} \quad \ell^e\big(\bar{y}(s), (\bar{y}(s), \bar{w}^0(s), \bar{w}(s), \alpha(s)\big) > 0, \quad \text{for a.e. } s \in \mathcal{J} \tag{33}$$

nor

$$p(s) = 0 \quad \text{and} \quad \frac{\partial \ell^e}{\partial x}\big(\bar{y}(s), \bar{w}^0(s), \bar{w}(s), \alpha(s)\big) \neq 0, \quad \text{for a.e. } s \in \mathcal{J}. \tag{34}$$

Proof. By hypothesis **(Hp1)** (i), $\Gamma = \mathbb{R} \times \hat{\Gamma}$, with $\hat{\Gamma}$ a cone of $\mathbb{R}^n$. Because of **(Hp1)** (ii) and of the identity $\Gamma^\perp = \{0\} \times \hat{\Gamma}^\perp$, the non-transversality condition yields $p_0 = -\lambda \frac{\partial \Psi}{\partial t}\big(\bar{y}^0(\bar{S}), \bar{y}(\bar{S})\big) + 0 = 0$.

First, let us assume by contradiction that (33) is verified on a subset $\mathcal{J} \subseteq [0, \bar{S}]$ of positive measure. Since $(p_0, p(s), \pi) = (0, 0, 0)$ for all $s \in \mathcal{J}$, by (19) we obtain that $\lambda \ell^e\big(\bar{y}(s), \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\big) = 0$ for a.e. $s \in \mathcal{J}$, which by (33) implies that $\lambda = 0$.

Secondly, assume that (34) is verified on a subset $\mathcal{J} \subseteq [0, \bar{S}]$ of positive measure. We still have $(p_0, p(s), \pi) = (0, 0, 0)$ on $\mathcal{J}$ and, by the adjoint equation, we deduce $\lambda \frac{\partial \ell^e}{\partial x}\big(\bar{y}(s), \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\big) = 0$ for a.e. $s \in \mathcal{J}$ so that by (34) one gets again $\lambda = 0$.

Choose a point $\hat{s} \in \mathcal{J}$, so that $p(\hat{s}) = 0$. Since in both cases one has $\lambda = 0$, the adjoint equation is linear in p, which in turn implies that $p \equiv 0$ on $[0, \bar{S}]$. Therefore, $(p_0, p, \pi, \lambda) = 0$, which contradicts the non-triviality condition. $\square$

Proof of Theorem 4.3. Observe that, since $\beta(\bar{S}) < K$, one has $\pi = 0$.

Suppose first that hypothesis (a) is verified. For every $s \in [0, \bar{S}]$, by $(\mathbf{I}.1)_{\bar{y}(s)}$ there exist an integer $r \geq 0$ and Lie brackets $B_1, \dots, B_r \in \mathfrak{B}^0$ verifying the rank condition (31) and, in view of (24), (25), for all $s \in [0, \bar{S}]$, one has

$$p(s) \cdot g_i(\bar{y}(s)) = 0, \quad p(s) \cdot B_j(\bar{y}(s)) = 0$$

for all $i = 1, \dots, m_1$, $j = 1, \dots, r$. Therefore, we obtain $p(s) = 0$ for all $s \in [0, \bar{S}]$. Assume by contradiction that there exists a subset of positive measure $\mathcal{J} \subseteq [0, \bar{S}]$ such that $\bar{w}^0(s) > 0$ for a.e. $s \in \mathcal{J}$. By the positivity of the function ℓ, this implies that $\ell^e\big(\bar{y}^0(s), \bar{y}(s), \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\big) > 0$ for a.e. $s \in \mathcal{J}$, which in turn is ruled out by Lemma 4.4 above.

Assume now that (b) holds true. If $s \in [0, \bar{S}] \backslash J$, we get $p(s) = 0$ arguing as in the previous case. If J has zero-measure, this also implies that $p(s) = 0$ for all $s \in [0, \bar{S}]$. On the contrary, assume that J has positive measure. For almost every $s \in J$ and for $a := \bar{\alpha}(s)$, by $(\mathbf{I}.2)_{\bar{y}(s)}$ there exist integers $r, k \geq 0$ and Lie brackets $B_1, \dots, B_r \in \mathfrak{B}^0$, $\hat{B}_1, \dots, \hat{B}_k \in \mathfrak{B}^1$ verifying the rank condition (32). Moreover, for almost every $s \in J$, by (24), (25), and (28) one has

$$p(s) \cdot g_i(\bar{y}(s)) = 0, \quad p(s) \cdot B_j(\bar{y}(s)) = 0,$$
$$p(s) \cdot [f_{\bar{\alpha}(s)}, g_i](\bar{y}(s)) = 0, \quad p(s) \cdot [f_{\bar{\alpha}(s)}, \hat{B}_l](\bar{y}(s)) = 0,$$

for all $i = 1, \dots, m$, $j = 1, \dots, r$, $l = 1, \dots, k$. We then deduce that $p(s) = 0$ for almost every $s \in J$. Summing up the above occurrences, by the continuity of p we get $p(s) = 0$ for every $s \in [0, \bar{S}]$. Now assume by contradiction that there exists a subset $\mathcal{J} \subseteq [0, \bar{S}]$ of positive measure such that $\bar{w}^0(s) > 0$ for a.e. $s \in \mathcal{J}$. At this point, the thesis follows arguing exactly as in case (a).

Finally, suppose that (c) holds true. The linear relations (29), (30) imply $p \equiv 0$, so, in view of the hypothesis $\ell > 0$ one concludes as in cases (b) and (c). $\square$

Remark 4.6. As mentioned in the introduction, our conditions might be regarded as a generalization to impulsive trajectories of [18], where one assumes further that $\mathcal{C} = \mathbb{R}^m$, the vector fields $g_1, \dots, g_m$ are of class C^∞, and their Lie algebra has constant dimension. In [18] one considers the (non extended) minimum time problem with L^∞ controls taking values in an unbounded set. Now, since the dynamics is control-affine, an optimal control might fail to exist in this class or even in the class of L^1 functions. On the other hand, if such an optimal control existed, the corresponding cost might or might not coincide with the infimum value of the extended, impulsive problem. Actually, in [18] one assumes that the optimal process is a *normal extremal*, and this is similar to a sufficient condition established in [30] for the avoidance of infimum-gaps. One might conjecture that, for some reason,[12] a higher-order Maximum Principle valid for the impulsive system can be a necessary condition for the non-impulsive unbounded system as well.

5. Proof of Theorem 4.1

Let $(\bar S, \bar w^0, \bar w, \bar\alpha, \bar y^0, \bar y, \bar\beta)$ be a canonical local minimizer of ($\mathrm{P_{s\text{-}t}}$) verifying $\bar\beta(\bar S) < K$, that we will call the *reference process*. Throughout this section $\hat\ell_1(\cdot, 0, \cdot) \equiv 0$, as required in the statement of Thm. 4.1. Moreover, we set

$$F^e(x, w^0, w, a) := f(x, a)w^0 + \sum_{i=1}^m g_i(x)w^i, \quad \text{for all } (x, w^0, w, a) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{C} \times A,$$
$$\bar F^e(s) := F_e(\bar y, \bar w^0, \bar w, \bar\alpha)(s), \quad \bar\ell^e(s) := \ell^e(\bar y, \bar w^0, \bar w, \bar\alpha)(s), \quad \text{for a.e. } s \in [0, \bar S].$$

The proof will be divided in several steps. First, following a time-rescaling procedure, we transform problem ($\mathrm{P_{s\text{-}t}}$) into a problem on the fixed interval $[0, \bar S]$. At this point, we define two classes of variations, comprising standard *needle variations* and *bracket-like variations*, the latter being produced by suitable *instantaneous* perturbations of the reference process. By using appropriate powers of the perturbation parameter ε, all these variations turn out to be of the same order ε. Once this is done, the proof proceeds by some set-separation arguments.

5.1. Rescaling the problem.

Definition 5.1. *Fix $\rho > 0$. For any $(S, w^0, w, \alpha, \zeta) \in \mathcal{W} \times L^\infty([0, \bar S], [-\rho, \rho])$, we say that $(S, w^0, w, \alpha, \zeta, y^0, y, y^\ell, \beta)$ is a* rescaled (space-time) process *if (y^0, y, y^ℓ, β) is the unique Carathéodory solution of*

$$\begin{cases} \dfrac{dy^0}{ds} = w^0(1 + \zeta), \\ \dfrac{dy}{ds} = F^e(y, w^0, w, \alpha)(1 + \zeta), \\ \dfrac{dy^\ell}{ds} = \ell^e(y, w^0, w, \alpha)(1 + \zeta), & \qquad a.e.\ s \in [0, \bar S], \\ \dfrac{d\beta}{ds} = |w|(1 + \zeta), \\ (y^0, y, y^\ell, \beta)(0) = (0, \check x, 0, 0), \end{cases} \tag{35}$$

and $(S, w^0, w, \alpha, y^0, y, y^\ell, \beta)$ is called feasible *if $(y^0(S), y(S), \beta(S)) \in \mathfrak{T} \times [0, K]$.*

We define the *rescaled* space-time optimization problem as

$$(\mathrm{P_e}) \qquad \begin{cases} \text{minimize } \Big\{\Psi((y^0, y)(\bar S)) + y^\ell(\bar S)\Big\}, \\ \text{over feasible rescaled processes } (\bar S, w^0, w, \alpha, \zeta, y^0, y, y^\ell, \beta). \end{cases}$$

It is easy to see that, for $\rho > 0$ sufficiently small, the reference process, regarded as a process $(\bar S, \bar w^0, \bar w, \bar\alpha, 0, \bar y^0, \bar y, \bar y^\ell, \bar\beta)$ of (35), is a local minimizer for ($\mathrm{P_e}$), which is a fixed end-time problem.[13] Since the proof involves only space-time trajectories which are close

[12] E.g. because of the *abundantness* (see [42]) of the absolutely continuous trajectories in the set of extended, impulsive trajectories.

[13] *I.e.,* there exists $\delta > 0$ such that $\Psi((\bar y^0, \bar y)(\bar S)) + \bar y^\ell(\bar S) \le \Psi((y^0, y)(\bar S)) + y^\ell(\bar S)$ for all feasible processes $(\bar S, w^0, w, \alpha, \zeta, y^0, y, y^\ell, \beta)$ satisfying $\mathrm{d}\big((\bar S, y^0, y, y^\ell, \beta), (\bar S, \bar y^0, \bar y, \bar y^\ell, \bar\beta)\big) < \delta$.

to the reference space-time trajectory $(\bar y^0, \bar y)$, using standard truncation and mollification arguments, we can assume the following hypothesis:

(Hp)* *all the assumptions in* **(Hp)** *are verified and, moreover,* ℓ^e, f, *the* g_i, *their partial derivatives* $\dfrac{\partial \ell^e}{\partial x^j}$, $\dfrac{\partial f}{\partial x^j}$, $\dfrac{\partial g_i}{\partial x^j}$ *and all the iterated brackets* $B \in \mathfrak{B}^0$ *(as defined in Definition 4.1) are uniformly continuous and bounded.*

Hypothesis **(Hp)*** guarantees that for any $(w^0, w, \alpha, \zeta) \in L^\infty([0, \bar S], W \times A \times [-\rho, \rho])$ there exists a unique solution (y^0, y, y^ℓ, β) to (35), defined on the whole interval $[0, \bar S]$. Moreover, the input-output map

$$\Phi : L^\infty\big([0, \bar S], W \times A \times [-\rho, \rho]\big) \to C^0([0, \bar S], \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}), \tag{36}$$

which associates to any control the corresponding solution to (35), turns out to be Lipschitz continuous when one considers the sup-norm over the set of trajectories, and the distance $\tilde d\big((w^0, w, \alpha, \zeta), (\tilde w^0, \tilde w, \tilde \alpha, \tilde \zeta)\big) := \text{meas}\big\{(w^0, w, \alpha, \zeta)(s) \neq (\tilde w^0, \tilde w, \tilde \alpha, \tilde \zeta)(s) : \ s \in [0, \bar S]\big\}$ for every pair (w^0, w, α, ζ), $(\tilde w^0, \tilde w, \tilde \alpha, \tilde \zeta)$ of controls.

5.2. **Needle and bracket-like approximations.**

Definition 5.2 (Variation generator)**.** *Let us define the* set of variation generators *as*

$$\mathfrak{V} := (W \times A \times [-\rho, \rho]) \bigcup \mathfrak{B}^0.$$ [14]

More specifically, any $\mathbf{c} = (w^0, w, a, \zeta) \in W \times A \times [-\rho, \rho]$ *will be called a* needle variation generator*, or a* variation generator of length 1*, while any bracket* $\mathbf{c} = B \in \mathfrak{B}^0$ *of length* h (≥ 2) *will be called a* bracket-like variation generator of length h*.*

To every variation generator $\mathbf{c}$ and to each instant $\bar s \in (0, \bar S)$, we now associate an infinitesimal space-time variation of the reference trajectory $(\bar y^0, \bar y, \bar y^\ell, \bar \beta)$, whose y-component coincides with either a standard needle variation or a Lie bracket. As usual, the needle variations will be considered at *Lebesgue points* of an appropriate associated function as given in next definition.[15]

Definition 5.3. *We will use* $(0, \bar S)_{\text{Leb}}$ *to denote the full measure subset of* $(0, \bar S)$ *consisting of the Lebesgue points of the function* $s \mapsto \big(\bar w^0(s), \bar F^e(s), \bar\ell^e(s), |\bar w|(s)\big)$, $s \in [0, \bar S]$.

Definition 5.4. (Needle variation)*. For every* $\bar s \in (0, \bar S)_{\text{Leb}}$ *and every needle variation generator* $\mathbf{c} = (w^0, w, a, \zeta)$*, consider the vector*

$$\begin{pmatrix} \mathbf{v}^0_{\mathbf{c},\bar s} \\ \mathbf{v}_{\mathbf{c},\bar s} \\ \mathbf{v}^\ell_{\mathbf{c},\bar s} \\ \mathbf{v}^v_{\mathbf{c},\bar s} \end{pmatrix} := \begin{pmatrix} w^0(1+\zeta) - \bar w^0(\bar s) \\ F^e(\bar y(\bar s), w^0, w, a)(1+\zeta) - \bar F^e(\bar s) \\ \ell^e(\bar y(\bar s), w^0, w, a)(1+\zeta) - \bar \ell^e(\bar s) \\ |w|(1+\zeta) - |\bar w(\bar s)| \end{pmatrix}. \tag{37}$$

(Bracket-like variation)*. For every* $\bar s \in (0, \bar S)$ *and every bracket-like variation generator* $\mathbf{c} = B \in \mathfrak{B}^0$*, one sets*

$$\begin{pmatrix} \mathbf{v}^0_{\mathbf{c},\bar s} \\ \mathbf{v}_{\mathbf{c},\bar s} \\ \mathbf{v}^\ell_{\mathbf{c},\bar s} \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{B(\bar y(\bar s))}{r^h_B} \\ 0 \end{pmatrix}, \tag{38}$$

where r_B *is defined as in Subsection 1.1.*

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[14]We recall that $W = \{(w^0, w) \in \mathcal{R}_+ \times \mathcal{C} : \ w^0 + |w| = 1\}$ and $\mathfrak{B}^0$ is the set of C^0-admissible iterated Lie brackets of length ≥ 2, as in Def. 4.1.

[15]Given $F \in L^1([a, b], \mathbb{R}^N)$, $s \in (a, b)$ is called a *Lebesgue point* if $\lim\limits_{\delta \to 0} \frac{1}{\delta} \int_{s-\delta}^{s+\delta} |F(\sigma) - F(s)| d\sigma = 0$. By the Lebesgue Differentiation Theorem, the set of Lebesgue points has measure $b - a$.

Definition 5.5 (Needle approximation)**.** *Let $\mathbf{c}=(w^0,w,a,\zeta)$ be a needle variation generator and let $\bar s\in(0,\bar S)$. For any control $(\tilde w^0,\tilde w,\tilde\alpha,\tilde\zeta)$ belonging to the set $L^\infty\left([0,\bar S],W\times A\times[-\rho,\rho]\right)$, the family $\left\{(\tilde w^0,\tilde w,\tilde\alpha,\tilde\zeta)^\varepsilon_{\mathbf{c},\bar s}:\ \varepsilon\in(0,\bar s)\right\}$, defined by*

$$(\tilde w^0,\tilde w,\tilde\alpha,\tilde\zeta)^\varepsilon_{\mathbf{c},\bar s}(s):=\begin{cases}(w^0,w,a,\zeta), & \text{if } s\in[\bar s-\varepsilon,\bar s],\\ (\tilde w^0,\tilde w,\tilde\alpha,\tilde\zeta)(s), & \text{if } s\in[0,\bar s-\varepsilon)\cup(\bar s,\bar S],\end{cases}\tag{39}$$

is called a needle control approximation of $(\tilde w^0,\tilde w,\tilde\alpha,\tilde\zeta)$ at $\bar s$ associated to $\mathbf{c}$.

In order to state Lemma 5.1 below –which is a standard result (see e.g. [33]) –, for any $\tilde y:=(y^0,y,y^\ell,\beta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ and any $(w^0,w,a)\in W\times A$, let us set

$$\tilde F(\tilde y,w^0,w,a):=(w^0,F^e(y,w^0,w,a),\ell^e(y,w^0,w,a),|w|)$$

and use $\tilde M(\cdot,\cdot)$ to denote the *fundamental matrix* of the *variational equation*

$$\frac{d\tilde V}{ds}(s)=\frac{\partial\tilde F}{\partial\tilde x}(\bar y^0(s),\bar y(s),\bar y^\ell(s),\bar\beta(s),\bar w^0(s),\bar w(s),\bar\alpha(s))\cdot\tilde V(s),\quad \text{a.e. } s\in[0,\bar S].\tag{40}$$

Namely, for each vector $\tilde v:=\left(v^0,v,v^\ell,v^v\right)\in\mathbb{R}^{1+n+1+1}$ and each $s_1\in[0,\bar S]$, the function $\tilde V(\cdot):=\tilde M(\cdot,s_1)\tilde v$ is the solution of (40) with initial condition $\tilde V(s_1)=\left(V^0,V,V^\ell,V^v\right)(s_1)=\tilde v$. It is straightforward to check that, for all $s\in[0,\bar S]$ one has:

- $\tilde M_{0,j}(s,s_1)=\tilde M_{j,0}(s,s_1)=\delta_{0,j},\quad$ for $j=0,\dots,n+2$,
- $\tilde M_{n+2,j}(s,s_1)=M_{j,n+2}(s,s_1)=\delta_{n+2,j},\quad$ for $j=0,\dots,n+2$,
- $\tilde M_{i,r}(s,s_1)=M_{i,r}(s,s_1)$, for $i,r=1,\dots,n$,
- $\tilde M_{r,n+1}(s,s_1)=\mu_r(s,s_1):=\displaystyle\int_{s_1}^s\sum_{j=1}^n\frac{\partial\ell^e}{\partial x^j}((\bar y,\bar w^0,\bar w,\bar\alpha)(\sigma))\cdot M_{j,r}(s,\sigma)d\sigma$, for $r=1,\dots,n$,
- $\tilde M_{n+1,n+1}(s,s_1)=1$,

where $M(\cdot,\cdot)$ denotes the fundamental matrix of the state-variational equation in $\mathbb{R}^n$

$$\frac{dV}{ds}(s)=\frac{\partial F^e}{\partial x}(\bar y(s),\bar w^0(s),\bar w(s),\bar\alpha(s))\cdot V(s),\quad \text{a.e. } s\in[0,\bar S].\tag{41}$$

Lemma 5.1 (Asymptotics of needle variations)**.** *Assume that $\bar s\in(0,\bar S)_{\rm Leb}$. For every needle variation generator $\mathbf{c}=(w^0,w,a,\zeta)\in W\times A\times[-\rho,\rho]$ and for every $s\in(\bar s,\bar S]$, setting $\mu(s,\bar s):=(\mu_1,\dots,\mu_n)(s,\bar s)$ we get*

$$\begin{pmatrix}y^{0\varepsilon}(s)-\bar y^0(s)\\ y^\varepsilon(s)-\bar y(s)\\ y^{\ell\varepsilon}(s)-\bar y^\ell(s)\\ \beta^\varepsilon(s)-\bar\beta(s)\end{pmatrix}=\varepsilon\tilde M(s,\bar s)\cdot\begin{pmatrix}\mathbf{v}^0_{\mathbf{c},\bar s}\\ \mathbf{v}_{\mathbf{c},\bar s}\\ \mathbf{v}^\ell_{\mathbf{c},\bar s}\\ \mathbf{v}^v_{\mathbf{c},\bar s}\end{pmatrix}+o(\varepsilon)=\varepsilon\begin{pmatrix}\mathbf{v}^0_{\mathbf{c},\bar s}\\ M(s,\bar s)\cdot\mathbf{v}_{\mathbf{c},\bar s}\\ \mu(s,\bar s)\cdot\mathbf{v}_{\mathbf{c},\bar s}+\mathbf{v}^\ell_{\mathbf{c},\bar s}\\ \mathbf{v}^v_{\mathbf{c},\bar s}\end{pmatrix}+o(\varepsilon),\tag{42}$$

where $\left(y^{0\varepsilon},y^\varepsilon,y^{\ell\varepsilon},\beta^\varepsilon\right)$ denotes the solution of system (35) corresponding to the needle control approximation $(\bar w^0,\bar w,\bar\alpha,0)^\varepsilon_{\mathbf{c},\bar s}$ of $(\bar w^0,\bar w,\bar\alpha,0)$ at $\bar s$ associated to $\mathbf{c}$.

Bracket-like approximations, which can be performed in various ways (see e.g. [2, 25, 12, 11, 18] and references therein), are here based on the following result:

Lemma 5.2. *Assume $\mathbf{(Hp)}^*$ with $\hat\ell_1(\cdot,0,\cdot)\equiv0$. Fix a point $(\tilde y^0,\tilde y,\tilde y^\ell,\tilde\beta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ and some $a\in A$. For every Lie bracket $B\in\mathfrak{B}^0$ of length h, there is $\bar\varepsilon>0$ such that, for any $s\in(0,\bar\varepsilon^{1/h}]$, there exists a piecewise constant control $(w^0_{\mathbf{c},s},w_{\mathbf{c},s})$, with $w^0_{\mathbf{c},s}(\sigma)=0$ for* *all* $\sigma \in [0,s]$, $w_{\mathbf{c},s} : [0,s] \to \Big\{ \pm\mathbf{e}_1, \dots, \pm\mathbf{e}_{m_1} \Big\}$, *verifying*

$$(y^0, y^\ell)(\sigma) = (\tilde{y}^0, \tilde{y}^\ell), \qquad \boldsymbol{\beta}(\sigma) = \tilde{\beta} + \sigma, \quad \text{for all } \sigma \in [0,s], \tag{43}$$

$$y(s) = \tilde{y} + \left(\frac{s}{r_{_B}}\right)^h B(\tilde{y}) + o(s^h), \tag{44}$$

where $r_{_B}$ *is the switch-number introduced in Subsect.1.1 and* $(y^0, y, y^\ell, \boldsymbol{\beta})$ *denotes the solution to the space-time control system in* (35) *corresponding to the control* $(w^0_{\mathbf{c},s}, w_{\mathbf{c},s}, a, 0)$ [16] *and the initial condition* $(y^0, y, y^\ell, \boldsymbol{\beta})(0) = (\tilde{y}^0, \tilde{y}, \tilde{y}^\ell, \tilde{\beta})$.

Proof. While the first relation in (43) is trivial, in that $w^0_{\mathbf{c},s} \equiv 0$ and the Lagrangian $\ell^e(y, a, w^0_{\mathbf{c},s}, w_{\mathbf{c},s}) = \ell_0(y,a) w^0_{\mathbf{c},s} \equiv 0$, a proof of (44) can be found in [21]. Finally, the second relation in (43) is trivial as well, for $|w_{\mathbf{c},s}| \equiv 1$ on $[0,s]$. □

Definition 5.6 (Bracket-like approximation). *Fix* $\bar{s} \in (0, \bar{S})$ *and let* $\mathbf{c} = B \in \mathfrak{B}^0$ *be a bracket-like variation generator of length* h. *For each* $\varepsilon > 0$ *such that* $\varepsilon < \bar{\varepsilon}$ *and* $2\varepsilon^{1/h} < \bar{s}$, *where* $\bar{\varepsilon}$ *is as in Lemma 5.2, consider the dilation*

$$\begin{aligned} &\gamma^\varepsilon : [\bar{s} - 2\varepsilon^{1/h}, \bar{s} - \varepsilon^{1/h}] \to [\bar{s} - 2\varepsilon^{1/h}, \bar{s}] \\ &\gamma^\varepsilon(\sigma) := (\bar{s} - 2\varepsilon^{1/h}) + 2\big(\sigma - (\bar{s} - 2\varepsilon^{1/h})\big). \end{aligned} \tag{45}$$

For any control $(\tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{\zeta}) \in L^\infty\big([0,\bar{S}], W \times A \times [-\rho,\rho]\big)$, *let us set*

$$(\tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{\zeta})^\varepsilon_{\mathbf{c},\bar{s}}(s) := \begin{cases} \Big(2\,\tilde{w}^0, 2\,\tilde{w}, \tilde{\alpha}, \tilde{\zeta}\Big) \circ \gamma^\varepsilon(s), & \text{if } s \in [\bar{s} - 2\varepsilon^{1/h}, \bar{s} - \varepsilon^{1/h}), \\ \Big(0, w_{\mathbf{c},\varepsilon^{1/h}}(s - (\bar{s} - \varepsilon^{1/h})), a\Big), & \text{if } s \in [\bar{s} - \varepsilon^{1/h}, \bar{s}], \\ \big(\tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{\zeta}\big)(s), & \text{if } s \in [0, \bar{s} - 2\varepsilon^{1/h}) \cup (\bar{s}, S], \end{cases} \tag{46}$$

where $a \in A$ *is arbitrary and* $w_{\mathbf{c},\varepsilon^{1/h}}$ *is as in Lemma 5.2. We refer to the family of controls* $\big\{(\tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{\zeta})^\varepsilon_{\mathbf{c},\bar{s}} : \varepsilon \in (0,\bar{\varepsilon}),\, 2\varepsilon^{1/h} < \bar{s}\big\}$ *as a* bracket-like control approximation of $(\tilde{w}^0, \tilde{w}, \tilde{\alpha}, \tilde{\zeta})$ at $\bar{s}$ associated to $\mathbf{c} = B$.

Lemma 5.3 (Asymptotics of bracket-like variations). *Let us consider a bracket-like variation generator* $\mathbf{c} = B \in \mathfrak{B}^0$, *with* B *of length* h. *For every point* $\bar{s} \in (0, \bar{S})$ *and for each* $\varepsilon > 0$ *as in Def. 5.6, let* $(\bar{w}^0, \bar{w}, \bar{\alpha}, 0)^\varepsilon_{\mathbf{c},\bar{s}}$ *be a bracket-like control approximation of* $(\bar{w}^0, \bar{w}, \bar{\alpha}, 0)$ *at* $\bar{s}$ *associated to* $\mathbf{c} = B$, *and let* $(y^{0\varepsilon}, y^\varepsilon, y^{\ell\varepsilon}, \beta^\varepsilon)$ *be the corresponding solution of system* (35). *Then, for every* $s \in (\bar{s}, \bar{S}]$ *one has*

$$\begin{pmatrix} y^{0\varepsilon}(s) - \bar{y}^0(s) \\ y^\varepsilon(s) - \bar{y}(s) \\ y^{\ell\varepsilon}(s) - \bar{y}^\ell(s) \end{pmatrix} = \varepsilon \begin{pmatrix} \mathbf{v}^0_{\mathbf{c},\bar{s}} \\ M(s,\bar{s}) \cdot \mathbf{v}_{\mathbf{c},\bar{s}} \\ \mu(s,\bar{s}) \cdot \mathbf{v}_{\mathbf{c},\bar{s}} + \mathbf{v}^\ell_{\mathbf{c},\bar{s}} \end{pmatrix} + \begin{pmatrix} 0 \\ o(\varepsilon) \\ o(\varepsilon) \end{pmatrix} = \varepsilon \begin{pmatrix} 0 \\ M(s,\bar{s}) \cdot \dfrac{B(\bar{y}(\bar{s}))}{(r_{_B})^h} \\ \mu(s,\bar{s}) \cdot \dfrac{B(\bar{y}(\bar{s}))}{(r_{_B})^h} \end{pmatrix} + \begin{pmatrix} 0 \\ o(\varepsilon) \\ o(\varepsilon) \end{pmatrix},$$

and $\beta^\varepsilon(s) - \bar{\beta}(s) = \varepsilon^{\frac{1}{h}}$.

Proof. By the rate-independence of the control system (35), $(y^{0\varepsilon}, y^\varepsilon, y^{\ell\varepsilon}) = (\bar{y}^0, \bar{y}, \bar{y}^\ell) \circ \gamma^\varepsilon$ on $[\bar{s} - 2\varepsilon^{1/h}, \bar{s} - \varepsilon^{1/h}]$, so that $(y^{0\varepsilon}, y^\varepsilon, y^{\ell\varepsilon})(\bar{s} - \varepsilon^{1/h}) = (\bar{y}^0, \bar{y}, \bar{y}^\ell)(\bar{s})$. Hence $y^{0\varepsilon}(\bar{s}) - \bar{y}^0(\bar{s}) = 0$, $y^{\ell\varepsilon}(\bar{s}) - \bar{y}^\ell(\bar{s}) = 0$, while

$$\begin{aligned} y^\varepsilon(\bar{s}) - \bar{y}(\bar{s}) &= \int_{\bar{s}-\varepsilon^{1/h}}^{\bar{s}} \sum_{i=1}^m g_i(y^\varepsilon(s)) w^i_{\mathbf{c},\varepsilon^{1/h}}(s - (\bar{s} - \varepsilon^{1/h}))\,\mathrm{d}s \\ &= \int_0^{\varepsilon^{1/h}} \sum_{i=1}^m g_i(y^\varepsilon(s + (\bar{s} - \varepsilon^{1/h}))\, w^i_{\mathbf{c},\varepsilon^{1/h}}(s)\,\mathrm{d}s, \end{aligned} \tag{47}$$

[16] Note that the choice of the element a is irrelevant.

where $w_{\mathbf{c},\varepsilon^{1/h}}$ is the control associated to the bracket B as in Lemma 5.2. It follows that $y^\varepsilon(\bar s) = y^\varepsilon\big(s + (\bar s - \varepsilon^{1/h})\big)\Big|_{s=\varepsilon^{1/h}} = y^\varepsilon(\varepsilon^{1/h})$, where we have used y^ε to denote the solution to the Cauchy problem $\dfrac{dy}{d\sigma}(\sigma) = \displaystyle\sum_{i=1}^m g_i(y(\sigma))w^i_{\mathbf{c},\varepsilon^{1/h}}(\sigma)$, $y(0) = \bar y(\bar s)$, so that, by Lemma 5.2, we get

$$y^\varepsilon(\bar s) - \bar y(\bar s) - \left(\frac{\varepsilon^{1/h}}{r_{_B}}\right)^h B(\bar y(\bar s)) = y^\varepsilon(\varepsilon^{1/h}) - \bar y(\bar s) - \varepsilon\,\frac{B(\bar y(\bar s))}{(r_{_B})^h} = o(\varepsilon).$$

Therefore, $y^\varepsilon(\bar s) - \bar y(\bar s) = \varepsilon\,\dfrac{B(\bar y(\bar s))}{(r_{_B})^h} + o(\varepsilon)$, and the proof of the first relation of the thesis is concluded, since for every $s \in (\bar s, \bar S]$, the fundamental matrix $\tilde M(s, \bar s)$ is the differential of the flow map from $\bar s$ to s. Finally, by the second relation in (43), one has $\beta^\varepsilon(s) - \bar\beta(s) = \beta^\varepsilon(\bar s) - \bar\beta(\bar s) = \varepsilon^{\frac{1}{h}}$. $\square$

5.3. **Composition of variations.** Let $\mathbf{c} \in \mathfrak{V}$ be a variation generator of length $h \geq 1$, and let $\bar s \in (0, \bar S)$. For any $\varepsilon > 0$ small enough [17], let us introduce the operator $\mathcal{A}^\varepsilon_{\mathbf{c},\bar s}$: $L^\infty\big([0,\bar S], \mathbb{R}_+ \times \mathcal{C} \times A \times [-\rho,\rho]\big) \to L^\infty\big([0,\bar S], \mathbb{R}_+ \times \mathcal{C} \times A \times [-\rho,\rho]\big)$ given by

$$\mathcal{A}^\varepsilon_{\mathbf{c},\bar s}(w^0, w, \alpha, \zeta) := (w^0, w, \alpha, \zeta)^\varepsilon_{\mathbf{c},\bar s}, \tag{48}$$

Lemma 5.4 (Multiple variations at different times)**.** *Let $N > 0$ be an integer and let $\vec{\mathbf{c}} := (\mathbf{c}_1, \dots, \mathbf{c}_N) \in \mathfrak{V}^N$ be an N-uple of variations of lengths $\vec{\mathbf{h}} := (h_1, \dots, h_N) \in \mathbb{N}^N$. Fix $\vec{\mathbf{s}} := (\bar s_1, \dots, \bar s_N) \in (0, \bar S)^N$, where $0 =: \bar s_0 < \bar s_1 < \dots < \bar s_N < \bar S$ and $\bar s_j \in (0, \bar S)_{\mathrm{Leb}}$ as soon as $h_j = 1$. For each $\vec\epsilon := (\varepsilon_1, \dots, \varepsilon_N) \in (0, +\infty)^N$ small enough, let us set*

$$(w^{0\vec\epsilon}, w^{\vec\epsilon}, \alpha^{\vec\epsilon}, \zeta^{\vec\epsilon}) := \mathcal{A}^{\varepsilon_N}_{\mathbf{c}_N,\bar s_N} \circ \dots \circ \mathcal{A}^{\varepsilon_j}_{\mathbf{c}_j,\bar s_j} \circ \dots \circ \mathcal{A}^{\varepsilon_1}_{\mathbf{c}_1,\bar s_1}(\bar w^0, \bar w, \bar\alpha, 0), \tag{49}$$

and let $\big(\bar S, w^{0\vec\epsilon}, w^{\vec\epsilon}, \alpha^{\vec\epsilon}, \zeta^{\vec\epsilon}, y^{0\vec\epsilon}, y^{\vec\epsilon}, y^{\ell\vec\epsilon}, \beta^{\vec\epsilon}\big)$ denote the corresponding process of (35).

Then, for every $s \in (\bar s_N, \bar S]$, one has

$$\begin{pmatrix} y^{0\vec\epsilon}(s) - \bar y^0(s) \\ y^{\vec\epsilon}(s) - \bar y(s) \\ y^{\ell\vec\epsilon}(s) - \bar y^\ell(s) \end{pmatrix} = \sum_{j=1}^N \varepsilon_j \begin{pmatrix} \mathbf{v}^0_{\mathbf{c}_j,\bar s_j} \\ M(s, \bar s_j)\mathbf{v}_{\mathbf{c}_j,\bar s_j} \\ \mu(s, \bar s_j) \cdot \mathbf{v}_{\mathbf{c}_j,\bar s_j} + \mathbf{v}^\ell_{\mathbf{c}_j,\bar s_j} \end{pmatrix} + o(|\vec\epsilon|), \tag{50}$$

and

$$\beta^{\vec\epsilon}(s) - \bar\beta(s) = \sum_{j \in I_1} \varepsilon_j\,(|w_j|(1+\zeta_j) - |\bar w(\bar s_j)|) + o(|\vec\epsilon|) + \sum_{j \in \{1,\dots,N\}\setminus I_1} (\varepsilon_j)^{\frac{1}{h_j}}, \tag{51}$$

where $I_1 := \{j = 1, \dots, N : \;\; h_j = 1\}$. In particular, if all $\mathbf{c}_j$ are needle variations, i.e. $\mathbf{c}_j := (w^0_j, w_j, a_j, \zeta_j)$ for every $j = 1, \dots, N$, one gets

$$\beta^{\vec\epsilon}(s) - \bar\beta(s) = \sum_{j=1}^N \varepsilon_j\big(|w_j|(1+\zeta_j) - |\bar w(\bar s_j)|\big) + o(|\vec\epsilon|). \tag{52}$$

Proof. Let us prove the result by induction on N, the number of composed variations. For $N = 1$, the result is proved in Lemmas 5.1 and 5.3. If $N \geq 2$, let us assume that the result holds true for $N - 1$ and let us show that it is valid for N as well. Let us use $(y^0, y, y^\ell, \beta)^N$ and $(y^0, y, y^\ell, \beta)^{N-1}$ to denote the trajectories associated to the N variations and to the

[17]Precisely, if $h \geq 2$, we require $0 < \varepsilon < \bar\varepsilon$, $2\varepsilon^{1/h} < \bar s$, as in Def. 5.6, while, in case $h = 1$, $\varepsilon < \bar s$.

first $N-1$ variations, respectively (we omit the dependence on $\vec{\epsilon}$ for brevity). Then one has

$$\begin{pmatrix} y^{0,N}(\bar{s}_N)-\bar{y}^0(\bar{s}_N) \\ y^N(\bar{s}_N)-\bar{y}(\bar{s}_N) \\ y^{\ell,N}(\bar{s}_N)-\bar{y}^\ell(\bar{s}_N) \\ \beta^N(\bar{s}_N)-\bar{\beta}(\bar{s}_N) \end{pmatrix} = \begin{pmatrix} y^{0,N}(\bar{s}_N)-y^{0,(N-1)}(\bar{s}_N) \\ y^N(\bar{s}_N)-y^{N-1}(\bar{s}_N) \\ y^{\ell,N}(\bar{s}_N)-y^{\ell,(N-1)}(\bar{s}_N) \\ \beta^N(\bar{s}_N)-\beta^{N-1}(\bar{s}_N) \end{pmatrix} + \begin{pmatrix} y^{0,(N-1)}(\bar{s}_N)-\bar{y}^0(\bar{s}_N) \\ y^{N-1}(\bar{s}_N)-\bar{y}(\bar{s}_N) \\ y^{\ell,(N-1)}(\bar{s}_N)-\bar{y}^\ell(\bar{s}_N) \\ \beta^{N-1}(\bar{s}_N)-\bar{\beta}(\bar{s}_N) \end{pmatrix}. \tag{53}$$

By the inductive hypothesis, we get that

$$\begin{pmatrix} y^{0,(N-1)}(\bar{s}_N)-\bar{y}^0(\bar{s}_N) \\ y^{N-1}(\bar{s}_N)-\bar{y}(\bar{s}_N) \\ y^{\ell,(N-1)}(\bar{s}_N)-\bar{y}^\ell(\bar{s}_N) \end{pmatrix} = \sum_{j=1}^{N-1} \varepsilon_j \begin{pmatrix} \mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j} \\ M(\bar{s}_N,\bar{s}_j)\mathbf{v}_{\mathbf{c}_j,\bar{s}_j} \\ \mu(\bar{s}_N,\bar{s}_j)\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}+\mathbf{v}^\ell_{\mathbf{c}_j,\bar{s}_j} \end{pmatrix} + o(|(\varepsilon_1,\dots,\varepsilon_{N-1})|), \tag{54}$$

and, setting $I_1^{N-1} := \{j=1,\dots,N-1 :\ h_j = 1\}$,

$$\begin{aligned} \beta^{N-1}(\bar{s}_N)-\bar{\beta}(\bar{s}_N) = & \sum_{j\in I_1^{N-1}} \varepsilon_j\left(|w_j|(1+\zeta_j)-|\bar{w}(\bar{s}_j)|\right) \\ & + o(|(\varepsilon_1,\dots,\varepsilon_{N-1})|) + \sum_{j\in\{1,\dots,N-1\}\setminus I_1^{N-1}} (\varepsilon_j)^{\frac{1}{h_j}}. \end{aligned} \tag{55}$$

We claim that

$$\begin{pmatrix} y^{0,N}(\bar{s}_N)-y^{0,(N-1)}(\bar{s}_N) \\ y^N(\bar{s}_N)-y^{N-1}(\bar{s}_N) \\ y^{\ell,N}(\bar{s}_N)-y^{\ell,(N-1)}(\bar{s}_N) \end{pmatrix} = \varepsilon_N \begin{pmatrix} \mathbf{v}^0_{\mathbf{c}_N,\bar{s}_N} \\ \mathbf{v}_{\mathbf{c}_N,\bar{s}_N} \\ \mathbf{v}^\ell_{\mathbf{c}_N,\bar{s}_N} \end{pmatrix} + o(|\vec{\epsilon}|), \tag{56}$$

and

$$\beta^N(\bar{s}_N)-\beta^{N-1}(\bar{s}_N) = \begin{cases} \varepsilon_N\big(|w_N|(1+\zeta_N)-|\bar{w}(\bar{s}_N)|\big)+o(|\varepsilon_N|), & \text{if } h_N=1, \\ (\varepsilon_N)^{\frac{1}{h_N}}, & \text{if } h_N\geq 2. \end{cases} \tag{57}$$

Once one has proven the claim, the validity of (50) and (51) follows easily by (53)-(55), by the properties of the fundamental matrix $\tilde{M}(s,\bar{s}_N)$. To prove (56), (57) we first consider the case when the length h_N of the Nth variation $\mathbf{c}_N$ is ≥ 2.

Case $h_N \geq 2$. Here $\mathbf{c}_N = B_N \in \mathfrak{B}^0$ is a bracket-like variation and one has

$$\begin{pmatrix} y^{0,N} \\ y^N \\ y^{\ell,N} \\ \beta^N \end{pmatrix}(\bar{s}_N-\varepsilon_N^{1/h_N}) = \begin{pmatrix} y^{0,N-1} \\ y^{N-1} \\ y^{\ell,N-1} \\ \beta^{N-1} \end{pmatrix}(\bar{s}_N),$$

so that $y^{0,N}(\bar{s}_N)-y^{0,N-1}(\bar{s}_N)=0$, $y^{\ell,N}(\bar{s}_N)-y^{\ell,N-1}(\bar{s}_N)=0$, $\beta^N(\bar{s}_N)-\beta^{N-1}(\bar{s}_N)=\varepsilon_N^{1/h_N}$, while

$$\begin{aligned} y^N(\bar{s}_N)-y^{N-1}(\bar{s}_N) &= \int_{\bar{s}_N-\varepsilon_N^{1/h_N}}^{\bar{s}_N} \sum g_i(y^N(s))\,w^i_{\mathbf{c}_N,\varepsilon_N^{1/h_N}}\Big(s-\big(\bar{s}_N-\varepsilon_N^{1/h_N}\big)\Big)\mathrm{d}s \\ &= \int_0^{\varepsilon^{1/h}} \sum_{i=1}^m g_i(y^N(s+(\bar{s}_N-\varepsilon_N^{1/h_N})))\,w^i_{\mathbf{c}_N,\varepsilon_N^{1/h_N}}(s)\,\mathrm{d}s, \end{aligned}$$

where the control $w_{\mathbf{c}_N,\varepsilon_N^{1/h_N}}$ is as in Lemma 5.2. If y^N denotes the solution to the Cauchy problem $\dfrac{dy}{d\sigma}(\sigma) = \displaystyle\sum_{i=1}^m g_i(y(\sigma))\,w^i_{\mathbf{c}_N,\varepsilon_N^{1/h_N}}(\sigma)$, $y(0) = y^{N-1}(\bar{s}_N)$, then $y^N(\bar{s}_N) = y^N\big(s+$

$(\bar{s}_N - \varepsilon_N^{1/h_N}))\Big|_{s=\varepsilon_N^{1/h_N}} = y^N(\varepsilon_N^{1/h_N})$ and, by Lemma 5.2, we get

$$
\begin{aligned}
y^N(\bar{s}_N) - y^{N-1}(\bar{s}_N) - & \left(\frac{\varepsilon_N^{1/h_N}}{r_{B_N}}\right)^{h_N} B_N(\bar{y}(\bar{s}_N)) \\
&= y^N(\varepsilon_N^{1/h_N}) - y^{N-1}(\bar{s}_N) - \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(y^{N-1}(\bar{s}_N)) \\
&\quad + \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(y^{N-1}(\bar{s}_N)) - \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(\bar{y}(\bar{s}_N)) \\
&= o(\varepsilon_N) + \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(y^{N-1}(\bar{s}_N)) - \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(\bar{y}(\bar{s}_N)).
\end{aligned}
$$

Now by the continuity of B_N and the inductive hypothesis (54), it follows that

$$
\begin{aligned}
&\left| \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(y^{N-1}(\bar{s}_N)) - \frac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(\bar{y}(\bar{s}_N)) \right| \\
&\qquad \le \frac{\varepsilon_N}{(r_{B_N})^{h_N}}\, \omega_{B_N}\left(|y^{N-1}(\bar{s}_N) - \bar{y}(\bar{s}_N)|\right) \le \frac{\varepsilon_N}{(r_{B_N})^{h_N}}\, \omega_{B_N}\left(\mathcal{O}\left(\varepsilon_1 + \cdots + \varepsilon_{N-1}\right)\right),
\end{aligned}
$$

where ω_{B_N} denotes the modulus of continuity of B_N and we use $\mathcal{O}$ to mean a nonnegative function such that $\mathcal{O}(r) \le Cr$ for all $r \ge 0$, for some constant $C > 0$. Therefore, $y^N(\bar{s}_N) - y^{N-1}(\bar{s}_N) = \dfrac{\varepsilon_N}{(r_{B_N})^{h_N}} B_N(\bar{y}(\bar{s}_N)) + o(|\vec{\epsilon}|)$, which concludes the proof in this case.

Case $h_N = 1$. Here $\mathbf{c}_N = (w_N^0, w_N, a_N, \zeta_N)$ and the aimed estimate is rather standard. Nonetheless, we perform it for the sake of self-consistency. One has

$$
\begin{aligned}
y^N(\bar{s}_N) - y^{N-1}(\bar{s}_N) = \int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} & \big[F^e(y^N(s), w_N^0, w_N, a_N)(1+\zeta_N) \\
& - F^e(y^{N-1}(s), \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s))\big]\, \mathrm{d}s = \int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} \big(r_1(s) + r_2 + r_3(s)\big) \mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
r_1(s) &:= F^e(y^N(s), w_N^0, w_N, a_N)(1+\zeta_N) - F^e(\bar{y}(\bar{s}_N), w_N^0, w_N, a_N)(1+\zeta_N), \\
r_2 &:= F^e(\bar{y}(\bar{s}_N), w_N^0, w_N, a_N)(1+\zeta_N) - \bar{F}^e(\bar{s}_N), \\
r_3(s) &:= \bar{F}^e(\bar{s}_N) - F^e(y^{N-1}(s), \bar{w}^0(\bar{s}_N), \bar{w}(\bar{s}_N), \bar{\alpha}(\bar{s}_N)).
\end{aligned}
$$

Let us start by estimating r_1. Observe that, for $s \in [\bar{s}_N - \varepsilon_N, \bar{s}_N]$,

$$
|y^N(s) - \bar{y}(\bar{s}_N)| \le |y^N(s) - y^{N-1}(s)| + |y^{N-1}(s) - \bar{y}(s)| + |\bar{y}(s) - \bar{y}(\bar{s}_N)|.
$$

Moreover, on $[\bar{s}_N - \varepsilon_N, \bar{s}_N]$, one has $\|y^N - y^{N-1}\|_\infty = \mathcal{O}\left(\varepsilon_N\right)$ by the Lipschitz continuity of the input-output map Φ defined in (36); $\|y^{N-1} - \bar{y}\|_\infty = \mathcal{O}\left(\varepsilon_1 + \cdots + \varepsilon_{N-1}\right)$ by the inductive hypothesis (54); and $\|\bar{y}(s) - \bar{y}(\bar{s}_N)\|_\infty = \mathcal{O}\left(\varepsilon_N\right)$ by the Lipschitz continuity of the reference trajectory. Hence $\|y^N(s) - \bar{y}(\bar{s}_N)\|_\infty = \mathcal{O}(|\vec{\epsilon}|)$, so that

$$
\left| \int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} r_1(s) \mathrm{d}s \right| \le \int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} L\, |y^N(s) - \bar{y}(\bar{s}_N)| \mathrm{d}s = \varepsilon_N\, \mathcal{O}(|\vec{\epsilon}|),
$$

where L is a suitable positive constant. By the previous estimates and recalling that $\bar{s}_N$ is a Lebesgue point of the map in Def. 5.3, we get

$$\begin{aligned}\Big|\int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} r_3(s)\mathrm{d}s\Big| \leq & \Big|\int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} \left[\bar{F}^e(\bar{s}_N)-\bar{F}^e(s)\right] ds\Big| \\ & + \Big|\int_{\bar{s}_N-\varepsilon_N}^{\bar{s}_N} \left[\bar{F}^e(s)-F^e(y^{N-1}(s),\bar{w}^0(s),\bar{w}(s),\bar{\alpha}(s))\right] ds\Big| \\ & \leq o(\varepsilon_N)+\varepsilon_N\,\mathcal{O}(|(\varepsilon_1,\dots,\varepsilon_{N-1})|).\end{aligned}$$

Therefore, $y^N(\bar{s}_N)-y^{N-1}(\bar{s}_N)=\varepsilon_N\, r_2+o(|\vec{\epsilon}|)$, and the relation in (56) concerning the state variables is proven. The proofs of the other relations are similar and actually easier, so we omit them. □

5.4. **Set separation.** Given a process $(\bar{S},w^0,w,\alpha,\zeta,y^0,y,y^\ell,\beta)$ of the rescaled problem $(\mathrm{P_e})$, let us introduce the total cost component

$$y^c(s):=\Psi(y^0(s),y(s))+y^\ell(s),\quad s\in[0,\bar{S}].\ ^{18} \tag{58}$$

Setting $\bar{y}^c(s):=\Psi(\bar{y}^0(s),\bar{y}(s))+\bar{y}^\ell(s)$, $s\in[0,\bar{S}]$, for any $\delta>0$ we define the δ-*reachable set* $\mathcal{R}_\delta$ and its *projection* $\mathcal{R'}_\delta$ as

$$\begin{aligned}\mathcal{R}_\delta &:= \left\{\begin{array}{c}(y^0,y,y^c,\beta)(\bar{S}):\ (S,w^0,w,\alpha,\zeta,y^0,y,y^\ell,\beta)\text{ verifies}\\ \mathrm{d}\left((\bar{S},y^0,y,y^c,\beta),(\bar{S},\bar{y}^0,\bar{y},\bar{y}^c,\bar{\beta})\right)<\delta\end{array}\right\}\subseteq\mathbb{R}^{1+n+1+1},\\ \mathcal{R'}_\delta &:= \left\{(y^0,y,y^c)(\bar{S}):\ (y^0,y,y^c,\beta)(\bar{S})\in\mathcal{R}_\delta\right\}\subseteq\mathbb{R}^{1+n+1}.\end{aligned}$$

When *all* $\mathbf{c}_j=(w_j^0,w_j,a_j,\zeta_j)$, $j=1,\dots N$, *are needle variations,* we define the set

$$E:=\left\{\begin{array}{c}\left(\begin{array}{c}\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j}\\ M(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}\\ \frac{\partial\bar{\Psi}}{\partial t}((\bar{S})\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j}+\frac{\partial\bar{\Psi}}{\partial x}(\bar{S})\cdot M(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}+\mu(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}+\mathbf{v}^\ell_{\mathbf{c}_j,\bar{s}_j}\\ |w|(1+\zeta_j)-|\bar{w}(\bar{s}_j)|\end{array}\right),\\ j=1,\dots N\end{array}\right\}$$

where $\dfrac{\partial\bar{\Psi}}{\partial t}(\bar{S}):=\dfrac{\partial\Psi}{\partial t}((\bar{y}^0,\bar{y})(\bar{S}))$, $\frac{\partial\bar{\Psi}}{\partial x}(\bar{S}):=\frac{\partial\Psi}{\partial x}((\bar{y}^0,\bar{y})(\bar{S}))$, and its projection E',

$$E':=\left\{\begin{array}{c}\left(\begin{array}{c}\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j}\\ M(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}\\ \frac{\partial\bar{\Psi}}{\partial t}((\bar{S})\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j}+\frac{\partial\bar{\Psi}}{\partial x}(\bar{S})\cdot M(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}+\mu(\bar{S},\bar{s}_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar{s}_j}+\mathbf{v}^\ell_{\mathbf{c}_j,\bar{s}_j}\end{array}\right),\\ j=1,\dots N\end{array}\right\}.$$

Finally, let us define the convex cones

$$R:=\mathrm{span}^+(E)\subset\mathbb{R}^{1+n+1+1},\quad R':=\mathrm{span}^+(E')\subset\mathbb{R}^{1+n+1}, \tag{59}$$

where, for a given subset Θ of a vector space, $\mathrm{span}^+(\Theta)$ denotes its *positive span.*

Lemma 5.5. (i) *The set R' is a Boltyanski approximating cone of the set $\mathcal{R'}_\delta$ at the point $(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})$.*

(ii) *When all $\mathbf{c}_j=(w_j^0,w_j,a_j,\zeta_j)$, for $j=1,\dots,N$, are needle variations, the set R is a Boltyanski approximating cone of the set $\mathcal{R}$ at $(\bar{y}^0,\bar{y},\bar{y}^c,\bar{\beta})(\bar{S})$.*

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[18] The function y^c can be obviously regarded as the solution of
$\frac{dy^c}{ds}=\Big(\frac{\partial\Psi}{\partial t}w^0+\frac{\partial\Psi}{\partial x}\left(f(y,\alpha)w^0+\sum_{i=1}^m g_i(y)w^i\right)+\ell^e(y,w^0,w,\alpha)\Big)(1+\zeta),\quad y^c(0)=\Psi(0,\check{x},0).$

Proof. Let us set $y^{c\vec{\epsilon}}(s) := \Psi((y^{0\vec{\epsilon}}, y^{\vec{\epsilon}})(s)) + y^{\ell\vec{\epsilon}}(s)$, where $y^{\ell\vec{\epsilon}}$, $y^{0\vec{\epsilon}}$, and $y^{\vec{\epsilon}}$ are as in Lemma 5.4. By (50) we get

$$y^{c\vec{\epsilon}}(\bar{S}) - \bar{y}^c(\bar{S}) = \sum_{j=1}^{N} \varepsilon_j \Big(\frac{\partial \bar{\Psi}}{\partial t}(\bar{S})\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j} + \frac{\partial \bar{\Psi}}{\partial x}(\bar{S}) \cdot M(\bar{S},\bar{s}_j) \cdot \mathbf{v}_{\mathbf{c}_j,\bar{s}_j} \\ + \mu(\bar{S},\bar{s}_j) \cdot \mathbf{v}_{\mathbf{c}_j,\bar{s}_j} + \mathbf{v}^\ell_{\mathbf{c}_j,\bar{s}_j} \Big) + o(|\vec{\epsilon}|).$$

Therefore, part (ii) of the statement follows from Lemma 5.4.

To prove part (i), for some $\tilde{\varepsilon} > 0$ sufficiently small, let us define the function $F : (0,+\infty)^N \cap \tilde{\varepsilon}\,\mathbb{B}_N \to \mathbb{R}^{1+n+2}$ by setting $F(\vec{\epsilon}) := \big(y^{0\vec{\epsilon}}(\bar{S}), y^{\vec{\epsilon}}(\bar{S}), y^{c\vec{\epsilon}}(\bar{S})\big)$. It is straightforward to prove that $F(\vec{\epsilon}) = \big(y^0(\bar{S}), y(\bar{S}), y^c(\bar{S}))\big) + L\cdot\vec{\epsilon} + o(|\vec{\epsilon}|)$, where the linear operator $L \in \mathrm{Hom}(\mathbb{R}^N, \mathbb{R}^{1+n+1})$ is defined by

$$L\cdot\vec{\epsilon} := \sum_{j=1}^{N} \varepsilon_j \begin{pmatrix} \mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j} \\ M(\bar{S},\bar{s}_j)\mathbf{v}_{\mathbf{c}_j,\bar{s}_j} \\ \frac{\partial \bar{\Psi}}{\partial t}(\bar{S})\mathbf{v}^0_{\mathbf{c}_j,\bar{s}_j} + \frac{\partial \bar{\Psi}}{\partial x}(\bar{S}) \cdot M(\bar{S},\bar{s}_j) \cdot \mathbf{v}_{\mathbf{c}_j,\bar{s}_j} + \mu(\bar{S},\bar{s}_j) \cdot \mathbf{v}_{\mathbf{c}_j,\bar{s}_j} + \mathbf{v}^\ell_{\mathbf{c}_j,\bar{s}_j} \end{pmatrix}.$$

Hence (i) is proved, in that $R' = L\cdot(0,+\infty)^N$. □

Let us consider the *profitable set* $\mathcal{P}$ and its projection $\mathcal{P}'$, defined as

$$\mathcal{P} := \mathfrak{T} \times \big(-\infty, \bar{y}^c(\bar{S})\big) \times [0,K] \bigcup \left\{(\bar{y}^0,\bar{y},\bar{y}^c,\bar{\beta})(\bar{S})\right\},$$
$$\mathcal{P}' := \mathfrak{T} \times \big(-\infty, \bar{y}^c(\bar{S})\big) \bigcup \left\{(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})\right\},$$

and let Γ be a Boltyanski approximating cone for the target $\mathfrak{T}$ at $(\bar{y}^0,\bar{y})(\bar{S})$. Recalling that $\bar{\beta}(\bar{S}) < K$, one trivially checks that the sets

$$P := \Gamma \times \mathbb{R}_- \times \{0\}, \qquad P' := \Gamma \times \mathbb{R}_-,$$

are Boltyanski approximating cones of $\mathcal{P}$ at $(\bar{y}^0,\bar{y},\bar{y}^c,\bar{\beta})(\bar{S})$ and of $\mathcal{P}'$ at $(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})$, respectively. We will need the following elementary result:

Lemma 5.6. *There exists $\delta > 0$ such that the sets $\mathcal{P}'$ and $\mathcal{R}'_\delta$ are locally separated at $(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})$.*

Proof. Suppose by contradiction that for every $\delta > 0$ the sets $\mathcal{P}'$ and $\mathcal{R}'_\delta$ are not locally separated at $(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})$. Then, given $\delta \in (0, K - \bar{\beta}(\bar{S}))$,[19] there exists a process $(\bar{S}, w^0, w, \alpha, \zeta, y^0, y, y^\ell, \beta)$ of (35) verifying

$$(y^0,y,y^c)(\bar{S}) \in \mathcal{R}'_\delta \cap \mathcal{P}', \quad \mathrm{d}((y^0,y,y^c,\beta),(\bar{y}^0,\bar{y},\bar{y}^c,\bar{\beta})) < \delta.$$

This implies that $\beta(\bar{S}) \le \delta + \bar{\beta}(\bar{S}) < K$, thus the final point $(y^0,y,y^c,\beta)(\bar{S}) \in \mathcal{R}_\delta \cap \mathcal{P}$. Hence, for every $\delta \in (0, K - \bar{\beta}(\bar{S}))$ the sets $\mathcal{P}$ and $\mathcal{R}_\delta$ are not locally separated, which contradicts the local optimality of the reference process. □

By Lemma 5.6 the projected reachable set $\mathcal{R}'_\delta$ is locally separated from the projected profitable set $\mathcal{P}'$ at $(\bar{y}^0,\bar{y},\bar{y}^c)(\bar{S})$, for some $\delta > 0$. Therefore, since R' and P' are approximating cones to $\mathcal{R}'_\delta$ and $\mathcal{P}'$, respectively, and P' is not a subspace, in view of Theorem 1.1 there exists a vector $(\xi_0,\xi,\xi_c) \in \mathbb{R}^{1+n+1}$ verifying

$$0 \ne (\xi_0,\xi,\xi_c) \in R'^{\perp} \cap (-P'^{\perp}).$$

Since $P'^{\perp} = \Gamma^\perp \times \mathbb{R}_+$, one gets $(\xi_0,\xi) \in -\Gamma^\perp$, $\xi_c = -\lambda \le 0$, and

$$\xi_0\mathbf{v}^0 + \xi\cdot\mathbf{v} + \xi_c\mathbf{v}^c \le 0 \qquad \forall (\mathbf{v}^0,\mathbf{v},\mathbf{v}^c) \in R'.$$

[19]This interval is not empty, for $\bar{\beta}(\bar{S}) < K$.

By the definition of R' given in (59), the latter relation is verified if and only if

$$\Big(\xi_0-\lambda\frac{\partial\bar\Psi}{\partial t}(\bar S)\Big)\mathbf{v}^0_{\mathbf{c}_j,\bar s_j}+\Big(\xi-\lambda\frac{\partial\bar\Psi}{\partial x}(\bar S)\Big)\cdot M(\bar S,\bar s_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar s_j}\\ -\lambda\left(\mu(\bar S,\bar s_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar s_j}+\mathbf{v}^\ell_{\mathbf{c}_j,\bar s_j}\right)\le 0,\quad\text{for all } j=1,\dots,N.$$

Therefore, setting

$$(p_0,p)(s):=\left(\xi_0-\lambda\frac{\partial\bar\Psi}{\partial t}(\bar S)\ ,\ \left(\xi-\lambda\frac{\partial\bar\Psi}{\partial x}(\bar S)\right)\cdot M(\bar S,s)-\lambda\mu(\bar S,s)\right),$$

we obtain that the multiplier $(p_0,p,\lambda)\in\mathbb{R}\times AC\left([0,\bar S],\mathbb{R}^n\right)\times\mathbb{R}_+$ verifies

$$p_0\mathbf{v}^0_{\mathbf{c}_j,\bar s_j}+p(\bar s_j)\cdot\mathbf{v}_{\mathbf{c}_j,\bar s_j}-\lambda\mathbf{v}^\ell_{\mathbf{c}_j,\bar s_j}\le 0,\quad\text{for every } j=1,\dots,N, \tag{60}$$

the non-triviality condition (13), and (by $M(\bar S,\bar S)=\mathrm{Id}$, $\mu(\bar S,\bar S)=0$) the non-transversality condition (15). Moreover, by the definitions of $M(\bar S,\cdot)$ and $\mu(\bar S,\cdot)$, the path p solves the adjoint equation (17). Finally, for a needle variation generator $\mathbf{c}_j=(w^0_j,w_j,a_j,\zeta_j)$, by (60) we get

$$H\Big(\bar y(\bar s_j),p_0,p(\bar s_j),0,\lambda,w^0_j(1+\zeta_j),w_j(1+\zeta_j),a_j\Big)\\ -H\Big(\bar y(\bar s_j),p_0,p(\bar s_j),0,\lambda,\bar w^0(\bar s_j),\bar w(\bar s_j),\bar\alpha(\bar s_j)\Big)\le 0,$$

while, for a bracket-like variation generator $\mathbf{c}_j=B_j$, we obtain $p(\bar s_j)\cdot B_j(\bar y(\bar s_j))\le 0$.

5.5. **Conclusion of the proof.** To conclude the proof we need to extend the previous inequalities to almost all $s\in[0,\bar S]$ and all variations generators $\mathbf{c}\in\mathfrak{V}$. This will be achieved via density arguments coupled with infinite intersection criteria. Though this is a quite standard procedure, we give the details for the sake of completeness. By Lusin's Theorem, one has that $(0,\bar S)_{\mathrm{Leb}}=\displaystyle\bigcup_{k=0}^{+\infty}E_k$, where E_0 has null measure and, for every $k\in\mathbb{N}$, the set E_k is compact and the restriction to E_k of the measurable map considered in Definition 5.3 is continuous. For every k, let $D_k\subseteq E_k$ be the set of density points[20] of E_k. Since D_k and E_k have the same measure, by the Lebesgue density Theorem, $D=\displaystyle\bigcup_{k=0}^{+\infty}D_k\subset[0,\bar S]$ has full measure.

Definition 5.7. *Let F be an arbitrary subset of $D\times\mathfrak{V}$. We say that a triple $(\bar p_0,\bar p,\lambda)\in\mathbb{R}^{1+n+1}$ verifies* (P)$_F$ *if $\lambda\ge 0$ and, setting $p_0:=\bar p_0$, $p(\cdot):=\bar p\cdot M(\bar S,\cdot)$, one has that:*

(i) $(p_0,p(\bar S))+\lambda\left(\dfrac{\partial\bar\Psi}{\partial t}(\bar S)\,,\ \dfrac{\partial\bar\Psi}{\partial x}(\bar S)\right)\in-\Gamma^\perp$*;*

(ii) *for every $(s,\mathbf{c})\in F$ with $\mathbf{c}=(w^0,w,a,\zeta)$, the following inequality $H\Big(\bar y(s),p_0,p(s),0,\lambda,w^0(1+\zeta),w(1+\zeta),a\Big)\le H\Big(\bar y(s),p_0,p(s),0,\lambda,\bar w^0(s),\bar w(s),\bar\alpha(s)\Big)$ holds true, while for every $(s,\mathbf{c})\in F$ such that $\mathbf{c}=B\in\mathfrak{B}^0$, $p(s)\cdot B(\bar y(s))\le 0$.*

For any given subset $F\subset D\times\mathfrak{V}$, let us set

$$\Lambda(F):=\Big\{(\bar p_0,\bar p,\lambda)\in\mathbb{R}^{1+n+1}:\ |(\bar p_0,\bar p,\lambda)|=1,\ (\bar p_0,\bar p,\lambda)\text{ verifies }(\mathrm{P})_F\Big\}.$$

Our goal consists in showing that $\Lambda(F)\neq\emptyset$ for some F comprising pairs $(s,\mathbf{c})$, such that the union of all times s is a full measure subset of $[0,\bar S]$ and $\mathbf{c}$ can range over all $\mathfrak{V}$. Clearly, for arbitrary subsets F_1, F_2 of $D\times\mathfrak{V}$ the sets $\Lambda(F_1),\Lambda(F_2)$, if not empty, are compact and

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[20] We recall that $t\in\tilde E\subset\mathbb{R}$ is a *density point* for $\tilde E$ if $\displaystyle\lim_{\delta\to0^+}\frac{\mathrm{meas}\big([t-\delta,t+\delta]\cap\tilde E\big)}{2\delta}=1$.

$\Lambda(F_1 \cup F_2) = \Lambda(F_1) \cap \Lambda(F_2)$. By the previous step, $\Lambda(F) \neq \emptyset$ as soon as F is *finite* and of the form

$$\Big\{(\bar{s}_1, \mathbf{c}_1), \ldots, (\bar{s}_N, \mathbf{c}_N)\Big\}, \quad \text{with } 0 =: \bar{s}_0 < \bar{s}_1 < \cdots < \bar{s}_N < \bar{S}. \tag{61}$$

In order to prove that $\Lambda(F) \neq \emptyset$ for an arbitrary *finite* set $F \subset D \times \mathfrak{V}$, we have to show that it is non-empty even when $F = \Big\{(\bar{s}_1, \mathbf{c}_1), \ldots, (\bar{s}_N, \mathbf{c}_N)\Big\}$ with $0 =: \bar{s}_0 \leq \bar{s}_1 \leq \cdots \leq \bar{s}_N < \bar{S}$ and one allows that $\bar{s}_j = \bar{s}_{j+1}$ for some $j = 0, \ldots, N-1$. To this end, observe that every $\bar{s}_j$ belongs to some set of density points D_k, that we denote $D_{k(j)}$. Hence, there exist sequences $(\bar{s}_{j,i})_{i \in \mathbb{N}}$, for $j = 1, \ldots, N$, such that

$$\bar{s}_{j,i} \in D_{k(j)} \quad \text{and} \quad \bar{s}_{1,i} < \cdots < \bar{s}_{N,i}, \quad \text{for all } i \in \mathbb{N}, \quad \text{and} \quad \lim_{i \to +\infty} \bar{s}_{j,i} = \bar{s}_j,$$

For each $i \in \mathbb{N}$, set $F_i := \Big\{(\bar{s}_{1,i}, \mathbf{c}_1), \ldots, (\bar{s}_{N,i}, \mathbf{c}_N)\Big\}$, so that F_i has the form (61) and hence $\Lambda(F_i) \neq \emptyset$. For each $i \in \mathbb{N}$, let us select $(\bar{p}_{0_i}, \bar{p}_i, \lambda_i) \in \Lambda(F_i)$. Since $|(\bar{p}_{0_i}, \bar{p}_i, \lambda_i)| = 1$, by possibly taking a subsequence, we can assume that $(\bar{p}_{0_i}, \bar{p}_i, \lambda_i)$ converges to a point $(\bar{p}_0, \bar{p}, \lambda)$ with $|(\bar{p}_0, \bar{p}, \lambda)| = 1$. By the definition of $D_{k(j)} (\subseteq E_{k(j)})$, passing to the limit as $i \to +\infty$ one obtains that $(\bar{p}_0, \bar{p}, \lambda) \in \Lambda(F)$. Hence we have proved that $\Lambda(F) \neq \emptyset$ as soon as $\text{card}(F) < +\infty$ [21]. In particular, if we take a finite family of subsets $F_1, \ldots, F_M \subset D \times \mathfrak{V}$ with $\text{card}(F_i) < +\infty$ for all $i = 1, \ldots, M$, we get $\Lambda(F_1) \cap \cdots \cap \Lambda(F_M) = \Lambda\left(\cup_{i=1}^M F_i\right) \neq \emptyset$. Hence $\Big\{\Lambda(F) \,:\, F \subset D \times \mathfrak{V},\ \text{card}(F) < +\infty\Big\}$ is a family of compact subsets such that the intersection of each finite subfamily is non-empty. This implies that also the (infinite) intersection of all $\Lambda(F)$ over finite sets F is non-empty. Therefore $\Lambda(D \times \mathfrak{V}) = \Lambda\left(\bigcup_{\text{card}(F) < +\infty} F\right) = \bigcap_{\text{card}(F) < +\infty} \Lambda(F) \neq \emptyset$. This means that there exists some covector $(\bar{p}_0, \bar{p}, \lambda) \neq 0$ such that, setting $p_0 := \bar{p}_0$, $p(\cdot) := \bar{p} \cdot M(\bar{S}, \cdot)$, for all time s in the full-measure set D, one gets

$$\begin{aligned}
&\mathbf{H}\big(\bar{y}(s), p_0, p(s), 0, \lambda\big) = H\Big(\bar{y}(s), p_0, p(s), 0, \lambda, \bar{w}^0(s), \bar{w}(s), \bar{\alpha}(s)\Big) \\
&\quad = \max_{(w^0, w, a, \zeta) \in W \times A \times \left[-\frac{1}{2}, \frac{1}{2}\right]} H\Big(\bar{y}(s), p_0, p(s), 0, \lambda, w^0(1+\zeta), w(1+\zeta), a\Big) \\
&\quad = \max_{\zeta \in \left[-\frac{1}{2}, \frac{1}{2}\right]} (1+\zeta)\, \mathbf{H}\big(\bar{y}(s), p_0, p(s), 0, \lambda\big),
\end{aligned} \tag{62}$$

$$p(s) \cdot B(\bar{y}(s)) \leq 0, \quad \text{for all } B \in \mathfrak{B}^0. \tag{63}$$

The first relation in (62) coincides with (18), while the last one immediately implies (19). Finally, observe that $B \in \mathfrak{B}^0$ if and only if $-B \in \mathfrak{B}^0$, so that (63) yields (25). This concludes the proof, since, in case $\bar{y}(\bar{S}) > 0$, the strengthened non-triviality condition (14) can be obtained as in the proof of the First Order Maximum Principle.

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[21] Here $\text{card}(Q)$ denotes the cardinality of the set Q

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