# ASYMPTOTICS FOR THE ELECTRIC FIELD CONCENTRATION IN THE PERFECT CONDUCTIVITY PROBLEM 

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#### Abstract

In the perfect conductivity problem of composite material, the electric field concentrates in a narrow region in between two inclusions and always becomes arbitrarily large when the distance between inclusions tends to zero. To characterize such singular behavior, we capture the leading term of the gradient and reveal that the blow-up rates are determined by their relative convexity of the two adjacent inclusions. On the other hand, a blow-up factor, which is a linear functional of boundary data, is found to determine the blowup will occur or not.


## 1. Introduction

1.1. Background. In composite materials, the inclusion are frequently located very closely and even touching. Especially, in high-contrast fiber-reinforced composites, it is a common phenomenon that high concentration of extreme electric field or stress field occurs in the narrow regions between two adjacent inclusions. The purpose of this paper is to investigate the asymptotic behavior of the electric field in the perfect conductivity problem when the distance between inclusions tends to zero. The conductivity problem can be modeled by the following boundary problem of the scalar equation with piecewise constant coefficients

$$
\begin{cases}\operatorname{div}\left(a_{k}(x) \nabla u_{k}\right)=0 & \text { in } D  \tag{1.1}\\ u_{k}=\varphi(x) & \text { on } \partial D\end{cases}
$$

where $D$ is a bounded open set in $\mathbb{R}^{n}, n \geq 2$, including two inclusions $D_{1}$ and $D_{2}$ with $\varepsilon$ apart, $\varphi \in C^{2}(\partial D)$ is given, and

$$
a_{k}(x)= \begin{cases}k \in[0,1) \cup(1, \infty] & \text { in } D_{1} \cup D_{2} \\ 1 & \text { in } \Omega=D \backslash \overline{D_{1} \cup D_{2}} .\end{cases}
$$

The gradient of the potential $u$ represents the electric field, $a_{k}(x)$ is the conductivity, which is a constant on the fibers, and a different constant on the matrix. When the conductivity of inclusions degenerate into infinity, we call it as the perfect conductivity problem. It is important from a practical point of view to know whether $|D u|$ can be arbitrarily large as the inclusions get closer to each other. Motivated by the celebrated work of Babuška, Andersson, Smith, and Levin [5] where they numerically analyzed the initiation and growth of damage in composite materials, in which the inclusions are frequently spaced very closely and even touching, there

[^0]have been many important works on the gradient estimates for solutions of elliptic and parabolic equations and systems arising from composite materials; see, for instance, $12,15-17,21,26,28,32,33$ and the references therein.

When $k$ is away from 0 and $\infty$, the gradient of the solution of (1.1), $\left|\nabla u_{k}\right|$, is bounded independently of the distance $\varepsilon$. Bonnetier and Vogelius [12] first obtained the $W^{1, \infty}$ estimate of $u_{k}$ for two touching disks $D_{1}$ and $D_{2}$ in dimension two, which improved a classical regularity result due to De Giorgi and Nash [14, 36, which asserts that the $H^{1}$ weak solution is in the Hölder class for $L^{\infty}$ coefficients. Of course, the bound in [12] depends on the value of $k$. Li and Vogelius [33] and Li and Nirenberg [32] extended such boundedness result to general divergence form second order elliptic equations and systems with piecewise Hölder continuous coefficients, and they proved that $\left|\nabla u_{k}\right|$ remains bounded when $\varepsilon$ tends to zero.

Actually, this is a bi-parameter problem, including two independent parameters: the contrast $k$ and the distance $\varepsilon$. In order to study the role of $\varepsilon$ played in such kind of concentration phenomenon, we consider another limit case with $k=+\infty$, the perfect conductivity problem:

$$
\begin{cases}\Delta u=0 & \text { in } \Omega  \tag{1.2}\\ u=C_{i} & \text { on } \bar{D}_{i}, i=1,2 \\ \int_{\partial D_{i}} \frac{\partial u}{\partial \nu^{-}}=0 & i=1,2 \\ u=\varphi(x) & \text { on } \partial D\end{cases}
$$

where $C_{1}$ and $C_{2}$ are some constants to be uniquely determined, $\varphi \in C^{2}(\partial D)$, and for $x \in \partial D_{i}$

$$
\frac{\partial u}{\partial \nu^{-}}(x):=\lim _{t \rightarrow 0^{+}} \frac{u(x)-u(x+t \nu)}{t} .
$$

Here and throughout this paper $\nu$ is the outward unit normal to the domain. It has been proved that the generic blow-up rate of $|\nabla u|$ is $\varepsilon^{-1 / 2}$ in two dimensions [1, 3, 4, 34, 37, 38, $(\varepsilon|\log \varepsilon|)^{-1}$ in three dimensions [6, 24, 34, and $\varepsilon^{-1}$ in higher dimensions [6]. Similar results for Lamé system with partially inifinite coefficients were established in [7] 9 , for $p$-Laplace equation in dimension two in [19]. More earlier work for the blow-up rate of a special solution with two identical circular inclusions was shown to be $\varepsilon^{-1 / 2}$, see [13, 25, 35].
$\mathrm{Bao}, \mathrm{Li}$ and Yin [6] introduced a linear functional $Q_{\varepsilon}[\varphi]$ and obtained the optimal bounds

$$
\frac{\rho_{n}(\varepsilon)\left|Q_{\varepsilon}[\varphi]\right|}{C \varepsilon} \leq\|\nabla u\|_{L^{\infty}(\Omega)} \leq \frac{C \rho_{n}(\varepsilon)\left|Q_{\varepsilon}[\varphi]\right|}{\varepsilon}+C\|\varphi\|_{C^{2}(\partial D)}
$$

where

$$
\rho_{n}(\varepsilon)= \begin{cases}\sqrt{\varepsilon} & \text { for } n=2  \tag{1.3}\\ |\log \varepsilon|^{-1} & \text { for } n=3\end{cases}
$$

If $\left|Q_{\varepsilon}[\varphi]\right|$ has a strictly positive lower bound independent of $\varepsilon$, then these inequality will show these blow-up rates are optimal. From the view of practical application in engineering, it is desirous and more important to know how to capture the leading term of such blow-up. Recently a better understanding of the stress concentration has been obtained in [2,23] that an asymptotic behavior of $\nabla u$ has been characterized by the singular function $q_{\varepsilon}$ associated with $D_{1}$ and $D_{2}$ in dimension two, and the asymptotic behavior of the stress concentration factor is also considered in [23].

Ammari, Ciraolo, Kang, Lee, Yun [2] extend the result in [23] to the case that inclusions $D_{1}, D_{2}$ are strictly convex simply connected domain in $\mathbb{R}^{2}$. For two adjacent spherical inclusions in $\mathbb{R}^{3}$ was studied by Kang, Lim and Yun [24] and Li, Wang and Xu [31]. Bonnetier and Triki [11] derived the asymptotics of the eigenvalues of the Poincaré variational problem as the distance between the inclusions tends to zero. Here it is also worth mentioning that Berlyand, Gorb and Novikov [10] used a network approximation to estimate the global stress in a composite with densely packed spherical inclusions.

In this paper, we give an essentially complete description of the gradient asymptotic expansion for arbitrary convex inclusions in all dimensions. The method is quite different with that used in [2, 23, 24]. Motivated by the decomposition in 30 , for the boundary estimates, we here decompose the solution $u$ of (1.2) as follows

$$
\begin{equation*}
u(x)=\left(C_{1}-C_{2}\right) v_{1}(x)+v_{b}(x), \quad \text { in } \Omega \tag{1.4}
\end{equation*}
$$

where $v_{1}$ and $v_{b}$ are, respectively, the solutions of

$$
\left\{\begin{array} { l l } 
{ \Delta v _ { 1 } = 0 } & { \text { in } \Omega , }  \tag{1.5}\\
{ v _ { 1 } = 1 } & { \text { on } \partial D _ { 1 } , } \\
{ v _ { 1 } = 0 } & { \text { on } \partial D _ { 2 } \cup \partial D , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{ll}
\Delta v_{b}=0 & \text { in } \Omega \\
v_{b}=C_{2} & \text { on } \partial D_{1} \cup \partial D_{2}, \\
v_{b}=\varphi(x) & \text { on } \partial D .
\end{array}\right.\right.
$$

It follows from (1.4) that

$$
\begin{equation*}
\nabla u=\left(C_{1}-C_{2}\right) \nabla v_{1}+\nabla v_{b} \tag{1.6}
\end{equation*}
$$

This decomposition comes with a significant advantage: $\nabla v_{1}$ is a singular part with an intuitive singularity $\varepsilon^{-1}$, while $\nabla v_{b}$ is a bounded part. Thus, the main reason to cause the difference of the rate of the blow-up lies in the term $\left|C_{1}-C_{2}\right|$. It turns out that it depends on the dimension $n$ and the geometry of the inclusions. On the other hand, the bounded term $\nabla v_{b}$ is also important, because it is closely related to the blow-up factor $\mathcal{B}_{0}[\varphi]$, which decides whether the blow-up will occur or not. For more details, see Proposition 1.10 below.
1.2. Notations and Main Results. We now proceed to state the main results of this paper. To do so we need to make our notation and assumptions more precise. We use $x=\left(x^{\prime}, x_{n}\right)$ to denote a point in $\mathbb{R}^{n}, n \geq 2, x^{\prime}=\left(x_{1}, x_{2}, \cdots, x_{n-1}\right)$. We assume that $\partial D$ is of $C^{2, \alpha}, 0<\alpha<1$. Let $D_{1}^{0}$ and $D_{2}^{0}$ be a pair of (touching) convex subdomains of $D$ and far away from $\partial D$, such that

$$
D_{1}^{0} \subset\left\{\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n} \mid x_{n}>0\right\}, \quad D_{2}^{0} \subset\left\{\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n} \mid x_{n}<0\right\}
$$

with $x_{n}=0$ as their common tangent plane, and

$$
\partial D_{1}^{0} \cap \partial D_{2}^{0}=\left\{\left(0^{\prime}, 0\right)\right\}, \quad \operatorname{dist}\left(D_{1}^{0} \cup D_{2}^{0}, \partial D\right)>\kappa_{0}
$$

where $\kappa_{0}>1$ is a constant. We further assume that the $C^{2, \alpha}$ norms of $\partial D_{i}(i=1,2)$ are bounded by some constants. By translating $D_{1}^{0}$ by a positive number $\varepsilon$ along $x_{n}$-axis, while $D_{2}^{0}$ is fixed, we obtain $D_{1}^{\varepsilon}$, that is,

$$
D_{1}^{\varepsilon}:=D_{1}^{0}+\left(0^{\prime}, \varepsilon\right)
$$

When there is no possibility of confusion, we drop the superscripts and denote

$$
D_{1}:=D_{1}^{\varepsilon}, \quad D_{2}:=D_{2}^{0}, \quad \text { and } \Omega:=D \backslash \overline{D_{1} \cup D_{2}} .
$$

We may assume that the points $P_{1} \in \partial D_{1}$ and $P_{2} \in \partial D_{2}$ satisfy

$$
P_{1}=\left(0^{\prime}, \varepsilon\right), \quad P_{2}=\left(0^{\prime}, 0\right)
$$

Fix a small universal constant $R_{0}<1$ such that the portions of $\partial D_{i}$ near $P_{i}$ can be parameterized by $\left(x^{\prime}, \varepsilon+h_{1}\left(x^{\prime}\right)\right)$ and $\left(x^{\prime}, h_{2}\left(x^{\prime}\right)\right)$, respectively, that is,

$$
x_{n}=\varepsilon+h_{1}\left(x^{\prime}\right), \text { and } x_{n}=h_{2}\left(x^{\prime}\right), \quad \text { for } x^{\prime} \in B_{2 R_{0}}^{\prime}:=\left\{x^{\prime} \in \mathbb{R}^{n-1}| | x^{\prime} \mid<2 R_{0}\right\} .
$$

Moreover, by the convexity assumptions on $\partial D_{i}$, we further assume that functions $h_{1}$ and $h_{2}$ satisfy

$$
\begin{gather*}
\varepsilon+h_{1}\left(x^{\prime}\right)>h_{2}\left(x^{\prime}\right), \quad \text { for }\left|x^{\prime}\right|<2 R_{0}  \tag{1.7}\\
h_{1}\left(0^{\prime}\right)=h_{2}\left(0^{\prime}\right)=0, \quad \nabla_{x^{\prime}} h_{1}\left(0^{\prime}\right)=\nabla_{x^{\prime}} h_{2}\left(0^{\prime}\right)=0 \tag{1.8}
\end{gather*}
$$

and for some constant $\kappa_{1}>0$, and for any $\xi \in \mathbb{R}^{n-1} \backslash\left\{0^{\prime}\right\}$,

$$
\begin{equation*}
\xi^{T} \nabla_{x^{\prime}}^{2} h_{1}\left(0^{\prime}\right) \xi \geq \kappa_{1}|\xi|^{2}>0, \quad \xi^{T} \nabla_{x^{\prime}}^{2} h_{2}\left(0^{\prime}\right) \xi \leq-\kappa_{1}|\xi|^{2}<0 \tag{1.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|h_{1}\right\|_{C^{3,1}\left(B_{2 R_{0}}^{\prime}\right)}+\left\|h_{2}\right\|_{C^{3,1}\left(B_{2 R_{0}}^{\prime}\right)} \leq C \tag{1.10}
\end{equation*}
$$

More generally, after a rotation of the coordinates if necessary, we assume that

$$
\begin{equation*}
\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)=\sum_{j=1}^{n-1} \frac{\lambda_{j}}{2} x_{j}^{2}+O\left(\left|x^{\prime}\right|^{2+\alpha}\right), \quad\left|x^{\prime}\right| \leq 2 R_{0} \tag{1.11}
\end{equation*}
$$

where $\operatorname{diag}\left(\lambda_{1}, \cdots, \lambda_{n-1}\right)=\nabla_{x^{\prime}}^{2}\left(h_{1}-h_{2}\right)\left(0^{\prime}\right)$. For $0 \leq r \leq 2 R_{0}$, let

$$
\Omega_{r}:=\left\{\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n}\left|h_{2}\left(x^{\prime}\right)<x_{n}<\varepsilon+h_{1}\left(x^{\prime}\right),\left|x^{\prime}\right|<r\right\} .\right.
$$

We introduce an auxiliary function $\bar{v}_{1} \in C^{2, \alpha}\left(\mathbb{R}^{n}\right)$, such that $\bar{v}_{1}=1$ on $\partial D_{1}$, $\bar{v}_{1}=0$ on $\partial D_{2} \cup \partial D$,

$$
\begin{equation*}
\bar{v}_{1}(x)=\frac{x_{n}-h_{2}\left(x^{\prime}\right)}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}, \quad \text { in } \Omega_{2 R_{0}} \tag{1.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\bar{v}_{1}\right\|_{C^{2, \alpha}\left(\mathbb{R}^{n} \backslash \Omega_{R_{0}}\right)} \leq C \tag{1.13}
\end{equation*}
$$

In view of (1.8)-1.10), a direct calculation gives

$$
\begin{align*}
\left|\partial_{x_{j}} \bar{v}_{1}(x)\right| & \leq \frac{C\left|x^{\prime}\right|}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}, j=1,2, \cdots, n-1,  \tag{1.14}\\
\partial_{x_{n}} \bar{v}_{1}(x) & =\frac{1}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}
\end{align*}
$$

Here and throughout this paper, unless otherwise stated, $C$ denotes a constant, whose values may vary from line to line, depending only on $n, \kappa_{0}, \kappa_{1},\|\partial \Omega\|_{C^{2, \alpha}}$, $\left\|\partial D_{1}\right\|_{C^{2, \alpha}}$ and $\left\|\partial D_{2}\right\|_{C^{2, \alpha}}$, but not on $\varepsilon$. Also, we call a constant having such dependence a universal constant.

Consider the following limit problem

$$
\begin{cases}\Delta u_{0}=0 & \text { in } \Omega^{0}:=D \backslash \overline{D_{1}^{0} \cup D_{2}^{0}}  \tag{1.15}\\ u_{0}=C_{0} & \text { on } \overline{D_{1}^{0} \cup D_{2}^{0}}, \\ \int_{\partial D_{1}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}}=0 & \\ u_{0}=\varphi(x) & \text { on } \partial D\end{cases}
$$

It will be shown later that $u_{0}$ is the limit of $u$. We use $u_{0}$ to define a linear functional of $\varphi$, which determines whether $\nabla u$ blows up or not,

$$
\begin{equation*}
\mathcal{B}_{0}[\varphi]:=-\int_{\partial D_{1}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}}=\int_{\partial D_{2}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}} . \tag{1.16}
\end{equation*}
$$

This factor was first introduced by Gorb and Novikov in [19] for $p$-Laplace equation, denoted by $\mathcal{R}_{0}$. It turns out there that $\mathcal{R}_{0}$ is the key characteristic parameter of the $W^{1, \infty}$ blow-up of $u$, see also 18 .

In the following, we use $O(1)$ to denote some quantity satisfying $|O(1)| \leq C$, for some constant $C$ independent of $\varepsilon$. We have the asymptotic expression of $\nabla u$ in the narrow region between $D_{1}$ and $D_{2}$ as follows:

Theorem 1.1. For $n=2,3$, let $D, D_{1}, D_{2}$ be defined as the above and satisfy (1.7)-(1.11), $\varphi \in C^{2}(\partial D)$. Assume that $u \in H^{1}(D) \cap C^{1}(\bar{\Omega})$ is the solution to (1.2). Then for $\varphi$ such that $\mathcal{B}_{0}[\varphi] \neq 0$, we have
(i) for $n=2$,

$$
\begin{equation*}
\nabla u=\frac{\mathcal{B}_{0}[\varphi] \sqrt{\varepsilon}}{\kappa_{2}} \nabla \bar{v}_{1}+O(1)\|\varphi\|_{C^{2}(\partial D)}, \quad \text { in } \Omega_{R_{0}} \tag{1.17}
\end{equation*}
$$

(ii) for $n=3$,

$$
\begin{equation*}
\nabla u=\frac{\mathcal{B}_{0}[\varphi]}{\kappa_{3}|\log \varepsilon|}\left(1+O\left(|\log \varepsilon|^{-1}\right)\right) \nabla \bar{v}_{1}+O(1)\|\varphi\|_{C^{2}(\partial D)}, \quad \text { in } \Omega_{R_{0}} \tag{1.18}
\end{equation*}
$$

where

$$
\kappa_{n}:= \begin{cases}\frac{\sqrt{2} \pi}{\sqrt{\lambda_{1}}} & n=2  \tag{1.19}\\ \frac{\pi}{\sqrt{\lambda_{1} \lambda_{2}}} & n=3\end{cases}
$$

$\lambda_{1}\left(\right.$ or $\lambda_{1}$ and $\left.\lambda_{2}\right)$ is the relatively principal curvature of $\partial D_{1}$ and $\partial D_{2}$, defined in (1.11).

Remark 1.2. We would like to point out that from (1.14) $\nabla \bar{v}_{1}$ is explicit. So the singularity of $\nabla u$ in the narrow region $\Omega_{R}$ can be calculated provided $\mathcal{B}_{0}[\varphi]$ is known for a given $\varphi$. The computation of $\mathcal{B}_{0}[\varphi]$ is an interesting numerical problem, because there is no singularity in $\nabla u_{0}$. We leave it to the interested readers.
Remark 1.3. This blow-up factor $\mathcal{B}_{0}[\varphi]$ is more natural than $Q_{\varepsilon}[\varphi]$ defined in [6, and it is much easier to check whether or not it equals zero, since $\nabla u_{0}$ is regular, namely, always bounded. While in the definition of $Q_{\varepsilon}[\varphi]$, the singular terms $\nabla v_{1}$ and $\nabla v_{2}$ are used. In fact, there may exist a boundary data $\varphi$ such that $\mathcal{B}_{0}[\varphi]=0$, but it is easy to find another $\varphi$ such that $\mathcal{B}_{0}[\varphi] \neq 0$ by a perturbation argument.
Remark 1.4. We would like to point out that from (1.14) our asymptotic formula (1.17) and (1.18) are actually pointwise expressions near the origin. This is different with the results in [18, 19], where the norm $\|\nabla u\|_{L^{\infty}\left(\Omega_{\delta}\right)}$ is considered.

From (1.19), one can see the constant $\kappa_{n}$ depends on the curvature of $\partial D_{1}^{0}$ and $\partial D_{2}^{0}$ at the origin. For example, if the mean curvature $\lambda_{1} \lambda_{2} \rightarrow 0$, then the quatity $\frac{1}{\kappa_{3}}$ in (1.18) tends to zero as well. While, when $\partial D_{1}^{0}$ and $\partial D_{2}^{0}$ are relatively convex of order $m>2$, especially when there exist a constant $\lambda>0$ such that

$$
\begin{equation*}
\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)=\lambda\left|x^{\prime}\right|^{m}, \quad m>2, \text { for }\left|x^{\prime}\right|<R_{0} \tag{1.20}
\end{equation*}
$$

that is, their relative curvature vanishes. This will cause the blow-up rate to change. In order to reveal the relation between the convexity and the blow-up rate for particles with zero curvature at the point of the closest distance, we here restrict our consideration only to this symmetric case (1.20). For more generalized $m$ convex inclusions cases, the same assertions should also be true. For simplicity, we also assume that

$$
\begin{equation*}
\left|\nabla_{x^{\prime}} h_{1}\right|,\left|\nabla_{x^{\prime}} h_{2}\right| \leq C\left|x^{\prime}\right|^{m-1}, \text { for }\left|x^{\prime}\right|<R_{0} \tag{1.21}
\end{equation*}
$$

We define

$$
\rho_{n}^{m}(\varepsilon)= \begin{cases}\varepsilon^{1-\frac{n-1}{m}} & \text { for } m>n-1, n \geq 3 \\ |\log \varepsilon|^{-1} & \text { for } m=n-1, n \geq 3 \\ \varepsilon^{1-\frac{1}{m}} & \text { for } m \geq 2, n=2\end{cases}
$$

Theorem 1.5. Let $D, D_{1}, D_{2}$ be of $C^{2, \alpha}$ and satisfy (1.20) and (1.21) with $m \geq 2$ if $n=2, m \geq n-1$ if $n \geq 3, \varphi \in C^{2}(\partial D)$. Assume that $u \in H^{1}(D) \cap C^{1}(\bar{\Omega})$ is the solution to (1.2). Then for $\varphi$ such that $\mathcal{B}_{0}[\varphi] \neq 0$, we have
(i) if $m \geq 2(n-1)$, $n \geq 2$,

$$
\begin{equation*}
\nabla u=\frac{\mathcal{B}_{0}[\varphi] \rho_{n}^{m}(\varepsilon)}{\mathcal{L} \lambda^{\frac{n-1}{m}}} \nabla \bar{v}_{1}+O(1)\|\varphi\|_{C^{2}(\partial D)}, \quad \text { in } \Omega_{R_{0}} \tag{1.22}
\end{equation*}
$$

(ii) if $n-1 \leq m<2(n-1)$ and $n \geq 3$,

$$
\begin{equation*}
\nabla u=\mathcal{B}_{0}[\varphi] \frac{\rho_{n}^{m}(\varepsilon)}{\mathcal{L} \lambda^{\frac{n-1}{m}}}\left(1+O\left(\rho_{n}^{m}(\varepsilon)\right)\right) \nabla \bar{v}_{1}+O(1)\|\varphi\|_{C^{2}(\partial D)}, \quad \text { in } \Omega_{R_{0}} \tag{1.23}
\end{equation*}
$$

where $\mathcal{L}$ is a constant depending only on $m$ and $n$.
Remark 1.6. In some sense Theorem 1.5 could be regard as an extension of an 2 D asymptotic formula (21) in [22],

$$
\begin{equation*}
\nabla u=\alpha_{0} \nabla q_{\varepsilon}+O(1) \tag{1.24}
\end{equation*}
$$

where $q_{\varepsilon}$ is a singular function in dimension two, with $\nabla q_{\varepsilon} \sim \frac{1}{\sqrt{\varepsilon}}$. The conclusions in Theorem 1.5 hold in dimensions two and three. Moreover, they show that the blowup rate of $|\nabla u|$ at the origin is $\frac{\rho_{n}^{m}(\varepsilon)}{\varepsilon}$, which depends on the space dimension and the order of the convexity of the inclusions. Especially, in $\mathbb{R}^{n}$, when the convexity of inclusions is different, the blow-up rate is different. In this sense, when we use a ball (with 2-convexity) to approximate an arbitrary convex inclusion, the error in general will be large, unless its convexity is also of order 2 .
1.3. The outline of the proof of Theorems $\mathbf{1 . 1}$ and 1.5. In this section we list the strategy and main ingredients of the proof of Theorem 1.1 and 1.5. Without loss of generality, we assume that $\|\varphi\|_{C^{2}(\partial D)}=1$, by considering $u /\|\varphi\|_{C^{2}(\partial D)}$ if $\|\varphi\|_{C^{2}(\partial D)}>0$. If $\left.\varphi\right|_{\partial D}=0$ then $u \equiv 0$.

Using the trace embedding theorem and $\|u\|_{H^{1}(\Omega)} \leq C$ (independent of $\varepsilon$ ), we have

$$
\begin{equation*}
\left|C_{1}\right|+\left|C_{2}\right| \leq C \tag{1.25}
\end{equation*}
$$

In view of (1.4) and the third line of (1.2), the constants $C_{1}-C_{2}$ is determined by the following linear system

$$
\begin{equation*}
\left(C_{1}-C_{2}\right) \int_{\partial D_{i}} \frac{\partial v_{1}}{\partial \nu^{-}}+\int_{\partial D_{i}} \frac{\partial v_{b}}{\partial \nu^{-}}=0, \quad i=1,2 \tag{1.26}
\end{equation*}
$$

If $\int_{\partial D_{1}} \frac{\partial v_{1}}{\partial \nu^{-}} \neq 0$, then from (1.26),

$$
\begin{equation*}
C_{1}-C_{2}=\frac{-\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}}}{\int_{\partial D_{1}} \frac{\partial v_{1}}{\partial \nu^{-}}} \tag{1.27}
\end{equation*}
$$

In the following we estimate the two terms $\int_{\partial D_{1}} \frac{\partial v_{1}}{\partial \nu}$ and $\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}}$, respectively. First, by the definition of $v_{1}$ and integration by parts, we have

$$
\int_{\partial D_{1}} \frac{\partial v_{1}}{\partial \nu^{-}}=\int_{\partial \Omega} v_{1} \frac{\partial v_{1}}{\partial \nu}=\int_{\Omega}\left|\nabla v_{1}\right|^{2}
$$

Theorem A. ([29]) For $n=2,3$, assume $D_{1}, D_{2}$ are of $C^{k, 1}, k \geq 3$ and satisfy (1.11). Then there exists a constant $M$, depending only on $D_{1}^{0}, D_{2}^{0}$ and $\Omega$, such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{1}\right|^{2}-\left(\frac{\kappa_{n}}{\rho_{n}(\varepsilon)}+M\right)=O\left(E_{n}(\varepsilon)\right) \tag{1.28}
\end{equation*}
$$

where

$$
E_{n}(\varepsilon)= \begin{cases}\varepsilon^{\frac{1}{4}-\frac{1}{2 k}} & n=2 \\ \varepsilon^{\frac{k-1}{2 k}}|\log \varepsilon| & n=3\end{cases}
$$

For $m$-convexity inclusions with zero-curvature, in order to extend Theorem A to all dimensions, we need the following proposition, which shows that $\nabla \bar{v}_{1}$ is the main singular part of $\nabla v_{1}$ in $\Omega_{R_{0}}$.
Proposition 1.7. For $n \geq 2$, assume $D_{1}, D_{2}$ are of $C^{2, \alpha}$ and satisfy

$$
\begin{equation*}
\frac{1}{C}\left|x^{\prime}\right|^{m} \leq\left(h_{1}-h_{2}\right)\left(x^{\prime}\right) \leq C\left|x^{\prime}\right|^{m}, \quad m \geq \max \{2, n-1\} \tag{1.29}
\end{equation*}
$$

and (1.21). Let $v_{1} \in H^{1}(D)$ be the weak solution of (1.5). Then

$$
\begin{equation*}
\left\|\nabla\left(v_{1}-\bar{v}_{1}\right)\right\|_{L^{\infty}(\Omega)} \leq C \tag{1.30}
\end{equation*}
$$

Theorem 1.8. For $n \geq 2$, assume $D_{1}, D_{2}$ are of $C^{2, \alpha}$, and satisfy (1.29) and (1.20) -(1.21). Then there exists a constant $M$, depending only on $D_{1}^{0}, D_{2}^{0}$ and $\Omega$, such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\frac{\mathcal{L} \lambda^{\frac{n-1}{m}}}{\rho_{n}^{m}(\varepsilon)}+M+O\left(E_{n}^{m}(\varepsilon)\right) \tag{1.31}
\end{equation*}
$$

where $\mathcal{L}$ is a constant depending only on $m$ and $n$, and

$$
E_{n}^{m}(\varepsilon)= \begin{cases}\varepsilon^{\frac{1}{4 m}} & \text { if } m \geq 2, n=2  \tag{1.32}\\ \max \left\{\varepsilon^{\frac{1}{n-1}}, \varepsilon^{\frac{1}{4}}|\log \varepsilon|\right\} & \text { if } m=n-1, n \geq 3 \\ \varepsilon^{\frac{n-1}{4 m}} & \text { if } m>n-1, n \geq 3\end{cases}
$$

From (1.31), one can see that the energy aggregation of $v_{1}$ depends on the local geometry of the inclusions, such as $\lambda$, and the order of convexity $m$. The proof of Theorem 1.8 will be given in Section 3

On the other hand, since $\Delta v_{b}=0$ in $D$ with $v_{b}=C_{2}$ on $\partial D_{1} \cup \partial D_{2}$, it follows from the standard elliptic theory that

Theorem 1.9. Suppose that $0<\varepsilon<1 / 2$ sufficiently small. There are two positive constants $A, C$, independent of $\varepsilon$, such that

$$
\begin{equation*}
\left|\nabla v_{b}\left(x^{\prime}, x_{n}\right)\right| \leq C \exp \left(-\frac{A}{\left(\varepsilon+\left|x^{\prime}\right|^{m}\right)^{1-1 / m}}\right)\left\|v_{b}\right\|_{L^{2}(\Omega)}, \quad \forall\left(x^{\prime}, x_{n}\right) \in \Omega_{R_{0}} \tag{1.33}
\end{equation*}
$$

Theorem 1.9 implies that

$$
\begin{equation*}
\left\|\nabla v_{b}\right\|_{L^{\infty}\left(\Omega_{R_{0}}\right)} \leq C \tag{1.34}
\end{equation*}
$$

So that, combining with the classical elliptic theory,

$$
\begin{equation*}
\left\|\nabla v_{b}\right\|_{L^{\infty}(\Omega)} \leq C \tag{1.35}
\end{equation*}
$$

Denote

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}[\varphi]:=-\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}} \tag{1.36}
\end{equation*}
$$

Substituting (1.31) and (1.35) into (1.27), we have

$$
\begin{equation*}
\left|C_{1}-C_{2}\right| \leq C \rho_{n}^{m}(\varepsilon) \tag{1.37}
\end{equation*}
$$

By using (1.37), we prove, see Lemma 4.2 below, that

$$
\begin{equation*}
\left|C_{i}-C_{0}\right| \leq C \rho_{n}^{m}(\varepsilon), \quad i=1,2 \tag{1.38}
\end{equation*}
$$

This shows that $u_{0}$ defined by (1.15) is the limit of $u$ defined by (1.2). Furthermore, as for the convergent rate of $\mathcal{B}_{\varepsilon}[\varphi]$ to $\mathcal{B}_{0}[\varphi]$, we have the following estimate.
Proposition 1.10. Let $\mathcal{B}_{\varepsilon}[\varphi]$ and $\mathcal{B}_{0}[\varphi]$ be defined by (1.36) and (1.16), respectively. Then
(i) under the assumptions of Theorem 1.1, we have

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}[\varphi]-\mathcal{B}_{0}[\varphi]=O\left(\rho_{n}^{2}(\varepsilon)\right), \quad n=2,3 \tag{1.39}
\end{equation*}
$$

(ii) under the assumptions of Theorem 1.5, we have

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}[\varphi]-\mathcal{B}_{0}[\varphi]=O\left(\rho_{n}^{m}(\varepsilon)\right) \quad m \geq \max \{2, n-1\}, n \geq 2 \tag{1.40}
\end{equation*}
$$

This convergence rate is optimal because of (1.38). The proof of Proposition 1.10 will be given in Section 4 We are now in position to prove Theorems 1.1 and 1.5

Proof of Theorem 1.1. By using (1.6), (1.30) and (1.34),

$$
\nabla u=\left(C_{1}-C_{2}\right) \nabla \bar{v}_{1}+O(1), \quad \text { in } \Omega_{R_{0}}
$$

It follows from Proposition 1.10 that

$$
C_{1}-C_{2}=\frac{-\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}}}{\int_{\partial D_{1}} \frac{\partial v_{1}}{\partial \nu^{-}}}=\frac{\mathcal{B}_{\varepsilon}[\varphi]}{\int_{\Omega}\left|\nabla v_{1}\right|^{2}}
$$

Thus, using (1.28) and (1.40),

$$
\begin{align*}
\nabla u(x) & =\left(C_{1}-C_{2}\right) \nabla \bar{v}_{1}(x)+O(1) \\
& =\frac{\mathcal{B}_{0}[\varphi]+O\left(\rho_{n}(\varepsilon)\right)}{\frac{\kappa_{n}}{\rho_{n}(\varepsilon)}+M+O\left(E_{n}(\varepsilon)\right)} \nabla \bar{v}_{1}(x)+O(1) . \tag{1.41}
\end{align*}
$$

In view of the definition of $\rho_{n}(\varepsilon)$, (1.3), Theorem 1.1 follows easily from the above.

Proof of Theorem 1.5. Replacing (1.28) by (1.31) in (1.41), we have

$$
\begin{aligned}
\nabla u(x) & =\left(C_{1}-C_{2}\right) \nabla \bar{v}_{1}(x)+O(1) \\
& =\frac{\mathcal{B}_{0}[\varphi]+O\left(\rho_{n}^{m}(\varepsilon)\right)}{\frac{\mathcal{L} \lambda^{\frac{n-1}{m}}}{\rho_{n}^{m}(\varepsilon)}+M+O\left(E_{n}^{m}(\varepsilon)\right)} \nabla \bar{v}_{1}(x)+O(1) .
\end{aligned}
$$

The proof is completed by a direct computation.
The rest of this paper is organized as follows. We establish the pointwise upper and lower bound estimates of $\left|\nabla v_{1}\right|$ in Section 2 The asymptotics of the energy of $v_{1}$ for $m$-convex inclusions is proved in Section 3. The proof of Theorem 1.9 and Proposition 1.10 is given in Section 4.

## 2. The gradient estimates of $v_{1}$

This section is devoted to the estimates of $\left|\nabla v_{1}\right|$ for $m$-convexity inclusions with zero-curvature.

Proof of Proposition 1.7. For simplicity, denote

$$
w:=v_{1}-\bar{v}_{1} .
$$

By the definition of $v_{1}$ in (1.5), and $v_{1}=\bar{v}_{1}$ on $\partial D_{1} \cup \partial D_{2} \cup \partial D$, we have

$$
\begin{cases}-\Delta w=\Delta \bar{v}_{1} & \text { in } \Omega  \tag{2.1}\\ w=0 & \text { on } \partial \Omega\end{cases}
$$

In view of (1.12) and (1.13),

$$
\begin{equation*}
\left\|\bar{v}_{1}\right\|_{C^{2, \alpha}\left(\Omega \backslash \Omega_{R_{0} / 3}\right)} \leq C \tag{2.2}
\end{equation*}
$$

Using the standard elliptic theory, we have

$$
\begin{equation*}
|w|+|\nabla w|+\left|\nabla^{2} w\right| \leq C, \quad \text { in } \Omega \backslash \Omega_{R_{0} / 2} \tag{2.3}
\end{equation*}
$$

Thus, to show (1.30), we only need to prove

$$
\|\nabla w\|_{L^{\infty}\left(\Omega_{R_{0} / 2}\right)} \leq C
$$

First, we claim that

$$
\begin{equation*}
\int_{\Omega}|\nabla w|^{2} \leq C \tag{2.4}
\end{equation*}
$$

Indeed, by the maximum principle, we have $0<v_{1}<1$. Becuase $\bar{v}_{1}$ is also bounded,

$$
\begin{equation*}
\|w\|_{L^{\infty}(\Omega)} \leq C \tag{2.5}
\end{equation*}
$$

A direct computation yields,

$$
\begin{equation*}
\left|\Delta \bar{v}_{1}(x)\right| \leq \frac{C}{\delta\left(x^{\prime}\right)}, \quad \text { where } \delta\left(x^{\prime}\right)=\varepsilon+h_{1}\left(x^{\prime}\right)-h_{2}\left(x^{\prime}\right), \quad x \in \Omega_{R_{0}} \tag{2.6}
\end{equation*}
$$

Now multiplying the equation in (2.1) by $w$, integrating by parts, and making use of (2.2), (2.5) and (2.6),

$$
\int_{\Omega}|\nabla w|^{2}=\int_{\Omega} w\left(\Delta \bar{v}_{1}\right) \leq\|w\|_{L^{\infty}(\Omega)}\left(\int_{\Omega_{R_{0}}}\left|\Delta \bar{v}_{1}\right|+C\right) \leq C
$$

Thus, (2.4) is proved.
For $0<t<s<R_{0}$, let $\eta$ be a smooth cutoff function satisfying $\eta\left(x^{\prime}\right)=1$ if $\left|x^{\prime}-z^{\prime}\right|<t, \eta\left(x^{\prime}\right)=0$ if $\left|x^{\prime}-z^{\prime}\right|>s, 0 \leq \eta\left(x^{\prime}\right) \leq 1$ if $t \leq\left|x^{\prime}-z^{\prime}\right| \leq s$, and $\left|\nabla_{x^{\prime}} \eta\left(x^{\prime}\right)\right| \leq \frac{2}{s-t}$. Multiplying the equation in (2.1) by $w \eta^{2}$ and integrating by parts leads to the following Caccioppolli's type inequality

$$
\begin{equation*}
\int_{\Omega_{t}\left(z^{\prime}\right)}|\nabla w|^{2} \leq \frac{C}{(s-t)^{2}} \int_{\Omega_{s}\left(z^{\prime}\right)}|w|^{2}+C(s-t)^{2} \int_{\Omega_{s}\left(z^{\prime}\right)}\left|\Delta \bar{v}_{1}\right|^{2} \tag{2.7}
\end{equation*}
$$

where

$$
\Omega_{r}\left(z^{\prime}\right):=\left\{\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n}\left|h_{2}\left(x^{\prime}\right)<x_{n}<\varepsilon+h_{1}\left(x^{\prime}\right),\left|x^{\prime}-z^{\prime}\right|<r\right\}\right.
$$

The rest of the proof is divided into two steps. By an iteration technique developed in [8], we first have

STEP 1. Proof of

$$
\begin{equation*}
\int_{\Omega_{\delta}\left(z^{\prime}\right)}|\nabla w|^{2} d x \leq C \delta\left(z^{\prime}\right)^{n}, \quad \text { for } n \geq 2 \tag{2.8}
\end{equation*}
$$

where

$$
\delta=\delta\left(z^{\prime}\right):=\varepsilon+h_{1}\left(z^{\prime}\right)-h_{2}\left(z^{\prime}\right), \quad \text { for }\left(z^{\prime}, z_{n}\right) \in \Omega_{R_{0}}
$$

We adapt the iteration technique developed in [8] and give a unified iteration process. For $0<s<\left|z^{\prime}\right| \leq R_{0} / 2$, we note that by using Hölder inequality,

$$
\int_{\Omega_{s}\left(z^{\prime}\right)}|w|^{2}=\int_{\Omega_{s}\left(z^{\prime}\right)}\left(\int_{h_{2}\left(x^{\prime}\right)}^{x_{n}} \partial_{x_{n}} w\right)^{2} \leq C \delta\left(z^{\prime}\right)^{2} \int_{\Omega_{s}\left(z^{\prime}\right)}|\nabla w|^{2}, \quad \text { if } 0<s<\frac{2\left|z^{\prime}\right|}{3} .
$$

Substituting it into (2.7) and denoting

$$
F(t):=\int_{\Omega_{t}\left(z^{\prime}\right)}|\nabla w|^{2}
$$

we have

$$
\begin{equation*}
F(t) \leq\left(\frac{C_{0} \delta\left(z^{\prime}\right)}{s-t}\right)^{2} F(s)+C(s-t)^{2} \int_{\Omega_{s}\left(z^{\prime}\right)}\left|\Delta \bar{v}_{1}\right|^{2}, \quad \forall 0<t<s<\frac{2\left|z^{\prime}\right|}{3} \tag{2.9}
\end{equation*}
$$

where $C_{0}$ is a fixed positive universal constant.
Let $k=\left[\frac{\max \left\{\varepsilon^{1 / m},\left|z^{\prime}\right|\right\}}{4 C_{0} \delta\left(z^{\prime}\right)}\right]$ and $t_{i}=\delta+2 C_{0} i \delta\left(z^{\prime}\right), i=0,1,2, \cdots, k$. Taking $s=t_{i+1}$ and $t=t_{i}$ in (2.9), and in view of (2.6),

$$
\begin{equation*}
\int_{\Omega_{t_{i+1}}\left(z^{\prime}\right)}\left|\Delta \bar{v}_{1}\right|^{2} \leq \int_{\left|x^{\prime}-z^{\prime}\right|<t_{i+1}} \frac{C}{\delta\left(x^{\prime}\right)} d x^{\prime} \leq \frac{C t_{i+1}^{n-1}}{\delta\left(z^{\prime}\right)} \leq C(i+1)^{n-1} \delta\left(z^{\prime}\right)^{(n-2)} \tag{2.10}
\end{equation*}
$$

We obtain an iteration formula

$$
F\left(t_{i}\right) \leq \frac{1}{4} F\left(t_{i+1}\right)+C(i+1)^{n-1} \delta\left(z^{\prime}\right)^{n}
$$

After $k$ iterations, using (2.4),

$$
F\left(t_{0}\right) \leq\left(\frac{1}{4}\right)^{k} F\left(t_{k}\right)+C \delta\left(z^{\prime}\right)^{n} \sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l^{n-1} \leq C \delta\left(z^{\prime}\right)^{n}
$$

This implies that (2.8).
STEP 2. Next, we use Sobolev embedding theorem and classical $L^{p}$ estimates for elliptic equations to prove (1.30).

By using the following scaling and translating of variables

$$
\left\{\begin{array}{l}
x^{\prime}-z^{\prime}=\delta\left(z^{\prime}\right) y^{\prime} \\
x_{n}=\delta\left(z^{\prime}\right) y_{n}
\end{array}\right.
$$

then $\Omega_{\delta\left(z^{\prime}\right)}\left(z^{\prime}\right)$ becomes $Q_{1}$, where for $r \leq 1$,
$Q_{r}=\left\{y \in \mathbb{R}^{n}\left|\frac{1}{\delta\left(z^{\prime}\right)} h_{2}\left(\delta\left(z^{\prime}\right) y^{\prime}+z^{\prime}\right)<y_{n}<\frac{\varepsilon}{\delta\left(z^{\prime}\right)}+\frac{1}{\delta\left(z^{\prime}\right)} h_{1}\left(\delta\left(z^{\prime}\right) y^{\prime}+z^{\prime}\right),\left|y^{\prime}\right|<r\right\}\right.$,
and the top and bottom boundaries respectively become

$$
y_{n}=\hat{h}_{1}\left(y^{\prime}\right):=\frac{1}{\delta\left(z^{\prime}\right)}\left(\varepsilon+h_{1}\left(\delta\left(z^{\prime}\right) y^{\prime}+z^{\prime}\right)\right), \quad\left|y^{\prime}\right|<1
$$

and

$$
y_{n}=\hat{h}_{2}\left(y^{\prime}\right):=\frac{1}{\delta\left(z^{\prime}\right)} h_{2}\left(\delta\left(z^{\prime}\right) y^{\prime}+z^{\prime}\right), \quad\left|y^{\prime}\right|<1
$$

Then

$$
\hat{h}_{1}\left(0^{\prime}\right)-\hat{h}_{2}\left(0^{\prime}\right):=\frac{1}{\delta\left(z^{\prime}\right)}\left(\varepsilon+h_{1}\left(z^{\prime}\right)-h_{2}\left(z^{\prime}\right)\right)=1
$$

and by (1.8),

$$
\left|\nabla_{x^{\prime}} \hat{h}_{1}\left(0^{\prime}\right)\right|+\left|\nabla_{x^{\prime}} \hat{h}_{2}\left(0^{\prime}\right)\right| \leq C\left|z^{\prime}\right|^{m-1}, \quad\left|\nabla_{x^{\prime}}^{2} \hat{h}_{1}\left(0^{\prime}\right)\right|+\left|\nabla_{x^{\prime}}^{2} \hat{h}_{2}\left(0^{\prime}\right)\right| \leq C\left|z^{\prime}\right|^{m-2}
$$

Since $R_{0}$ is small, $\left\|\hat{h}_{1}\right\|_{C^{1,1}\left((-1,1)^{n-1}\right)}$ and $\left\|\hat{h}_{2}\right\|_{C^{1,1}\left((-1,1)^{n-1}\right)}$ are small and $Q_{1}$ is essentially a unit square (or a unit cylinder for $n=3$ ) as far as applications of the Sobolev embedding theorem and classical $L^{p}$ estimates for elliptic equations are concerned. Let
$\bar{V}_{1}\left(y^{\prime}, y_{n}\right):=\bar{v}_{1}\left(z^{\prime}+\delta\left(z^{\prime}\right) y^{\prime}, \delta\left(z^{\prime}\right) y_{n}\right), \quad W\left(y^{\prime}, y_{n}\right):=w\left(z^{\prime}+\delta\left(z^{\prime}\right) y^{\prime}, \delta\left(z^{\prime}\right) y_{n}\right), \quad y \in Q_{1}^{\prime}$,
then by (2.1),

$$
-\Delta W=\Delta \bar{V}_{1}, \quad y \in Q_{1}
$$

where

$$
\left|\Delta \bar{V}_{1}\right|=\delta\left(z^{\prime}\right)^{2}\left|\Delta \bar{v}_{1}\right|
$$

Since $W=0$ on the top and bottom boundaries of $Q_{1}$, using the Poincaré inequality,

$$
\|W\|_{H^{1}\left(Q_{1}\right)} \leq C\|\nabla W\|_{L^{2}\left(Q_{1}\right)}
$$

By $W^{2, p}$ estimates for elliptic equations (see e.g. [20]), the Sobolev embedding theorems, and using the bootstrap argument, with $p>n$,

$$
\|\nabla W\|_{L^{\infty}\left(Q_{1 / 2}\right)} \leq C\|W\|_{W^{2, p}\left(Q_{1 / 2}\right)} \leq C\left(\|\nabla W\|_{L^{2}\left(Q_{1}\right)}+\left\|\Delta \bar{V}_{1}\right\|_{L^{\infty}\left(Q_{1}\right)}\right)
$$

It follows from $\nabla W=\delta \nabla w$ and (2.6), (2.8) that

$$
\begin{align*}
& \|\nabla w\|_{L^{\infty}\left(\Omega_{\delta\left(z^{\prime}\right) / 2}\left(z^{\prime}\right)\right)} \\
& \leq C\left(\delta\left(z^{\prime}\right)^{-n / 2}\|\nabla w\|_{L^{2}\left(\Omega_{\delta\left(z^{\prime}\right)}^{\prime}\left(z^{\prime}\right)\right)}+\delta\left(z^{\prime}\right)\left\|\Delta \bar{v}_{1}\right\|_{L^{\infty}\left(\Omega_{\delta\left(z^{\prime}\right)}\left(z^{\prime}\right)\right)}\right) \leq C \tag{2.11}
\end{align*}
$$

Proposition 1.7 is established.
Remark 2.1. We point out that the estimate involving $\Delta \bar{v}_{1}$ is very crucial in the above proof, such as (2.10) and (2.11), for $\int_{\Omega_{t_{i+1}}\left(z^{\prime}\right)}\left|\Delta \bar{v}_{1}\right|^{2}$ and $\delta\left(z^{\prime}\right)\left\|\Delta \bar{v}_{1}\right\|_{L^{\infty}\left(\Omega_{\delta\left(z^{\prime}\right)}\left(z^{\prime}\right)\right)}$.

An immediate consequence of Proposition 1.7 is that
Corollary 2.2. Under the assumption as in Proposition 1.7.

$$
\begin{equation*}
\frac{1}{C\left(\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)\right)} \leq\left|\nabla v_{1}\left(x^{\prime}, x_{n}\right)\right| \leq \frac{C}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)},\left(x^{\prime}, x_{n}\right) \in \Omega_{R_{0}} \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\nabla v_{1}\right\|_{L^{\infty}\left(\Omega \backslash \Omega_{R_{0}}\right)} \leq C \tag{2.13}
\end{equation*}
$$

## 3. Proof of Theorem 1.8

Define $v_{1}^{0}$ to be the solution of the limiting problem

$$
\begin{cases}\Delta v_{1}^{0}=0 & \text { in } \Omega^{0}  \tag{3.1}\\ v_{1}^{0}=1 & \text { on } \partial D_{1}^{0} \backslash\{0\} \\ v_{1}^{0}=0 & \text { on } \partial D_{2}^{0} \cup \partial D\end{cases}
$$

Similarly as $\bar{v}_{1}$, we construct an auxiliary function $\bar{v}_{1}^{0}$, such that $\bar{v}_{1}^{0}=1$ on $\partial D_{1}^{0} \backslash\{0\}$, $\bar{v}_{1}^{0}=0$ on $\partial D_{2}^{0} \cup \partial D$,

$$
\begin{equation*}
\bar{v}_{1}^{0}=\frac{x_{n}-h_{2}\left(x^{\prime}\right)}{\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)} \quad \text { in } \Omega_{R_{0}}^{0}:=\left\{\left(x^{\prime}, x_{n}\right)\left|h_{2}\left(x^{\prime}\right) \leq x_{n} \leq h_{1}\left(x^{\prime}\right),\left|x^{\prime}\right| \leq R_{0}\right\},\right. \tag{3.2}
\end{equation*}
$$

and $\left\|\bar{v}_{1}^{0}\right\|_{C^{2, \alpha}\left(\Omega^{0} \backslash \Omega_{R_{0}}^{0}\right)} \leq C$. It is easy to see that

$$
\begin{equation*}
\left|\partial_{x^{\prime}} \bar{v}_{1}^{0}(x)\right| \leq \frac{C}{\left|x^{\prime}\right|}, \quad \partial_{x_{n}} \bar{v}_{1}^{0}(x)=\frac{1}{\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}, x \in \Omega_{R_{0}}^{0} \backslash\{0\} . \tag{3.3}
\end{equation*}
$$

It follows from the proof of Proposition 1.7 that

$$
\begin{equation*}
\left\|\nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)\right\|_{L^{\infty}\left(\Omega^{0}\right)} \leq C \tag{3.4}
\end{equation*}
$$

This shows that $\nabla \bar{v}_{1}^{0}$ is also the main term of $\nabla v_{1}^{0}$.
Lemma 3.1. Let $v_{1}$ and $v_{1}^{0}$ be defined by (1.5) and (3.1), respectively. Then

$$
\begin{equation*}
\left\|v_{1}-v_{1}^{0}\right\|_{L^{\infty}\left(\Omega \backslash\left(D_{1} \cup D_{2} \cup D_{1}^{0} \cup \Omega_{\varepsilon^{1 /(2 m)}}\right)\right)} \leq C \varepsilon^{1 / 2}, \quad i=1,2 \tag{3.5}
\end{equation*}
$$

Proof. We will first consider the difference $v_{1}-v_{1}^{0}$ on the boundary of $\Omega \backslash\left(D_{1} \cup\right.$ $\left.D_{2} \cup D_{1}^{0} \cup \Omega_{\varepsilon^{1 /(2 m)}}\right)$, then use the maximum principle to obtain (3.5).

STEP 1. Obviously,

$$
\begin{equation*}
v_{1}-v_{1}^{0}=0, \quad \text { on } \partial D_{2} \cup \partial D \tag{3.6}
\end{equation*}
$$

In the following we only need to deal with the boundary $\partial\left(D_{1} \cup D_{1}^{0}\right)$. We divide it into two parts: (a) $\partial D_{1}^{0} \backslash D_{1}$ and (b) $\partial D_{1} \backslash D_{1}^{0}$.
(a) When $x \in \partial D_{1}^{0} \backslash D_{1}$, we introduce a cylinder

$$
\mathcal{C}_{r}:=\left\{x \in \mathbb{R}^{n}\left|2 \min _{\left|x^{\prime}\right|=r} h_{2}\left(x^{\prime}\right) \leq x_{n} \leq \varepsilon+2 \max _{\left|x^{\prime}\right|=r} h_{1}\left(x^{\prime}\right),\left|x^{\prime}\right|<r\right\}, \quad r \leq R_{0}\right.
$$

(a1) For $x \in \partial D_{1}^{0} \cap\left(\mathcal{C}_{R_{0}} \backslash \mathcal{C}_{\varepsilon^{1 /(2 m)}}\right)$, using $v_{1}^{0}=1$ on $\partial D_{1}^{0}$ and $v_{1}=1$ on $\partial D_{1}$, by mean value theorem and estimate (2.12), we have, for some $\theta_{\varepsilon} \in(0,1)$

$$
\begin{aligned}
\left|v_{1}(x)-v_{1}^{0}(x)\right|=\left|v_{1}(x)-1\right| & =\left|v_{1}\left(x^{\prime}, h_{1}\left(x^{\prime}\right)\right)-v_{1}\left(x^{\prime}, \varepsilon+h_{1}\left(x^{\prime}\right)\right)\right| \\
& =\left|\partial_{x_{n}} v_{1}\left(x^{\prime}, \theta_{\varepsilon} \varepsilon+h_{1}\left(x^{\prime}\right)\right)\right| \cdot \varepsilon \\
& \leq \frac{C \varepsilon}{\varepsilon+\left|x^{\prime}\right|^{m}} \leq C \varepsilon^{1 / 2}
\end{aligned}
$$

(a2) For $x \in \partial D_{1}^{0} \cap\left(\Omega \backslash \Omega_{R_{0}}\right)$, there exists $y_{\varepsilon} \in \partial D_{1} \cap \overline{\Omega \backslash \Omega_{R_{0} / 2}}$ such that $\left|x-y_{\varepsilon}\right|<$ $C \varepsilon$ (note that $v_{1}\left(y_{\varepsilon}\right)=1$ ). By (2.13) and mean value theorem again, for some $\theta_{\varepsilon} \in(0,1)$
$\left|v_{1}(x)-v_{1}^{0}(x)\right|=\left|v_{1}(x)-1\right|=\left|v_{1}(x)-v_{1}\left(y_{\varepsilon}\right)\right| \leq\left|\nabla v_{1}\left(\left(1-\theta_{\varepsilon}\right) x+\theta_{\varepsilon} y_{\varepsilon}\right)\right|\left|x-y_{\varepsilon}\right| \leq C \varepsilon$.
(b) When $x \in \partial D_{1} \backslash D_{1}^{0}$, since $0<v_{1}<1$ in $\Omega$ and $\Delta v_{1}=0$ in $\Omega$, it follows from the boundary estimates of harmonic function that there exists $y_{x} \in \Omega,\left|y_{x}-x\right| \leq C \varepsilon$ such that $v_{1}\left(y_{x}\right)=v_{1}^{0}(x)$. Using (2.13) again,

$$
\left|v_{1}(x)-v_{1}^{0}(x)\right|=\left|v_{1}(x)-v_{1}\left(y_{x}\right)\right| \leq\left\|\nabla v_{1}\right\|_{L^{\infty}\left(\Omega \backslash \Omega_{R_{0}}\right)}\left|x-y_{x}\right| \leq C \varepsilon
$$

Therefore,

$$
\begin{equation*}
\left|v_{1}(x)-v_{1}^{0}(x)\right| \leq C \varepsilon^{1 / 2}, \quad \text { for } x \in \partial\left(D_{1} \cup D_{1}^{0}\right) \backslash \mathcal{C}_{\varepsilon^{1 /(2 m)}} \tag{3.7}
\end{equation*}
$$

STEP 2. We consider the lateral boundary of $\Omega_{\varepsilon^{1 /(2 m)}}^{0}$,

$$
S_{1 /(2 m)}:=\left\{\left(x^{\prime}, x_{n}\right)\left|h_{2}\left(x^{\prime}\right) \leq x_{n} \leq h_{1}\left(x^{\prime}\right),\left|x^{\prime}\right|=\varepsilon^{1 /(2 m)}\right\}\right.
$$

By using (1.30) and $\left(v_{1}-\bar{v}_{1}\right)=0$ on $\partial D_{2}$, we have, for $x \in S_{1 /(2 m)}$,

$$
\begin{equation*}
\left|\left(v_{1}-\bar{v}_{1}\right)(x)\right| \leq\left\|\nabla\left(v_{1}-\bar{v}_{1}\right)\right\|_{L^{\infty}\left(S_{1 /(2 m)}\right)}\left|\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)\right| \leq C\left|x^{\prime}\right|^{m} \leq C \varepsilon^{1 / 2} \tag{3.8}
\end{equation*}
$$

Similarly, since $\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)=0$ on $\partial D_{2}$, it follows from (3.4) and mean value theorem that for $x \in S_{1 /(2 m)}$,

$$
\begin{equation*}
\left|\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)(x)\right| \leq\left\|\nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)\right\|_{L^{\infty}\left(S_{1 / m-\beta}\right)}\left|\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)\right| \leq C\left|x^{\prime}\right|^{m} \leq C \varepsilon^{1 / 2} \tag{3.9}
\end{equation*}
$$

Since $\bar{v}_{1}=\bar{v}_{1}^{0} \equiv 0$ on $\partial D_{2}$, then for $x \in S_{1 /(2 m)}$,

$$
\begin{align*}
\left|\left(\bar{v}_{1}-\bar{v}_{1}^{0}\right)(x)\right| & \leq\left\|\partial_{x_{n}}\left(\bar{v}_{1}-\bar{v}_{1}^{0}\right)\right\|_{L^{\infty}\left(S_{1 /(2 m)}\right)}\left|\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)\right| \\
& \leq C \max _{\left|x^{\prime}\right|=\varepsilon^{1 /(2 m)}}\left\{\frac{1}{\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}-\frac{1}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}\right\}\left|x^{\prime}\right|^{m} \\
& \leq \frac{C \varepsilon}{\left|x^{\prime}\right|^{m}\left(\varepsilon+\left|x^{\prime}\right|^{m}\right)}\left|x^{\prime}\right|^{m} \leq C \varepsilon^{1 / 2} \tag{3.10}
\end{align*}
$$

Thus, combining (3.8), (3.9) with (3.10), we have, for $x \in S_{1 /(2 m)}$,

$$
\begin{equation*}
\left|\left(v_{1}-v_{1}^{0}\right)(x)\right| \leq\left|\left(v_{1}-\bar{v}_{1}\right)(x)\right|+\left|\left(\bar{v}_{1}-\bar{v}_{1}^{0}\right)(x)\right|+\left|\left(\bar{v}_{1}^{0}-v_{1}^{0}\right)(x)\right| \leq C \varepsilon^{1 / 2} \tag{3.11}
\end{equation*}
$$

Finally, by (3.6), (3.7) and (3.11), and applying the maximum principle to ( $v_{1}-$ $\left.v_{1}^{0}\right)$ on $\Omega \backslash\left(D_{1} \cup D_{2} \cup D_{1}^{0} \cup \Omega_{\varepsilon^{1 /(2 m)}}\right)$, we obtain (3.5).

If $\partial D_{1}$ and $\partial D_{2}$ are assumed to be fo $C^{2, \alpha}$ and satisfy (1.20) and (1.21), then we have an improvement of Lemma 3.1 by interpolation.

Lemma 3.2. Assume that $v_{1}$ and $v_{1}^{0}$ are solution of (1.5) and (3.1), respectively. If $\partial D_{1}^{0}$ and $\partial D_{2}^{0}$ are of $C^{2, \alpha}$ and satisfy (1.20) -(1.21), then

$$
\begin{align*}
& \left|\nabla v_{1}(x)\right| \leq C\left|x^{\prime}\right|^{-m}, \quad x \in \Omega_{R_{0}} \backslash \Omega_{\varepsilon^{1 /(2 m)}} \\
& \left|\nabla v_{1}^{0}(x)\right| \leq C\left|x^{\prime}\right|^{-m}, \quad x \in \Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon^{1 /(2 m)}}^{0} \tag{3.12}
\end{align*}
$$

and

$$
\begin{equation*}
\left|\nabla\left(v_{1}-v_{1}^{0}\right)(x)\right| \leq C \varepsilon^{1 / 4}\left|x^{\prime}\right|^{-m}, \quad \text { in } \Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon^{1 /(2 m)}}^{0} \tag{3.13}
\end{equation*}
$$

Proof. For $\varepsilon^{1 /(2 m)} \leq\left|z^{\prime}\right| \leq R_{0}$, we make use of the change of variable

$$
\left\{\begin{array}{l}
x^{\prime}-z^{\prime}=\left|z^{\prime}\right|^{m} y^{\prime} \\
x_{n}=\left|z^{\prime}\right|^{m} y_{n}
\end{array}\right.
$$

to rescale $\Omega_{\left|z^{\prime}\right|+\left|z^{\prime}\right|^{m}} \backslash \Omega_{\left|z^{\prime}\right|}$ into an approximate unit-size cube (or cylinder) $Q_{1}$, and $\Omega_{\left|z^{\prime}\right|+\left|z^{\prime}\right|^{m}}^{0} \backslash \Omega_{\left|z^{\prime}\right|}^{0}$ into $Q_{1}^{0}$. Let

$$
V_{1}(y)=v_{1}\left(z^{\prime}+\left|z^{\prime}\right|^{m} y^{\prime},\left|z^{\prime}\right|^{m} y_{n}\right), \quad \text { in } Q_{1}
$$

and

$$
V_{1}^{0}(y)=v_{1}^{0}\left(z^{\prime}+\left|z^{\prime}\right|^{m} y^{\prime},\left|z^{\prime}\right|^{m} y_{n}\right), \quad \text { in } Q_{1}^{0}
$$

Since $0<V_{1}, V_{1}^{0}<1$, using the standard elliptic theory, we have

$$
\left|\nabla^{2} V_{1}\right| \leq C, \quad \text { in } Q_{1}, \quad \text { and }\left|\nabla^{2} V_{1}^{0}\right| \leq C, \quad \text { in } Q_{1}^{0}
$$

Interpolating it with (3.5) yields

$$
\left|\nabla\left(V_{1}-V_{1}^{0}\right)\right| \leq C \varepsilon^{\frac{1}{2}\left(1-\frac{1}{2}\right)} \leq C \varepsilon^{1 / 4}, \quad \text { in } Q_{1}^{0}
$$

Thus, rescaling it back to $v_{1}-v_{1}^{0}$, we have (3.13) holds.
By the way, we have

$$
\left|\nabla v_{1}(x)\right| \leq C\left|z^{\prime}\right|^{-m}, \quad x \in \Omega_{\left|z^{\prime}\right|+\left|z^{\prime}\right|^{m}} \backslash \Omega_{\left|z^{\prime}\right|}
$$

and

$$
\left|\nabla v_{1}^{0}(x)\right| \leq C\left|z^{\prime}\right|^{-m}, \quad x \in \Omega_{\left|z^{\prime}\right|+\left|z^{\prime}\right| m}^{0} \backslash \Omega_{\left|z^{\prime}\right|}^{0}
$$

so (3.12) follows.
Proof of Theorem 1.8. To prove (1.31), we divid the integral into two parts:

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\int_{\Omega \backslash \Omega_{\varepsilon} \gamma}\left|\nabla v_{1}\right|^{2}+\int_{\Omega_{\varepsilon} \gamma}\left|\nabla v_{1}\right|^{2}=: \mathrm{I}+\mathrm{II} \tag{3.14}
\end{equation*}
$$

where we take $\gamma=\frac{1}{4 m}$, for convenience.
STEP 1. We first prove

$$
\begin{equation*}
\mathrm{I}=\int_{\Omega \backslash \Omega_{\varepsilon} \gamma}\left|\nabla v_{1}\right|^{2}=\int_{\Omega^{0} \backslash \Omega_{\varepsilon^{0}}^{0}}\left|\nabla v_{1}^{0}\right|^{2}+O\left(E_{n}^{m}(\varepsilon)\right), \tag{3.15}
\end{equation*}
$$

where $E_{n}^{m}(\varepsilon)$ is defined in (1.32). We divide term I further as follows:

$$
\mathrm{I}=\int_{\Omega \backslash \Omega_{R_{0}}}\left|\nabla v_{1}\right|^{2}+\int_{\Omega_{R_{0} \backslash \Omega_{\varepsilon} \gamma}}\left|\nabla v_{1}\right|^{2}=: \mathrm{I}_{1}+\mathrm{I}_{2} .
$$

First, for term $\mathrm{I}_{1}$, we claim that

$$
\begin{equation*}
\mathrm{I}_{1}=M_{1}+O\left(\varepsilon^{1 / 4}\right), \quad M_{1}:=\int_{\Omega^{0} \backslash \Omega_{R_{0}}^{0}}\left|\nabla v_{1}^{0}\right|^{2} \tag{3.16}
\end{equation*}
$$

Indeed, since

$$
\Delta\left(v_{1}-v_{1}^{0}\right)=0, \quad \text { in } \Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0} / 2}\right)
$$

and

$$
0<v_{1}, v_{1}^{0}<1, \quad \text { in } \Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0} / 2}\right)
$$

it follows that provided $\partial D_{1}^{0}, \partial D_{2}^{0}$ and $\partial \Omega$ are of $C^{2, \alpha}$,

$$
\left|\nabla^{2}\left(v_{1}-v_{1}^{0}\right)\right| \leq\left|\nabla^{2} v_{1}\right|+\left|\nabla^{2} v_{1}^{0}\right| \leq C, \quad \text { in } \Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0}}\right)
$$

where $C$ is independent of $\varepsilon$. By using an interpolation with (3.5), we have

$$
\left|\nabla\left(v_{1}-v_{1}^{0}\right)\right| \leq C \varepsilon^{1 / 2\left(1-\frac{1}{2}\right)} \leq C \varepsilon^{1 / 4}, \quad \text { in } \Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0}}\right)
$$

In view of the boundedness of $\left|\nabla v_{1}\right|$ in $D_{1}^{0} \backslash\left(D_{1} \cup \Omega_{R_{0}}\right)$ and $D_{1} \backslash D_{1}^{0}$, and $\mid D_{1}^{0} \backslash$ $\left(D_{1} \cup \Omega_{R_{0}}\right) \mid$ and $\left|D_{1} \backslash D_{1}^{0}\right|$ are less than $C \varepsilon$,

$$
\mathrm{I}_{1}-M_{1}
$$

$$
=\int_{\Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0}}\right)}\left(\left|\nabla v_{1}\right|^{2}-\left|\nabla v_{1}^{0}\right|^{2}\right)+\int_{D_{1}^{0} \backslash\left(D_{1} \cup \Omega_{R_{0}}\right)}\left|\nabla v_{1}\right|^{2}+\int_{D_{1} \backslash D_{1}^{0}}\left|\nabla v_{1}^{0}\right|^{2}
$$

$$
=2 \int_{\Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0}}\right)} \nabla v_{1}^{0} \nabla\left(v_{1}-v_{1}^{0}\right)+\int_{\Omega \backslash\left(D_{1} \cup D_{1}^{0} \cup D_{2} \cup \Omega_{R_{0}}\right)}\left|\nabla\left(v_{1}-v_{1}^{0}\right)\right|^{2}+O(\varepsilon)
$$

$$
=O\left(\varepsilon^{1 / 4}\right)
$$

Thus, (3.16) is proved.
For $\mathrm{I}_{2}$, we will prove that

$$
\begin{equation*}
\mathrm{I}_{2}=\mathrm{I}_{2}^{(0)}+E_{n, m}(\varepsilon), \quad \mathrm{I}_{2}^{(0)}:=\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\nabla v_{1}^{0}\right|^{2} \tag{3.17}
\end{equation*}
$$

Indeed,

$$
\begin{align*}
\mathrm{I}_{2}-\mathrm{I}_{2}^{(0)}= & \int_{\left(\Omega_{R_{0}} \backslash \Omega_{\varepsilon \gamma}\right) \backslash\left(\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}\right)}\left|\nabla v_{1}\right|^{2}+\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\nabla\left(v_{1}-v_{1}^{0}\right)\right|^{2} \\
& +2 \int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0} \gamma} \nabla v_{1}^{0} \cdot \nabla\left(v_{1}-v_{1}^{0}\right) . \tag{3.18}
\end{align*}
$$

For the first term in the right hand side of (3.18), because the thickness of $\left(\Omega_{R_{0}} \backslash\right.$ $\left.\Omega_{\varepsilon^{\gamma}}\right) \backslash\left(\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon^{\gamma}}^{0}\right)$ is $\varepsilon$, using Lemma 3.2,

$$
\begin{aligned}
\int_{\left(\Omega_{R_{0}} \backslash \Omega_{\varepsilon} \gamma\right) \backslash\left(\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}\right)}\left|\nabla v_{1}\right|^{2} & \leq C \varepsilon \int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\left|x^{\prime}\right|^{2 m}} \\
& \leq C \varepsilon^{1+(n-2 m-1) \gamma} \leq C \varepsilon^{\frac{1}{2}+\frac{n-1}{4 m}} \leq C E_{n, m}(\varepsilon)
\end{aligned}
$$

For the second and third terms, for any $\varepsilon^{\gamma} \leq\left|z^{\prime}\right| \leq R_{0}, \gamma=\frac{1}{4 m}$, if $\partial D_{1}^{0}$ and $\partial D_{2}^{0}$ are of $C^{2, \alpha}$, then by Lemma 3.2 ,

$$
\begin{aligned}
\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\nabla\left(v_{1}-v_{1}^{0}\right)\right|^{2} & \leq C \varepsilon^{1 / 2} \int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|x^{\prime}\right|^{-2 m} d x^{\prime} d x_{n} \\
& \leq C \varepsilon^{1 / 2} \int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\left|x^{\prime}\right|^{m}} \leq C \varepsilon^{1 / 4} E_{n, m}(\varepsilon),
\end{aligned}
$$

and

$$
\left|2 \int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}} \nabla v_{1}^{0} \cdot \nabla\left(v_{1}-v_{1}^{0}\right)\right| \leq C \varepsilon^{1 / 4} \int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\left|x^{\prime}\right|^{m}} \leq C E_{n, m}(\varepsilon) .
$$

Thus, (3.17) holds, so does (3.15) with (3.16).
STEP 2. Next, we use the explicit functions $\bar{v}_{1}^{0}$ and $\bar{v}_{1}$ to approximate $v_{1}^{0}$ and $v_{1}$, respectively.

Denote

$$
M_{2}:=2 \int_{\Omega_{R_{0}}^{0}} \nabla \bar{v}_{1}^{0} \cdot \nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)+\int_{\Omega_{R_{0}}^{0}}\left(\left|\nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)\right|^{2}+\left|\partial_{x^{\prime}} \bar{v}_{1}^{0}\right|^{2}\right)
$$

which is a constant, depending on $R_{0}$ but not on $\varepsilon$. Using (3.3) and (3.4), a similar argument as in Step 1 yields

$$
\begin{aligned}
\mathrm{I}_{2}^{(0)} & =\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\nabla \bar{v}_{1}^{0}\right|^{2}+2 \int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}} \nabla \bar{v}_{1}^{0} \cdot \nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)+\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\nabla\left(v_{1}^{0}-\bar{v}_{1}^{0}\right)\right|^{2} \\
& =\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\partial_{x_{n}} \bar{v}_{1}^{0}\right|^{2}+M_{2}+O\left(\varepsilon^{\frac{n-1}{4 m}}\right) .
\end{aligned}
$$

For term II in (3.14),

$$
\begin{equation*}
\mathrm{II}=\int_{\Omega_{\varepsilon} \gamma}\left|\nabla v_{1}\right|^{2}=\int_{\Omega_{\varepsilon} \gamma}\left|\nabla \bar{v}_{1}\right|^{2}+2 \int_{\Omega_{\varepsilon} \gamma} \nabla \bar{v}_{1} \cdot \nabla\left(v_{1}-\bar{v}_{1}\right)+\int_{\Omega_{\varepsilon} \gamma}\left|\nabla\left(v_{1}-\bar{v}_{1}\right)\right|^{2} . \tag{3.19}
\end{equation*}
$$

By Proposition 1.7, we have

$$
2 \int_{\Omega_{\varepsilon \gamma} \gamma} \nabla \bar{v}_{1} \cdot \nabla\left(v_{1}-\bar{v}_{1}\right)+\int_{\Omega_{\varepsilon} \gamma}\left|\nabla\left(v_{1}-\bar{v}_{1}\right)\right|^{2}=O\left(\varepsilon^{\frac{n-1}{4 m}}\right)
$$

Recalling the assumption (1.20)-(1.21) and (1.12), we have

$$
\begin{equation*}
\left|\partial_{x^{\prime}} \bar{v}_{1}(x)\right| \leq \frac{C\left|x^{\prime}\right|^{m-1}}{\varepsilon+\left|x^{\prime}\right|^{m}}, \quad \partial_{x_{n}} \bar{v}_{1}(x)=\frac{1}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}, x \in \Omega_{\varepsilon^{\gamma}} \tag{3.20}
\end{equation*}
$$

Therefore

$$
\int_{\Omega_{\varepsilon} \gamma}\left|\partial_{x^{\prime}} \bar{v}_{1}\right|^{2} \leq C \int_{\left|x^{\prime}\right|<\varepsilon^{\gamma}} \frac{\left|x^{\prime}\right|^{2 m-2}}{\varepsilon+\left|x^{\prime}\right|^{m}} d x^{\prime} \leq C \int_{\left|x^{\prime}\right|<\varepsilon^{\gamma}}\left|x^{\prime}\right|^{m-2} d x^{\prime}=O\left(\varepsilon^{\frac{n+m-3}{4 m}}\right)
$$

Since $m \geq 2$, then $n+m-3 \geq n-1$. Hence, it follows from (3.19) and $m \geq 2$ that

$$
\mathrm{II}=\int_{\Omega_{\varepsilon \gamma}}\left|\partial_{x_{n}} \bar{v}_{1}\right|^{2}+O\left(\varepsilon^{\frac{n-1}{4 m}}\right)
$$

Now combining Step 1 with the above, using $\gamma=1 /(4 m)$, we obtain

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\int_{\Omega_{\varepsilon} \gamma}\left|\partial_{x_{n}} \bar{v}_{1}\right|^{2}+\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon \gamma}^{0}}\left|\partial_{x_{n}} \bar{v}_{1}^{0}\right|^{2}+M_{1}+M_{2}+O\left(E_{n}^{m}(\varepsilon)\right) . \tag{3.21}
\end{equation*}
$$

STEP 3. Next, we will calculate the first two terms in the right hand side of (3.21). It follows from (3.3) and (3.20) that

$$
\begin{align*}
& \int_{\Omega_{\varepsilon} \gamma}\left|\partial_{x_{n}} \bar{v}_{1}\right|^{2}+\int_{\Omega_{R_{0}}^{0} \backslash \Omega_{\varepsilon^{\gamma}}^{0}}\left|\partial_{x_{n}} \bar{v}_{1}^{0}\right|^{2} \\
= & \int_{R_{0}>\left|x^{\prime}\right|>\varepsilon^{\gamma}} \frac{d x^{\prime}}{\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)}+\int_{\left|x^{\prime}\right|<\varepsilon^{\gamma}} \frac{d x^{\prime}}{\varepsilon+\left(h_{1}-h_{2}\right)\left(x^{\prime}\right)} \\
= & \int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\lambda\left|x^{\prime}\right|^{m}}+\int_{\left|x^{\prime}\right|<\varepsilon^{\gamma}} \frac{d x^{\prime}}{\varepsilon+\lambda\left|x^{\prime}\right|^{m}} \\
= & \int_{\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\varepsilon+\lambda\left|x^{\prime}\right|^{m}}+O\left(\varepsilon^{\frac{1}{2}+\frac{n-1}{4 m}}\right) \tag{3.22}
\end{align*}
$$

we here used that

$$
\begin{aligned}
\left|\int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}}\left(\frac{1}{\lambda\left|x^{\prime}\right|^{m}}-\frac{1}{\varepsilon+\lambda\left|x^{\prime}\right|^{m}}\right) d x^{\prime}\right| & \leq C \varepsilon \int_{\varepsilon^{\gamma}<\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\left|x^{\prime}\right|^{2 m}} \\
& \leq C \varepsilon^{1+(n-2 m-1) \gamma} \leq C \varepsilon^{\frac{1}{2}+\frac{n-1}{4 m}}
\end{aligned}
$$

Finally, we calculate the first term in the line of (3.22).
(i) For $n=2$, we have

$$
2 \int_{0}^{R_{0}} \frac{d x_{1}}{\varepsilon+\lambda x_{1}^{m}}=\frac{\mathcal{L}_{m, 2}}{\varepsilon^{1-\frac{1}{m}} \lambda^{\frac{1}{m}}}+M_{3}^{(1)}+O\left(\varepsilon^{\frac{1}{4 m}}\right)
$$

where

$$
\mathcal{L}_{m, 2}:=\int_{0}^{+\infty} \frac{1}{1+y^{m}} d y, \quad M_{3}^{(1)}:=\frac{2}{\lambda} \frac{m-1}{R_{0}^{m-1}}, \quad m \geq 2
$$

Therefore, from (3.21),

$$
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\frac{\mathcal{L}_{m, 2}}{\varepsilon^{1-\frac{1}{m}} \lambda^{\frac{1}{m}}}+M+O\left(\varepsilon^{\frac{1}{4 m}}\right), \quad M=M_{1}+M_{2}+M_{3}^{(1)}
$$

(ii) For $n \geq 3, m=n-1$, for the first term of (3.22),

$$
\begin{align*}
\int_{\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\varepsilon+\lambda\left|x^{\prime}\right|^{m}} & =\frac{\omega_{n-1}}{\lambda} \int_{0}^{R_{0}\left(\frac{\lambda}{\varepsilon}\right)^{1 / m}} \frac{r^{m-1} d r}{1+r^{m}}  \tag{3.23}\\
& =\frac{\mathcal{L}_{m, n}}{\lambda|\log \varepsilon|}+M_{3}^{(2)}+O\left(\varepsilon^{\frac{1}{m}}\right)
\end{align*}
$$

where

$$
\mathcal{L}_{m, n}:=\frac{\omega_{n-1}}{m}, \quad M_{3}^{(2)}:=\frac{\omega_{n-1}}{\lambda}\left(\log R_{0}+\frac{1}{m} \log \lambda\right) .
$$

Therefore, from (3.21),

$$
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\frac{\mathcal{L}_{m, n}}{\lambda|\log \varepsilon|}+M+O\left(\varepsilon^{\frac{1}{n-1}}\right)+O\left(E_{n}^{n-1}(\varepsilon)\right)
$$

where

$$
M=M_{1}+M_{2}+M_{3}^{(2)}
$$

(iii) For $n \geq 3, m>n-1$

$$
\begin{aligned}
\int_{\left|x^{\prime}\right|<R_{0}} \frac{d x^{\prime}}{\varepsilon+\lambda\left|x^{\prime}\right|^{m}} & =\frac{\omega_{n-1}}{\lambda^{\frac{n-1}{m}} \varepsilon^{1-\frac{n-1}{m}}} \int_{0}^{R_{0}\left(\frac{\lambda}{\varepsilon}\right)^{1 / m}} \frac{r^{n-2} d r}{1+r^{m}} \\
& =\frac{\mathcal{L}_{m, n}}{\lambda^{\frac{n-1}{m}} \varepsilon^{1-\frac{n-1}{m}}}+M_{3}^{(3)}+O\left(\varepsilon^{2-\frac{n-1}{m}}\right)
\end{aligned}
$$

where

$$
\mathcal{L}_{m, n}:=\omega_{n-1} \int_{0}^{+\infty} \frac{r^{n-2} d r}{1+r^{m}}, \quad M_{3}^{(3)}:=\frac{\omega_{n-1}}{\lambda} R_{0}^{n-1-m}
$$

Therefore, from (3.21),

$$
\int_{\Omega}\left|\nabla v_{1}\right|^{2}=\frac{\mathcal{L}_{m, n}}{\lambda^{\frac{n-1}{m}} \varepsilon^{1-\frac{n-1}{m}}}+M+O\left(\varepsilon^{\frac{1}{4 m}}\right)
$$

where

$$
M=M_{1}+M_{2}+M_{3}^{(3)}
$$

It is not difficult to prove that these $M$ are some constants independent of $R_{0}$. If not, suppose that there exist $M\left(R_{0}\right)$ and $M\left(\tilde{R}_{0}\right)$, both independent of $\varepsilon$, such that (1.31) holds, then

$$
M\left(R_{0}\right)-M\left(\tilde{R}_{0}\right) \rightarrow 0, \quad \text { as } \varepsilon \rightarrow 0
$$

which implies that $M\left(R_{0}\right)=M\left(\tilde{R}_{0}\right)$.

## 4. The proof of Theorem 1.9 and Proposition 1.10

### 4.1. Estimates for $\left|\nabla v_{b}\right|$.

Proof of Theorem 1.9. First, by the trace theorem, we have $\left|C_{2}\right| \leq C$. Recall that $v_{b}$ satisfies that

$$
\begin{cases}\Delta\left(v_{b}-C_{2}\right)=0 & \text { in } \Omega  \tag{4.1}\\ v_{b}-C_{2}=0 & \text { on } \partial D_{1} \cup \partial D_{2} \\ v_{b}-C_{2}=\varphi(x)-C_{2} & \text { on } \partial D\end{cases}
$$

For any $0<t<s<R_{0}$, we introduce a cutoff function $\eta \in C^{\infty}\left(\Omega_{R_{0}}\right)$ satisfying $0 \leq \eta \leq 1, \eta=1$ in $\Omega_{t}\left(z^{\prime}\right), \eta=0$ in $\Omega_{R_{0}} \backslash \Omega_{s}\left(z^{\prime}\right)$, and $|\nabla \eta| \leq \frac{2}{s-t}$. Multiplying $\eta^{2}\left(v_{b}-C_{2}\right)$ on the both sides of the equation in (4.1) and applying the integration by parts, we have

$$
\int_{\Omega_{s}\left(z^{\prime}\right)}\left|\nabla\left(v_{b}-C_{2}\right)\right|^{2} \eta^{2} d x \leq \frac{C}{(s-t)^{2}} \int_{\Omega_{s}\left(z^{\prime}\right)}\left|v_{b}-C_{2}\right|^{2} d x
$$

Since $v_{b}-C_{2}=0$ on $\partial D_{2}$, by Hölder inequality, we have

$$
\int_{\Omega_{s}\left(z^{\prime}\right)}\left|v_{b}-C_{2}\right|^{2} \leq C \delta\left(z^{\prime}\right)^{2} \int_{\Omega_{s}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x
$$

Thus, we have

$$
\begin{equation*}
\int_{\Omega_{t}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x \leq C\left(\frac{\delta\left(z^{\prime}\right)}{s-t}\right)^{2} \int_{\Omega_{s}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x \tag{4.2}
\end{equation*}
$$

For simplicity, denote

$$
F(t):=\int_{\Omega_{t}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x
$$

then (4.2) can be written as

$$
F(t) \leq\left(\frac{C_{0} \delta\left(z^{\prime}\right)}{s-t}\right)^{2} F(s)
$$

here we fix the universal constant $C_{0}$. Let $t_{0}=\delta, t_{i+1}=t_{i}+2 C_{0} \delta$, then we have the following iteration formula

$$
F\left(t_{i}\right) \leq \frac{1}{4} F\left(t_{i+1}\right) .
$$

After $k=\left[\frac{\delta\left(z^{\prime}\right)^{1 / m}}{2 C_{0} \delta\left(z^{\prime}\right)}\right]$ times, we have

$$
F\left(t_{0}\right) \leq\left(\frac{1}{4}\right)^{k} \int_{\Omega_{\left|z^{\prime}\right|}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x \leq C\left(\frac{1}{4}\right)^{\left[\frac{1}{2 C_{0} \delta\left(z^{\prime}\right)^{1-1 / m}}\right]} .
$$

So that

$$
\int_{\Omega_{\delta\left(z^{\prime}\right)}\left(z^{\prime}\right)}\left|\nabla v_{b}\right|^{2} d x \leq C \exp \left(-\frac{1}{2 C_{0} \delta\left(z^{\prime}\right)^{1-1 / m}}\right)
$$

A similar procedure as Step 2 in the proof of Proposition 1.7 yields (1.33).
4.2. Proof of Propostion 1.10. We recall the decomposition as in 6]

$$
\begin{equation*}
u(x)=C_{1} v_{1}(x)+C_{2} v_{2}(x)+v_{0}(x), \quad \text { in } \Omega \tag{4.3}
\end{equation*}
$$

where $v_{1}$ is defined in (1.5), $v_{2}$ and $v_{0}$ are, respectively, the solutions of

$$
\left\{\begin{array} { l l } 
{ \Delta v _ { 2 } = 0 } & { \text { in } \Omega , }  \tag{4.4}\\
{ v _ { 2 } = 1 } & { \text { on } \partial D _ { 2 } , } \\
{ v _ { 2 } = 0 } & { \text { on } \partial D _ { 1 } \cup \partial D , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{ll}
\Delta v_{0}=0 & \text { in } \Omega \\
v_{0}=0 & \text { on } \partial D_{1} \cup \partial D_{2} \\
v_{0}=\varphi(x) & \text { on } \partial D
\end{array}\right.\right.
$$

Then $\left(v_{1}+v_{2}\right)$ satisfies

$$
\begin{cases}\Delta\left(v_{1}+v_{2}\right)=0, & \text { in } \Omega  \tag{4.5}\\ v_{1}+v_{2}=1, & \text { on } \partial D_{1} \cup \partial D_{2} \\ v_{1}+v_{2}=0, & \text { on } \partial D\end{cases}
$$

We decompose $u_{0}$ into

$$
u_{0}=C_{0} u_{0}^{1}+u_{0}^{0}, \quad \text { in } \Omega^{0}
$$

where $u_{0}^{1}, u_{0}^{0} \in C^{1}(\bar{\Omega})$ are, respectively, the solutions of

$$
\left\{\begin{array} { l l } 
{ \Delta u _ { 0 } ^ { 1 } = 0 , } & { \text { in } \Omega ^ { 0 } , }  \tag{4.6}\\
{ u _ { 0 } ^ { 1 } = 1 , } & { \text { on } \partial D _ { 1 } ^ { 0 } \cup \partial D _ { 2 } ^ { 0 } , } \\
{ u _ { 0 } ^ { 1 } = 0 , } & { \text { on } \partial D , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{ll}
\Delta u_{0}^{0}=0, & \text { in } \Omega^{0}, \\
u_{0}^{0}=0, & \text { on } \partial D_{1}^{0} \cup \partial D_{2}^{0}, \\
u_{0}^{0}=\varphi, & \text { on } \partial D
\end{array}\right.\right.
$$

To prove Proposition 1.10 we need the following lemmas.

## Lemma 4.1.

$$
\begin{equation*}
\left|\int_{\partial D_{i}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu^{-}}-\int_{\partial D_{i}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}\right| \leq C \varepsilon^{1^{-}}, \quad i=1,2 \tag{4.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\int_{\partial D_{i}} \frac{\partial v_{0}}{\partial \nu^{-}}-\int_{\partial D_{i}^{0}} \frac{\partial u_{0}^{0}}{\partial \nu^{-}}\right| \leq C \varepsilon^{1^{-}}, \quad i=1,2 \tag{4.8}
\end{equation*}
$$

where $\varepsilon^{1^{-}}$means $\varepsilon^{1-\eta}$ for any small positive constant $\eta$.
Proof. We only prove (4.7) with $i=1$ for instance, the others are the same. It follows from Theorem 1.9 with $\varphi(x)=C_{2}=1$ that

$$
\begin{equation*}
\left|\nabla\left(v_{1}+v_{2}\right)\right| \leq C, \quad \text { in } \Omega \tag{4.9}
\end{equation*}
$$

Because of the same reason,

$$
\begin{equation*}
\left|\nabla u_{0}^{1}\right| \leq C, \quad \text { in } \Omega^{0} \tag{4.10}
\end{equation*}
$$

Letting

$$
\phi_{1}:=\left(v_{1}+v_{2}\right)-u_{0}^{1}
$$

then $\Delta \phi_{1}=0$ in $V=D \backslash \overline{D_{1} \cup D_{1}^{0} \cup D_{2}}$, and $\phi_{1}=0$ on $\partial D$. It is obvious that $\left(v_{1}+v_{2}\right)=u_{0}^{1}=1$ on $\partial D_{2}$, that is, $\phi_{1}=0$ on $\partial D_{2}$. On $\partial D_{1}^{0} \backslash D_{1}$, by using mean
value theorem and (4.9), we have

$$
\begin{aligned}
\left.\left|\phi_{1}\right|\right|_{\partial D_{1}^{0} \backslash D_{1}} & =\left.\left|\left(v_{1}+v_{2}\right)-u_{0}^{1}\right|\right|_{\partial D_{1}^{0} \backslash D_{1}}=\left.\left|\left(v_{1}+v_{2}\right)-1\right|\right|_{\partial D_{1}^{0} \backslash D_{1}} \\
& =\left.\left|\left(v_{1}+v_{2}\right)\left(x^{\prime}, x_{d}\right)-\left(v_{1}+v_{2}\right)\left(x^{\prime}, x_{d}+\varepsilon\right)\right|\right|_{\partial D_{1}^{0} \backslash D_{1}} \\
& \leq\left|\nabla\left(v_{1}+v_{2}\right)(\xi)\right| \varepsilon \leq C \varepsilon
\end{aligned}
$$

for some $\xi \in \Omega$; similarly, using (4.10),

$$
\begin{aligned}
\left.\left|\phi_{1}\right|\right|_{\partial D_{1} \backslash D_{1}^{0}} & =\left.\left|\left(v_{1}+v_{2}\right)-u_{0}^{1}\right|\right|_{\partial D_{1} \backslash D_{1}^{0}}=\left.\left|1-u_{0}^{1}\right|\right|_{\partial D_{1} \backslash D_{1}^{0}} \\
& =\left.\left|u_{0}^{1}\left(x^{\prime}, x_{d}-\varepsilon\right)-u_{0}^{1}\left(x^{\prime}, x_{d}\right)\right|\right|_{\partial D_{1} \backslash D_{1}^{0}}=\left|\nabla u_{0}^{1}(\xi)\right| \varepsilon \leq C \varepsilon,
\end{aligned}
$$

for some another $\xi \in \Omega^{0}$. Applying the maximum principle to $\phi_{1}$ on $V$, we have

$$
\begin{equation*}
\left|\phi_{1}\right| \leq C \varepsilon, \quad \text { on } V . \tag{4.11}
\end{equation*}
$$

Denote

$$
\Omega^{+}:=V \cap\left\{x \in \Omega \mid x_{d}>0\right\},(\partial D)^{+}:=\left\{x \in \partial D \mid x_{d}>0\right\}, \text { and } \gamma=\left\{x_{d}=0\right\} \cap \Omega .
$$

Since $\left(v_{1}+v_{2}\right)$ and $u_{0}^{1}$ are harmonic in $\Omega^{+} \backslash D_{1}$ and $\Omega^{+} \backslash D_{1}^{0}$, respectively, by using integration by parts,

$$
0=\int_{\Omega^{+} \backslash D_{1}} \Delta\left(v_{1}+v_{2}\right)=\int_{\partial D_{1}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu^{-}}+\int_{(\partial D)^{+}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu}+\int_{\gamma} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu}
$$

and

$$
0=\int_{\Omega^{+} \backslash D_{1}^{0}} \Delta u_{0}^{1}=\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+\int_{(\partial D)^{+}} \frac{\partial u_{0}^{1}}{\partial \nu}+\int_{\gamma} \frac{\partial u_{0}^{1}}{\partial \nu} .
$$

Thus,

$$
\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}-\int_{\partial D_{1}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu^{-}}=\int_{(\partial D)^{+}} \frac{\partial \phi_{1}}{\partial \nu}+\int_{\gamma} \frac{\partial \phi_{1}}{\partial \nu} .
$$

First, using the standard boundary gradient estimates for $\phi_{1}$ and (4.11), we have

$$
\left|\int_{(\partial D)^{+}} \frac{\partial \phi_{1}}{\partial \nu}\right| \leq C \varepsilon
$$

Divide $\gamma$ into three pieces: $\gamma=\gamma_{1} \cup \gamma_{2} \cup \gamma_{3}$, where

$$
\begin{gathered}
\gamma_{1}:=\left\{\left(x^{\prime}, 0\right)| | x^{\prime} \left\lvert\, \leq \frac{A}{2|\log \varepsilon|}\right.\right\}, \quad \gamma_{2}:=\left\{\left(x^{\prime}, 0\right)\left|\frac{A}{2|\log \varepsilon|}<\left|x^{\prime}\right|<R_{0}\right\}\right. \\
\gamma_{3}:=\gamma \backslash\left(\gamma_{1} \cup \gamma_{2}\right)
\end{gathered}
$$

the constant $A$ is determined in (1.33). Write

$$
\int_{\gamma} \frac{\partial \phi_{1}}{\partial \nu}=\int_{\gamma_{1}}+\int_{\gamma_{2}}+\int_{\gamma_{3}} \frac{\partial \phi_{1}}{\partial \nu}:=\mathrm{I}+\mathrm{II}+\mathrm{III}
$$

For $\left(y^{\prime}, 0\right) \in \gamma_{1}$, by Theorem 1.9

$$
\left|\nabla\left(v_{1}+v_{2}\right)\right|,\left|\nabla u_{0}^{1}\right| \leq C \exp \left(-\frac{A}{\left(\varepsilon+\left|x^{\prime}\right|^{m}\right)^{1-\frac{1}{m}}}\right), \quad \text { in } \Omega_{R_{0}}
$$

Hence

$$
|\mathrm{I}| \leq C \varepsilon
$$

For $\left(y^{\prime}, 0\right) \in \gamma_{2}$, there exists a $r>\frac{1}{C}\left|y^{\prime}\right|^{m}$ for some $C>1$ such that $B_{r}\left(y^{\prime}, 0\right) \subset V$. It then follows from the standard gradient estimates for harmonic function and (4.11) that

$$
\left|\nabla \phi_{1}\left(y^{\prime}, 0\right)\right| \leq \frac{C \varepsilon}{\left|y^{\prime}\right|^{m}}
$$

and

$$
|\mathrm{II}| \leq C \varepsilon \int_{\frac{A}{2|\log \varepsilon|}<\left|y^{\prime}\right|<R_{0}} \frac{1}{\left|y^{\prime}\right|^{m}} d S \leq C \begin{cases}\varepsilon|\log \varepsilon|^{m-n+1} & \text { for } m>n-1, n \geq 3 \\ \varepsilon \log |\log \varepsilon| & \text { for } m=n-1, n \geq 3 \\ \varepsilon|\log \varepsilon|^{m-1} & \text { for } m \geq 2, n=2\end{cases}
$$

For $\left(y^{\prime}, 0\right) \in \gamma_{3}$, there is a universal constant $r>0$ such that $B_{r}(x) \subset V$ for all $x \in \gamma_{3}$. So we have from (4.19) that for any $x \in \gamma_{3}$,

$$
\left|\nabla \phi_{1}\right| \leq \frac{C \varepsilon}{r} \leq C \varepsilon
$$

and

$$
|\mathrm{III}| \leq C \varepsilon
$$

Thus, we have (4.7) with $i=1$.
Lemma 4.2. Let $C_{1}$ and $C_{2}$ be defined in (1.2) and $C_{0}$ be in (1.15). We have

$$
\begin{equation*}
\left|\frac{C_{1}+C_{2}}{2}-C_{0}\right| \leq C\left(\rho_{n}^{m}(\varepsilon)\right) \tag{4.12}
\end{equation*}
$$

As a consequence, combining it with (1.37), we have

$$
\begin{equation*}
\left|C_{i}-C_{0}\right| \leq\left|C_{i}-\frac{C_{1}+C_{2}}{2}\right|+\left|\frac{C_{1}+C_{2}}{2}-C_{0}\right| \leq C\left(\rho_{n}^{m}(\varepsilon)\right), \quad i=1,2 \tag{4.13}
\end{equation*}
$$

Proof. In view of the decomposition (4.3), the third line of (1.2), we have

$$
\begin{equation*}
C_{1} \int_{\partial D_{i}} \frac{\partial v_{1}}{\partial \nu^{-}}+C_{2} \int_{\partial D_{i}} \frac{\partial v_{2}}{\partial \nu^{-}}+\int_{\partial D_{i}} \frac{\partial v_{0}}{\partial \nu^{-}}=0, \quad i=1,2 \tag{4.14}
\end{equation*}
$$

Let

$$
a_{i j}=\int_{\partial D_{i}} \frac{\partial v_{j}}{\partial \nu^{-}}, \quad b_{i}=-\int_{\partial D_{i}} \frac{\partial v_{0}}{\partial \nu^{-}} .
$$

That is,

$$
\left\{\begin{array}{l}
a_{11} C_{1}+a_{12} C_{2}=b_{1} \\
a_{21} C_{1}+a_{22} C_{2}=b_{2}
\end{array}\right.
$$

So that

$$
\left(a_{11}+a_{21}\right) C_{1}+\left(a_{12}+a_{22}\right) C_{2}=b_{1}+b_{2}
$$

Since $a_{12}=a_{21}$, it follows that

$$
\left(a_{11}+a_{21}\right)\left(C_{1}+C_{2}\right)+\left(a_{22}-a_{11}\right) C_{2}=b_{1}+b_{2}
$$

Similarly,

$$
\left(a_{12}+a_{22}\right)\left(C_{1}+C_{2}\right)-\left(a_{22}-a_{11}\right) C_{1}=b_{1}+b_{2}
$$

Adding these two equations together and dividing by two yields

$$
\left(a_{11}+a_{21}+a_{12}+a_{22}\right) \frac{C_{1}+C_{2}}{2}+\left(a_{22}-a_{11}\right) \frac{\left(C_{2}-C_{1}\right)}{2}=b_{1}+b_{2}
$$

That is,

$$
\begin{align*}
& \left(\int_{\partial D_{1}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu^{-}}+\int_{\partial D_{2}} \frac{\partial\left(v_{1}+v_{2}\right)}{\partial \nu^{-}}\right) \frac{C_{1}+C_{2}}{2} \\
= & \left(-\int_{\partial D_{1}} \frac{\partial v_{0}}{\partial \nu^{-}}-\int_{\partial D_{2}} \frac{\partial v_{0}}{\partial \nu^{-}}\right)+\left(\int_{\Omega}\left|\nabla v_{1}\right|^{2}-\int_{\Omega}\left|\nabla v_{2}\right|^{2}\right) \frac{\left(C_{2}-C_{1}\right)}{2} . \tag{4.15}
\end{align*}
$$

Recalling that $\bar{v}_{2}=1-\bar{v}_{1}$ in $\Omega_{R}$, we have $\left|\nabla \bar{v}_{1}\right|=\left|\nabla \bar{v}_{2}\right|$. By (1.30) and (2.12),

$$
\begin{aligned}
& \left.\left|\int_{\Omega_{R}}\right| \nabla v_{1}\right|^{2}-\int_{\Omega_{R}}\left|\nabla v_{2}\right|^{2} \mid \\
= & \left|\int_{\Omega_{R}}\right| \nabla\left(v_{1}-\bar{v}_{1}\right)+\left.\nabla \bar{v}_{1}\right|^{2}-\int_{\Omega_{R}}\left|\nabla\left(v_{2}-\bar{v}_{2}\right)+\nabla \bar{v}_{2}\right|^{2} \mid \\
\leq & \left.\sum_{i=1}^{2}\left|\int_{\Omega_{R}}\right| \nabla\left(v_{i}-\bar{v}_{i}\right)\right|^{2}+2 \nabla \bar{v}_{i} \nabla\left(v_{i}-\bar{v}_{i}\right) \mid \leq C .
\end{aligned}
$$

Hence,

$$
\left.\left|\int_{\Omega}\right| \nabla v_{2}\right|^{2}-\int_{\Omega}\left|\nabla v_{1}\right|^{2} \mid \leq C .
$$

By using Lemma 4.1 and (1.37), (4.15) can be written as

$$
\begin{align*}
& \left(\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+O\left(\varepsilon^{1^{-}}\right)\right) \frac{C_{1}+C_{2}}{2} \\
= & -\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{0}}{\partial \nu^{-}}-\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{0}}{\partial \nu^{-}}+O\left(\varepsilon^{1^{-}}\right)+O\left(\rho_{n}^{m}(\varepsilon)\right) . \tag{4.16}
\end{align*}
$$

On the other hand, from the third line of (1.15), we have

$$
\begin{equation*}
C_{0}\left(\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}\right)+\left(\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{0}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{0}}{\partial \nu^{-}}\right)=0 . \tag{4.17}
\end{equation*}
$$

Comparing it with (4.16), and in view of $\left|C_{i}\right| \leq C$, we have

$$
\left(\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}\right)\left(\frac{C_{1}+C_{2}}{2}-C_{0}\right)=O\left(\rho_{n}^{m}(\varepsilon)\right)
$$

Using the integration by parts and recalling the definition of $u_{0}^{1}$, we have

$$
0<\int_{\partial D_{1}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}+\int_{\partial D_{2}^{0}} \frac{\partial u_{0}^{1}}{\partial \nu^{-}}=\int_{\Omega^{0}}\left|\nabla u_{0}^{1}\right|^{2} \leq C .
$$

The proof of (4.12) is finished.
Proof of Proposition 1.10. Let

$$
\phi(x):=C_{2}-C_{0}-\left(v_{b}(x)-u_{0}(x)\right),
$$

then $\Delta \phi=0$ in $V=D \backslash \overline{D_{1} \cup D_{1}^{0} \cup D_{2}}$. It is easy to see that $\phi=0$ on $\partial D_{2}$ and from Lemma 4.2

$$
\left.|\phi|\right|_{\partial D}=\left|C_{2}-C_{0}\right| \leq C \rho_{n}^{m}(\varepsilon) .
$$

On $\partial D_{1}^{0} \backslash D_{1}$, by mean value theorem, (1.34) and (4.13), we have

$$
\begin{equation*}
\left.|\phi(x)|\right|_{\partial D_{1}^{0} \backslash D_{1}}=\left.\left|C_{2}-v_{b}(x)\right|\right|_{\partial D_{1}^{0} \backslash D_{1}}=\left|\partial_{x_{n}} v_{b}\left(x^{\prime}, \xi_{n}\right)\right| \varepsilon \leq C \varepsilon \tag{4.18}
\end{equation*}
$$

where $\xi_{n} \in(0, \varepsilon)$. Similarly,

$$
\left.|\phi|\right|_{\partial D_{1} \backslash D_{1}^{0}}=\left.\left|C_{0}-u_{0}\right|\right|_{\partial D_{1} \backslash D_{1}^{0}}=\left|\nabla u_{0}(\xi)\right| \varepsilon \leq C \varepsilon
$$

for some $\xi \in D_{1} \backslash D_{1}^{0}$. We now apply the maximum principle to $\phi$ on $V$,

$$
\begin{equation*}
|\phi| \leq C \rho_{n}^{m}(\varepsilon), \quad \text { on } V \tag{4.19}
\end{equation*}
$$

Similarly as in the proof of Lemma 4.1 since $v_{b}$ and $u_{0}$ are harmonic in $\Omega^{+} \backslash D_{1}$ and $\Omega^{+} \backslash D_{1}^{0}$, respectively, by using integration by parts, we have

$$
\mathcal{B}_{\varepsilon}[\varphi]=-\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}}=\int_{(\partial D)^{+}} \frac{\partial v_{b}}{\partial \nu}+\int_{\gamma} \partial_{x_{n}} v_{b}
$$

and

$$
\mathcal{B}_{0}[\varphi]=-\int_{\partial D_{1}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}}=\int_{(\partial D)^{+}} \frac{\partial u_{0}}{\partial \nu}+\int_{\gamma} \partial_{x_{n}} u_{0}
$$

Thus,

$$
\mathcal{B}_{0}[\varphi]-\mathcal{B}_{\varepsilon}[\varphi]=\int_{\partial D_{1}^{0}} \frac{\partial u_{0}}{\partial \nu^{-}}-\int_{\partial D_{1}} \frac{\partial v_{b}}{\partial \nu^{-}}=\int_{(\partial D)^{+}} \frac{\partial \phi}{\partial \nu}+\int_{\gamma} \partial_{x_{n}} \phi
$$

First, as before, using the standard boundary gradient estimates for $\phi$ and (4.19), we have

$$
\left|\int_{(\partial D)^{+}} \frac{\partial \phi}{\partial \nu}\right| \leq C \rho_{n}^{m}(\varepsilon)
$$

Next, similarly in the proof of Lemma 4.1, we divide $\gamma$ into three pieces: $\gamma=$ $\gamma_{1} \cup \gamma_{2} \cup \gamma_{3}$, with a minor modification, where

$$
\begin{gathered}
\gamma_{1}:=\left\{\left(x^{\prime}, 0\right)| | x^{\prime} \left\lvert\, \leq \varepsilon^{\frac{n-1}{m(m-n+1)}}\right.\right\}, \quad \gamma_{2}:=\left\{\left(x^{\prime}, 0\right)\left|\varepsilon^{\frac{n-1}{m(m-n+1)}}<\left|x^{\prime}\right|<R_{0}\right\}\right. \\
\gamma_{3}:=\gamma \backslash\left(\gamma_{1} \cup \gamma_{2}\right)
\end{gathered}
$$

Write

$$
\int_{\gamma} \partial_{x_{n}} \phi=\int_{\gamma_{1}}+\int_{\gamma_{2}}+\int_{\gamma_{3}} \partial_{x_{n}} \phi:=\mathrm{I}+\mathrm{II}+\mathrm{III}
$$

As in the proof of Lemma 4.1 replacing (4.11) by (4.19), it is easy to see that

$$
|\mathrm{III}| \leq C \rho_{n}^{m}(\varepsilon)
$$

Now consider term II. On $\bar{\Omega}_{R_{0}^{0}}$, since $\phi=0$ on $\partial D_{2}$ and $\phi=\varepsilon \partial_{x_{n}} v_{b}\left(x^{\prime}, \xi_{n}\right)$ on $\partial D_{1}^{0} \backslash D_{1}$ from (4.18), then we choose $\bar{\phi}=\varepsilon \partial_{x_{n}} v_{b}\left(x^{\prime}, \xi_{n}\right) \bar{v}_{1}^{0}$ to approximate $\phi$ in $\Omega_{R_{0}^{0}}$, where $\bar{v}_{1}^{0}$ is defined in (3.2). Thus, $\phi-\bar{\phi}=0$ on $\partial \Omega_{R_{0}}^{0} \backslash \Omega$. Let $w_{\phi}=\phi-\bar{\phi}$. Since $\left\|v_{b}\right\|_{C^{3}} \leq C$ by theoerem 1.1 in [28], it follows from the proof of Proposition 1.7, we have $\left\|\nabla w_{\phi}\right\| \leq C \varepsilon$. From the definition of $\bar{v}_{1}^{0}$, (3.2), and (3.3), we have, for $\left(y^{\prime}, 0\right) \in \gamma_{2}$,

$$
\left|\partial_{x_{n}} \bar{v}_{0}^{1}\left(y^{\prime}, 0\right)\right| \leq \frac{C}{\left|y^{\prime}\right|^{m}}, \quad \text { and } \quad\left|\partial_{x_{n}} \bar{\phi}\left(y^{\prime}, 0\right)\right| \leq \frac{C \varepsilon}{\left|y^{\prime}\right|^{m}}+C \varepsilon
$$

Hence

$$
\left|\int_{\gamma_{2}} \partial_{x_{n}} \bar{\phi}\right| \leq \int_{\gamma_{2}} \frac{C \varepsilon}{\left|y^{\prime}\right|^{m}}+C \varepsilon \leq C \rho_{n}^{m}(\varepsilon) .
$$

Together with $\left|\int_{\gamma_{2}} \partial_{x_{n}} w_{\phi}\right| \leq C \varepsilon$, yields

$$
|\mathrm{II}| \leq C \rho_{n}^{m}(\varepsilon)
$$

For $\left(y^{\prime}, 0\right) \in \gamma_{1}$, by Theorem 1.9 ,

$$
\left|\nabla v_{b}\right|,\left|\nabla u_{0}\right| \leq C \exp \left(-\frac{A}{\left(\varepsilon+\left|x^{\prime}\right|^{m}\right)^{1-\frac{1}{m}}}\right), \quad \text { in } \Omega_{R}^{0}
$$

Hence

$$
|\mathrm{I}| \leq C \varepsilon
$$

Thus, the proof of Propostion 1.10 is completed.

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## References

[1] H. Ammari; H. Kang; M. Lim, Gradient estimates to the conductivity problem. Math. Ann. 332 (2005), 277-286.
[2] H. Ammari; G. Ciraolo; H. Kang; H. Lee; K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity. Arch. Ration. Mech. Anal. 208 (2013), 275-304.
[3] H. Ammari; H. Kang; H. Lee; J. Lee; M. Lim, Optimal estimates for the electrical field in two dimensions. J. Math. Pures Appl. 88 (2007), 307-324.
[4] H. Ammari; H. Kang; H. Lee; M. Lim; H. Zribi, Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions. J. Differential Equations 247 (2009), 2897-2912.
[5] I. Babuška; B. Andersson; P. Smith; K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale. Comput. Methods Appl. Mech. Engrg. 172 (1999), 27-77.
[6] E. Bao; Y.Y. Li; B. Yin, Gradient estimates for the perfect conductivity problem. Arch. Ration. Mech. Anal. 193 (2009), 195-226.
[7] J.G. Bao; H.J. Ju; H.G. Li, Optimal boundary gradient estimates for Lamé systems with partially infinite coefficients. Adv. Math. 314 (2017), 583-629.
[8] J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients, Arch. Ration. Mech. Anal. 215 (2015), no. 1, 307-351.
[9] J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two. Adv. Math. 305 (2017), 298-338.
[10] L. Berlyand; Y. Gorb; A. Novikov, Discrete network approximation for highly-packed composites with irregular geometry in three dimensions. Multiscale methods in science and engineering, 21-57, Lect. Notes Comput. Sci. Eng., 44, Springer, Berlin, 2005.
[11] E. Bonnetier; F. Triki, On the spectrum of the Poincaré variational problem for two close-totouching inclusions in 2D. Arch. Ration. Mech. Anal. 209 (2013), no. 2, 541-567.
[12] E. Bonnetier; M. Vogelius, An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section. SIAM J. Math. Anal. 31 (2000), 651-677.
[13] B. Budiansky; G.F. Carrier, High shear stresses in stiff fiber composites. J. App. Mech. 51 (1984), 733-735.
[14] E. De Giorgi, Sulla differenziabilit'a e l'analiticit'a delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 3 (1957), pp. 25-43.
[15] H.J. Dong, Gradient estimates for parabolic and elliptic systems from linear laminates. Arch. Ration. Mech. Anal. 205 (2012), no. 1, 119-149.
[16] H.J. Dong; H.G. Li, Optimal Estimates for the Conductivity Problem by Green's Function Method. Arch. Ration. Mech. Anal. 231 (2019), no. 3, 1427-1453.
[17] J.S. Fan; K. Kim; S. Nagayasu; G. Nakamura, A gradient estimate for solutions to parabolic equations with discontinuous coefficients. Electron. J. Differential Equations 2013, No. 93, 24 pp.
[18] Y. Gorb, Singular behavior of electric field of high-contrast concentrated composites. Multiscale Model. Simul. 13 (2015), no. 4, 1312-1326.
[19] Y. Gorb; A. Novikov, Blow-up of solutions to a $p$-Laplace equation. Multiscale Model. Simul. 10 (2012), no. 3, 727-743.
[20] D. Gilbarg, N.S. Trudinger: Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp. ISBN: 3-540-41160-7.
[21] H.J. Ju; H.G. Li; L.J. Xu, Estimates for elliptic systems in a narrow region arising from composite materials. Quart. Appl. Math. 77 (2019), 177-199.
[22] H. Kang; H. Lee; K. Yun, Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions. Math. Ann. 363 (2015), no. 3-4, 1281-1306.
[23] H. Kang; M. Lim; K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities. J. Math. Pures Appl. (9) 99 (2013), 234-249.
[24] H. Kang; M. Lim; K. Yun, Characterization of the electric field concentration between two adjacent spherical perfect conductors. SIAM J. Appl. Math. 74 (2014), 125-146.
[25] J.B. Keller, Stresses in narrow regions, Trans. ASME J. Appl. Mech. 60 (1993), 1054-1056.
[26] Y. Kim, Gradient estimates for elliptic equations with measurable nonlinearities. J. Math. Pures Appl. (9) 114 (2018), 118-145.
[27] H.G. Li; Y.Y. Li, Gradient estimates for parabolic systems from composite material. Sci. China Math. 60 (2017), no. 11, 2011-2052.
[28] H.G. Li; Y.Y. Li; E.S. Bao; B. Yin, Derivative estimates of solutions of elliptic systems in narrow regions. Quart. Appl. Math. 72 (2014), 589-596.
[29] H.G. Li; Y.Y. Li; Z.L. Yang, Asymptotics of the gradient of solutions to the perfect conductivity problem. Multiscale Model. Simul. 17 (2019), no. 3, 899-925.
[30] H.G. Li; L.J. Xu, Optimal estimates for the perfect conductivity problem with inclusions close to the boundary. SIAM J. Math. Anal. 49 (2017), no. 4, 3125-3142.
[31] H.G. Li; F. Wang; L.J. Xu, Characterization of electric fields between two spherical perfect conductors with general radii in 3D. J. Differential Equations 267 (2019), no. 11, 6644-6690.
[32] Y.Y. Li; L. Nirenberg, Estimates for elliptic system from composite material. Comm. Pure Appl. Math. 56 (2003), 892-925.
[33] Y.Y. Li; M. Vogelius, Gradient stimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Rational Mech. Anal. 153 (2000), 91-151.
[34] M. Lim; K. Yun, Lim, Blow-up of electric fields between closely spaced spherical perfect conductors. Comm. Partial Differential Equations 34 (2009), no. 10-12, 1287-1315.
[35] X. Markenscoff, Stress amplification in vanishingly small geometries. Computational Mechanics 19 (1996), 77-83.
[36] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. J. Math., 80 (1958), pp. 931-954.
[37] K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. SIAM J. Appl. Math. 67 (2007), 714-730.
[38] K. Yun, Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections. J. Math. Anal. Appl. 350 (2009), 306-312.
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