Asymptotic Behavior for Textiles

Georges Griso, Julia Orlik, Stephan Wackerle

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Abstract

The paper is dedicated to the asymptotic investigation of textiles as elasticity problem on beam structures. The structure is subjected to a simultaneous homogenization and dimension reduction with respect to the asymptotic behavior of the beams thickness and periodicity. Important for the problem are the contact conditions between the beams, which yield multiple limits depending on the order. In this paper two limiting cases are presented: a linear and a Leray-Lions-type problem.

Keyword: Homogenization, periodic unfolding method, dimension reduction, linear elasticity, variational inequality, Leray-Lions problem, contact, plates, structure of beams
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1 Introduction

This paper investigates the simultaneous homogenization and dimension reduction of textiles as periodic beam structures, here a woven canvas structure. While textile structures are very complicated if fully resolved, the homogenization takes the local configuration into account and gives rise to a representative homogeneous plate model. For more literature on homogenization we refer to [7, 10, 22] and the references therein. Moreover, since woven textiles are very thin with respect to the in-plane dimensions, an additional dimension reduction further reduces the complexity of the model. For dimension reduction of plates see for instance [4, 5, 6]. The combination of both is part of current investigations [7, Ch. 11] and [16].

For this paper the textile consists of long individual fibers, modeled by beams, being in contact with each other. Due to the contact, which is modeled via a gap-function g_ε, the problem is stated as variational inequality. The applied forces are scaled in order to get $\|e(u_\varepsilon)\|_{L^2}$ of order $\varepsilon^{5/2}$. Although linear elasticity is assumed and the total energy remains in the linear regime, the order of the contact will determine the limit. This corresponds to the fact that the contact characterizes the stability of the structure. Hereafter we consider only two cases, namely $g_\varepsilon \sim \varepsilon^4$ and $g_\varepsilon \sim \varepsilon^3$ leading to a typical full linear problem or to a Leray-Lions problem (see [12, 19]) respectively. Another case, though, assumes $g_\varepsilon \sim \varepsilon^2$, it will be presented in a forthcoming paper. This last case is out of scope for this paper, yet it may be very interesting to model for instance shearing of textiles with loose weave, see e.g. [2, 20, 23, 24].

The strategy in this paper is to use the displacements of every single beam to compose, by extension, global displacement fields of the whole structure. Then Korn-estimates on all fields and the unfolding operator give rise to a general limit problem. Finally, the existence of solutions in the case of the Leray-Lions problem is shown and additionally for the linear case the uniqueness is deduced.

Specifically, we start with a single periodically curved beam of radius r and periodicty ε. With the help of the decomposition of didsplacements, which yields elementary and residual displacements for the beam, some basic results are shown. For the general definition and properties of this decomposition see [7, 13, 14, 15]. The resulting elementary displacements are further modified to account for the curved behavior and to simplify estimates on the full structure. These general results for one beam are transferred on the whole textile structure and global fields are introduced, which are defined on the 2D mid-plane of the limit plate. The definition of the global and local displacement fields is similar to the method of the scale-splitting operators in [11], where the Q_1-interpolation is used to obtain global fields. For all fields, local and global, Korn-like estimates are established. Here the contact condition plays an important role, as they give rise to better estimations by linking the different displacements coming from the beams. However, the estimations already show the plate character when comparing the displacements to the ones in [1, 7, 15].

The main tool for the homogenization and dimension reduction is the unfolding operator, which was introduced in [11] and further developed in [7, 8]. Note that the weak convergence in the unfolding method is equivalent to the typical two-scale convergence in homogenization [11, 21]. For the dimension reduction of

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plates we refer to [1, 3, 4, 5, 7] as it is common asymptotic model. However, for the textile the simultaneous homogenization and dimension reduction is achieved by a combination of the unfolding operator and the rescaling operator (for the latter see [7, 15]). It is important to note that we consider the limit of both simultaneously, since it is known that homogenization and dimension reduction do not commute, see [7, Chapter 11]. Specifically, we introduce an adapted unfolding operator for this textiles structure and show fundamental properties. This allows to derive the unfolded limits of displacements and gradients as well as the strain tensor. To capture the unfolded limit contact condition it is necessary to modify and restrict the unfolding operator, similar to the boundary unfolding operator, see [7, 9, 17].

The end of the paper is dedicated to the derivation and investigation of the limit problem. To obtain the limit variational inequality specific test-functions are defined, which always satisfy the contact for every ε. For the limit problem two different orders of the contact are investigated, namely $g_\varepsilon \sim \varepsilon^3$ and $g_\varepsilon \sim \varepsilon^4$. For both cases the homogenized problem is a variational equality, but for $g_\varepsilon \sim \varepsilon^3$ the cell problems remain variational inequalities yielding a Leray-Lions problem. Though, in the linear case, which corresponds to $g_\varepsilon \sim \varepsilon^4$, the cell problems degenerate to variational equalities. The difference between these two cases is also obvious in the structure of correctors, where the Leray-Lions problem has an additional corrector accounting for the nonlinearity coming from the contact. Eventually, the existence of solutions for the Leray-Lions problem is shown, but due to the lack of strict monotonicity of the homogenized operator the uniqueness is not provable. However, in the full linear setting both is easily shown.

In this paper, all the constants in the estimates are independent of ε.

2 Notations

- $\Omega = (0, L)^2$,
- Choose ε such that $2\varepsilon N_\varepsilon = L$, $N_\varepsilon \in \mathbb{N}^*$. Hence no cell intersects the boundary and Λ_ε is a null-set,
- $\mathcal{K}_\varepsilon = \Big\{(p, q) \in \mathbb{N}^* \times \mathbb{N}^* \;\mid\; (p\varepsilon, q\varepsilon) \in \overline{\Omega}\Big\} = \{0, \dots, 2N_\varepsilon\}^2$ the set of knots,
- $\dot{\mathcal{K}}_\varepsilon = \big\{(p, q) \mid (p, q) \in \{1, \dots, 2N_\varepsilon - 1\} \times \{1, \dots, 2N_\varepsilon - 1\}\big\}$,
- κ a fixed constant belonging to $(0, 1/3]$ and $r = \kappa\varepsilon$; for simplicity, sometimes we will write r instead of $\kappa\varepsilon$,
- $\omega_\kappa = (-\kappa, \kappa)^2$ the reference beam cross-section and $\omega_r = (-r, r)^2 = (-\kappa\varepsilon, \kappa\varepsilon)^2$ the rescaled cross-section,
- $\mathcal{U}_i = \mathcal{U} \cdot \mathbf{e}_i$, $\mathcal{R}_i = \mathcal{R} \cdot \mathbf{e}_i$, $i = 1, 2, 3$,
- $d_i = \dfrac{d}{dz_i}$, $\partial_i = \dfrac{\partial}{\partial z_i}$, $d_{X_i} = \dfrac{d}{dX_i}$, $\partial_{X_i} = \dfrac{\partial}{\partial X_i}$.

3 Preliminary Results: Curved Beams

3.1 The parameterization of the curved beam

To describe the reference domain, define the 2-periodic function

$$\Phi(z) = \begin{cases} -\kappa, & \text{if } z \in [0, \kappa], \\ \kappa\Big(6\frac{(z-\kappa)^2}{(1-2\kappa)^2} - 4\frac{(z-\kappa)^3}{(1-2\kappa)^3} - 1\Big) & \text{if } z \in [\kappa, 1-\kappa], \\ \kappa & \text{if } z \in [1-\kappa, 1], \\ \Phi(2-z) & \text{if } z \in [1, 2]. \end{cases} \tag{3.1}$$

Then, rescaling $\Phi_\varepsilon(z) = \varepsilon\Phi(\frac{z}{\varepsilon})$ gives the oscillation of the middle line. This function is piecewise in $\mathcal{C}^2(\mathbb{R})$ and overall in $\mathcal{C}^1(\mathbb{R})$. Note that by definition this function satisfies

$$\|\Phi_\varepsilon\|_{L^\infty(0,2\varepsilon)} \le C\varepsilon \qquad \|\Phi'_\varepsilon\|_{L^\infty(0,2\varepsilon)} \le C. \tag{3.2}$$

Dealing with curved beams, we consider the centerline of the beam parameterized by the function

$$M_\varepsilon(z_1) = z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad z_1 \in [0, L],$$

which is obviously a beam with $\mathbf{e}_1$ as 'mean'-direction with an oscillation in the third direction. Its arc length s_1 is given by

$$s_1(0) = 0, \qquad \boldsymbol{\gamma}_\varepsilon(z_1) \doteq \frac{ds_1}{dz_1}(z_1) = \sqrt{1 + \big[\Phi'_\varepsilon(z_1)\big]^2}, \qquad z_1 \in [0, L]. \tag{3.3}$$

The beam is referred to the corresponding Frenet-Serret frame $\big(\mathbf{t}_\varepsilon(z_1), \mathbf{e}_2, \mathbf{n}_\varepsilon(z_1)\big)$, $z_1 \in [0, L]$ where

$$\mathbf{t}_\varepsilon = \frac{dM_\varepsilon}{ds_1} = \frac{1}{\gamma_\varepsilon}\big(\mathbf{e}_1 + \Phi_\varepsilon' \mathbf{e}_3\big) \qquad \text{and} \qquad \mathbf{n}_\varepsilon = \mathbf{t}_\varepsilon \wedge \mathbf{e}_2 = \frac{1}{\gamma_\varepsilon}\big(- \Phi_\varepsilon' \mathbf{e}_1 + \mathbf{e}_3\big),$$
$$\frac{d\mathbf{t}_\varepsilon}{ds_1} = c_\varepsilon \mathbf{n}_\varepsilon, \qquad \frac{d\mathbf{t}_\varepsilon}{dz_1} = c_\varepsilon \gamma_\varepsilon \mathbf{n}_\varepsilon, \qquad \frac{d\mathbf{n}_\varepsilon}{ds_1} = -c_\varepsilon \mathbf{t}_\varepsilon, \qquad \frac{d\mathbf{n}_\varepsilon}{dz_1} = -c_\varepsilon \gamma_\varepsilon \mathbf{t}_\varepsilon,$$

where $c_\varepsilon = \dfrac{\Phi_\varepsilon''}{\gamma_\varepsilon^3}$ is the curvature. The vector fields $\mathbf{t}_\varepsilon$ and $\mathbf{n}_\varepsilon$ belong to $\mathcal{C}^1([0, L]; \mathbb{R}^3)$, c_ε is piecewise continuous. The cross-section of the beam is the square ω_r. Together this results in two sets in which the beams can be expressed

$$P_r \doteq (0, L) \times \omega_r \quad \text{and} \quad \mathcal{P}_\varepsilon \doteq \Big\{ x \in \mathbb{R}^3 \;\Big|\; x = \psi_\varepsilon(z) = M_\varepsilon(z_1) + z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon(z_1), \quad z \in P_r \Big\}.$$

The first one being a straight reference beam and the second the corresponding curved one. The function ψ_ε is the transition map from the straight to the curved beams, i.e. formally we say $\mathcal{P}_\varepsilon = \psi_\varepsilon(P_r)$. Consequently, this also results in two frames in which the beam is referred to: first one is fixed $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and the second one $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$ is mobile.

Note that, although the parametrization with respect to the arc-length (3.3) has some advantages, we choose here to parametrize the beams with respect to the length of the textile $(0, L)$ instead, to simplify the limiting behavior.

3.2 The decomposition of displacement

Let u be a displacement in $H^1(\mathcal{P}_\varepsilon; \mathbb{R}^3)$; we recall the decomposition obtained in [15, Theorem 3.1] and in [14, Lemma 3.2]

$$u(x) = \mathcal{U}(z_1) + \mathcal{R}(z_1) \wedge (z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon(z_1)) + \overline{u}(z), \quad \text{for a.e. } x = \psi_\varepsilon(z) \in \mathcal{P}_\varepsilon, \;\; z \in P_r \tag{3.4}$$

where $\mathcal{U}$ and $\mathcal{R}$ belong to $H^1(0, L; \mathbb{R}^3)$ and $\overline{u} \in H^1(\mathcal{P}_\varepsilon; \mathbb{R}^3)$. The warping $\overline{u}$ satisfies for a.e. $z_1 \in (0, L)$ (see [15])

$$\int_{\omega_r} \overline{u}(z_1, z_2, z_3) dz_2 dz_3 = 0, \qquad \int_{\omega_r} \overline{u}(z_1, z_2, z_3) \wedge \big(z_2 \mathbf{e}_2 + z_3 \mathbf{n}_\varepsilon(z_1)\big) dz_2 dz_3 = 0. \tag{3.5}$$

Remark 3.1. *Consider the functional determinant* $\det(\nabla \psi_\varepsilon)$ *and define*

$$\eta_\varepsilon(z) \doteq \det\big(\nabla \psi_\varepsilon(z)\big) = \gamma_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big), \qquad \forall z \in \overline{P_r}$$

where c_ε is the curvature.

Furthermore, there exists a constant $\widehat{\kappa} \in (0, 1/3]$ depending on the curvature of the parametrization Φ such that for $\kappa \le \widehat{\kappa}$ the transformation ψ_ε from P_r onto $\mathcal{P}_\varepsilon$ is a diffeomorphism with

$$\nabla \psi_\varepsilon = \big(\eta_\varepsilon \mathbf{t}_\varepsilon | \mathbf{e}_2 | \mathbf{n}_\varepsilon\big) = \mathbf{C}_\varepsilon \begin{pmatrix} \eta_\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \big(\nabla \psi_\varepsilon\big)^{-1} = \Big(\frac{1}{\eta_\varepsilon} \mathbf{t}_\varepsilon | \mathbf{e}_2 | \mathbf{n}_\varepsilon\Big)^T = \mathbf{C}_\varepsilon^T \begin{pmatrix} \frac{1}{\eta_\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\mathbf{C}_\varepsilon = \big(\mathbf{t}_\varepsilon \,|\, \mathbf{e}_2 \,|\, \mathbf{n}_\varepsilon\big)$ and where η_ε is bounded from below and above

$$\frac{1}{C} \le \|\eta_\varepsilon\|_{L^\infty(0,L)} \le C.$$

Corollary 3.2. *Suppose $s \in [1, +\infty]$. There exist two constant C_0, C_1 independent of ε such that for every $\varphi \in L^s(\mathcal{P}_\varepsilon)$*

$$C_0 \|\varphi \circ \psi_\varepsilon\|_{L^s(P_r)} \le \|\varphi\|_{L^s(\mathcal{P}_\varepsilon)} \le C_1 \|\varphi \circ \psi_\varepsilon\|_{L^s(P_r)}.$$

Consequently, we henceforth write indifferently φ in place of $\varphi \circ \psi_\varepsilon$ for all functions.

3.3 Estimates for one beam

For the displacements of a single curved beam we have from [15] the following estimates on the decomposed fields

$$\|\overline{u}\|_{L^2(P_r; \mathbb{R}^3)} \le Cr \|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\nabla \overline{u}\|_{L^2(P_r; \mathbb{R}^{3\times 3})} \le C \|e(u)\|_{L^2(\mathcal{P}_\varepsilon; \mathbb{R}^{3\times 3})}, \tag{3.6}$$
$$\Big\|\frac{d\mathcal{R}}{ds_1}\Big\|_{L^2(0,L)} \le \frac{C}{r^2} \|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \Big\|\frac{d\mathcal{U}}{ds_1} - \mathcal{R} \wedge \mathbf{t}_\varepsilon\Big\|_{L^2(0,L)} \le \frac{C}{r} \|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{3.7}$$

The constants are independent of r (here recall that $2r$ is the thickness of the beam).
In (3.6) we can consider the gradients with respect to both sets of variables (z_1, z_2, z_3) or (s_1, z_2, z_3) and since the Jacobian determinant η_ε of the change of variables is bounded, see Remark 3.1, the estimates only change in the constant.
Similarly, we are led to replace (3.7) by

$$\Big\|\frac{d\mathcal{R}}{dz_1}\Big\|_{L^2(0,L)} \leq \frac{C}{r^2}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \quad \Big\|\frac{d\mathcal{U}}{dz_1} - \mathcal{R}\wedge\frac{dM_\varepsilon}{dz_1}\Big\|_{L^2(0,L)} \leq \frac{C}{r}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{3.8}$$

the constants are independent of r. The above estimates (3.6)-(3.7) and (3.8) have not changed in the order of epsilon due to Corollary 3.2.

Additionally to the above decomposition we define another splitting of the displacement $\mathcal{U}$, cf. (3.4).

Definition 3.3. *The field $\mathbb{U}$ is defined via*

$$\mathcal{U} = \mathbb{U} + \Phi_\varepsilon \mathcal{R}\wedge \mathbf{e}_3. \tag{3.9}$$

The reason to define the additional field $\mathbb{U}$ for the beam is provided in the following Lemma and simplifies estimate $(3.8)_2$.

Lemma 3.4. *The field $\mathbb{U}$ satisfies*

$$\big\|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big\|_{L^2(0,L)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\mathcal{U}_\alpha - \mathbb{U}_\alpha\|_{L^2(0,L)} \leq C\varepsilon\|\mathcal{R}\|_{L^2(0,L)}. \tag{3.10}$$

Proof. Estimates (3.10) are the immediate consequences of the L^∞-norm of Φ_ε , i.e. (3.2), and (3.8).

Indeed, inserting the definition and using the estimates for the remaining parts yields

$$\begin{aligned}\|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\|_{L^2(0,L)} &\leq \Big\|d_1\mathcal{U} - \mathcal{R}\wedge\frac{dM_\varepsilon}{dz_1}\Big\|_{L^2(0,L)} + \|\Phi_\varepsilon d_1\mathcal{R}\wedge\mathbf{e}_3\|_{L^2(0,L)}\\ &\leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)} + \|\Phi_\varepsilon\|_{L^\infty(0,L)}\|d_1\mathcal{R}\|_{L^2(0,L)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.\end{aligned}$$

The second inequality is trivial. □

Note, that there exist discrete versions of the estimates $(3.8)_1$ and (3.10), which are necessary to establish global estimates.

Lemma 3.5. *The fields $\mathcal{R}$ and $\mathbb{U}$ defined above satisfy*

$$\sum_{p=0}^{2N_\varepsilon-1}\big|\mathcal{R}\big((p+1)\varepsilon\big) - \mathcal{R}(p\varepsilon)\big|^2 + \sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big) - \mathbb{U}(p\varepsilon)}{\varepsilon} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\Big|^2 \leq \frac{C}{\varepsilon^3}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}. \tag{3.11}$$

Proof. Consider the left hand side and transform using the fundamental theorem of calculus and the Jensen inequality:

$$\sum_{p=0}^{2N_\varepsilon-1}\big|\mathcal{R}\big((p+1)\varepsilon\big) - \mathcal{R}(p\varepsilon)\big|^2 \leq \varepsilon\sum_{p=0}^{2N_\varepsilon-1}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathcal{R}(z_1)\big|^2 dz_1 \leq \varepsilon\|d_1\mathcal{R}\|^2_{L^2(0,L)}.$$

Then use the estimate (3.8) to conclude the first inequality.
By the same means we obtain

$$\sum_{p=0}^{2N_\varepsilon-1}\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big) - \mathbb{U}(p\varepsilon)}{\varepsilon} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\Big|^2 \leq \frac{1}{\varepsilon}\sum_{p=0}^{2N_\varepsilon-1}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\big|^2 dz_1. \tag{3.12}$$

Additionally, note that by introducing now the function $\mathcal{R}$ we obtain

$$\begin{aligned}\int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U} - \mathcal{R}(p\varepsilon)\wedge\mathbf{e}_1\big|^2 dz &\leq \int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big|^2 dz_1 + \int_{p\varepsilon}^{(p+1)\varepsilon}\big|\mathcal{R} - \mathcal{R}(p\varepsilon)\big|^2 dz_1\\ &\leq \int_{p\varepsilon}^{(p+1)\varepsilon}\big|d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big|^2 dz_1 + \varepsilon^2\int_{p\varepsilon}^{(p+1)\varepsilon}\big|\partial_1\mathcal{R}\big|^2 dz_1,\end{aligned}$$

in every interval $(p\varepsilon, (p+1)\varepsilon)$. The second inequality is an application of the Poincaré inequality. Finally, we conclude the claim by inserting this into (3.12), where the remaining two terms are covered by the estimates (3.8) and (3.10). □

The end of this section is dedicated to another decomposition of the displacements. Specifically, any function can be decomposed into a piecewise linear function and an additional function capturing the remaining higher orders. To do so, note that a function φ defined on the set $\{p\varepsilon \mid p = 1, \dots, 2N_\varepsilon - 1\}$ is easily extended to $\varphi \in W^{1,\infty}$ by linear interpolation. Hence, define the piecewise linear interpolations $\mathcal{R}^{(nod)}, \mathbb{U}^{(nod)} \in W^{1,\infty}$ with the values in the vertices

$$\mathcal{R}^{(nod)}(p\varepsilon) = \mathcal{R}(p\varepsilon) \qquad \text{and} \qquad \mathbb{U}^{(nod)}(p\varepsilon) = \mathbb{U}(p\varepsilon).$$

Then, the original displacements admit the decomposition

$$\mathcal{R}(z) = \mathcal{R}^{(nod)}(z) + \mathcal{R}^{(0)}(z) \qquad \text{and} \qquad \mathbb{U}(z) = \mathbb{U}^{(nod)}(z) + \mathbb{U}^{(0)}(z). \tag{3.13}$$

Here the functions $\mathcal{R}^{(0)}$ and $\mathbb{U}^{(0)}$ capture the high oscillations and are by definition zero on the nodes, i.e. $\mathcal{R}^{(0)}(p\varepsilon) = \mathbb{U}^{(0)}(p\varepsilon) = 0$ for all $p \in 0, \dots, 2N_\varepsilon$.

Lemma 3.6. *The functions $\mathcal{R}^{(0)}$, $\mathbb{U}^{(0)}$, $\mathcal{R}^{(nod)}$ and $\mathbb{U}^{(nod)}$ satisfy for $i = 2, 3$*

$$\begin{aligned}
&\|\mathcal{R}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathcal{R}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathcal{R}^{(nod)}\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\
&\|\mathbb{U}^{(0)}\|_{L^2(0,L)} + \varepsilon\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} \le C\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)},\\
&\|d\mathbb{U}^{(nod)} - \mathcal{R}^{(nod)} \wedge \mathbf{e}_1\|_{L^2(0,L)} + \|d\mathbb{U}^{(0)} - \mathcal{R}^{(0)} \wedge \mathbf{e}_1\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.
\end{aligned} \tag{3.14}$$

Proof. Note that $d\mathcal{R}^{(nod)}$ is constant in every interval $\big(p\varepsilon, (p+1)\varepsilon\big)$ and that $\mathcal{R}^{(0)}$ and $\mathbb{U}^{(0)}$ are zero on the nodes. Thus, $d\mathcal{R}^{(nod)}$ and $d\mathcal{R}^{(0)}$ are orthogonal to each other in the L^2-sense. Indeed, we have

$$\langle d\mathcal{R}^{(nod)}, d\mathcal{R}^{(0)}\rangle_{L^2(p\varepsilon,(p+1)\varepsilon)} = d\mathcal{R}^{(nod)}_p \int_{p\varepsilon}^{(p+1)\varepsilon} d\mathcal{R}^{(0)} dz_1 = 0,$$

where $d\mathcal{R}^{(nod)}_p = \dfrac{\mathcal{R}^{(nod)}((p+1)\varepsilon) - \mathcal{R}^{(nod)}(p\varepsilon)}{\varepsilon}$ is constant. The integral is zero since $\mathcal{R}^{(0)}(p\varepsilon) = \mathcal{R}^{(0)}((p+1)\varepsilon) = 0$. By this orthogonality and summing over all cells we obtain

$$\|d\mathcal{R}^{(nod)}\|^2_{L^2(0,L)} + \|d\mathcal{R}^{(0)}\|^2_{L^2(0,L)} = \|d\mathcal{R}\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^4}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

Then, the Poincaré-inequality yields

$$\|\mathcal{R} - \mathcal{R}^{(nod)}\|_{L^2(0,L)} = \|\mathcal{R}^{(0)}\|_{L^2(0,L)} \le \varepsilon\|d\mathcal{R}^{(0)}\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.$$

For the estimate $(3.14)_2$, similar considerations lead first to

$$\|d\mathbb{U}^{(nod)}_1\|^2_{L^2(0,L)} + \|d\mathbb{U}^{(0)}_1\|^2_{L^2(0,L)} = \|d\mathbb{U}_1\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.$$

Then, it is easy to obtain

$$\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} \le \|d\mathbb{U} - \mathcal{R} \wedge \mathbf{e}_1\|_{L^2(0,L)} + \|d\mathbb{U}^{(nod)} - \mathcal{R}^{(nod)} \wedge \mathbf{e}_1\|_{L^2(0,L)} + \|\mathcal{R} - \mathcal{R}^{(nod)}\|_{L^2(0,L)}$$

which together with

$$\begin{aligned}
\|d\mathbb{U}^{(nod)} - \mathcal{R}^{(nod)} \wedge \mathbf{e}_1\|^2_{L^2(0,L)} \le \sum_{p=0}^{2N_\varepsilon - 1} \varepsilon\Big|\frac{\mathbb{U}\big((p+1)\varepsilon\big) - \mathbb{U}(p\varepsilon)}{\varepsilon} - \mathcal{R}(p\varepsilon) \wedge \mathbf{e}_1\Big|^2&\\
+ C\varepsilon^2\|d\mathcal{R}\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e(u)\|^2_{L^2(\mathcal{P}_\varepsilon)}.&
\end{aligned} \tag{3.15}$$

yields

$$\|\mathbb{U}^{(0)}\|_{L^2(0,L)} \le \varepsilon\|d\mathbb{U}^{(0)}\|_{L^2(0,L)} \le C\|e(u)\|_{L^2(\mathcal{P}_\varepsilon)}.$$

The last estimate in (3.14) is a consequence of (3.15) as well. $\square$

3.4 Symmetric gradient for one beam

The gradient with respect to the set of variables (z_1, z_2, z_3) of the whole displacement u is split

$$\nabla_z u = \nabla_z U^e + \nabla_z \overline{u},$$

with the elementary displacement $U^e = u - \overline{u}$ and the warping $\overline{u}$. First, consider only the gradient of the elementary displacement:

$$\nabla_z U^e = \big(\partial_{z_1} U^e \mid \partial_{z_2} U^e \mid \partial_{z_3} U^e\big) = \Big(d_1\mathcal{U} + d_1\mathcal{R} \wedge \big(z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon\big) - z_3 c_\varepsilon \gamma_\varepsilon \mathcal{R} \wedge \mathbf{t}_\varepsilon \;\Big|\; \mathcal{R} \wedge \mathbf{e}_2 \;\Big|\; \mathcal{R} \wedge \mathbf{n}_\varepsilon\Big).$$

Obviously, this is in the local coordinate system of the parametrized beam $\mathcal{P}_\varepsilon$, which is not sufficient for the problem, where the Cartesian system of the composed textile is needed.

The transition between the reference systems comes by the change of variables and basis. First, one has $\nabla_z u = \nabla_x u \nabla \psi_\varepsilon$. Hence

$$\mathbf{C}_\varepsilon^T \, \nabla_x u \, \mathbf{C}_\varepsilon = \mathbf{C}_\varepsilon^T \, \nabla_z u \begin{pmatrix} \frac{1}{\eta_\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{t}_\varepsilon & \frac{\partial u}{\partial z_2} \cdot \mathbf{t}_\varepsilon & \frac{\partial u}{\partial z_3} \cdot \mathbf{t}_\varepsilon \\ \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{e}_2 & \frac{\partial u}{\partial z_2} \cdot \mathbf{e}_2 & \frac{\partial u}{\partial z_3} \cdot \mathbf{e}_2 \\ \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{n}_\varepsilon & \frac{\partial u}{\partial z_2} \cdot \mathbf{n}_\varepsilon & \frac{\partial u}{\partial z_3} \cdot \mathbf{n}_\varepsilon \end{pmatrix}.$$

Recall that $e_x(u) = \frac{1}{2}\Big(\nabla_x u + (\nabla_x u)^T\Big)$ and define the symmetric tensor $e_z(u)$ by

$$e_z(u) = \mathbf{C}_\varepsilon^T \, e_x(u) \, \mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\big(\frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{e}_2 + \frac{\partial u}{\partial z_2} \cdot \mathbf{t}_\varepsilon\big) & \frac{\partial u}{\partial z_2} \cdot \mathbf{e}_2 & * \\ \frac{1}{2}\big(\frac{1}{\eta_\varepsilon}\frac{\partial u}{\partial z_1} \cdot \mathbf{n}_\varepsilon + \frac{\partial u}{\partial z_3} \cdot \mathbf{t}_\varepsilon\big) & \frac{1}{2}\big(\frac{\partial u}{\partial z_2} \cdot \mathbf{n}_\varepsilon + \frac{\partial u}{\partial z_3} \cdot \mathbf{e}_2\big) & \frac{\partial u}{\partial z_3} \cdot \mathbf{n}_\varepsilon \end{pmatrix}. \tag{3.16}$$

Now, we change the notation of the symmetric strain tensor. Due to the symmetry of this tensor, it can be written as a vector with six entries. Write

$$E_x(u) = \Big(e_{x,11}, e_{x,22}, e_{x,33}, \sqrt{2}e_{x,12}, \sqrt{2}e_{x,13}, \sqrt{2}e_{x,23}\Big)^T,$$
$$E_z(u) = \Big(e_{z,11}, e_{z,22}, e_{z,33}, \sqrt{2}e_{z,12}, \sqrt{2}e_{z,13}, \sqrt{2}e_{z,23}\Big)^T.$$

There exists a matrix $\widetilde{\mathbf{C}}_\varepsilon \in C^1(\overline{\mathcal{P}_\varepsilon})^{(6\times 6)}$ such that

$$E_z(u) = \widetilde{\mathbf{C}}_\varepsilon \, E_x(u) \quad \text{where} \quad \widetilde{\mathbf{C}}_\varepsilon = \begin{pmatrix} \frac{1}{\gamma_\varepsilon^2} & 0 & \frac{(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 & \frac{\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 & \frac{1}{\gamma_\varepsilon^2} & 0 & \frac{-\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_\varepsilon} & 0 & \frac{\Phi_\varepsilon'}{\gamma_\varepsilon} \\ \frac{-\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 & \frac{\sqrt{2}\Phi_\varepsilon'}{\gamma_\varepsilon^2} & 0 & \frac{1-(\Phi_\varepsilon')^2}{\gamma_\varepsilon^2} & 0 \\ 0 & 0 & 0 & \frac{-\Phi_\varepsilon'}{\gamma_\varepsilon} & 0 & \frac{1}{\gamma_\varepsilon} \end{pmatrix}. \tag{3.17}$$

Observe that $\widetilde{\mathbf{C}}_\varepsilon$ is an orthogonal matrix.

The gradient for the elementary displacement is a straight forward computation and composed of

$$\begin{aligned} &\frac{\partial U^e}{\partial z_1} \cdot \mathbf{t}_\varepsilon = d_1\mathcal{U} \cdot \mathbf{t}_\varepsilon - d_1\mathcal{R} \cdot (z_2\mathbf{n}_\varepsilon - z_3\mathbf{e}_2), && \frac{\partial U^e}{\partial z_2} \cdot \mathbf{t}_\varepsilon = -\mathcal{R} \cdot \mathbf{n}_\varepsilon, && \frac{\partial U^e}{\partial z_3} \cdot \mathbf{t}_\varepsilon = \mathcal{R} \cdot \mathbf{e}_2, \\ &\frac{\partial U^e}{\partial z_1} \cdot \mathbf{e}_2 = d_1\mathcal{U} \cdot \mathbf{e}_2 - z_3 d_1\mathcal{R} \cdot \mathbf{t}_\varepsilon - z_3 c_\varepsilon \gamma_\varepsilon \mathcal{R} \cdot \mathbf{n}_\varepsilon, && \frac{\partial U^e}{\partial z_2} \cdot \mathbf{e}_2 = 0, && \frac{\partial U^e}{\partial z_3} \cdot \mathbf{e}_2 = -\mathcal{R} \cdot \mathbf{t}_\varepsilon, \\ &\frac{\partial U^e}{\partial z_1} \cdot \mathbf{n}_\varepsilon = d_1\mathcal{U} \cdot \mathbf{n}_\varepsilon + z_2 d_1\mathcal{R} \cdot \mathbf{t}_\varepsilon + z_3 c_\varepsilon \gamma_\varepsilon \mathcal{R} \cdot \mathbf{e}_2, && \frac{\partial U^e}{\partial z_2} \cdot \mathbf{n}_\varepsilon = \mathcal{R} \cdot \mathbf{t}_\varepsilon, && \frac{\partial U^e}{\partial z_3} \cdot \mathbf{n}_\varepsilon = 0. \end{aligned}$$

To compute the complete strain tensor note, that it is a linear operation as well and we can consider the elementary displacement and the warping again separately. The symmetric gradient for the elementary displacement (given by (3.16)) is obtained by combining the respective terms and yields $e_{z,22}(U^e) = e_{z,33}(U^e) = e_{z,23}(U^e) = 0$ and the nonzero components

$$\begin{aligned} e_{z,11}(U^e) &= \frac{1}{\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{t}_\varepsilon - d_1\mathcal{R}\cdot(z_2\mathbf{n}_\varepsilon - z_3\mathbf{e}_2)\right], \\ e_{z,12}(U^e) &= \frac{1}{2\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{e}_2 - z_3 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon\right], \\ e_{z,13}(U^e) &= \frac{1}{2\eta_\varepsilon}\left[\big(d_1\mathcal{U} - \boldsymbol{\gamma}_\varepsilon\mathcal{R}\wedge\mathbf{t}_\varepsilon\big)\cdot\mathbf{n}_\varepsilon + z_2 d_1\mathcal{R}\cdot\mathbf{t}_\varepsilon\right]. \end{aligned}$$

In the following, we pass over to the new displacement defined in Definition 3.3 and with the identity

$$d_1\mathcal{U} - \gamma_\varepsilon\mathcal{R}\cdot\mathbf{t}_\varepsilon = \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big) + \Phi_\varepsilon d_1\mathcal{R}\wedge\mathbf{e}_3, \qquad \text{a.e. in } (0,L)$$

the strain tensor is transformed to

$$\begin{aligned} \eta_\varepsilon e_{z,11} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{t}_\varepsilon + d_1\mathcal{R}\cdot\big(\big(\frac{\Phi_\varepsilon}{\gamma_\varepsilon} + z_3\big)\mathbf{e}_2 - z_2\mathbf{n}_\varepsilon\big), \\ 2\eta_\varepsilon e_{z,12} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_2 - d_1\mathcal{R}\cdot\big(z_3\mathbf{t}_\varepsilon + \Phi_\varepsilon\mathbf{e}_1\big), \\ 2\eta_\varepsilon e_{z,13} &= \big(d_1\mathbb{U} - \mathcal{R}\wedge\mathbf{e}_1\big)\cdot\mathbf{n}_\varepsilon + d_1\mathcal{R}\cdot\big(z_2\mathbf{t}_\varepsilon - \frac{\Phi_\varepsilon\Phi'_\varepsilon}{\gamma_\varepsilon}\mathbf{e}_2\big). \end{aligned} \tag{3.18}$$

The completion of the strain tensor for the full displacement $e_z(u) = e_z(U^e) + e_z(\overline{u})$ includes the warping terms again and with $e_z(\overline{u}) = \mathbf{C}_\varepsilon^T\, e_x(\overline{u})\, \mathbf{C}_\varepsilon$ given by (3.16).

4 The textile structure

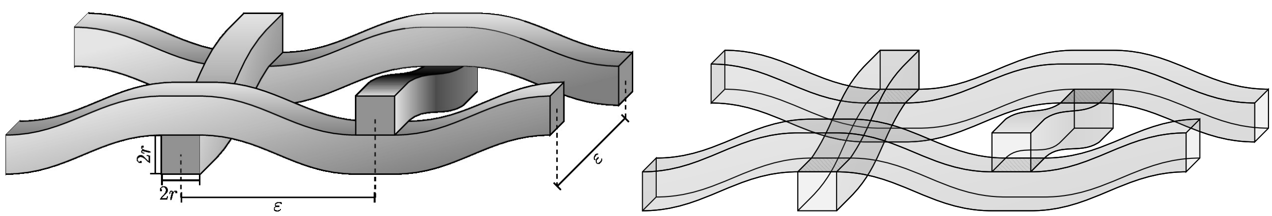

Figure 1: A portion of the textile structure containing a part of the periodicity cell (which is 2ε). The right picture shows the contact areas $\mathbf{C}_{pq}$ of the beams.

In the subsequent work we omit the indication r for r-dependent functions, if there is a dependence on ε as well. This is for the sake of comprehensibility and prevention of index-overloading.

Set

$$P_r^{(1)} \doteq \big\{z\in\mathbb{R}^3 \mid z_1\in(0,L),\ (z_2,z_3)\in\omega_r\big\}, \qquad P_r^{(2)} \doteq \big\{z\in\mathbb{R}^3 \mid z_2\in(0,L),\ (z_1,z_3)\in\omega_r\big\},$$

for the reference beams in the two directions. Then the curved beams are defined by

$$\mathcal{P}_\varepsilon^{(1,q)} \doteq \Big\{x\in\mathbb{R}^3 \mid x = \psi_\varepsilon^{(1,q)}(z),\ z\in P_r^{(1)}\Big\}, \qquad \mathcal{P}_\varepsilon^{(2,p)} \doteq \Big\{x\in\mathbb{R}^3 \mid x = \psi_\varepsilon^{(2,p)}(z),\ z\in P_r^{(2)}\Big\},$$

with the diffeomorphisms

$$\psi_\varepsilon^{(1,q)}(z) \doteq M_\varepsilon^{(1,q)}(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1), \quad \psi_\varepsilon^{(2,p)}(z) \doteq M_\varepsilon^{(2,p)}(z_2) + z_1\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2),$$

and the corresponding middle line parametrizations

$$M_\varepsilon^{(1,q)}(z_1) \doteq z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + (-1)^{q+1}\Phi_\varepsilon(z_1)\mathbf{e}_3, \quad M_\varepsilon^{(2,p)}(z_2) \doteq p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + (-1)^p\Phi_\varepsilon(z_2)\mathbf{e}_3.$$

Then, the whole textile structure is given by

$$\mathcal{S}_\varepsilon \doteq \bigcup_{q=1}^{2N_\varepsilon}\mathcal{P}_\varepsilon^{(1,q)} \cup \bigcup_{p=0}^{2N_\varepsilon}\mathcal{P}_\varepsilon^{(2,p)}. \tag{4.1}$$

Moreover, observe the respective local Frenet-frames $\left(\mathbf{t}_\varepsilon^{(1,q)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1,q)}\right)$ and $\left(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2,p)}, \mathbf{n}_\varepsilon^{(2,p)}\right)$ with

$$\begin{aligned}
\mathbf{t}_\varepsilon^{(1,q)}(z_1) &= \frac{dM_\varepsilon^{(1,q)}}{ds_1}(z_1), \qquad \mathbf{n}_\varepsilon^{(1,q)}(z_1) = \mathbf{t}_\varepsilon^{(1,q)}(z_1) \wedge \mathbf{e}_2, \qquad z_1 \in [0, L],\\
\mathbf{t}_\varepsilon^{(2,p)}(z_2) &= \frac{dM_\varepsilon^{(2,p)}}{ds_2}(z_2), \qquad \mathbf{n}_\varepsilon^{(2,p)}(z_2) = \mathbf{e}_1 \wedge \mathbf{t}_\varepsilon^{(2,p)}(z_2), \qquad z_2 \in [0, L].
\end{aligned}$$

Denote by $\mathbf{C}_\varepsilon^{(1,q)} = \left(\mathbf{t}_\varepsilon^{(1,q)}, \mathbf{e}_2, \mathbf{n}_\varepsilon^{(1,q)}\right)$ and $\mathbf{C}_\varepsilon^{(2,p)} = \left(\mathbf{e}_1, \mathbf{t}_\varepsilon^{(2,p)}, \mathbf{n}_\varepsilon^{(2,p)}\right)$ analogously the respective basis-transformation matrix. Note, that we work mostly on the straight reference beams, i.e., with respect to (z_1, z_2, z_3). Thus set

$$\mathbf{P}_r^{[1]} \doteq \bigcup_{q=1}^{2N_\varepsilon} \left(q\varepsilon\mathbf{e}_2 + P_r^{(1)}\right) \quad \text{and} \quad \mathbf{P}_r^{[2]} \doteq \bigcup_{p=0}^{2N_\varepsilon} \left(p\varepsilon\mathbf{e}_1 + P_r^{(2)}\right).$$

Then, for every $\varphi \in L^1(\mathcal{S}_\varepsilon)$, the couple $\left(\varphi^{[1]}, \varphi^{[2]}\right) \in L^1\left(\mathbf{P}_r^{[1]}\right) \times L^1\left(\mathbf{P}_r^{[2]}\right)$ is associated, with

$$\begin{aligned}
\varphi^{[1]}(q\varepsilon\mathbf{e}_2 + z) &= \varphi\big(q\varepsilon\mathbf{e}_2 + \psi_\varepsilon^{(1,q)}(z)\big), \qquad \text{for } q \in \{1, \dots, 2N_\varepsilon\} \text{ and a.e. } z \in P_r^{(1)},\\
\varphi^{[2]}(p\varepsilon\mathbf{e}_1 + z) &= \varphi\big(p\varepsilon\mathbf{e}_1 + \psi_\varepsilon^{(2,p)}(z)\big), \qquad \text{for } p \in \{0, \dots, 2N_\varepsilon\} \text{ and a.e. } z \in P_r^{(2)}.
\end{aligned}$$

Then, the integral over the whole structure is easily split

$$\begin{aligned}
\int_{\mathcal{S}_\varepsilon} \varphi(x)dx &= \int_{\mathbf{P}_r^{[1]}} \varphi^{[1]}(z)|\det\left(\nabla\psi_\varepsilon^{[1]}(z)\right)|dz + \int_{\mathbf{P}_r^{[2]}} \varphi^{[2]}(z)|\det\left(\nabla\psi_\varepsilon^{[2]}(z)\right)|dz\\
&= \sum_{q=1}^{2N_\varepsilon} \int_{P_r^{(1)}} \varphi^{[1]}(q\varepsilon\mathbf{e}_2 + z)|\det\left(\nabla\psi_\varepsilon^{(1,q)}(z)\right)|dz + \sum_{p=0}^{2N_\varepsilon} \int_{P_r^{(2)}} \varphi^{[2]}(p\varepsilon\mathbf{e}_1 + z)|\det\left(\nabla\psi_\varepsilon^{(2,p)}(z)\right)|dz. \quad (4.2)
\end{aligned}$$

4.1 Boundary conditions

The only assumption applied on the textile-structure is a clamp-condition on its lateral boundary $z_2 = 0$ such that every displacement there equals zero. In fact, due to the structure (4.1) only the displacements $u^{(2,p)}$ are affected by this condition, i.e., $u^{(2,p)}_{|z_2=0} = 0$ for every $p \in \{0, \dots, 2N_\varepsilon\}$.

4.2 The contact conditon

The contact between the fibers is restricted to the portions, where the beams are right above each other. So define the contact domains as small areas included in the lateral boundary of the beams

$$\mathbf{C}_{pq} \doteq C_{pq} \times \{0\}, \qquad C_{pq} \doteq (p\varepsilon - r, p\varepsilon + r) \times (q\varepsilon - r, q\varepsilon + r) = (p\varepsilon, q\varepsilon) + \omega_r, \qquad (p,q) \in \mathcal{K}_\varepsilon.$$

Observe, that in these contact domains the centerlines of the beams reduce for a.e. $(z_1, z_2) \in C_{pq}$ to

$$M^{(1,q)}(z_1) = z_1\mathbf{e}_1 + q\varepsilon\mathbf{e}_2 + (-1)^{p+q}r\mathbf{e}_3 \quad \text{and} \quad M^{(2,p)}(z_2) = p\varepsilon\mathbf{e}_1 + z_2\mathbf{e}_2 + (-1)^{p+q+1}r\mathbf{e}_3.$$

Then, the beam-to-beam interaction is characterized by the non-negative gap-function $g_\varepsilon : \mathcal{K}_\varepsilon \to [0, +\infty)^3$ and the condition

$$|u_\alpha^{(1,q)} - u_\alpha^{(2,p)}| \leq g_{\varepsilon,\alpha}, \qquad \text{a.e in } \mathbf{C}_{pq}, \ (p,q) \in \mathcal{K}_\varepsilon$$

for in-plane displacements, while the third direction

$$0 \leq \left(u^{(1,q)} - u^{(2,p)}\right)(-1)^{p+q} \leq g_{\varepsilon,3}, \qquad \text{a.e in } \mathbf{C}_{pq}, \ (p,q) \in \mathcal{K}_\varepsilon.$$

needs to account for the oscillating manner of the beams switching the vertical positions. Further restrictions and specifications on the contact are given later in the work.

4.3 The admissible displacements of the structure

Given the structure, the boundary condition and the contact, the convex set of the admissible displacements is denoted by

$$\begin{aligned}\mathcal{V}_\varepsilon \doteq \Big\{ u = \big(u^{(1,1)},\dots,u^{(1,2N_\varepsilon)},u^{(2,0)}\dots,u^{(2,2N_\varepsilon)}\big) \in \prod_{q=1}^{2N_\varepsilon} H^1(\mathcal{P}_\varepsilon^{(1,q)})^3 \times \prod_{p=0}^{2N_\varepsilon} H^1(\mathcal{P}_\varepsilon^{(2,p)})^3 \ \mid \\ \text{such that } 0 \leq \big(u_3^{(1,q)}(x) - u_3^{(2,p)}(x)\big)(-1)^{p+q} \leq g_{\varepsilon,3}(p\varepsilon,q\varepsilon), \\ |u_\alpha^{(1,q)}(x) - u_\alpha^{(2,p)}(x)| \leq g_{\varepsilon,\alpha}(p\varepsilon,q\varepsilon), \quad \text{for a.e } x \in \mathbf{C}_{pq}, \quad (p,q) \in \dot{\mathcal{K}}_\varepsilon, \\ u^{(2,0)}_{|z_2=0} = u^{(2,1)}_{|z_2=0} = \dots = u^{(2,2N_\varepsilon)}_{|z_2=0} = 0 \Big\},\end{aligned} \tag{4.3}$$

where $g_{\varepsilon,i},\ i \in \{1,2,3\}$, is a non-negative function belonging to $\mathcal{C}(\overline{\Omega})$. The space $\mathcal{V}_\varepsilon$ is equipped with the semi-norm

$$\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)} = \sum_{q=1}^{2N_\varepsilon} \|e(u^{(1,q)})\|^2_{L^2(\mathcal{P}_\varepsilon^{(1,q)})} + \sum_{p=0}^{2N_\varepsilon} \|e(u^{(2,p)})\|^2_{L^2(\mathcal{P}_\varepsilon^{(1,p)})}, \qquad \forall u \in \mathcal{V}_\varepsilon.$$

4.4 The elasticity problem

The original problem of the textile is stated in the three dimensional setting as typical elasticity problem on the given space. Thus for a complete description a material law a is needed. Hereafter, we consider the usual Hooks law satisfying

- a_ε is bounded: $a_{\varepsilon,ijkl} \in L^\infty(\mathcal{S}_\varepsilon)$
- a_ε is symmetric: $a_{\varepsilon,ijkl} = a_{\varepsilon,jikl} = a_{\varepsilon,klij}$
- a_ε is positive definite: $\exists c_0,\ C_0 > 0\ :\ c_0\xi_{ij}\xi_{kl} \leq a_{\varepsilon,jikl}(x)\xi_{ij}\xi_{kl} \leq C_0\xi_{ij}\xi_{kl}$ for a.e. $x \in \mathcal{S}_\varepsilon$, where ξ is a 3×3 symmetric matrix

It is also convenient to use the stress tensor σ_ε instead of the material law a_ε where $\sigma_{\varepsilon,ij}(u) = a_{\varepsilon,ijkl}e_{kl}(u)$. The textile problem in variational form reads as

$$\begin{cases}\text{Find } u_\varepsilon \in \mathcal{V}_\varepsilon \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} a_\varepsilon e(u_\varepsilon) : e(u_\varepsilon - \varphi)\,dx - \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - \varphi)\,dx \leq 0, \qquad \forall \varphi \in \mathcal{V}_\varepsilon.\end{cases} \tag{4.4}$$

Moreover, later the vectorial notation of the problem is used. Thus, recall

$$\begin{cases}\text{Find } u_\varepsilon \in \mathcal{V}_\varepsilon \text{ such that:} \\ \displaystyle\int_{\mathcal{S}_\varepsilon} A_\varepsilon E_x(u_\varepsilon) \cdot E_x(u_\varepsilon - \varphi)\,dx - \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - \varphi)\,dx \leq 0, \qquad \forall \varphi \in \mathcal{V}_\varepsilon.\end{cases} \tag{4.5}$$

where $A_\varepsilon \in L^\infty(\mathcal{S}_\varepsilon)^{6\times 6}$ is bounded, symmetric and positive definite, which is easily deduced from the properties of a_ε. Furthermore, it satisfies

$$c_0|\zeta|^2 \leq A_\varepsilon(x)\,\zeta\cdot\zeta \leq C_0|\zeta|^2, \qquad \text{for a.e. } x \in \mathcal{S}_\varepsilon \text{ and } \forall \zeta \in \mathbb{R}^6.$$

Note that the problem in the current form is solvable but not unique. This comes from the boundary conditions which allows rigid motions, i.e. motions in the kernel of the symmetric strain tensor. Namely the displacements $u^{(1,q)}$ can have an in-plane rigid motion, since they are only subjected to the rather loose contact condition in $\mathcal{V}_\varepsilon$. To circumvent this ambiguity equip the space with a glued contact at $z_1 = 0$ whereby the $\mathbf{e}_1$-directed beams inherit the clamped condition at $z_2 = 0$. This does not change the limit behavior in the following hence w.l.o.g. we omit this condition below in the estimates and the limit and just use it for the uniqueness of the original problem.
With the additional condition (glued contact at $z_1 = 0$) existence and uniqueness of this problem is ensured by Stampacchia-Lemma (see [18]).

5 Preliminary estimates

This section is dedicated to the derivation of estimates on local and global fields. Furthermore, the extension onto the plate-domain $[0,L]^2$ and the scale splitting of the fields is discussed.

5.1 An extension operator

The definition of global fields on $\Omega = (0, L)^2$ is characterized by an extension of the fields between the contact midpoints $(p\varepsilon, q\varepsilon)$. To characterize the extension, let φ be a function defined on $\mathcal{K}_\varepsilon$. We extend φ as a function belonging to $W^{1,\infty}(\Omega)$, denoted $\boldsymbol{\varphi}$, in the following way: in the cell $\varepsilon(p,q) + \varepsilon Y$, $(p,q) \in \{0, \dots, 2N_\varepsilon - 1\}^2$, we define $\boldsymbol{\varphi}$ as the Q_1-interpolate of its values on the vertices of the cell $\varepsilon(p,q) + \varepsilon Y$.

Lemma 5.1. *Let φ be a function defined on $\mathcal{K}_\varepsilon$ and extended as above in a function denoted $\boldsymbol{\varphi}$ and belonging to $W^{1,\infty}(\Omega)$. One has*

$$\|\boldsymbol{\varphi}\|^2_{L^2(\Omega)} \leq C\varepsilon^2 \sum_{(p,q)\in\mathcal{K}_\varepsilon} |\varphi(p\varepsilon, q\varepsilon)|^2. \tag{5.1}$$

Moreover $\boldsymbol{\varphi}$ satisfies

$$\begin{aligned}
&\boldsymbol{\varphi}(z_1, z_2) = \boldsymbol{\varphi}(z_1, q\varepsilon) + (z_2 - q\varepsilon)\frac{\partial \boldsymbol{\varphi}}{\partial z_2}(z_1, z_2)\\
&\qquad\qquad \forall z_1 \in [0, L], \ \ \textit{for a.e.} \ z_2 \in \big((q-1)\varepsilon, (q+1)\varepsilon\big) \cap [0, L], \ \ q \in \{0, \dots, 2N_\varepsilon\},\\
&\boldsymbol{\varphi}(z_1, z_2) = \boldsymbol{\varphi}(p\varepsilon, z_2) + (z_1 - p\varepsilon)\frac{\partial \boldsymbol{\varphi}}{\partial z_1}(z_1, z_2)\\
&\qquad\qquad \forall z_2 \in [0, L], \ \ \textit{for a.e.} \ z_1 \in \big((p-1)\varepsilon, (p+1)\varepsilon\big) \cap [0, L], \ \ p \in \{0, \dots, 2N_\varepsilon\}.
\end{aligned} \tag{5.2}$$

Proof. Since the function $\boldsymbol{\varphi}$ is a Q_1-interpolate, it is decomposed using the four Q_1-basis functions $\{N_i(x,y)\}$ for $i = 1, \dots, 4$ in the cell Y. Then, with $\boldsymbol{\varphi}_i$ denoting the four values on the vertices, we obtain

$$\int_Y |\boldsymbol{\varphi}|^2 dxdy = \int_Y \Big|\sum_{i=1}^4 \boldsymbol{\varphi}_i N_i(x,y)\Big|^2 dxdy \leq 4\sum_{i=1}^4 |\boldsymbol{\varphi}_i|^2 \|N_i\|^2_{L^2(Y)} = \frac{4}{9}\sum_{i=1}^4 |\boldsymbol{\varphi}_i|^2$$

Consequently, with a rescaling argument transfer this to the cell εY and the fact that every node is part of four cells we obtain the claim by summing over all the cells.
A straightforward calculation gives (5.2). □

This estimation of the interpolant is crucial for the upcoming estimates of the extended fields. Furthermore, the defined extension leaves a function on Ω linear on the edges of the cells $\varepsilon(p,q) + \varepsilon Y$ and thereby on the middle lines of the beams, which is a desirable property for the next section.

We denote by $\mathbf{g}_\varepsilon$ the extension of g_ε.

5.2 Decomposition of the displacements of the beams structure $\mathcal{S}_\varepsilon$

We decompose the displacements $u^{(1,q)}$, $q \in \{1, \dots, 2N_\varepsilon\}$, $u^{(2,p)}$, $p \in \{0, \dots, 2N_\varepsilon\}$, as in Section 3 (see (3.4))

$$\begin{aligned}
u^{(1,q)}(x) &= \mathcal{U}^{(1,q)}(z_1) + \mathcal{R}^{(1,q)}(z_1) \wedge (z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon^{(1,q)}(z_1)) + \overline{u}^{(1,q)}(z),\\
&\qquad\qquad\qquad\qquad \text{for a.e. } x = \psi_\varepsilon^{(1,q)}(z) \in \mathcal{P}_\varepsilon^{(1,q)}, \ \ z \in P_r^{(1)},\\
u^{(2,p)}(x) &= \mathcal{U}^{(2,p)}(z_2) + \mathcal{R}^{(2,p)}(z_2) \wedge (z_1\mathbf{e}_1 + z_3\mathbf{n}_\varepsilon^{(2,p)}(z_2)) + \overline{u}^{(2,p)}(z),\\
&\qquad\qquad\qquad\qquad \text{for a.e. } x = \psi_\varepsilon^{(2,p)}(z) \in \mathcal{P}_\varepsilon^{(2,p)}, \ \ z \in P_r^{(2)}.
\end{aligned}$$

Following (3.9), set $((p_\varepsilon, q_\varepsilon) \in \{0, \dots, 2N_\varepsilon\} \times \{1, \dots, 2N\varepsilon\})$

$$\mathbb{U}^{(1,q)} = \mathcal{U}^{(1,q)} - (-1)^{q+1}\Phi_\varepsilon \mathcal{R}^{(1,q)} \wedge \mathbf{e}_3 \qquad \text{and} \qquad \mathbb{U}^{(2,p)} = \mathcal{U}^{(2,p)} - (-1)^p \Phi_\varepsilon \mathcal{R}^{(2,p)} \wedge \mathbf{e}_3.$$

Denote $\mathcal{U}^{(\alpha)}$, $\mathbb{U}^{(\alpha)}$ and $\mathcal{R}^{(\alpha)}$, $\alpha = 1, 2$ the functions defined on every $(p,q) \in \mathcal{K}_\varepsilon$, by

$$\begin{aligned}
&\mathcal{U}^{(1)}(p\varepsilon, q\varepsilon) = \mathcal{U}^{(1,q)}(p\varepsilon), \qquad \mathbb{U}^{(1)}(p\varepsilon, q\varepsilon) = \mathbb{U}^{(1,q)}(p\varepsilon), \qquad \mathcal{R}^{(1)}(p\varepsilon, q\varepsilon) = \mathcal{R}^{(1,q)}(p\varepsilon),\\
&\mathcal{U}^{(2)}(p\varepsilon, q\varepsilon) = \mathcal{U}^{(2,p)}(q\varepsilon), \qquad \mathbb{U}^{(2)}(p\varepsilon, q\varepsilon) = \mathbb{U}^{(2,p)}(q\varepsilon), \qquad \mathcal{R}^{(2)}(p\varepsilon, q\varepsilon) = \mathcal{R}^{(2,p)}(q\varepsilon),
\end{aligned} \tag{5.3}$$

and then extended to functions belonging to $W^{1,\infty}(\Omega)$ as defined in the previous section 5.1.

Moreover, it is necessary to identify the remaining displacement covering the fast oscillations on the middle lines. Thus, similar to (3.13) set

$$\begin{aligned}
&\widetilde{\mathcal{U}}^{(1)}(\cdot, q\varepsilon) = \mathcal{U}^{(1,q)}(\cdot) - \mathcal{U}^{(1)}(\cdot, q\varepsilon), \qquad \widetilde{\mathcal{U}}^{(2)}(p\varepsilon, \cdot) = \mathcal{U}^{(2,p)}(\cdot) - \mathcal{U}^{(2)}(p\varepsilon, \cdot),\\
&\widetilde{\mathbb{U}}^{(1)}(\cdot, q\varepsilon) = \mathbb{U}^{(1,q)}(\cdot) - \mathbb{U}^{(1)}(\cdot, q\varepsilon), \qquad \widetilde{\mathbb{U}}^{(2)}(p\varepsilon, \cdot) = \mathbb{U}^{(2,p)}(\cdot) - \mathbb{U}^{(2)}(p\varepsilon, \cdot), \qquad (p,q) \in \mathcal{K}_\varepsilon.\\
&\widetilde{\mathcal{R}}^{(1)}(\cdot, q\varepsilon) = \mathcal{R}^{(1,q)}(\cdot) - \mathcal{R}^{(1)}(\cdot, q\varepsilon), \qquad \widetilde{\mathcal{R}}^{(2)}(p\varepsilon, \cdot) = \mathcal{R}^{(2,p)}(\cdot) - \mathcal{R}^{(2)}(p\varepsilon, \cdot),
\end{aligned}$$

These fields denoted with $\widetilde{\ }$ are only defined on the lines

$$L_\varepsilon^{(1)} = \bigcup_q \{q\varepsilon\mathbf{e}_2 + z_1 \mid z_1 \in (0,L)\}, \qquad L_\varepsilon^{(2)} = \bigcup_p \{p\varepsilon\mathbf{e}_1 + z_2 \mid z_2 \in (0,L)\},$$

and are equal to zero on every knot $(p\varepsilon, q\varepsilon) \in \mathcal{K}_\varepsilon$. Furthermore, they coincide with the fields $\mathbb{U}^{(0)}$ and $\mathcal{R}^{(0)}$ for every single beam in (3.13).

Note, that $\mathbf{P}_\varepsilon^{[\alpha]} = L_\varepsilon^{(\alpha)} \times \omega_r$ and that $\widetilde{\mathcal{U}}^{(\alpha)}, \widetilde{\mathcal{R}}^{(\alpha)}, \widetilde{\mathbb{U}}^{(\alpha)} \in H^1(L_\varepsilon^{(\alpha)})$. The following lemma, recalls the results of Lemma 3.6 for the new setting.

Lemma 5.2. *The fields $\widetilde{\mathbb{U}}^{(\alpha)}$, $\widetilde{\mathcal{R}}^{(\alpha)}$ satisfy the estimates*

$$\begin{aligned} &\|\widetilde{\mathbb{U}}^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\widetilde{\mathbb{U}}^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} \le C\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}, \\ &\|\widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\widetilde{\mathcal{R}}^{(\alpha)}\|_{L^2(L_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}, \end{aligned} \tag{5.4}$$

Proof. A direct consequence of Lemma 3.6. □

Concluding this section, note that the estimate on the warping (3.6) is easily ported onto the complete structure and we obtain

$$\sum_{q=1}^{2N_\varepsilon}\Big(\|\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})}+\varepsilon^2\|\nabla\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})}\Big)+\sum_{p=0}^{2N_\varepsilon}\Big(\|\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})}+\varepsilon^2\|\nabla\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})}\Big)\le C\varepsilon^2\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}. \tag{5.5}$$

5.3 First global estimates

Below we estimate the extended fields $\mathbb{U}^{(1)}$, $\mathbb{U}^{(2)}$, $\mathcal{U}^{(1)}$, $\mathcal{U}^{(2)}$, $\mathcal{R}^{(1)}$, $\mathcal{R}^{(2)}$. Lemma 3.4 and estimate (3.8) give rise to the following Lemma.

Lemma 5.3. *One has*

$$\|\partial_\alpha\mathcal{R}^{(\alpha)}\|_{L^2(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \|\partial_\alpha\mathbb{U}^{(\alpha)} - \mathcal{R}^{(\alpha)} \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}. \tag{5.6}$$

Proof. Both estimates are easily obtained from $(3.11)_1$ and $(3.10)_1$. Indeed, recall the definition (5.3) of the global fields and consider for instance $(5.6)_1$ for $\alpha = 1$:

$$\big\|\partial_1\mathcal{R}^{(1)}\big\|^2_{L^2(\Omega)}\le 2\varepsilon^2\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big|\frac{\mathcal{R}^{(1)}((p+1)\varepsilon,q\varepsilon) - \mathcal{R}^{(1)}(p\varepsilon,q\varepsilon)}{\varepsilon}\Big|^2\le 2\sum_{(p,q)\in\mathcal{K}_\varepsilon}\Big|\mathcal{R}^{(1)}((p+1)\varepsilon,q\varepsilon) - \mathcal{R}^{(1)}(p\varepsilon,q\varepsilon)\Big|^2$$

where upon we apply $(3.11)_1$ and obtain $(5.6)_1$. The case $\alpha = 2$ is analogous.
The second estimate $(5.6)_2$ is a consequence $(3.11)_2$ by the same means as for $(5.6)_1$. □

Corollary 5.4. *Furthermore, the fields satisfy*

$$\|\mathbb{U}_2^{(2)}\|_{L^2(\Omega)} + \varepsilon\big(\big\|\mathcal{R}^{(2)}\big\|_{L^2(\Omega)} + \big\|\mathbb{U}_3^{(2)}\big\|_{L^2(\Omega)}\big) \le \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}. \tag{5.7}$$

Proof. Use (5.6), the Poincaré inequality and the boundary condition in order to get (5.7). □

5.4 Estimates on contact

The contact between the fibers gives rise to estimates on the global fields and their difference. The latter is very important to determine the limits and whether or not they coincide. First recall, that for a.e. $x \in \mathbf{C}_{pq}$ (note that $|z_3| = r$ in $\mathbf{C}_{pq}$) the displacements reduce to

$$\begin{aligned} u^{(1,q)}(x) &= \mathcal{U}^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}^{(1,q)}(p\varepsilon + z_1) \wedge \big(z_2\mathbf{e}_2 + (-1)^{p+q+1}r\mathbf{e}_3\big) + \overline{u}^{(1,q)}(x), \\ &= \mathbb{U}^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}^{(1,q)}(p\varepsilon + z_1) \wedge z_2\mathbf{e}_2 + \overline{u}^{(1,q)}(x), \\ u^{(2,p)}(x) &= \mathcal{U}^{(2,p)}(q\varepsilon + z_2) + \mathcal{R}^{(2,p)}(q\varepsilon + z_2) \wedge \big(z_1\mathbf{e}_1 + (-1)^{p+q}r\mathbf{e}_3\big) + \overline{u}^{(2,p)}(x), \\ &= \mathbb{U}^{(2,p)}(q\varepsilon + z_2) + \mathcal{R}^{(2,p)}(q\varepsilon + z_2) \wedge z_1\mathbf{e}_1 + \overline{u}^{(2,p)}(x). \end{aligned} \tag{5.8}$$

with $z = (z_1, z_2) \in \omega_r$.

To estimate the fields $\mathbb{U}$ and $\mathcal{R}$ independently, it is necessary to start with the warping.

Lemma 5.5. *Let u be in $\mathcal{V}_\varepsilon$, then we have*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \big(\|\overline{u}^{(1,q)}\|^2_{L^2(\mathbf{C}_{pq})} + \|\overline{u}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\big) \le C\varepsilon\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}. \tag{5.9}$$

Proof. Recall the classical inequality: for any $\phi \in H^1(0,r)$ we have

$$r|\phi(0)|^2 \le 2\|\phi\|^2_{L^2(0,r)} + r^2\|\phi'\|^2_{L^2(0,r)}.$$

This inequality applied in the third direction yields

$$\begin{aligned}\sum_{(p,q)\in\mathcal{K}_\varepsilon} r\left(\|\overline{u}^{(1,q)}\|^2_{L^2(\mathbf{C}_{pq})} + \|\overline{u}^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\right) &\le C\sum_{q=1}^{2N_\varepsilon}\left(\|\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})} + r^2\|\nabla\overline{u}^{(1,q)}\|^2_{L^2(P_r^{(1)})}\right)\\ &\quad + C\sum_{p=0}^{2N_\varepsilon}\left(\|\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})} + r^2\|\nabla\overline{u}^{(2,p)}\|^2_{L^2(P_r^{(2)})}\right).\end{aligned}$$

Together with (5.5) this gives (5.9). □

Lemma 5.6. *The global fields satisfy*

$$\big\|\mathbb{U}^{(1)} - \mathbb{U}^{(2)}\big\|_{L^2(\Omega)} + \varepsilon\big(\big\|\mathcal{R}^{(1)} - \mathcal{R}^{(2)}\big\|_{L^2(\Omega)}\big) \le C\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \sqrt{\varepsilon}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big). \tag{5.10}$$

The constant does not depend on ε and r.

Proof. From the equalities (5.8), the jump conditions in the definition of $\mathcal{V}_\varepsilon$ (see Subsection 4.3) one first obtains

$$\begin{aligned}\sum_{p,\,q}\int_{C_{pq}} \big|\mathbb{U}^{(1,q)}(p\varepsilon + z_1) + \mathcal{R}^{(1,q)}(p\varepsilon+z_1)\wedge z_2\mathbf{e}_2 &\\ - \mathbb{U}^{(2,p)}(q\varepsilon+z_2) - \mathcal{R}^{(2,p)}(q\varepsilon+z_2)\wedge z_1\mathbf{e}_1\big|^2 dz_1dz_2 &\\ \le \sum_{p,\,q}\big(\|\overline{u}^{(1,q)}\|^2_{L^2(C_{pq})} + \|\overline{u}^{(2,p)}\|^2_{L^2(C_{pq})} + \varepsilon^2\big|g_\varepsilon(p\varepsilon,q\varepsilon)\big|^2\big).&\end{aligned}$$

Then estimates (3.11) and (5.9) lead to

$$\sum_{p,\,q}\big|\mathbb{U}^{(1,q)}(p\varepsilon) - \mathbb{U}^{(2,p)}(q\varepsilon)\big|^2 + \varepsilon^2\sum_{p,\,q}\big|\mathcal{R}^{(1,q)}(p\varepsilon) - \mathcal{R}^{(2,p)}(q\varepsilon)\big|^2 \le C\sum_{p,\,q}\big|g_\varepsilon(p\varepsilon,q\varepsilon)\big|^2 + \frac{C}{\varepsilon}\|e(u)\|^2_{L^2(\mathcal{S}_\varepsilon)}.$$

Now, applying Lemma 5.1 yields the claim. □

These estimates give enough information about the relation between the field, such that the estimates from Lemma 5.4 can be transferred to the corresponding fields $\mathbb{U}_2^{(1)}$, $\mathbb{U}_3^{(1)}$, $\mathcal{R}^{(1)}$.

Corollary 5.7. *The global fields $\mathbb{U}_2^{(1)}$, $\mathbb{U}_3^{(1)}$, $\mathcal{R}^{(1)}$ satisfy*

$$\begin{aligned}&\|\mathbb{U}_2^{(1)}\|_{L^2(\Omega)} + \varepsilon\|\mathcal{R}^{(1)}\|_{L^2(\Omega)} \le C\Big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\Big),\\ &\|\mathbb{U}_3^{(1)}\|_{L^2(\Omega)} \le C\Big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\Big).\end{aligned} \tag{5.11}$$

Proof. A direct consequence of Lemma 5.6 and Corollary 5.4. □

5.5 The bending and membrane displacements

In this section, the estimates on the global displacements are completed. For this we use the estimates from above, where especially the contact-driven estimates are important.

Lemma 5.8. *The rotations $\mathcal{R}^{(\alpha)}$ and bending displacement $\mathbb{U}_3^{(\alpha)}$ fulfill*

$$\begin{aligned}&\|\mathcal{R}^{(\alpha)}\|_{H^1(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon^2}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\ &\|\mathbb{U}_3^{(\alpha)}\|_{H^1(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}.\end{aligned} \tag{5.12}$$

Proof. First, $(5.1)_2$ and (5.10) give

$$\|\partial_\beta \mathbb{U}_3^{(1)} - \partial_\beta \mathbb{U}_3^{(2)}\|_{L^2(\Omega)} + \varepsilon\big(\|\partial_\beta \mathcal{R}^{(1)} - \partial_\beta \mathcal{R}^{(2)}\|_{L^2(\Omega)}\big) \le \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}.$$

Then, using $(5.10)_1$ one obtains $(5.12)_1$. Estimate $(5.12)_1$ together with (5.6), (5.11) yield $(5.12)_2$. □

The next lemma estimates the components $e_{12}(\mathbb{U}^{(1)})$ and $e_{12}(\mathbb{U}^{(2)})$ in the strain tensor.

Lemma 5.9. *One has*

$$\|e_{12}(\mathbb{U}^{(1)})\|_{L^2(\Omega)} + \|e_{12}(\mathbb{U}^{(2)})\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}\big(\sqrt{\varepsilon}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\big).$$

Proof. Observe, that

$$\begin{aligned}\|\partial_1 \mathbb{U}_2^{(1)} + \partial_2 \mathbb{U}_1^{(1)}\|_{L^2(\Omega)} \le & \|\big(\partial_1 \mathbb{U}^{(1)} - \mathcal{R}^{(1)}\wedge \mathbf{e}_1\big)\cdot \mathbf{e}_2\|_{L^2(\Omega)} + \|\big(\partial_2 \mathbb{U}^{(2)} - \mathcal{R}^{(2)}\wedge \mathbf{e}_2\big)\cdot \mathbf{e}_1\|_{L^2(\Omega)}\\ &+ \|\partial_2 \mathbb{U}_1^{(2)} - \partial_2 \mathbb{U}_1^{(1)}\|_{L^2(\Omega)} + \|\mathcal{R}_3^{(1)} - \mathcal{R}_3^{(2)}\|_{L^2(\Omega)}\end{aligned}$$

Then from $(5.6)_2$ and $(5.10)_{1,2}$, it yields

$$\|e_{12}(\mathbb{U}^{(1)})\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}\big(\sqrt{\varepsilon}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\big).$$

In the same way one estimates $\|e_{12}(\mathbb{U}^{(2)})\|_{L^2(\Omega)}$. □

The estimate on the symmetric gradient allows to transfer the estimation onto the membrane displacements itself. The next corollary uses the 2D-Korn-inequality to obtain the H^1-estimates on $\mathbb{U}^{(\alpha)}$.

Corollary 5.10. *The membrane displacements and $\mathcal{R}_3^{(\alpha)}$ satisfy ($(\alpha,\beta)\in\{1,2\}^2$)*

$$\|\mathbb{U}_\beta^{(\alpha)}\|_{H^1(\Omega)} + \|\mathcal{R}_3^{(\alpha)}\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}\big(\sqrt{\varepsilon}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \|\mathbf{g}_\varepsilon\|_{L^2(\Omega)}\big). \tag{5.13}$$

Proof. By the clamp-condition at $z_2 = 0$, the estimates in Lemmas 5.3-5.9 and the 2D-Korn inequality we deduce the estimate on $\mathbb{U}_\beta^{(\alpha)}$. The estimate for $\mathcal{R}_3^{(\alpha)}$ is a consequence of the first one and $(5.6)_2$. □

5.6 Final decomposition

Since $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ (respectively $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$) converge to the same limit with a contact $\|\mathbf{g}_\varepsilon\| \le C\varepsilon^3$. Hence, it is convenient to define a combined field and we set

$$\mathbb{U} = \frac{1}{2}\big(\mathbb{U}^{(1)} + \mathbb{U}^{(2)}\big),\quad \mathcal{R} = \frac{1}{2}\big(\mathcal{R}^{(1)} + \mathcal{R}^{(2)}\big),\quad \mathbb{U}^{(g)} = \frac{1}{2}\big(\mathbb{U}^{(1)} - \mathbb{U}^{(2)}\big),\quad \mathcal{R}^{(g)} = \frac{1}{2}\big(\mathcal{R}^{(1)} - \mathcal{R}^{(2)}\big). \tag{5.14}$$

Observe that these fields vanish on $z_2 = 0$ by definition and moreover one has

$$\mathbb{U}^{(1)} = \mathbb{U} + \mathbb{U}^{(g)},\qquad \mathbb{U}^{(2)} = \mathbb{U} - \mathbb{U}^{(g)},\qquad \mathcal{R}^{(1)} = \mathcal{R} + \mathcal{R}^{(g)},\qquad \mathcal{R}^{(2)} = \mathcal{R} - \mathcal{R}^{(g)},$$

and for the original beam-displacements

$$\begin{aligned}\mathbb{U}^{(1,q)} &= \mathbb{U}(\cdot,q\varepsilon) + \mathbb{U}^{(g)}(\cdot,q\varepsilon) + \widetilde{\mathbb{U}}^{(1,q)}, &\qquad \mathbb{U}^{(2,p)} &= \mathbb{U}(p\varepsilon,\cdot) - \mathbb{U}^{(g)}(p\varepsilon,\cdot) + \widetilde{\mathbb{U}}^{(2,p)},\\ \mathcal{R}^{(1,q)} &= \mathcal{R}(\cdot,q\varepsilon) + \mathcal{R}^{(g)}(\cdot,q\varepsilon) + \widetilde{\mathcal{R}}^{(1,q)}, &\qquad \mathcal{R}^{(2,p)} &= \mathcal{R}(p\varepsilon,\cdot) - \mathcal{R}^{(g)}(p\varepsilon,\cdot) + \widetilde{\mathcal{R}}^{(2,p)}.\end{aligned} \tag{5.15}$$

The Lemma below is an immediate consequence of the above results for global fields.

Lemma 5.11. *One has*

$$\begin{aligned}&\|\mathcal{R}\|_{H^1(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon^2}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\ &\|\mathbb{U}_3\|_{H^1(\Omega)} \le \frac{C}{\varepsilon\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\ &\|\mathbb{U}_1\|_{H^1(\Omega)} + \|\mathbb{U}_2\|_{H^1(\Omega)} + \|\mathcal{R}_3\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\\ &\|\partial_\alpha \mathbb{U} - \mathcal{R}\wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)},\end{aligned}$$

and

$$\big\|\mathbb{U}^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\big\|\nabla \mathbb{U}^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\big\|\mathcal{R}^{(g)}\big\|_{L^2(\Omega)} + \varepsilon^2\|\nabla \mathcal{R}^{(g)}\|_{L^2(\Omega)} \le C\big(\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)}\big).$$

Proof. The estimates on the gradient on $\mathbb{U}^{(g)}$ and $\mathcal{R}^{(g)}$ are the consequences of the fact that all these fields are piecewise linear between two knots $(p\varepsilon, q\varepsilon)$. □

5.7 Assumption on the right hand side

The elastic energy corresponds directly to the force applied to the structure and with the estimates on the displacements, one can show that the force $f^{(\alpha)} \in H^1(\Omega)$ ($\alpha = 1, 2$) with

$$f_\varepsilon^{(\alpha)} = \varepsilon^\tau f_{1,\varepsilon}^{(\alpha)} \mathbf{e}_1 + \varepsilon^\tau f_{2,\varepsilon}^{(\alpha)} \mathbf{e}_2 + \varepsilon^{\tau+1} f_{3,\varepsilon}^{(\alpha)} \mathbf{e}_3 \tag{5.16}$$

and then restricting to the middle-line of every beam, i.e. for $(p, q) \in \mathcal{K}_\varepsilon$:

$$f_\varepsilon^{(1,q)}(z_1) = f_\varepsilon^{(1)}(z_1, q\varepsilon) \quad \text{for a.e. } z_1 \in (0, L), \qquad f_\varepsilon^{(2,p)}(z_2) = f_\varepsilon^{(2)}(p\varepsilon, z_2) \quad \text{for a.e. } z_2 \in (0, L).$$

is sufficient to estimate the elastic energy. Henceforth, write indifferently f_ε for the collection of the forces $f^{(1,q)}$ and $f^{(2,p)}$ in the beams, since the difference is in most cases obvious and a distinction is not necessary.

Indeed, the estimates from Section 5 lead to

$$\Big| \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot u_\varepsilon dx \Big| \leq C\varepsilon^{\tau+1/2}(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}) \Big[\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} + \Big(\frac{1}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} \Big) \|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} \Big].$$

Then by the coercivity of the problem, we obtain for $\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{\tau+1}$:

$$C_0 \|e(u_\varepsilon)\|^2_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\tau+1/2}(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}) \Big[\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} + \varepsilon^{\tau+1/2} \Big]$$

and thus

$$\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{\tau+1/2}(\|f^{(1)}\|_{H^1(\Omega)} + \|f^{(2)}\|_{H^1(\Omega)}) \leq C\varepsilon^{\tau+1/2}. \tag{5.17}$$

6 Asymptotic behavior of the macroscopic fields

From now on, we assume that the gap-function $g_\varepsilon = \varepsilon^3 g$ with $g \in \mathcal{C}(\overline{\Omega})^3$ satisfies

$$g_\varepsilon = \varepsilon^3 g, \quad g \in \mathcal{C}(\overline{\Omega})^3, \text{ hence } \mathbf{g}_\varepsilon \in \mathcal{C}(\overline{\Omega})^3 \text{ and } \|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^3 \|g\|_{L^\infty(\Omega)}. \tag{6.1}$$

This condition bequeath much information and regularity for the whole problem.

Furthermore, we assume that the elastic energy satisfies

$$\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \leq C\varepsilon^{5/2}, \tag{6.2}$$

which is achieved by estimate (5.17) for a right-hand side in the form (5.16) and $\tau = 2$.

6.1 First limit of the macroscopic fields

Lemma 6.1. *Let $\{u_\varepsilon\}$ be a sequence of displacements belonging to $\mathcal{V}_\varepsilon$ and satisfying* (6.2). *Then there exist a subsequence of ε, still denoted ε, and functions $\mathbb{U}_1, \mathbb{U}_2 \in H^1(\Omega)$, $\mathbb{U}_3 \in H^2(\Omega)$, $\mathcal{R}_\alpha \in H^1(\Omega)$ and $\mathcal{Z}_\alpha \in L^2(\Omega)^3$ such that the following convergences hold ($(\alpha, \beta) \in \{1, 2\}^2$):*

$$\frac{1}{\varepsilon^2} \mathbb{U}_\varepsilon^{(g)} \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^3, \qquad \frac{1}{\varepsilon} \mathcal{R}_\varepsilon^{(g)} \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^3, \tag{6.3}$$

and

$$\begin{aligned} &\frac{1}{\varepsilon^2}\mathbb{U}_{\varepsilon,\alpha},\ \frac{1}{\varepsilon^2}\mathbb{U}_{\varepsilon,\alpha}^{(\beta)} \rightharpoonup \mathbb{U}_\alpha \text{ weakly in } H^1(\Omega),\\ &\frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3},\ \frac{1}{\varepsilon}\mathbb{U}_{\varepsilon,3}^{(\beta)} \rightharpoonup \mathbb{U}_3 \text{ weakly in } H^1(\Omega),\\ &\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha},\ \frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha}^{(\beta)} \rightharpoonup \mathcal{R}_\alpha \text{ weakly in } H^1(\Omega),\\ &\frac{1}{\varepsilon^2}\Big(\partial_\alpha \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha\Big) \rightharpoonup \mathcal{Z}_\alpha \text{ weakly in } L^2(\Omega)^3. \end{aligned} \tag{6.4}$$

The fields satisfy the boundary conditions $\mathbb{U}(\cdot, 0) = \mathcal{R}(\cdot, 0) = 0$. Moreover, in the limit the identity

$$\partial_1 \mathbb{U}_3 = -\mathcal{R}_2, \qquad \partial_2 \mathbb{U}_3 = \mathcal{R}_1 \tag{6.5}$$

holds true.

Proof. Lemma 5.11 gives

$$\|\mathcal{R}_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon, \quad \|\mathcal{R}_{\varepsilon,3}\|_{L^2(\Omega)} \le C\varepsilon^2, \quad \|\partial_\alpha \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha\|_{L^2(\Omega)} \le C\varepsilon^2,$$
$$\|\mathbb{U}_{\varepsilon,1}\|_{H^1(\Omega)} + \|\mathbb{U}_{\varepsilon,2}\|_{H^1(\Omega)} \le C\varepsilon^2, \quad \|\mathbb{U}_3\|_{H^1(\Omega)} \le C\varepsilon$$

and

$$\big\|\mathcal{R}_\varepsilon^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\|\nabla\mathcal{R}_\varepsilon^{(g)}\|_{L^2(\Omega)} \le C\varepsilon^2, \qquad \big\|\mathbb{U}_\varepsilon^{(g)}\big\|_{L^2(\Omega)} + \varepsilon\big\|\nabla\mathbb{U}_\varepsilon^{(g)}\big\|_{L^2(\Omega)} \le C\varepsilon^3.$$

Hence, there exist a subsequence of ε, still denoted ε, and functions $\mathbb{U}_1$, $\mathbb{U}_2$, $\mathbb{U}_3$, $\mathcal{R}_1$ and $\mathcal{R}_2$ in $H^1(\Omega)$ such that the convergences (6.3)-(6.4) hold. Moreover, one has

$$\frac{1}{\varepsilon}(\partial_\alpha \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_\alpha)\cdot \mathbf{e}_3 \to 0 \;\text{ strongly in } L^2(\Omega),$$

from which we obtain (6.5). As byproduct this shows that $\mathbb{U}_3$ belongs to $H^2(\Omega)$. For the boundary conditions we refer to the definition of the fields, then $\mathbb{U}_i(\cdot,0) = \mathcal{R}_\alpha(\cdot,0) = 0$ is an immediate consequence. □

Beside the weak convergence of the rotation field $\mathcal{R}$, we have in the limit that $\mathcal{R}_3 = 0$ by estimate $(5.13)_2$ in Corollary 5.10.

6.2 The unfolding operator for the middle-lines

In this section, we introduce the unfolding operator especially for the global fields $\mathbb{U}$, $\mathcal{R}$, $\mathbb{U}^{(g)}$, $\mathcal{R}^{(g)}$, $\mathbb{U}^{(\alpha)}$ and $\mathcal{R}^{(\alpha)}$. Therefore, set $\mathcal{Y} = (0,2)^2$, the periodicity cell of the global fields. Furthermore, set

$$\mathcal{Y}^{\ell s} = \bigcup_{(a,b)\in\{0,1\}^2} \Big\{(z_1,b) \mid z_1 \in (a,a+1)\Big\} \cup \Big\{(a,z_2) \mid z_2 \in (b,b+1)\Big\},$$
$$\mathcal{Y}^K = \Big\{(a,b) \mid (a,b) \in \{0,1,2\}^2\Big\},$$

for set of lines and set of knots in $\mathcal{Y}$. Note, that the cell $\mathcal{Y}$ contains nine points, i.e. $\mathcal{Y}^K$, where the beams are in contact and the local fields are equal to zero.

Definition 6.2. *For every measurable function φ in the domain Ω define the measurable function $\mathcal{T}_\varepsilon(\varphi)$ on $\Omega\times\mathcal{Y}$ by*

$$\mathcal{T}_\varepsilon(\varphi)(s,X') = \varphi\big(2p\varepsilon\mathbf{e}_1 + 2q\varepsilon\mathbf{e}_2 + \varepsilon X'\big) \quad \text{for a.e. } \; s \in (2p\varepsilon, 2q\varepsilon) + \varepsilon\mathcal{Y}, \quad X' \in \mathcal{Y}.$$

Note, that $\mathcal{T}_\varepsilon$ maps $L^p(\Omega)$ into $L^p(\Omega\times\mathcal{Y})$. The properties of the unfolding operator can be found in [11]. The most important one is in the next Lemma.

Lemma 6.3. *The unfolding operator $\mathcal{T}_\varepsilon : L^s(\Omega) \to L^s(\Omega\times\mathcal{Y})$ satisfies for every $\varphi \in L^s(\Omega)$*

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^s(\Omega\times\mathcal{Y})} \le C\|\varphi\|_{L^s(\Omega)},$$

where C is a constant only depending on $\mathcal{Y}$

Proof. This is a consequence of the results in [11]. □

For the determination of the limits, especially for the limit-contact, a special property of the unfolding operator is needed.

Lemma 6.4 (see [7, Lemma 11.11])**.** *Let $\{(u_\varepsilon, v_\varepsilon)\}_\varepsilon$ be a sequence converging weakly to (u,v) in the space $H^1(\Omega)\times H^1(\Omega)^2$. Assume furthermore that there exist $\mathcal{Z}$ in $L^2(\Omega)^2$ and $\widehat{v}$ in $L^2(\Omega; H^1_{per,0}(\mathcal{Y}))^2$ such that*

$$\frac{1}{\varepsilon}\big(\nabla u_\varepsilon + v_\varepsilon\big) \rightharpoonup \mathcal{Z} \quad \textit{weakly in } \; L^2(\Omega)^2,$$
$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \rightharpoonup \nabla v + \nabla_X \widehat{v} \quad \textit{weakly in } \; L^2(\Omega\times\mathcal{Y})^{2\times 2}.$$

Then u belongs to $H^2(\Omega)$ and there exists $\mathfrak{u} \in L^2(\Omega; H^1_{per,0}(\mathcal{Y}))$ such that, up to a subsequence,

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\nabla u_\varepsilon + v_\varepsilon\big) \rightharpoonup \mathcal{Z} + \nabla_X \mathfrak{u} + \widehat{v} \quad \textit{weakly in } \; L^2(\Omega\times\mathcal{Y})^2.$$

To conclude this subsection define the spaces of special Q^1-interpolates by

$$Q^1(\mathcal{Y}) = \Big\{\varphi \in W^{1,\infty}(\mathcal{Y}) \mid \varphi \text{ is the } Q_1 \text{ interpolated of the values on the points in } \mathcal{Y}^K\Big\},$$
$$Q^1_{per}(\mathcal{Y}) = Q^1(\mathcal{Y}) \cap H^1_{per}(\mathcal{Y}).$$

6.3 Unfolded limits of the macroscopic fields

Lemma 6.5. *There exist $\widehat{\mathbb{U}}_\alpha$, $\widehat{\mathcal{R}}_\alpha \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$, $\widehat{\mathbb{U}}_3 \in L^2(\Omega; H^1_{per}(\mathcal{Y}))$, $\widehat{\mathbb{U}}^{(g)}, \mathcal{R}^{(g)} \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))^3$ and $\widehat{\mathcal{R}}_3 \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$ ($\alpha \in \{1,2\}$)*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\nabla \mathbb{U}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathbb{U}_\alpha + \nabla_X \widehat{\mathbb{U}}_\alpha \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y})^2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\nabla \mathcal{R}_{\varepsilon,\alpha}) \rightharpoonup \nabla \mathcal{R}_\alpha + \nabla_X \widehat{\mathcal{R}}_\alpha \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y})^2,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\mathcal{R}_{\varepsilon,3}) \rightharpoonup \widehat{\mathcal{R}}_3 \;\; \textit{weakly in } L^2(\Omega; Q^1(\mathcal{Y})),\\
&\frac{1}{\varepsilon^3}\mathcal{T}_\varepsilon(\mathbb{U}^{(g)}_\varepsilon) \rightharpoonup \widehat{\mathbb{U}}^{(g)} \;\; \textit{weakly in } L^2(\Omega; Q^1(\mathcal{Y}))^3,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon(\mathcal{R}^{(g)}_\varepsilon) \rightharpoonup \widehat{\mathcal{R}}^{(g)} \;\; \textit{weakly in } L^2(\Omega; Q^1(\mathcal{Y}))^3,
\end{aligned}\tag{6.6}$$

Moreover, one has

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_1\big)\cdot \mathbf{e}_1 \rightharpoonup \partial_1 \mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_1 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_1\big)\cdot \mathbf{e}_2 \rightharpoonup \partial_1 \mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_2 - \widehat{\mathcal{R}}_3 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_1\big)\cdot \mathbf{e}_3 \rightharpoonup \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_3 + \widehat{\mathcal{R}}_2 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}\tag{6.7}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_2\big)\cdot \mathbf{e}_1 \rightharpoonup \partial_2 \mathbb{U}_1 + \partial_{X_2}\widehat{\mathbb{U}}_1 + \widehat{\mathcal{R}}_3 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_2\big)\cdot \mathbf{e}_2 \rightharpoonup \partial_2 \mathbb{U}_2 + \partial_{X_2}\widehat{\mathbb{U}}_2 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2 \mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon \wedge \mathbf{e}_2\big)\cdot \mathbf{e}_3 \rightharpoonup \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_3 - \widehat{\mathcal{R}}_1 \;\; \textit{weakly in } L^2(\Omega\times\mathcal{Y}).
\end{aligned}\tag{6.8}$$

Proof. Convergences (6.6) are the consequences of the estimates in Lemma 5.11 and the convergences in Lemma 6.1 (see [11]). Convergences $(6.7)_{1,2}$ and $(6.8)_{1,2}$ are the immediate consequences of (6.6), while convergences $(6.7)_3$ and $(6.8)_3$ come from the convergence $(6.4)_4$, Lemma 6.4 and denoting $\widehat{\mathbb{U}}_3 = \mathfrak{u}$. $\square$

Set

$$\mathcal{Y}_{ab} = (a, a+1)\times(b, b+1), \qquad (a,b)\in\{0,1\}^2.$$

In Lemma below we precise the function $\widehat{\mathbb{U}}_3$.

Lemma 6.6. *There exists $\widetilde{\mathbb{U}}_3 \in L^2(\Omega; Q^1_{per}(\mathcal{Y}))$ such that*

$$\widehat{\mathbb{U}}_3(\cdot, X_1, X_2) = \widetilde{\mathbb{U}}_3(\cdot, X_1, X_2) - \frac{1}{2}(X_1-1)^2\partial_{11}\mathbb{U}_3 - \frac{1}{2}(X_2-1)^2\partial_{22}\mathbb{U}_3. \tag{6.9}$$

Proof. Write

$$\mathcal{R}_{\varepsilon,\alpha} = \mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha}) + \big(\mathcal{R}_{\varepsilon,\alpha} - \mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})\big)$$

where

$$\mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha}) = \frac{1}{|\mathcal{Y}|}\int_{\mathcal{Y}} \mathcal{T}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})(\cdot, X_1, X_2)dX_1dX_2.$$

One has from the estimate of $\mathcal{R}_{\varepsilon,\alpha}$, convergence $(6.6)_2$ and Theorem 3.5 in [11]

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\mathcal{R}_{\varepsilon,\alpha} - \mathcal{M}_\varepsilon(\mathcal{R}_{\varepsilon,\alpha})\big) \rightharpoonup (X_1-1)\frac{\partial \mathcal{R}_\alpha}{\partial z_1} + (X_2-1)\frac{\partial \mathcal{R}_\alpha}{\partial z_2} + \widehat{\mathcal{R}}_\alpha \quad \text{weakly in } \; L^2(\Omega\times\mathcal{Y}).$$

Hence, due to $(6.7)_3$ and $(6.8)_3$ together with the above convergence, we obtain the following weak convergences in $L^2(\Omega\times\mathcal{Y})$ (recall that $\partial_1\mathbb{U}_3 = -\mathcal{R}_2$, and $\partial_2\mathbb{U}_3 = \mathcal{R}_1$):

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_1\mathbb{U}_\varepsilon - \mathcal{M}_\varepsilon(\mathcal{R}_\varepsilon)\wedge\mathbf{e}_1\big)\cdot\mathbf{e}_3 &\rightharpoonup \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_3 - (X_1-1)\frac{\partial \mathcal{R}_2}{\partial z_1} - (X_2-1)\frac{\partial \mathcal{R}_2}{\partial z_2},\\
&= \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_3 + (X_1-1)\partial_{11}\mathbb{U}_3 + (X_2-1)\partial_{12}\mathbb{U}_3,\\
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon\big(\partial_2\mathbb{U}_\varepsilon - \mathcal{M}_\varepsilon(\mathcal{R}_\varepsilon)\wedge\mathbf{e}_2\big)\cdot\mathbf{e}_3 &\rightharpoonup \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_3 + (X_1-1)\frac{\partial \mathcal{R}_1}{\partial z_1} + (X_2-1)\frac{\partial \mathcal{R}_1}{\partial z_2},\\
&= \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_3 + (X_1-1)\partial_{12}\mathbb{U}_3 + (X_2-1)\partial_{22}\mathbb{U}_3.
\end{aligned}$$

Set

$$\widetilde{\mathbb{U}}_3(\cdot, X_1, X_2) = \widehat{\mathbb{U}}_3(\cdot, X_1, X_2) + \frac{1}{2}(X_1 - 1)^2 \partial_{11}\mathbb{U}_3 + \frac{1}{2}(X_2 - 1)^2 \partial_{22}\mathbb{U}_3.$$

This function belongs to $L^2(\Omega; H^1_{per}(\mathcal{Y}))$.
Now, observe that by construction $\mathcal{T}_\varepsilon(\partial_1 \mathbb{U}_\varepsilon)(\cdot, X_1, X_2)$ is piecewise constant with respect to X_1 and linear with respect to X_2 in each domain $\mathcal{Y}_{ab}$, conversely $\mathcal{T}_\varepsilon(\partial_2 \mathbb{U}_\varepsilon)(\cdot, X_1, X_2)$ is piecewise constant with respect to X_2 and linear with respect to X_1 in each domain $\mathcal{Y}_{ab}$. As a consequence, the function $\partial_1 \widetilde{\mathbb{U}}_3(\cdot, X_1, X_2)$ is piecewise constant with respect to X_1 and linear with respect to X_2 in each domain $\mathcal{Y}_{ab}$, and $\partial_2 \widetilde{\mathbb{U}}_3(\cdot, X_1, X_2)$ is piecewise constant with respect to X_2 and linear with respect to X_1 in each domain $\mathcal{Y}_{ab}$. It means that $\widetilde{\mathbb{U}}_3$ belongs to $L^2(\Omega; Q^1_{per}(\mathcal{Y}))$. $\square$

7 Asymptotic behavior of the unfolded fields

7.1 Unfolding for the Textile

Set

$$Cyl^{(1,b)} \doteq b\mathbf{e}_2 + (0,2) \times (-\kappa, \kappa)^2, \qquad Cyl^{(2,a)} \doteq a\mathbf{e}_1 + (-\kappa, \kappa) \times (0,2) \times (-\kappa, \kappa),$$

$$Cyls^{(1)} \doteq Cyl^{(1,0)} \cup Cyl^{(1,1)}, \qquad Cyls^{(2)} \doteq Cyl^{(2,0)} \cup Cyl^{(2,1)}, \qquad Cyls \doteq Cyls^{(1)} \cup Cyls^{(2)}.$$

Definition 7.1. *For every φ measurable function in the domain $\mathbf{P}^{[\alpha]}_\varepsilon$, one defines the measurable function $\Pi^{[\alpha]}_\varepsilon(\varphi)$ in $\Omega \times Cyls^{(\alpha)}$ by ($\alpha \in \{1,2\}$)*

$$\Pi^{[\alpha]}_\varepsilon(\varphi)(z, X) = \varphi\big(2p\varepsilon\mathbf{e}_1 + 2q\varepsilon\mathbf{e}_2 + \varepsilon X\big), \qquad z \in 2p\varepsilon\mathbf{e}_1 + 2q\varepsilon\mathbf{e}_2 + \varepsilon\mathcal{Y}, \quad X \in Cyls^{(\alpha)}.$$

Furthermore, for every $\varphi \in L^s(\mathcal{S}_\varepsilon)$ ($s \in [1, +\infty)$) we define the unfolding operator

$$\Pi_\varepsilon(\varphi) = \big(\Pi^{(1)}_\varepsilon(\varphi^{[1]}), \Pi^{(2)}_\varepsilon(\varphi^{[2]})\big)$$

as a mapping from $L^s(\mathcal{S}_\varepsilon)$ into $L^s(\Omega \times Cyls^{(1)}) \times L^s(\Omega \times Cyls^{(2)})$ and we set

$$\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega \times Cyls)} = \Big(\|\Pi^{(1)}_\varepsilon(\varphi^{(1)})\|^s_{L^s(\Omega \times Cyls^{(1)})} + \|\Pi^{(2)}_\varepsilon(\varphi^{(2)})\|^s_{L^s(\Omega \times Cyls^{(2)})}\Big)^{1/s}.$$

In fact, the unfolding operator $\mathcal{T}_\varepsilon$, defined in 6.2, is a restriction of the unfolding operator $\Pi^{[\alpha]}$ of the complete textile. Indeed, we find for a function φ defined on $\mathcal{K}_\varepsilon$ and extended as in Subsection 5.1 into a function belonging to $W^{1,\infty}(\Omega)$, denoted $\boldsymbol{\varphi}$, then

$$\begin{aligned}\Pi^{[1]}_\varepsilon(\boldsymbol{\varphi}_{|\mathcal{Y}^{\ell s}})(s, X) = \boldsymbol{\varphi}\big((2p\mathbf{e}_1 + 2q\mathbf{e}_2)\varepsilon + \varepsilon b\mathbf{e}_2 + \varepsilon X_1\mathbf{e}_1\big) = \mathcal{T}_\varepsilon(\boldsymbol{\varphi})(s, X_1, b),\\ s \in (2p\mathbf{e}_1 + 2q\mathbf{e}_2) + \varepsilon\mathcal{Y}, \quad b \in \{0,1\}, \quad X_1 \in (0,2), \quad (p,q) \in \{0, \dots N_\varepsilon - 1\}^2.\end{aligned} \tag{7.1}$$

The second direction

$$\begin{aligned}\Pi^{[2]}_\varepsilon(\boldsymbol{\varphi}_{|\mathcal{Y}^{\ell s}})(s, X) = \boldsymbol{\varphi}\big((2p\mathbf{e}_1 + 2q\mathbf{e}_2)\varepsilon + \varepsilon a\mathbf{e}_1 + \varepsilon X_2\mathbf{e}_2\big) = \mathcal{T}_\varepsilon(\boldsymbol{\varphi})(s, a, X_2),\\ s \in (2p\mathbf{e}_1 + 2q\mathbf{e}_2) + \varepsilon\mathcal{Y}, \quad a \in \{0,1\}, \quad X_2 \in (0,2), \quad (p,q) \in \{0, \dots N_\varepsilon - 1\}^2,\end{aligned} \tag{7.2}$$

is derived analogously.

To characterize the unfolded functions, it is necessary to give a relation to the original function. Note, that this unfolding operator changes the convergence-rate, since a dimension reduction is directly incorporated. To address this individually, it is possible to define it as composition of an unfolding operator and a rescaling operator, see e.g. [7, 16].

Lemma 7.2. *For every $\varphi \in L^1(\mathbf{P}^{[\alpha]}_r)$, one has*

$$\int_{\mathbf{P}^{[\alpha]}_r} \varphi(z)dz = \frac{\varepsilon}{4}\int_\Omega \int_{Cyls^{[\alpha]}} \Pi^{[\alpha]}_\varepsilon(\varphi)(s, X)dsdX, \qquad \alpha \in \{1,2\}.$$

Proof. It is an easy consequence of the transformation of integrals and the definitions above. Indeed, we have for $\alpha = 1$

$$\begin{aligned}
\int_{\mathbf{P}_r^{[1]}} \varphi(z)dz &= \sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon}\int_{\varepsilon Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + z)dz = \varepsilon^3 \sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon}\int_{Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + \varepsilon X)dX \\
&= \frac{\varepsilon^3}{4\varepsilon^2}\sum_q^{N_\varepsilon}\sum_p^{N_\varepsilon}\int_{2p\varepsilon\mathbf{e}_1 + 2q\varepsilon\mathbf{e}_2 + (0,2\varepsilon)^2}\int_{Cyls^{(1)}} \varphi(2q\varepsilon\mathbf{e}_2 + 2p\varepsilon\mathbf{e}_1 + \varepsilon X)dXdz \\
&= \frac{\varepsilon}{4}\int_\Omega\int_{Cyls^{(1)}} \Pi_\varepsilon^{[1]}(\varphi)(z,X)dXdz.
\end{aligned}$$

Analogously for $\alpha = 2$ which yields then the claim. $\square$

Lemma 7.3. *For every $\varphi \in L^s(\mathcal{S}_\varepsilon)$, $s \in [1, +\infty]$ one has*

$$C_0\varepsilon^{1/s}\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega\times Cyls)} \le \|\varphi\|_{L^s(\mathcal{S}_\varepsilon)} \le C_1\varepsilon^{1/s}\|\Pi_\varepsilon(\varphi)\|_{L^s(\Omega\times Cyls)}. \tag{7.3}$$

Proof. First assume $s \in [1, +\infty)$. As consequence of the above Lemma and (4.2), one gets for every $\varphi \in L^s(\mathcal{S}_\varepsilon)$

$$\begin{aligned}
\|\varphi\|^s_{L^s(\mathcal{S}_\varepsilon)} = \int_{\mathcal{S}_\varepsilon}|\varphi(x)|^s dx &= \sum_{q=0}^{2N_\varepsilon}\int_{\mathbf{P}_r^{[1]}} |\varphi^{[1]}(q\varepsilon\mathbf{e}_2 + z)|^s|\det\big(\nabla\psi_\varepsilon^{(1,q)}(z)\big)|dz \\
&+ \sum_{p=1}^{2N_\varepsilon}\int_{\mathbf{P}_r^{[2]}} |\varphi^{[2]}(p\varepsilon\mathbf{e}_1 + z)|^s|\det\big(\nabla\psi_\varepsilon^{(2,p)}(z)\big)|dz. \\
&\le C\varepsilon\Big(\|\Pi_\varepsilon^{(1)}(\varphi^{[1]})\|^s_{L^s(\Omega\times Cyls^{(1)})} + \|\Pi_\varepsilon^{(2)}(\varphi^{[2]})\|^s_{L^s(\Omega\times Cyls^{(2)})}\Big).
\end{aligned}$$

Since the Jacobian's are bounded from below, we also obtain

$$C\varepsilon\Big(\|\Pi_\varepsilon^{(1)}(\varphi^{[1]})\|^s_{L^s(\Omega\times Cyls^{(1)})} + \|\Pi_\varepsilon^{(2)}(\varphi^{[2]})\|^s_{L^s(\Omega\times Cyls^{(2)})}\Big) \le \|\varphi\|^s_{L^s(\mathcal{S}_\varepsilon)}.$$

Hence (7.3) is proved for any $s \in [1, +\infty)$. The case $s = +\infty$ is obvious. $\square$

Actually, the most important in the following is the case $s = 2$ where

$$\|\Pi_\varepsilon(\varphi)\|_{L^2(\Omega\times Cyls)} \le \frac{C}{\sqrt{\varepsilon}}\|\varphi\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \forall\varphi \in L^2(\mathcal{S}_\varepsilon).$$

7.2 Limits of the unfolded elementary displacements

Lemma 7.4. *Under the assumptions of Lemma 6.1, the following convergences hold ($(\alpha,\beta) \in \{1,2\}^2$):*

$$\begin{aligned}
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}\big(\mathbb{U}^{(\alpha)}_{\varepsilon,3\,|\mathcal{Y}^{\ell s}}\big) \rightharpoonup \mathbb{U}_3 \;\; \textit{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),\\
&\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}\big(\mathcal{R}^{(\alpha)}_{\varepsilon,\beta\,|\mathcal{Y}^{\ell s}}\big) \rightharpoonup \mathcal{R}_\beta \;\; \textit{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),\\
&\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(\mathbb{U}^{(\alpha)}_{\varepsilon,\beta\,|\mathcal{Y}^{\ell s}}\big) \longrightarrow \mathbb{U}_\beta \;\; \textit{weakly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),
\end{aligned}$$

where the limit fields are given by Lemma 6.1.

Proof. These convergences are the consequences of the definitions (5.14), the convergences in Lemmas 6.1 and the definitions of the unfolding operators $\Pi_\varepsilon^{[\alpha]}$, $\mathcal{T}_\varepsilon$ and (7.1)-(7.2). $\square$

Denote

$$H^1_{00}(0,2) \doteq \Big\{\psi \in H^1(0,2) \;\Big|\; \psi(0) = \psi(1) = \psi(2) = 0\Big\}.$$

Recall, that the fields $\mathbb{U}_\varepsilon$, $\mathcal{R}_\varepsilon$ and $\mathbb{U}_\varepsilon^{(g)}$, $\mathcal{R}_\varepsilon^{(g)}$ have to be restricted to $L^{(\alpha)}$, the center lines of the beams, to build the actual beam displacements, cf. (5.15).

Lemma 7.5. *Under the assumptions of Lemma 6.1, there exist a subsequence of* $\{\varepsilon\}$ *(still denoted* $\{\varepsilon\}$*) and* $\widehat{\mathcal{R}}^{(\alpha,b)}$, $\widehat{\mathbb{U}}^{(\alpha,b)} \in L^2(\Omega; H^1_{00}(0,2))^3$ *such that the following convergences hold (*$(a,b) \in \{0,1\}^2$, $(\alpha,\beta) \in \{1,2\}^2$*):*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi^{[1]}_\varepsilon\big(\mathcal{R}^{[1]}_{\varepsilon,3}\big) \rightharpoonup \widehat{\mathcal{R}}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(1,b)}_3 \quad \textit{weakly in } L^2(\Omega; H^1(Cyl^{(1,b)})),\\
&\frac{1}{\varepsilon^2}\Pi^{[2]}_\varepsilon\big(\mathcal{R}^{[2]}_{\varepsilon,3}\big) \rightharpoonup \widehat{\mathcal{R}}_{3|X_1=a} - \widehat{\mathcal{R}}^{(g)}_{3|X_1=a} + \widehat{\mathcal{R}}^{(2,a)}_3 \quad \textit{weakly in } L^2(\Omega; H^1(Cyl^{(2,a)})),\\
&\frac{1}{\varepsilon}\Pi^{[1]}_\varepsilon\big(\partial_1\mathcal{R}^{[1]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_1\mathcal{R}_\beta + \partial_{X_1}\widehat{\mathcal{R}}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(1,b)}_\beta \ \textit{weakly in } L^2(\Omega\times Cyl^{(1,b)}),\\
&\frac{1}{\varepsilon}\Pi^{[2]}_\varepsilon\big(\partial_2\mathcal{R}^{[2]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_2\mathcal{R}_\beta + \partial_{X_2}\widehat{\mathcal{R}}_{\beta|X_1=a} - \partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}_\beta \ \textit{weakly in } L^2(\Omega\times Cyl^{(2,a)}),
\end{aligned}\tag{7.4}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi^{[1]}_\varepsilon\big(\partial_1\,\mathbb{U}^{[1]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_1\mathbb{U}_\beta + \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathbb{U}}_{\beta|X_2=b} + \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_\beta \ \textit{weakly in } L^2(\Omega\times Cyl^{(1,b)}),\\
&\frac{1}{\varepsilon^2}\Pi^{[2]}_\varepsilon\big(\partial_2\,\mathbb{U}^{[2]}_{\varepsilon,\beta}\big) \rightharpoonup \partial_2\mathbb{U}_\beta - \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathbb{U}}_{\beta|X_1=a} + \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_\beta \ \textit{weakly in } L^2(\Omega\times Cyl^{(2,a)}).
\end{aligned}\tag{7.5}$$

Moreover

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi^{[1]}_\varepsilon(\partial_1\mathbb{U}^{[1]}_\varepsilon - \mathcal{R}^{[1]}_\varepsilon\wedge\mathbf{e}_1) \rightharpoonup\\
&\begin{pmatrix}\partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b}\\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b}\\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b}\end{pmatrix} + \begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_1\\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_2 - \widehat{\mathcal{R}}^{(1,b)}_3\\ \partial_{X_1}\widehat{\mathbb{U}}^{(1,b)}_3 + \widehat{\mathcal{R}}^{(1,b)}_2\end{pmatrix} + \begin{pmatrix}\partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b}\\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b}\\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b}\end{pmatrix}\\
&\hspace{8cm}\textit{weakly in } L^2(\Omega\times Cyl^{(1,b)})^3,
\end{aligned}\tag{7.6}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi^{[2]}_\varepsilon(\partial_2\mathbb{U}^{[2]}_\varepsilon - \mathcal{R}^{[2]}_\varepsilon\wedge\mathbf{e}_2) \rightharpoonup\\
&\begin{pmatrix}\partial_2\mathbb{U}_1 + \partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a} + \widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2 + \partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a} - \widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix} + \begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1 + \widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3 - \widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix} - \begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a} + \widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a} - \widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\\
&\hspace{8cm}\textit{weakly in } L^2(\Omega\times Cyl^{(2,a)})^3.
\end{aligned}\tag{7.7}$$

Proof. First, as a consequence of estimates (5.4), there exist a subsequence of $\{\varepsilon\}$ (still denoted $\{\varepsilon\}$) and $\widehat{\mathcal{R}}^{(\alpha,c)}$, $\widehat{\mathbb{U}}^{(\alpha,c)} \in L^2(\Omega; H^1_{00}(0,2))^3$ such that the following convergences hold ($c \in \{0,1\}$, $(\alpha,\beta) \in \{1,2\}^2$):

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Pi^{[\alpha]}_\varepsilon\big(\widetilde{\mathcal{R}}^{(\alpha)}_\varepsilon\big) \rightharpoonup \widehat{\mathcal{R}}^{(\alpha,c)} \quad \text{weakly in } L^2(\Omega; H^1(Cyl^{(\alpha,b)}))^3,\\
&\frac{1}{\varepsilon^3}\Pi^{[\alpha]}_\varepsilon\big(\widetilde{\mathbb{U}}^{(\alpha)}_\varepsilon\big) \rightharpoonup \widehat{\mathbb{U}}^{(\alpha,c)} \quad \text{weakly in } L^2(\Omega; H^1(Cyl^{(\alpha,b)}))^3.
\end{aligned}\tag{7.8}$$

Furthermore, note that the displacements are split according to (5.15). Hence, with Lemma 10.4 we obtain the restrictions onto the beam centerlines for the limit fields.

In fact, it is a priori not clear if the limit functions admit a trace. Actually, to obtain this result note that due to the piecewise-linear character of the functions, one has

$$\|[\partial_1\mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\|_{L^2(\Omega)} + \varepsilon\,\|\partial_2\,[(\partial_1\mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon\wedge\mathbf{e}_1)\cdot\mathbf{e}_3]\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}\|e(u)\|_{L^2(\mathcal{S}_\varepsilon)} + \frac{C}{\varepsilon}\|\mathbf{g}_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^2.$$

As a consequence the restricted unfolded function equals the unfolded restricted function, i.e.

$$\Pi^{[1]}_\varepsilon\big(\,[\partial_1\mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\big)_{|X_2=b} = \Pi^{[1]}_\varepsilon\big(\,([\partial_1\mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3)_{|L^{(1)}_\varepsilon}\,\big)$$

and we have by Lemma 10.4

$$\|\Pi^{[1]}_\varepsilon\big(\,[\partial_1\mathbb{U}_\varepsilon - \mathcal{R}_\varepsilon\wedge\mathbf{e}_1]\cdot\mathbf{e}_3\big)_{|X_2=b}\|_{L^2(\Omega\times(\overline{\mathcal{Y}}\cap\{X_2=b\})} \le C\varepsilon^2.$$

The second direction is analogously.

Observe that the resulting restrictions only apply to the variable in the "lateral" direction, i.e. $X_2 = b$ for the fields corresponding to the fields with index $(1,b)$ (or $X_1 = a$ for $(2,a)$ respectively).

Then, convergences (7.4)-(7.5) are the consequences of the above, those in Lemma 6.5 and (7.8). From (7.4)-(7.5) we also derive $(7.6)_{1,2}$-$(7.7)_{1,2}$. For the convergences $(7.6)_3$-$(7.7)_3$ we use Lemma 6.5 and (7.4)-(7.5). ☐

Remark 7.6. *The limit displacements itself converge strongly*

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}(u_{\beta,\varepsilon}^{[\alpha]}) \to \mathbb{U}_\beta - \left(\frac{X_3}{\gamma}+\Phi\right)\partial_\beta \mathbb{U}_3, \qquad \textit{strongly in } L^2(\Omega; H^1(Cyls^{(\alpha)})),$$
$$\frac{1}{\varepsilon}\Pi_\varepsilon^{[\alpha]}(u_{3,\varepsilon}^{[\alpha]}) \to \mathbb{U}_3, \qquad \textit{strongly in } L^2(\Omega; H^1(Cyls^{(\alpha)})).$$

Note, that for $\Phi \equiv 0$ they coincide with the usual Kirchoff-Love *displacement for a plate.*

7.3 The limit of the warping

Now, set for the convergences of the warpings

$$\widehat{\mathbf{W}}^{(1)} \doteq \{v^{(1)} = (v^{(1,0)}, v^{(1,1)}) \in H^1(Cyls^{(1)}) \ | \ v^{(1,b)}(2, X_2-b, X_3) = v^{(1,b)}(0, X_2-b, X_3)\},$$
$$\widehat{\mathbf{W}}^{(2)} \doteq \{v^{(2)} = (v^{(2,0)}, v^{(2,1)}) \in H^1(Cyls^{(2)}) \ | \ v^{(2,a)}(X_1-a, 2, X_3) = v^{(2,a)}(X_1-a, 0, X_3)\}.$$

Lemma 7.7. *There exists a subsequence, still denoted by ε, and $\overline{u}^{(\alpha)} \in L^2(\Omega; \widehat{\mathbf{W}}^{(\alpha)})^3$ such that the following convergence holds*

$$\frac{1}{\varepsilon^3}\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]}) \rightharpoonup \overline{u}^{(\alpha,c)} \ \ \textit{weakly in} \quad L^2(\Omega; H^1(Cyl^{(\alpha,c)}))^3, \quad \alpha \in \{1,2\}, \ \ c \in \{0,1\}.$$

Furthermore, the fields $\overline{u}^{(1,b)}$, $b \in \{0,1\}$, satisfy a.e. in $\Omega \times (0,L)$

$$\int_\omega \overline{u}^{(1,b)}(\cdot, X)dX_2dX_3 = 0, \qquad \int_\omega \overline{u}^{(1,b)}(\cdot, X) \wedge \big((X_2-b)\mathbf{e}_2 + X_3\mathbf{n}(X_1)\big)dX_2dX_3 = 0. \tag{7.9}$$

or respectively for $\overline{u}^{(2,a)}$, $a \in \{0,1\}$

$$\int_\omega \overline{u}^{(2,a)}(\cdot, X)dX_1dX_3 = 0, \qquad \int_\omega \overline{u}^{(2,a)}(\cdot, X) \wedge \big((X_1-a)\mathbf{e}_1 + X_3\mathbf{n}(X_2)\big)dX_1dX_3 = 0. \tag{7.10}$$

Proof. From (3.6)-(7.3) we have for the warping terms the estimates

$$\begin{aligned} \|\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]})\|_{L^2(\Omega\times Cyls^{(\alpha)})} &= \frac{2}{\sqrt{\varepsilon}}\|\overline{u}_\varepsilon^{[\alpha]}\|_{L^2(\mathcal{S}_\varepsilon)} \le C\frac{r}{\sqrt{\varepsilon}}\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \le C\varepsilon^3 \\ \Big\|\frac{\partial}{\partial X_i}\Pi_\varepsilon^{[\alpha]}(\overline{u}_\varepsilon^{[\alpha]})\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} &= \varepsilon\Big\|\Pi_\varepsilon^{[\alpha]}\Big(\frac{\partial}{\partial z_i}\overline{u}_\varepsilon^{[\alpha]}\Big)\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} \\ &\le C\frac{\varepsilon}{\sqrt{\varepsilon}}\Big\|\frac{\partial}{\partial z_i}\overline{u}_\varepsilon^{[\alpha]}\Big\|_{L^2(\mathcal{S}_\varepsilon)} \le C\sqrt{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \le C\varepsilon^3. \end{aligned} \tag{7.11}$$

The conditions (7.9) and (7.10) are the result of the conditions (3.5) on the warping. □

For simplification define the spaces

$$\overline{\mathbf{W}}^{(1)} \doteq \{v \in \widehat{\mathbf{W}}^{(1)} \ | \ v \text{ satisfies (7.9)}\}, \qquad \overline{\mathbf{W}}^{(2)} \doteq \{v \in \widehat{\mathbf{W}}^{(2)} \ | \ v \text{ satisfies (7.10)}\}.$$

To conclude this section, note that the limit of the warping strain tensor is directly inherited of (3.16), i.e. the symmetric gradient of one beam, resulting in

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[\alpha]}\big(e_z(\overline{u}_\varepsilon^{[\alpha]})\big) \rightharpoonup \mathcal{E}_X^{(\alpha,c)}(\overline{u}^{(\alpha,c)}) \quad \text{weakly in} \quad L^2(\Omega \times Cyl^{(\alpha,c)})^{3\times 3}, \quad c \in \{0,1\}$$

with

$$\mathcal{E}_X^{(1,b)}(\varphi) = \begin{pmatrix} \frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{t}^{(1,b)} & * & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{e}_2 + \partial_{X_2}\varphi\cdot\mathbf{t}^{(1,b)}\big) & \partial_{X_2}\varphi\cdot\mathbf{e}_2 & * \\ \frac{1}{2}\big(\frac{1}{\eta^{(1,b)}}\partial_{X_1}\varphi\cdot\mathbf{n}^{(1,b)} + \partial_{X_3}\varphi\cdot\mathbf{t}^{(1,b)}\big) & \frac{1}{2}\big(\partial_{X_2}\varphi\cdot\mathbf{n}^{(1,b)} + \partial_{X_3}\varphi\cdot\mathbf{e}_2\big) & \partial_{X_3}\varphi\cdot\mathbf{n}^{(1,b)} \end{pmatrix} \tag{7.12}$$

$$\mathcal{E}_X^{(2,a)}(\varphi) = \begin{pmatrix} \partial_{X_1}\varphi\cdot\mathbf{e}_1 & * & * \\ \frac{1}{2}\big(\partial_{X_1}\varphi\cdot\mathbf{t}^{(2,a)} + \frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{e}_1\big) & \frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{t}^{(2,a)} & * \\ \frac{1}{2}\big(\partial_{X_1}\varphi\cdot\mathbf{n}^{(2,a)} + \partial_{X_3}\varphi\cdot\mathbf{e}_1\big) & \frac{1}{2}\big(\frac{1}{\eta^{(2,a)}}\partial_{X_2}\varphi\cdot\mathbf{n}^{(2,a)} + \partial_{X_3}\varphi\cdot\mathbf{t}^{(2,a)}\big) & \partial_{X_3}\varphi\cdot\mathbf{n}^{(2,a)} \end{pmatrix} \tag{7.13}$$

for the first and second direction respectively.

7.4 The limit strain tensor for the elementary displacement

For the strong limits $\Phi^{(\alpha,b)}$, $\boldsymbol{\gamma}$, $\mathbf{t}^{(\alpha,b)}$ and $\mathbf{n}^{(\alpha,b)}$, see Appendix 10.

First note that the strain-tensor admits a weak limit in form of a weak convergent subsequence. Indeed assumption (6.2) gives rise to the estimate:

$$\Big\| \frac{1}{\varepsilon^2} \Pi_\varepsilon^{[\alpha]}\big(e_z(u_\varepsilon^{[\alpha]})\big)\Big\|_{L^2(\Omega\times Cyls^{(\alpha)})} \le \frac{1}{\varepsilon^{5/2}} \|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \le C.$$

Hence, there exists a weak convergent subsequence. However, to state the actual limit problem this is not enough. For this all convergences in the section above are needed. To simplify the representation of the limit strain tensor we split the limit into two main parts

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(e_z(u_\varepsilon)\big) \rightharpoonup \mathbf{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\overline{u}^{(1,b)}) \quad \text{weakly in } \ L^2(\Omega; H^1(Cyl^{(1,b)}))^9, \quad b\in\{0,1\},$$

where $\mathbf{E}^{(1,b)}$ denotes the limit of the strain tensor for the elementary displacements.

Recall, the form of the strain-tensor for one beam (3.18). Then for every field use the decomposition developed in Sections 3-5 and with the convergences above we find for the elementary displacement the limit strain tensor entries $\mathbf{E}^{(1,b)}_{z,23} = \mathbf{E}^{(1,b)}_{z,22} = \mathbf{E}^{(1,b)}_{z,33} = 0$ and

$$\begin{aligned}
\eta \mathbf{E}^{(1,b)}_{z,11} =& \Bigg[\begin{pmatrix} \partial_1 \mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}_1^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_2^{(1,b)} - \widehat{\mathcal{R}}_3^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_3^{(1,b)} + \widehat{\mathcal{R}}_2^{(1,b)}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b}\end{pmatrix}\Bigg]\cdot \mathbf{t}^{(1,b)} \\
&+ \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\Big(\big(\frac{\Phi^{(1,b)}}{\boldsymbol{\gamma}} + X_3\big)\mathbf{e}_2 - (X_2-b)\mathbf{n}^{(1,b)}\Big)
\end{aligned}$$

$$\begin{aligned}
2\eta \mathbf{E}^{(1,b)}_{z,12} =& \Bigg[\begin{pmatrix} \partial_1 \mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}_1^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_2^{(1,b)} - \widehat{\mathcal{R}}_3^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_3^{(1,b)} + \widehat{\mathcal{R}}_2^{(1,b)}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b}\end{pmatrix}\Bigg]\cdot \mathbf{e}_2 \\
&- \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\big(X_3\mathbf{t}^{(1,b)} + \Phi^{(1,b)}\mathbf{e}_1\big)
\end{aligned}$$

$$\begin{aligned}
2\eta \mathbf{E}^{(1,b)}_{z,13} =& \Bigg[\begin{pmatrix} \partial_1 \mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_{1|X_2=b} \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_{2|X_2=b} - \widehat{\mathcal{R}}_{3|X_2=b} \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_{3|X_2=b} + \widehat{\mathcal{R}}_{2|X_2=b}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}_1^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_2^{(1,b)} - \widehat{\mathcal{R}}_3^{(1,b)} \\ \partial_{X_1}\widehat{\mathbb{U}}_3^{(1,b)} + \widehat{\mathcal{R}}_2^{(1,b)}\end{pmatrix} + \begin{pmatrix} \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{1|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{2|X_2=b} - \widehat{\mathcal{R}}^{(g)}_{3|X_2=b} \\ \partial_{X_1}\widehat{\mathbb{U}}^{(g)}_{3|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{2|X_2=b}\end{pmatrix}\Bigg]\cdot \mathbf{n}^{(1,b)} \\
&+ \big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}_{|X_2=b} + \partial_{X_1}\widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \partial_{X_1}\widetilde{\mathcal{R}}^{(1,b)}\big)\cdot\Big((X_2-b)\mathbf{t}^{(1,b)} - \frac{\Phi^{(1,b)}d_{X_1}\Phi^{(1,b)}}{\boldsymbol{\gamma}}\mathbf{e}_2\Big).
\end{aligned}$$

To simplify this tensor field $\mathbf{E}^{(1,b)}$ define the purely microscopic displacement

$$\begin{aligned}
\widehat{u}^{(1,b)} =& \big(\widehat{\mathbb{U}}_{|X_2=b} + \widehat{\mathbb{U}}^{(g)}_{|X_2=b} + \widehat{\mathbb{U}}^{(1,b)}\big) \\
&+ \big(\mathbf{Z} + \widehat{\mathcal{R}}_{|X_2=b} + \widehat{\mathcal{R}}^{(g)}_{|X_2=b} + \widehat{\mathcal{R}}^{(1,b)}\big)\wedge\big(\Phi^{(1,b)}\mathbf{e}_3 + X_3\mathbf{n}^{(1,b)} + (X_2-b)\mathbf{e}_2\big) + \overline{u}^{(1,b)} \qquad (7.14)
\end{aligned}$$

where

$$\mathbf{Z} = -\mathcal{Z}_{23}\mathbf{e}_1 + \mathcal{Z}_{13}\mathbf{e}_2 - \frac{1}{2}\big(\partial_1\mathbb{U}_2 - \partial_2\mathbb{U}_1\big)\mathbf{e}_3.$$

Then the strain tensor limit for the elementary displacement can be rewritten

$$\mathbf{E}_z^{(1,b)} + \mathcal{E}_X^{(1,b)}(\overline{u}^{(1,b)}) = \mathcal{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)})$$

or equivalently write directly

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{[1]}\big(e_z(u_\varepsilon)\big) \rightharpoonup \mathcal{E}^{(1,b)} + \mathcal{E}_X^{(1,b)}(\widehat{u}^{(1,b)}), \qquad \text{weakly in } \ L^2(\Omega; H^1(Cyl^{(1,b)}))^9, \quad b\in\{0,1\}, \qquad (7.15)$$

where

$$
\begin{aligned}
\mathcal{E}^{(1,b)}_{11}(\mathbb{U}) =& \frac{1}{\eta^{(1,b)}}\Bigg[\begin{pmatrix} e_{11}(\mathbb{U})\\ e_{12}(\mathbb{U})\\ 0\end{pmatrix}\cdot \mathbf{t}^{(1,b)} + \begin{pmatrix}\partial_{12}\mathbb{U}_3\\ -\partial_{11}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\Big(\Big(X_3+\frac{\Phi^{(1,b)}}{\gamma}\Big)\mathbf{e}_2-(X_2-b)\,\mathbf{n}^{(1,b)}\Big)\Bigg]\\
\mathcal{E}^{(1,b)}_{12}(\mathbb{U}) =& \frac{1}{2\eta^{(1,b)}}\Bigg[\begin{pmatrix} e_{11}(\mathbb{U})\\ e_{12}(\mathbb{U})\\ 0\end{pmatrix}\cdot \mathbf{e}_2 - \begin{pmatrix}\partial_{12}\mathbb{U}_3\\ -\partial_{11}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\Big(X_3\mathbf{t}^{(1,b)}+\Phi^{(1,b)}\mathbf{e}_1\Big)\Bigg],\\
\mathcal{E}^{(1,b)}_{13}(\mathbb{U}) =& \frac{1}{2\eta^{(1,b)}}\Bigg[\begin{pmatrix} e_{11}(\mathbb{U})\\ e_{12}(\mathbb{U})\\ 0\end{pmatrix}\cdot \mathbf{n}^{(1,b)} + \begin{pmatrix}\partial_{12}\mathbb{U}_3\\ -\partial_{11}\mathbb{U}_3\\ 0\end{pmatrix}\cdot\Big((X_2-b)\,\mathbf{t}^{(1,b)}-\frac{\Phi^{(1,b)}d_1\Phi^{(1,b)}}{\gamma}\mathbf{e}_2\Big)\Bigg].
\end{aligned}
\tag{7.16}
$$

and $\mathcal{E}^{(1,b)}_{22}=\mathcal{E}^{(1,b)}_{33}=\mathcal{E}^{(1,b)}_{23}=0$ include all macroscopic fields. Note, that for this representation the identities (6.5) were used.

7.4.1 The limit strain tensor for the $\mathbf{e}_2$-direction

For the sake of completeness, the limit strain tensor for the $\mathbf{e}_2$-directed beams is adressed hereafter. Nevertheless, due to the very similar character only the end result for the elementary displacement is shown. Besides $\mathbf{E}^{(2,a)}_{z,11}=\mathbf{E}^{(2,a)}_{z,13}=\mathbf{E}^{(2,a)}_{z,33}=0$, one has

$$
\begin{aligned}
\eta\mathbf{E}^{(2,a)}_{z,22} =& \Bigg[\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1+\widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3-\widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix}-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\Bigg]\cdot\mathbf{t}^{(2,a)}\\
&-\big(\partial_2\mathcal{R}+\partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}-\partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\partial_{X_2}\widetilde{\mathcal{R}}^{(2,a)}\big)\cdot\big(\big(\frac{\Phi^{(2,a)}}{\gamma}+X_3\big)\mathbf{e}_1-(X_1-a)\mathbf{n}^{(2,a)}\big),
\end{aligned}
$$

$$
\begin{aligned}
2\eta\mathbf{E}^{(2,a)}_{z,12} =& \Bigg[\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1+\widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3-\widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix}-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\Bigg]\cdot\mathbf{e}_1\\
&+\big(\partial_2\mathcal{R}+\partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}-\partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}\big)\cdot\big(X_3\mathbf{t}^{(2,a)}+\Phi^{(2,a)}\mathbf{e}_2\big),
\end{aligned}
$$

$$
\begin{aligned}
2\eta\mathbf{E}^{(2,a)}_{z,23} =& \Bigg[\begin{pmatrix}\partial_2\mathbb{U}_1+\partial_{X_2}\widehat{\mathbb{U}}_{1|X_1=a}+\widehat{\mathcal{R}}_{3|X_1=a}\\ \partial_2\mathbb{U}_2+\partial_{X_2}\widehat{\mathbb{U}}_{2|X_1=a}\\ \mathcal{Z}_{23}+\partial_{X_2}\widehat{\mathbb{U}}_{3|X_1=a}-\widehat{\mathcal{R}}_{1|X_1=a}\end{pmatrix}+\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_1+\widehat{\mathcal{R}}^{(2,a)}_3\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_2\\ \partial_{X_2}\widehat{\mathbb{U}}^{(2,a)}_3-\widehat{\mathcal{R}}^{(2,a)}_1\end{pmatrix}-\begin{pmatrix}\partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{1|X_1=a}+\widehat{\mathcal{R}}^{(g)}_{3|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{2|X_1=a}\\ \partial_{X_2}\widehat{\mathbb{U}}^{(g)}_{3|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{1|X_1=a}\end{pmatrix}\Bigg]\cdot\mathbf{n}^{(2,a)}\\
&-\big(\partial_1\mathcal{R}+\partial_{X_2}\widehat{\mathcal{R}}_{|X_1=a}-\partial_{X_2}\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\partial_{X_2}\widehat{\mathcal{R}}^{(2,a)}\big)\cdot\big((X_1-a)\mathbf{t}^{(2,a)}-\frac{\Phi^{(2,a)}d_{X_2}\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\big).
\end{aligned}
$$

Define analogously to (7.14), the microscopic displacement

$$
\begin{aligned}
\widehat{u}^{(2,a)}=&\big(\widehat{\mathbb{U}}_{|X_1=a}-\widehat{\mathbb{U}}^{(g)}_{|X_1=a}+\widehat{\mathbb{U}}^{(2,a)}\big)\\
&+\big(\mathbf{Z}+\widehat{\mathcal{R}}_{|X_1=a}-\widehat{\mathcal{R}}^{(g)}_{|X_1=a}+\widehat{\mathcal{R}}^{(2,a)}\big)\wedge\big(\Phi^{(2,a)}\mathbf{e}_3+X_3\mathbf{n}^{(2,a)}+(X_1-a)\mathbf{e}_1\big)+\overline{u}^{(2,a)}.
\end{aligned}
\tag{7.17}
$$

For the same reason the limit strain tensor splits into two parts

$$
\frac{1}{\varepsilon^2}\Pi^{[2]}_\varepsilon\big(e_z(u_\varepsilon)\big)\rightharpoonup\mathcal{E}^{(2,a)}+\mathcal{E}^{(2,a)}_X(\widehat{u}^{(2,a)})\quad\text{weakly in}\ \ L^2(\Omega\times Cyl^{(2,a)})^9
$$

collecting global

$$
\begin{aligned}
\mathcal{E}_{22}^{(2,a)}(\mathbb{U}) =& \frac{1}{\eta^{(2,a)}}\left[\begin{pmatrix} e_{12}(\mathbb{U}) \\ e_{22}(\mathbb{U}) \\ 0 \end{pmatrix}\cdot \mathbf{t}^{(2,a)} - \begin{pmatrix} \partial_{22}\mathbb{U}_3 \\ -\partial_{12}\mathbb{U}_3 \\ 0 \end{pmatrix}\cdot\left(\left(X_3+\frac{\Phi^{(2,a)}}{\gamma}\right)\mathbf{e}_1-(X_1-a)\,\mathbf{n}^{(2,a)}\right)\right],\\
\mathcal{E}_{12}^{(2,a)}(\mathbb{U}) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix} e_{12}(\mathbb{U}) \\ e_{22}(\mathbb{U}) \\ 0 \end{pmatrix}\cdot \mathbf{e}_1 + \begin{pmatrix} \partial_{22}\mathbb{U}_3 \\ -\partial_{12}\mathbb{U}_3 \\ 0 \end{pmatrix}\cdot\left(X_3\mathbf{t}^{(2,a)}+\Phi^{(2,a)}\mathbf{e}_2\right)\right],\\
\mathcal{E}_{23}^{(2,a)}(\mathbb{U}) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix} e_{12}(\mathbb{U}) \\ e_{22}(\mathbb{U}) \\ 0 \end{pmatrix}\cdot \mathbf{n}^{(2,a)} - \begin{pmatrix} \partial_{22}\mathbb{U}_3 \\ -\partial_{12}\mathbb{U}_3 \\ 0 \end{pmatrix}\cdot\left((X_1-a)\,\mathbf{t}^{(2,a)}-\frac{\Phi^{(2,a)}d\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\right)\right].
\end{aligned}
\tag{7.18}
$$

and local displacements $\mathcal{E}_X^{(2,a)}(\widehat{u}^{(2,a)})$ in the form (7.13).

7.5 The limit contact conditions

Recall the decomposition in the contact parts, see (5.8), and note that it reduces to

$$
\begin{aligned}
u_\varepsilon^{(1,q)}(x) &= \mathbb{U}_\varepsilon^{(1,q)}(p\varepsilon+z_1)+\mathcal{R}_\varepsilon^{(1,q)}(p\varepsilon+z_1)\wedge z_2\mathbf{e}_2+\overline{u}_\varepsilon^{(1,q)}(x),\\
u_\varepsilon^{(2,p)}(x) &= \mathbb{U}_\varepsilon^{(2,p)}(q\varepsilon+z_2)+\mathcal{R}_\varepsilon^{(2,p)}(q\varepsilon+z_2)\wedge z_1\mathbf{e}_1+\overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned}
$$

for a.e. $x\in\mathbf{C}_{pq}$ or equivalently $z=(z_1,z_2)\in\omega_r$ and $|z_3|=r$. Using additionally the splitting (5.15), we obtain for the displacements in the contact parts

$$
\begin{aligned}
u_\varepsilon^{(1,q)}(x) =& \mathbb{U}_\varepsilon(p\varepsilon+z_1,q\varepsilon)+\mathbb{U}_\varepsilon^{(g)}(p\varepsilon+z_1,q\varepsilon)+\widetilde{\mathbb{U}}_\varepsilon^{(1,q)}(p\varepsilon+z_1)\\
&+\Big[\mathcal{R}_\varepsilon(p\varepsilon+z_1,q\varepsilon)+\mathcal{R}_\varepsilon^{(g)}(p\varepsilon+z_1,q\varepsilon)+\widetilde{\mathcal{R}}_\varepsilon^{(1,q)}(p\varepsilon+z_1)\Big]\wedge z_2\mathbf{e}_2+\overline{u}_\varepsilon^{(1,q)}(x),\\
u_\varepsilon^{(2,p)}(x) =& \mathbb{U}_\varepsilon(p\varepsilon,q\varepsilon+z_2)-\mathbb{U}_\varepsilon^{(g)}(p\varepsilon,q\varepsilon+z_2)+\widetilde{\mathbb{U}}_\varepsilon^{(2,p)}(q\varepsilon+z_2)\\
&+\Big[\mathcal{R}_\varepsilon(p\varepsilon,q\varepsilon+z_2)-\mathcal{R}_\varepsilon^{(g)}(p\varepsilon,q\varepsilon+z_2)+\widetilde{\mathcal{R}}_\varepsilon^{(2,p)}(q\varepsilon+z_2)\Big]\wedge z_1\mathbf{e}_1+\overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned}
$$

From (5.2), one obtains (same identities for $\mathcal{R}$)

$$
\left.
\begin{aligned}
\mathbb{U}_\varepsilon(p\varepsilon+z_1,q\varepsilon+z_2) &= \mathbb{U}_\varepsilon(p\varepsilon+z_1,q\varepsilon)+z_2\frac{\partial\mathbb{U}_\varepsilon}{\partial z_2}(p\varepsilon+z_1,q\varepsilon+z_2),\\
\mathbb{U}_\varepsilon(p\varepsilon+z_1,q\varepsilon+z_2) &= \mathbb{U}_\varepsilon(p\varepsilon,q\varepsilon+z_2)+z_1\frac{\partial\mathbb{U}_\varepsilon}{\partial z_1}(p\varepsilon+z_1,q\varepsilon+z_2),
\end{aligned}
\right.\qquad \forall(z_1,z_2)\in\omega_r.
$$

These identities yield for a.e. $x\in\mathbf{C}_{pq}$ that the difference between two beam-displacements in contact can be written as

$$
\begin{aligned}
u_\varepsilon^{(1,q)}(x)-u_\varepsilon^{(2,p)}(x) =& -z_2\Big(\frac{\partial\mathbb{U}_\varepsilon}{\partial z_2}-\mathcal{R}_\varepsilon\wedge\mathbf{e}_2\Big)(p\varepsilon+z_1,q\varepsilon+z_2)+z_1\Big(\frac{\partial\mathbb{U}_\varepsilon}{\partial z_1}-\mathcal{R}_\varepsilon\wedge\mathbf{e}_1\Big)(p\varepsilon+z_1,q\varepsilon+z_2)\\
&+\mathbb{U}^{(g)}(p\varepsilon+z_1,q\varepsilon)+\mathbb{U}^{(g)}(p\varepsilon,q\varepsilon+z_2)+\widetilde{\mathbb{U}}^{(1,q)}(p\varepsilon+z_1)-\widetilde{\mathbb{U}}^{(2,p)}(q\varepsilon+z_2)\\
&+\Big[-z_2\frac{\partial\mathcal{R}_\varepsilon}{\partial z_2}(p\varepsilon+z_1,q\varepsilon+z_2)+\mathcal{R}_\varepsilon^{(g)}(p\varepsilon+z_1,q\varepsilon)+\widetilde{\mathcal{R}}_\varepsilon^{(1,q)}(P\varepsilon+z_1)\Big]\wedge z_2\mathbf{e}_2\\
&+\Big[z_1\frac{\partial\mathcal{R}_\varepsilon}{\partial z_1}(p\varepsilon+z_1,q\varepsilon+z_2)+\mathcal{R}_\varepsilon^{(g)}(p\varepsilon,q\varepsilon+z_2)-\widetilde{\mathcal{R}}_\varepsilon^{(2,q)}(q\varepsilon+z_2)\Big]\wedge z_1\mathbf{e}_1+\overline{u}_\varepsilon^{(1,q)}(x)-\overline{u}_\varepsilon^{(2,p)}(x).
\end{aligned}
\tag{7.19}
$$

This expansion allows to estimate the jump and obtain the correct convergences via the following Lemma.

Lemma 7.8. *The difference of* $u_\varepsilon^{(1,q)}$ *and* $u_\varepsilon^{(2,p)}$ *satisfies*

$$
\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|u_\varepsilon^{(1,q)}-u_\varepsilon^{(2,p)}\|^2_{L^2(\mathbf{C}_{pq})}\le C\varepsilon^6.
$$

Proof. The estimate of the Lemma is an immediate consequence of (7.19) and the Lemmas 5.2, 5.11 as well as the estimates (7.11). ☐

To obtain the limit, it is necessary to introduce a third unfolding operator. Therefore, let

$$\mathbf{C} \doteq \bigcup_{a,b=0}^{1} \mathbf{C}_{ab}, \qquad \mathbf{C}_{ab} = \omega_\kappa + a\,\mathbf{e}_1 + b\,\mathbf{e}_2, \qquad (a,b) \in \{0,1\}^2,$$

denote the limit contact domain. Then, the unfolding operator for the contact is defined for every $\varphi \in L^p\left(\bigcup_{(p,q)\in\mathcal{K}_\varepsilon} \mathbf{C}_{pq}\right)$ by

$$T_\varepsilon^C(\varphi)(z_1,z_2,X_1,X_2) = \varphi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon\begin{pmatrix} a \\ b\end{pmatrix} + \varepsilon\left(X' - \begin{pmatrix} a \\ b\end{pmatrix}\right)\right),$$

with $\mathcal{T}_\varepsilon(\varphi) \in L^p(\Omega\times\mathbf{C})$. Note, that this operator is related to the previous defined unfolding operators via the identities

$$\begin{aligned}
&T_\varepsilon^C(\varphi)(\cdot,X_1,X_2) = \mathcal{T}_\varepsilon(\varphi)_{|\Omega\times\mathbf{C}}(\cdot,X_1,X_2), \qquad \text{for } \varphi\in L^p(\Omega), && (7.20)\\
&T_\varepsilon^C(\varphi)(\cdot,X_1,X_2) = \Pi_\varepsilon^{(\alpha)}(\varphi)_{|\Omega\times\mathbf{C}} = \Pi_\varepsilon^{(\alpha)}(\varphi)(\cdot,X_1,X_2,(-1)^{\alpha+a+b}\kappa), \quad \text{for } \varphi\in L^p(\mathbf{P}^{[1]}), && (7.21)
\end{aligned}$$

The following Lemma gives the main property of T_ε^C

Lemma 7.9. *The unfolding operator T_ε^C satisfies*

$$\|T_\varepsilon^C(\varphi)\|_{L^p(\Omega\times\mathbf{C})} \le C\|\varphi\|_{L^p(\bigcup_{(p,q)\in\mathcal{K}_\varepsilon}\mathbf{C}_{pq})}, \qquad \textit{for every } \varphi\in L^p\Big(\bigcup_{(p,q)\in\mathcal{K}_\varepsilon}\mathbf{C}_{pq}\Big)$$

Proof. Follows directly from $(7.20)_1$ and Lemma 6.3. □

Due to Lemmas 6.5, 7.4, 7.5, 7.8 and 7.9 the following weak convergence is obtained ($\cdot$ represents the macroscopic variable $z=(z_1,z_2)$):

$$\begin{aligned}
\frac{1}{\varepsilon^3}\big[T_\varepsilon^C(u^{[1]}) - T_\varepsilon^C(u^{[2]})\big] \rightharpoonup\; &-(X_2-b)\begin{pmatrix} \partial_2\mathbb{U}_1 + \partial_{X_2}\widehat{\mathbb{U}}_1 + \widehat{\mathcal{R}}_3 \\ \partial_2\mathbb{U}_2 + \partial_{X_2}\widehat{\mathbb{U}}_2 \\ \mathcal{Z}_{23} + \partial_{X_2}\widehat{\mathbb{U}}_3 - \widehat{\mathcal{R}}_1\end{pmatrix} + (X_1-a)\begin{pmatrix} \partial_1\mathbb{U}_1 + \partial_{X_1}\widehat{\mathbb{U}}_1 \\ \partial_1\mathbb{U}_2 + \partial_{X_1}\widehat{\mathbb{U}}_2 - \widehat{\mathcal{R}}_3 \\ \mathcal{Z}_{13} + \partial_{X_1}\widehat{\mathbb{U}}_3 + \widehat{\mathcal{R}}_2\end{pmatrix}\\
&+ \widehat{\mathbb{U}}^{(g)}(\cdot,X_1,b) + \widehat{\mathbb{U}}^{(g)}(\cdot,a,X_2) + \widehat{\mathbb{U}}^{(1,b)}(\cdot,X_1) - \widehat{\mathbb{U}}^{(2,a)}(\cdot,X_2)\\
&-(X_2-b)^2\big(\partial_2\mathcal{R} + \partial_{X_2}\widehat{\mathcal{R}}\big)(\cdot,X_1,X_2)\wedge\mathbf{e}_2 + \big[\widehat{\mathcal{R}}^{(g)}(\cdot,X_1,b) + \widehat{\mathcal{R}}^{(1,b)}(\cdot,X_1)\big]\wedge(X_2-b)\mathbf{e}_2\\
&+(X_1-a)^2\big(\partial_1\mathcal{R} + \partial_{X_1}\widehat{\mathcal{R}}\big)(\cdot,X_1,X_2)\wedge\mathbf{e}_1 + \big[\widehat{\mathcal{R}}^{(g)}(\cdot,a,X_2) - \widehat{\mathcal{R}}^{(2,a)}(\cdot,X_2)\big]\wedge(X_1-a)\mathbf{e}_1\\
&+\overline{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa) - \overline{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa) \qquad \text{weakly in } L^2(\Omega\times\mathbf{C}_{ab})^3.
\end{aligned}$$

Now, since $\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2)$ belongs to $Q^1_{per}(\mathcal{Y})$, one has in $\omega\times\big([a,a+1]\times[b,b+1]\big)$

$$\begin{aligned}
&(X_1-a)\partial_{X_1}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - \widehat{\mathbb{U}}_\alpha(\cdot,a,X_2),\\
&(X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - \widehat{\mathbb{U}}_\alpha(\cdot,X_1,b),\\
\Longrightarrow\; &(X_1-a)\partial_{X_1}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) - (X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_\alpha(\cdot,X_1,X_2) = \widehat{\mathbb{U}}_\alpha(\cdot,X_1,b) - \widehat{\mathbb{U}}_\alpha(\cdot,a,X_2)
\end{aligned}$$

and similar ones for $\widehat{\mathcal{R}}_\alpha$ and $\widetilde{\mathbb{U}}_3$. Using (6.9), one obtains in $\omega\times\big([a,a+1]\times[b,b+1]\big)$

$$\begin{aligned}
(X_1-a)\partial_{X_1}&\widehat{\mathbb{U}}_3(\cdot,X_1,X_2) - (X_2-b)\partial_{X_2}\widehat{\mathbb{U}}_3(\cdot,X_1,X_2)\\
&= \widehat{\mathbb{U}}_3(\cdot,X_1,b) - \widehat{\mathbb{U}}_3(\cdot,a,X_2) - \frac{1}{2}(X_1-a)^2\partial_{11}\mathbb{U}_3 + \frac{1}{2}(X_2-b)^2\partial_{22}\mathbb{U}_3.
\end{aligned}$$

Taking into account the fact that $\mathcal{R} = \partial_2\mathbb{U}_3\mathbf{e}_1 - \partial_1\mathbb{U}_3\mathbf{e}_2$, equalities (7.14)-(7.17) and the above identities the limit is equal to

$$\begin{aligned}
&-(X_2-b)\begin{pmatrix}\partial_2\mathbb{U}_1 \\ \partial_2\mathbb{U}_2 \\ \mathcal{Z}_{23}\end{pmatrix} + (X_1-a)\begin{pmatrix}\partial_1\mathbb{U}_1 \\ \partial_1\mathbb{U}_2 \\ \mathcal{Z}_{13}\end{pmatrix} - \frac{1}{2}(X_2-b)^2\partial_{22}\mathbb{U}_3\mathbf{e}_3 + \frac{1}{2}(X_1-a)^2\partial_{11}\mathbb{U}_3\mathbf{e}_3\\
&+\widehat{\mathbb{U}}^{(1,b)}(\cdot,X_1) - \widehat{\mathbb{U}}^{(2,a)}(\cdot,X_2) + \widehat{\mathbb{U}}(\cdot,X_1,b) - \widehat{\mathbb{U}}(\cdot,a,X_2) + \widehat{\mathbb{U}}^{(g)}(\cdot,X_1,b) + \widehat{\mathbb{U}}^{(g)}(\cdot,a,X_2)\\
&+\big[\widehat{\mathcal{R}}(\cdot,X_1,b) + \widehat{\mathcal{R}}^{(g)}(\cdot,X_1,b) + \widehat{\mathcal{R}}^{(1,b)}(\cdot,X_1)\big]\wedge(X_2-b)\mathbf{e}_2 + \overline{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa)\\
&-\big[\widehat{\mathcal{R}}(\cdot,a,X_2) - \widehat{\mathcal{R}}^{(g)}(\cdot,a,X_2) + \widehat{\mathcal{R}}^{(2,a)}(\cdot,X_2)\big]\wedge(X_1-a)\mathbf{e}_1 - \overline{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa)\\
&= \mathbf{M}_{ab}(\mathbb{U})(X_1,X_2) + \widehat{u}^{(1,b)}(\cdot,X_1,X_2,(-1)^{a+b+1}\kappa) - \widehat{u}^{(2,a)}(\cdot,X_1,X_2,(-1)^{a+b}\kappa).
\end{aligned}$$

with the macroscopic part

$$\mathbf{M}_{ab}(\mathbb{U})(X_1,X_2)=\begin{pmatrix}(X_1-a)e_{11}(\mathbb{U})-(X_2-b)e_{12}(\mathbb{U})\\(X_1-a)e_{12}(\mathbb{U})-(X_2-b)e_{22}(\mathbb{U})\\\frac12(X_1-a)^2\partial_{11}\mathbb{U}_3-\frac12(X_2-b)^2\partial_{22}\mathbb{U}_3\end{pmatrix}. \tag{7.22}$$

Now, remember that $g_\varepsilon=\varepsilon^3 g$ with $g\in\mathcal{C}(\overline{\Omega})^3$ (see assumption (6.1)). Then, the unfolded limit contact condition for $(\widehat{u}^{(1)},\widehat{u}^{(2)})\in L^2(\Omega;\widehat{\mathbf{W}}^{(1)})\times L^2(\Omega;\widehat{\mathbf{W}}^{(2)})$ is defined by

$$|\mathbf{M}_{\alpha,ab}(\mathbb{U})+\widehat{u}_\alpha^{(1,b)}-\widehat{u}_\alpha^{(2,a)}|\le g_\alpha\quad\text{a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2, \tag{7.23}$$

$$0\le(-1)^{a+b}\big(\mathbf{M}_{3,ab}(\mathbb{U})+\widehat{u}_3^{(1,b)}-\widehat{u}_3^{(2,a)}\big)\le g_3\quad\text{a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2, \tag{7.24}$$

for the in-plane and outer-plane components respectively.

7.6 The limit space

Consequently, after investigating the limit displacements it is possible to define the limit space for the unfolded problem. Thus, set

$$\mathcal{H}^1(\Omega)\doteq\big\{V\in H^1(\Omega)\mid V=0\text{ on }z_2=0\big\},\qquad \mathcal{H}^2(\Omega)\doteq\big\{V\in H^2(\Omega)\mid V=\partial_2V=0\text{ on }z_2=0\big\}.$$

Then, the limit fields $\big(\mathbb{U}_1,\mathbb{U}_2,\mathbb{U}_3,\widehat{u}^{(1)},\widehat{u}^{(2)}\big)$ belong to the convex set

$$\mathbf{X}\doteq\mathcal{H}^1(\Omega)^2\times\mathcal{H}^2(\Omega)\times L^2(\Omega;\widehat{\mathbf{W}}^{(1)})\times L^2(\Omega;\widehat{\mathbf{W}}^{(2)}).$$

In fact, together with the contact condition one has

$$\begin{aligned}\mathcal{X}\doteq\Big\{\big(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\widehat{v}^{(1)},\widehat{v}^{(2)}\big)\in\mathbf{X}\ \Big|\ &|\mathbf{M}_{ab,\alpha}(\mathbb{V})+\widehat{v}_\alpha^{(1,b)}-\widehat{v}_\alpha^{(2,a)}|\le g_\alpha\ \text{ a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\\ 0\le(-1)^{a+b}\big(M_{ab,3}(\mathbb{V})+\widehat{v}_3^{(1,b)}-\widehat{v}_3^{(2,a)}\big)\le g_3\ &\text{ a.e. in }\ \ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\Big\}\end{aligned} \tag{7.25}$$

The space $\mathcal{X}$ is a closed subset of the space

$$\mathcal{X}\subset\mathcal{H}^1(\Omega)^2\times\mathcal{H}^2(\Omega)\times L^2(\Omega;H^1(Cyls^{(1)})^3)\times L^2(\Omega;H^1(Cyls^{(2)})^3)$$

endowed with the product norm. However, $\mathcal{X}$ is not a subspace and hence not a Hilbert space.

8 The test-functions

In this section the used variables have to be split according to the splitting in of the unfolding operator, i.e., the "cell-number" and the local variable in the cell. Hence, note that for $z\in\mathbb{R}^2$ there exits a unique decomposition

$$z=[z]+\{z\},\qquad z\in\mathbb{Z}^2,\quad\{z\}\in(0,1)^2,\quad\text{for a.e. }z\in\mathbb{R}^2. \tag{8.1}$$

The composition of the test-functions has to take the contact into account, i.e., the test-functions have to satisfy the contact condition in (4.3) for every ε and the limit conditions (7.23)-(7.24). To ensure this behavior, it is necessary to choose the test-functions in a special way. First, we split the cell-domain further according to Figure 8.

Let $\big(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\widehat{v}^{(1)},\widehat{v}^{(2)}\big)$ be in the space $\mathcal{X}\cap\mathcal{C}^2(\overline{\Omega})^2\times\mathcal{C}^3(\overline{\Omega})\times\mathcal{C}^1(\overline{\Omega};\widehat{\mathbf{W}}^{(1)})\times\mathcal{C}^1(\overline{\Omega};\widehat{\mathbf{W}}^{(2)})$ such that $\widehat{v}^{(2)}(\cdot,0)=0$ vanishes at the boundary $z_2=0$. Now, we replace $\widehat{v}$ by $\widehat{v}'$ where

$$\begin{aligned}\widehat{v}'^{(1,b)}&=\widehat{v}^{(1,b)}+\frac12\big(\partial_1\mathbb{V}_2-\partial_2\mathbb{V}_1\big)\mathbf{e}_3\wedge\big(\Phi^{(1,b)}\mathbf{e}_3+X_3\mathbf{n}^{(1,b)}+(X_2-b)\mathbf{e}_2\big),\\ \widehat{v}'^{(2,a)}&=\widehat{v}^{(2,a)}+\frac12\big(\partial_1\mathbb{V}_2-\partial_2\mathbb{V}_1\big)\mathbf{e}_3\wedge\big(\Phi^{(2,a)}\mathbf{e}_3+X_3\mathbf{n}^{(2,a)}+(X_1-a)\mathbf{e}_1\big).\end{aligned}$$

We easily check that $\big(\mathbb{V}_1,\mathbb{V}_2,\mathbb{V}_3,\widehat{v}'^{(1)},\widehat{v}'^{(2)}\big)$ satisfies the following contact conditions

$$\begin{aligned}&|M'_{ab,\alpha}(\mathbb{V})+\widehat{v}'^{(1,b)}_\alpha-\widehat{v}'^{(2,a)}_\alpha|\le g_\alpha\quad\text{a.e. in }\ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\\ &0\le(-1)^{a+b}\big(M'_{ab,3}(\mathbb{V})+\widehat{v}'^{(1,b)}_3-\widehat{v}'^{(2,a)}_3\big)\le g_3\quad\text{a.e. in }\ \ \Omega\times\mathbf{C}_{ab},\ (a,b)\in\{0,1\}^2,\\ &\mathbf{M}'_{ab}(\mathbb{V})(X_1,X_2)=\begin{pmatrix}(X_1-a)e_{11}(\mathbb{V})-(X_2-b)\partial_2\mathbb{V}_1\\(X_1-a)\partial_1\mathbb{V}_2-(X_2-b)e_{22}(\mathbb{V})\\\frac12(X_1-a)^2\partial_{11}\mathbb{V}_3-\frac12(X_2-b)^2\partial_{22}\mathbb{V}_3\end{pmatrix}.\end{aligned}$$

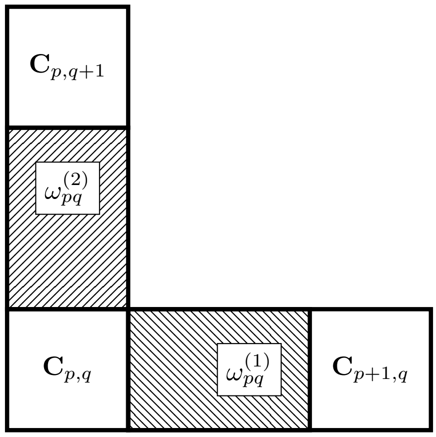

Figure 2: The 2D-cells for test-functions, with the different areas.

Hereafter, we describe how to define the functions $\mathbb{V}^{(1)}_{\varepsilon,\alpha}(\cdot, q\varepsilon) \in W^{1,\infty}(0, L)$ and $\mathbb{V}^{(1)}_{\varepsilon,3}(\cdot, q\varepsilon) \in W^{2,\infty}(0, L)$, $q \in \{0, \dots, 2N_\varepsilon\}$ (resp. $\mathbb{V}^{(2)}_{\varepsilon,\alpha}(p\varepsilon, \cdot) \in W^{1,\infty}(0, L)$ and $\mathbb{V}^{(2)}_{\varepsilon,3}(p\varepsilon, \cdot) \in W^{2,\infty}(0, L)$, $p \in \{1, \dots, 2N_\varepsilon\}$).
Below denote $z' = (z_1, z_2)$ and $p = \left[\dfrac{z_1}{\varepsilon}\right]$, $q = \left[\dfrac{z_2}{\varepsilon} + \dfrac{1}{2}\right]$. Then define

$$\mathbb{V}^{(1)}_{\varepsilon,\alpha}(z') = \begin{cases} \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1 \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \mathbb{V}^{(1)}_{\varepsilon,\alpha} \text{ linear interpolated in the stripe } \omega^{(1)}_{pq}, \end{cases}$$

$$\big(\partial_2 \mathbb{V}_3\big)^{(1)}_\varepsilon(z') = \begin{cases} \partial_2 \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_{12} \mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \big(\partial_2 \mathbb{V}_3\big)^{(1)}_\varepsilon \text{ linear interpolated in the stripe } \omega^{(1)}_{pq}, \end{cases}$$

and

$$\mathbb{V}^{(1)}_{\varepsilon,3}(z') = \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_1 - p\varepsilon)\partial_1 \mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_1 - p\varepsilon)^2 \partial_{11} \mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \mathbb{V}^{(1)}_{\varepsilon,3} \text{ cubic interpolated in the stripe } \omega^{(1)}_{pq}. \end{cases}$$

On the strips in direction $\mathbf{e}_2$ define $\mathbb{V}^{(2)}_{\varepsilon,\beta}$ accordingly by

$$\mathbb{V}^{(2)}_{\varepsilon,\alpha}(z') = \begin{cases} \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2 \mathbb{V}_\alpha(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \text{linear interpolated in the stripe } \omega^{(2)}_{pq}, \end{cases}$$

$$\big(\partial_1 \mathbb{V}_3\big)^{(2)}_\varepsilon(z') = \begin{cases} \partial_1 \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_{12} \mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \big(\partial_1 \mathbb{V}_3\big)^{(2)}_\varepsilon \text{ linear interpolated in the stripe } \omega^{(2)}_{pq}, \end{cases}$$

and

$$\mathbb{V}^{(2)}_{\varepsilon,3}(z') = \begin{cases} \mathbb{V}_3(p\varepsilon, q\varepsilon) + (z_2 - q\varepsilon)\partial_2 \mathbb{V}_3(p\varepsilon, q\varepsilon) + \dfrac{1}{2}(z_2 - q\varepsilon)^2 \partial_{22} \mathbb{V}_3(p\varepsilon, q\varepsilon) & \text{in } C_{pq}, \\ \mathbb{V}^{(2)}_{\varepsilon,3} \text{ cubic interpolated in the stripe } \omega^{(2)}_{pq}. \end{cases}$$

At last, the remaining displacements $\widehat{v}^{(1)}$ and $\widehat{v}^{(2)}$ are subjected to an analogous transformation. Hence, define

$$\widehat{v}'^{(1,b)}_\varepsilon(z', X) = \begin{cases} \widehat{v}'^{(1,b)}(p\varepsilon, q\varepsilon, X) \quad \text{a.e. in } C_{pq} \times (-\kappa, \kappa), \\ \text{linear interpolated with respect to } z_1 \text{ in the stripe } \omega^{(1)}_{pq}, \end{cases}$$

$$\widehat{v}'^{(2,a)}_\varepsilon(z', X) = \begin{cases} \widehat{v}'^{(2,a)}(p\varepsilon, q\varepsilon, X) \quad \text{in } C_{pq} \times (-\kappa, \kappa), \\ \text{linear interpolated with respect to } z_2 \text{ in the stripe } \omega^{(2)}_{pq}. \end{cases}$$

Lemma 8.1. *The test-functions satisfy the following strong convergences:*

$$\begin{aligned}
&\Pi_\varepsilon^{[\alpha]}(\mathbb{V}_\varepsilon^{(\alpha)}) \to \mathbb{V} \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)})^3,\\
&\Pi_\varepsilon^{[\alpha]}(\partial_\alpha\mathbb{V}_\varepsilon^{(\alpha)}) \to \partial_\alpha\mathbb{V} \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)})^3,\\
&\Pi_\varepsilon^{[\alpha]}(\partial_{\alpha\alpha}\mathbb{V}_{\varepsilon,3}^{(\alpha)}) \to \partial_{\alpha\alpha}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(\alpha)}),\\
&\Pi_\varepsilon^{[1]}\big(\big(\partial_2\mathbb{V}_3\big)_\varepsilon^{(1)}\big) \to \partial_2\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(1)}),\\
&\Pi_\varepsilon^{[1]}\big(\partial_1\big(\partial_2\mathbb{V}_3\big)_\varepsilon^{(1)}\big) \to \partial_{12}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(1)}),\\
&\Pi_\varepsilon^{[2]}\big(\big(\partial_1\mathbb{V}_3\big)_\varepsilon^{(2)}\big) \to \partial_1\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(2)}),\\
&\Pi_\varepsilon^{[2]}\big(\partial_2\big(\partial_1\mathbb{V}_3\big)_\varepsilon^{(2)}\big) \to \partial_{21}\mathbb{V}_3 \quad \textit{strongly in } L^2(\Omega\times Cyls^{(2)}),\\
&\Pi_\varepsilon^{[\alpha]}(\widehat{v}_\varepsilon^{\prime(\alpha,c)}) \to \widehat{v}^{\prime(\alpha,c)} \quad \textit{strongly in } L^2(\Omega;H^1(Cyls^{(\alpha)}))^3.
\end{aligned}$$

Proof. The proof of this Lemma is an easy consequence of the unfolding properties and the regularity of the test-functions. □

Then, compose the full test displacements in the respective directions $\mathbf{e}_1$ and $\mathbf{e}_2$ by (recall that $z'=(z_1,z_2)$ and $z=(z_1,z_2,z_3)$)

$$V_\varepsilon^{(1,b)}(z) = V_\varepsilon^{e(1,b)}(z) + \widehat{v}_\varepsilon^{\prime(1,b)}\Big(z', 2\Big\{\frac{z_1}{\varepsilon}\Big\}, 2\Big\{\frac{z_2}{2\varepsilon}\Big\} - b, \frac{z_3}{\varepsilon}\Big), \tag{8.2}$$

$$V_\varepsilon^{(2,a)}(z) = V_\varepsilon^{e(2,a)}(z) + \widehat{v}_\varepsilon^{\prime(2,a)}\Big(z', 2\Big\{\frac{z_1}{\varepsilon}\Big\} - a, 2\Big\{\frac{z_2}{2\varepsilon}\Big\}, \frac{z_3}{\varepsilon}\Big). \tag{8.3}$$

The elementary displacements for the $\mathbf{e}_1$-directed and the $\mathbf{e}_2$-directed beams are defined respectively by

$$V_\varepsilon^{e(1,b)} = \begin{pmatrix}\varepsilon^2\mathbb{V}_{\varepsilon,1}^{(1)}\\ \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(1)}\\ \varepsilon\mathbb{V}_{\varepsilon,3}^{(1)}\end{pmatrix} + \begin{pmatrix}\varepsilon^2\big(\partial_2\mathbb{V}_3\big)_\varepsilon^{(1)}\\ -\varepsilon^2\partial_1\mathbb{V}_{\varepsilon,3}^{(1)}\\ 0\end{pmatrix} \wedge \Big(\Phi^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + \Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\} - b\Big)\mathbf{e}_2 + \frac{z_3}{\varepsilon}\mathbf{n}^{(1,b)}\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}\Big)\Big),$$

$$V_\varepsilon^{e(2,a)} = \begin{pmatrix}\varepsilon^2\mathbb{V}_{\varepsilon,1}^{(2)}\\ \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(2)}\\ \varepsilon\mathbb{V}_{\varepsilon,3}^{(2)}\end{pmatrix} + \begin{pmatrix}\varepsilon^2\partial_2\mathbb{V}_{\varepsilon,3}^{(1)}\\ -\varepsilon^2\big(\partial_1\mathbb{V}_3\big)_\varepsilon^{(2)}\\ 0\end{pmatrix} \wedge \Big(\Phi^{(2,a)}\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\mathbf{e}_3 + \Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\} - a\Big)\mathbf{e}_1 + \frac{z_3}{\varepsilon}\mathbf{n}^{(2,a)}\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}\Big)\Big).$$

The test-functions are build to satisfy the contact-conditions before and after the limit and to yield the same strain tensor in the limit, which we show hereafter.

The unfolded limiting strain tensor of the test-functions is an immediate consequence of their definition and the convergences in Lemma 8.1 and the limit is written in the same way as in Section 7.4.

Corollary 8.2. *The unfolded strain tensor of the test-functions 8.2 satisfies*

$$\frac{1}{\varepsilon^2}\Pi^{(1)}\Big(e_z(V_\varepsilon^{(1,b)})\Big) \to \mathcal{E}^{(1,b)}(\mathbb{V}) + \mathcal{E}_X^{(1,b)}(\widehat{v}^{(1,b)}), \qquad \textit{strongly in } L^2(\Omega\times Cyls^{(1)})^{3\times3},$$

$$\frac{1}{\varepsilon^2}\Pi^{(2)}\Big(e_z(V_\varepsilon^{(2,a)})\Big) \to \mathcal{E}^{(2,a)}(\mathbb{V}) + \mathcal{E}_X^{(2,a)}(\widehat{v}^{(2,a)}), \qquad \textit{strongly in } L^2(\Omega\times Cyls^{(2)})^{3\times3},$$

where $\mathcal{E}^{(1,b)}$ *and* $\mathcal{E}_X^{(1,b)}$*, respectively* $\mathcal{E}^{(2,a)}$ *and* $\mathcal{E}_X^{(2,a)}$ *are the same as in* (7.16), (7.12), (7.18) *and* (7.13).

Proof. Easy consequence of Lemma 8.1 and the properties of the unfolding operator. □

8.1 The contact condition of the test-functions

It is necessary to check the contact condition for the test-functions, as they must satisfy this cone-condition to be in the $\mathcal{V}_\varepsilon$. Due to the special choice of test-function in the section before, this is an immediate consequence. Indeed, since the function on the contact parts $\mathbf{C}_{pq}$ are chosen such that a Taylor expansion of the macroscopic fields is exact and does not admit any remainder terms. To check this, note that, on the contact area $\mathbf{C}_{pq}$ the elementary test-functions reduce to

$$\widetilde{\mathbb{V}}_\varepsilon^{e(1,b)}(z',(-1)^{a+b+1}r)=\begin{pmatrix}\varepsilon^2\mathbb{V}^{(1)}_{\varepsilon,1}(z')\\ \varepsilon^2\mathbb{V}^{(1)}_{\varepsilon,2}(z')\\ \varepsilon\mathbb{V}^{(1)}_{\varepsilon,3}(z')\end{pmatrix}+\begin{pmatrix}\varepsilon^2\big(\partial_2\mathbb{V}_3\big)^{(1)}_\varepsilon(z')\\ -\varepsilon^2\partial_1\mathbb{V}^{(1)}_{\varepsilon,3}(z')\\ 0\end{pmatrix}\wedge\Big(2\Big\{\frac{z_2}{2\varepsilon}\Big\}-b\Big)\mathbf{e}_2$$

$$\widetilde{\mathbb{V}}_\varepsilon^{e(2,a)}(z',(-1)^{a+b}r)=\begin{pmatrix}\varepsilon^2\mathbb{V}^{(2)}_{\varepsilon,1}(z')\\ \varepsilon^2\mathbb{V}^{(2)}_{\varepsilon,2}(z')\\ \varepsilon\mathbb{V}^{(2)}_{\varepsilon,3}(z')\end{pmatrix}+\begin{pmatrix}\varepsilon^2\partial_2\mathbb{V}^{(1)}_{\varepsilon,3}(z')\\ -\varepsilon^2\big(\partial_1\mathbb{V}_3\big)^{(2)}_\varepsilon(z')\\ 0\end{pmatrix}\wedge\Big(2\Big\{\frac{z_1}{2\varepsilon}\Big\}-a\Big)\mathbf{e}_1.$$

Now, consider the difference of the two test displacements in $\mathbf{C}_{pq}$:

$$\begin{aligned}\widetilde{\mathbb{V}}_\varepsilon^{(1,b)}(z',(-1)^{a+b+1}r)-\widetilde{\mathbb{V}}^{(2,a)}(z',(-1)^{a+b}r)=\varepsilon^3&\begin{pmatrix}\frac{z_1-p\varepsilon}{\varepsilon}\partial_1\mathbb{V}_1(p\varepsilon,q\varepsilon)-\frac{z_2-q\varepsilon}{\varepsilon}\partial_2\mathbb{V}_1(p\varepsilon,q\varepsilon)\\ \frac{z_1-p\varepsilon}{\varepsilon}\partial_1\mathbb{V}_2(p\varepsilon,q\varepsilon)-\frac{z_2-q\varepsilon}{\varepsilon}\partial_2\mathbb{V}_2(p\varepsilon,q\varepsilon)\\ \frac{(z_1-p\varepsilon)^2}{2\varepsilon^2}\partial_{11}\mathbb{V}_3(p\varepsilon,q\varepsilon)-\frac{(z_2-q\varepsilon)^2}{2\varepsilon^2}\partial_{22}\mathbb{V}_3(p\varepsilon,q\varepsilon)\end{pmatrix}\\ +\varepsilon^3\widehat{v}'^{(1,b)}\Big(p\varepsilon,q\varepsilon,\frac{z_1-p\varepsilon}{\varepsilon},\frac{z_2-q\varepsilon}{\varepsilon},(-1)^{a+b+1}\kappa\Big)&-\varepsilon^3\widehat{v}'^{(2,a)}\Big(p\varepsilon,q\varepsilon,\frac{z_1-p\varepsilon}{\varepsilon},\frac{z_2-q\varepsilon}{\varepsilon},(-1)^{a+b}\kappa\Big).\end{aligned}\tag{8.4}$$

Note that by the conditions on the test functions, see (7.25), the microscopic contact in (4.3) as well as in the unfolded limit (7.22)-(7.24) immediately and even includes the case of a rigid contact where $\mathbf{g}\equiv 0$. Hence, the contact conditions are satisfied for every ε by the definition of the test-functions.

Finally, we conclude this section by density of the spaces

$$\mathcal{C}^1(\overline{\Omega})\cap\mathcal{H}^1(\Omega)\subset\mathcal{H}^1(\Omega),\quad \mathcal{C}^2(\overline{\Omega})\cap\mathcal{H}^2(\Omega)\subset\mathcal{H}^2(\Omega),\quad \mathcal{C}^1(\overline{\Omega};\widehat{\mathbf{W}}^{(\alpha)})\subset L^2(\Omega;\widehat{\mathbf{W}}^{(\alpha)}).$$

Hence, the convergences of the unfolded strain tensor and the contact condition hold for all functions in $\mathcal{X}$.

9 The limit problem

In this section, all tools and results developed in this paper are summarized and lead to the homogenization of the textile elasticity problem. Thus, recall the initial variational inequality in the vectorial notation:

$$\begin{aligned}&\text{Find } u_\varepsilon\in V_\varepsilon \text{ such that:}\\ &\int_{\mathcal{S}_\varepsilon}A_\varepsilon\,E_x(u_\varepsilon)\cdot E_x(u_\varepsilon-\varphi)\,dx-\int_{\mathcal{S}_\varepsilon}f_\varepsilon\cdot(u_\varepsilon-\varphi)\,dx\le 0,\qquad\forall\varphi\in V_\varepsilon.\end{aligned}\tag{9.1}$$

Let us denote by $\widetilde{\mathbf{C}}_\varepsilon^{(\alpha,c)}$ the orthogonal matrices as in (3.17) for the different beam directions, such that $E_z^{(\alpha,c)}(u^{(\alpha,c)})=\widetilde{\mathbf{C}}_\varepsilon^{(\alpha,c)}E_x(u^{(\alpha,c)})$ and define the matrices $\widetilde{A}_\varepsilon^{(1,q)}=(\widetilde{\mathbf{C}}^{(1,q)})^{-1}A_\varepsilon^{(1,q)}\widetilde{\mathbf{C}}^{(1,q)}$ and $\widetilde{A}_\varepsilon^{(2,p)}=(\widetilde{\mathbf{C}}^{(2,p)})^{-1}A_\varepsilon^{(2,p)}\widetilde{\mathbf{C}}^{(2,p)}$ respectively. Then, unfolding the problem (9.1) we obtain (for every $\varphi\in V_\varepsilon$)

$$\begin{aligned}&\sum_{q=1}^{N_\varepsilon}\Big(\int_{P_r^{(1,q)}}\widetilde{A}_\varepsilon^{(1,q)}E_z(u_\varepsilon^{(1,q)})\cdot E_z(u_\varepsilon^{(1,q)}-\varphi^{(1,q)})\,\big|\eta_\varepsilon^{(1,q)}\big|dz-\int_{P_r^{(1,q)}}f_\varepsilon^{(1)}\cdot(u_\varepsilon^{(1,q)}-\varphi^{(1,q)})\,\big|\eta_\varepsilon^{(1,q)}\big|dz\Big)\\ &+\sum_{p=0}^{N_\varepsilon}\Big(\int_{P_r^{(2,p)}}\widetilde{A}_\varepsilon^{(2,p)}E_z(u_\varepsilon^{(2,p)})\cdot E_z(u_\varepsilon^{(2,p)}-\varphi^{(2,p)})\,\big|\eta_\varepsilon^{(1,q)}\big|dz-\int_{P_r^{(2,p)}}f_\varepsilon^{(2)}\cdot\big(u_\varepsilon^{(2,p)}-\varphi^{(2,p)}\big)\,\big|\eta_\varepsilon^{(2,p)}\big|dz\Big)\le 0.\end{aligned}\tag{9.2}$$

For the following analysis we introduce a new notation in order to simplify the expressions of the different microscopic and macroscopic problems.

Notation 9.1. *Set*

$$\mathbf{M}_{ab}(\zeta)(X_1,X_2)=\begin{pmatrix}(X_1-a)\zeta_1-(X_2-b)\zeta_2\\ (X_1-a)\zeta_2-(X_2-b)\zeta_3\\ \frac{1}{2}(X_1-a)^2\zeta_4-\frac{1}{2}(X_2-b)^2\zeta_5\end{pmatrix}.$$

Moreover, define the displacements

$$\begin{aligned}\widehat{W}^{(1,b)}(\zeta)(X)=\theta_1(X_1)\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}+\begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\wedge(\theta_2(X_1)\mathbf{e}_1+\theta_1(X_1)(X_2-b)\mathbf{e}_2)&\\ +\begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\wedge\Big(\phi^{(1,b)}(X_1)\mathbf{e}_3+X_3\mathbf{n}^{(1,b)}(X_1)\Big),&\end{aligned}$$

$$\widehat{W}^{(2,a)}(\zeta)(X) = \theta_1(X_2)\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix} + \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix} \wedge \left(\theta_2(X_2)\mathbf{e}_2 + (X_1-a)\theta_1(X_2)\mathbf{e}_1\right)$$
$$+ \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix} \wedge \left(\phi^{(2,a)}(X_2)\mathbf{e}_3 + X_3\mathbf{n}^{(2,a)}(X_2)\right).$$

where $\theta_1 \in \mathcal{C}^1_{per}(0,2)$ (resp. $\theta_2 \in \mathcal{C}^1_{per}(0,2)$) is 2-periodic and satisfies

$$\theta_1(t) = t - c \quad (resp.\ \ \theta_2(t) = \frac{1}{2}(t-c)^2)\ \ a.e.\ in\ [c-\kappa, c+\kappa], \quad c \in \{0,1\}.$$

Then the difference on the contact area can be expressed by

$$\widehat{W}^{(1,b)}(\zeta)(X) - \widehat{W}^{(2,a)}(\zeta)(X) = \mathbf{M}_{ab}(\zeta)(X_1,X_2) \qquad a.e.\ on\ \ \mathbf{C}_{ab} \tag{9.3}$$

and hence resembles the original contact condition.
Similarly, define the strain tensor in vectorial notation in the according form

$$\mathcal{E}(\zeta) = \mathcal{E}^{(1,b)}(\zeta)\mathbb{1}_{Cyls^{(1,b)}} + \mathcal{E}^{(2,a)}(\zeta)\mathbb{1}_{Cyls^{(2,a)}} = \sum_{i=1}^{6}\zeta_n\mathcal{E}(\mathbf{e}_n),$$

with

$$\mathcal{E}^{(\alpha,c)}(\zeta) = \left(\mathcal{E}^{(\alpha,c)}_{11}, \mathcal{E}^{(\alpha,c)}_{22}, \mathcal{E}^{(\alpha,c)}_{33}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{12}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{13}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{23}\right)^T \tag{9.4}$$

and

$$\begin{aligned}
\mathcal{E}^{(1,b)}_{11}(\zeta) =& \frac{1}{\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot\mathbf{t}^{(1,b)} + \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left(\left(X_3 + \frac{\Phi^{(1,b)}}{\gamma}\right)\mathbf{e}_2 - (X_2-b)\,\mathbf{n}^{(1,b)}\right)\right],\\
\mathcal{E}^{(1,b)}_{12}(\zeta) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot\mathbf{e}_2 - \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left(X_3\mathbf{t}^{(1,b)} + \Phi^{(1,b)}\mathbf{e}_1\right)\right],\\
\mathcal{E}^{(1,b)}_{13}(\zeta) =& \frac{1}{2\eta^{(1,b)}}\left[\begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\cdot\mathbf{n}^{(1,b)} + \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}\cdot\left((X_2-b)\,\mathbf{t}^{(1,b)} - \frac{\Phi^{(1,b)}d_1\Phi^{(1,b)}}{\gamma}\mathbf{e}_2\right)\right],
\end{aligned}$$

and $\mathcal{E}^{(1,b)}_{22}(\zeta) = \mathcal{E}^{(1,b)}_{33}(\zeta) = \mathcal{E}^{(1,b)}_{23}(\zeta) = 0$. Accordingly, the tensor $\mathcal{E}^{(2,a)}(\zeta)$ is defined by

$$\begin{aligned}
\mathcal{E}^{(2,a)}_{22}(\zeta) =& \frac{1}{\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot\mathbf{t}^{(2,a)} - \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left(\left(X_3 + \frac{\Phi^{(2,a)}}{\gamma}\right)\mathbf{e}_1 - (X_1-a)\,\mathbf{n}^{(2,a)}\right)\right],\\
\mathcal{E}^{(2,a)}_{12}(\zeta) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot\mathbf{e}_1 + \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left(X_3\mathbf{t}^{(2,a)} + \Phi^{(2,a)}\mathbf{e}_2\right)\right],\\
\mathcal{E}^{(2,a)}_{23}(\zeta) =& \frac{1}{2\eta^{(2,a)}}\left[\begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\cdot\mathbf{n}^{(2,a)} - \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix}\cdot\left((X_1-a)\,\mathbf{t}^{(2,a)} - \frac{\Phi^{(2,a)}d\Phi^{(2,a)}}{\gamma}\mathbf{e}_1\right)\right].
\end{aligned}$$

and $\mathcal{E}^{(2,a)}_{11}(\zeta) = \mathcal{E}^{(2,a)}_{33}(\zeta) = \mathcal{E}^{(2,a)}_{13}(\zeta) = 0$.
Additionally, without renaming rewrite the local strain tensor in vectorial form, i.e.

$$\mathcal{E}^{(\alpha,c)}_X = \left(\mathcal{E}^{(\alpha,c)}_{X,11}, \mathcal{E}^{(\alpha,c)}_{X,22}, \mathcal{E}^{(\alpha,c)}_{X,33}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{X,12}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{X,13}, \sqrt{2}\mathcal{E}^{(\alpha,c)}_{X,23}\right)^T \tag{9.5}$$

Furthermore, for the sake of comprehensibility and readability define

$$\begin{aligned}
\widetilde{\mathbf{A}}(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\widetilde{A}^{(\alpha,c)}(X)\mathbb{1}_{Cyls^{(\alpha,c)}}(X), & \mathcal{E}_X(\varphi) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\mathcal{E}^{(\alpha,c)}_X(\varphi)\mathbb{1}_{Cyls^{(\alpha,c)}}(X)\\
\boldsymbol{\eta}(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\boldsymbol{\eta}^{(\alpha,c)}(X)\mathbb{1}_{Cyls^{(\alpha,c)}}(X), & \rho(X) &= \sum_{\alpha=1}^{2}\sum_{c=0}^{1}\frac{1}{|Cyls^{(\alpha,c)}|}\mathbb{1}_{Cyls^{(\alpha,c)}}(X)
\end{aligned} \tag{9.6}$$

Finally, define different function spaces accounting for different contact conditions. First, for every $\widehat{w} = (\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$ *set*

$$\sum_{(\alpha,c)\in\{1,2\}\times\{0,1\}} \int_{Cyls^{(\alpha,c)}} \widehat{w}^{(\alpha,c)}\, dX = \int_{Cyls} \widehat{w}\, dX.$$

Then, define

$$\widehat{\mathbf{W}}_{lin} \doteq \Big\{ \widehat{w} = (\widehat{w}^{(1)}, \widehat{w}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \mid \widehat{w}^{(1)} = \widehat{w}^{(2)} \text{ a.e. on } \mathbf{C}_{ab},\ (a,b) \in \{0,1\}^2,\ \int_{Cyls} \widehat{w}\, dX = 0 \Big\}$$

and $K_{\zeta,z}$ *and* K_z *the convex subsets of* $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$, *(*$(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega}$*)*

$$\begin{aligned} K_{\zeta,z} \doteq \Big\{ & \widehat{v} = (\widehat{v}^{(1)}, \widehat{v}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \mid |\mathbf{M}_{ab,\alpha}(\zeta) + \widehat{v}_\alpha^{(1,b)} - \widehat{v}_\alpha^{(2,a)}| \le g_\alpha(z) \text{ a.e. on } \mathbf{C}_{ab} \\ & 0 \le (-1)^{a+b}\big(\mathbf{M}_{ab,3}(\zeta) + \widehat{v}_3^{(1,b)} - \widehat{v}_3^{(2,a)}\big) \le g_3(z) \text{ a.e. on } \mathbf{C}_{ab}, \text{ and } \int_{Cyls} \widehat{w}\, dX = 0 \Big\}, \\ K_z \doteq \Big\{ & \widehat{v} = (\widehat{v}^{(1)}, \widehat{v}^{(2)}) \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \mid |\widehat{v}_\alpha^{(1,b)} - \widehat{v}_\alpha^{(2,a)}| \le g_\alpha(z) \text{ a.e. on } \mathbf{C}_{ab}, \\ & 0 \le (-1)^{a+b}\big(\widehat{v}_3^{(1,b)} - \widehat{v}_3^{(2,a)}\big) \le g_3(z) \text{ a.e. on } \mathbf{C}_{ab}, \text{ and } \int_{Cyls} \widehat{w}\, dX = 0 \Big\}. \end{aligned}$$

9.1 The unfolded limit problem

Lemma 9.2. *Let* $(\widehat{w}, \widehat{v})$ *be in* $\widehat{\mathbf{W}}_{lin} \times \widehat{\mathbf{W}}_{lin}$ *(resp. in* $K_z \times K_z$, $K_{\zeta,z} \times K_{\zeta,z}$*) satisfying*

$$\mathcal{E}_X(\widehat{w}) = \mathcal{E}_X(\widehat{v}) \quad \text{a.e. in} \quad Cyls \tag{9.7}$$

then

$$\widehat{v} = \widehat{w} \quad \text{a.e. in} \quad Cyls.$$

We also have

$$\begin{aligned} &\forall\, \widehat{w} \in \widehat{\mathbf{W}}_{lin}, && \|\widehat{w}\|_{H^1(Cyls)} \le C\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)}, \\ &\forall\, \widehat{w} \in K_z, && \|\widehat{w}\|_{H^1(Cyls)} \le C\big(\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)} + \|g\|_{L^\infty(\Omega)}\big), \\ &\forall\, \widehat{w} \in K_{\zeta,z}, && \|\widehat{w}\|_{H^1(Cyls)} \le C\big(\|\mathcal{E}_X(\widehat{w})\|_{L^2(Cyls)} + \|g\|_{L^\infty(\Omega)} + |\zeta|\big). \end{aligned} \tag{9.8}$$

Proof. Let $(\widehat{w}, \widehat{v})$ be in $\widehat{\mathbf{W}}_{lin} \times \widehat{\mathbf{W}}_{lin}$ satisfying (9.7). Set $r^{(1,b)} = \widehat{w}^{(1,b)} - \widehat{v}^{(1,b)}$ and $r^{(2,a)} = \widehat{w}^{(2,a)} - \widehat{v}^{(2,a)}$ with $(a,b) \in \{0,1\}^2$. These displacements are rigid motions, so we write

$$\begin{aligned} r^{(1,b)}(X) &= A^{(1,b)} + B^{(1,b)} \wedge (X_1\mathbf{e}_1 + \Phi^{(1,b)}(X_1)\mathbf{e}_3) + B^{(1,b)} \wedge \big((X_2 - b)\mathbf{e}_2 + X_3\mathbf{n}^{(1,b)}(X_1)\big), \\ r^{(2,a)}(X) &= A^{(2,a)} + B^{(2,a)} \wedge (X_2\mathbf{e}_2 + \Phi^{(2,a)}(X_2)\mathbf{e}_3) + B^{(2,a)} \wedge \big((X_1 - a)\mathbf{e}_1 + X_3\mathbf{n}^{(2,a)}(X_2)\big), \end{aligned} \tag{9.9}$$

Furthermore, they are periodic in the respective direction. Hence $r^{(1,b)}(0, X_2, X_3) = r^{(1,b)}(2, X_2, X_3)$ yields $B^{(1,b)} \wedge \mathbf{e}_1 = 0$ and $r^{(2,a)}(X_1, 0, X_3) = r^{(2,a)}(X_1, 2, X_3)$ analogously $B^{(2,a)} \wedge \mathbf{e}_2 = 0$.
They also satisfy the contact conditions, hence

$$r^{(1,\alpha)}(X_1, X_2, (-1)^{\alpha+\beta}\kappa) = r^{(2,\beta)}(X_1, X_2, (-1)^{\alpha+\beta+1}\kappa), \qquad \forall\,(X_1, X_2) \in \mathbf{C}_{\alpha\beta}$$

The first condition yields

$$A^{(1,0)} + B^{(1,0)} \wedge X_2\mathbf{e}_2 = A^{(2,0)} + B^{(2,0)} \wedge X_1\mathbf{e}_1 \qquad \forall (X_1, X_2) \in (-\kappa, \kappa)^2.$$

That gives $B^{(1,0)} \wedge \mathbf{e}_2 = B^{(2,0)} \wedge \mathbf{e}_1 = 0$ and then taking into account the preceding equalities one has $B^{(1,0)} = B^{(2,0)} = 0$ and also $A^{(1,0)} = A^{(2,0)}$. In the same way, we obtain

$$B^{(1,b)} = B^{(2,b)} = 0, \qquad A^{(1,b)} = A^{(2,a)} \qquad \forall (a,b) \in \{0,1\}^2.$$

Thus the difference of $\widehat{w}$ and $\widehat{v}$ is constant and there exits $A \in \mathbb{R}^3$ such that $\widehat{w} - \widehat{v} = A$. The last condition in the definition of $\widehat{\mathbf{W}}_{lin}$ implies $A = 0$. The Korn inequality gives $(9.8)_1$.
Now, if $(\widehat{w}, \widehat{v})$ belongs to $K_z \times K_z$. The periodicity yields again $B^{(1,b)} \wedge \mathbf{e}_1 = 0$ and $B^{(2,a)} \wedge \mathbf{e}_2 = 0$. The difference between the two cases lies in the contact conditions and we obtain e.g. on $\mathbf{C}_{00}$

$$\begin{aligned} &|A_1^{(1,0)} - A_1^{(2,0)}| \le g_1(z), \quad |A_2^{(1,0)} - A_2^{(2,0)}| \le g_2(z), \\ &0 \le A_3^{(1,0)} + X_2 B_1^{(1,0)} - A_3^{(2,0)} + X_1 B_2^{(2,0)} \le g_3(z), \end{aligned} \qquad \forall\,(X_1, X_2) \in \mathbf{C}_{00} \tag{9.10}$$

Since $(X_1, X_2) \in (-\kappa, \kappa)^2$, the third condition in (9.10) gives $0 \le A_3^{(1,0)} - A_3^{(2,0)} \le g_3(z)$ as well as $2\kappa\big(|B_1^{(1,0)}| + |B_2^{(2,0)}|\big) \le g_3(z)$. In the same way, we get similar conditions for the other contact parts:

$$|A^{(1,a)} - A^{(2,b)}| + |B^{(1,a)}| + |B^{(2,b)}| \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a,b) \in \{0,1\}^2.$$

Set

$$A = \frac{1}{4}\big(A^{(1,0)} + A^{(1,1)} + A^{(2,0)} + A^{(2,1)}\big).$$

One has

$$|A^{(1,a)} - A| \le C\|g\|_{L^\infty(\Omega)}, \quad |A^{(2,b)} - A| \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a,b) \in \{0,1\}^2.$$

That leads to

$$\|r^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|r^{(2,0)} - A\|_{L^2(Cyls^{(2,b)})} \le C\|g\|_{L^\infty(\Omega)} \qquad \forall (a,b) \in \{0,1\}^2. \tag{9.11}$$

Finally, the above inequalities and the Korn inequality give

$$\|(\widehat{w} - \widehat{v})^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|(\widehat{w} - \widehat{v})^{(2,b)} - A\|_{L^2(Cyls^{(2,b)})} \le C\|g\|_{L^\infty(\Omega)}, \quad \forall (a,b) \in \{0,1\}^2.$$

The last condition in the definition of K_z implies $A = 0$. The Korn inequality gives $(9.8)_2$.
In the last case $(\widehat{w}, \widehat{v})$ in $K_{\zeta,z} \times K_{\zeta,z}$ we replace (9.11) by

$$\|r^{(1,a)} - A\|_{L^2(Cyls^{(1,a)})} + \|r^{(2,0)} - A\|_{L^2(Cyls^{(2,b)})} \le C\big(|\zeta| + \|g\|_{L^\infty(\Omega)}\big) \qquad \forall (a,b) \in \{0,1\}^2.$$

The conclusion is analogously. □

Theorem 9.3. *Suppose that $f_\varepsilon^{(\alpha)}$ is defined as in* (5.16) *and that*

$$g_\varepsilon = \varepsilon^3 g, \qquad g \in \mathcal{C}(\overline{\Omega})^3. \tag{9.12}$$

Moreover, assume that $A_\varepsilon^{(\alpha)} = A^{(\alpha)}\Big(\dfrac{\cdot}{\varepsilon}\Big)$ satisfies assumptions of Section 4.4 with $A^{(\alpha)} \in [L^\infty(Cyls^{(\alpha)})]^{6\times 6}$.

Let $u_\varepsilon = (u^{(1,1)}, \dots, u^{(1,2N_\varepsilon)}, u^{(2,0)}, \dots, u^{(2,2N_\varepsilon)}) \in \mathcal{V}_\varepsilon$ be a solution to problem (9.1). *Then there exists a subsequence, still denoted by ε, and $(\mathbb{U}, \widehat{u}) \in \mathcal{X}$ such that the fields satisfy the unfolded limit problem*

Find $(\mathbb{U}, \widehat{u}) \in \mathcal{X}$ such that for every $(\mathbb{V}, \widehat{v}) \in \mathcal{X}$:

$$\int_{\Omega \times Cyls} \rho \widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{u})\Big] \cdot \Big[\mathcal{E}(\zeta - \xi) + \mathcal{E}_X(\widehat{u} - \widehat{v})\Big]|\boldsymbol{\eta}|\, dzdX \le \int_\Omega F \cdot (\mathbb{U} - \mathbb{V})\, dz, \tag{9.13}$$

where

$$\zeta = (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3), \quad \xi = (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3).$$

The tensor-fields $\mathcal{E}$ and $\mathcal{E}_X$ are defined in (9.6) *(and in Section 7.4) and $F = f^{(1)} + f^{(2)}$.*

Proof. Choose the test-functions according to Section 8. Then the limit (9.13) is a consequence of unfolding for integrals, the assumptions and the convergences in Sections 7 and 8. The form of the right-hand side follows by integrating over the other parts of the displacement, which vanish due to symmetry reasons.

Note, that until now the test-functions are in the space

$$(\mathbb{V}, \widehat{v}^{(\alpha)}) \in \mathcal{X} \cap \big[\mathcal{C}^1(\overline{\Omega})^2 \times \mathcal{C}^2(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega}, \widehat{\mathbf{W}}^{(\alpha)})\big].$$

The density-argument of this space in $\mathcal{X}$ is a bit more involved due to the cone-condition coming from the contact. This issue is resolved by truncation and regularization of the functions, which then allow together with the typical density argument to conclude the claim. □

Before investigating the existence and uniqueness, it is necessary to describe the homogenized problem completely. Hence introduce the correctors and their respective problems. In fact, since the problem (9.13) is nonlinear, it is split into multiple problems, of which most are linear but one remaining problem captures the non-linearity.

The corrector-problem for the field $\widehat{u}$ is obtained by choosing $V = U$ in 9.13 leading to the following microscopic problem:

For (ζ, z) in $\mathbb{R}^6 \times \overline{\Omega}$, find $\widehat{v}_{\zeta,z} \in K_{\zeta,z}$,

$$\int_{Cyls} \rho \widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{w})|\boldsymbol{\eta}|\, dX \le 0, \qquad \forall \widehat{w} \in K_{\zeta,z}. \tag{9.14}$$

This variational inequality admits solutions by the Stampacchia-Theorem (see [18]). Two solutions $\widehat{v}_{\zeta,z}$ and $\widehat{w}_{\zeta,z}$ of this problem satisfy

$$\mathcal{E}_X(\widehat{v}_{\zeta,z}) = \mathcal{E}_X(\widehat{w}_{\zeta,z}).$$

Indeed, consider the problems of the two solutions with specific test-functions

$$\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{w}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0,$$
$$\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{w}_{\zeta,z})\Big] \cdot \mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0.$$

Adding these two inequalities yields

$$\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z}) \cdot \mathcal{E}_X(\widehat{w}_{\zeta,z} - \widehat{v}_{\zeta,z})|\boldsymbol{\eta}|\, dX \le 0. \tag{9.15}$$

wherefrom it follows that $\mathcal{E}_X(\widehat{v}_{\zeta,z}) = \mathcal{E}_X(\widehat{w}_{\zeta,z})$ since by coercivity (9.15) is also non-negative. Thus solutions of (9.14) differ only from rigid motions, see Lemma 9.2 and there exist rigid displacements $r^{(\alpha,c)}_{\zeta,z}$, $(\alpha,c) \in \{1,2\}\times\{0,1\}$ such that

$$\widehat{w}^{(\alpha,c)}_{\zeta,z} - \widehat{v}^{(\alpha,c)}_{\zeta,z} = r^{(\alpha,c)}_{\zeta,z}, \qquad (\alpha,c) \in \{1,2\}\times\{0,1\}.$$

One has

$$\|r^{(\alpha,c)}_{\zeta,z}\|_{L^2(Cyls^{(\alpha,c)})} \le C\big(|\zeta| + \|g\|_{L^\infty(\Omega)}\big), \qquad (\alpha,c) \in \{1,2\}\times\{0,1\}.$$

Now, we introduce the six typical linear corrector problems as the solution of the following variational problems:

$$\begin{aligned}
&\text{Find } \widehat{\chi}_n \in \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)} \text{ such that } \widehat{\chi}^{(1,b)}_n + \widehat{W}^{(1,b)}(\mathbf{e}_n) = \widehat{\chi}^{(2,a)}_n + \widehat{W}^{(2,a)}(\mathbf{e}_n) \text{ a.e. on } \mathbf{C}_{ab},\\
&\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\mathbf{e}_n) + \mathcal{E}_X(\widehat{\chi}_n)\Big] \cdot \mathcal{E}_X(\widehat{w})|\boldsymbol{\eta}|\, dX = 0, \qquad \forall\, \widehat{w} \in \widehat{\mathbf{W}}_{lin}, \quad n \in \{1,\dots,6\}.
\end{aligned} \tag{9.16}$$

with $n \in \{1,\dots,6\}$ and the unit-vectors $\mathbf{e}_n \in \mathbb{R}^6$. Due to the definition of $\widehat{\mathbf{W}}_{lin}$ the problems (9.16) admit unique solutions.

Denote $\mathbf{S}$ the vector space generated by $\{\widehat{\chi}_1,\dots,\widehat{\chi}_6\}$. Every function $\widehat{v} \in K_{\zeta,z}$ is uniquely written as

$$\widehat{v} = \sum_{i=1}^6 \zeta_i\widehat{\chi}_i + \widehat{w}, \qquad \sum_{i=1}^6 \zeta_i\widehat{\chi}_i \in \mathbf{S}, \;\; \widehat{w} \in K_z.$$

Hence, the solution of (9.14) is uniquely decomposed as

$$\widehat{v}_{\zeta,z} = \widehat{v}_{\zeta,\,lin} + \widehat{\chi}_{\zeta,z}, \qquad \widehat{v}_{\zeta,\,lin} = \sum_{i=1}^6 \zeta_i\widehat{\chi}_i \in \mathbf{S}, \quad \widehat{\chi}_{\zeta,z} \in K_z. \tag{9.17}$$

The additional corrector $\widehat{\chi}_{\zeta,z}$ takes into account the nonlinearity and is the solution of the variational problem

$$\begin{aligned}
&\text{For } (\zeta,z) \text{ in } \mathbb{R}^6\times\overline{\Omega}, \text{ find } \widehat{\chi}_{\zeta,z} \in K_z,\\
&\int_{Cyls} \rho\widetilde{\mathbf{A}}\Big[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,lin}) + \mathcal{E}_X(\widehat{\chi}_{\zeta,z})\Big] \cdot \mathcal{E}_{X,kl}(\widehat{\chi}_{\zeta,z} - \widehat{w})|\boldsymbol{\eta}|\, dX \le 0, \qquad \forall \widehat{w} \in K_z.
\end{aligned} \tag{9.18}$$

This variational inequality admits solutions by the Stampacchia-Theorem [18]. Here also, two solutions $\widehat{\chi}_{\zeta,z}$ and $\widetilde{\chi}_{\zeta,z}$ of (9.18) differ only by rigid motions (see Lemma 9.2). Hence ,there exist rigid displacements $r^{(\alpha,c)}_{\zeta,z}$, $(\alpha,c) \in \{1,2\}\times\{0,1\}$ such that

$$\widehat{\chi}^{(\alpha,c)}_{\zeta,z} - \widetilde{\chi}^{(\alpha,c)}_{\zeta,z} = r^{(\alpha,c)}_{\zeta,z}, \qquad (\alpha,c) \in \{1,2\}\times\{0,1\}$$

and one has

$$\|r^{(\alpha,c)}_{\zeta,z}\|_{L^2(Cyls^{(\alpha,c)})} \le C\|g\|_{L^\infty(\Omega)}, \qquad (\alpha,c) \in \{1,2\}\times\{0,1\}.$$

Lemma 9.4. *The map $(\zeta,z) \in \mathbb{R}^6\times\overline{\Omega} \longmapsto \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$ is continuous. Moreover, one has*

$$\forall(\zeta,z) \in \mathbb{R}^6\times\overline{\Omega}, \qquad \begin{aligned} &\big\|\mathcal{E}_X(\widehat{\chi}_{\zeta,z})\big\|_{L^2(Cyls)} \le C|\zeta|,\\ &\big\|\widehat{\chi}_{\zeta,z}\big\|_{H^1(Cyls)} \le C\big(|\zeta| + \|g\|_{L^\infty(\Omega)}\big). \end{aligned} \tag{9.19}$$

The constants do not depend on (ζ,z).

Proof. First, choose $\widehat{w} = 0$ in (9.18), that leads to the estimate $(9.19)_1$. Then $(9.19)_2$ is a consequence of Lemma 9.2.
Now, we prove that the map is continuous. Let (ζ, z) be in $\mathbb{R}^6 \times \overline{\Omega}$ and $\{(\zeta_n, z_n)\}_{n\in\mathbb{N}^*}$ a sequence satisfying

$$(\zeta_n, z_n) \in \mathbb{R}^6 \times \overline{\Omega}, \qquad \zeta_n \longrightarrow \zeta, \quad \text{and} \quad z_n \longrightarrow z.$$

Due to (9.19) and Lemma 9.2, the sequence $\{\widehat{\chi}_{\zeta_n,z_n}\}_{n\in\mathbb{N}^*}$ is uniformly bounded in $H^1(Cyls)^3$. Hence, there exist a subsequence $\{n'\}$ and $\widehat{\chi}_0 \in H^1(Cyls)^3$ such that

$$\begin{aligned} &\widehat{\chi}_{\zeta_{n'},z_{n'}} \rightharpoonup \widehat{\chi}_0 \quad \text{weakly in } H^1(Cyls)^3,\\ &\widehat{\chi}_{\zeta_{n'},z_{n'}} \longrightarrow \widehat{\chi}_0 \quad \text{strongly in } L^2(Cyls)^3 \end{aligned} \qquad \text{and} \qquad \widehat{\chi}_{\zeta_{n'},z_{n'}}(X) \longrightarrow \widehat{\chi}_0(X) \quad \text{for a.e. } X \in Cyls. \tag{9.20}$$

First, using the definition of K_z and passing to the limit gives ($(a,b) \in \{0,1\}^2$)

$$|\widehat{\chi}^{(1,b)}_{0,\alpha} - \widehat{\chi}^{(2,a)}_{0,\alpha}| \le g_\alpha(z), \quad 0 \le (-1)^{a+b}\big(\widehat{\chi}^{(1,b)}_{0,3} - \widehat{\chi}^{(2,a)}_{0,3}\big) \le g_3(z), \quad \text{a.e. on } \mathbf{C}_{ab}, \ \text{ and } \ \int_{Cyls} \widehat{\chi}_0 \, dX = 0,$$

which implies that $\widehat{\chi}_0 \in K_z$. Then, from (9.18) one has for all $\widehat{w}_{n'} \in K_{z_{n'}}$ that

$$\begin{aligned} \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \cdot \mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\, dX &\le -\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}(\zeta_{n'}) \cdot \mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}} - \widehat{w}_{n'})|\boldsymbol{\eta}|\, dX\\ -\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{v}_{\zeta_{n'},lin}) \cdot \mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}} - \widehat{w}_{n'})|\boldsymbol{\eta}|\, dX &+ \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \cdot \mathcal{E}_X(\widehat{w}_{n'})|\boldsymbol{\eta}|\, dX. \end{aligned} \tag{9.21}$$

Now, for every $\widehat{w} \in K_z$, we build a sequence $\widehat{w}_{n'}$ of admissible test-displacements strongly converging to $\widehat{w}$ in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Set ($i \in \{1,2,3\}$, $\alpha \in \{1,2\}$)

$$\Big(\widehat{w}^{(\alpha)}_{n'}\Big)_i = \frac{g_i(z_{n'})}{g_i(z)}\Big(\widehat{w}^{(\alpha)}\Big)_i \quad \text{if } g_i(z) \neq 0, \qquad \Big(\widehat{w}^{(\alpha)}_{n'}\Big)_i = \Big(\widehat{w}^{(\alpha)}\Big)_i \quad \text{if } g_i(z) = 0.$$

Clearly, due to the continuity of g, the sequence $\{(\widehat{w}^{(1)}_{n'}, \widehat{w}^{(2)}_{n'})\}_{n'\in\mathbb{N}}$ strongly converges to $(\widehat{w}^{(1)}, \widehat{w}^{(2)})$ in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Then, observe that the left-hand side of (9.21) is converging by weak lower semi-continuity of the integral and the weak convergence of $\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})$. In the right-hand side we have a sum of integrals with a product of a weakly L^2-convergent term with another which converges strongly. Hence for all $\widehat{w} \in K_z$ one has

$$\begin{aligned} \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_0) \cdot \mathcal{E}_X(\widehat{\chi}_0)|\boldsymbol{\eta}|\, dX &\le \liminf_{n'\to 0} \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \cdot \mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\, dX\\ &\le \limsup_{n'\to 0} \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \cdot \mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}})|\boldsymbol{\eta}|\, dX \le -\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}(\zeta) \cdot \mathcal{E}_X(\widehat{\chi}_0 - \widehat{w})|\boldsymbol{\eta}|\, dX\\ &\quad -\int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{v}_{\zeta,lin}) \cdot \mathcal{E}_X(\widehat{\chi}_0 - \widehat{w})|\boldsymbol{\eta}|\, dX + \int_{Cyls} \rho\widetilde{\mathbf{A}}\mathcal{E}_X(\widehat{\chi}_0) \cdot \mathcal{E}_X(\widehat{w})|\boldsymbol{\eta}|\, dX. \end{aligned}$$

Therefore, the field $\widehat{\chi}_0$ solves the problem (9.18). Recall that

$$\mathcal{E}_X(\widehat{\chi}_{\zeta_{n'},z_{n'}}) \rightharpoonup \mathcal{E}_X(\widehat{\chi}_0) \quad \text{weakly in } L^2(Cyls)^6.$$

Due to the uniqueness of the strain tensor of the solution to problem (9.18), one has $\mathcal{E}_X(\widehat{\chi}_0) = \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. As a consequence the whole sequence $\{\mathcal{E}_X(\widehat{\chi}_{\zeta_n,z_n})\}_{n\in\mathbb{N}^*}$ converges to $\mathcal{E}_X(\widehat{\chi}_0) = \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. That gives the continuity of the map $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto \mathcal{E}_X(\widehat{\chi}_{\zeta,z})$. ☐

Remark 9.5. *Denote $\widehat{v}_\zeta$ the solution of (9.14) with $g_i = 1$, $i = 1,2,3$. Then consider the variational inequality (9.14) with $g_1 = g_2 = g_3 = G$. In this case one has*

$$\mathcal{E}_X(\widehat{\chi}_{\zeta,z}) = G(z)\mathcal{E}_X(\widehat{\chi}_{\zeta/G(z)}).$$

Proposition 9.6. *Under the assumptions of Theorem 9.3, the function A^{hom} defined by ($n \in \{1,\dots,6\}$)*

$$A^{hom}_n(z,\zeta) = \int_{Cyls} \rho\widetilde{\mathbf{A}}\left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\right] \cdot \big(\mathcal{E}(\mathbf{e}_n) + \mathcal{E}_X(\widehat{\chi}_n)\big)|\boldsymbol{\eta}|\, dX \tag{9.22}$$

with $\widehat{v}_{\zeta,z}$ the solution of problem (9.14) is of Caratheodory type and monotone.

Proof. First note that from Lemma 9.4 the map $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto \mathcal{E}_X(\widehat{v}_{\zeta,z}) \in \mathcal{C}(\mathbb{R}^6 \times \overline{\Omega}; L^2(Cyls))^6$ is continuous. Hence, the map $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega} \longmapsto A^{hom}(\zeta, z) \in \mathcal{C}(\mathbb{R}^6 \times \overline{\Omega}; \mathbb{R}^6)$ is continuous. Moreover, due to (9.17) and $(9.19)_1$, it satisfies

$$|A^{hom}(z,\zeta)| \le C|\zeta| \quad \text{for every } (z,\zeta) \in \overline{\Omega} \times \mathbb{R}^6. \tag{9.23}$$

Monotonicity is easily shown by

$$\begin{aligned}
&\big(A^{hom}(z,\zeta) - A^{hom}(z,\xi)\big) \cdot (\zeta - \xi) \\
&= \int_{Cyls} \rho \widetilde{\mathbf{A}} \left[\mathcal{E}(\zeta-\xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})\right] \cdot \left[\mathcal{E}(\zeta-\xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})\right] |\boldsymbol{\eta}| \, dX \\
&\quad + \int_{Cyls} \rho \widetilde{\mathbf{A}} \left[\mathcal{E}(\zeta-\xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})\right] \cdot \mathcal{E}_X(\widehat{\chi}_{\xi,z} - \widehat{\chi}_{\zeta,z}) |\boldsymbol{\eta}| \, dX
\end{aligned}$$

The last integral is non-negative by problem (9.18) and the first one by coercivity of the matrix A. Hence, using the above Lemma 9.7 we arrive at

$$\big(A^{hom}(z,\zeta) - A^{hom}(z,\xi)\big) \cdot (\zeta - \xi) \ge C \int_{Cyls} \left|\mathcal{E}(\zeta-\xi) + \mathcal{E}_X(\widehat{v}_{\zeta,z} - \widehat{v}_{\xi,z})\right|^2 \, dX \ge 0 \tag{9.24}$$

with constants independent of ζ, ξ, z and $C > 0$. $\square$

Lemma 9.7. *There exist two constant C_1, $C' > 0$ such that*

$$\forall (z,\zeta) \in \Omega \times \mathbb{R}^9, \qquad |\zeta| \ge C_1 \|g\|_{L^\infty(\Omega)} \implies A^{hom}(z,\zeta) \cdot \zeta \ge C'|\zeta|^2.$$

Proof. Step 1. In this step we show that there exists a constant $C_0 > 0$ such that, if the equation

$$\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}) = 0$$

admits a solution in $K_{\zeta,z}$, $(\zeta, z) \in \mathbb{R}^6 \times \overline{\Omega}$, then $|\zeta| \le C_0 \|g\|_{L^\infty(\Omega)}$.
The solution of the above equation is given by

$$\widehat{v}^{(1,b)} = \mathcal{A}^{(1,b)} + \mathcal{B}^{(1,b)} \wedge \big((X_2 - b)\mathbf{e}_2 + X_3 \mathbf{n}(X_1)\big), \qquad \widehat{v}^{(2,a)} = \mathcal{A}^{(2,a)} + \mathcal{B}^{(2,a)} \wedge \big((X_1 - a)\mathbf{e}_1 + X_3 \mathbf{n}(X_2)\big)$$

with

$$\begin{aligned}
&\mathcal{B}^{(1,b)}(X_1) = \mathbf{b}^{(1,b)} - (X_1 - 1)\begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix}, \qquad \mathcal{B}^{(2,a)}(X_1) = \mathbf{b}^{(2,a)} - (X_2 - 1)\begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix},\\
&\mathcal{A}^{(1,b)}(X_1) = \mathbf{a}^{(1,b)} + (X_1 - 1)\left[\mathbf{b}^{(1,b)} \wedge \mathbf{e}_1 - \begin{pmatrix}\zeta_1\\ \zeta_2\\ 0\end{pmatrix}\right] - \frac{1}{2}(X_1-1)^2 \begin{pmatrix}\zeta_6\\ -\zeta_4\\ 0\end{pmatrix} \wedge \mathbf{e}_1 - \Phi^{(1,b)}(X_1)\mathcal{B}^{(1,b)}(X_1) \wedge \mathbf{e}_3,\\
&\mathcal{A}^{(2,a)}(X_2) = \mathbf{a}^{(2,a)} + (X_2 - 1)\left[\mathbf{b}^{(2,a)} \wedge \mathbf{e}_2 - \begin{pmatrix}\zeta_2\\ \zeta_3\\ 0\end{pmatrix}\right] - \frac{1}{2}(X_2-1)^2 \begin{pmatrix}\zeta_5\\ -\zeta_6\\ 0\end{pmatrix} \wedge \mathbf{e}_2 - \Phi^{(2,a)}(X_2)\mathcal{B}^{(2,a)}(X_2) \wedge \mathbf{e}_3,
\end{aligned}$$

where $\mathbf{b}^{(1,b)}$, $\mathbf{a}^{(1,b)}$, $\mathbf{b}^{(2,a)}$, $\mathbf{a}^{(2,a)}$ belong to $\mathbb{R}^3$.
First, note that the functions $X_1 \longmapsto (X_1-1)^2$ and $X_2 \longmapsto (X_2-1)^2$ can be extended in 2-periodic functions. Then, the periodicity of $\mathcal{A}^{(1,b)}$ and $\mathcal{B}^{(1,b)}$ (resp. $\mathcal{A}^{(2,a)}$ and $\mathcal{B}^{(2,a)}$) with respect to X_1 (resp. X_2) yields $\zeta_1 = \zeta_3 = \zeta_4 = \zeta_5 = \zeta_6 = 0$ and

$$\mathbf{b}^{(1,b)} = \begin{pmatrix}\mathbf{b}_1^{(1,b)}\\ 0\\ \zeta_2\end{pmatrix}, \quad \mathbf{b}^{(2,a)} = \begin{pmatrix}0\\ \mathbf{b}_2^{(2,a)}\\ -\zeta_2\end{pmatrix}.$$

This reduces the displacements tremendously to

$$\begin{aligned}
\widehat{v}^{(1,b)}(X) &= \mathbf{a}^{(1,b)} - \begin{pmatrix}\mathbf{b}_1^{(1,b)}\\ 0\\ \zeta_2\end{pmatrix} \wedge \Big[\Phi^{(1,b)}(X_1)\mathbf{e}_3 + (X_2 - b)\mathbf{e}_2 + X_3 \mathbf{n}^{(1,b)}(X_1)\Big],\\
\widehat{v}^{(2,a)}(X) &= \mathbf{a}^{(2,a)} - \begin{pmatrix}0\\ \mathbf{b}_2^{(2,a)}\\ -\zeta_2\end{pmatrix} \wedge \Big[\Phi^{(2,a)}(X_2)\mathbf{e}_3 + (X_1 - a)\mathbf{e}_1 + X_3 \mathbf{n}^{(2,a)}(X_2)\Big].
\end{aligned}$$

Then the displacements on the contact parts read as

$$\widehat{v}^{(1,b)}(X) = \mathbf{a}^{(1,b)} - \begin{pmatrix} \mathbf{b}_1^{(1,b)} \\ 0 \\ \zeta_2 \end{pmatrix} \wedge (X_2 - b)\mathbf{e}_2, \qquad \widehat{v}^{(2,a)}(X) = \mathbf{a}^{(2,a)} - \begin{pmatrix} 0 \\ \mathbf{b}_2^{(2,a)} \\ -\zeta_2 \end{pmatrix} \wedge (X_1 - a)\mathbf{e}_1,$$

Hence

$$\mathbf{M}_{ab}(\zeta) + \widehat{v}^{(1,b)} - \widehat{v}^{(2,a)} = \mathbf{a}^{(1,b)} - \mathbf{a}^{(2,a)} + \begin{pmatrix} -2(X_2 - b)\zeta_2 \\ 2(X_1 - a)\zeta_2 \\ (X_2 - b)\mathbf{b}_1^{(1,b)} + (X_1 - a)\mathbf{b}_2^{(2,a)} \end{pmatrix}$$

and thereby $2\kappa|\zeta_2| \leq \|g\|_{L^\infty(\Omega)}$.

Step 2. In this step, we prove by contradiction that there exists a constant $C_1\|g\|_{L^\infty(\Omega)} > 0$ such that for all $(z, \zeta) \in \overline{\Omega} \times \mathbb{R}^9$ and all $\widehat{v} \in K_{\zeta,z}$ it holds

$$|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)} = 2C_0\|g\|_{L^\infty(\Omega)} \implies \int_{Cyls} \big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v})\big)^2\, dX \geq C_1\|g\|_{L^\infty(\Omega)}|\zeta|^2. \tag{9.25}$$

Suppose (9.25) not satisfied. Then, for every $n \in \mathbb{N}^*$ there exists $(z_n, \zeta_n) \in \overline{\Omega} \times \mathbb{R}^9$ with $|\zeta_n| \geq C_1\|g\|_{L^\infty(\Omega)}$ and $\widehat{v}_n \in K_{\zeta_n, z_n}$ such that

$$\int_{Cyls} \big(\mathcal{E}(\zeta_n) + \mathcal{E}_X(\widehat{v}_n)\big)^2\, dX \leq \frac{1}{n}|\zeta_n|^2, \qquad n \in \mathbb{N}^*. \tag{9.26}$$

• Case 1: a subsequence of $\{|\zeta_n|\}$ is bounded. From (9.26) and $(9.8)_3$ the sequence $\{\widehat{v}_n\}_n$ is bounded in $\widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}$. Then, there exists a subsequence of $\{n\}$ (still denoted $\{n\}$) such that

$$\zeta_n \longrightarrow \zeta, \qquad \widehat{v}_n \rightharpoonup \widehat{v} \quad \text{weakly in} \quad \widehat{\mathbf{W}}^{(1)} \times \widehat{\mathbf{W}}^{(2)}.$$

One has $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$. Now, passing to the limit in (9.26) gives

$$\int_{Cyls} \big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v})\big)^2\, dX \leq \liminf_{n\to+\infty} \int_{Cyls} \big(\mathcal{E}(\zeta_n) + \mathcal{E}_X(\widehat{v}_n)\big)^2\, dX \leq 0.$$

Hence

$$\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}) = 0.$$

Using Step 1, this yields that $|\zeta| \leq \dfrac{C_1\|g\|_{L^\infty(\Omega)}}{2} = C_0\|\mathbf{g}\|_{L^\infty(\Omega)}$, wich obviously contradicts the fact that $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$. Hence $\lim\limits_{n\to+\infty} |\zeta_n| = +\infty$.

• Case 2: $\lim\limits_{n\to+\infty} |\zeta_n| = +\infty$. Set $\zeta_n' = \frac{C_1\|g\|_{L^\infty(\Omega)}\zeta_n}{|\zeta_n|}$ and $\widehat{v}_n' = \frac{C_1\|g\|_{L^\infty(\Omega)}}{|\zeta_n|}\widehat{v}_n$. Then one has $|\zeta_n'| = C_1\|g\|_{L^\infty(\Omega)}$ and from (9.26)

$$\int_{Cyls} \big(\mathcal{E}(\zeta_n') + \mathcal{E}_X(\widehat{v}_n')\big)^2\, dX \leq \frac{1}{n}|\zeta_n'|^2 \leq \frac{C_1^2}{n}\|g\|^2_{L^\infty(\Omega)}.$$

Then, proceeding as in the first case one obtains a contradiction and (9.25) is proved.

Step 3. In this step we show

$$\forall (z, \zeta) \in \Omega \times \mathbb{R}^9, \qquad |\zeta| \geq C_1\|g\|_{L^\infty(\Omega)} \implies A^{hom}(z, \zeta)\cdot\zeta \geq C'|\zeta|^2.$$

For every $(z, \zeta) \in \Omega \times \mathbb{R}^9$ such that $|\zeta| \geq C_1\|g\|_{L^\infty(\Omega)}$ one has

$$\begin{aligned} A^{hom}(z, \zeta)\cdot\zeta &= \int_{Cyls} \rho\widetilde{\mathbf{A}}\,[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,\,lin}) + \mathcal{E}_X(\widehat{\chi}_{\zeta,z})]\cdot\big(\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,\,lin})\big)|\boldsymbol{\eta}|\, dX \\ &= \int_{Cyls} \rho\widetilde{\mathbf{A}}\,[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})]\cdot[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z}]\,|\boldsymbol{\eta}|\, dX - \int_{Cyls} \rho\widetilde{\mathbf{A}}\,[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z}]\cdot\mathcal{E}_X(\widehat{\chi}_{\xi,z})|\boldsymbol{\eta}|\, dX. \end{aligned}$$

The last integral is non-negative by problem (9.18). For the first term we apply Step 2 and the coercivity of the matrix A. We conclude thereby that

$$A^{hom}(z, \zeta)\cdot\zeta \geq C'|\zeta|^2. \qquad \square$$

Theorem 9.8. *Under the assumptions of Theorem 9.3 the homogenized problem*

$$\begin{aligned} &\textit{Find } \mathbb{U} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega) \textit{ such that:} \\ &\int_\Omega A^{hom}\big(z,\zeta(z)\big) \cdot \xi(z)\, dz = \int_\Omega F(z)\, \mathbb{V}(z)\, dz, \quad \forall\, \mathbb{V} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega) \end{aligned} \tag{9.27}$$

with

$$\zeta = (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3), \quad \xi = (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3)$$

and the nonlinear differential operator ($m \in \{1, \dots, 6\}$)

$$A_m^{hom}(\cdot, \zeta) = \int_{Cyls} \rho \widetilde{\mathbf{A}} \left[\mathcal{E}(\zeta) + \mathcal{E}_X(\widehat{v}_{\zeta,z})\right] \cdot \left[\mathcal{E}(\mathbf{e}_m) + \mathcal{E}_X(\widehat{\chi}_m)\right] |\boldsymbol{\eta}|\, dX, \tag{9.28}$$

admits solutions.

Proof. The solvability of the problem (9.27) is a direct consequence of the Caratheodory-type, monotonicity, coercivity and boundedness (9.23) of the function A^{hom}. □

The operator-structure of the homogenized problem is known as *Leray-Lions*-operator.

9.2 The Linear Limit

As seen in the section before the limit-problem is a overall non-linear problem due to the contact. In particular, this corresponds to the contact $\mathbf{g}_\varepsilon \sim \varepsilon^3$ but in the case where $\mathbf{g}_\varepsilon = 0$ or at least $\mathbf{g}_\varepsilon \sim \varepsilon^{3+\delta}$ with $\delta > 0$ the problem reduces to a linear problem in both the microscopic and the macroscopic level. Indeed, in this case the limiting contact-condition is just

$$\mathbf{M}_{ab}(\zeta) + \widehat{u}^{(1,b)} - \widehat{u}^{(1,b)} = 0 \tag{9.29}$$

with $\zeta = (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3)$ as above. Thus, we find that the corrector problem (9.14) reduces to (9.16). Hence, all necessary information is already captured by the linear correctors and the nonlinear corrector vanishes $\widehat{\chi}_{\zeta,z} = 0$. This reduces the homogenized operator to a matrix with the entries

$$A_{nm}^{hom,lin} = \int_{Cyls} \rho \widetilde{\mathbf{A}} \left[\mathcal{E}(\mathbf{e}_n) + \mathcal{E}_X(\widehat{\chi}_n)\right] \cdot \left[\mathcal{E}(\mathbf{e}_m) + \mathcal{E}_X(\widehat{\chi}_m)\right] |\boldsymbol{\eta}|\, dX, \quad m, n \in \{1, \dots, 6\} \tag{9.30}$$

and leads to the homogenized problem

$$\begin{aligned} &\text{Find } \mathbb{U} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega) \text{ such that:} \\ &\int_\Omega A^{hom,lin} \zeta \cdot \xi\, dz = \int_\Omega F\, \mathbb{V}\, dz, \quad \forall\, \mathbb{V} \in [\mathcal{H}^1(\Omega)]^2 \times \mathcal{H}^2(\Omega) \end{aligned} \tag{9.31}$$

with

$$\zeta = (e_{11}(\mathbb{U}), e_{12}(\mathbb{U}), e_{22}(\mathbb{U}), \partial_{11}\mathbb{U}_3, \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3), \quad \xi = (e_{11}(\mathbb{V}), e_{12}(\mathbb{V}), e_{22}(\mathbb{V}), \partial_{11}\mathbb{V}_3, \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3).$$

Theorem 9.9. *Under the assumptions of 9.8 and additionally that the contact satisfies* $\|\mathbf{g}_\varepsilon\| \le \varepsilon^{3+\delta}$ *with* $\delta > 0$ *the problem* (9.31) *is uniquely solvable.*

Proof. The existence is a direct consequence of Theorem 9.8. The uniqueness is a consequence of being a coercive bilinear form and the Lax-Milgram-Lemma. □

10 Appendix

Remark 10.1 (Transformation ψ_ε and its Jacobian determinant)**.** *Recall that*

$$\psi_\varepsilon(z) = M_\varepsilon(z_1) + z_2\mathbf{e}_2 + z_3\mathbf{n}_\varepsilon(z_1), \qquad M_\varepsilon(z_1) = z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3.$$

Then by differentiating and the Frenet formulas the Jacobian

$$\nabla\psi_\varepsilon(z) = \Big(\frac{dM(z_1)}{dz_1} + z_3\frac{d\mathbf{n}_\varepsilon(z_1)}{dz_1}\Big|\mathbf{e}_2\Big|\mathbf{n}_\varepsilon(z_1)\Big) = \big(\boldsymbol{\gamma}_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big)\mathbf{t}_\varepsilon(z_1)\big|\mathbf{e}_2\big|\mathbf{n}_\varepsilon(z_1)\big)$$

is easily obtained as well as the Jacobian determinant

$$\det(\nabla\psi_\varepsilon)(z) = \eta_\varepsilon(z) = \boldsymbol{\gamma}_\varepsilon(z_1)\big(1 - z_3 c_\varepsilon(z_1)\big).$$

One has $1 \leq \boldsymbol{\gamma}_\varepsilon(z) \leq C$ *for every* $z \in \overline{P_r}$ *(see* (3.2)*).*
To show that ψ_ε *is a diffeomorphism, it is left to show that* $0 < 1 - z_3 c_\varepsilon(z_1) \leq C$*. But the boundedness is immediately clear form the boundedness of* Φ_ε' *and* $r\Phi_\varepsilon''$ *in* L^∞ *for fixed* κ *small enough. Recall that*

$$c_\varepsilon(z_1) = \frac{\Phi_\varepsilon''(z_1)}{(\boldsymbol{\gamma}_\varepsilon(z_1))^3}.$$

Hence

$$1 - z_3 c_\varepsilon(z_1) \geq 1 - r\|\Phi_\varepsilon''\|_{L^\infty(0,L)} \geq 1 - \frac{12\kappa^2}{(1-2\kappa)^2}$$

and thereby $\kappa < \widehat{\kappa} = \frac{\sqrt{3}-1}{4}$*. While for the upper it suffices that* η_ε *is piecewise* $\mathcal{C}^2$ *on a compact interval.*

For the beams in $\mathbf{e}_2$*-direction we have similarly*

$$\nabla\psi_\varepsilon^{(2)}(z) = \Big(\mathbf{e}_1 \;\big|\; \boldsymbol{\gamma}_\varepsilon(z_2)\big(1 - z_3 c_\varepsilon(z_2)\big)\mathbf{t}_\varepsilon^{(2)}(z_2) \;\big|\; \mathbf{n}_\varepsilon^{(2)}(z_2)\Big) \tag{10.1}$$

where we denoted all functions with the index (2) *to distinguish them from the beam considered before.*

Lemma 10.2 (The unfolded limit of $\nabla\psi_\varepsilon$)**.** *The oscillating function* Φ_ε *converges strongly*

$$\frac{1}{\varepsilon}\Pi^{(\alpha)}(\Phi_\varepsilon^{[\alpha]}) \to \Phi^{(\alpha,b)} \quad \text{strongly in } L^2(\Omega; H^1(\mathcal{Y}^{\ell s,\alpha})) \tag{10.2}$$

with $\Phi^{(\alpha,b)}(X_\alpha) = (-1)^{\alpha+b}\Phi(X_\alpha)$*. Moreover, we have the following strong convergences*

$$\begin{aligned}
&\Pi_\varepsilon^{[\alpha]}(\mathbf{t}_\varepsilon^{[\alpha]}) \to \mathbf{t}^{(\alpha,c)}(X_\alpha - c) = \frac{1}{\boldsymbol{\gamma}(X_\alpha - c)}\big(\mathbf{e}_\alpha + d_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha - c)\mathbf{e}_3\big) && \text{strongly in } [L^2(\Omega; H^1(0,2))]^3\\
&\Pi_\varepsilon^{[\alpha]}(\mathbf{n}_\varepsilon^{[\alpha]}) \to \mathbf{n}^{(\alpha,c)}(X_\alpha - c) = \frac{1}{\boldsymbol{\gamma}(X_\alpha - c)}\big(-d_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha - c)\mathbf{e}_\alpha + \mathbf{e}_3\big) && \text{strongly in } [L^2(\Omega; H^1(0,2))]^3\\
&\Pi_\varepsilon^{[\alpha]}(c_\varepsilon^{[\alpha]}) \to \widehat{c}^{(\alpha,c)}(X_\alpha - c) = \frac{d^2_{X_\alpha}\Phi^{(\alpha,c)}(X_\alpha - c)}{\boldsymbol{\gamma}(X_\alpha - c)^3} && \text{strongly in } [L^2(\Omega; H^1(0,2))]\\
&\Pi_\varepsilon^{[\alpha]}(\eta_\varepsilon^{[\alpha]}) \to \eta^{(\alpha,c)}(X_\alpha - c, X_3) = \boldsymbol{\gamma}(X_\alpha - c)\big(1 - X_3\widehat{c}^{(\alpha,c)}(X_\alpha - c)\big) && \text{strongly in } L^2(\Omega \times Cyls^{(\alpha)})
\end{aligned}$$

with $\boldsymbol{\gamma}(t) = \sqrt{1 + \big(\Phi'(t)\big)^2}$ *and thereby*

$$\Pi_\varepsilon^{[1]}(\nabla\psi_\varepsilon^{[1]}) \to \big(\eta^{(1,b)}\mathbf{t}^{(1,b)} \mid \mathbf{e}_2 \mid \mathbf{n}^{(1,b)}\big), \qquad \Pi_\varepsilon^{[2]}(\nabla\psi_\varepsilon^{[2]}) \to \big(\mathbf{e}_1 \mid \eta^{(2,a)}\mathbf{t}^{(2,a)} \mid \mathbf{n}^{(2,a)}\big).$$

Remark 10.3. *Note that the identity* $\mathbf{n}_\varepsilon = \mathbf{t}_\varepsilon \wedge \mathbf{e}_2$ *remains for all the original, the unfolded and the limit vectors. Additionally also the curvature fulfills its identity* $\dfrac{d\mathbf{n}^{(1,b)}}{dX_1} = -\widehat{c}^{(1,b)}\,\boldsymbol{\gamma}\,\mathbf{t}^{(1,b)}$ *at the limit.*

Lemma 10.4. *Let* $\{\phi_\varepsilon\}_\varepsilon$ *be a sequence of functions in* $L^2(\Omega)$ *satisfying*

$$\|\phi_\varepsilon\|_{L^2(\Omega)} + \varepsilon\Big\|\frac{\partial\phi_\varepsilon}{\partial z_1}\Big\|_{L^2(\Omega)} \leq C$$

where C *does not depend on* ε*. Then, up to a subsequence, there exists* $\widehat{\phi} \in L^2(\Omega \times \mathcal{Y})$ *such that* $\widehat{\phi}$ *is* 2*-periodic with respect to* X_1 *and* $\dfrac{\partial\widehat{\phi}}{\partial X_1} \in L^2(\Omega \times \mathcal{Y})$*. Moreover*

$$\begin{aligned}
&\mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}),\\
&\varepsilon\mathcal{T}_\varepsilon\Big(\frac{\partial\phi_\varepsilon}{\partial z_1}\Big) \rightharpoonup \frac{\partial\widehat{\phi}}{\partial X_1} \quad \text{weakly in } L^2(\Omega \times \mathcal{Y}),\\
&\mathcal{T}_\varepsilon(\phi_\varepsilon)_{|X_1=a} \rightharpoonup \widehat{\phi}_{|X_1=a} \quad \text{weakly in } L^2\big(\Omega \times (\overline{\mathcal{Y}} \cap \{X_1 = a\})\big).
\end{aligned}$$

Proof. These convergences are easy consequences of the properties of the unfolding operator $\mathcal{T}_\varepsilon$. ☐

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