KIPPENHAHN'S THEOREM FOR JOINT NUMERICAL RANGES AND QUANTUM STATES

DANIEL PLAUMANN, RAINER SINN, AND STEPHAN WEIS

Abstract. Kippenhahn's Theorem asserts that the numerical range of a matrix is the convex hull of a certain algebraic curve. Here, we show that the joint numerical range of finitely many Hermitian matrices is similarly the convex hull of a semi-algebraic set. We discuss an analogous statement regarding the dual convex cone to a hyperbolicity cone and prove that the class of bases of these dual cones is closed under linear operations. The result offers a new geometric method to analyze quantum states.

1. Introduction

Let H_d be the real vector space of Hermitian $d \times d$ matrices. We denote the set of *density matrices* by

$$\mathcal{B} = \mathcal{B}_d = \{\rho \in H_d \colon \rho \succeq 0, \mathrm{tr}(\rho) = 1\}, \tag{1}$$

where $A \succeq 0$ means that $A \in H_d$ is positive semi-definite. The letter $\mathcal{B}$ underlines that the set $\mathcal{B}$ is a base of the cone of positive semi-definite matrices (see Section 5). We use the bilinear form $\langle A, B\rangle = \mathrm{tr}(AB)$, $A, B \in H_d$, to identify H_d and its dual space. Let $A_1, A_2, \ldots, A_n \in H_d$, $n \in \mathbb{N}$, and define

$$W = W_{A_1,A_2,\ldots,A_n} = \{(\langle \rho, A_1\rangle, \ldots, \langle \rho, A_n\rangle) \colon \rho \in \mathcal{B}_d\}, \tag{2}$$

a convex compact subset of the dual space $(\mathbb{R}^n)^*$ to $\mathbb{R}^n$. The set W has been called *joint numerical range* in operator theory, see Section 2 of [8] (also joint algebraic numerical range [43]).

Our motivation for this paper is matrix theory and quantum mechanics. Physicists call density matrices *quantum states*, as density matrices describe the physical states of a quantum system including all statistical properties [33]. Hence, our results contribute to the geometry of quantum states [2, 5]. Section 2 presents numerical range methods in quantum mechanics.

Perhaps more commonly, the term joint numerical range refers to

$$W^{\sim} = W^{\sim}_{A_1,A_2,\ldots,A_n} = \{(\langle\psi|A_1\psi\rangle, \ldots, \langle\psi|A_n\psi\rangle) \colon \psi \in \mathbb{C}^d, \langle\psi|\psi\rangle = 1\}, \tag{3}$$

where $\langle\varphi|\psi\rangle$ is the standard inner product of $\varphi, \psi \in \mathbb{C}^d$, see [14, 41]. A *pure state* is a projection ρ onto the span of a unit vector $\psi \in \mathbb{C}^d$. Since $\langle\rho, A\rangle = \langle\psi|A\psi\rangle$ holds for all $A \in H_d$, the set $W^{\sim}$ is a linear image of the set of pure states. As the pure states are the extreme points of the set of density matrices $\mathcal{B}$, the joint numerical range W is the convex hull of $W^{\sim}$.

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Thus, $W = W^{\sim}$ holds when $W^{\sim}$ is convex. This is the case if $n = 2$ by the Toeplitz-Hausdorff theorem [28, 59], whose 100th anniversary we celebrated at the time of writing. Note that $W^{\sim}_{A_1,A_2}$ is the standard *numerical range* $W(A) = \{\langle\psi|A\psi\rangle : \psi \in \mathbb{C}^d, \langle\psi|\psi\rangle = 1\} \subset \mathbb{C}$ of $A = A_1 + \mathrm{i}\, A_2$. Also, $W^{\sim}$ is convex if $n = 3$ and $d \geq 3$ [1]. The convexity of $W^{\sim}$ is an open problem for $n > 3$, see [26, 41]. Numerical ranges are generally nonconvex if the complex field is replaced with the skew field of the quaternions [50, p. 39].

Algebraic geometry has been employed to study numerical ranges since the 1930s. We consider the determinant

$$p = \det(x_0 \mathbb{1} + x_1 A_1 + \cdots + x_n A_n) \in \mathbb{R}[x_0, x_1, \ldots, x_n],$$

where $\mathbb{1}$ denotes the $d \times d$ identity matrix, and the complex projective hypersurface

$$\mathcal{V}(p) = \{x \in \mathbb{P}^n \mid p(x) = 0\}.$$

If $n = 2$, then $\mathcal{V}(p) \subset \mathbb{P}^2$ is an algebraic curve. Murnaghan [45] showed that the eigenvalues of the matrix $A_1 + \mathrm{i}\, A_2$ are the foci of the curve

$$T = \{y_1 + \mathrm{i}\, y_2 \mid y_1, y_2 \in \mathbb{R}, (1 : y_1 : y_2) \in \mathcal{V}(p)^*\} \subset \mathbb{C},$$

where $X^* \subset (\mathbb{P}^n)^*$ denotes the dual variety parametrizing hyperplanes tangent to a variety $X \subset \mathbb{P}^n$ (cf. Section 4 for more details). Kippenhahn recognized the meaning of the convex hull of the curve T.

Theorem 1.1 (Kippenhahn [38])**.** *The numerical range of $A_1 + \mathrm{i}\, A_2$ is the convex hull of the curve T, in other words, $W(A_1 + \mathrm{i}\, A_2) = \mathrm{conv}(T)$.*

Theorem 1.1 is a well-known tool in matrix analysis [4, 19, 23, 34]. The curve $T \subset \mathbb{C}$ is called the *Kippenhahn curve* or the *boundary generating curve* of the numerical range $W(A_1 + \mathrm{i}\, A_2)$. The curve T has proven useful in classifications of numerical ranges of 3-by-3 matrices [35, 38] and 4-by-4 matrices [10, 15], and it has been studied for special matrices [16, 21, 22, 42].

We sketch a proof of Theorem 1.1 in a manner that may help to explain the geometry behind the proof of Theorem 4.5 later on: Consider the convex set $S = \{(x_1, x_2)^T \in \mathbb{R}^2 \mid \mathbb{1} + x_1 A_1 + x_2 A_2 \succeq 0\}$ (a *spectrahedron*). Assume for simplicity that S is compact, $X = \mathcal{V}(p)$ is smooth, and that the degree d of p is even. The curve X is hyperbolic, i.e. its real points consist of $\frac{d}{2}$ nested ovals in the real projective plane. The innermost oval is the boundary of S. All but finitely many points of the dual curve X^* correspond to simple tangent lines to X. The set of real points of the dual curve X^* (of degree $d(d-1)$) again consists of $\frac{d}{2}$ nested connected components, together with at most finitely many isolated real (singular) points. The tangent lines to the boundary of S now correspond to the outermost oval of X^*, since all other tangent lines to X do not pass through S (see Corollary 4.3). The outermost oval therefore bounds the convex dual S° of S, which is exactly the numerical range $W(A_1 + \mathrm{i}\, A_2)$. The claim of Kippenhahn's theorem follows if we can show that none of the isolated real singularities of X^* lie outside of W; see Theorem 4.9.

Chien and Nakazato [14] provided a more rigorous proof of Theorem 1.1 compared to Kippenhahn's. They also found a triple of Hermitian 3×3-matrices for which the literal analogue of Theorem 1.1 fails in dimension $n = 3$. We will see that the last part of the above sketch in the plane case, the position of singular points of the dual curve, is exactly what causes the failure of the theorem in higher dimensions. This can also be seen in the counterexample by Chien and Nakazato, Example 6.4.

By removing all singular points from the projective dual variety $\mathcal{V}(p)^*$ and taking Euclidean closure, we will prove that a modified version of Theorem 1.1 is valid in all dimensions. Let $X_1, \dots, X_r$ denote the irreducible components of the hypersurface $\mathcal{V}(p)$. We consider the set $(X_i^*)_{\text{reg}}$ of the regular points of the dual variety X_i^*, the set

$$T_i = \{(y_1, \dots, y_n) \in (\mathbb{R}^n)^* \mid (1 : y_1 : \dots : y_n) \in (X_i^*)_{\text{reg}}\}, \qquad i = 1, \dots, r,$$

and the Euclidean closure $T^\sim = \text{clos}(T_1 \cup \dots \cup T_r)$ of the union $T_1 \cup \dots \cup T_r$. Our main result is as follows.

Theorem 1.2. *The joint numerical range W is the convex hull of $T^\sim$.*

Remark 1.3. 1) Theorem 1.2 implies that the set $T^\sim$ contains all extreme points of the compact, convex set W. We show in the proof of Theorem 4.5 that $T^\sim$ contains the exposed points of W, and hence all of the extreme points by a limiting argument (Straszewicz's Theorem). But just as in Kippenhahn's original theorem, $T^\sim$ is not necessarily contained in the boundary of W, only in W.

2) We point out that Theorem 4.5, which implies the theorem stated here, holds more generally for hyperbolic hypersurfaces rather than just determinantal hypersurfaces. While this makes no difference in the plane, by the Helton-Vinnikov theorem, the statement is indeed more general in higher dimensions, and the proof relies purely on the real geometry of hyperbolic polynomials.

3) The joint numerical range W is a semi-algebraic set as it is a linear image of the semi-algebraic set $\mathcal{B}$ by quantifier elimination (see e.g. [7, Thm. 2.2.1]). The set $T_1 \cup \dots \cup T_r$ and hence its Euclidean closure $T^\sim$ are semi-algebraic sets as well.

4) Theorem 1.2 not only describes a semi-algebraic set that contains the extreme points of W but more precisely the Zariski closure of the set of extreme points: The union of the dual varieties X_i^* of the irreducible components X_i of the algebraic boundary of the hyperbolicity cone of p is the Zariski-closure of the set of extreme points of W, see Remark 4.15 for details.

We organize the article as follows. Section 3 collects preliminaries from convex geometry and real algebraic geometry. Section 4 presents a detailed discussion of the remarkable fact, proved by the second author in [53], that the dual convex cone $C^\vee$ to a hyperbolicity cone C is the closed convex cone generated by a particular semi-algebraic set. This implies that every base of $C^\vee$ is the closed *convex hull* of a section of that semi-algebraic set. The same is true for linear images of the bases as we show in Section 5, because (up to isomorphism) they are bases of dual convex cones to sections of C, which are hyperbolicity cones themselves. Returning to the cone of positive semidefinite matrices in Section 6, we obtain a proof of Theorem 1.2 and discuss examples. We analyze the case $n = 2$ separately in Theorem 4.9, which yields a proof of Kippenhahn's original result as stated in Theorem 1.1.

2. Connections to Quantum Mechanics

Physicists refer to linear images of certain subsets of the set of quantum states $\mathcal{B}_d$ as *numerical ranges*. Often they consider images under a map $\mathcal{B}_d \to \mathbb{R}^n$, $\rho \mapsto \langle \rho, A_i \rangle_{i=1}^n$, where $A_1, A_2, \ldots, A_n \in H_d$ are Hermitian matrices. We discuss examples where algebraic geometry could help solving problems of quantum mechanics in the context of numerical ranges. In this paper, we ignore numerical ranges outside the pattern of linear images of subsets of $\mathcal{B}_d$, for example higher-rank numerical ranges [17].

2.1. **Linear images of the set of all quantum states.** The joint numerical range $W_{A_1,A_2,\ldots,A_n}$, as defined in Equation (2), appears in problems of experimental and theoretical physics.

The geometry of the joint numerical range has been of direct interest to the experimentalists Xie et al. [61]. Their drawings of data from photonic experiments show ellipses on the boundary of the joint numerical range W_{A_1,A_2,A_3} of three 3-by-3 matrices $A_1, A_2, A_3 \in H_3$, clearly in agreement with the classification: the exposed faces of positive dimensions are ellipses and segments assembling one of ten configurations [56]. The possibility to carry out experiments with 4-level quantum systems calls for a similar classification of W_{A_1,A_2,A_3} for 4-by-4 matrices $A_1, A_2, A_3 \in H_4$.

Using Theorem 1.1 and analyzing the Kippenhahn curve, Kippenhahn [38] obtained a classification of the numerical range $W(A_1 + \mathrm{i}\, A_2) = W_{A_1,A_2}$ in terms of flat portions on the boundary of W_{A_1,A_2} for all Hermitian 3-by-3 matrices $A_1, A_2 \in H_3$, see also [35]. A similar approach has been taken for 4-by-4 matrices [10, 15]. We are invited to describe the exposed faces of positive dimensions of the joint numerical range W_{A_1,A_2,A_3}, employing Theorem 1.2 and studying the semi-algebraic set $T^{\sim}$, whose convex hull is W_{A_1,A_2,A_3}. This should reproduce the classification of the set W_{A_1,A_2,A_3} for 3-by-3 matrices [56] and lead to a classification for 4-by-4 matrices.

Quantum thermodynamics [62] describes equilibrium states with multiple conserved quantities $A_1, A_2, \ldots, A_n$ in terms of *generalized Gibbs states*

$$\rho_x = \frac{e^{x_1 A_1 + \cdots + x_n A_n}}{\operatorname{tr} e^{x_1 A_1 + \cdots + x_n A_n}} \in \mathcal{B}_d, \qquad x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n.$$

The "boundary at infinity" ($|x| \to \infty$) to the manifold $\{\rho_x : x \in \mathbb{R}^n\}$ may not be closed in the Euclidean topology if the matrices $A_1, A_2, \ldots, A_n$ fail to commute [60]. The physical meaning of this topological problem remains mysterious [12]. Mathematically, the discontinuity depends on the geometry of the joint numerical range $W_{A_1,A_2,\ldots,A_n} = \operatorname{conv}(T^{\sim})$ [12, 51, 54]. Hence, the semi-algebraic set $T^{\sim}$ explains infinitesimal properties of the manifold of generalized Gibbs states near the boundary at infinity.

A connection between functional analysis and algebraic geometry awaits further investigation. The *Wigner distribution* of a quantum state $\rho \in \mathcal{B}_d$ with respect to $A_1, A_2, \ldots, A_n$ is the tempered distribution $\mathcal{W}_\rho$ on $\mathbb{R}^n$ that satisfies

$$\int \mathrm{d}a\, \mathcal{W}_\rho(a_1, \ldots, a_n) f(x_1 a_1 + \cdots + x_n a_n) = \langle \rho, f(x_1 A_1 + \cdots + x_n A_n) \rangle$$

for all $x \in \mathbb{R}^n$ and all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$. The Wigner distribution is a common tool in quantum optics and theoretical physics. Schwonnek and Werner [52] showed that the distribution $\mathcal{W}_\rho$ is compactly supported on the joint numerical range $W_{A_1,A_2,\dots,A_n}$ and that the singularities of $\mathcal{W}_\rho$ lie in the semi-algebraic set $T^\sim$.

2.2. **Linear images of subsets of the set of quantum states.** Linear images of semi-algebraic subsets of $\mathcal{B}_d$ are amenable to algebraic geometry as well. We mention examples relevant to quantum mechanics.

Algebraic geometry [18] has proven helpful in the theory of pure state tomography [29], the reconstruction of a pure state $\rho \in \mathcal{B}_d$ from its expected value tuple $\langle \rho, A_i \rangle_{i=1}^n \in W^\sim$ in the linear image $W^\sim$ of the set of pure states defined in Equation (3). A different topic, for example, in density functional theory [11], is describing the extreme points of the joint numerical range W. Both $W^\sim$ and $T^\sim$ are semi-algebraic subsets of W that contain the extreme points of W. The set $T^\sim$ is especially suitable to study the extreme points of W, see Remark 4.15.

Many-particle systems are fascinating due to interaction and correlation between the units. The simplest example in the quantum domain is the two-qubit system with Hilbert space $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. Physicists [13, 24] have studied the *joint product numerical range* of $A_1, A_2, \dots, A_n \in H_4$,

$$\Pi = \{\langle \psi \otimes \varphi | A_i \psi \otimes \varphi \rangle_{i=1}^n \mid \psi, \varphi \in \mathbb{C}^2, \langle \psi|\psi\rangle = \langle \varphi|\varphi\rangle = 1\},$$

a linear image of the set of product states $\sigma \otimes \tau$, where $\sigma, \tau \in \mathcal{B}_2$ are pure states. The convex hull of the product states is the set of *separable* states, the states that lack the genuine quantum correlation called *entanglement* [2, 5]. Hence, the set Π and its convex hull allow us to study quantum correlations.

Two-qubit density matrices offer insights into statistical mechanics. As per the *quantum de Finetti theorem* [40, 55], the two-particle marginals of an infinite bosonic qubit-system are convex combinations of symmetric product states $\sigma \otimes \sigma$, where $\sigma \in \mathcal{B}_2$ is a pure state. The ground state energy of an energy operator with two-party interactions on an infinite bosonic qubit-system is the distance of the origin from a supporting hyperplane to the set

$$\Pi^{\mathrm{sym}}_{A_1,A_2,\dots,A_n} = \{\langle \psi \otimes \psi | A_i \psi \otimes \psi \rangle_{i=1}^n \mid \psi \in \mathbb{C}^2, \langle \psi|\psi\rangle = 1\}$$

for suitable matrices $A_1, A_2, \dots, A_n \in H_4$. Notably, ruled surfaces on the boundary of the convex hull of $\Pi^{\mathrm{sym}}_{A_1,A_2,A_3} \subset \mathbb{R}^3$ are expressions of phase transitions [13, 63]. An analogue to Theorem 1.2 regarding the set $\Pi^{\mathrm{sym}}_{A_1,A_2,A_3}$ would be helpful for the analysis of bosonic qubit-systems.

3. Preliminaries

We collect terms, basic results, and references to the literature regarding convex geometry and (real) algebraic geometry.

3.1. **Convex Geometry.** We discuss various notions of cones in a finite-dimensional real vector space V: cones (which may be nonconvex), convex cones, and normal cones. Additionally, we define affine cones over complex projective varieties in Section 3.2. As a general reference for convex geometry, we recommend [3, 49].

A subset C of V is a *cone* if $\lambda x \in C$ whenever $\lambda > 0$ and $x \in C$. A subset K of V is a *convex cone* if K is a nonempty convex set and if $\lambda x \in K$ whenever $\lambda \geq 0$ and $x \in K$. The *affine hull* $\mathrm{aff}(S)$, *convex hull* $\mathrm{conv}(S)$, *cone*, *convex cone* $\mathrm{cc}(S)$, and *closed convex cone* generated by a subset $S \subset V$ is the smallest affine space, convex set, cone, convex cone, and closed convex cone, respectively, containing S.

A subset $B \subset V$ is a *base* of a cone $C \subset V$ if B is the intersection of C with an affine hyperplane, $0 \notin \mathrm{aff}(B)$, and for all nonzero points $x \in C$ there exists $y \in B$ and $\lambda > 0$ such that $x = \lambda y$. A convex cone $K \subset V$ is *pointed* if $K \cap (-K) = \{0\}$, i.e. if K contains no lines.

We denote the dual vector space of V by V^*. The *annihilator* of a subset $S \subset V$ is

$$S^{\perp} = \{\ell \in V^* \colon \ell(x) = 0 \text{ for all } x \in S\},$$

the *dual convex cone* to S is

$$S^{\vee} = \{\ell \in V^* \colon \ell(x) \geq 0 \text{ for all } x \in S\},$$

and the *dual convex set* to S is

$$S^{\circ} = \{\ell \in V^* \colon 1 + \ell(x) \geq 0 \text{ for all } x \in S\}.$$

We denote intersections of S with affine hyperplanes avoiding the origin by

$$\mathrm{h}_\ell(S) = \{x \in S \colon \ell(x) = 1\}, \qquad \ell \in V^*, \ell \neq 0. \tag{4}$$

Lemma 3.1. *Let $C \subset V$ be a cone and let $B \neq \emptyset$ be a base of C. Then there exists a nonzero functional $\ell \in V^*$ such that $B = \mathrm{h}_\ell(C)$. If C admits a compact base, then $C \cup \{0\}$ is closed.*

Proof. Let B be a nonempty base of the cone C and let X be the linear span of C. It follows from the definition of a base that there is a nonzero linear functional $\hat{\ell} \in X^*$ such that $B \subset \mathrm{h}_{\hat{\ell}}(X)$, hence $B = \mathrm{h}_{\hat{\ell}}(C)$. Identifying the dual space X^* with any subspace complementary to the annihilator $X^{\perp}$ in V^* and extending $\hat{\ell}$ to a functional $\ell \in V^*$ by the Hahn-Banach theorem, we obtain $B = \mathrm{h}_\ell(C)$.

Let $(x_i) \subset C$ be a sequence converging to a nonzero point $x \in V$. If the base $\mathrm{h}_\ell(C)$ of C is compact, then the sequence $(\frac{x_i}{\ell(x_i)}) \subset \mathrm{h}_\ell(C)$ has a converging subsequence with limit $y \in \mathrm{h}_\ell(C)$. As $\ell(x) = \lim_{i\to\infty} \ell(x_i) \geq 0$ and as $x = \lim_{i\to\infty} \ell(x_i)\frac{x_i}{\ell(x_i)} = \ell(x)y$, the point x lies in C. □

The biduality theorem for closed convex cones follows from the separation theorem in convex geometry, see for example [49, Theorem 14.1].

Theorem 3.2. *Let $K \subset V$ be a closed convex cone. Then $(K^{\vee})^{\vee} = K$.*

We describe the family of bases of a closed convex cone.

Lemma 3.3. *Let $K \subset V$ be a closed convex cone. The following assertions are equivalent for all nonzero points $x \in V$.*

1) *The point x is an interior point of K.*
2) *The set $\mathrm{h}_x(K^{\vee})$ is a base of $K^{\vee}$.*

If one of these equivalent assertions is true, then the set $\mathrm{h}_x(K^{\vee})$ is compact.

Proof. Let $x \in V$ be nonzero. By Theorem 13.1 in [49] the point x is an interior point of K if and only if $\ell(x) < \delta^*(\ell|K)$ holds for all $\ell \in V^* \setminus \{0\}$, where

$$\delta^*(\ell|K) = \sup_{x\in K} \ell(x) = \begin{cases} 0 & \text{if } \ell \in -K^{\vee}, \\ \infty & \text{if } \ell \notin -K^{\vee}, \end{cases} \qquad \ell \in V^*,$$

is the support function of K. Hence the assertion (1) is equivalent to $\ell(x) > 0$ for all $\ell \in K^\vee \setminus \{0\}$, which is equivalent to $\mathrm{h}_x(K^\vee)$ being a base of $K^\vee$.

The compactness follows from properties of the *recession cone* $0^+(\mathrm{h}_x(K^\vee))$, the set of vectors $\ell' \in V^*$ such that $\ell + \lambda\ell'$ lies in $\mathrm{h}_x(K^\vee)$ for all $\ell \in \mathrm{h}_x(K^\vee)$ and $\lambda \geq 0$. We can assume that the set $\mathrm{h}_x(K^\vee)$ is nonempty. In this case we have $0^+(\mathrm{h}_x(K^\vee)) = x^\perp \cap K^\vee$ by Coro. 8.3.2 and 8.3.3 of [49] as the recession cone of $\mathrm{h}_x(V^*)$ is $x^\perp$. Since $\mathrm{h}_x(K^\vee)$ is a base of $K^\vee$, we have $\ell(x) > 0$ for all $\ell \in K^\vee \setminus \{0\}$, hence $x^\perp \cap K^\vee = \{0\}$. Now [49, Thm. 8.4] shows that $\mathrm{h}_x(K^\vee)$ is bounded, hence compact. □

Cones that are contained in a pointed closed convex cone behave nicely.

Lemma 3.4. *Let $K \subset V$ be a closed convex cone with nonempty interior. Let $C \subset K^\vee$ be a cone and let x be a nonzero interior point of K. Then*

$$\mathrm{h}_x(\mathrm{cc}(\mathrm{clos}(C))) = \mathrm{conv}(\mathrm{h}_x(\mathrm{clos}(C))) = \mathrm{conv}(\mathrm{clos}(\mathrm{h}_x(C))) \tag{5}$$

is a compact base of the pointed, closed convex cone $\mathrm{cc}(\mathrm{clos}(C)) = \mathrm{clos}(\mathrm{cc}(C))$.

Proof. Let $x \neq 0$ be an interior point of K. Then $\mathrm{h}_x(K^\vee)$ is a compact base of $K^\vee$ by Lemma 3.3. *A fortiori*, $\mathrm{h}_x(C)$ is a base of the cone $C \subset K^\vee$; thus $\mathrm{h}_x(\mathrm{cc}(C)) \subset \mathrm{conv}(\mathrm{h}_x(C))$. The converse inclusion is clear and proves the first equality sign in Equation (5) after replacing C with $\mathrm{clos}(C)$. The inclusion $\mathrm{h}_x(\mathrm{clos}(C)) \subset \mathrm{clos}(\mathrm{h}_x(C))$ holds, again as $\mathrm{h}_x(C)$ is a base of C. The converse inclusion is clear and proves the second equality sign in Equation (5). As $\mathrm{h}_x(\mathrm{clos}(C))$ is compact, its convex hull is compact by Theorem 17.2 in [49]. Hence $\mathrm{h}_x(\mathrm{cc}(\mathrm{clos}(C)))$ is a compact base of the convex cone $\mathrm{cc}(\mathrm{clos}(C))$, which is closed by Lemma 3.1. This proves $\mathrm{clos}(\mathrm{cc}(C)) \subset \mathrm{cc}(\mathrm{clos}(C))$; the converse inclusion is clear. □

Proposition 3.5 allows us to focus on irreducible varieties in Section 4.

Proposition 3.5. *Let $K_1, \ldots, K_r \subset V$ be convex cones and let $e \neq 0$ be an interior point of $K = K_1 \cap K_2 \cap \cdots \cap K_r$. Let $C_i \subset V^*$ be a cone such that $K_i^\vee = \mathrm{clos}(\mathrm{cc}(C_i))$, that is to say, the dual convex cone to K_i is the closed convex cone generated by C_i for $i = 1, \ldots, r$. Then*

$$K^\vee = \mathrm{cc}\left(\,\mathrm{clos}(C_1 \cup C_2 \cup \cdots \cup C_r)\right) \tag{6}$$

and

$$\mathrm{h}_e(K^\vee) = \mathrm{conv}\Big(\,\mathrm{clos}\left(\,\mathrm{h}_e(C_1) \cup \mathrm{h}_e(C_2) \cup \cdots \cup \mathrm{h}_e(C_r)\right)\Big). \tag{7}$$

Proof. Corollary 16.4.2 of [49] shows $K^\vee = K_1^\vee + K_2^\vee + \cdots + K_r^\vee$ as e is an interior point of K_i for all $i = 1, \ldots, r$, hence $K^\vee = \mathrm{cc}(K_1^\vee \cup K_2^\vee \cup \cdots \cup K_r^\vee)$. This proves Equation (6), as we have, by assumption and by Lemma 3.4,

$$K_i = \mathrm{clos}(\mathrm{cc}(C_i)) = \mathrm{cc}(\mathrm{clos}(C_i)), \qquad i = 1, \ldots, r.$$

Equation (7) follows from the equation $\mathrm{h}_e(\mathrm{cc}(\mathrm{clos}(C))) = \mathrm{conv}(\mathrm{clos}(\mathrm{h}_e(C)))$ provided in Lemma 3.4 using $C = \bigcup_{i=1}^r C_i$ and from Equation (6). □

We discuss faces and normal cones. Let $C \subset V$ be a convex subset. A subset $F \subset C$ is a *face* of C if F is convex and whenever $(1-\lambda)x + \lambda y \in F$ for some $\lambda \in (0,1)$ and $x, y \in C$, then x and y are also in F. A subset $F \subset C$ is an *exposed face* of C if there is a linear functional $\ell \in V^*$ such that

$x \in F$ if and only if $\ell(x) = \inf_{y \in C} \ell(y)$ for all $x \in C$. A point $x \in C$ is an *extreme point* (resp. *exposed point*) of C if $\{x\}$ is a face (resp. exposed face) of C. Let $\mathbb{R}_+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}$. If $x \in C$ is a nonzero point, and $\mathbb{R}_+ x$ is a face (resp. exposed face) of C, then the set $\mathbb{R}_+ x$ is called an *extreme ray* (resp. *exposed ray*) of C.

We denote the set of all faces of C by $\mathcal{F}(C)$. The family $\mathcal{F}(C)$ is a complete lattice of finite length under the partial ordering of set inclusion [57]. The *duality operator* of the closed convex cone C is the map

$$N_C : \mathcal{F}(C) \to \mathcal{F}(C^\vee), \qquad F \mapsto F^\perp \cap C^\vee.$$

The map N_C is antitone, that is to say, $F \subset G$ implies $N_C(F) \supset N_C(G)$ for all faces $F, G \in \mathcal{F}(C)$. The image of N_C is the set of nonempty exposed faces of $C^\vee$ and [57, Prop. 2.4] shows $F \subset N_{C^\vee} \circ N_C(F)$ for all faces $F \in \mathcal{F}(C)$. We have just confirmed that the pair of duality operators $N_C, N_{C^\vee}$ defines a *Galois connection* between $\mathcal{F}(C)$ and $\mathcal{F}(C^\vee)$. Theorem 20 in Section V.8 of [6] then proves the following assertion, which we use in Theorem 4.5.

Lemma 3.6. *Let $C \subset V$ be a closed convex cone. The duality operator N_C restricts to an antitone lattice isomorphism from the set of nonempty exposed faces of C to the set of nonempty exposed faces of $C^\vee$. The inverse isomorphism is the restricted duality operator $N_{C^\vee}$.*

It is easy to see that the exposed face $N_C(F)$ of $C^\vee$ is the (inner) *normal cone* to the closed convex cone C at F,

$$N_C(F) = \{\ell \in V^* \colon \ell(y - x) \geq 0 \ \forall y \in C \ \forall x \in F\},$$

for all nonempty faces $F \subset C$. In addition, for every relative interior point x of F, we have

$$N_C(F) = x^\perp \cap C^\vee = \{\ell \in V^* \colon \ell(y - x) \geq 0 \ \forall y \in C\}, \tag{8}$$

hence we also refer to $N_C(F)$ as the (inner) normal cone to C at x.

3.2. **Real Algebraic Geometry.** We are working in the setup of real algebraic geometry. A *(real) affine variety* for us is a subset of $\mathbb{C}^n$ (for some $n \in \mathbb{N}$) that is defined by a finite number of polynomial equations $p_1 = p_2 = \cdots = p_r = 0$, $r \in \mathbb{N}$, with real coefficients $p_i \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, $i = 1, \ldots, r$. The affine varieties in $\mathbb{C}^n$ are the closed sets of the *Zariski topology* on $\mathbb{C}^n$. So the Zariski closure of a set $S \subset \mathbb{C}^n$ is the smallest real affine variety containing S. A *(real) projective variety* for us is a subset of projective space $\mathbb{P}^n$ that is defined by a finite number of homogeneous polynomial equations $p_1 = p_2 = \cdots = p_r = 0$, $r \in \mathbb{N}$, with real coefficients $p_i \in \mathbb{R}[x_0, x_1, \ldots, x_n]$, $i = 1, \ldots, r$. The projective varieties in $\mathbb{P}^n$ are the closed sets of the *Zariski topology* on $\mathbb{P}^n$. Identifying points in $\mathbb{P}^n$ with lines in $\mathbb{C}^{n+1}$, a projective variety can be seen as an affine variety in $\mathbb{C}^{n+1}$ which is an algebraic cone. The *affine cone* $\widehat{S}$ over a subset $S \subset \mathbb{P}^n$ is the union of all lines in $\mathbb{C}^{n+1}$ spanned by a vector $(x_0, x_1, \ldots, x_n)^T$ such that $(x_0 : x_1 : \cdots : x_n) \in S$. Conversely, the projective variety $\mathbb{P}(X) \subset \mathbb{P}^n$ associated with an algebraic cone $X \subset \mathbb{C}^{n+1}$ consists of the points $(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n$ for which the vector $(x_0, x_1, \ldots, x_n)^T$ is included in X. These notions are explained in introductory textbooks on algebraic geometry like [27] with the caveat that affine and projective varieties are usually complex varieties, i.e. defined by finitely many polynomial equations with complex coefficients. A point $x \in \mathbb{P}^n$ is *real* if the line $\widehat{\{x\}} \subset \mathbb{C}^{n+1}$ contains a nonzero real point. We denote the set of real points of a subset $S \subset \mathbb{C}^n$ or $S \subset \mathbb{P}^n$ by $S(\mathbb{R})$. The dual projective space $(\mathbb{P}^n)^* = \mathbb{P}((\mathbb{C}^{n+1})^*)$ is the projective space over the dual vector space so that the hyperplanes in $\mathbb{P}^n$ are in one-to-one correspondence with points in $(\mathbb{P}^n)^*$. We specify a functional $\ell \in (\mathbb{P}^n)^*$ in terms of its hyperplane

$$H \;=\; \mathcal{V}(\ell) \;\subset\; \mathbb{P}^n$$

by writing $\ell = [H]$. Identifying $((\mathbb{P}^n)^*)^* = \mathbb{P}^n$, a point $x \in \mathbb{P}^n$ defines a hyperplane in $(\mathbb{P}^n)^*$ which we denote by

$$\mathcal{V}(x) \;=\; \{y \in (\mathbb{P}^n)^* \colon x_0y_0 + x_1y_1 + \cdots + x_ny_n = 0\}.$$

Definition 3.7. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. We define the *projective dual variety* X^* of X as the Zariski closure of the set of hyperplanes that are tangent to X at some regular point, i.e. the closure of

$$\{[H] \in (\mathbb{P}^n)^* \colon T_xX \subset H \text{ for some } x \in X_{\text{reg}}\}.$$

An instructive and especially nice case of duality occurs for hypersurfaces defined by quadratic forms of full rank.

Example 3.8. Let $q \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be a quadratic form and let M_q be the real symmetric $(n+1) \times (n+1)$ matrix representing q, i.e. with

$$q = (x_0, x_1, \ldots, x_n)M_q(x_0, x_1, \ldots, x_n)^T.$$

The projective variety $X = \mathcal{V}(q) \subset \mathbb{P}^n$ is smooth if and only if the rank of M_q is $n+1$. We compute the dual variety of X under the assumption that X is smooth. Let $x = (x_0 : x_1 : \cdots : x_n) \in X$ be a point. The differential $\ell_x = 2x^TM_q \in (\mathbb{P}^n)^*$ of q at x defines the tangent hyperplane $T_xX = \{y \in \mathbb{P}^n \colon \ell_x(y) = 0\}$ to X at x. In other words, the dual variety to X is the Zariski closure of the set $\{\ell_x \colon x \in X\} \subset (\mathbb{P}^n)^*$. The condition $x \in X$ is $0 = x^TM_qx = (\ell_x^TM_q^{-1})M_q(M_q^{-1}\ell_x)$. We conclude that X^* is the quadratic hypersurface defined by M_q^{-1}.

For irreducible algebraic varieties, the famous biduality theorem holds.

Theorem 3.9 (Biduality Theorem; see [25, Ch. 1, Thm. 1.1])**.** *If $X \subset \mathbb{P}^n$ is an irreducible projective variety, then $(X^*)^* = X$ under the canonical identification of the bidual of $\mathbb{P}^n$ with $\mathbb{P}^n$ itself.*

This theorem has several useful consequences like the following.

Remark 3.10. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. For all points x of X in a dense subset in the Euclidean topology of X, the point x is regular, the hyperplane $\mathcal{V}(x) \subset (\mathbb{P}^n)^*$ is tangent to X^* at a regular point ℓ, and the hyperplane $\mathcal{V}(\ell) \subset \mathbb{P}^n$ is tangent to X at x. This is an application of the conormal variety $CN(X)$, defined as the Zariski closure of

$$CN_0(X) \;=\; \{(x, [H]) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \colon x \in X_{\text{reg}}, T_xX \subset H\}.$$

The projection $\pi_1 \colon CN_0(X) \to X_{\text{reg}}$ is the conormal bundle of X, which shows that $CN_0(X)$ is an irreducible and smooth variety. The biduality theorem is often proven as a consequence of the fact that $CN(X) = CN(X^*)$, see [25, Chapter 1]. The biduality theorem implies that the subset

$$U = \{(x, [H]) \in CN_0(X) \colon [H] \in (X^*)_{\mathrm{reg}}\}$$

is a non-empty Zariski open subset of $CN(X)$. If $x \in X_{\mathrm{reg}}$ and $[H] \in (X^*)_{\mathrm{reg}}$ are regular points, then H is tangent to X at x if and only if $\mathcal{V}(x)$ is tangent to X^* at $[H]$, see [58, Thm. 1.7(b)]. This shows

$$U = CN_0(X) \cap CN_0(X^*).$$

By definition, the right-hand side consists of pairs $(x, [H])$ of regular points $x \in X_{\mathrm{reg}}$ and $[H] \in (X^*)_{\mathrm{reg}}$ such that $\mathcal{V}(x)$ is tangent to X^* at $[H]$ and H is tangent to X at x. Since U is dense in $CN(X)$ in the Euclidean topology [44, Thm. 2.33], the claim follows as the projection from $CN(X)$ to the first factor X is continuous and surjective.

When passing to real points, the direct analogue of Remark 3.10 fails: The set of regular real points $X_{\mathrm{reg}}(\mathbb{R})$ of an irreducible projective variety $X \subset \mathbb{P}^n$ may not be dense in $X(\mathbb{R})$ with respect to the Euclidean topology, even if it is non-empty, see for example [7, Section 3.1] or Example 6.4 below. This is addressed in the following remark.

Remark 3.11. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. For all real regular points x of X in a dense subset of $X_{\mathrm{reg}}(\mathbb{R})$ in the Euclidean topology, the hyperplane $\mathcal{V}(x) \subset (\mathbb{P}^n)^*$ is tangent to X^* at a real regular point ℓ and the hyperplane $\mathcal{V}(\ell) \subset \mathbb{P}^n$ is tangent to X at x.

This claim is trivial if X has no regular real points. We resume the discussion from Remark 3.10 assuming that X does contain a regular real point. The variety $CN_0(X)$ is smooth and contains real points, since X contains smooth real points. Since $CN_0(X) \setminus CN_0(X^*) = CN_0(X) \setminus U$ is a Zariski closed proper subset relative to $CN_0(X)$, it is of lower dimension. As $CN_0(X)(\mathbb{R})$ is a real analytic manifold of dimension $\dim(CN_0(X))$, see [7, Prop. 3.3.11], the set $U(\mathbb{R})$ is dense in $CN_0(X)(\mathbb{R})$ in the Euclidean topology. This proves the claim, because the projection of $CN_0(X)(\mathbb{R})$ onto the first factor X is onto $X_{\mathrm{reg}}(\mathbb{R})$.

Definition 3.12. We call a real point x of an algebraic variety $X \subset \mathbb{C}^n$ *central* if it is in the Euclidean closure of the set of regular and real points of X, i.e. if x is in the Euclidean closure of $X_{\mathrm{reg}}(\mathbb{R})$.

Remark 3.13. For an irreducible algebraic variety $X \subset \mathbb{C}^n$, a point $x \in X(\mathbb{R})$ is central if and only if the local dimension of x in $X(\mathbb{R})$ is equal to $\dim(X)$, see [7, Prop. 7.6.2].

4. Dual Hyperbolicity Cones

We discuss a result by the second author [53] more explicitly. The result is that the dual convex cone to a hyperbolicity cone is the convex cone generated by a particular semi-algebraic cone. This semi-algebraic cone is the Euclidean closure of the cone of regular real points on the dual variety to the hyperbolic hypersurface that lie in the right half-space. The algebraic boundary of the hyperbolicity cone allows us to simplify this semi-algebraic cone. We prove a stronger result for three-dimensional hyperbolicity cones.

A homogeneous polynomial $p \in \mathbb{R}[x_0, x_1, \dots, x_n]$ of degree d is called *hyperbolic* with respect to a fixed point $e \in \mathbb{R}^{n+1}$ if $p(e) \neq 0$ and the polynomial $p(te-a)$ in one variable t has only real roots for every point $a \in \mathbb{R}^{n+1}$. Without loss of generality, we fix the sign at e and always assume $p(e) > 0$. The roots of $p(te-a)$ are sometimes called the eigenvalues of a with respect to p and e, in analogy with characteristic polynomials of Hermitian matrices. Given any such polynomial p, the set

$$C_e(p) \;=\; \{a \in \mathbb{R}^{n+1} \colon \text{all roots of } p(te-a) \text{ are non-negative}\}$$

is a closed convex cone called the *(closed) hyperbolicity cone of p with respect to e*, and e is an interior point of $C_e(p)$, see [48]. Our goal is to describe the dual convex cone

$$C_e(p)^\vee \;=\; \{\ell \in (\mathbb{R}^{n+1})^* \colon \ell(x) \geq 0 \text{ for all } x \in C_e(p)\}.$$

An essential technique is projective duality. A general approach is described in the paper [53] by the second author. The goal of this section is to explain this method more explicitly for the special case of hyperbolicity cones.

An important example of a hyperbolic polynomial is the determinant of a matrix pencil, i.e. $p = \det(x_0A_0 + \dots + x_nA_n)$ for Hermitian $d \times d$ matrices $A_0, \dots, A_n \in H_d$, which is hyperbolic with respect to $e = (e_0, \dots, e_n)^T$ provided the matrix $e_0A_0 + \dots + e_nA_n$ is positive definite. In this case, $C_e(p)$ is the *spectrahedral cone* defined by $A_0, \dots, A_n$, see Section 6. However, the discussion in this section does not require such a determinantal representation and we consider general hyperbolic polynomials.

The proof of our main result, Theorem 4.5, makes use of the following Lemma 4.2 on hyperbolic polynomials. For the sake of completeness, we include a short proof based on the Helton-Vinnikov theorem on determinantal representations of hyperbolic curves; see [47, Lemma 2.4] for a direct proof of a special case.

Definition 4.1. Let $f \in \mathbb{R}[x_0, x_1, \dots, x_n]$ be a polynomial and $x \in \mathbb{C}^{n+1}$ be a point. The *multiplicity* of x on $\mathcal{V}(f) \subset \mathbb{C}^{n+1}$ is the smallest degree of a non-zero homogeneous term in the Taylor expansion of f around x.

Lemma 4.2. *Let $p \in \mathbb{R}[x_0, \dots, x_n]$ be hyperbolic with respect to e and let $x \in \mathbb{R}^{n+1}$. If x has multiplicity m on $\mathcal{V}(p)$, the hyperbolic hypersurface defined by p, then $t = 0$ is a root of multiplicity m of $p(x + te) \in \mathbb{R}[t]$. Moreover, $t = 0$ is also a root of multiplicity m of $p(x + t(e - x)) \in \mathbb{R}[t]$.*

Proof. If the multiplicity of x on $\mathcal{V}(p)$ is m, then $\frac{\partial^m}{\partial s^m} p(x + sy)|_{s=0} \neq 0$ for generic $y \in \mathbb{R}^{n+1}$. Fix such y in the interior of $C_e(p)$ and consider the hyperbolic polynomial $p(rx+sy+te)$ in three variables r, s, t. By the Helton-Vinnikov theorem ([31, Thm. 2.2]), this polynomial has a determinantal representation

$$p(rx + sy + te) = \det(rA + sB + tC)$$

with real symmetric matrices A, B, C, where B and C are positive definite, hence factor as $B = UU^T$ and $C = VV^T$, with U and V invertible. Now $s = 0$ is a root of $p(x+sy) = \det(A+sB)$ of multiplicity m, which means that $U^{-1}A(U^T)^{-1}$ has m-dimensional kernel. But then so does $V^{-1}A(V^T)^{-1}$, hence the root $t = 0$ of $p(x + te) = \det(A + tC)$ has multiplicity m as well.

The second part of the claim follows from the part we have just proved. Indeed, write $f(s,t) = p(sx+te) \in \mathbb{R}[s,t]$ which is the homogenization of $p(x+te)$ because $p(e) \neq 0$. Since $t = 0$ is a root of multiplicity m of $p(x+te)$, we can write $p(x+te) = t^m q(t)$ with $q(0) \neq 0$. Since p is hyperbolic with respect to e, the polynomial q factors into linear terms over $\mathbb{R}$, say $q = c(1-\lambda_1 t)\cdot \ldots \cdot (1-\lambda_{d-m}t)$ for some nonzero $c \in \mathbb{R}$. Here we used $q(0) \neq 0$ because the λ_i are the reciprocals of the roots of q. So we get that $f(s,t) = c \cdot t^m \prod(s-\lambda_i t)$. The polynomial $p(sx+t(e-x))$ is the same as $f(s-t,t) = c\cdot t^m \prod(s-(\lambda_i+1)t)$. Dehomogenizing this again shows that $t=0$ is a root of multiplicity m of $p(x+t(e-x))$. □

Corollary 4.3. *If x is a regular real point of a hyperbolic hypersurface $\mathcal{V}(p)$, then the line incident with x and the hyperbolicity direction e is not tangent to $\mathcal{V}(p)$ at x, i.e. is not contained in $T_x(\mathcal{V}(p))$.*

Proof. If a line L is tangent to $\mathcal{V}(p)$ at x, then the multiplicity of x in $L \cap \mathcal{V}(p)$ is greater than the multiplicity of x in $\mathcal{V}(p)$. This is impossible if $e \in L$, by the previous Lemma 4.2. □

We prove basics from differential and convex geometry regarding hyperbolicity cones.

Lemma 4.4. *Let $p \in \mathbb{R}[x_0,\ldots,x_n]$ be an irreducible hyperbolic polynomial with respect to $e \in \mathbb{R}^{n+1}$. Then $M = \partial C_e(p) \cap \mathcal{V}(p)_{\mathrm{reg}}$ is an n-dimensional real analytic manifold, which is open and dense in the Euclidean boundary $\partial C_e(p)$ of $C_e(p)$ in the Euclidean topology. The (inner) normal cone to $C_e(p)$ at any point $x \in M$ is the ray $\mathbb{R}_+\ell$, where $\ell = \nabla p(x)^T \in C_e(p)^\vee \subset (\mathbb{R}^{n+1})^*$.*

Proof. Since $\partial C_e(p) \subset \mathcal{V}(p)(\mathbb{R})$ and since the set of singular points of $\mathcal{V}(p)(\mathbb{R})$ is a variety of dimension at most $n-1$, see [7, Prop. 3.3.14], the complement $M = \partial C_e(p) \cap \mathcal{V}(p)_{\mathrm{reg}}$ is open and dense in $\partial C_e(p)$ in the Euclidean topology. As $\mathcal{V}(p)_{\mathrm{reg}}(\mathbb{R})$ is an analytic manifold of dimension n, see [7, Prop. 3.3.11], and as the eigenvalues depend continuously on $x \in \mathbb{R}^{n+1}$, see [48], the set M is an analytic manifold of dimension n.

Let $x \in M$. As x is a regular point of $\mathcal{V}(p)$, the functional $\ell = \nabla p(x)^T$ is non-zero. Hence, Lemma 4.2, case $m = 1$, shows $\ell(e) \neq 0$. Since M is an n-dimensional analytic manifold included in $C_e(p)$, the normal cone $N(x)$ of $C_e(p)$ at x is a subset of the line $\mathbb{R}\ell$, hence $N(x) = \mathbb{R}_+\ell$ or $N(x) = -\mathbb{R}_+\ell$ as $x \in \partial C_e(p)$. The derivative polynomial $p' \in \mathbb{R}[x_0,\ldots,x_n]$, defined by

$$p'(y) = \tfrac{\partial}{\partial t} p(y+te)|_{t=0} = \nabla p(y)^T e, \qquad y \in \mathbb{R}^{n+1},$$

is hyperbolic with respect to e and $C_e(p) \subset C_e(p')$ holds, see [48]. This proves $\ell(e) = p'(x) > 0$ and rules out $N(x) = -\mathbb{R}_+\ell$, as $N(x) \subset C_e(p)^\vee$ by Equation (8). This proves the claim. □

The second author obtained Theorem 4.5 and Corollary 4.14 below in [53, Example 3.15]. We slightly abuse notation in the following statements. If $X \subset \mathbb{C}^{n+1}$ is an algebraic cone, then we write $X^* \subset (\mathbb{C}^{n+1})^*$ for the affine cone over the projective dual variety $\mathbb{P}(X)^*$ of $\mathbb{P}(X)$. Let $H_{e,+} \subset (\mathbb{R}^{n+1})^*$ denote the half-space $H_{e,+} = \{\ell \in (\mathbb{R}^{n+1})^* \colon \ell(e) \geq 0\}$ for nonzero $e \in \mathbb{R}^{n+1}$.

Theorem 4.5 (Sinn [53])**.** *Let $p \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be an irreducible hyperbolic polynomial with respect to $e \in \mathbb{R}^{n+1}$. Then we have*

$$C_e(p)^\vee \;=\; \operatorname{clos}\Big(\operatorname{cc}\big((\mathcal{V}(p)^*)_{\mathrm{reg}}(\mathbb{R}) \cap H_{e,+}\big)\Big).$$

Proof. We prove the inclusion "$\supset$". Since $C_e(p)^\vee$ is a closed convex cone, it is enough to show that $S = (\mathcal{V}(p)^*)_{\mathrm{reg}}(\mathbb{R}) \cap H_{e,+}$ is contained in $C_e(p)^\vee$. By Remark 3.11, it is enough to prove $\ell \in C_e(p)^\vee$ for all $\ell \in S$ such that the hyperplane $\mathcal{V}(\ell)$ is tangent to $\mathcal{V}(p)$ at a regular real point x, i.e. $\ell = \nabla p(x)^T \in (\mathbb{R}^{n+1})^*$. By Lemma 4.2 above, case $m = 1$, we have $0 \neq \frac{\partial}{\partial t} p(x + te)|_{t=0} = \ell(e)$ for such an ℓ and hence $e \notin \mathcal{V}(\ell)$. As p is hyperbolic with respect to every interior point of the hyperbolicity cone [48], it follows that the interior of $C_e(p)$ is disjoint from $\mathcal{V}(\ell)$. This means that ℓ has constant sign on $C_e(p)$. Since $\ell \in H_{e,+}$, it follows that $\ell \in C_e(p)^\vee$.

The inclusion "$\subset$" follows from [53, Coro. 3.14]. We repeat the argument, adapted to our setup, for completeness. Since $C_e(p)^\vee$ is the convex cone generated by its extreme rays and the right-hand side is a convex cone, it suffices to prove that every extreme ray of $C_e(p)^\vee$ is contained in the right-hand side. By Straszewicz's Theorem [49, Thm. 18.6], which says that every extreme ray is a limit of exposed rays, it suffices to prove the claim for every exposed ray of $C_e(p)^\vee$, because the right-hand side is closed.

Let $\mathbb{R}_+\ell$ be an exposed ray of the convex cone $C_e(p)^\vee$. It is enough to show that ℓ is a central point of $\mathcal{V}(p)^*(\mathbb{R})$, i.e. ℓ lies in the (Euclidean) closure of $(\mathcal{V}(p)^*)_{\mathrm{reg}}(\mathbb{R})$. Let $F_\ell = \{x \in C_e(p) \colon \ell(x) = 0\}$ be the exposed face of $C_e(p)$ corresponding to ℓ and let x be a point in the relative interior of F_ℓ. As proven in Lemma 4.4, the analytic manifold $M = \partial C_e(p) \cap \mathcal{V}(p)_{\mathrm{reg}}$ is dense in the Euclidean boundary $\partial C_e(p)$ of the hyperbolicity cone $C_e(p)$. Hence, there is a sequence $(y_j) \subset M$ converging to x. Remark 3.11 shows that after slightly moving the members of the sequence (y_j) within M without changing the limit x, the hyperplane $\mathcal{V}(y_j)$ is tangent to $\mathcal{V}(p)^*$ at a regular real point $\ell_j \in (\mathcal{V}(p)^*)_{\mathrm{reg}}(\mathbb{R})$ and $\mathcal{V}(\ell_j)$ is tangent to $\mathcal{V}(p)$ at y_j for all $j \in \mathbb{N}$.

After scaling ℓ_j with a nonzero real number, we have $\ell_j = \nabla p(y_j)^T$. Hence, the ray $\mathbb{R}_+\ell_j$ lies in the dual convex cone $C_e(p)^\vee$ by Lemma 4.4 for all $j \in \mathbb{N}$. After normalizing and passing to a subsequence, the sequence (ℓ_j) converges to a point ℓ' in the compact unit sphere of $(\mathbb{R}^{n+1})^*$. We have $\ell' \in C_e(p)^\vee$ and $\ell'(x) = 0$, the latter as

$$\ell'(x) = \ell_j(x - y_j) + (\ell' - \ell_j)y_j + (\ell' - \ell_j)(x - y_j)$$

holds for all $j \in \mathbb{N}$ and since $\ell_j \to \ell'$ and $y_j \to x$ as $j \to \infty$. Since $\mathbb{R}_+\ell$ is an exposed ray of $C_e(p)^\vee$, the lattice isomorphism of Lemma 3.6 and Equation (8) show $\mathbb{R}_+\ell = F_\ell^\perp \cap C_e(p)^\vee = x^\perp \cap C_e(p)^\vee$. This proves $\mathbb{R}_+\ell = \mathbb{R}_+\ell'$. Hence ℓ is a central point of $\mathcal{V}(p)^*(\mathbb{R})$. $\square$

In other words, Theorem 4.5 says that the dual convex cone to the hyperbolicity cone $C_e(p)$ is the closed convex cone generated by the regular real points of the dual variety $\mathcal{V}(p)^*$ lying in the appropriate half-space $H_{e,+}$.

The well-known Steiner surface explains why singular points of the dual variety $\mathcal{V}(p)^*$ have to be excluded from the statement of Theorem 4.5.

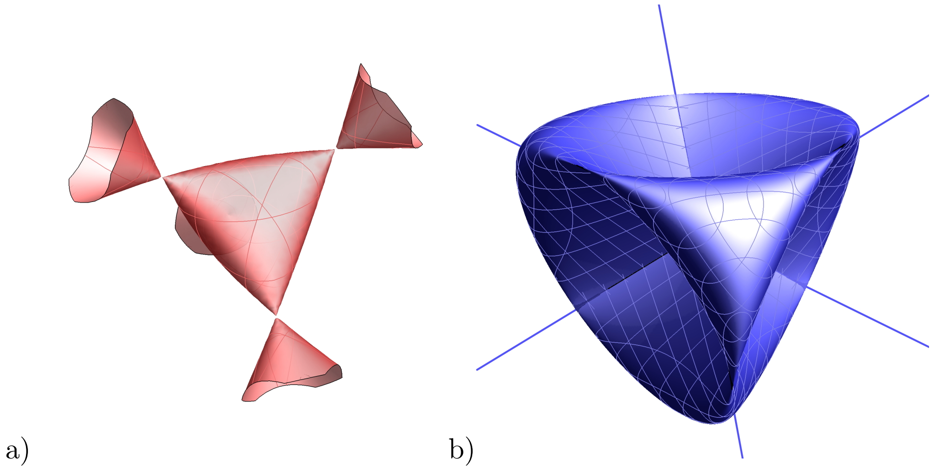


FIGURE 1. a) Cayley cubic. b) Steiner surface with three singular lines.

Example 4.6. The Cayley cubic is the cubic hypersurface in $\mathbb{P}^3$ defined by the polynomial

$$p = \det \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_0 & x_2 \\ x_3 & x_2 & x_0 \end{pmatrix}.$$

This polynomial is irreducible and hyperbolic with respect to the point $(1,0,0,0)$ in $\mathbb{R}^4 = \{(x_0,x_1,x_2,x_3)\}$. Its hyperbolicity cone is the homogenization of the elliptope $\mathcal{E}_3$, which is the feasible set of the Goemans-Williamson semidefinite relaxation of the MAX-CUT problem (for graphs with three vertices).

The dual convex cone is the closed convex cone generated by the regular real points with $y_0 > 0$ on the Steiner surface given by the equation

$$q = y_1^2 y_2^2 + y_1^2 y_3^2 + y_2^2 y_3^2 - 2y_0 y_1 y_2 y_3.$$

The singular locus of this quartic surface is the union of three real lines in $(\mathbb{P}^3)^*$, which are not contained in the dual convex cone. See Figure 1, where we draw the real affine parts of the varieties $\mathcal{V}(p)$ and $\mathcal{V}(q)$, that is to say, the set of points $(x_1,x_2,x_3)^T \in \mathbb{R}^3$ for which $(1 : x_1 : x_2 : x_3)$ lies in $\mathcal{V}(p)$ in drawing a) and the set of points $(y_1,y_2,y_3) \in (\mathbb{R}^3)^*$ for which $(1 : y_1 : y_2 : y_3)$ lies in $\mathcal{V}(q)$ in drawing b).

Convex geometry suffices to generalize Theorem 4.5 from irreducible polynomials to arbitrary hyperbolic polynomials.

Corollary 4.7. *Let $p \in \mathbb{R}[x_0,x_1,\dots,x_n]$ be a hyperbolic polynomial with respect to $e \in \mathbb{R}^{n+1}$. Let $X_1, X_2, \dots, X_r$ be the irreducible components of the hyperbolic hypersurface $\mathcal{V}(p)$. Then we have*

$$C_e(p)^\vee = \mathrm{cc}\left(\mathrm{clos}(S_1 \cup S_2 \cup \cdots \cup S_r)\right), \tag{9}$$

where $S_i = (X_i^*)_{\mathrm{reg}}(\mathbb{R}) \cap H_{e,+}$ for $i = 1,\dots,r$.

Proof. If $p = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ is a factorization of p into irreducible factors, then $C_e(p) = C_e(p_1) \cap C_e(p_2) \cap \cdots \cap C_e(p_r)$. The claim follows from Theorem 4.5 and from Equation (6) in Proposition 3.5. $\square$

Remark 4.8. 1) The cones S_i in Corollary 4.7, their union $S_1 \cup S_2 \cup \cdots \cup S_r$, and the Euclidean closure $\operatorname{clos}(S_1 \cup S_2 \cup \cdots \cup S_r)$, see [7, Prop. 2.2.2], are semi-algebraic sets. Hence, the closed convex cone $C_e(p)^\vee$ is the convex cone generated by the semi-algebraic cone $\operatorname{clos}(S_1 \cup S_2 \cup \cdots \cup S_r)$.

2) The Euclidean closure of the set S_i in Corollary 4.7 is the set of central points of $X_i^*(\mathbb{R})$ lying in $H_{e,+}$ for all $i = 1, 2, \dots, r$. Hence, writing $\operatorname{cent}(X)$ for the set of central real points of X, we can rephrase Equation (9) as

$$C_e(p)^\vee \;=\; \operatorname{cc}\left(\bigcup_{i=1}^{r} \operatorname{cent}(X_i^*(\mathbb{R})) \cap H_{e,+}\right).$$

We can strengthen Theorem 4.5 to the homogeneous version of Kippenhahn's Theorem, the statement of Theorem 4.9, if the hyperbolicity cone has dimension three. Chien and Nakazato [14] observed that this stronger version is false in higher dimensions, see Example 6.4.

Theorem 4.9. *Let $p \in \mathbb{R}[x_0, x_1, x_2]$ be an irreducible hyperbolic polynomial with respect to $e \in \mathbb{R}^3$. Then $C_e(p)^\vee = \operatorname{cc}\big(\mathcal{V}(p)^*(\mathbb{R}) \cap H_{e,+}\big)$.*

Proof. The inclusion $C_e(p)^\vee \subset \operatorname{cc}\big(\mathcal{V}(p)^*(\mathbb{R}) \cap H_{e,+}\big)$ follows from Theorem 4.5 and Lemma 3.4. We prove the opposite inclusion by contradiction based on two observations. Let ℓ be a nonzero functional that lies in $\operatorname{cc}\big(\mathcal{V}(p)^*(\mathbb{R}) \cap H_{e,+}\big)$ but not in $C_e(p)^\vee$.

First, the hyperplane $\mathcal{V}(\ell) \subset \mathbb{C}^3$ intersects the interior of the hyperbolicity cone $C_e(p)$, as it holds for all functionals $\hat\ell \in H_{e,+} \setminus C_e(p)^\vee$. Since $\hat\ell \in H_{e,+}$, we have $\hat\ell(e) \geq 0$, and since $\hat\ell \notin C_e(p)^\vee$ there is a point $x \in C_e(p)$ such that $\hat\ell(x) < 0$. Hence, there is $\lambda \in [0, 1)$ such that the point $y = (1-\lambda)e + \lambda x$ lies on $\mathcal{V}(\hat\ell)$. As e is an interior point of $C_e(p)$ so is y [49, Thm. 6.1].

Secondly, the line $\mathcal{V}(\ell) \subset \mathbb{P}^2$ is tangent to $\mathcal{V}(p)$ at a point $x \in \mathcal{V}(p)$. Applying Remark 3.10 to the projective variety $X = \mathcal{V}(p)^*$, we can choose a sequence of regular points (ℓ_j) in $\mathcal{V}(p)^*$ converging to ℓ and a sequence of regular points (x_j) of $\mathcal{V}(p)$ such that the line $\mathcal{V}(\ell_j)$ is tangent to $\mathcal{V}(p)$ at x_j for all $j \in \mathbb{N}$. Because of the compactness of the projective space $\mathbb{P}^2$, we can assume (x_j) converges to a point $x \in \mathcal{V}(p)$. Since $\mathcal{V}(\ell_j)$ is tangent to $\mathcal{V}(p)$ at x_j, the line $\mathcal{V}(\ell)$ is tangent to $\mathcal{V}(p)$ at x, see [20, Sec. 8.2].

The real line $\mathcal{V}(\ell) \subset \mathbb{P}^2$ intersects the hyperbolicity cone $C_e(p)$ in an interior point by the first observation. Since the polynomial p is hyperbolic with respect to this interior point and since the point $x \in \mathcal{V}(p)$ constructed above lies on the line $\mathcal{V}(\ell)$, it follows that x is a real point. Lemma 4.2 then shows that $\mathcal{V}(\ell)$ is not tangent to $\mathcal{V}(p)$ at x, which contradicts the second observation. □

Again, convex geometry suffices to generalize Theorem 4.9 from irreducible polynomials to arbitrary hyperbolic polynomials.

Corollary 4.10. *Let $p \in \mathbb{R}[x_0, x_1, x_2]$ be a hyperbolic polynomial with respect to $e \in \mathbb{R}^3$. Then $C_e(p)^\vee = \operatorname{cc}(\mathcal{V}(p)^*(\mathbb{R}) \cap H_{e,+})$.*

Proof. If $p = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ is a factorization of p into irreducible factors, then

$$C_e(p) = C_e(p_1) \cap C_e(p_2) \cap \cdots \cap C_e(p_r)$$

and

$$\mathcal{V}(p)^* = \mathcal{V}(p_1)^* \cup \mathcal{V}(p_2)^* \cup \cdots \cup \mathcal{V}(p_r)^*.$$

The claim follows from Theorem 4.9 and Equation (6) in Proposition 3.5. □

Not all components in the union $\mathcal{V}(p)^* = \mathcal{V}(p_1)^* \cup \mathcal{V}(p_2)^* \cup \cdots \cup \mathcal{V}(p_r)^*$ are needed in the statements of Corollary 4.10 and Corollary 4.7. The selection can be described as follows.

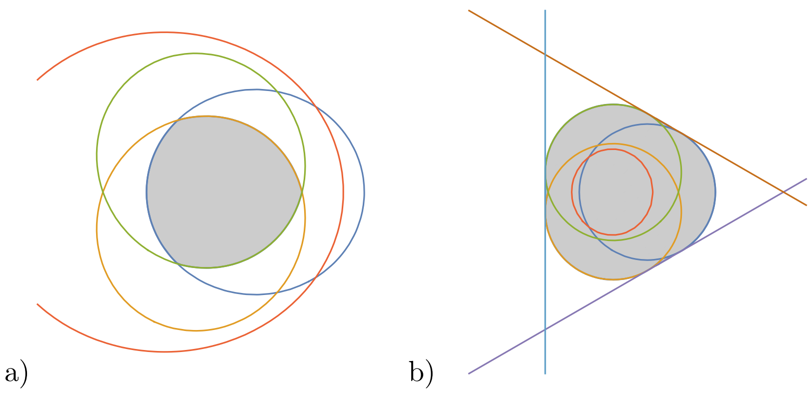
Definition 4.11. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. The *algebraic boundary* of S, denoted $\partial_a S$, is the Zariski closure in $\mathbb{C}^n$ of the Euclidean boundary ∂S of S.

Determining the algebraic boundary of the hyperbolicity cone of a hyperbolic polynomial $p \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ amounts to computing a factorization of p into irreducible factors and picking the correct subset of the factors.

Remark 4.12. 1) Let $p \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be irreducible and hyperbolic with respect to e. The algebraic boundary of the hyperbolicity cone $C_e(p)$ is the algebraic hypersurface $\mathcal{V}(p) = \{x \in \mathbb{C}^{n+1} : p(x) = 0\}$.

2) If p is hyperbolic with respect to e, but factors as $p = p_1 p_2 \ldots p_r$ into irreducible factors, then $\mathcal{V}(p) = \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \cdots \cup \mathcal{V}(p_r)$ is the decomposition of the hypersurface $\mathcal{V}(p)$ into its irreducible components. The algebraic boundary of $C_e(p)$ is a union of some, not necessarily all, of the irreducible hypersurfaces $\mathcal{V}(p_i)$. The hypersurfaces in this union are the irreducible components of $\partial_a C_e(p)$.

3) If p is squarefree, i.e. $p_j \neq p_i$ for $i \neq j$, then the factors p_i that contribute to the algebraic boundary of $C_e(p)$ are exactly those with the property that the hyperbolicity cone of $\prod_{j \neq i} p_j$ is strictly larger than $C_e(p)$, because $C_e(p) = \bigcap_i C_e(p_i)$.



a) b)

FIGURE 2. a) Intersection of four filled ellipses (gray area). b) Convex hull of the dual ellipses (gray area).

Example 4.13. Consider the four ellipses depicted in Figure 2 a). The intersection of the filled ellipses is an area with nonempty interior, which is isometric to the base of a hyperbolicity cone *via* the embedding $\mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2)^T \mapsto (1, x_1, x_2)^T$. Note that the red ellipse does not contribute to the algebraic boundary of the hyperbolicity cone, but the other ellipses do. Homogenizing the polynomials, we obtain four conics in $\mathbb{P}^2$ from the ellipses. The real affine parts of the dual conics, *via* the embedding $(\mathbb{R}^2)^* \to (\mathbb{P}^2)^*$, $(y_1, y_2) \mapsto (1 : y_1 : y_2)$, are depicted in Figure 2 b). Their convex hull is a base of the dual convex cone to the hyperbolicity cone. We return to these bases in Section 5. Note that the red ellipse in Figure 2 b) is redundant, as the other three ellipses generate the same convex hull as all four together.

Corollary 4.14 (Sinn [53]). *Let $p \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ be a hyperbolic polynomial with respect to $e \in \mathbb{R}^{n+1}$. Let $X_1, X_2, \ldots, X_r$ be the irreducible components of the algebraic boundary $\partial_a C_e(p)$. Then we have*

$$C_e(p)^\vee = \operatorname{cc}\big(\operatorname{clos}(S_1 \cup S_2 \cup \cdots \cup S_r)\big),$$

where $S_i = (X_i^*)_{\mathrm{reg}}(\mathbb{R}) \cap H_{e,+}$ for $i = 1, \ldots, r$.

Proof. This follows from Part 3) of Remark 4.12 and Corollary 4.7. □

Remark 4.15. The algebraic boundary of the hyperbolicity cone $C_e(p)$ is algebraically the optimal description of the set of extreme rays of $C_e(p)^\vee$. Using the same techniques and ideas that are presented in this paper, one can argue that the Zariski closure of the set of extreme rays of $C_e(p)^\vee$ is the union of the dual varieties X_i^* for which the varieties X_i belong to the algebraic boundary $\partial_a C_e(p)$, see [53, Corollary 3.5].

The statements above in this section require employing the duality theory of real algebraic geometry, which is more subtle than the duality theory of algebraic geometry. The reason is that the set of Hermitian matrices is just a real vector space, not a complex vector space.

Remark 4.16. The image of the set $H_d^{(1)}$ of Hermitian matrices of rank 1 under the projection map

$$\pi(M) = (\langle M, A_1\rangle, \langle M, A_2\rangle, \ldots, \langle M, A_n\rangle)$$

from H_d to $\mathbb{R}^n$ is of interest to us (see Section 2), as its convex hull is the joint numerical range. The complexification of the real vector space H_d is the complex vector space of all complex $d \times d$ matrices. The Zariski closure of $H_d^{(1)}$ is the variety $R_d^{(1)}$ of complex $d \times d$ matrices of rank at most 1. Therefore, the images of $H_d^{(1)}$ and $R_d^{(1)}$ under the map π have the same Zariski closure. To evaluate $\langle M, A_i\rangle$ at a complex matrix M, we write $M = \operatorname{Re}(M) + \mathrm{i}\operatorname{Im}(M)$, where $\operatorname{Re}(M)$ and $\operatorname{Im}(M)$ are Hermitian, and define $\langle M, A_i\rangle = \operatorname{tr}(\operatorname{Re}(M)A_i) + \mathrm{i}\operatorname{tr}(\operatorname{Im}(M)A_i)$ (which is $\mathbb{C}$-linear). If the map π restricted to $R_d^{(1)}$ is an isomorphism, then Prop. 4.1 in [25] implies roughly speaking that $\pi(R_d^{(1)})$ is projectively dual to the intersection of the orthogonal complement of the kernel of π and the dual variety of $R_d^{(1)}$. This dual variety is the determinantal hypersurface in the space of complex $d \times d$ matrices. The assumption that π restricted to $R_d^{(1)}$ is an isomorphism is generically satisfied provided that n is sufficiently large relative to the size d of the matrices; specifically the kernel of π must not intersect the secant variety of $R_d^{(1)}$, which is the variety of matrices of rank at most 2.

5. Bases of Dual Hyperbolicity Cones

The results of Section 4 carry over from the dual convex cone of a hyperbolicity cone to all its bases, by replacing the convex cone generated by a semi-algebraic cone with the convex hull of a base of this semi-algebraic cone. Remarkably, the results also apply to all linear images of these bases, because we can interpret these linear images as the bases of the dual convex cones to linear sections of the original hyperbolicity cone.

$$\begin{tikzcd}
e \in \mathcal{K} \subset \ V & \mathbb{R} \oplus \mathbb{R}^n \arrow[l, "\phi_0 = \pi_0^*"'] & \mathbb{R}^n \arrow[l, "\phi' = \pi_2^*"'] \arrow[ll, bend right, "\phi = \pi^*"']
\end{tikzcd}$$

$$\cup$$

$$C = \phi_0^{-1}(\mathcal{K})$$

$$C^\vee = \pi_0(\mathcal{K}^\vee)$$

$$\cap$$

$$\begin{tikzcd}
\mathcal{B} = \mathrm{h}_e(\mathcal{K}^\vee) \subset \mathcal{K}^\vee \subset \ V^* \arrow[r, "\pi_0"] \arrow[rr, bend right, "\pi"'] & (\mathbb{R} \oplus \mathbb{R}^n)^* \arrow[r, "\pi_2"] & (\mathbb{R}^n)^* \ \supset \pi(\mathcal{B})
\end{tikzcd}$$

Figure 3. The point e is an interior point of the convex cone $\mathcal{K}$. We study the linear image $\pi(\mathcal{B})$ of a base $\mathcal{B}$ of the dual convex cone $\mathcal{K}^\vee$. The cone $C = \phi_0^{-1}(\mathcal{K})$ allows us to identify $\pi(\mathcal{B})$ with a base of the dual convex cone $C^\vee$.

Before returning to hyperbolicity cones at the end of the section, we discuss the necessary convex geometry. See Figure 3 for a summary of our setup.

Definition 5.1. Let V be a finite-dimensional real vector space. Let $\mathcal{K} \subset V$ be a closed convex cone and let $e \neq 0$ be an interior point of $\mathcal{K}$. The set

$$\mathcal{B} = \mathrm{h}_e(\mathcal{K}^\vee) = \{\ell \in \mathcal{K}^\vee : \ell(e) = 1\}$$

introduced in Equation (4) is a compact convex base of the dual convex cone $\mathcal{K}^\vee$ by Lemma 3.3. We study the image of $\mathcal{B}$ under an arbitrary linear map

$$\pi \colon V^* \to (\mathbb{R}^n)^*.$$

Let $\pi_0 \colon V^* \to (\mathbb{R} \oplus \mathbb{R}^n)^*$, $\ell \mapsto (\ell(e), \pi(\ell))$ and let $\pi_2 \colon (\mathbb{R} \oplus \mathbb{R}^n)^* \to (\mathbb{R}^n)^*$, $(y_0, y) \mapsto y$ denote the projection onto the second summand. We denote the dual map to π, π_0, and π_2 by $\phi : \mathbb{R}^n \to V$, $\phi_0 : \mathbb{R} \oplus \mathbb{R}^n \to V$, and $\phi' : \mathbb{R}^n \to \mathbb{R} \oplus \mathbb{R}^n$, respectively. We will describe $\pi(\mathcal{B})$ in terms of the set

$$C = \phi_0^{-1}(\mathcal{K}) \subset \mathbb{R} \oplus \mathbb{R}^n.$$

Let $b_0, b_1, \ldots, b_n$ denote the standard basis of $\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{R}^n$.

It is easy to see that the set C is a closed convex cone containing b_0 as an interior point. In addition, C can be seen as a linear section of the convex cone $\mathcal{K}$ through the interior point e. To be more precise, let $v_1, v_2, \ldots, v_n \in V$ and let $\pi_{v_1,\ldots,v_n} : V^* \to (\mathbb{R}^n)^*$ be defined by

$$\pi_{v_1,\ldots,v_n}(\ell) = (\ell(v_1), \ldots, \ell(v_n)), \qquad \ell \in V^*. \tag{10}$$

If $\pi = \pi_{v_1,\dots,v_n}$, then

$$\phi_0((x_0, x)^T) = x_0 e + x_1 v_1 + \cdots + x_n v_n$$

holds for all $(x_0, x)^T \in \mathbb{R} \oplus \mathbb{R}^n$ where $x = (x_1, \dots, x_n)^T$. The image of ϕ_0 is the subspace $\operatorname{im}(\phi_0) = \operatorname{span}(e, v_1, \dots, v_n)$, which intersects the convex cone $\mathcal{K}$ in the interior point e. If $e, v_1, v_2, \dots, v_n$ are linearly independent, then the monomorphism ϕ_0 identifies C with a linear section of $\mathcal{K}$ through e.

Lemma 3.3 shows that the set $\mathrm{h}_{b_0}(C^\vee) = \{(y_0, y) \in C^\vee : y_0 = 1\}$ is a (compact, convex) base of the dual convex cone $C^\vee$, as b_0 is an interior point of C. Obviously, the set $\mathrm{h}_{b_0}(C^\vee)$ is isometric under π_2 to the set

$$\pi_2 \circ \mathrm{h}_{b_0}(C^\vee) = \{y \in (\mathbb{R}^n)^* : (1, y) \in C^\vee\}.$$

We show this set is a linear image of the base $\mathcal{B}$ of $\mathcal{K}^\vee$.

Proposition 5.2. *The image of the convex cone $\mathcal{K}^\vee$ under the linear map π_0 is the closed convex cone $\pi_0(\mathcal{K}^\vee) = C^\vee$. The image of the set $\mathcal{B}$ is the compact base $\pi_0(\mathcal{B}) = \mathrm{h}_{b_0}(C^\vee)$ of $C^\vee$. In addition we have $\pi(\mathcal{B}) = \pi_2 \circ \mathrm{h}_{b_0}(C^\vee)$.*

Proof. To prove the equation $C^\vee = \pi_0(\mathcal{K}^\vee)$, first let $y = \pi_0(\ell) \in \pi_0(\mathcal{K}^\vee)$ for some $\ell \in \mathcal{K}^\vee$. For all $x \in C$ we have $\phi_0(x) \in \mathcal{K}$ and hence

$$y(x) = \pi_0(\ell)(x) = \ell(\phi_0(x)) \geq 0.$$

This shows that $y \in C^\vee$ and therefore $\pi_0(\mathcal{K}^\vee) \subset C^\vee$. We prove a partial converse by duality, i.e. we show $(\pi_0(\mathcal{K}^\vee))^\vee \subset C$. Let $x \in (\pi_0(\mathcal{K}^\vee))^\vee$. Then

$$\ell(\phi_0(x)) = \pi_0(\ell)(x) \geq 0 \qquad \text{for all } \ell \in \mathcal{K}^\vee,$$

which implies $\phi_0(x) \in \mathcal{K}$ or in other words $x \in \phi_0^{-1}(\mathcal{K}) = C$.

We finish proving $C^\vee = \pi_0(\mathcal{K}^\vee)$ by showing that the convex cone $\pi_0(\mathcal{K}^\vee)$ is closed. By Lemma 3.1 it suffices to prove that $\pi_0(\mathcal{B})$ is a compact base of $\pi_0(\mathcal{K}^\vee)$. The set $\pi_0(\mathcal{B})$ is compact as $\mathcal{B}$ is compact. The set $\mathcal{K}^\vee \setminus \{0\}$ is the cone generated by $\mathcal{B}$, hence the set $\pi_0(\mathcal{K}^\vee) \setminus \{0\}$ is the cone generated by $\pi_0(\mathcal{B})$. It remains to show that the origin does not lie in the affine hull of $\pi_0(\mathcal{B})$. Since $\ell(e) = 1$ holds for all $\ell \in \mathcal{B}$, the affine hull of $\mathcal{B}$ does not intersect the annihilator $e^\perp = \{\ell \in V^* : \ell(e) = 0\}$. As $e \in \operatorname{im}(\phi_0)$ and $\ker(\pi_0) = \operatorname{im}(\phi_0)^\perp$, we have $\ker \pi_0 \subset e^\perp$. This shows $\operatorname{aff}(\mathcal{B}) \cap \ker \pi_0 = \emptyset$, which proves that the origin does not lie in $\pi_0(\operatorname{aff}(\mathcal{B})) = \operatorname{aff}(\pi_0(\mathcal{B}))$.

Using the equation $\pi_0(\mathcal{K}^\vee) = C^\vee$ and the projection $\pi_1 : (\mathbb{R} \oplus \mathbb{R}^n)^* \to \mathbb{R}^*$, $(y_0, y) \mapsto y_0$ onto the first summand, we get

$$\begin{aligned}
\pi_0(\mathcal{B}) &= \{\pi_0(\ell) \in (\mathbb{R} \oplus \mathbb{R}^n)^* : \ell \in \mathcal{K}^\vee, \ell(e) = 1\} \\
&= \{\pi_0(\ell) \in (\mathbb{R} \oplus \mathbb{R}^n)^* : \ell \in \mathcal{K}^\vee, \pi_1(\pi_0(\ell)) = 1\} \\
&= \{(y_0, y) \in C^\vee : y_0 = 1\} = \mathrm{h}_{b_0}(C^\vee).
\end{aligned}$$

Finally, $\pi(\mathcal{B}) = \pi_2 \circ \pi_0(\mathcal{B}) = \pi_2 \circ \mathrm{h}_{b_0}(C^\vee)$ completes the proof. □

Proposition 5.2 induces a duality of convex sets. Let $b_0^*, b_1^*, \dots, b_n^*$ denote the standard basis of $(\mathbb{R}^{n+1})^* = (\mathbb{R} \oplus \mathbb{R}^n)^*$. Let $\widetilde{\pi_2} \colon \mathbb{R} \oplus \mathbb{R}^n \to \mathbb{R}^n$, $(x_0, x)^T \mapsto x$ denote the projection onto the second summand. Clearly, the affine slice $\mathrm{h}_{b_0^*}(C) = \{(x_0, x)^T \in C : x_0 = 1\}$ of the convex cone C is isometric under $\widetilde{\pi_2}$ to the set

$$\widetilde{\pi_2} \circ \mathrm{h}_{b_0^*}(C) = \{x \in \mathbb{R}^n : (1, x)^T \in C\}.$$

We show this set is the dual convex set to a linear image of the set $\mathcal{B}$.

Corollary 5.3. *The dual convex set to $\pi(\mathcal{B})$ is $\widetilde{\pi_2} \circ \mathrm{h}_{b_0^*}(C) = \big(\pi(\mathcal{B})\big)^\circ$. If the set $\pi(\mathcal{B})$ contains the origin, then $\pi(\mathcal{B}) = \big(\widetilde{\pi_2} \circ \mathrm{h}_{b_0^*}(C)\big)^\circ$.*

Proof. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
x \in (\pi(\mathcal{B}))^\circ &\iff x \in (\pi_2 \circ \mathrm{h}_{b_0}(C^\vee))^\circ \\
&\iff 1 + y(x) \geq 0 && \forall y \in \pi_2 \circ \mathrm{h}_{b_0}(C^\vee) \\
&\iff (y_0, y)(1, x)^T \geq 0 && \forall (y_0, y) \in \mathrm{h}_{b_0}(C^\vee) \\
&\iff (1, x)^T \in C \\
&\iff x \in \widetilde{\pi_2} \circ \mathrm{h}_{b_0^*}(C).
\end{aligned}$$

The first equivalence follows from Proposition 5.2. Regarding the fourth equivalence, note that $\mathrm{h}_{b_0}(C^\vee)$ is a base of the convex cone $C^\vee$. The remaining equivalences follow immediately from the definitions. As $\pi(\mathcal{B})$ is a closed convex set, the second statement follows from the equation $S = (S^\circ)^\circ$, which holds for all closed convex subsets $S \subset (\mathbb{R}^n)^*$ containing the origin [49]. □

The convex duality of Corollary 5.3 has been used earlier [14, 32, 30] in the context of the joint numerical range. The conic duality of Proposition 5.2 has advantages in our situation.

Remark 5.4. If the convex set $\pi(\mathcal{B})$ contains the origin, we could use the convex duality

$$\pi(\mathcal{B}) = (\widetilde{\pi_2} \circ \mathrm{h}_{b_0^*}(C))^\circ \tag{11}$$

of Corollary 5.3 to describe the set $\pi(\mathcal{B})$. Example 5.5 and Figure 4 present an example where Equation (11) fails as $0 \in \pi(\mathcal{B})$ fails. A remedy would be to translate $\pi(\mathcal{B})$ along a vector $(\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^n)^*$ and apply Equation (11) to the translated set. Indeed, if $v_1, \dots, v_n \in V$, then the map in Equation (10) yields

$$\pi_{v_1+\lambda_1 e, \dots, v_n+\lambda_n e}(\mathcal{B}) = \pi_{v_1,\dots,v_n}(\mathcal{B}) + (\lambda_1, \dots, \lambda_n).$$

The translation is unnecessary if we use the conic duality

$$\pi(\mathcal{B}) = \pi_2 \circ \mathrm{h}_{b_0}(C^\vee) \tag{12}$$

of Proposition 5.2, which is valid even if $0 \in \pi(\mathcal{B})$ fails.

The main reason why we employ the conic duality of Equation (12) is that it conveys the algebraic geometry of the dual hyperbolicity cone $C^\vee$ described in Section 4 to the convex set $\pi(\mathcal{B})$ directly. The algebraic geometry of the hyperbolicity cone C is hidden behind a convex duality if we use the Equation (11) to describe the set $\pi(\mathcal{B})$.

Example 5.5. Let $\mathcal{K} = \{(\xi_1, \xi_2)^T \in \mathbb{R}^2 : \xi_1, \xi_2 \geq 0\}$ be the nonnegative quadrant of the plane $V = \mathbb{R}^2$ and let $e = (1,1)^T \in V$. The segment

$$\mathcal{B} = \mathrm{h}_e(\mathcal{K}^\vee) = \big[(0,1),(1,0)\big]$$

is a base of the dual convex cone $\mathcal{K}^\vee = \{z^T : z \in \mathcal{K}\}$. Let $v = (2, \frac{1}{3})^T \in V$. The image of $\mathcal{B}$ under the map $\pi : V^* \to \mathbb{R}^*$, $z \mapsto z(v)$ is the interval $[\frac{1}{3}, 2]$, which does not contain the origin, $0 \notin \pi(\mathcal{B})$. We visualize Corollary 5.3, Equation (12), and the failure of Equation (11) in Figure 4.

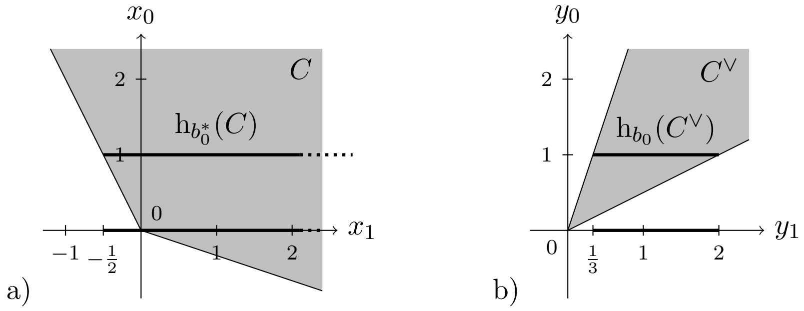


FIGURE 4. a) The projection of the affine slice $h_{b_0^*}(C)$ of the convex cone C to the x_1-axis is the interval $[-\frac{1}{2},\infty) = [\frac{1}{3},2]^\circ$. b) The projection of the base $h_{b_0}(C^\vee)$ of the dual convex cone $C^\vee$ to the y_1-axis is the interval $\pi(\mathcal{B}) = [\frac{1}{3},2] \neq [-\frac{1}{2},\infty)^\circ$.

We formulate the results from Section 4 in terms of bases of cones, taking the convex cone $\mathcal{K} \subset V$ in Definition 5.1 equal to a hyperbolicity cone.

Theorem 5.6. *Let p be a hyperbolic polynomial on V with hyperbolicity direction $e \in V$ and hyperbolicity cone $C_e(p) \subset V$. The image of the compact convex base $h_e(C_e(p)^\vee)$ of the dual convex cone $C_e(p)^\vee$ under the linear map π is*

$$\pi\Big(\, h_e\left(C_e(p)^\vee\right)\Big) = \mathrm{conv}\left(\, \mathrm{clos}(T_1 \cup T_2 \cup \cdots \cup T_r)\right).$$

Here, $p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}$ is a factorization of the pullback $p\circ\phi_0 \in \mathbb{R}[x_0, x_1, \ldots, x_n]$ into irreducible polynomials, and

$$T_i = \{(y_1,\ldots,y_n) \in (\mathbb{R}^n)^* \mid (1 : y_1 : \cdots : y_n) \in (\mathcal{V}(p_i)^*)_{\mathrm{reg}}\}, \quad i = 1,\ldots,r.$$

Proof. The pullback $p \circ \phi_0$ is a hyperbolic polynomial on $\mathbb{R} \oplus \mathbb{R}^n$ with hyperbolicity direction b_0. As $\phi_0((x_0,x)^T) = x_0 e + \phi(x)$ holds for all points $(x_0,x)^T \in \mathbb{R} \oplus \mathbb{R}^n$, we have $p \circ \phi_0(b_0) = p(e) > 0$. The equation

$$p \circ \phi_0(tb_0 - a) = p(te - \phi_0(a)) \tag{13}$$

shows that the polynomial $p\circ\phi_0(tb_0 - a)$ in one variable t has only real roots for every point $a \in \mathbb{R} \oplus \mathbb{R}^n$ so that $p \circ \phi_0$ is indeed hyperbolic with respect to the point b_0. Equation (13) also shows

$$C_{b_0}(p \circ \phi_0) = \phi_0^{-1}\left(C_e(p)\right).$$

Using the hyperbolicity cone $\mathcal{K} = C_e(p)$ in Proposition 5.2, we get

$$\pi\Big(\, h_e\left(C_e(p)^\vee\right)\Big) = \pi_2 \circ h_{b_0}\left(C_{b_0}(p \circ \phi_0)^\vee\right).$$

If $p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r} = p \circ \phi_0$ is a factorization into irreducible polynomials, then

$$C_{b_0}(p \circ \phi_0) = C_{b_0}(p_1) \cap C_{b_0}(p_2) \cap \cdots \cap C_{b_0}(p_r)$$

holds. Theorem 4.5 proves $C_{b_0}(p_i)^\vee = \mathrm{clos}(\mathrm{cc}(C_i))$, $i = 1,\ldots,r$, where

$$C_i = \Big\{(y_0,y_1,\ldots,y_n) \in (\mathbb{R}\oplus\mathbb{R}^n)^* \mid (y_0 : y_1 : \cdots : y_n) \in (\mathcal{V}(p_i)^*)_{\mathrm{reg}}, y_0 \geq 0\Big\}.$$

Now Equation (7) in Proposition 3.5 yields

$$h_{b_0}\Big(C_{b_0}(p\circ\phi_0)^\vee\Big) = \mathrm{conv}\,\Big(\,\mathrm{clos}\left(\, h_{b_0}(C_1) \cup h_{b_0}(C_2) \cup \cdots \cup h_{b_0}(C_r)\right)\Big).$$

The claim follows since the map $\pi_2 : (\mathbb{R}\oplus\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$ restricts to an affine isomorphism from $\mathrm{h}_{b_0}((\mathbb{R}\oplus\mathbb{R}^n)^*)$ onto $(\mathbb{R}^n)^*$, and since $\pi_2 \circ \mathrm{h}_{b_0}(C_i) = T_i$ holds for $i = 1, \dots, r$. $\square$

Like in Corollary 4.14, some of the factors of the pullback $p \circ \phi_0 = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$ are redundant in Theorem 5.6. Let $I \subset \{1, 2, \dots, r\}$ be the subset such that $i \in I$ if and only if the hypersurface $\mathcal{V}(p_i)$ belongs to the algebraic boundary $\partial_a C_{b_0}(p \circ \phi_0)$ of the hyperbolicity cone $C_{b_0}(p \circ \phi_0)$ for $i = 1, \dots, r$. Then $\pi(\mathcal{B}) = \operatorname{conv}\Big(\operatorname{clos}\big(\bigcup_{i\in I} T_i\big)\Big)$.

6. The Cone of Positive-Semidefinite Matrices

Here we prove the theorems stated in the introduction by applying our results to the hyperbolicity cone of positive semi-definite matrices. We discuss block diagonalization *versus* factorization of the determinant. We present examples, among them the promised example by Chien and Nakazato.

Let $V = H_d$ be the space of Hermitian $d \times d$ matrices. The determinant is a hyperbolic polynomial on V with respect to any positive definite matrix. The hyperbolicity cone is the set of positive semi-definite matrices

$$\mathcal{K} = C_{\mathbb{1}}(\det) = \{A \in H_d : A \succeq 0\}.$$

This convex cone is a self-dual convex cone, i.e. $\mathcal{K}^\vee = \mathcal{K}$, as we identify the dual space V^* with V using the scalar product $\langle A, B\rangle = \operatorname{tr}(AB)$, $A, B \in H_d$. In the notation of Definition 5.1, the base $\mathrm{h}_{\mathbb{1}}(\mathcal{K}^\vee)$ of $\mathcal{K}^\vee$ is the set of density matrices introduced in Equation (1),

$$\mathcal{B} = \mathrm{h}_{\mathbb{1}}(\mathcal{K}^\vee) = \{\rho \in H_d \colon \rho \succeq 0, \operatorname{tr}(\rho) = 1\}.$$

Let $A_1, \dots, A_n \in H_d$ be Hermitian matrices and let $\pi : H_d \to (\mathbb{R}^n)^*$ be the linear map defined by $\pi(A) = \langle A, A_i\rangle_{i=1}^n$ for all $A \in H_d$. The image of the set $\mathcal{B}$ under π is the joint numerical range defined in Equation (2),

$$W = \pi(\mathcal{B}) = \{\langle \rho, A_i\rangle_{i=1}^n : \rho \in \mathcal{B}\}.$$

The dual map $\phi = \pi^*$ has the values $\phi(x) = x_1 A_1 + \dots + x_n A_n$ and the map $\phi_0 : \mathbb{R}\oplus\mathbb{R}^n \to V$, see Definition 5.1, has the values $\phi_0[(x_0, x)^T] = x_0\mathbb{1} + \phi(x)$ for all $x_0 \in \mathbb{R}$ and $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. Hence, the pullback $\det\circ\phi_0$ takes the form

$$p = \det\circ\phi_0 = \det(x_0\mathbb{1} + x_1A_1 + \dots + x_nA_n) \in \mathbb{R}[x_0, x_1, \dots, x_n]. \tag{14}$$

With this preparation, Theorem 1.2 follows from Theorem 5.6 immediately. To obtain Kippenhahn's original result, as stated in Theorem 1.1, we use Corollary 4.10 instead of Corollary 4.7 in the proof of Theorem 5.6.

Remark 6.1. In the matrix setting of this section, the convex cone $\phi_0^{-1}(\mathcal{K})$ is the *spectrahedral cone* [46]

$$\{x \in \mathbb{R}^{n+1} : x_0\mathbb{1} + x_1A_1 + \dots + x_nA_n \succeq 0\},$$

which is the hyperbolicity cone $C_{b_0}(p)$ of the polynomial in Equation (14), as we observed more generally in the proof of Theorem 5.6 above.

Researchers have studied the relationships between the possibility to block diagonalize the Hermitian matrices $A_1, \dots, A_n \in H_d$ simultaneously and to factor the determinant p in Equation (14). The matrices $A_1, \dots, A_n$ are called *unitarily reducible* if there is a d-by-d unitary U and an integer $0 < j < d$ such that $UA_iU^* = A_{i,1} \oplus A_{i,2}$ is a block diagonal matrix, where $A_{i,1} \in H_j$ and $A_{i,2} \in H_{d-j}$ for all $i = 1, \dots, n$. Otherwise the matrices $A_1, \dots, A_n$ are *unitarily irreducible.*

Clearly, the polynomial $p \in \mathbb{R}[x_0, x_1, x_2]$ is the product of the polynomials corresponding to the two diagonal blocks for every pair of unitarily reducible matrices $A_1, A_2 \in H_d$. Interestingly, p may even factor when the matrices A_1, A_2 are unitarily irreducible.

Remark 6.2 (Kippenhahn's conjecture)**.** Kippenhahn [38, Prop. 28a] conjectured that if p has a repeated factor, then the matrices A_1, A_2 are unitarily reducible. Laffey [39] found a first counterexample to the conjecture by presenting a pair of unitarily irreducible hermitian 8-by-8 matrices $A_1, A_2 \in H_8$, where p is the square of a polynomial of degree four. The paper [9] summarizes more recent work on the topic and shows that every pair of hermitian 6-by-6 matrices $A_1, A_2 \in H_6$ is unitarily reducible if p is the square of a polynomial of degree three that defines a smooth cubic.

Similarly, one may ask whether the polynomial $p \in \mathbb{R}[x_0, x_1, x_2]$ can be a product of several irreducible factors of multiplicity one if the hermitian matrices $A_1, A_2 \in H_d$ are unitarily irreducible. Such questions have been studied in detail by Kerner and Vinnikov ([36, 37]), resulting in a number of necessary and sufficient conditions. For example, the union of three lines in the projective plane admits a unitarily irreducible determinantal representation (see [36, Remark 3.6]).

Here is a simple example demonstrating that the dual varieties to the irreducible components of the hypersurface $\mathcal{V}(p)$ may have different dimensions.

Example 6.3. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The irreducible components of $\mathcal{V}(p)$ are $X_1 = \{x \in \mathbb{P}^3 : x_1^2 + x_2^2 + x_3^2 = x_0^2\}$ and $X_2 = \{x \in \mathbb{P}^3 : x_0 + 2x_1 = 0\}$. The dual varieties are $X_1^* = \{y \in (\mathbb{P}^3)^* : y_1^2 + y_2^2 + y_3^2 = y_0^2\}$, by Example 3.8, and $X_2^* = \{(1 : 2 : 0 : 0)\}$. Both X_1^* and X_2^* are smooth, so $T_1 = \{y \in (\mathbb{R}^3)^* : y_1^2 + y_2^2 + y_3^2 = 1\}$ and $T_2 = \{(2, 0, 0)\}$. The joint numerical range $W = \operatorname{conv}(T_1 \cup T_2)$ is the convex hull of a sphere and a point outside the sphere.

In our last example, we discuss the original example given by Chien and Nakazato from the point of view of real algebraic geometry in detail. The example explains why singular points of the dual varieties $\mathcal{V}(p_i)^*$ have to be excluded from the statement of Theorem 5.6.

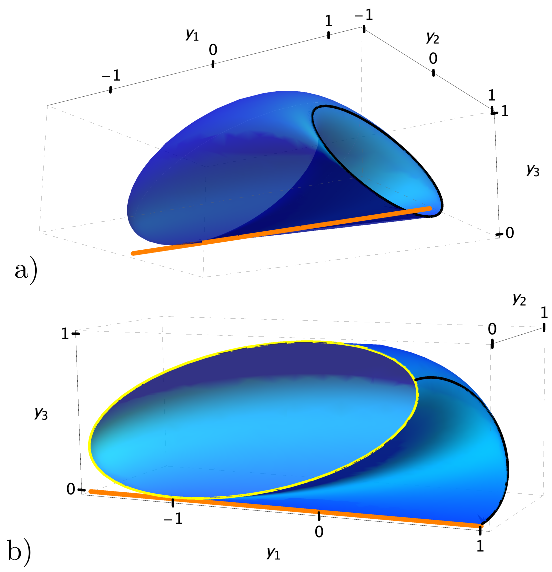


FIGURE 5. Pictures for Example 6.4: a) Surface T (blue), singular locus y_1-axis (orange), and ellipse on the boundary of the joint numerical range (black). b) The hyperplane $y_2 = 0$ intersects T in the union of an ellipse (yellow) and the y_1-axis (orange).

Example 6.4 (Chien and Nakazato [14]). Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The hyperbolic cubic form

$$\begin{aligned} p &= \det(x_0\mathbb{1} + x_1A_1 + x_2A_2 + x_3A_3) \\ &= x_0^3 + x_0^2x_3 - 2x_0x_1^2 - x_0x_2^2 - x_1^3 - x_1^2x_3 + x_1x_2^2 \end{aligned}$$

is irreducible. The dual variety to $X = \{x \in \mathbb{P}^3 : p(x) = 0\}$ is the hypersurface $X^* = \{y \in (\mathbb{P}^3)^* : q(y) = 0\}$ defined by the homogeneous quartic form

$$\begin{aligned} q = {} & 4y_0^2y_3^2 + 8y_0y_1y_3^2 - 4y_0y_2^2y_3 - 24y_0y_3^3 + 4y_1^2y_3^2 - 4y_1y_2^2y_3 - 8y_1y_3^3 \\ & + y_2^4 + 8y_2^2y_3^2 + 20y_3^4. \end{aligned}$$

It is easy to show that the singular locus $X^* \setminus (X^*)_{\text{reg}}$ of X^* is the line $\{(y_0, y_1, y_2, y_3) \in (\mathbb{P}^3)^* : y_2 = y_3 = 0\}$.

The surface $T = \{(y_1, y_2, y_3) \in (\mathbb{R}^3)^* \mid (1 : y_1 : y_2 : y_3) \in X^*\}$, depicted in Figure 5, is the zero-set of the polynomial $Q(y_1, y_2, y_3) = q(1, y_1, y_2, y_3)$. All points $(y_1, 0, 0) \in T$ on the line of singular points $y_2 = y_3 = 0$ in the interval $|y_1| \leq 1$ are central points of X^*. This can be seen from a parametrization of X described in [52]. In addition, the points $(y_1, 0, 0) \in T$ with $|y_1| > 1$ are not central. This can be seen from the roots of the quadratic polynomial $R_{y_1,y_3} \in \mathbb{R}[y_2]$ satisfying $R_{y_1,y_3}(y_2^2) = Q(y_1, y_2, y_3)$. In fact, if $(y_1, y_3) \in (\mathbb{R}^2)^*$, then the line $\{(y_1, \lambda, y_3) \colon \lambda \in \mathbb{R}\}$ intersects T if and only if R_{y_1,y_3} has a non-negative root y_2. It is not hard to see (for example using Sturm's theorem [7]) that, under the assumption $y_3 \neq 0$, the polynomial R_{y_1,y_3} has a non-negative root if and only if (y_1, y_3) lies in the convex hull of the union of the singleton $\{(1, 0, 0)\}$ with the ellipse given by $y_1^2 + 5y_3^2 - 2y_1y_3 + 2y_1 - 6y_3 + 1 = y_2 = 0$, the yellow curve in Figure 5 b). Therefore, the points $(y_1, 0, 0)$ with $|y_1| > 1$ are not central. These are the points which are removed from $X^*(\mathbb{R})$ in the statement of Theorem 1.2. The convex hull of the remainder of T is the joint numerical range W.

For a general, regular point $(y_0 : y_1 : y_2 : y_3)$ on X^*, the dual hyperplane in $\mathbb{P}^3$ defined by $x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0$ intersects X in an irreducible but singular cubic. The hyperplane sections of the hypersurface X corresponding to points of the line of singular points of X^* are special in the following way: The sections are reducible curves in $\mathbb{P}^2$ that factor into a conic and a line, which is tangent to the conic at a real point. Not all of the points on this line are central points of X^*. The central points of X^* on this line are exactly those that we can perturb such that the dual hyperplane section of X deforms to a real cubic with a real singularity. The other points on this line can only be perturbed on X^* to complex points, which means that the dual hyperplane section of X only deforms to complex singular hyperplane sections that are complex cubics with complex singularities.

To see this explicitly, we compute the restrictions of p to hyperplanes in $\mathbb{P}^3$ defined by $x_0y_0 + x_1y_1 = 0$. Let us assume for simplicity for now that $y_0 \neq 0$ so that $x_0 = ax_1$ for some $a \in \mathbb{R}$. Then p factors

$$p(ax_1, x_1, x_2, x_3) = x_1 \left((a^3 - 2a - 1)x_1^2 + (1 - a)x_2^2 + (a^2 - 1)x_1x_3 \right).$$

So in the plane defined by $x_0y_0 + x_1y_1 = 0$, we see the conic

$$(a^3 - 2a - 1)x_1^2 + (1 - a)x_2^2 + (a^2 - 1)x_1x_3 = 0$$

and the line defined by $x_0 = x_1 = 0$, which is tangent to the conic at the point $(0 : 0 : 0 : 1)$, which is a singular point of X. (The computation for the case $y_0 = 0$ is similar.) For every real value of the parameter a, the conic is real and indefinite of full rank except for $a = -1$ and $a = 1$. These two values of a bound the interval of central points on the line of singular points on X^*; see Figure 5.

Section 6 of [56] presents further examples of surfaces analogous to the surface T in Example 6.4 that contain straight lines. These lines are identified in the paper `arXiv:1603.06569v3 [math.FA]` on the preprint server arXiv.

So why do naive generalizations of Kippenhahn's Theorem fail in higher dimensions? The reason is that hyperplanes and lines are only the same in $\mathbb{P}^2$. A central point of the argument in the proof of Kippenhahn's Theorem is Lemma 4.2, which argues that there are no lines through the interior of the hyperbolicity cone that are tangent to the hyperbolic hypersurface (Corollary 4.3). The hyperplane corresponding to a real point of the dual variety may well meet an interior point of the hyperbolicity cone as long as the hyperplane contains no tangent line to the hyperbolic hypersurface incident with this interior point. Our generalization holds essentially because this can only happen for hyperplanes that are not central points of the dual variety.

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Daniel Plaumann
Technische Universität Dortmund
Fakultät für Mathematik
Vogelpothsweg 87
44227 Dortmund
Germany
e-mail `Daniel.Plaumann@tu-dortmund.de`

Rainer Sinn
Freie Universität Berlin
Fachbereich Mathematik und Informatik
Arnimallee 2
14195 Berlin
Germany
e-mail `rainer.sinn@fu-berlin.de`

Stephan Weis
Centro de Matemática da Universidade de Coimbra
Apartado 3008
EC Santa Cruz
3001 - 501 Coimbra
Portugal
e-mail `maths@weis-stephan.de`