# The global well-posedness for the compressible fluid model of Korteweg type 

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#### Abstract

In this paper, we consider the compressible fluid model of Korteweg type which can be used as a phase transition model. It is shown that the system admits a unique, global strong solution for small initial data in $\mathbb{R}^{N}, N \geq 3$. In this study, the main tools are the maximal $L_{p}$ - $L_{q}$ regularity and $L_{p}-L_{q}$ decay properties of solutions to the linearized equations.


## 1 Introduction

We consider the following compressible viscous fluid model of Korteweg type in the $N$ dimensional Euclidean space $\mathbb{R}^{N}, N \geq 3$.

$$
\begin{cases}\partial_{t} \rho+\operatorname{div}(\rho \mathbf{u})=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T)  \tag{1.1}\\ \rho\left(\partial_{t} \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}\right)-\operatorname{Div} \mathbf{T}+\nabla P(\rho)=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T) \\ \left.(\rho, \mathbf{u})\right|_{t=0}=\left(\rho_{*}+\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N},\end{cases}
$$

where $\partial_{t}=\partial / \partial t, t$ is the time variable, $\rho=\rho(x, t), x=\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{R}^{N}$ and $\mathbf{u}=\mathbf{u}(x, t)=$ $\left(u_{1}(x, t), \ldots, u_{N}(x, t)\right)$ are respective unknown density field and velocity field, $P(\rho)$ is the pressure field satisfying a $C^{\infty}$ function defined on $\rho>0$, where $\rho_{*}$ is a positive constant. Moreover, $\mathbf{T}=\mathbf{S}(\mathbf{u})+\mathbf{K}(\rho)$ is the stress tensor, where $\mathbf{S}(\mathbf{u})$ and $\mathbf{K}(\rho)$ are respective the viscous stress tensor and Korteweg stress tensor given by

$$
\begin{aligned}
\mathbf{S}(\mathbf{u}) & =\mu_{*} \mathbf{D}(\mathbf{u})+\left(\nu_{*}-\mu_{*}\right) \operatorname{div} \mathbf{u} \mathbb{I} \\
\mathbf{K}(\rho) & =\frac{\kappa_{*}}{2}\left(\Delta \rho^{2}-|\nabla \rho|^{2}\right) \mathbb{I}-\kappa_{*} \nabla \rho \otimes \nabla \rho
\end{aligned}
$$

Here, $\mathbf{D}(\mathbf{u})$ denotes the deformation tensor whose $(j, k)$ components are $D_{j k}(\mathbf{u})=\partial_{j} u_{k}+\partial_{k} u_{j}$ with $\partial_{j}=\partial / \partial x_{j}$. For any vector of functions $\mathbf{v}=\left(v_{1}, \ldots, v_{N}\right)$, we set $\operatorname{div} \mathbf{v}=\sum_{j=1}^{N} \partial_{j} v_{j}$, and also for any $N \times N$ matrix field $\mathbf{L}$ with $(j, k)^{\text {th }}$ components $L_{j k}$, the quantity $\operatorname{Div} \mathbf{L}$ is an $N$-vector with $j^{\text {th }}$ component $\sum_{k=1}^{N} \partial_{k} L_{j k} . \mathbb{I}$ is the $N \times N$ identity matrix and $\mathbf{a} \otimes \mathbf{b}$ denotes an $N \times N$ matrix with $(j, k)^{\text {th }}$ component $a_{j} b_{k}$ for any two $N$-vectors $\mathbf{a}=\left(a_{1}, \ldots, a_{N}\right)$ and $\mathbf{b}=\left(b_{1}, \ldots, b_{N}\right)$. We assume that the viscosity coefficients $\mu_{*}, \nu_{*}$, the capillary coefficient $\kappa_{*}$, and the mass density $\rho_{*}$ of the reference body satisfy the conditions:

$$
\begin{equation*}
\mu_{*}>0, \quad \mu_{*}+\nu_{*}>0, \quad \kappa_{*}>0, \quad P^{\prime}\left(\rho_{*}\right)>0, \quad \text { and } \quad \frac{1}{4}\left(\frac{\mu_{*}+\nu_{*}}{\rho_{*}}\right)^{2} \neq \rho_{*} \kappa_{*} \tag{1.2}
\end{equation*}
$$

[^0]Under the condition (1.2), we can prove the suitable decay properties of solutions to the linearized equations in addition to the maximal $L_{p}-L_{q}$ regularity, which enable us to prove the global wellposedness, cf. Theorem 4.1, below. The system (1.1) governs the motion of the compressible fluids with capillarity effects, which was proposed by Korteweg [16] as a diffuse interface model for liquid-vapor flows based on Van der Waals's approach [26] and derived rigorously by Dunn and Serrin in [8]. There are many mathematical results on Korteweg model. Bresch, Desjardins, and Lin 3] proved the existence of global weak solution, and then Haspot improved their result in [10. Hattori and Li [11, 12, first showed the local and global unique existence in Sobolev space. They assumed the initial data ( $\rho_{0}, \mathbf{u}_{0}$ ) belong to $H^{s+1}\left(\mathbb{R}^{N}\right) \times H^{s}\left(\mathbb{R}^{N}\right)^{N}(s \geq[N / 2]+3)$. Hou, Peng, and Zhu [13] improved the results [11, 12] when the total energy is small. Wang and Tan [27, Tan and Wang [22], Tan, Wang, and Xu [23], and Tan and Zhang [24] established the optimal decay rates of the global solutions in Sobolev space. Li [17] and Chen and Zhao [4] considerd Navier-Stokes-Korteweg system with external force. Bian, Yeo, and Zhu [1] obtained the vanishing capillarity limit of the smooth solution. In particular, we refer to the existence and uniqueness results in critical Besov space proved by Danchin and Desjardins in 6. Their initial data $\left(\rho_{0}, \mathbf{u}_{0}\right)$ are assumed to belong to $\dot{B}_{2,1}^{N / 2}\left(\mathbb{R}^{N}\right) \cap \dot{B}_{2,1}^{N / 2-1}\left(\mathbb{R}^{N}\right) \times \dot{B}_{2,1}^{N / 2-1}\left(\mathbb{R}^{N}\right)^{N}$. It is not clear about the decay estimates for the solutions in $[6]$. In this paper, we discuss the global existence and uniqueness of the strong solutions for (1.1) in the maximal $L_{p}-L_{q}$ regularity class. We also prove the decay estimates of the solutions to (1.1). We assume that the initial data, $\left(\rho_{0}, \mathbf{u}_{0}\right)$, belong to the following Besov space:

$$
D_{q, p}\left(\mathbb{R}^{N}\right)=B_{q, p}^{3-2 / p}\left(\mathbb{R}^{N}\right) \times B_{q, p}^{2(1-1 / p)}\left(\mathbb{R}^{N}\right)^{N}
$$

where regularity of the initial data is independent of the dimension comparing with [6]. In oder to establish the unique existence theorem of global in time strong solutions in Sobolev space, we take the exponents $p$ large enough freely to guarantee $L_{p}$ summability in time, because we can expect only polynomially in time decay properties in unbounded domains. This is one of the important aspects of the maximal $L_{p}-L_{q}$ regularity approach to the mathematical study of the viscous fluid flows. Since the Korteweg model was drived by using on Van der Waals potential, we also have to consider the cases where $P^{\prime}\left(\rho_{*}\right)=0$ and $P^{\prime}\left(\rho_{*}\right)<0$ unlike the Navier-Stokes-Fourier model. We know the local wellposedness for thses two cases, but for the global well-posedness, our approach does not work. On this point, we refer [5] and [15].

Finally, we summarize several symbols and functional spaces used throughout the paper. $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$ and $\mathbb{R}_{+}=(0, \infty)$. Let $q^{\prime}$ be the dual exponent of $q$ defined by $q^{\prime}=q /(q-1)$ for $1<q<\infty$. For any multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{N}\right) \in \mathbb{N}_{0}^{N}$, we write $|\alpha|=\alpha_{1}+\cdots+\alpha_{N}$ and $\partial_{x}^{\alpha}=\partial_{1}^{\alpha_{1}} \cdots \partial_{N}^{\alpha_{N}}$ with $x=\left(x_{1}, \ldots, x_{N}\right)$. For scalar function $f$ and $N$-vector of functions $\mathbf{g}$, we set

$$
\begin{gathered}
\nabla f=\left(\partial_{1} f, \ldots, \partial_{N} f\right), \quad \nabla \mathbf{g}=\left(\partial_{i} g_{j} \mid i, j=1, \ldots, N\right) \\
\nabla^{2} f=\left\{\partial_{i} \partial_{j} f \mid i, j=1, \ldots, N\right\}, \quad \nabla^{2} \mathbf{g}=\left\{\partial_{i} \partial_{j} g_{k} \mid i, j, k=1, \ldots, N\right\}
\end{gathered}
$$

where $\partial_{i}=\partial / \partial x_{i}$. For scalar functions, $f, g$, and $N$-vectors of functions, $\mathbf{f}, \mathbf{g}$, we set $(f, g)_{\mathbb{R}^{N}}=\int_{\mathbb{R}^{N}} f g d x$, and $(\mathbf{f}, \mathbf{g})_{\mathbb{R}^{N}}=\int_{\mathbb{R}^{N}} \mathbf{f} \cdot \mathbf{g} d x$, respectively. For Banach spaces $X$ and $Y, \mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from $X$ into $Y$ and $\operatorname{Hol}(\mathrm{U}, \mathcal{L}(\mathrm{X}, \mathrm{Y}))$ the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a domain $U$ in $\mathbb{C}$. For any $1 \leq p, q \leq \infty, L_{q}\left(\mathbb{R}^{N}\right), W_{q}^{m}\left(\mathbb{R}^{N}\right)$ and $B_{q, p}^{s}\left(\mathbb{R}^{N}\right)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_{q}\left(\mathbb{R}^{N}\right)},\|\cdot\|_{W_{q}^{m}\left(\mathbb{R}^{N}\right)}$ and $\|\cdot\|_{B_{q, p}^{s}\left(\mathbb{R}^{N}\right)}$ denote their norms, respectively. We set $W_{q}^{0}\left(\mathbb{R}^{N}\right)=L_{q}\left(\mathbb{R}^{N}\right)$ and $W_{q}^{s}\left(\mathbb{R}^{N}\right)=B_{q, q}^{s}\left(\mathbb{R}^{N}\right) . C^{\infty}\left(\mathbb{R}^{N}\right)$ denotes the set all $C^{\infty}$ functions defined on $\mathbb{R}^{N} . L_{p}((a, b), X)$ and $W_{p}^{m}((a, b), X)$ denote the usual Lebesgue space and Sobolev space of $X$-valued function defined on an interval $(a, b)$, respectively. The $d$-product space of $X$ is defined by $X^{d}=\left\{f=\left(f, \ldots, f_{d}\right) \mid f_{i} \in X(i=1, \ldots, d)\right\}$, while its norm is denoted by $\|\cdot\|_{X}$ instead of $\|\cdot\|_{X^{d}}$ for the sake of simplicity. We set

$$
W_{q}^{m, \ell}\left(\mathbb{R}^{N}\right)=\left\{(f, \mathbf{g}) \mid f \in W_{q}^{m}\left(\mathbb{R}^{N}\right), \mathbf{g} \in W_{q}^{\ell}\left(\mathbb{R}^{N}\right)^{N}\right\},\|(f, \mathbf{g})\|_{W_{q}^{m, \ell}\left(\mathbb{R}^{N}\right)}=\|f\|_{W_{q}^{m}\left(\mathbb{R}^{N}\right)}+\|\mathbf{g}\|_{W_{q}^{\ell}\left(\mathbb{R}^{N}\right)}
$$

Furthermore, we set

$$
\begin{aligned}
L_{p, \delta}\left(\mathbb{R}_{+}, X\right) & =\left\{f(t) \in L_{p, \text { loc }}\left(\mathbb{R}_{+}, X\right) \mid e^{-\delta t} f(t) \in L_{p}\left(\mathbb{R}_{+}, X\right)\right\} \\
W_{p, \delta}^{1}\left(\mathbb{R}_{+}, X\right) & =\left\{f(t) \in L_{p, \delta}\left(\mathbb{R}_{+}, X\right) \mid e^{-\delta t} \partial_{t}^{j} f(t) \in L_{p}\left(\mathbb{R}_{+}, X\right)(j=0,1)\right\}
\end{aligned}
$$

for $1<p<\infty$ and $\delta>0$. Let $\mathcal{F}_{x}=\mathcal{F}$ and $\mathcal{F}_{\xi}^{-1}=\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by setting

$$
\hat{f}(\xi)=\mathcal{F}_{x}[f](\xi)=\int_{\mathbb{R}^{N}} e^{-i x \cdot \xi} f(x) d x, \quad \mathcal{F}_{\xi}^{-1}[g](x)=\frac{1}{(2 \pi)^{N}} \int_{\mathbb{R}^{N}} e^{i x \cdot \xi} g(\xi) d \xi
$$

The letter $C$ denotes generic constants and the constant $C_{a, b, \ldots}$ depends on $a, b, \ldots$. The values of constants $C$ and $C_{a, b, \ldots}$ may change from line to line. We use small boldface letters, e.g. u to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{H}$ to denote matrix-valued functions, respectively. In order to state our main theorem, we set a solution space and several norms:

$$
\begin{align*}
X_{p, q, t} & =\left\{(\theta, \mathbf{u}) \mid \theta \in L_{p}\left((0, t), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right) \cap W_{p}^{1}\left((0, t), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)\right. \\
\mathbf{u} & \left.\in L_{p}\left((0, t), W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right) \cap W_{p}^{1}\left((0, t), L_{q}\left(\mathbb{R}^{N}\right)^{N}\right), \quad \rho_{*} / 4 \leq \rho_{*}+\theta(t, x) \leq 4 \rho_{*}\right\} \\
{[U]_{q, \ell, t} } & =\sup _{0 \leq s \leq t}<s>^{\ell}\|U(\cdot, s)\|_{L_{q}\left(\mathbb{R}^{N}\right)}(U=\theta, \mathbf{u},(\theta, \mathbf{u})), \\
{[\nabla U]_{q, \ell, t} } & =\sup _{0 \leq s \leq t}<s>^{\ell}\|\nabla U(\cdot, s)\|_{L_{q}\left(\mathbb{R}^{N}\right)}(U=\theta,(\theta, \mathbf{u})), \\
\mathcal{N}(\theta, \mathbf{u})(t) & =\sum_{j=0}^{1} \sum_{i=1}^{2}\left\{\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{\infty, \frac{N}{q_{1}}+\frac{j}{2}, t}\right. \\
& +\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{1}, \frac{N}{2 q_{1}}+\frac{j}{2}, t}+\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{2}, \frac{N}{2 q_{2}}+1+\frac{j}{2}, t}  \tag{1.3}\\
& +\|\left(<s>^{\ell_{i}}(\theta, \mathbf{u})\left\|_{L_{p}\left((0, t), W_{q_{i}}^{3,2}\left(\mathbb{R}^{N}\right)\right)}+\right\|<s>^{\ell_{i}}\left(\partial_{s} \theta, \partial_{s} \mathbf{u}\right) \|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\right\},
\end{align*}
$$

where $\langle s\rangle=(1+s), \ell_{1}=N / 2 q_{1}-\tau, \ell_{2}=N / 2 q_{2}+1-\tau$, and $\tau$ is given in Theorem1.1 below.
We now state our main theorem.
Theorem 1.1. Assume that condition (1.2) holds and that $N \geq 3$. Let $q_{1}, q_{2}$ and $p$ be numbers such that

$$
2<p<\infty, q_{1}<N<q_{2}, \frac{1}{q_{1}}=\frac{1}{q_{2}}+\frac{1}{N}, \frac{2}{p}+\frac{N}{q_{2}}<1
$$

Let $\tau$ be a number such that

$$
\frac{1}{p}<\tau<\frac{N}{q_{2}}+\frac{1}{p}
$$

Then, there exists a small number $\epsilon>0$ such that for any initial data $\left(\rho_{0}, \mathbf{u}_{0}\right) \in \cap_{i=1}^{2} D_{q_{i}, p}\left(\mathbb{R}^{N}\right) \cap$ $L_{q_{1} / 2}\left(\mathbb{R}^{N}\right)^{N+1}$ with

$$
\mathcal{I}:=\sum_{i=1}^{2}\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q_{i}, p}\left(\mathbb{R}^{N}\right)}+\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{L_{q_{1} / 2}\left(\mathbb{R}^{N}\right)}<\epsilon
$$

problem (1.1) admits a solution ( $\rho, \mathbf{u}$ ) with $\rho=\rho_{*}+\theta$ and

$$
(\theta, \mathbf{u}) \in X_{p, q_{2}, \infty}
$$

satisfying the estimate

$$
\mathcal{N}(\theta, \mathbf{u})(\infty) \leq L \epsilon
$$

with some constant $L$ independent of $\epsilon$.
Remark 1.2. (1) In theorem [1.1, the constant $L$ is defined from several constants appearing in the estimates for the linearized equations and the constant $\epsilon$ will be chosen in such a way that $L^{2} \epsilon<1$.
(2) We only consider the dimension $N \geq 3$. In fact, in the case $N=2, q_{1}<2$, and so $q_{1} / 2<1$. In this case, our argument does not work.

## 2 Maximal $L_{p}-L_{q}$ regularity

In this section, we show the maximal $L_{p}-L_{q}$ regularity for problem:

$$
\begin{cases}\partial_{t} \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f & \text { in } \mathbb{R}^{N} \text { for } t>0  \tag{2.1}\\ \gamma_{0} \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g} & \text { in } \mathbb{R}^{N} \text { for } t>0 \\ \left.(\rho, \mathbf{u})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

where $\gamma_{i}(i=0,1,2)$ are functions of $x \in \mathbb{R}^{N}$ satisfying the following assumption:
Assumption 2.1. Let $\gamma_{k}=\gamma_{k}(x)(k=0,1,2)$ be uniformly continuous functions on $\mathbb{R}^{N}$. Moreover, there exist positive constants $\rho_{1}$ and $\rho_{2}$ such that

$$
\begin{equation*}
\rho_{1} \leq \gamma_{k}(x) \leq \rho_{2}, \quad\left|\nabla \gamma_{k}(x)\right| \leq \rho_{2} \quad \text { for any } x \in \mathbb{R}^{N} . \tag{2.2}
\end{equation*}
$$

We now state the maximal $L_{p}-L_{q}$ regularity theorem.
Theorem 2.2. Let $1<p, q<\infty$ and suppose that Assumption 2.1 holds. Then, there exists a constant $\delta_{0} \geq 1$ such that the following assertion holds: For any initial data $\left(\rho_{0}, \mathbf{u}_{0}\right) \in D_{q, p}\left(\mathbb{R}^{N}\right)$ and functions in the right-hand sides $(f, \mathbf{g}) \in L_{p, \delta_{0}}\left(\mathbb{R}_{+}, W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right.$ ), problem (2.1) admits unique solutions $\rho$ and $\mathbf{u}$ with

$$
\begin{aligned}
& \rho \in W_{p, \delta_{0}}^{1}\left(\mathbb{R}_{+}, W_{q}^{1}\left(\mathbb{R}^{N}\right)\right) \cap L_{p, \delta_{0}}\left(\mathbb{R}_{+}, W_{q}^{3}\left(\mathbb{R}^{N}\right)\right), \\
& \mathbf{u} \in W_{p, \delta_{0}}^{1}\left(\mathbb{R}_{+}, L_{q}\left(\mathbb{R}^{N}\right)^{N}\right) \cap L_{p, \delta_{0}}\left(\mathbb{R}_{+}, W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right),
\end{aligned}
$$

possessing the estimate

$$
\begin{align*}
& \left\|e^{-\delta t} \partial_{t} \rho\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \rho\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|e^{-\delta t} \partial_{t} \mathbf{u}\right\|_{L_{p}\left(\mathbb{R}_{+}, L_{q}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \mathbf{u}\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{2}\left(\mathbb{R}^{N}\right)\right)}  \tag{2.3}\\
& \leq C_{p, q, N, \delta_{0}}\left(\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}+\left\|\left(e^{-\delta t} f, e^{-\delta t} \mathbf{g}\right)\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\right)
\end{align*}
$$

for any $\delta \geq \delta_{0}$.

## 2.1 $\mathcal{R}$-boundedness of solution operators

In this subsection, we analyze the following resolvent problem in order to prove Theorem 2.2

$$
\begin{cases}\lambda \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f & \text { in } \mathbb{R}^{N}  \tag{2.4}\\ \gamma_{0} \lambda \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g} & \text { in } \mathbb{R}^{N}\end{cases}
$$

where $\mu_{*}, \nu_{*}, \kappa_{*}$ and $\gamma_{k}=\gamma_{k}(x)$ are satisfying (1.2) and (2.2). Here, $\lambda$ is the resolvent parameter varying in a sector

$$
\Sigma_{\epsilon, \lambda_{0}}=\left\{\lambda \in \mathbb{C}| | \arg \lambda\left|<\pi-\epsilon,|\lambda| \geq \lambda_{0}\right\}\right.
$$

for $0<\epsilon<\pi / 2$ and $\lambda_{0} \geq 1$.
We introduce the definition of the $\mathcal{R}$-boundedness of operator families.
Definition 2.3. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C>0$ and $p \in[1, \infty)$ such that for any $n \in \mathbb{N},\left\{T_{j}\right\}_{j=1}^{n} \subset \mathcal{T},\left\{f_{j}\right\}_{j=1}^{n} \subset X$ and sequences $\left\{r_{j}\right\}_{j=1}^{n}$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:

$$
\left\{\int_{0}^{1}\left\|\sum_{j=1}^{n} r_{j}(u) T_{j} f_{j}\right\|_{Y}^{p} d u\right\}^{1 / p} \leq C\left\{\int_{0}^{1}\left\|\sum_{j-1}^{n} r_{j}(u) f_{j}\right\|_{X}^{p} d u\right\}^{1 / p}
$$

The smallest such $C$ is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T})$.
The following theorem is the main result of this subsection.

Theorem 2.4. Let $1<q<\infty, 0<\epsilon<\pi / 2$ and suppose that Assumption 2.1 holds. Then, there exist a positive constant $\lambda_{0} \geq 1$ and operator families

$$
\begin{aligned}
& \mathcal{A}(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon, \lambda_{0}}, \mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)\right) \\
& \mathcal{B}(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon, \lambda_{0}}, \mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right)\right)
\end{aligned}
$$

such that for any $\lambda=\delta+i \tau \in \Sigma_{\epsilon, \lambda_{0}}$ and $F=(f, \mathbf{g}) \in W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$,

$$
\rho=\mathcal{A}(\lambda) F, \quad \mathbf{u}=\mathcal{B}(\lambda) F
$$

are unique solutions of problem (2.4), and

$$
\begin{align*}
& \mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), A_{q}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\left(\tau \partial_{\tau}\right)^{\ell} \mathcal{S}_{\lambda} \mathcal{A}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq 2 \kappa_{0},  \tag{2.5}\\
& \mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), B_{q}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\left(\tau \partial_{\tau}\right)^{\ell} \mathcal{T}_{\lambda} \mathcal{B}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq 2 \kappa_{0}
\end{align*}
$$

for $\ell=0,1$, where $\mathcal{S}_{\lambda} \rho=\left(\nabla^{3} \rho, \lambda^{1 / 2} \nabla^{2} \rho, \lambda \rho\right), \mathcal{T}_{\lambda} \mathbf{u}=\left(\nabla^{2} \mathbf{u}, \lambda^{1 / 2} \nabla \mathbf{u}, \lambda \mathbf{u}\right), A_{q}\left(\mathbb{R}^{N}\right)=L_{q}\left(\mathbb{R}^{N}\right)^{N^{3}+N^{2}} \times$ $W_{q}^{1}\left(\mathbb{R}^{N}\right), B_{q}\left(\mathbb{R}^{N}\right)=L_{q}\left(\mathbb{R}^{N}\right)^{N^{3}+N^{2}+N}$, and $\kappa_{0}$ is a constant independent of $\lambda$.

Postponing the proof of Theorem 2.4 we are concerned with time dependent problem (2.1). Let $\mathcal{A}$ be a linear operator defined by

$$
\mathcal{A}(\rho, \mathbf{u})=\left(-\gamma_{2} \operatorname{div} \mathbf{u}, \gamma_{0}^{-1} \mu_{*} \Delta \mathbf{u}+\gamma_{0}^{-1} \nu_{*} \nabla \operatorname{div} \mathbf{u}-\gamma_{0}^{-1} \nabla\left(\gamma_{1} \rho\right)+\gamma_{0}^{-1} \kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)\right)
$$

for $(\rho, \mathbf{u}) \in W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$. Since Definition 2.3 with $n=1$ implies the uniform boundedness of the operator family $\mathcal{T}$, solutions $\rho$ and $\mathbf{u}$ of equations (2.4) satisfy the resolvent estimate:

$$
\begin{equation*}
|\lambda|\|(\rho, \mathbf{u})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}+\|(\rho, \mathbf{u})\|_{W_{q}^{3,2}\left(\mathbb{R}^{N}\right)} \leq C_{\kappa_{0}}\|(f, \mathbf{g})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \tag{2.6}
\end{equation*}
$$

for any $\lambda \in \Sigma_{\epsilon, \lambda_{0}}$ and $(f, \mathbf{g}) \in W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$. By (2.6), we have the following theorem.
Theorem 2.5. Let $1<q<\infty$ and suppose that Assumption 2.1 holds. Then, the operator $\mathcal{A}$ generates an analytic semigroup $\left\{e^{\mathcal{A} t}\right\}_{t \geq 0}$ on $W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$. Moreover, there exists constants $\delta_{1} \geq 1$ and $C_{q, N, \delta_{1}}>0$ such that $\left\{e^{\mathcal{A} t}\right\}_{t \geq 0}$ satisfies the estimates:

$$
\begin{aligned}
&\left\|e^{\mathcal{A} t}\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C_{q, N, \delta_{1}} e^{\delta_{1} t}\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}, \\
&\left\|\partial_{t} e^{\mathcal{A} t}\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C_{q, N, \delta_{1}} e^{\delta_{1} t} t^{-1}\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \\
&\left\|\partial_{t} e^{\mathcal{A} t}\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C_{q, N, \delta_{1}} e^{\delta_{1} t}\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{W_{q}^{3,2}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

for any $t>0$.
Combining Theorem 2.5 with a real interpolation method (cf. Shibata and Shimizu 21, Proof of Theorem 3.9]), we have the following result for the equation (2.1) with $(f, \mathbf{g})=(0,0)$.

Theorem 2.6. Let $1<p, q<\infty$, and suppose that Assumption 2.1 holds. Then, for any $\left(\rho_{0}, \mathbf{u}_{0}\right) \in$ $D_{q, p}\left(\mathbb{R}^{N}\right)$, problem (2.1) with $(f, \mathbf{g})=(0,0)$ admits a unique solution $(\rho, \mathbf{u})=e^{\mathcal{A} t}\left(\rho_{0}, \mathbf{u}_{0}\right)$ possessing the estimate:

$$
\begin{align*}
& \left\|e^{-\delta t} \partial_{t} \rho\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \rho\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|e^{-\delta t} \partial_{t} \mathbf{u}\right\|_{L_{p}\left(\mathbb{R}_{+}, L_{q}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \mathbf{u}\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{2}\left(\mathbb{R}^{N}\right)\right)}  \tag{2.7}\\
& \leq C_{p, q, N, \delta_{1}}\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}
\end{align*}
$$

for any $\delta \geq \delta_{1}$.
The remaining part of this subsection is devoted to proving Theorem [2.4. For this purpose, we use the following lemmas.

Lemma 2.7. (1) Let $X$ and $Y$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T}+\mathcal{S}=\{T+S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and

$$
\mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{S})
$$

(2) Let $X, Y$ and $Z$ be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S} \mathcal{T}=\{S T \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and

$$
\mathcal{R}_{\mathcal{L}(X, Z)}(\mathcal{S T}) \leq \mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T}) \mathcal{R}_{\mathcal{L}(Y, Z)}(\mathcal{S})
$$

(3) Let $1<p, q<\infty$ and let $D$ be domain in $\mathbb{R}^{N}$. Let $m(\lambda)$ be a bounded function defined on a subset $\Lambda$ in a complex plane $\mathbb{C}$ and let $M_{m}(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_{m}(\lambda) f=m(\lambda) f$ for any $f \in L_{q}(D)$. Then,

$$
\mathcal{R}_{\mathcal{L}\left(L_{q}(D)\right)}\left(\left\{M_{m}(\lambda) \mid \lambda \in \Lambda\right\}\right) \leq C_{N, q, D}\|m\|_{L_{\infty}(\Sigma)}
$$

Proof. For the assertions (1) and (2) we refer [7, Proposition 3.4], and for the assertions (3) we refer [7] Remarks 3.2 (4)] (also see [2]).

Proof of Theorem 2.4. We first construct $\mathcal{R}$-bounded solution operators. According to Theorem 3.1 in [18, we have the operator families

$$
\begin{aligned}
& \mathcal{A}_{0}(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon, \lambda_{0}}, \mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)\right) \\
& \mathcal{B}_{0}(\lambda) \in \operatorname{Hol}\left(\Sigma_{\epsilon, \lambda_{0}}, \mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right)\right)
\end{aligned}
$$

such that for any $\lambda \in \Sigma_{\epsilon, \lambda_{0}}$ and $F \in W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$,

$$
\begin{equation*}
\rho=\mathcal{A}_{0}(\lambda) F, \quad \mathbf{u}=\mathcal{B}_{0}(\lambda) F \tag{2.8}
\end{equation*}
$$

uniquely solve the equations

$$
\begin{cases}\lambda \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f & \text { in } \mathbb{R}^{N}  \tag{2.9}\\ \gamma_{0} \lambda \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g} & \text { in } \mathbb{R}^{N}\end{cases}
$$

which is the case where (2.4) with $\gamma_{1}=0$. Moreover, we know that

$$
\begin{align*}
& \mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), A_{q}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\left(\tau \partial_{\tau}\right)^{\ell} \mathcal{S}_{\lambda} \mathcal{A}_{0}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq \kappa_{0} \\
& \mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right), B_{q}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\left(\tau \partial_{\tau}\right)^{\ell} \mathcal{T}_{\lambda} \mathcal{B}_{0}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq \kappa_{0} \quad(\ell=0,1) \tag{2.10}
\end{align*}
$$

with some constant $\kappa_{0}$. Inserting (2.8) into the left-hand sides of (2.4), we have

$$
\begin{cases}\lambda \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f & \text { in } \mathbb{R}^{N}  \tag{2.11}\\ \gamma_{0} \lambda \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g}+\nabla\left(\gamma_{1} \mathcal{A}_{0}(\lambda) F\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

Set $\mathcal{F}(\lambda) F=\left(0,-\nabla\left(\gamma_{1} \mathcal{A}_{0}(\lambda) F\right)\right)$. Let $n \in \mathbb{N},\left\{\lambda_{\ell}\right\}_{\ell=1}^{n} \subset\left(\Sigma_{\epsilon, \lambda_{0}}\right)^{n}$, and $\left\{F_{\ell}\right\}_{\ell=1}^{n} \subset\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)^{n}$.
By Lemma 2.7 and (2.2), we have

$$
\begin{aligned}
& \int_{0}^{1}\left\|\sum_{\ell=1}^{n} r_{\ell}(u) \mathcal{F}\left(\lambda_{\ell}\right) F_{\ell}\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}^{q} d u \\
& =\int_{0}^{1}\left\|\sum_{\ell=1}^{n} r_{\ell}(u) \nabla\left(\gamma_{1} \mathcal{A}_{0}\left(\lambda_{\ell}\right) F_{\ell}\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)}^{q} d u \\
& \leq C_{\rho_{2}}^{q}\left(\int_{0}^{1}\left\|\sum_{\ell=1}^{n} r_{\ell}(u) \mathcal{A}_{0}\left(\lambda_{\ell}\right) F_{\ell}\right\|_{L_{q}\left(\mathbb{R}^{N}\right)}^{q} d u+\int_{0}^{1}\left\|\sum_{\ell=1}^{n} r_{\ell}(u) \nabla \mathcal{A}_{0}\left(\lambda_{\ell}\right) F_{\ell}\right\|_{L_{q}\left(\mathbb{R}^{N}\right)}^{q} d u\right) \\
& \leq C_{\rho_{2}}^{q} \kappa_{0}^{q}\left(\lambda_{0}^{-3 q / 2}+\lambda_{0}^{-q}\right) \int_{0}^{1}\left\|\sum_{\ell=1}^{n} r_{\ell}(u) F_{\ell}\right\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}^{q} d u .
\end{aligned}
$$

Choosing $\lambda_{0} \geq 1$ so large that $C_{\rho_{2}}^{q} \kappa_{0}^{q}\left(\lambda_{0}^{-3 q / 2}+\lambda_{0}^{-q}\right) \leq(1 / 2)^{q}$, we have

$$
\begin{equation*}
\mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\mathcal{F}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq 1 / 2 \tag{2.12}
\end{equation*}
$$

Analogously, we have

$$
\begin{equation*}
\mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\tau \partial \tau \mathcal{F}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq 1 / 2 \tag{2.13}
\end{equation*}
$$

By (2.12) and (2.13), for each $\lambda \in \Sigma_{\epsilon, \lambda_{0}},(\mathbb{I}-\mathcal{F}(\lambda))^{-1}=\mathbb{I}+\sum_{k=1}^{\infty} \mathcal{F}(\lambda)^{k}$ exists and

$$
\begin{equation*}
\mathcal{R}_{\mathcal{L}\left(W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\left(\left\{\left(\tau \partial_{\tau}\right)^{\ell}(\mathbb{I}-\mathcal{F}(\lambda))^{-1} \mid \lambda \in \Sigma_{\epsilon, \lambda_{0}}\right\}\right) \leq 2 \quad(\ell=0,1) \tag{2.14}
\end{equation*}
$$

where $\mathbb{I}$ is the identity operator. Setting $\mathcal{A}(\lambda)=\mathcal{A}_{0}(\lambda)(\mathbb{I}-\mathcal{F}(\lambda))^{-1}, \mathcal{B}(\lambda)=\mathcal{B}_{0}(\lambda)(\mathbb{I}-\mathcal{F}(\lambda))^{-1}$, by (2.10), (2.14) and Lemma 2.7 we see that $(\rho, \mathbf{u})=(\mathcal{A}(\lambda) F, \mathcal{B}(\lambda) F)$ is a solution to (2.4) and $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ possess the estimates (2.5).

We next show the uniqueness of solutions. Let $B_{d}\left(x_{0}\right) \subset \mathbb{R}^{N}$ be the ball of radius $d>0$ centered at $x_{0} \in \mathbb{R}^{N}$. In view of (2.2), we may assume that

$$
\begin{equation*}
\left|\gamma_{k}(x)-\gamma_{k}\left(x_{0}\right)\right| \leq \rho_{2} M_{1} \text { for any } x \in B_{d}\left(x_{0}\right) \quad(k=0,1,2) \tag{2.15}
\end{equation*}
$$

where we set $M_{1}=d$. We will choose $M_{1}$ small enough eventually, so that we may assume that $0<M_{1}<1$ below.

Let $(\rho, \mathbf{u})$ be a solution of the homogeneous equations:

$$
\begin{cases}\lambda \rho+\gamma_{2} \operatorname{div} \mathbf{u}=0 & \text { in } \mathbb{R}^{N}  \tag{2.16}\\ \gamma_{0} \lambda \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=0 & \text { in } \mathbb{R}^{N}\end{cases}
$$

By (2.16), ( $\rho, \mathbf{u}$ ) satisfies the following equations:

$$
\begin{cases}\lambda \rho+\gamma_{2}\left(x_{0}\right) \operatorname{div} \mathbf{u}=F(\rho, \mathbf{u}) & \text { in } \mathbb{R}^{N}, \\ \gamma_{0}\left(x_{0}\right) \lambda \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1}\left(x_{0}\right) \rho\right)-\kappa_{*} \nabla\left(\gamma_{2}\left(x_{0}\right) \Delta \rho\right)=G(\rho, \mathbf{u}) & \text { in } \mathbb{R}^{N}\end{cases}
$$

where

$$
\begin{aligned}
& F(\rho, \mathbf{u})=\left(\gamma_{2}\left(x_{0}\right)-\gamma_{2}\right) \operatorname{div} \mathbf{u} \\
& G(\rho, \mathbf{u})=\left(\gamma_{0}-\gamma_{0}\left(x_{0}\right)\right) \lambda \mathbf{u}+\nabla\left(\left(\gamma_{1}\left(x_{0}\right)-\gamma_{1}\right) \rho\right)-\kappa_{*} \nabla\left(\left(\gamma_{2}\left(x_{0}\right)-\gamma_{2}\right) \Delta \rho\right)
\end{aligned}
$$

By (2.2) and (2.15), we have

$$
\begin{equation*}
\|(F(\rho, \mathbf{u}), G(\rho, \mathbf{u}))\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C_{\rho_{2}} M_{1}\left\|\left(\nabla^{3} \rho, \nabla^{2} \mathbf{u}, \lambda \mathbf{u}\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)}+C_{\rho_{2}}\left\|\left(\rho, \nabla \rho, \nabla^{2} \rho, \nabla \mathbf{u}\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)} \tag{2.17}
\end{equation*}
$$

On the other hand, by (2.5), we have

$$
\begin{equation*}
\|(\rho, \mathbf{u})\|_{W_{q}^{3,2}\left(\mathbb{R}^{N}\right)}+\lambda_{0}^{1 / 2}\left\|\left(\rho, \nabla \rho, \nabla^{2} \rho, \nabla \mathbf{u}\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)} \leq 2 \kappa_{0}\|(F(\rho, \mathbf{u}), G(\rho, \mathbf{u}))\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \tag{2.18}
\end{equation*}
$$

for any $\lambda \in \Sigma_{\epsilon, \lambda_{0}}$. Combining (2.17) and (2.18), we have

$$
\left(1-2 \kappa_{0} C_{\rho_{2}} M_{1}\right)\|(\rho, \mathbf{u})\|_{W_{q}^{3,2}\left(\mathbb{R}^{N}\right)}+\left(\lambda_{0}^{1 / 2}-2 \kappa_{0} C_{\rho_{2}}\right)\left\|\left(\rho, \nabla \rho, \nabla^{2} \rho, \nabla \mathbf{u}\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)} \leq 0
$$

Choosing $M_{1}$ so small that $1-2 \kappa_{0} C_{\rho_{2}} M_{1}>0$ and $\lambda_{0}$ so large that $\lambda_{0}^{1 / 2}-2 \kappa_{0} C_{\rho_{2}}>0$, we have $(\rho, \mathbf{u})=(0,0)$, which proves the uniqueness. This completes the proof of Theorem 2.4,

### 2.2 A proof of Theorem 2.2

To prove Theorem 2.2 the key tool is the Weis operator valued Fourier multiplier theorem. Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all $X$ valued $C^{\infty}$ functions having compact support and the Schwartz space of rapidly decreasing $X$ valued functions, respectively, while $\mathcal{S}^{\prime}(\mathbb{R}, X)=\mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in$ $L_{1, \text { loc }}(\mathbb{R} \backslash\{0\}, X)$, we define the operator $T_{M}: \mathcal{F}^{-1} \mathcal{D}(\mathbb{R}, X) \rightarrow \mathcal{S}^{\prime}(\mathbb{R}, Y)$ by

$$
\begin{equation*}
T_{M} \phi=\mathcal{F}^{-1}[M \mathcal{F}[\phi]], \quad(\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)) . \tag{2.19}
\end{equation*}
$$

Theorem 2.8 (Weis [28). Let $X$ and $Y$ be two UMD Banach spaces and $1<p<\infty$. Let $M$ be $a$ function in $C^{1}(\mathbb{R} \backslash\{0\}, \mathcal{L}(X, Y))$ such that

$$
\mathcal{R}_{\mathcal{L}(X, Y)}\left(\left\{\left.\left(\tau \frac{d}{d \tau}\right)^{\ell} M(\tau) \right\rvert\, \tau \in \mathbb{R} \backslash\{0\}\right\}\right) \leq \kappa<\infty \quad(\ell=0,1)
$$

with some constant $\kappa$. Then, the operator $T_{M}$ defined in (2.19) is extended to a bounded linear operator from $L_{p}(\mathbb{R}, X)$ into $L_{p}(\mathbb{R}, Y)$. Moreover, denoting this extension by $T_{M}$, we have

$$
\left\|T_{M}\right\|_{\mathcal{L}\left(L_{p}(\mathbb{R}, X), L_{p}(\mathbb{R}, Y)\right)} \leq C \kappa
$$

for some positive constant $C$ depending on $p, X$ and $Y$.
We now prove Theorem 2.2 In view of Theorem [2.6, we prove the existence of solutions to problem (2.1) with $\left(\rho_{0}, \mathbf{u}_{0}\right)=(0,0)$. Let $(f, \mathbf{g}) \in L_{p, \delta_{0}}\left(\mathbb{R}_{+}, W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)$. Let $f_{0}$ and $\mathbf{g}_{0}$ be the zero extension of $f$ and $\mathbf{g}$ to $t<0$. We consider problem:

$$
\begin{align*}
\partial_{t} \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f_{0} & \text { in } \mathbb{R}^{N} \text { for } t \in \mathbb{R}, \\
\gamma_{0} \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g}_{0} & \text { in } \mathbb{R}^{N} \text { for } t \in \mathbb{R} . \tag{2.20}
\end{align*}
$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ be the Laplace transform and its inverse transform. Let $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$ be the operators given in Theorem 2.4. Then, we have

$$
\begin{aligned}
\rho & =\mathcal{L}^{-1}\left[\mathcal{A}(\lambda)\left(\mathcal{L}\left[f_{0}\right], \mathcal{L}\left[\mathbf{g}_{0}\right]\right)\right], \\
\mathbf{u} & =\mathcal{L}^{-1}\left[\mathcal{B}(\lambda)\left(\mathcal{L}\left[f_{0}\right], \mathcal{L}\left[\mathbf{g}_{0}\right]\right)\right] .
\end{aligned}
$$

with $\lambda=\delta+i \tau \in \mathbb{C}$. Applying Theorem[2.4] and Theorem 2.8, we see that $\rho$ and $\mathbf{u}$ satisfy the equations (2.20) and the estimate:

$$
\begin{align*}
& \left\|e^{-\delta t} \partial_{t} \rho\right\|_{L_{p}\left(\mathbb{R}, W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \rho\right\|_{L_{p}\left(\mathbb{R}, W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|e^{-\delta t} \partial_{t} \mathbf{u}\right\|_{L_{p}\left(\mathbb{R}, L_{q}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C_{N, p, q, \delta_{0}}\left\|\left(e^{-\delta t} f_{0}, e^{-\delta t} \mathbf{g}_{0}\right)\right\|_{L_{p}\left(\mathbb{R}, W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}  \tag{2.21}\\
& =C_{N, p, q, \delta_{0}}\left\|\left(e^{-\delta t} f, e^{-\delta t} \mathbf{g}\right)\right\|_{L_{p}\left(\mathbb{R}_{+}, W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}
\end{align*}
$$

for any $\delta \geq \delta_{0}$.
We now prove that $\rho=0$ and $\mathbf{u}=0$ for $t \leq 0$, we consider the dual problem. Let $T$ be a real number. By Theorem [2.6] we see that for any $\left(\theta_{0}, \mathbf{v}_{0}\right) \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)^{N+1}$, there exists a solution $(\theta, \mathbf{v})$ such that

$$
\begin{cases}\partial_{t} \theta+\gamma_{2} \operatorname{div} \mathbf{v}=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(-T, \infty) \\ \gamma_{0} \partial_{t} \mathbf{v}-\mu_{*} \Delta \mathbf{v}-\nu_{*} \nabla \operatorname{div} \mathbf{v}+\nabla\left(\gamma_{1} \theta\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \theta\right)=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(-T, \infty) \\ \left.(\theta, \mathbf{v})\right|_{t=-T}=\left(\theta_{0}, \mathbf{v}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

satisfying

$$
\begin{aligned}
& \left\|e^{-\delta t} \partial_{t} \theta\right\|_{L_{p}\left((-T, \infty), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \theta\right\|_{L_{p}\left((-T, \infty), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|e^{-\delta t} \partial_{t} v\right\|_{L_{p}\left((-T, \infty), L_{q}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{-\delta t} \mathbf{v}\right\|_{L_{p}\left((-T, \infty), W_{q}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C_{N, p, q, \delta_{1}}\left\|\left(\theta_{0}, \mathbf{v}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)} .
\end{aligned}
$$

Setting $\omega(x, t)=\theta(x,-t)$ and $\mathbf{w}(x, t)=\mathbf{v}(x,-t)$, we see that $(\omega, \mathbf{w})$ satisfies

$$
\begin{cases}\partial_{t} \omega-\gamma_{2} \operatorname{div} \mathbf{w}=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(-T, \infty) \\ \gamma_{0} \partial_{t} \mathbf{w}+\mu_{*} \Delta \mathbf{w}+\nu_{*} \nabla \operatorname{div} \mathbf{w}-\nabla\left(\gamma_{1} \omega\right)+\kappa_{*} \nabla\left(\gamma_{2} \Delta \omega\right)=0 & \text { in } \mathbb{R}^{N} \text { for } t \in(-T, \infty) \\ \left.(\omega, \mathbf{w})\right|_{t=T}=\left(\theta_{0}, \mathbf{v}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

satisfying

$$
\begin{align*}
& \left\|e^{\delta t} \partial_{t} \omega\right\|_{L_{p}\left((-\infty, T), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{\delta t} \omega\right\|_{L_{p}\left((-\infty, T), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|e^{\delta t} \partial_{t} \mathbf{w}\right\|_{L_{p}\left((-\infty, T), L_{q}\left(\mathbb{R}^{N}\right)\right)}+\left\|e^{\delta t} \mathbf{w}\right\|_{L_{p}\left((-\infty, T), W_{q}^{2}\left(\mathbb{R}^{N}\right)\right)}  \tag{2.22}\\
& \leq C_{N, p, q, \delta_{1}}\left\|\left(\theta_{0}, \mathbf{v}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)} .
\end{align*}
$$

By integration by parts, we have

$$
\begin{align*}
& \left(\mathbf{g}_{0}, \mathbf{w}\right)_{\mathbb{R}^{N} \times(-\infty, T)}-\left(\gamma_{1} \gamma_{2}^{-1} f_{0}, \omega\right)_{\mathbb{R}^{N} \times(-\infty, T)}+\left(\kappa_{*} f_{0}, \Delta \omega\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& =\left(\gamma_{0} \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right), \mathbf{w}\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& \quad-\left(\gamma_{1} \gamma_{2}^{-1}\left(\partial_{t} \rho+\gamma_{2} \operatorname{div} \mathbf{u}\right), \omega\right)_{\mathbb{R}^{N} \times(-\infty, T)}+\left(\kappa_{*}\left(\partial_{t} \rho+\gamma_{2} \operatorname{div} \mathbf{u}\right), \Delta \omega\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& =\left(\gamma_{0} \mathbf{u}(T), \mathbf{w}(T)\right)_{\mathbb{R}^{N}}-\left(\gamma_{1} \gamma_{2}^{-1} \rho(T), \omega(T)\right)_{\mathbb{R}^{N}}+\left(\kappa_{*} \rho(T), \Delta \omega(T)\right)_{\mathbb{R}^{N}} \\
& \quad-\left(\mathbf{u}, \gamma_{0} \partial_{t} \mathbf{w}+\mu_{*} \Delta \mathbf{w}+\nu_{*} \nabla \operatorname{div} \mathbf{w}\right)_{\mathbb{R}^{N} \times(-\infty, T)}-\left(\rho, \gamma_{1} \operatorname{div} \mathbf{w}\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& \quad+\kappa_{*}\left(\rho, \Delta\left(\gamma_{2} \operatorname{div} \mathbf{w}\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)}+\left(\rho, \partial_{t}\left(\gamma_{1} \gamma_{2}^{-1} \omega\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)}+\left(\mathbf{u}, \nabla\left(\gamma_{1} \omega\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)}  \tag{2.23}\\
& \quad-\kappa_{*}\left(\rho, \partial_{t}(\Delta \omega)\right)_{\mathbb{R}^{N} \times(-\infty, T)}-\kappa_{*}\left(\mathbf{u}, \nabla\left(\gamma_{2} \Delta \omega\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& =\left(\gamma_{0} \mathbf{u}(T), \mathbf{v}_{0}\right)_{\mathbb{R}^{N}}-\left(\gamma_{1} \gamma_{2}^{-1} \rho(T), \theta_{0}\right)_{\mathbb{R}^{N}}+\left(\kappa_{*} \rho(T), \Delta \theta_{0}\right)_{\mathbb{R}^{N}} \\
& \quad-\left(\mathbf{u}, \gamma_{0} \partial_{t} \mathbf{w}+\mu_{*} \Delta \mathbf{w}+\nu_{*} \nabla \operatorname{div} \mathbf{w}-\nabla\left(\gamma_{1} \omega\right)+\kappa_{*} \nabla\left(\gamma_{2} \Delta \omega\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& \quad+\left(\rho, \gamma_{1} \gamma_{2}^{-1}\left(\partial_{t} \omega-\gamma_{2} \operatorname{div} \mathbf{w}\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)}+\kappa_{*}\left(\rho, \Delta\left(\partial_{t} \omega-\gamma_{2} \operatorname{div} \mathbf{w}\right)\right)_{\mathbb{R}^{N} \times(-\infty, T)} \\
& =\left(\gamma_{0} \mathbf{u}(T), \mathbf{v}_{0}\right)_{\mathbb{R}^{N}}-\left(\gamma_{1} \gamma_{2}^{-1} \rho(T), \theta_{0}\right)_{\mathbb{R}^{N}}+\left(\kappa_{*} \rho(T), \Delta \theta_{0}\right)_{\mathbb{R}^{N}} .
\end{align*}
$$

Let $T$ be any negative number. Since $f_{0}=0, \mathbf{g}_{0}=0$ for $t<0$, we have

$$
\left(\gamma_{0} \mathbf{u}(T), \mathbf{v}_{0}\right)_{\mathbb{R}^{N}}-\left(\gamma_{1} \gamma_{2}^{-1} \rho(T), \theta_{0}\right)_{\mathbb{R}^{N}}+\left(\kappa_{*} \rho(T), \Delta \theta_{0}\right)_{\mathbb{R}^{N}}=0
$$

Choosing $\mathbf{v}_{0}$ and $\theta_{0}$ arbitrarily, we see that $(\rho(T), \mathbf{u}(T))=(0,0)$ for any $T \leq 0$. Finally, $(\rho, \mathbf{u})+$ $e^{\mathcal{A} t}\left(\rho_{0}, \mathbf{u}_{0}\right)$ is a solution of equations (2.1), which completes the existence proof.

Finally, we show the uniqueness of solutions. Let $\rho$ and $\mathbf{u}$ satisfy the equation (2.20) with $f_{0}=0$, $\mathbf{g}_{0}=0$. By (2.23), we have $\left(\gamma_{0} \mathbf{u}(T), \mathbf{v}_{0}\right)_{\mathbb{R}^{N}}-\left(\gamma_{1} \gamma_{2}^{-1} \rho(T), \theta_{0}\right)_{\mathbb{R}^{N}}+\left(\kappa_{*} \rho(T), \Delta \theta_{0}\right)_{\mathbb{R}^{N}}=0$ for any $\left(\theta_{0}, \mathbf{v}_{0}\right) \in$ $C_{0}^{\infty}\left(\mathbb{R}^{N}\right)^{N+1}$ and $T \in \mathbb{R}$, which implies that $(\rho(T), \mathbf{u}(T))=(0,0)$ for any $T \in \mathbb{R}$. This completes the proof of Theorem 2.2.

## 3 Local well-posedness for (1.1)

This section is devoted to proving the local wellposedness stated as follows.
Theorem 3.1. Let $1<p, q<\infty, 2 / p+N / q<1$ and $R>0$. Then, there exists a time $T$ depending on $R$ such that for any initial data $\left(\rho_{0}, \mathbf{u}_{0}\right) \in D_{q, p}\left(\mathbb{R}^{N}\right)$ with $\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)} \leq R$ satisfying the range condition (3.3), problem (1.1) admits a unique solution $(\rho, \mathbf{u})$ with $\rho=\rho_{*}+\theta$ and $(\theta, \mathbf{u}) \in X_{p, q, T}$.

To prove Theorem 3.1 we linearize nonlinear problem (1.1) at $\left(\rho_{*}+\rho_{0}(x), 0\right)$, and then we have the equations:

$$
\begin{cases}\partial_{t} \theta+\left(\rho_{*}+\rho_{0}(x)\right) \operatorname{div} \mathbf{u}=f(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T)  \tag{3.1}\\ \left(\rho_{*}+\rho_{0}(x)\right) \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u} & \\ \quad+P^{\prime}\left(\rho_{*}\right) \nabla \theta-\kappa_{*} \nabla\left(\left(\rho_{*}+\rho_{0}(x)\right) \Delta \theta\right)=\mathbf{g}(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T) \\ \left.(\theta, \mathbf{u})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

where

$$
\begin{aligned}
f(\theta, \mathbf{u})= & -\int_{0}^{t} \partial_{s} \theta d s \operatorname{div} \mathbf{u}-\mathbf{u} \cdot \nabla \theta \\
\mathbf{g}(\theta, \mathbf{u})= & -\int_{0}^{t} \partial_{s} \theta d s \partial_{t} \mathbf{u}-\left(\rho_{*}+\theta\right) \mathbf{u} \cdot \nabla \mathbf{u}-\nabla\left(\int_{0}^{1} P^{\prime \prime}\left(\rho_{*}+\tau \theta\right)(1-\tau) d \tau \theta^{2}\right) \\
& +\nabla\left(\kappa_{*} \int_{0}^{t} \partial_{s} \theta d s \Delta \theta\right)+\kappa_{*} \operatorname{Div}\left(\frac{1}{2}|\nabla \theta|^{2} \mathbb{I}-\nabla \theta \otimes \nabla \theta\right)
\end{aligned}
$$

To solve problem (3.1) in the maximal regularity class, we consider the following time local linear problem:

$$
\begin{cases}\partial_{t} \rho+\gamma_{2} \operatorname{div} \mathbf{u}=f & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T)  \tag{3.2}\\ \gamma_{0} \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u}+\nabla\left(\gamma_{1} \rho\right)-\kappa_{*} \nabla\left(\gamma_{2} \Delta \rho\right)=\mathbf{g} & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T) \\ \left.(\rho, \mathbf{u})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

If we extend $f$ and $\mathbf{g}$ by zero outside of $(0, T)$, by Theorem 2.2 and the uniquness of solutions, we have the following result.

Theorem 3.2. Let $T, R>0,1<p, q<\infty$ and suppose that Assumption 2.1 holds. Then, there exists a constant $\delta_{0} \geq 1$ such that the following assertion holds: For any initial data $\left(\rho_{0}, \mathbf{u}_{0}\right) \in D_{q, p}\left(\mathbb{R}^{N}\right)$ with $\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)} \leq R$ satisfying the range condition:

$$
\begin{equation*}
\rho_{*} / 2<\rho_{*}+\rho_{0}(x)<2 \rho_{*} \quad\left(x \in \mathbb{R}^{N}\right) \tag{3.3}
\end{equation*}
$$

and right members $(f, \mathbf{g}) \in L_{p}\left((0, T), W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)$, problem (3.2) admits a unique solution $(\rho, \mathbf{u}) \in X_{p, q, T}$ possessing the estimate

$$
\begin{equation*}
E_{p, q}(\rho, \mathbf{u})(t) \leq C_{p, q, N, \delta_{0}, R} e^{\delta t}\left(\left\|\left(\rho, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}+\|(f, \mathbf{g})\|_{L_{p}\left((0, t), W_{q}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\right) \tag{3.4}
\end{equation*}
$$

for any $t \in(0, T]$ and $\delta \geq \delta_{0}$, where we set

$$
\begin{aligned}
E_{p, q}(\rho, \mathbf{u})(t) & =\left\|\partial_{s} \rho\right\|_{L_{p}\left((0, t), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)}+\|\rho\|_{L_{p}\left((0, t), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)} \\
& +\left\|\partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q}\left(\mathbb{R}^{N}\right)^{N}\right)}+\|\mathbf{u}\|_{L_{p}\left((0, t), W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right)}
\end{aligned}
$$

and constant $C_{p, q, N, \delta_{0}, R}$ is independent of $\delta$ and $t$.
To prove Theorem 3.1, we use the following Lemma.
Lemma 3.3. Let $\mathbf{u} \in W_{p}^{1}\left((0, T), L_{q}\left(\mathbb{R}^{N}\right)^{N}\right) \cap L_{p}\left((0, T), W_{q}^{2}\left(\mathbb{R}^{N}\right)^{N}\right)$ and $\rho \in W_{p}^{1}\left((0, T), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right) \cap$ $L_{p}\left((0, T), W_{q}^{3}\left(\mathbb{R}^{N}\right)\right)$, with $2<p<\infty, 1<q<\infty$ and $T>0$. Then,

$$
\begin{equation*}
\sup _{0<s<T}\|(\rho(\cdot, s), \mathbf{u}(\cdot, s))\|_{D_{q, p}\left(\mathbb{R}^{N}\right)} \leq C\left\{\|(\rho(\cdot, 0), \mathbf{u}(\cdot, 0))\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}+E_{p, q}(\rho, \mathbf{u})(T)\right\} \tag{3.5}
\end{equation*}
$$

with the constant $C$ independent of $T$.
If we assume that $2 / p+N / q<1$ in addition, then

$$
\begin{equation*}
\sup _{0<s<S}\|(\rho(\cdot, s), \mathbf{u}(\cdot, s))\|_{W_{\infty}^{2,1}\left(\mathbb{R}^{N}\right)} \leq C\left\{\|(\rho(\cdot, 0), \mathbf{u}(\cdot, 0))\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}+E_{p, q}(\rho, \mathbf{u})(S)\right\} \tag{3.6}
\end{equation*}
$$

for any $S \in(0, T)$ with the constant $C$ independent of $S$ and $T$.
Proof. Employing the same argument as in the proof of Lemma 1 in [19], we see that inequality (3.5) follows from real interpolation theorem. By $2 / p+N / q<1$, we see that $B_{q, p}^{2(1-1 / p)}\left(\mathbb{R}^{N}\right)$ and $B_{q, p}^{3-2 / p}\left(\mathbb{R}^{N}\right)$ are continuously imbedded into $W_{\infty}^{1}\left(\mathbb{R}^{N}\right)$ and $W_{\infty}^{2}\left(\mathbb{R}^{N}\right)$, respectively, and so by (3.5), we have (3.6).

Proof of Theorem 3.1. Let $T$ and $L$ be two positive numbers determined later and let $\mathcal{I}_{L, T}$ be a space defined by setting

$$
\begin{equation*}
\mathcal{I}_{L, T}=\left\{(\theta, \mathbf{u}) \in X_{p, q, T}|(\theta, \mathbf{u})|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right), E_{p, q}(\theta, \mathbf{u})(T) \leq L\right\} . \tag{3.7}
\end{equation*}
$$

Given $(\omega, \mathbf{v}) \in \mathcal{I}_{L, T}$, let $\theta$ and $\mathbf{u}$ be solutions to the following problem:

$$
\begin{cases}\partial_{t} \theta+\left(\rho_{*}+\rho_{0}(x)\right) \operatorname{div} \mathbf{u}=f(\omega, \mathbf{v}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T)  \tag{3.8}\\ \left(\rho_{*}+\rho_{0}(x)\right) \partial_{t} \mathbf{u}-\mu_{*} \Delta \mathbf{u}-\nu_{*} \nabla \operatorname{div} \mathbf{u} & \\ \quad+P^{\prime}\left(\rho_{*}\right) \nabla \theta-\kappa_{*} \nabla\left(\left(\rho_{*}+\rho_{0}(x)\right) \Delta \theta\right)=\mathbf{g}(\omega, \mathbf{v}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T) \\ \left.(\theta, \mathbf{u})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

We first consider the estimate for the right-hand sides of (3.8). Since $2(1-1 / p)>1$ by Lemma 3.3, we have

$$
\begin{equation*}
\sup _{t \in(0, T)}\|\mathbf{v}(\cdot, t)\|_{W_{q}^{1}\left(\mathbb{R}^{N}\right)} \leq C\left(\left\|\mathbf{u}_{0}\right\|_{\left.B_{q, p}^{2(1-1 / p)}\left(\mathbb{R}^{N}\right)\right)}+E_{p, q}(\omega, \mathbf{v})(T)\right) \leq C(R+L) \tag{3.9}
\end{equation*}
$$

where $C$ is a constant independent of $T$. Moreover, we have $B_{q, p}^{3-2 / p}\left(\mathbb{R}^{N}\right) \subset W_{q}^{2}\left(\mathbb{R}^{N}\right)$, and then

$$
\begin{equation*}
\sup _{t \in(0, T)}\|\omega(\cdot, t)\|_{W_{q}^{2}\left(\mathbb{R}^{N}\right)} \leq C\left(\left\|\rho_{0}\right\|_{B_{q, p}^{3-2 / p}\left(\mathbb{R}^{N}\right)}+E_{p, q}(\omega, \mathbf{v})(T)\right) \leq C(R+L) \tag{3.10}
\end{equation*}
$$

Since

$$
\begin{equation*}
W_{q}^{1}\left(\mathbb{R}^{N}\right) \subset L_{\infty}\left(\mathbb{R}^{N}\right) \tag{3.11}
\end{equation*}
$$

as follows from the assumption $N<q<\infty$, by (3.3) and Hölder's inequality, we have

$$
\begin{equation*}
\sup _{t \in(0, T)}\|\omega(\cdot, t)\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}=\sup _{t \in(0, T)}\left\|\int_{0}^{t} \partial_{s} \omega(\cdot, s) d s+\rho_{0}\right\|_{L_{\infty}} \leq C T^{1 / p^{\prime}} L+\frac{\rho_{*}}{2} . \tag{3.12}
\end{equation*}
$$

Choosing $T$ so small that $C T^{1 / p^{\prime}} L \leq \rho_{*} / 4$, we have $\rho_{*} / 4 \leq \rho_{*}+\tau \omega \leq 7 \rho_{*} / 4(\tau \in[0,1])$, so that

$$
\begin{align*}
& \sup _{t \in(0, T)}\left\|\nabla \int_{0}^{1} P^{\prime \prime}\left(\rho_{*}+\tau \omega\right)(1-\tau) d \tau\right\|_{L_{\infty}\left(\mathbb{R}^{N}\right)} \\
& \leq \sup _{t \in(0, T)}\left\|\nabla \int_{0}^{1} P^{\prime \prime \prime}\left(\rho_{*}+\tau \omega\right)(1-\tau) d \tau \nabla \omega(\cdot, t)\right\|_{L_{\infty}\left(\mathbb{R}^{N}\right)} \\
& \leq C \sup _{t \in(0, T)}\|\nabla \omega(\cdot, t)\|_{L_{\infty}\left(\mathbb{R}^{N}\right) \leq C(R+L)} . \tag{3.13}
\end{align*}
$$

By (3.5), (3.11), (3.13) and Hölder's inequality, we have

$$
\begin{align*}
& \|f(\omega, \mathbf{v})\|_{L_{p}\left((0, T), W_{q}^{1}\left(\mathbb{R}^{N}\right)\right)} \leq C\left\{T^{1 / p}(R+L)^{2}+T^{1 / p^{\prime}} L^{2}\right\}  \tag{3.14}\\
& \|\mathbf{g}(\omega, \mathbf{v})\|_{L_{p}\left((0, T), L_{q}\left(\mathbb{R}^{N}\right)\right)} \leq C\left\{T^{1 / p}(R+L)^{2}+T^{1 / p}(R+L)^{3}+T^{1 / p^{\prime}} L^{2}\right\}, \tag{3.15}
\end{align*}
$$

where $C$ is a constant independent of $T, L$ and $R$. By Theorem 3.2, (3.14) and (3.15), we have

$$
\begin{equation*}
E_{p, q}(\theta, \mathbf{u})(T) \leq C_{R} e^{\delta T}\left\{\left\|\left(\rho, \mathbf{u}_{0}\right)\right\|_{D_{q, p}\left(\mathbb{R}^{N}\right)}+C(R, L, T)\right\} \tag{3.16}
\end{equation*}
$$

where $C(R, L, T)=T^{1 / p}(R+L)^{2}+T^{1 / p}(R+L)^{3}+T^{1 / p^{\prime}} L^{2}$. Choosing $T \in(0,1)$ so small that $C(R, L, T) \leq$ $R$, by (3.16), we have

$$
E_{p, q}(\theta, \mathbf{u})(T) \leq 2 C_{R} e^{\delta T} R
$$

Choosing $T$ in such a way that $\delta T \leq 1$ in addition, and setting $L=2 C_{R} R$, we have

$$
\begin{equation*}
E_{p, q}(\theta, \mathbf{u})(T) \leq L \tag{3.17}
\end{equation*}
$$

Let $\Phi$ be a map defined by $\Phi(\omega, \mathbf{v})$, and then by (3.17) $\Phi$ is a map from $\mathcal{I}_{L, T}$ into itself. Let $\left(\omega_{i}, \mathbf{v}_{i}\right) \in \mathcal{I}_{L, T}(i=1,2),\left(\theta_{1}-\theta_{2}, \mathbf{u}_{1}-\mathbf{u}_{2}\right)$ with $\left(\theta_{i}, \mathbf{u}_{i}\right)=\Phi\left(\omega_{i}, \mathbf{v}_{i}\right)$ satisfies (3.8) with zero initial data. By Theorem 3.2 we have

$$
E_{p, q}\left(\theta_{1}-\theta_{2}, \mathbf{u}_{1}-\mathbf{u}_{2}\right)(T) \leq C e^{\delta T}\left(L+L^{2}\right)\left(T^{1 / p}+T^{1 / p^{\prime}}\right) E_{p, q}\left(\omega_{1}-\omega_{2}, \mathbf{v}_{1}-\mathbf{v}_{2}\right)(T)
$$

Choosing $T$ so small that $C e^{\delta T}\left(L+L^{2}\right)\left(T^{1 / p}+T^{1 / p^{\prime}}\right) \leq 1 / 2, \Phi$ is contraction on $\mathcal{I}_{L, T}$, so that by the Banach contraction mapping theorem, there exists a unique fixed point $(\theta, \mathbf{u}) \in \mathcal{I}_{L, T}$ such that $(\theta, \mathbf{u})=\Phi(\theta, \mathbf{u})$, which uniquely solves (3.1). This completes the proof of Theorem 3.1]

## 4 Global well-posedness for (1.1) with small initial data

In this section, we show the global well-posedness for (1.1), that is, we prove Theorem 1.1. Setting $\rho=\rho_{*}+\theta, \alpha_{*}=\mu_{*} / \rho_{*}, \beta_{*}=\nu_{*} / \rho_{*}$ and $\gamma_{*}=P^{\prime}\left(\rho_{*}\right) / \rho_{*}$, we write (1.1) as follows:

$$
\begin{cases}\partial_{t} \theta+\rho_{*} \operatorname{div} \mathbf{u}=f(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T)  \tag{4.1}\\ \partial_{t} \mathbf{u}-\alpha_{*} \Delta \mathbf{u}-\beta_{*} \nabla \operatorname{div} \mathbf{u}-\kappa_{*} \nabla \Delta \theta+\gamma_{*} \nabla \theta=\mathbf{g}(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t \in(0, T) \\ \left.(\theta, \mathbf{u})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

where

$$
\begin{aligned}
f(\theta, \mathbf{u})= & -(\theta \operatorname{div} \mathbf{u}+\mathbf{u} \cdot \nabla \theta) \\
\mathbf{g}(\theta, \mathbf{u})= & -\mathbf{u} \cdot \nabla \mathbf{u}+\left(\frac{1}{\rho_{*}+\theta}-\frac{1}{\rho_{*}}\right) \operatorname{Div} \mathbf{S}+\frac{\kappa_{*}}{\rho_{*}+\theta}\left(\nabla \theta \Delta \theta+\frac{1}{2} \operatorname{Div}|\nabla \theta|^{2}-\operatorname{Div}(\nabla \theta \otimes \nabla \theta)\right) \\
& -\left(\frac{P^{\prime}\left(\rho_{*}\right)}{\rho_{*}+\theta}-\frac{P^{\prime}\left(\rho_{*}\right)}{\rho_{*}}\right) \nabla \theta-\frac{P^{\prime}\left(\rho_{*}+\theta\right)-P^{\prime}\left(\rho_{*}\right)}{\rho_{*}+\theta} \nabla \theta .
\end{aligned}
$$

To prove Theorem 1.1 the key issue is decay properties of solutions, and so we start with the following subsection.

### 4.1 Decay property of solutions to the linearized problem

In this subsection, we consider the following linearized problem:

$$
\begin{cases}\partial_{t} \theta+\rho_{*} \operatorname{div} \mathbf{u}=0 & \text { in } \mathbb{R}^{N} \text { for } t>0  \tag{4.2}\\ \partial_{t} \mathbf{u}-\alpha_{*} \Delta \mathbf{u}-\beta_{*} \nabla \operatorname{div} \mathbf{u}-\kappa_{*} \nabla \Delta \theta+\gamma_{*} \nabla \theta=0 & \text { in } \mathbb{R}^{N} \text { for } t>0 \\ \left.(\theta, \mathbf{u})\right|_{t=0}=(f, \mathbf{g}) & \text { in } \mathbb{R}^{N}\end{cases}
$$

Then, by taking Fourier transform of (4.2) and solving the ordinary differential equation with respect to $t$, we have

$$
\begin{align*}
S_{1}(t)(f, \mathbf{g}) & :=\theta=-\mathcal{F}_{\xi}^{-1}\left[\frac{\lambda_{-} e^{\lambda_{+} t}-\lambda_{+} e^{\lambda_{-} t}}{\lambda_{+}-\lambda_{-}} \hat{f}\right]-\sum_{k=1}^{N} \mathcal{F}_{\xi}^{-1}\left[\rho_{*} \frac{e^{\lambda_{+} t}-e^{\lambda_{-} t}}{\lambda_{+}-\lambda_{-}} i \xi_{k} \hat{g}_{k}\right] \\
S_{2}(t)(f, \mathbf{g}) & :=\mathbf{u}=\mathcal{F}_{\xi}^{-1}\left[e^{-\alpha_{*}|\xi|^{2} t} \hat{\mathbf{g}}\right]-\sum_{k=1}^{N} \mathcal{F}_{\xi}^{-1}\left[e^{-\alpha_{*}|\xi|^{2} t} \frac{\xi \xi_{k}}{|\xi|^{2}} \hat{g}_{k}\right]-\mathcal{F}_{\xi}^{-1}\left[i\left(\gamma_{*}+\kappa_{*}|\xi|^{2}\right) \frac{e^{\lambda_{+} t}-e^{\lambda_{-} t}}{\lambda_{+}-\lambda_{-}} \xi \hat{f}\right] \\
& -\sum_{k=1}^{N} \mathcal{F}_{\xi}^{-1}\left[\frac{\left\{\left(\alpha_{*}+\beta_{*}\right)|\xi|^{2}+\lambda_{-}\right\} e^{\lambda_{+} t}-\left\{\left(\alpha_{*}+\beta_{*}\right)|\xi|^{2}+\lambda_{+}\right\} e^{\lambda_{-} t}}{|\xi|^{2}\left(\lambda_{+}-\lambda_{-}\right)},\right. \tag{4.3}
\end{align*}
$$

where

$$
\lambda_{ \pm}=-\frac{\alpha_{*}+\beta_{*}}{2}|\xi|^{2} \pm \sqrt{\left(\frac{\left(\alpha_{*}+\beta_{*}\right)^{2}}{4}-\rho_{*} \kappa_{*}\right)|\xi|^{4}-\rho_{*} \gamma_{*}|\xi|^{2}}
$$

To show decay estimates of $\theta$ and $\mathbf{u}$, we use the following expansion formulae:

$$
\begin{align*}
& \lambda_{ \pm}=-\frac{\alpha_{*}+\beta_{*}}{2}|\xi|^{2} \pm i \sqrt{\rho_{*} \gamma_{*}}|\xi|+i O\left(|\xi|^{2}\right) \\
& \lambda_{ \pm}= \begin{cases}-\frac{\alpha_{*}+\beta_{*}}{2}|\xi|^{2} \pm \sqrt{\delta_{*}}|\xi|^{2}+O(1) & \delta_{*}>0 \\
-\frac{\alpha_{*}+\beta_{*}}{2}|\xi|^{2} \pm i \sqrt{\left|\delta_{*}\right|}|\xi|^{2}+O(1) & \delta_{*}<0, \quad \text { as }|\xi| \rightarrow \infty\end{cases} \tag{4.4}
\end{align*}
$$

where $\delta_{*}=\left(\alpha_{*}+\beta_{*}\right)^{2} / 4-\rho_{*} \kappa_{*}$.
Theorem 4.1. Let $S_{i}(t)(i=1,2)$ be the solution operators of (4.2) given (4.3) and let $S(t)(f, \mathbf{g})=$ $\left(S_{1}(t)(f, \mathbf{g}), S_{2}(t)(f, \mathbf{g})\right)$. Then, $S(t)$ has the following decay property

$$
\begin{equation*}
\left\|\partial_{x}^{j} S(t)(f, \mathbf{g})\right\|_{W_{p}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)} \tag{4.5}
\end{equation*}
$$

with $j \in \mathbb{N}_{0}$ and some constant $C$ depending on $j, p, q, \alpha_{*}, \beta_{*}$ and $\gamma_{*}$, where

$$
\begin{cases}1<q \leq p \leq \infty \text { and }(p, q) \neq(\infty, \infty) & \text { if } 0<t \leq 1,  \tag{4.6}\\ 1<q \leq 2 \leq p \leq \infty \text { and }(p, q) \neq(\infty, \infty) & \text { if } t \geq 1 .\end{cases}
$$

Proof. To prove (4.5), we divide the solution formula into the low frequency part and high frequency part. For this purpose, we introduce a cut off function $\varphi(\xi) \in C^{\infty}\left(\mathbb{R}^{N}\right)$ which equals 1 for $|\xi| \leq \epsilon$ and 0 for $|\xi| \geq 2 \epsilon$, where $\epsilon$ is a suitably small positive constant. Let $\Phi_{0}$ and $\Phi_{\infty}$ be operators acting on $(f, \mathbf{g}) \in W_{q}^{1,0}\left(\mathbb{R}^{N}\right)$ defined by setting

$$
\Phi_{0}(f, \mathbf{g})=\mathcal{F}_{\xi}^{-1}[\varphi(\xi)(\hat{f}(\xi), \hat{\mathbf{g}}(\xi))], \quad \Phi_{\infty}(f, \mathbf{g})=\mathcal{F}_{\xi}^{-1}[(1-\varphi(\xi))(\hat{f}(\xi), \hat{\mathbf{g}}(\xi))]
$$

Let $S_{i}^{0}(t)(f, \mathbf{g})=S_{i}(t) \Phi_{0}(f, \mathbf{g})$ and $S_{i}^{\infty}(t)(f, \mathbf{g})=S_{i}(t) \Phi_{\infty}(f, \mathbf{g})$. We first consider the low frequency part. Namely, we estimate $S^{0}(f, \mathbf{g})=\left(S_{1}^{0}(t)(f, \mathbf{g}), S_{2}^{0}(t)(f, \mathbf{g})\right)$. If $(p, q)$ satisfies the conditions (4.6), employing the same argument as in the proof of Theorem 3.1 in 14, we have

$$
\left\|\partial_{x}^{j} S^{0}(t)(f, \mathbf{g})\right\|_{W_{p}^{1,0}\left(\mathbb{R}^{N}\right)} \leq C t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}
$$

with $j \in \mathbb{N}_{0}$.
We next consider the high frequency part, that is we estimate $S^{\infty}(t)(f, \mathbf{g})=\left(S_{1}^{\infty}(t)(f, g), S_{2}^{\infty}(t)(f, \mathbf{g})\right)$. By the solution formulas (4.3), we have

$$
\begin{aligned}
S_{1}^{\infty}(t)(f, \mathbf{g}) & =\mathcal{F}_{\xi}^{-1}\left[e^{\lambda_{ \pm}(\xi) t} h(\xi)(\hat{f}, \hat{\mathbf{g}})\right](x) \\
\left(\partial_{x}^{j+1} S_{1}^{\infty}(t)(f, \mathbf{g}), \partial_{x}^{j} S_{2}^{\infty}(t)(f, \mathbf{g})\right) & =\mathcal{F}_{\xi}^{-1}\left[e^{\lambda_{ \pm}(\xi) t} h_{j}(\xi)(i \xi \hat{f}, \hat{\mathbf{g}})\right](x)
\end{aligned}
$$

where $h$ and $h_{j}$ satisfy the conditions:

$$
\begin{equation*}
\left|\partial_{\xi}^{\alpha} h(\xi)\right| \leq C|\xi|^{-|\alpha|}, \quad\left|\partial_{\xi}^{\alpha} h_{j}(\xi)\right| \leq C|\xi|^{j-|\alpha|} \tag{4.7}
\end{equation*}
$$

for $j \in \mathbb{N}_{0}$ and any multi-index $\alpha \in \mathbb{N}_{0}^{N}$ with some constant $C$ depending on $\alpha, \alpha_{*}, \beta_{*}$ and $\gamma_{*}$. Using the estimate $\left(|\xi| t^{1 / 2}\right)^{j} e^{-C_{*}|\xi|^{2} t} \leq C e^{-\left(C_{*} / 2\right)|\xi|^{2} t}$ and the following Bell's formula for the derivatives of the composite functions:

$$
\partial_{\xi}^{\alpha} f(g(\xi))=\sum_{k=1}^{|\alpha|} f^{(k)}(g(\xi)) \sum_{\substack{\alpha=\alpha_{1}+\ldots+\alpha_{k} \\\left|\alpha_{i}\right| \geq 1}} \Gamma_{\alpha_{1}, \ldots, \alpha_{k}}^{\alpha}\left(\partial_{\xi}^{\alpha_{1}} g(\xi)\right) \cdots\left(\partial_{\xi}^{\alpha_{k}} g(\xi)\right)
$$

with $f^{(k)}(t)=d^{k} f(t) / d t^{k}$ and suitable coefficients $\Gamma_{\alpha_{1}, \ldots, \alpha_{k}}^{\alpha}$, we see that

$$
\begin{equation*}
\left|\partial_{\xi}^{\alpha} e^{\lambda_{ \pm}(\xi) t}\right| \leq C e^{-C_{*}|\xi|^{2} t}|\xi|^{-|\alpha|} \tag{4.8}
\end{equation*}
$$

with some constant $C_{*}$ depending on $\alpha_{*}, \beta_{*}$ and $\gamma_{*}$. By (4.7) and (4.8), we have

$$
\left|\partial_{\xi}^{\alpha} e^{\lambda_{ \pm}(\xi) t} h(\xi)\right| \leq C e^{-\left(C_{*} / 2\right)|\xi|^{2} t}|\xi|^{-|\alpha|},\left|\partial_{\xi}^{\alpha} e^{\lambda_{ \pm}(\xi) t} h_{j}(\xi)\right| \leq C t^{-j / 2} e^{-\left(C_{*} / 2\right)|\xi|^{2} t}|\xi|^{-|\alpha|}
$$

Applying Fourier multiplier theorem, we have

$$
\begin{aligned}
\left\|S_{1}^{\infty}(t)(f, \mathbf{g})\right\|_{L_{q}\left(\mathbb{R}^{N}\right)} & \leq C_{q} e^{-c t}\|(f, \mathbf{g})\|_{L_{q}\left(\mathbb{R}^{N}\right)} \\
\left\|\left(\partial_{x}^{j+1} S_{1}^{\infty}(t)(f, \mathbf{g}), \partial_{x}^{j} S_{2}^{\infty}(t)(f, \mathbf{g})\right)\right\|_{L_{q}\left(\mathbb{R}^{N}\right)} & \leq C_{q} t^{-j / 2} e^{-c t}\|(f, \mathbf{g})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

with some positive constant $c$ when $1<q<\infty$, which together with Sobolev's imbedding theorem implies

$$
\begin{aligned}
\left\|S_{1}^{\infty}(t)(f, \mathbf{g})\right\|_{L_{p}\left(\mathbb{R}^{N}\right)} & \leq C_{q} t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|(f, \mathbf{g})\|_{L_{q}\left(\mathbb{R}^{N}\right)} \\
\left\|\left(\partial_{x}^{j+1} S_{1}^{\infty}(t)(f, \mathbf{g}), \partial_{x}^{j} S_{2}^{\infty}(t)(f, \mathbf{g})\right)\right\|_{L_{p}\left(\mathbb{R}^{N}\right)} & \leq C_{q} t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q}^{1,0}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

when $1<q \leq p \leq \infty$ and $(p, q) \neq(\infty, \infty)$, and therefore $S^{\infty}(f, \mathbf{g})$ satisfies (4.5). This completes the proof of Theorem 4.1.

### 4.2 A proof of Theorem 1.1

We prove Theorem 1.1 by the Banach fixed point argument. Let $p, q_{1}$ and $q_{2}$ be exponents given in Theorem 1.1 Let $\epsilon$ be a small positive number and let $\mathcal{N}(\theta, \mathbf{u})$ be the norm defined in (1.3). We define the underlying space $\mathcal{I}_{\epsilon}$ by setting

$$
\begin{equation*}
\mathcal{I}_{\epsilon}=\left\{(\theta, \mathbf{u}) \in X_{p, \frac{q_{1}}{2}, \infty} \cap X_{p, q_{2}, \infty}|(\theta, \mathbf{u})|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right), \quad \mathcal{N}(\theta, \mathbf{u})(\infty) \leq L \epsilon\right\} \tag{4.9}
\end{equation*}
$$

with some constant $L$ which will be determined later. Given $(\theta, \mathbf{u}) \in \mathcal{I}_{\epsilon}$, let $(\omega, \mathbf{w})$ be a solution to the equation:

$$
\begin{cases}\partial_{t} \omega+\rho_{*} \operatorname{div} \mathbf{w}=f(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t>0 \\ \rho_{*} \partial_{t} \mathbf{w}-\mu_{*} \Delta \mathbf{w}-\nu_{*} \nabla \operatorname{div} \mathbf{w}+P^{\prime}\left(\rho_{*}\right) \nabla \omega-\kappa_{*} \rho_{*} \nabla \Delta \omega=\mathbf{g}(\theta, \mathbf{u}) & \text { in } \mathbb{R}^{N} \text { for } t>0 \\ \left.(\omega, \mathbf{w})\right|_{t=0}=\left(\rho_{0}, \mathbf{u}_{0}\right) & \text { in } \mathbb{R}^{N}\end{cases}
$$

where

$$
\begin{aligned}
f(\theta, \mathbf{u})= & -\theta \operatorname{div} \mathbf{u}-\mathbf{u} \cdot \nabla \theta \\
\mathbf{g}(\theta, \mathbf{u})= & -\theta \partial_{t} \mathbf{u}-\left(\rho_{*}+\theta\right) \mathbf{u} \cdot \nabla \mathbf{u}-\nabla\left(\int_{0}^{1} P^{\prime \prime}\left(\rho_{*}+\tau \theta\right)(1-\tau) d \tau \theta^{2}\right) \\
& +\kappa_{*} \operatorname{div}(\theta \nabla \theta)+\kappa_{*} \operatorname{Div}\left(\frac{1}{2}|\nabla \theta|^{2} \mathbb{I}-\nabla \theta \otimes \nabla \theta\right)
\end{aligned}
$$

We shall prove

$$
\begin{equation*}
\mathcal{N}(\omega, \mathbf{w})(t) \leq C\left(\mathcal{I}+\mathcal{N}(\theta, \mathbf{u})(t)^{2}\right) \tag{4.10}
\end{equation*}
$$

where $\mathcal{I}$ is defined in Theorem 1.1
Since $(\theta, \mathbf{u}) \in X_{p, \frac{q_{1}}{2}, \infty} \cap X_{p, q_{2}, \infty}$, we have

$$
\begin{equation*}
\frac{\rho_{*}}{4} \leq \rho_{*}+\theta(t, x) \leq 4 \rho_{*} . \tag{4.11}
\end{equation*}
$$

We now estimate $(\omega, \mathbf{w})$ in the case that $t>2$. By Duhamel's principle, we write $(\omega, \mathbf{w})$ as

$$
\begin{equation*}
(\omega, \mathbf{w})=S(t)\left(\rho_{0}, \mathbf{u}_{0}\right)+\int_{0}^{t} S(t-s)(f(s), \mathbf{g}(s)) d s \tag{4.12}
\end{equation*}
$$

Since $S(t)\left(\rho_{0}, \mathbf{u}_{0}\right)$ can be estimated directly by Theorem 4.1, we only estimate the second term, below. We divide the second term into three parts as follows.

$$
\begin{equation*}
\int_{0}^{t}\left\|\partial_{x}^{j} S(t-s)(f(s), \mathbf{g}(s))\right\|_{X} d s=\left(\int_{0}^{t / 2}+\int_{t / 2}^{t-1}+\int_{t-1}^{t}\right)\left\|\partial_{x}^{j} S(t-s)(f(s), \mathbf{g}(s))\right\|_{X} d s=: \sum_{k=1}^{3} I_{X}^{k} \tag{4.13}
\end{equation*}
$$

for $t>2$, where $X=L_{\infty}, L_{q_{1}}$ and $L_{q_{2}}$.

## Estimates in $L_{\infty}$.

By (4.11) and Theorem 4.1 with $(p, q)=\left(\infty, q_{1} / 2\right)$ and Hölder's inequality under the condition $q_{1} / 2 \leq 2$, we have

$$
\begin{equation*}
I_{\infty}^{1} \leq C \int_{0}^{t / 2}(t-s)^{-\frac{N}{q_{1}}-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q_{1} / 2}^{1,0}\left(\mathbb{R}^{N}\right)} d s \leq C \int_{0}^{t / 2}(t-s)^{-\frac{N}{q_{1}}-\frac{j}{2}}\left(A_{1}+B_{1}\right) d s \tag{4.14}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{1}=\left(\|(\theta, \mathbf{u})\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}\right)\|(\nabla \theta, \nabla \mathbf{u})\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)} \\
& B_{1}=\|\theta\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\left\|\left(\nabla^{2} \theta, \nabla^{2} \mathbf{u}\right)\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}\right)+\left(\|\mathbf{u}\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}\right)\left\|\nabla^{2} \theta\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

Since $A_{1}$ has only lower order derivatives, we have

$$
\begin{align*}
A_{1} & \leq\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{1}{2}\right)}[(\theta, \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}+\left\langle s>^{-\left(\frac{N}{q_{1}}+1\right)}[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right. \\
& \leq\left\langle s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}\right)}\left([(\theta, \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right.\right. \tag{4.15}
\end{align*}
$$

On the other hand, since $B_{1}$ has higher order derivatives, we have

$$
\begin{align*}
B_{1} & \leq\langle s\rangle^{-\left(\frac{N}{q_{1}}-\tau\right)}[\theta]_{q_{1}, \frac{N}{2 q_{1}}, t}\left\langle s>^{\frac{N}{2 q_{1}}-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\|(\theta, \mathbf{u})\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right)\right. \\
& +\left\langle s>^{-\left(\frac{N}{q_{1}}-\tau\right)}[\mathbf{u}]_{q_{1}, \frac{N}{2 q_{1}}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right. \\
& +\left\langle s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}-\tau\right)}[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right. \\
& \leq\langle s\rangle^{-\left(\frac{N}{q_{1}}-\tau\right)}[\theta]_{q_{1}, \frac{N}{2 q_{1}}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\|(\theta, \mathbf{u})\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right) \\
& +\left\langle s>^{-\left(\frac{N}{q_{1}}-\tau\right)}\left([\mathbf{u}]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right)<s>^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)} .\right. \tag{4.16}
\end{align*}
$$

Since $1-\left(N / q_{1}+1 / 2\right)<0$ and $1-\left(N / q_{1}-\tau\right) p^{\prime}<0$ as follows from $q_{1}<N$ and $\tau<N / q_{2}+1 / p$, by (4.14), (4.15) and (4.16), we have

$$
\begin{align*}
I_{\infty}^{1} & \leq C t^{-\frac{N}{q_{1}}-\frac{j}{2}} \int_{0}^{t / 2}<s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}\right)} d s\left([(\theta, \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right)[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +C t^{-\frac{N}{q_{1}}-\frac{j}{2}}\left(\int_{0}^{t / 2}<s>^{-\left(\frac{N}{q_{1}}-\tau\right) p^{\prime}} d s\right)^{1 / p^{\prime}}[\theta]_{q_{1}, \frac{N}{2 q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{1}}\left(\mathbb{R}^{N}\right)\right)}\right. \\
& +\left\|<s>^{\frac{N}{2 q_{1}}-\tau}(\theta, \mathbf{u})\right\|_{\left.L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)\right\}} \\
& +C t^{-\frac{N}{q_{1}}-\frac{j}{2}}\left(\int_{0}^{t / 2}<s>^{-\left(\frac{N}{q_{1}}-\tau\right) p^{\prime}} d s\right)^{1 / p^{\prime}}\left([\mathbf{u}]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right)\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C t^{-\frac{N}{q_{1}}-\frac{j}{2}} E_{0}(t) \tag{4.17}
\end{align*}
$$

where

$$
\begin{aligned}
E_{0}(t)= & \left([(\theta, \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right)[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +[\theta]_{q_{1}, \frac{N}{2 q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{1}}\left(\mathbb{R}^{N}\right)\right)}+\left\|<s>^{\frac{N}{2 q_{1}}-\tau}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)}\right\} \\
& \left.+\left([\mathbf{u}]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right)\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\right.}\left(\mathbb{R}^{N}\right)\right) .
\end{aligned}
$$

Analogously, we have

$$
\begin{equation*}
I_{\infty}^{2} \leq C t^{-\frac{N}{q_{1}}-\frac{j}{2}} E_{0}(t) \tag{4.18}
\end{equation*}
$$

We now estimate $I_{\infty}^{3}$. By (4.11) and Theorem 4.1 with $(p, q)=\left(\infty, q_{2}\right)$, we have

$$
\begin{equation*}
I_{\infty}^{3} \leq C \int_{t-1}^{t}(t-s)^{-\frac{N}{2 q_{2}}-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q_{2}}^{1,0}\left(\mathbb{R}^{N}\right)} d s \leq C \int_{t-1}^{t}(t-s)^{-\frac{N}{2 q_{2}}-\frac{j}{2}}\left(A_{2}+B_{2}\right) d s \tag{4.19}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{2}=\left(\|(\theta, \mathbf{u})\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\right)\|(\nabla \theta, \nabla \mathbf{u})\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)}, \\
& B_{2}=\|\theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)}+\left\|\left(\nabla^{2} \theta, \nabla^{2} \mathbf{u}\right)\right\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)}\right)+\left(\|\mathbf{u}\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\right)\left\|\nabla^{2} \theta\right\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)} .
\end{aligned}
$$

satisfying

$$
\begin{align*}
A_{2} & \leq\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+\frac{3}{2}\right)}[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t} \\
& +\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+2\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t}  \tag{4.20}\\
B_{2} & \left.\leq\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+1-\tau\right)}[\theta]_{\infty, \frac{N}{q_{1}}, t}<s\right\rangle^{\frac{N}{2 q_{2}}+1-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)}+\|(\theta, \mathbf{u})\|_{W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)}\right) \\
& +\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+1-\tau\right)}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}\langle s\rangle^{\frac{N}{2 q_{2}}+1-\tau}\|\theta\|_{W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)} \\
& +\langle s\rangle^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+\frac{3}{2}-\tau\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\langle s\rangle^{\frac{N}{2 q_{2}}+1-\tau}\|\theta\|_{W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)} . \tag{4.21}
\end{align*}
$$

Since $1-\left(N / 2 q_{2}+j / 2\right)>0,1-\left(N / 2 q_{2}+j / 2\right) p^{\prime}>0$, and $N / 2 q_{2}+1 / 2-\tau>j / 2$ as follows from $N<q_{2}$, $2 / p+N / q_{2}<1$ and $\tau<N / q_{2}+1 / p$, by (4.19), (4.20) and (4.21), we have

$$
\begin{align*}
I_{\infty}^{3} \leq & C t^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+\frac{3}{2}\right)} \int_{t-1}^{t}(t-s)^{-\left(\frac{N}{2 q_{2}}+\frac{j}{2}\right)} d s[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t} \\
& +C t^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+2\right)} \int_{t-1}^{t}(t-s)^{-\left(\frac{N}{2 q_{2}}+\frac{j}{2}\right)} d s[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t} \\
& +C t^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+1-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\left(\frac{N}{2 q_{2}}+\frac{j}{2}\right) p^{\prime}} d s\right)^{1 / p^{\prime}}[\theta]_{\infty, \frac{N}{q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{2}}\left(\mathbb{R}^{N}\right)\right)}\right. \\
& +\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau}(\theta, \mathbf{u})\right\|_{\left.L_{p}\left((0, t), W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)\right)\right\}} \\
& +C t^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+1-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\left(\frac{N}{2 q_{2}}+\frac{j}{2}\right) p^{\prime}} d s\right)^{1 / p^{\prime}}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
& +C t^{-\left(\frac{N}{q_{1}}+\frac{N}{2 q_{2}}+\frac{3}{2}-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\left(\frac{N}{2 q_{2}}+\frac{j}{2}\right) p^{\prime}} d s\right)^{1 / p^{\prime}}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
\leq & C t^{-\frac{N}{q_{1}}-\frac{j}{2}} E_{2}(t), \tag{4.22}
\end{align*}
$$

where

$$
\begin{aligned}
E_{2}(t)= & \left\{[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}+[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\right\}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t} \\
& +[\theta]_{\infty, \frac{N}{q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{2}}\left(\mathbb{R}^{N}\right)\right)}+\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)\right)}\right\} \\
& +\left\{[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}+[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\right\}\left\|<s>^{\frac{N}{2 q_{2}}+1-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)\right)} .
\end{aligned}
$$

By (4.12), (4.17), (4.18) and (4.22), we have

$$
\begin{equation*}
\sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{\infty, \frac{N}{q_{1}}+\frac{j}{2},(2, t)} \leq C\left(\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{L_{q_{1} / 2}\left(\mathbb{R}^{N}\right)}+E_{0}(t)+E_{2}(t)\right) \tag{4.23}
\end{equation*}
$$

## Estimates in $L_{q_{1}}$.

Using (4.11) and Theorem 4.1 with $(p, q)=\left(q_{1}, q_{1} / 2\right)$ and employing the same calculation as in the estimate in $L_{\infty}$, we have

$$
\begin{equation*}
I_{q_{1}}^{1}+I_{q_{1}}^{2} \leq C t^{-\frac{N}{2 q_{1}}-\frac{j}{2}} E_{0}(t) \tag{4.24}
\end{equation*}
$$

By Theorem 4.1 with $(p, q)=\left(q_{1}, q_{1}\right)$, we have

$$
\begin{equation*}
I_{q_{1}}^{3} \leq C \int_{t-1}^{t}(t-s)^{-\frac{j}{2}}\|(f, \mathbf{g})\|_{W_{q_{1}}^{1,0}\left(\mathbb{R}^{N}\right)} d s \leq C \int_{t-1}^{t}(t-s)^{-\frac{j}{2}}\left(A_{3}+B_{3}\right) d s \tag{4.25}
\end{equation*}
$$

where

$$
\begin{aligned}
& A_{3}=\left(\|(\theta, \mathbf{u})\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\right)\|(\nabla \theta, \nabla \mathbf{u})\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)} \\
& B_{3}=\|\theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\left\|\left(\nabla^{2} \theta, \nabla^{2} \mathbf{u}\right)\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}\right)+\left(\|\mathbf{u}\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}+\|\nabla \theta\|_{L_{\infty}\left(\mathbb{R}^{N}\right)}\right)\left\|\nabla^{2} \theta\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}
\end{aligned}
$$

satisfying

$$
\begin{align*}
A_{3} & \leq\langle s\rangle^{-\left(\frac{3 N}{2 q_{1}}+\frac{1}{2}\right)}[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +\langle s\rangle^{-\left(\frac{3 N}{2 q_{1}}+1\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t},  \tag{4.26}\\
B_{3} & \leq\langle s\rangle^{-\left(\frac{3 N}{2 q_{1}}-\tau\right)}[\theta]_{\infty, \frac{N}{q_{1}}, t}\langle s\rangle^{\frac{N}{2 q_{1}}-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}}\left(\mathbb{R}^{N}\right)+\|(\theta, \mathbf{u})\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right) \\
& \left.+\langle s\rangle^{-\left(\frac{3 N}{2 q_{1}}-\tau\right)}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}<s\right\rangle^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)} \\
& +\langle s\rangle^{-\left(\frac{3 N}{2 q_{1}}+\frac{1}{2}-\tau\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\langle s\rangle^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)} . \tag{4.27}
\end{align*}
$$

Since $1-(j / 2) p^{\prime}>0$, and $3 N / 2 q_{1}-\tau>N / 2 q_{1}+j / 2$ as follows from $p>2$ and $\tau<N / q_{2}+1 / p$, by (4.25), (4.26) and (4.27), we have

$$
\begin{align*}
I_{\infty}^{3} & \leq C t^{-\left(\frac{3 N}{2 q_{1}}+\frac{1}{2}\right)} \int_{t-1}^{t}(t-s)^{-\frac{j}{2}} d s[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +C t^{-\left(\frac{3 N}{2 q_{1}}+1\right)} \int_{t-1}^{t}(t-s)^{-\frac{j}{2}} d s[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +C t^{-\left(\frac{3 N}{2 q_{1}}-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\frac{j}{2} p^{\prime}} d s\right)^{1 / p^{\prime}}[\theta]_{\infty, \frac{N}{q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{1}}\left(\mathbb{R}^{N}\right)\right)}\right. \\
& +\left\|<s>^{\frac{N}{2 q_{1}}-\tau}(\theta, \mathbf{u})\right\|_{\left.L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)\right\}} \\
& +C t^{-\left(\frac{3 N}{2 q_{1}}-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\frac{j}{2} p^{\prime}} d s\right)^{1 / p^{\prime}}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
& +C t^{-\left(\frac{3 N}{2 q_{1}}+\frac{1}{2}-\tau\right)}\left(\int_{t-1}^{t}(t-s)^{-\frac{j}{2} p^{\prime}} d s\right)^{1 / p^{\prime}}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\left\|<s>^{\frac{N}{2 q_{1}}-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C t^{-\frac{N}{2 q_{1}}-\frac{j}{2}} E_{1}(t), \tag{4.28}
\end{align*}
$$

where

$$
\begin{aligned}
E_{1}(t) & =\left\{[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}+[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\right\}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t} \\
& +[\theta]_{\infty, \frac{N}{q_{1}}, t}\left\{\left\|<s>^{\frac{N}{2 q_{1}}+\frac{1}{2}-\tau} \partial_{s} \mathbf{u}\right\|_{L_{p}\left((0, t), L_{q_{1}}\left(\mathbb{R}^{N}\right)\right)}+\left\|<s>^{\frac{N}{2 q_{1}}+\frac{1}{2}-\tau}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)}\right\} \\
& +\left\{[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}+[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}\right\}\left\|<s>^{\frac{N}{2 q_{1}}+\frac{1}{2}-\tau} \theta\right\|_{L_{p}\left((0, t), W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)\right)} .
\end{aligned}
$$

By (4.12), (4.24) and (4.28), we have

$$
\begin{equation*}
\sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{1}, \frac{N}{2 q_{1}}+\frac{j}{2},(2, t)} \leq C\left(\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{L_{q_{1} / 2}\left(\mathbb{R}^{N}\right)}+E_{0}(t)+E_{1}(t)\right) \tag{4.29}
\end{equation*}
$$

## Estimates in $L_{q_{2}}$.

Using (4.11) and Theorem 4.1 with $(p, q)=\left(q_{2}, q_{1} / 2\right)$ and $(p, q)=\left(q_{2}, q_{2}\right)$, we have

$$
\begin{equation*}
\sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{2}, \frac{N}{2 q_{2}}+1+\frac{j}{2},(2, t)} \leq C\left(\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{L_{q_{1} / 2}\left(\mathbb{R}^{N}\right)}+E_{0}(t)+E_{2}(t)\right) \tag{4.30}
\end{equation*}
$$

In the case that $t \in(0,2)$, we have estimates by the maximal $L_{p}-L_{q}$ regularity and the embedding property. In fact, by theorem 3.2 and (4.20), (4.21), (4.26) and (4.27), we have

$$
\begin{align*}
& \|(\theta, \mathbf{u})\|_{L_{p}\left((0,2), W_{q_{i}}^{3,2}\left(\mathbb{R}^{N}\right)\right)}+\left\|\left(\partial_{s} \theta, \partial_{s} \mathbf{u}\right)\right\|_{L_{p}\left((0,2), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C\left\{\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q_{i}, p}\left(\mathbb{R}^{N}\right)}+\|(f, \mathbf{g})\|_{L_{p}\left((0,2), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\right\} \\
& \leq C\left\{\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q_{i}, p}\left(\mathbb{R}^{N}\right)}+E_{i}(2)\right\} \tag{4.31}
\end{align*}
$$

for $i=1,2$.
By Lemma 3.3 we have

$$
\begin{equation*}
\|(\theta, \mathbf{u})\|_{L_{\infty}\left((0,2), W_{\infty}^{1}\left(\mathbb{R}^{N}\right)\right)} \leq C\left\{\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q_{2}, p}\left(\mathbb{R}^{N}\right)}+E_{2}(2)\right\} \tag{4.32}
\end{equation*}
$$

Combining (4.23), (4.29), (4.30), (4.31) and (4.32), we have

$$
\begin{align*}
& \sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{\infty, \frac{N}{q_{1}}+\frac{j}{2},(0, t)} \leq C\left(\mathcal{I}+E_{0}(t)+E_{2}(t)\right), \\
& \sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{1}, \frac{N}{2 q_{1}}+\frac{j}{2},(0, t)} \leq C\left(\mathcal{I}+E_{0}(t)+E_{1}(t)\right),  \tag{4.33}\\
& \sum_{j=0}^{1}\left[\left(\nabla^{j} \theta, \nabla^{j} \mathbf{u}\right)\right]_{q_{2}, \frac{N}{2 q_{2}}+1+\frac{j}{2},(0, t)} \leq C\left(\mathcal{I}+E_{0}(t)+E_{2}(t)\right) .
\end{align*}
$$

We next consider the estimates of the weighted norm in the maximal $L_{p}-L_{q}$ regularity class by the following time shifted equations, which is equivalent to the first and the second equations of (4.1):

$$
\begin{aligned}
& \partial_{s}\left(<s>^{\ell_{i}} \theta\right)+\delta_{0}<s>^{\ell_{i}} \theta+\rho_{*} \operatorname{div}\left(<s>^{\ell_{i}} \mathbf{u}\right) \\
& =<s>^{\ell_{i}} f(\theta, \mathbf{u})+\delta_{0}<s>^{\ell_{i}} \theta+\left(\partial_{s}<s>^{\ell_{i}}\right) \theta \\
& \partial_{s}\left(<s>^{\ell_{i}} \mathbf{u}\right)+\delta_{0}<s>^{\ell_{i}} \mathbf{u}-\alpha_{*} \Delta\left(<s>^{\ell_{i}} \mathbf{u}\right)-\beta_{*} \nabla\left(\operatorname{div}<s>^{\ell_{i}} \mathbf{u}\right) \\
& \quad+\kappa_{*} \nabla \Delta<s>^{\ell_{i}} \theta-\gamma_{*} \nabla<s>^{\ell_{i}} \theta \\
& =<s>^{\ell_{i}} \mathbf{g}(\theta, \mathbf{u})+\delta_{0}<s>^{\ell_{i}} \mathbf{u}+\left(\partial_{s}<s>^{\ell_{i}}\right) \mathbf{u}
\end{aligned}
$$

where $i=1,2, \ell_{1}=N / 2 q_{1}-\tau$ and $\ell_{2}=N / 2 q_{2}+1-\tau$. We estimate the left-hand sides of the time shifted equations. Since $1-\delta p<0$, by (4.33), we have

$$
\begin{align*}
& \left\|<s>^{\ell_{1}}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{1}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \leq\left(\int_{0}^{t}<s>^{-\delta p} d s\right)^{1 / p}\left([(\theta, \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right) \\
& \leq C\left(\mathcal{I}+E_{0}(t)+E_{1}(t)\right)  \tag{4.34}\\
& \left\|<s>^{\ell_{2}}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{2}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \leq\left(\int_{0}^{t}<s>^{-\delta p} d s\right)^{1 / p}\left([(\theta, \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+1, t}+[\nabla \theta]_{q_{1}, \frac{N}{2 q_{2}}+\frac{3}{2}, t}\right) \\
& \leq C\left(\mathcal{I}+E_{0}(t)+E_{2}(t)\right) . \tag{4.35}
\end{align*}
$$

Employing the same calculation as in (4.34) and (4.35), we have

$$
\begin{equation*}
\left\|\left(\partial_{s}<s>^{\ell_{i}}\right)(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \leq C\left(\mathcal{I}+E_{0}(t)+E_{i}(t)\right) \tag{4.36}
\end{equation*}
$$

By (4.26) and (4.27), we have

$$
\begin{aligned}
< & s>^{\ell_{1}}\|(f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))\|_{W_{q_{1}}^{1,0}\left(\mathbb{R}^{N}\right)} \\
& \leq C\left\{<s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}+\tau\right)}[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{1}}+\frac{1}{2}, t}\right. \\
& +<s>^{-\left(\frac{N}{q_{1}}+1+\tau\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{1}, \frac{N}{2 q_{2}}+\frac{1}{2}, t} \\
& +<s>^{-\frac{N}{q_{1}}}[\theta]_{\infty, \frac{N}{q_{1}}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{1}}\left(\mathbb{R}^{N}\right)}+\|(\theta, \mathbf{u})\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right) \\
& +<s>^{-\frac{N}{q_{1}}}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)} \\
& \left.+<s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}<s>^{\frac{N}{2 q_{1}}-\tau}\|\theta\|_{W_{q_{1}}^{2}\left(\mathbb{R}^{N}\right)}\right\},
\end{aligned}
$$

and so we have

$$
\begin{equation*}
\left\|<s>^{\ell_{1}}(f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))\right\|_{L_{p}\left((0, t), W_{q_{1}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \leq C E_{1}(t) \tag{4.37}
\end{equation*}
$$

By (4.20) and (4.21), we have

$$
\begin{aligned}
< & s>^{\ell_{2}}\|(f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))\|_{W_{q_{2}}^{1,0}\left(\mathbb{R}^{N}\right)} \\
& \leq C\left\{<s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}+\tau\right)}[(\theta, \mathbf{u})]_{\infty, \frac{N}{q_{1}}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t}\right. \\
& +<s>^{-\left(\frac{N}{q_{1}}+1+\tau\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}[(\nabla \theta, \nabla \mathbf{u})]_{q_{2}, \frac{N}{2 q_{2}}+\frac{3}{2}, t} \\
& +<s>^{-\frac{N}{q_{1}}}[\theta]_{\infty, \frac{N}{q_{1}}, t}<s>^{\frac{N}{2 q_{2}}+1-\tau}\left(\left\|\partial_{s} \mathbf{u}\right\|_{L_{q_{2}}\left(\mathbb{R}^{N}\right)}+\|(\theta, \mathbf{u})\|_{W_{q_{2}}^{2}\left(\mathbb{R}^{N}\right)}\right) \\
& +<s>^{-\frac{N}{q_{1}}}[\mathbf{u}]_{\infty, \frac{N}{q_{1}}, t}<s>^{\frac{N}{2 q_{2}}+1-\tau}\|\theta\|_{W_{q_{2}}^{2}}\left(\mathbb{R}^{N}\right) \\
& \left.+<s>^{-\left(\frac{N}{q_{1}}+\frac{1}{2}\right)}[\nabla \theta]_{\infty, \frac{N}{q_{1}}+\frac{1}{2}, t}<s>^{\frac{N}{2 q_{2}}+1-\tau}\|\theta\|_{W_{q_{2}}^{2}}\left(\mathbb{R}^{N}\right)\right\},
\end{aligned}
$$

and so we have

$$
\begin{equation*}
\left.\|<s>^{\ell_{2}}(f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))\right) \|_{L_{p}\left((0, t), W_{q_{2}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \leq C E_{2}(t) \tag{4.38}
\end{equation*}
$$

By Theorem (3.2, 4.34), (4.35), (4.36), (4.37) and (4.38), we have

$$
\begin{align*}
& \left\|<s>^{\ell_{i}}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{i}}^{3,2}\left(\mathbb{R}^{N}\right)\right)}+\left\|<s>^{\ell_{i}}\left(\partial_{s} \theta, \partial_{s} \mathbf{u}\right)\right\|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C\left(\left\|\left(\rho_{0}, \mathbf{u}_{0}\right)\right\|_{D_{q_{i}, p}\left(\mathbb{R}^{N}\right)}+\left\|<s>^{\ell_{i}}(f(\theta, \mathbf{u}), \mathbf{g}(\theta, \mathbf{u}))\right\|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)}\right. \\
& +\left\|<s>^{\ell_{i}}(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)}+\left\|\left(\partial_{s}<s>^{\ell_{i}}\right)(\theta, \mathbf{u})\right\|_{L_{p}\left((0, t), W_{q_{i}}^{1,0}\left(\mathbb{R}^{N}\right)\right)} \\
& \leq C\left(\mathcal{I}+E_{0}(t)+E_{i}(t)\right) . \tag{4.39}
\end{align*}
$$

Combining (4.33) and (4.39), we have (4.10). Recalling that $\mathcal{I} \leq \epsilon$, for $(\theta, \mathbf{u}) \in \mathcal{I}_{\epsilon}$, we have

$$
\begin{equation*}
\mathcal{N}(\omega, \mathbf{w})(\infty) \leq C\left(\mathcal{I}+\mathcal{N}(\theta, \mathbf{u})(\infty)^{2}\right) \leq C \epsilon+C L^{2} \epsilon^{2} \tag{4.40}
\end{equation*}
$$

Choosing $\epsilon$ so small that $L^{2} \epsilon \leq 1$ and setting $L=2 C$ in (4.40), we have

$$
\begin{equation*}
\mathcal{N}(\omega, \mathbf{w}) \leq L \epsilon . \tag{4.41}
\end{equation*}
$$

We define a map $\Phi$ acting on $(\theta, \mathbf{u}) \in \mathcal{I}_{\epsilon}$ by $\Phi(\theta, \mathbf{u})=(\omega, \mathbf{w})$, and then it follows from 4.41) that $\Phi$ is the map from $\mathcal{I}_{\epsilon}$ into itself. Considering the difference $\Phi\left(\theta_{1}, \mathbf{u}_{1}\right)-\Phi\left(\theta_{2}, \mathbf{u}_{2}\right)$ for $\left(\theta_{i}, \mathbf{u}_{i}\right) \in \mathcal{I}_{\epsilon}(i=1,2)$, employing the same argument as in the proof of (4.40) and choosing $\epsilon>0$ samller if necessary, we see that $\Phi$ is a consraction map on $\mathcal{I}_{\epsilon}$, and therefore there exists a fixed point $(\omega, \mathbf{w}) \in \mathcal{I}_{\epsilon}$ which solves the equation (4.1). Since the existence of solutions to (4.1) is proved by the contraction mapping principle, the uniqueness of solutions belonging to $\mathcal{I}_{\epsilon}$ follows immediately, which completes the proof of Theorem 1.1.

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