

Scattered classes of graphs

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November 5, 2020

Abstract

For a class \mathcal{C} of graphs G equipped with functions f_G defined on subsets of $E(G)$ or $V(G)$, we say that \mathcal{C} is *k-scattered* with respect to f_G if there exists a constant ℓ such that for every graph $G \in \mathcal{C}$, the domain of f_G can be partitioned into subsets of size at most k so that the union of every collection of the subsets has f_G value at most ℓ . We present structural characterizations of graph classes that are *k-scattered* with respect to several graph connectivity functions.

In particular, our theorem for cut-rank functions provides a rough structural characterization of graphs having no $mK_{1,n}$ vertex-minor, which allows us to prove that such graphs have bounded linear rank-width.

^{*}Supported by IBS-R029-C1 and the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (No. NRF-2018R1D1A1B07050294), and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC consolidator grant DISTRUCT, agreement No. 648527). Part of the research took place while Kwon was at Logic and Semantics, Technische Universität Berlin, Berlin, Germany.

[†]Supported by IBS-R029-C1 and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2017R1A2B4005020).

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1 Introduction

All graphs in this paper are undirected and simple. For a graph G , we write $V(G)$ and $E(G)$ to denote the vertex set and edge set of G , respectively.

In the theory of split decompositions, Cunningham [7] introduced the concept of a *brittle graph*. A *split* of a graph G is a partition (A, B) of the vertex set such that $|A|, |B| \geq 2$ and no two vertices in A have distinct nonempty sets of neighbors in B . Brittle graphs are connected graphs such that every vertex bipartition into two sets of size at least 2 is a split. All brittle graphs are complete graphs or stars. Brittle graphs form basic classes of graphs in canonical split decompositions.

Motivated by brittle graphs, we introduce the general concept of a partition (X_1, X_2, \dots, X_m) of the vertex set or the edge set of a graph such that each X_i has at most k elements, and for every $I \subseteq \{1, 2, \dots, m\}$, some connectivity measurement between $\bigcup_{i \in I} X_i$ and the rest is at most ℓ , for given integers k and ℓ . Brittle graphs then can be seen as graphs that admit a partition (X_1, X_2, \dots, X_m) , where X_1, X_2, \dots, X_m consist of distinct individual vertices, and for every $I \subseteq \{1, 2, \dots, m\}$, the cut-rank function of $\bigcup_{i \in I} X_i$ is at most 1. This concept trades off between the allowed sizes of parts in a partition and the allowed values for a selected connectivity measurement.

We formally define this concept and provide examples. Let X be a finite set and $f : 2^X \rightarrow \mathbb{Z}$. The *f-width* of a partition (X_1, X_2, \dots, X_m) of X , for some m , is

$$\max \left\{ f\left(\bigcup_{i \in I} X_i\right) : I \subseteq \{1, 2, \dots, m\} \right\}.$$

The *k-brittleness* of f is the minimum f -width of all partitions of X into parts of size at most k .

We are mainly interested in the following four functions arising from graphs naturally.

- For a subset F of $E(G)$, let $\kappa_G(F)$ be the number of vertices incident with both an edge in F and an edge not in F .
- For a subset S of $V(G)$, let $\eta_G(S)$ be the number of edges incident with both a vertex in S and a vertex not in S .
- For a subset S of $V(G)$, let $\nu_G(S)$ be the size of a maximum matching of a bipartite subgraph of G obtained by taking edges joining S and $V(G) \setminus S$.

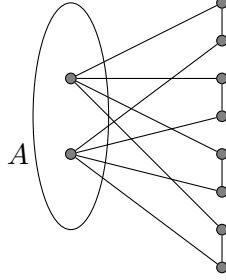


Figure 1: The graph $4P_4/A$ for a path $P_4 = abcd$ with $A = \{a, d\}$.

- For a subset S of $V(G)$, let $\rho_G(S)$ be the rank of the $S \times (V(G) \setminus S)$ 0-1 matrix over the binary field whose (a, b) -entry for $a \in S$, $b \notin S$ is 1 if a, b are adjacent and 0 otherwise. This function is called the *cut-rank* function of G . (See Oum [26] for more properties of the cut-rank functions.)

The k -brittleness of κ_G , η_G , ν_G , ρ_G are called the *vertex k -brittleness* $\beta_k^\kappa(G)$, the *edge k -brittleness* $\beta_k^\eta(G)$, the *matching k -brittleness* $\beta_k^\nu(G)$, the *rank k -brittleness* $\beta_k^\rho(G)$ of G , respectively. We say that a class \mathcal{C} of graphs is *vertex k -scattered* if the vertex k -brittleness of graphs in \mathcal{C} is bounded, *edge k -scattered* if the edge k -brittleness of graphs in \mathcal{C} is bounded, *matching k -scattered* if the matching k -brittleness of graphs in \mathcal{C} is bounded, and *rank k -scattered* if the rank k -brittleness of graphs in \mathcal{C} is bounded.

A class \mathcal{C} of graphs is called a *subgraph ideal* if it contains every graph isomorphic to a subgraph of a graph in \mathcal{C} . We characterize subgraph ideals which are vertex k -scattered, edge k -scattered, or matching k -scattered. We remark that corresponding k -brittleness parameters do not increase by taking a subgraph. Our first theorem characterizes a vertex k -scattered subgraph ideal. For a graph H , we write mH to denote the disjoint union of m copies of H . A set A of vertices is *independent* if no two vertices in A are adjacent. (Note that \emptyset is independent.) For a graph H and an independent set $A \subseteq V(H)$, we write mH/A to denote the graph obtained from mH by identifying all m copies of each vertex in A into one vertex. Note that the number of vertices of mH/A is $m(|V(H)| - |A|) + |A|$ and $1H/A = H$. See Figure 1 for an illustration.

Theorem 1.1. *Let k be a positive integer. A subgraph ideal \mathcal{C} is vertex k -scattered if and only if*

$$\{1H/A, 2H/A, 3H/A, 4H/A, \dots\} \not\subseteq \mathcal{C}$$

for every connected graph H with exactly $k + 1$ edges and each of its independent sets $A \subsetneq V(H)$ such that $H - A$ is connected.

Our second theorem characterizes an edge k -scattered subgraph ideal.

Theorem 1.2. *Let k be a positive integer. A subgraph ideal \mathcal{C} is edge k -scattered if and only if*

$$\{K_{1,1}, K_{1,2}, K_{1,3}, \dots\} \not\subseteq \mathcal{C}$$

and

$$\{T, 2T, 3T, 4T, \dots\} \not\subseteq \mathcal{C}$$

for every tree T on $k + 1$ vertices.

Our third theorem characterizes a matching k -scattered subgraph ideal.

Theorem 1.3. *Let k be a positive integer. A subgraph ideal \mathcal{C} is matching k -scattered if and only if*

$$\{T, 2T, 3T, \dots\} \not\subseteq \mathcal{C}$$

for every tree T on $k + 1$ vertices.

Finally we characterize rank k -scattered graph classes. As the cut-rank function may increase when we take a subgraph, subgraph ideals are not suitable for the study of rank k -scattered graph classes. For instance, complete graphs are rank 1-scattered and yet an arbitrary graph is a subgraph of a complete graph.

Instead of subgraphs, the containment relation called *vertex-minors* is more suitable for the study of rank k -scattered graph classes. A vertex-minor of a graph G is an induced subgraph of a graph that can be obtained from G by a sequence of *local complementations* [2, 3, 4, 26], where local complementation at a vertex v is an operation to flip the adjacency relations between every pair of neighbors of v . The precise definition will be presented in Section 2. The cut-rank function is preserved when applying local complementations [3, 26] and therefore, the rank k -brittleness of a graph does not increase when taking vertex-minors.

A class \mathcal{C} of graphs is called a *vertex-minor ideal* if it contains every graph isomorphic to a vertex-minor of a graph in \mathcal{C} . Our last theorem characterizes rank k -scattered vertex-minor ideals.

Theorem 1.4. *Let k be a positive integer. A vertex-minor ideal \mathcal{C} is rank k -scattered if and only if*

$$\{H, 2H, 3H, 4H, \dots\} \not\subseteq \mathcal{C}$$

for every connected graph H on $k + 1$ vertices.

There are lots of interesting open problems on vertex-minors. In particular, the conjecture of Oum [27], if true, implies that for every circle graph H , every graph G with sufficiently large rank-width has a vertex-minor isomorphic to H . This statement was known to be true when G is a bipartite graph, a circle graph, or a line graph [26, 27]. Very recently, Geelen, Kwon, McCarty, and Wollan [16] announced that they have a proof of this statement. Their proof uses our Theorem 1.4 as a starting point.

Kanté and Kwon [19] proposed the following analogous conjecture for linear rank-width.

Conjecture 1.5 (Kanté and Kwon [19]). *For every fixed forest T , there is an integer $f(T)$ such that every graph of linear rank-width at least $f(T)$ contains a vertex-minor isomorphic to T .*

By Ramsey’s theorem, every sufficiently large connected graph contains one of $K_{1,n}$, K_n , or P_n as an induced subgraph and if n is huge, then each of these graphs contains a large star graph as a vertex-minor. Therefore for each fixed n , each component of a graph having no $K_{1,n}$ vertex-minor has bounded number of vertices and thus it has bounded linear rank-width. Thus, Conjecture 1.5 is true when T is a star.

We can strengthen this observation using Theorem 1.4 and verify Conjecture 1.5 when T is the disjoint union of stars.

Theorem 1.6. *For positive integers m and n , the class of graphs having no vertex-minor isomorphic to $mK_{1,n}$ has bounded linear rank-width.*

Dahlberg, Helsen, and Wehner [8] showed that it is NP-complete to decide whether a graph G contains a vertex-minor isomorphic to a graph H , even if both H and G are restricted to circle graphs. However, we do not know the complexity of deciding whether a graph contains a vertex-minor isomorphic to a fixed graph H . By Theorem 1.6, we can recognize whether a graph contains a vertex-minor isomorphic to the fixed disjoint union of stars and complete graphs in polynomial time. This works as follows. By Theorem 1.6, if the input graph has large linear rank-width, then trivially it has a vertex-minor isomorphic to $mK_{1,n}$ for some large m and n where

$mK_{1,n}$ contains the disjoint union of stars and complete graphs as a vertex-minor. Otherwise, the input graph has bounded rank-width and so the theorem of Courcelle and Oum [6] provides a polynomial-time algorithm.

This paper is organized as follows. In Section 2, we present necessary definitions and notations. Section 3 proves Theorem 1.1 for vertex k -scattered subgraph ideals, Section 4 proves Theorem 1.2 for edge k -scattered subgraph ideals, Section 5 proves Theorem 1.3 for matching k -scattered subgraph ideals, and Section 6 proves Theorem 1.4 for rank k -scattered vertex-minor ideals. Section 7 compares our concepts with various other graph parameters. Section 8 discusses the application of Theorem 1.4 for linear rank-width, proving Theorem 1.6.

2 Preliminaries

For a graph G and a vertex set S of G , we write $G[S]$ to denote the subgraph of G induced by S . For $v \in V(G)$ and $S \subseteq V(G)$, $G - v$ is the graph obtained from G by removing v and all edges incident with v , and $G - S$ is the graph obtained by removing all vertices in S . For $F \subseteq E(G)$, $G - F$ is the subgraph of G with the vertex set $V(G)$ and the edge set $E(G) \setminus F$. For a vertex v of a graph G , $N_G(v)$ is the set of *neighbors* of v in G , and the *degree* of v is the number of edges incident with v . For two disjoint vertex subsets A and B of G , we write $G[A, B]$ to denote the bipartite subgraph on the bipartition (A, B) consisting of all edges of G having one end in A and the other end in B . For two graphs G and H , let $G \cup H$ be the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

A *matching* of a graph is a set of edges of which no two edges share an end. For a matching M , we write $V(M)$ to denote the set of all vertices incident with an edge in M . A *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* in a graph is a set of pairwise non-adjacent vertices.

The *adjacency matrix* of a graph $G = (V, E)$, denoted by $A(G)$, is a $V \times V$ 0-1 matrix whose (v, w) entry is 1 if and only if v and w are adjacent.

We write P_n and K_n to denote a path on n vertices and a complete graph on n vertices respectively. We write $K_{m,n}$ to denote a complete bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$. For a graph G , we denote by \overline{G} the *complement* of G .

We write $R(n; k)$ to denote the minimum number N such that every coloring of the edges of K_N into k colors induces a monochromatic complete subgraph on n vertices. Ramsey's theorem implies that $R(n; k)$ exists.

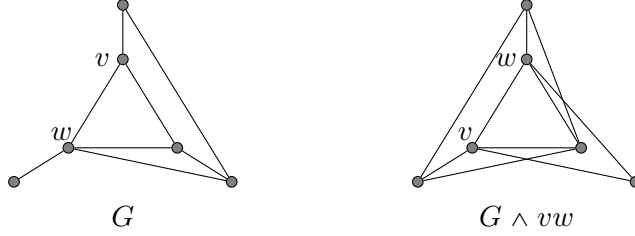


Figure 2: An example of pivoting.

Vertex-minors For a vertex v in a graph G , performing a *local complementation* at v is to replace the subgraph of G induced on $N_G(v)$ by its complement graph. We write $G * v$ to denote the graph obtained from G by applying a local complementation at v . Two graphs G and H are *locally equivalent* if G can be obtained from H by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G .

For an edge uv of a graph G , *pivoting* the edge uv in G is to take a series of three local complementations at u , v , and u . We write $G \wedge uv$ to denote the graph obtained by pivoting uv . In other words, $G \wedge uv = G * u * v * u$. Note that $G \wedge uv$ is identical to the graph obtained from G by flipping the adjacency relation between every pair of vertices x and y where x and y are contained in distinct sets of $N_G(u) \setminus (N_G(v) \cup \{v\})$, $N_G(v) \setminus (N_G(u) \cup \{u\})$, and $N_G(u) \cap N_G(v)$, and finally swapping the labels of u and v [26]. To *flip* the adjacency relation between two vertices, we delete the edge if it exists and add it otherwise. See Figure 2 for an example. For more details, see [26].

Graph operations For two graphs G and H on disjoint vertex sets, each having n vertices, we would like to introduce operations to construct graphs on $2n$ vertices by making the disjoint union of them and adding some edges between two graphs. Roughly speaking, $G \boxplus H$ will add a perfect matching, $G \boxtimes H$ will add the complement of a perfect matching, and $G \boxdot H$ will add a bipartite chain graph. Formally, for two n -vertex graphs G and H with fixed ordering on the vertex sets $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$ respectively, let $G \boxplus H$, $G \boxtimes H$, $G \boxdot H$ be graphs on the vertex set $V(G) \cup V(H)$ whose subgraph induced by $V(G)$ or $V(H)$ is G or H , respectively such that for all $i, j \in \{1, 2, \dots, n\}$,

- (i) $v_i w_j \in E(G \boxplus H)$ if and only if $i = j$,
- (ii) $v_i w_j \in E(G \boxtimes H)$ if and only if $i \neq j$,

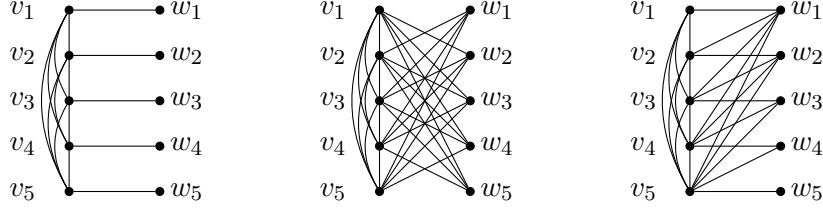


Figure 3: $K_5 \square \overline{K_5}$, $K_5 \otimes \overline{K_5}$, and $K_5 \square \overline{K_5}$.

(iii) $v_i w_j \in E(G \square H)$ if and only if $i \geq j$.

See Figure 3 for illustrations of $K_5 \square \overline{K_5}$, $K_5 \otimes \overline{K_5}$, and $K_5 \square \overline{K_5}$. In each of the constructed graphs, we say that v_i is *matched with* w_j when $i = j$.

3 Vertex k -scattered subgraph ideals

In this section, we characterize vertex k -scattered subgraph ideals.

Theorem 1.1. *Let k be a positive integer. A subgraph ideal \mathcal{C} is vertex k -scattered if and only if*

$$\{1H/A, 2H/A, 3H/A, 4H/A, \dots\} \not\subseteq \mathcal{C}$$

for every connected graph H with exactly $k + 1$ edges and each of its independent set $A \subsetneq V(H)$ such that $H - A$ is connected.

For the forward part, we show the following.

Lemma 3.1. *Let k, ℓ be positive integers. Let H be a connected graph with exactly $k + 1$ edges. If A is an independent set of H such that $H - A$ is connected, then the vertex k -brittleness of $(2\ell + 1)H/A$ is at least $\ell + 1$.*

Proof. Suppose not. Let $G = (2\ell + 1)H/A$. Let (X_1, X_2, \dots, X_t) be a partition of $E(G)$ such that its κ_G -width is at most ℓ and $|X_i| \leq k$ for all $1 \leq i \leq t$.

For a component C of $G - A$, let

$$Y_C = \{i \in \{1, \dots, t\} : \text{some vertex in } V(C) \text{ is incident with an edge in } X_i\}.$$

For each component C of $G - A$, $|Y_C| \geq 2$ because $|X_1|, |X_2|, \dots, |X_t| \leq k$ and vertices in C are incident with more than k edges in total.

Let us pick a random subset I of $\{1, 2, \dots, t\}$. For each component C of $G - A$, the probability that $Y_C \cap I \neq \emptyset$ and $Y_C \setminus I \neq \emptyset$ is $1 - 2^{1-|Y_C|} \geq 1/2$. By the linearity of the expectation, there exists a subset I' of $\{1, 2, \dots, t\}$ such that at least $\ell + 1$ components C of $G - A$ satisfy $Y_C \cap I' \neq \emptyset$ and $Y_C \setminus I' \neq \emptyset$. If $Y_C \cap I' \neq \emptyset$ and $Y_C \setminus I' \neq \emptyset$ for some component C , then $V(C)$ has a vertex incident with both an edge in $\bigcup_{i \in I'} X_i$ and an edge in $\bigcup_{i \notin I'} X_i$, because C is connected.

This means that $\kappa_G(\bigcup_{i \in I'} X_i) \geq \ell + 1$, contradicting our assumption. \square

For the converse direction of Theorem 1.1, we prove that for positive integers k and n , every graph with sufficiently large vertex k -brittleness must contain a subgraph isomorphic to nH/A for some connected graph H with $k + 1$ edges and some independent set $A \subsetneq V(H)$ such that $H - A$ is connected. We prove this statement by induction on k . The following lemma will be used in the induction step.

Lemma 3.2. *Let H be a connected graph with exactly k edges and let $A \subsetneq V(H)$ be an independent set such that $H - A$ is connected. Let m, n be positive integers such that $m \geq 4(k + 1)^2 n^2$. Let G be a graph containing mH/A as a subgraph. If for each component C of $(mH/A) - A$, G has an edge not in $E(mH/A)$ but incident with vertices in C , then G contains a subgraph isomorphic to nH'/A' for some connected graph H' with $k + 1$ edges and an independent set $A' \subsetneq V(H')$ such that $H' - A'$ is connected.*

Proof. It is trivial if $n = 1$. Thus we may assume that $n > 1$. Let us choose a minimal subgraph G' of G such that $V(G) = V(G')$, $E(G') \cap E(mH/A) = \emptyset$ and for every component C of $(mH/A) - A$, there is an edge in G' incident with some vertex of C . Then G' is a forest and $(V(G) \setminus V(mH/A)) \cup A$ is independent in G' by the minimality. Moreover, between two components of $(mH/A) - A$, G' has at most one edge and for each component C of $(mH/A) - A$, the graph $G'[A \cup V(C)]$ has at most one edge. Moreover if $G'[A \cup V(C)]$ has an edge, then no other edges of G' have exactly one end in $V(C)$. Let m' be the number of components C of $(mH/A) - A$ such that $G'[A \cup V(C)]$ has no edge.

Let G'' be the subgraph of G' obtained by deleting all edges e having both ends in $V(C) \cup A$ for some component C of $(mH/A) - A$. As one edge of G' is incident with at most two components, G'' has at least $m'/2$ edges and G' has at least $m'/2 + (m - m')$ edges.

If $m - m' > \binom{k+1}{2}(n - 1)$, then by the pigeon-hole principle, there exists a pair of vertices x and y in H such that at least n isomorphic copies of H in mH/A has the copies x', y' of x and y , respectively, such that

x', y' are adjacent in G' . Then let H' be the graph obtained from H by adding xy . Then G has nH'/A as a subgraph. So, we may assume that $m - m' \leq \binom{k+1}{2}(n-1)$.

Note that vertices in A are isolated in G'' . If a vertex v in $V(mH/A)$ has degree more than 1 in G'' , then no vertex in $G - V(mH/A)$ is adjacent to v in G'' because G' is chosen to be minimal. Therefore all neighbors of v in G'' are in distinct components of $(mH/A) - A$. Notice that the same holds for a vertex v outside of mH/A , because $(V(G) \setminus V(mH/A)) \cup A$ is independent in G'' .

If G'' has a vertex v of degree more than $(k+1)(n-1)$, then more than $(k+1)(n-1)$ components of $(mH/A) - A$ have vertices adjacent to v in G'' . By the pigeon-hole principle, there exists a vertex w of $H - A$ such that in at least n components of $(mH/A) - A$, the copies of w are adjacent to v in G'' . Let H' be the graph obtained from H by adding a new vertex v of degree 1 adjacent to w . Let $A' = A \cup \{v\}$. Then G has nH'/A' as a subgraph and both H' and $H' - A'$ are connected. So we may assume that the maximum degree of G'' is at most $(k+1)(n-1)$.

As G'' is a forest, G'' is bipartite. By König's theorem on the edge coloring, G'' is $(k+1)(n-1)$ -edge-colorable. So G'' has a matching M with

$$|M| \geq \frac{|E(G'')|}{(k+1)(n-1)} \geq \frac{m'}{2(k+1)(n-1)}.$$

Suppose that $m' > 4(k+1)^2(n-1)^2$. Let C_1, C_2, \dots, C_m be the components of $(mH/A) - A$. Let I be a random subset of $\{1, 2, \dots, m\}$ and $X = \bigcup_{i \in I} V(C_i)$. For each edge e in M , the probability that e has exactly one end in X is $1/2$, no matter whether e has one or two ends in $V(mH/A)$. Thus, there exist I and $M' \subseteq M$ such that $|M'| \geq |M|/2 > (k+1)(n-1)$ and every edge of M' has one end in X and the other end not in X . By the pigeon-hole principle, there exists a vertex u of $H - A$ such that at least n edges e of M' are incident with copies of u in mH/A . Then let H' be the graph obtained from H by adding a new vertex v and an edge from v to u and let $A' = A$. Then G has nH'/A' as a subgraph and both H' and $H' - A'$ are connected. Therefore we may assume that $m' \leq 4(k+1)^2(n-1)^2$.

Then $m = m' + (m - m') \leq 4(k+1)^2(n-1)^2 + \binom{k+1}{2}(n-1)$. As $n-1 < 2n-1$ and $k/2 < 4(k+1)$, we deduce that $m < 4(k+1)^2(n-1)^2 + 4(k+1)^2(2n-1) = 4(k+1)^2n^2$. This contradicts our assumption on m . \square

Lemma 3.3. *Every graph with vertex 1-brittleness at least $256n^4$ contains nP_3/A as a subgraph for some independent set $A \subsetneq V(P_3)$ such that $P_3 - A$ is connected.*

Proof. Let G be a graph with vertex 1-brittleness at least $256n^4$. We may assume that G has no components with at most 2 vertices. If G has at least n components, then each component has P_3 as a subgraph and therefore nP_3/\emptyset is a subgraph of G . So we may assume that G has less than n components.

Let G' be the induced subgraph of G obtained by deleting all degree-1 vertices. Then if a vertex of G' has degree less than 2, then it has its private neighbor in $V(G) \setminus V(G')$ of degree 1 in G .

If G' has a vertex v of degree at least $16n^2$, then G' has $mP_2/\{v\}$ as a subgraph where m is the degree of v in G' . By Lemma 3.2, G contains nP_3/A for some independent set $A \subsetneq V(P_3)$ where $P_3 - A$ is connected. So we may assume that every vertex of G' has degree less than $16n^2$.

If G' has a matching M of size at least $16n^2$, then G' has mP_2/\emptyset as a subgraph where $m = |M|$. By Lemma 3.2, G contains nP_3/A for some independent set $A \subsetneq V(P_3)$ where $P_3 - A$ is connected. So we may assume that every matching of G' has less than $16n^2$ edges.

Then by the theorem of Vizing, G' is $16n^2$ -edge-colorable and therefore $|E(G')| \leq 16n^2(16n^2 - 1) = 256n^4 - 16n^2$. As G' has at most $n - 1$ components, $|V(G')| \leq |E(G')| + n - 1 < 256n^4$. Then the vertex 1-brittleness of G is less than $256n^4$, which is a contradiction. \square

For a set A of vertices of a graph G , a *Tutte bridge* of A in G is either an edge joining two vertices in A or a connected subgraph of G consisting of one component C of $G - A$ and all edges joining C and A . Alternatively we may define a Tutte bridge as a connected subgraph of G induced by an equivalence class on $E(G)$ where two edges e and f are equivalent if and only if there is a path starting with e and ending with f such that no internal vertex is in A .

For a Tutte bridge B of A in G , *deleting* B from G is to remove all edges in B and remove all vertices in $V(B) \setminus A$. Note that every component of G is a Tutte bridge of \emptyset . The next lemma shows that the vertex k -brittleness does not decrease too much by deleting small Tutte bridges.

Lemma 3.4. *Let G be a graph and A be a set of vertices of G . If G' is the subgraph of G obtained by deleting all Tutte bridges of A having at most k edges, then $\beta_k^\kappa(G') \geq \beta_k^\kappa(G) - |A|$.*

Proof. Let $P' = (X_1, X_2, \dots, X_t)$ be a partition of $E(G')$ whose $\kappa_{G'}$ -width is equal to $\beta_k^\kappa(G')$. We extend P' to a partition P of $E(G)$ by adding $E(B)$ as one part for each Tutte bridge B of A in G with at most k edges.

Then the κ_G -width of P is at most $\beta_k^\kappa(G') + |A|$ and therefore $\beta_k^\kappa(G) \leq \beta_k^\kappa(G') + |A|$. \square

To complete our proof, we will iteratively find an independent set A_i and two Tutte bridges of A_i having at most k edges for each i . By combining two Tutte bridges, we will build a bigger connected subgraph, assuming that A_i is nonempty. Then we apply the sunflower lemma for the sets A_1, A_2, \dots , which will allow us to find what we wanted. The next lemma allows us to find two Tutte bridges to be combined later.

Lemma 3.5. *Let m, n, k be positive integers. Let H be a connected graph with k edges and let $A \subsetneq V(H)$ be an independent set of H such that $H - A$ is connected. Let G be a graph having mH/A as a subgraph such that no subgraph of G is isomorphic to nH'/A' for some connected graph H' with $k+1$ edges and an independent set $A' \subsetneq V(H')$ for which $H' - A'$ is connected. Let X be a set of vertices of G . If $m > 4(k+1)^2n^2 + |X|$, then G has two distinct Tutte bridges B_1, B_2 of A , satisfying the following.*

- (i) *Each B_i has exactly k edges.*
- (ii) *$V(B_1) \cap A = V(B_2) \cap A = A$.*
- (iii) *neither $B_1 - A$ nor $B_2 - A$ has a vertex in X .*

Proof. By Lemma 3.2, less than $4(k+1)^2n^2$ components C of $mH/A - A$ are incident with an edge in $E(G) \setminus E(mH/A)$. Therefore there are at least $|X| + 2$ components of $mH/A - A$ that form Tutte bridges of A in G isomorphic to H . Among them, at least two, say B_1 and B_2 , will not intersect X . Since $H, H - A$ are connected and A is independent in H , we deduce that $V(B_1) \cap A = V(B_2) \cap A = A$. \square

We need the sunflower lemma. Let \mathcal{F} be a family of sets. A subset $\{M_1, M_2, \dots, M_p\}$ of \mathcal{F} is a *sunflower* with *core* A (possibly an empty set) and p *petals* if for all distinct $i, j \in \{1, 2, \dots, p\}$, $M_i \cap M_j = A$.

Theorem 3.6 (Sunflower Lemma [12, Erdős and Rado]). *Let k and p be positive integers, and \mathcal{F} be a family of sets each of cardinality k . If $|\mathcal{F}| > k!(p-1)^k$, then \mathcal{F} contains a sunflower with p petals.*

Later we will apply Lemma 3.5 iteratively and take $F_i := B_1 \cup B_2$ and $S_i := A$ in the i -th round. Then we will apply the following lemma with $t := 2k$. Note that under this setting, $B_1 \cup B_2$ is connected if A is non-empty.

Lemma 3.7. *Let m, n, k, t be positive integers. Let G be a graph. For each $i \in \{1, 2, \dots, m\}$, let F_i be a connected subgraph of G with exactly t edges having an independent set $S_i \subsetneq V(F_i)$ such that $1 \leq |S_i| \leq k$ and $F_i - X$ is connected for all $X \subsetneq S_i$. If $V(F_i) \cap V(F_j) \subseteq S_i \cap S_j$ and $S_i \neq S_j$ for all $1 \leq i < j \leq m$ and $m > k \cdot k! \binom{(t+1)t/2}{t}^k (n-1)^k$, then G has a subgraph isomorphic to nH/A for some connected graph H with exactly t edges and an independent set $A \subsetneq V(H)$ such that $H - A$ is connected.*

Proof. We may assume $n > 1$ because otherwise we can take $H = F_1$ and $A = \emptyset$. Let $p = \binom{(t+1)t/2}{t} (n-1) + 1 \geq 2$. By the pigeonhole principle, more than $k!(p-1)^k$ of S_1, S_2, \dots, S_m have the same cardinality. By Theorem 3.6, there exist $i_1 < i_2 < \dots < i_p$ such that $\{S_{i_1}, S_{i_2}, \dots, S_{i_p}\}$ is a sunflower with p petals and $|S_{i_1}| = |S_{i_2}| = \dots = |S_{i_p}|$.

Let A be the core, that is $A = \bigcap_{j=1}^p S_{i_j}$. Since $S_i \neq S_j$ for all $i \neq j$, we have $A \neq S_{i_j}$ for all $j \in \{1, 2, \dots, p\}$ and therefore $F_{i_j} - A$ is connected.

Since $V(F_i) \cap V(F_j) \subseteq S_i \cap S_j$ for all $1 \leq i < j \leq m$, we deduce that $F_{i_1} - A, F_{i_2} - A, \dots, F_{i_p} - A$ are vertex-disjoint. There are at most $\binom{(t+1)t/2}{t}$ connected non-isomorphic graphs having exactly t edges and so at least n of $F_{i_1}, F_{i_2}, \dots, F_{i_p}$ are pairwise isomorphic with isomorphisms fixing A , by the pigeonhole principle. This proves the lemma. \square

Proposition 3.8. *For positive integers k and n , there exists an integer $\ell = \ell(k, n)$ such that every graph with vertex k -brittleness at least ℓ contains nH/A as a subgraph for some connected graph H with exactly $k+1$ edges and an independent set $A \subsetneq V(H)$ such that $H - A$ is connected.*

Proof. We define that

$$\ell(1, n) := 256n^4,$$

and for $k \geq 2$,

$$\begin{aligned} \ell(k, n) := & \ell \left(k-1, 4(k+1)^2 n^2 + k^2 \cdot k! \binom{(2k+1)k}{2k}^k (n-1)^k \right) \\ & + k^2 \cdot k! \binom{(2k+1)k}{2k}^k (n-1)^k. \end{aligned}$$

We prove the statement by induction on k . If $k = 1$, then it is true by Lemma 3.3. Now, we prove for $k \geq 2$. Suppose G has vertex k -brittleness at least $\ell = \ell(k, n)$ and no subgraph of G is isomorphic to nH'/A' for a

connected graph H' with $k+1$ edges having an independent set $A' \subsetneq V(H')$ such that $H' - A'$ is connected. Let $m = 4(k+1)^2 n^2 + k^2 \cdot k! \binom{(2k+1)k}{2k}^k (n-1)^k$. Let G_1 be the subgraph of G obtained by deleting all components with at most k edges. By Lemma 3.4, $\beta_k^\kappa(G_1) = \beta_k^\kappa(G)$. Since $\ell(k, n) \geq \ell(k-1, m)$, by the induction hypothesis, G_1 has mH_1/A_1 as a subgraph for some connected graph H_1 with k edges having an independent set $A_1 \subsetneq V(H_1)$ such that $H_1 - A_1$ is connected. Note that $|A_1| \leq k$. We may assume that $n \geq 2$, since G_1 has a component with more than k edges.

If $A_1 = \emptyset$, then each component of mH_1/A_1 has a vertex incident (in G_1) with an edge not in $E(mH_1/A_1)$ because every component of G_1 has more than k edges. By Lemma 3.2, G has a connected subgraph H with an independent set A having desired properties, contradicting our assumption. Therefore $A_1 \neq \emptyset$.

By Lemma 3.5, G_1 has two Tutte bridges $B_{1,1}$ and $B_{1,2}$ of A_1 , each having exactly k edges such that $V(B_{1,1}) \cap A_1 = V(B_{1,2}) \cap A_1 = A_1$. Let $F_1 = B_{1,1} \cup B_{1,2}$. Since $A_1 \neq \emptyset$, F_1 is a connected graph. Then $F_1 - X$ is connected for all $X \subsetneq A_1$.

For $i = \{2, \dots, k \cdot k! \binom{(2k+1)k}{2k}^k (n-1)^k + 1\}$, we define G_i as the subgraph of G_{i-1} obtained by deleting all Tutte bridges of A_{i-1} having at most k edges and then deleting all components having at most k edges. By applying Lemma 3.4 twice, we deduce that $\beta_k^\kappa(G_i) \geq \beta_k^\kappa(G_{i-1}) - |A_{i-1}| - |\emptyset| \geq \beta_k^\kappa(G_{i-1}) - k$. By induction,

$$\beta_k^\kappa(G_i) \geq \beta_k^\kappa(G_1) - (i-1)k \geq \ell(k-1, m),$$

and by the induction hypothesis, G_i has mH_i/A_i as a subgraph for some connected graph H_i with k edges and an independent set $A_i \subsetneq V(H_i)$ such that $H_i - A_i$ is connected. Note that $|A_i| \leq k$. If $A_i = \emptyset$, then each component of mH_i/A_i has a vertex incident (in G_i) with an edge not in $E(mH_i/A_i)$ because every component of G_i has more than k edges, contradicting the assumption by Lemma 3.2. Thus $A_i \neq \emptyset$. Since

$$m > 4(k+1)^2 n^2 + (i-1)k,$$

by Lemma 3.5, G_i has two Tutte bridges $B_{i,1}$ and $B_{i,2}$ of A_i , each having exactly k edges such that $V(B_{i,1}) \cap A_i = V(B_{i,2}) \cap A_i = A_i$ and neither $B_{i,1} - A_i$ nor $B_{i,2} - A_i$ has a vertex in $A_1 \cup A_2 \cup \dots \cup A_{i-1}$. Let $F_i = B_{i,1} \cup B_{i,2}$. As $A_i \neq \emptyset$, F_i is a connected graph. Then $F_i - X$ is connected for all $X \subsetneq A_i$.

We claim that for $i < j$, $V(F_i) \cap V(F_j) \subseteq A_i \cap A_j$. Suppose not. Let $x \in V(F_i) \cap V(F_j)$. When we construct G_{i+1} from G_i , we remove all Tutte

bridges of A_i with at most k edges, including all vertices of $F_i - A_i$. Since F_j is a subgraph of G_j , we deduce that $x \in A_i$. Because we choose F_j so that $F_j - A_j$ has no vertex in $A_1 \cup A_2 \cup \dots \cup A_{j-1}$ but $x \in A_i$, we conclude that $x \in A_j$. This proves the claim.

Suppose that $A_i = A_j$ for some $i < j$. By construction, $B_{j,1} - A_j$ has no vertex in $A_1 \cup A_2 \cup \dots \cup A_{j-1}$. Note that $B_{j,1}$ is not a Tutte bridge of A_i in G_i . So G_i has an edge e joining a vertex $v \in V(B_{j,1} - A_i)$ to a vertex w not in $V(B_{j,1})$. Note that $w \notin V(G_j)$ because $B_{j,1}$ is a Tutte bridge of A_j in G_j . Let p be the minimum integer such that $p \geq i$, $w \in V(G_p)$, and $w \notin V(G_{p+1})$. Since $B_{j,1}$ is a subgraph of G_p and no vertex of $B_{j,1} - A_j$ is in A_p , all edges of $B_{j,1}$ together with e are in the same Tutte bridge of A_p in G_p , which has more than k edges. Furthermore all edges of $B_{j,1}$ and e are in the same component in the graph obtained from G_p by deleting all Tutte bridges of A_p with at most k edges. So w is not deleted when constructing G_{p+1} , contradicting the assumption that $w \notin V(G_{p+1})$. Therefore $A_i \neq A_j$ for all $i < j$.

By applying Lemma 3.7 to F_i and A_i for all $1 \leq i \leq k \cdot k! \binom{(2k+1)k}{2k}^k (n-1)^k + 1$, we deduce that G has a subgraph isomorphic to nH/A for some connected graph H with $2k$ edges having an independent set $A \subsetneq V(H)$ such that $H - A$ is connected.

We claim that H contains a connected subgraph H' with exactly $k+1$ edges such that $H' - (A \cap V(H'))$ is connected. If $H - A$ has more than k edges, then we can simply take H' as a connected subgraph of $H - A$ with $k+1$ edges. If $H - A$ has at most k edges, then let H' be a connected subgraph of H containing $H - A$ as a subgraph such that H' has exactly $k+1$ edges. This proves the claim. However, this claim contradicts our assumption because G contains nH'/A' as a subgraph where $A' = A \cap V(H')$. \square

Lemma 3.1 and Proposition 3.8 imply Theorem 1.1.

4 Edge k -scattered subgraph ideals

In this section, we characterize edge k -scattered subgraph ideals.

Theorem 1.2. *Let k be a positive integer. A subgraph ideal \mathcal{C} is edge k -scattered if and only if*

$$\{K_{1,1}, K_{1,2}, K_{1,3}, \dots\} \not\subseteq \mathcal{C}$$

and

$$\{T, 2T, 3T, 4T, \dots\} \not\subseteq \mathcal{C}$$

for every tree T on $k + 1$ vertices.

First we prove that for some connected graph H on $k + 1$ vertices, the disjoint union of sufficiently many copies of H should have large edge k -brittleness. In fact, this is same for matching k -brittleness and rank k -brittleness, which we prove at the same time as follows.

Lemma 4.1. *Let m, n, k be positive integers with $n > 2m$ and H be a connected graph on $k + 1$ vertices. Then the following hold.*

- (i) nH has edge k -brittleness at least $m + 1$.
- (ii) nH has matching k -brittleness at least $m + 1$.
- (iii) nH has rank k -brittleness at least $m + 1$.

Proof. Let $G := nH$. Let (X_1, X_2, \dots, X_t) be a partition of $V(G)$ such that $|X_i| \leq k$. Let C_1, C_2, \dots, C_n be the components of G . Note that each C_i intersects at least two of X_1, X_2, \dots, X_t . Let I be a random subset of $\{1, 2, \dots, t\}$. For each ℓ , the probability that C_ℓ contains both a vertex in $\bigcup_{i \in I} X_i$ and a vertex in $\bigcup_{j \in \{1, 2, \dots, t\} \setminus I} X_j$ is at least $1/2$. Thus, by the linearity of expectation, there exists $I \subseteq \{1, 2, \dots, t\}$ such that more than m components of G have both a vertex in $\bigcup_{i \in I} X_i$ and a vertex in $\bigcup_{j \in \{1, 2, \dots, t\} \setminus I} X_j$. This implies that $\eta_G(\bigcup_{i \in I} X_i) > m$, $\nu_G(\bigcup_{i \in I} X_i) > m$, and $\rho_G(\bigcup_{i \in I} X_i) > m$. \square

For edge k -brittleness, a large star is also an obstruction.

Lemma 4.2. *For positive integers k and m , $K_{1, k+m}$ has edge k -brittleness at least $m + 1$.*

Proof. Let (X_1, X_2, \dots, X_t) be a partition of $V(K_{1, k+m})$ such that $|X_i| \leq k$. We may assume that X_1 contains the center of $K_{1, k+m}$. Then $\eta_{K_{1, k+m}}(X_1) \geq (k + m) - (k - 1)$. \square

Now, we show the backward direction of Theorem 1.2.

Proposition 4.3. *For all positive integers k and n , there exists an integer $\ell = \ell(k, n)$ such that every graph with edge k -brittleness more than ℓ contains a subgraph isomorphic to either $K_{1, n}$ or nT for some tree T on $k + 1$ vertices.*

Proof. Let $\ell(1, n) = n(n-1)$ and $\ell(k, n) = \ell(k-1, 4k(n-1)^2 + 1)$ for $k \geq 2$.

We proceed by induction on k . We may assume that every vertex has degree at most $n-1$. If $k = 1$, then by the theorem of Vizing, G has a matching of size at least $|E(G)|/n$. Since the edge 1-brittleness is less than or equal to $|E(G)|$, G has a matching of size more than $\ell(1, n)/n = n-1$, and so G contains a subgraph isomorphic to nK_2 . Thus, we may assume that $k > 1$.

We may assume that every component of G has more than k vertices, because otherwise removing them does not decrease the edge k -brittleness. By the induction hypothesis, G has a subgraph isomorphic to mT for a tree T on k vertices where $m = 4k(n-1)^2 + 1$. Let C_1, C_2, \dots, C_m be the disjoint copies of T in G .

Let G' be a minimal subgraph of G such that for all $1 \leq i \leq m$, G' has at least one edge joining C_i with a vertex not in C_i . Since each edge of G' is incident with at most two of C_1, C_2, \dots, C_m , we have $|E(G')| \geq \lceil m/2 \rceil > 2k(n-1)^2$. Note that G' is a forest. So by König's theorem on the edge coloring of bipartite graphs, G' is $(n-1)$ -edge-colorable and so it has a matching M with $|M| > 2k(n-1)$. Each edge of M is incident with at least one copy of some vertex of T in mT .

Let I be a random subset of $\{1, 2, \dots, m\}$. Let $X = \bigcup_{i \in I} V(C_i)$ and $Y = V(G) \setminus X$. The probability that an edge in M has one end in X and the other end in Y is $1/2$ and therefore there exist I and $M' \subseteq M$ such that $|M'| \geq |M|/2 > k(n-1)$ and each edge of M' has one end in X and the other end in Y .

Now M' has a subset M'' with $|M''| > n-1$ such that there exists a vertex w of T with the property that for every edge of M'' , its end in X is a copy of w in mT . Let T' be the tree obtained from T by adding a new vertex adjacent to w only. Then G has nT' as a subgraph. \square

Proposition 4.3 and Lemmas 4.1 and 4.2 imply Theorem 1.2.

5 Matching k -scattered subgraph ideals

In this section, we characterize matching k -scattered subgraph ideals. We already proved in Lemma 4.1 that for a connected graph H on $k+1$ vertices, the disjoint union of sufficiently many copies of H has large matching k -brittleness. Such obstructions exactly characterize matching k -scattered subgraph ideals.

Theorem 1.3. *Let k be a positive integer. A subgraph ideal \mathcal{C} is matching k -scattered if and only if*

$$\{T, 2T, 3T, \dots\} \not\subseteq \mathcal{C}$$

for every tree T on $k + 1$ vertices.

First let us prove that deleting a vertex does not decrease the matching k -brittleness a lot.

Lemma 5.1. *Let k be a positive integer. For each vertex v of a graph G ,*

$$\beta_k^\nu(G) \leq \beta_k^\nu(G - v) + 1.$$

Proof. Let $P' = (X_1, X_2, \dots, X_t)$ be a partition of $V(G - v)$ such that $|X_i| \leq k$ and the ν_{G-v} -width of P' is minimum, that is $\beta_k^\nu(G - v)$. Let $P = (X_1, X_2, \dots, X_t, \{v\})$. Then the ν_G -width of P is at most $\beta_k^\nu(G - v) + 1$. \square

The following proposition with Lemma 4.1 proves Theorem 1.3.

Proposition 5.2. *For all positive integers k and n , there exists $\ell = \ell(k, n)$ such that every graph with matching k -brittleness more than ℓ contains a subgraph isomorphic to nT for some tree T on $k + 1$ vertices.*

Proof. Let $\ell(k, n) = (k + 1)^k(n - 1)$. Let G be a graph with matching k -brittleness more than $\ell(k, n)$. Let $G_0 = G$ and $S_0 = \emptyset$. We claim that there exist disjoint subsets $S_1, S_2, \dots, S_{(k+1)^{k-1}(n-1)}, S_{(k+1)^{k-1}(n-1)+1}$ such that each S_i induces a connected subgraph of G with $k + 1$ vertices. For $i = 1, 2, \dots, (k + 1)^{k-1}(n - 1) + 1$, let G_i be the induced subgraph of $G_{i-1} - S_{i-1}$ obtained by deleting all components with at most k vertices. Notice that by Lemma 5.1, $\beta_k^\nu(G_i) \geq \beta_k^\nu(G_{i-1}) - |S_{i-1}| = \beta_k^\nu(G_{i-1}) - (k + 1)$. By induction, we deduce that $\beta_k^\nu(G_i) \geq \beta_k^\nu(G) - (k + 1)(i - 1) > 0$. Thus G_i contains a component with more than k vertices and therefore it has a vertex set S_i of size $k + 1$ inducing a connected subgraph. This proves the claim.

Let T_i be a spanning tree of $G[S_i]$ for each i . Since the number of labeled trees on $k + 1$ vertices is $(k + 1)^{k-1}$, there exist more than $n - 1$ of these spanning trees that are pairwise isomorphic. \square

6 Rank k -scattered vertex-minor ideals

We characterize rank k -scattered vertex-minor ideals. As we mentioned, the rank k -brittleness of a graph may increase when taking a subgraph. Instead we use vertex-minors because of the following lemma.

Lemma 6.1 (See Oum [26, Proposition 2.6]). *If G is locally equivalent to G' , then for every subset X of vertices of G , $\rho_G(X) = \rho_{G'}(X)$.*

Here is our main theorem for rank k -scattered vertex-minor ideals.

Theorem 1.4. *Let k be a positive integer. A vertex-minor ideal \mathcal{C} is rank k -scattered if and only if for every connected graph H on $k + 1$ vertices,*

$$\{H, 2H, 3H, 4H, \dots\} \not\subseteq \mathcal{C}.$$

First, it is easy to observe the following.

Proposition 6.2. *If H is a vertex-minor of G , then*

$$\beta_k^\rho(G) \leq \beta_k^\rho(H) + |V(G)| - |V(H)|.$$

Proof. Let G' be a graph locally equivalent to G such that H is an induced subgraph of G' . Note that applying local complementation does not change the rank k -brittleness of a graph by Lemma 6.1. Therefore, we have $\beta_k^\rho(G') = \beta_k^\rho(G)$. It is easy to observe that removing a vertex may decrease the rank k -brittleness by at most 1 by a proof analogous to the proof of Lemma 5.1. Therefore, $\beta_k^\rho(H) \geq \beta_k^\rho(G') - (|V(G')| - |V(H)|) = \beta_k^\rho(G) - (|V(G)| - |V(H)|)$, as required. \square

Lemma 4.1 states that for a connected graph H on $k + 1$ vertices, the disjoint union of sufficiently many copies of H has large rank k -brittleness. It means that if $\{H, 2H, 3H, 4H, \dots\} \subseteq \mathcal{C}$ for some connected graph H on $k + 1$ vertices, then \mathcal{C} is not rank k -scattered. Now we focus on the other direction of Theorem 1.4. We need the following Ramsey-type theorem for bipartite graphs without twins.

Theorem 6.3 (Ding, Oporowski, Oxley, Vertigan [11]). *For every positive integer n , there exists an integer $f(n)$ such that for every bipartite graph G with a bipartition (S, T) , if no two vertices in S have the same set of neighbors and $|S| \geq f(n)$, then S and T have n -element subsets S' and T' , respectively, such that $G[S', T']$ is isomorphic to $\overline{K_n} \sqcup \overline{K_n}$, $\overline{K_n} \boxtimes \overline{K_n}$, or $\overline{K_n} \boxtimes \overline{K_n}$.*

In several places of the proof, when we obtain $H_1 \sqcup H_2$ or $H_1 \boxtimes H_2$ where $H_1, H_2 \in \{\overline{K_n}, K_n\}$, we want to make each part an independent set. The following lemma describes how to reduce each of them to $\overline{K_{n'}} \sqcup \overline{K_{n'}}$ for some n' .

Lemma 6.4. *Let n be an integer.*

- (1) If $n \geq 2$, then $K_n \sqcup \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$.
- (2) If $n \geq 3$, then $K_n \sqcup K_n$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.
- (3) If $n \geq 3$, then $\overline{K_n} \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.
- (4) If $n \geq 3$, then $K_n \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.
- (5) If $n \geq 2$, then $K_n \boxtimes K_n$ has a vertex-minor isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$.

Proof. (1) Let $V(K_n) = \{v_i : 1 \leq i \leq n\}$ and $V(\overline{K_n}) = \{w_i : 1 \leq i \leq n\}$. The graph $(K_n \sqcup \overline{K_n} - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$.

(2) Let $\{v_i : 1 \leq i \leq n\}$ and $\{w_i : 1 \leq i \leq n\}$ be the vertex sets of two copies of $\overline{K_n}$. The graph $((K_n \sqcup K_n - \{v_1, w_2\}) * v_2 * w_1) - \{v_2, w_1\}$ is isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.

(3) Let $\{v_i : 1 \leq i \leq n\}$ and $\{w_i : 1 \leq i \leq n\}$ be the vertex sets of two copies of $\overline{K_n}$. The graph $((\overline{K_n} \boxtimes \overline{K_n} - \{v_1, w_2\}) \wedge v_2 w_1) - \{v_2, w_1\}$ is isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.

(4) Let $V(K_n) = \{v_i : 1 \leq i \leq n\}$ and $V(\overline{K_n}) = \{w_i : 1 \leq i \leq n\}$. The graph $(K_n \boxtimes \overline{K_n} - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$. Thus, by (1), it contains a vertex-minor isomorphic to $\overline{K_{n-2}} \sqcup \overline{K_{n-2}}$.

(5) Let $\{v_i : 1 \leq i \leq n\}$ and $\{w_i : 1 \leq i \leq n\}$ be the vertex sets of two copies of K_n . The graph $(K_n \boxtimes K_n - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$. \square

From $H_1 \sqcup H_2$ with $H_1, H_2 \in \{\overline{K_n}, K_n\}$, we can obtain a long induced path as a vertex-minor. So, if n is sufficiently large, then this directly gives us mP_{k+1} for some large m .

Lemma 6.5 (Kwon and Oum [21]). *Let n be a positive integer.*

- (1) $\overline{K_n} \sqcup \overline{K_n}$ is locally equivalent to P_{2n} .
- (2) $K_n \sqcup \overline{K_n}$ is locally equivalent to P_{2n} .
- (3) If $n \geq 2$, then $K_n \sqcup K_n$ has a vertex-minor isomorphic to P_{2n-2} .

Proof. (1) and (2) are proved in [21]. To prove (3), let $\{v_i : 1 \leq i \leq n\}$ and $\{w_i : 1 \leq i \leq n\}$ be the vertex sets of two copies of K_n , where v_i is adjacent to w_j if and only if $i \geq j$. Then $(K_n \sqcup K_n - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \sqcup \overline{K_{n-1}}$. Thus, the result follows from (2). \square

We will prove the backward direction of Theorem 1.4 by induction on k . In the procedure, we find a vertex-minor containing a vertex set S which induces a subgraph isomorphic to mH for some connected graph H on k vertices. Generally, we meet two situations: the cut-rank of S is large or small. In the next lemma, we prove that if the cut-rank of S is large, then we can directly find a vertex-minor isomorphic to the disjoint union of many copies of some connected graph on $k + 1$ vertices. If the cut-rank is small, then we will recursively find another such set after excluding S .

Lemma 6.6. *For positive integers k and n , there exists a positive integer $m = f_1(k, n)$ such that if a graph G admits a set $W = \{w_1, \dots, w_m\}$ that is a clique or an independent set satisfying the following two properties, then G has a vertex-minor isomorphic to nH' for some connected graph H' on $k + 1$ vertices.*

- (i) $G - W = mH$ for some connected graph H on k vertices.
- (ii) For some vertex v of H and its copies v_1, v_2, \dots, v_m in mH , v_i is adjacent to w_j if and only if $i = j$. (In other words, the subgraph induced by $W \cup \{v_1, v_2, \dots, v_m\}$ is isomorphic to $K_m \boxtimes \overline{K_m}$ or $\overline{K_m} \boxtimes \overline{K_m}$.)

Proof. Let H_i be the i -th copy of H in $G - W$. We fix an isomorphism from H to H_i and isomorphisms between copies of H so that these isomorphisms are compatible.

Assume that $m > 2^{k-1}(m_1 - 1)$. For each w_i , there are at most 2^{k-1} possible sets of neighbors in H_i . So there exists a subset W_1 of W with $|W_1| = m_1$ such that the set of all neighbors of each $w_i \in W_1$ in H_i is identical up to isomorphisms between copies of H .

Assume that $m_1 \geq R(m_2; (2^{k-1})^2)$. For a vertex w_i and $j \neq i$, there are 2^{k-1} possible ways of having edges between the j -th copy of $H - v$ and w_i . By applying Ramsey's theorem, we deduce that there exists a subset $W_2 \subseteq W_1$ of size m_2 such that for all $i < j$ with $w_i, w_j \in W_2$, the set of all neighbors of w_i in H_j is identical up to isomorphisms between copies of H and the set of all neighbors of w_j in H_i is identical up to isomorphisms between copies of H .

Assume that

$$m_2 \geq \max \left(\left\lceil \frac{(k+2)n-1}{2} \right\rceil + 1, n+3 \right).$$

Suppose that there exist $i_1 < i_2 < i_3$ such that $w_{i_1}, w_{i_2}, w_{i_3} \in W_2$ and there exists a vertex u of H so that exactly one of the copies of u in H_{i_1} and H_{i_3}

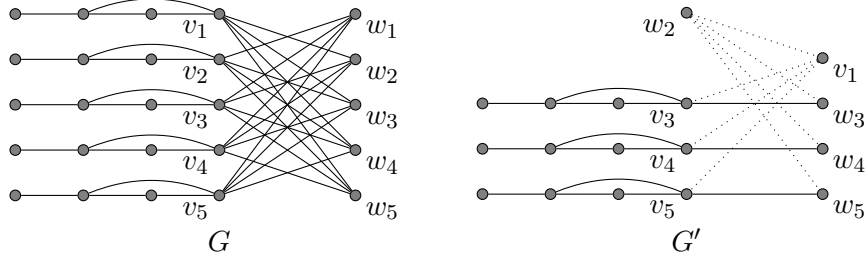


Figure 4: Obtaining $G' = (G \wedge v_1 w_2) - V(H_1) - V(H_2) - w_1 - w_2$ from G in the proof of Lemma 6.7.

is adjacent to w_{i_2} . Then G contains $\overline{K_{m_2-1}} \square \overline{K_{m_2-1}}$ or $\overline{K_{m_2-1}} \square K_{m_2-1}$ as an induced subgraph. By Lemma 6.5, G has a vertex-minor isomorphic to $P_{(k+2)n-1}$ and therefore G has nP_{k+1} as a vertex-minor.

Thus we may assume that there are no such $i_1 < i_2 < i_3$. Since $m_2 \geq 3$, for all $i \neq j$ with $w_i, w_j \in W_2$, the set of all neighbors of w_i in H_j is identical up to isomorphisms between copies of H .

Suppose that $w_i \in W_2$ has no neighbors in H_j when $j \neq i$ and $w_j \in W_2$. If W_2 is an independent set, then clearly G has an induced subgraph isomorphic to $m_2 H'$ for some connected graph H' on $k+1$ vertices. If W_2 is a clique, then for some $w_i \in W_2$, $G * w_i$ contains an induced subgraph isomorphic to $(m_2 - 1)H'$ for some connected graph H' on $k+1$ vertices.

Thus, we may assume that $w_i \in W_2$ has at least one neighbor u_j in H_j for some $j \neq i$ with $w_j \in W_2$. Let $G' = G \wedge w_i u_j - V(H_i) - V(H_j) - w_i - w_j$. If W_2 is an independent set, then G' has an induced subgraph isomorphic to $(m_2 - 2)H'$ for some connected graph H' on $k+1$ vertices. This is because, by (ii), in G , v_ℓ is adjacent to w_ℓ and non-adjacent to w_i and u_j for all ℓ with $w_\ell \in W_2$, $\ell \neq i, j$.

If W_2 is a clique, then let $w_{i_1} \in W_2 \setminus \{w_i, w_j\}$ and $G'' = G' * w_{i_1} - V(H_{i_1})$. Then G'' contains an induced subgraph isomorphic to $(m_2 - 3)H'$ for some connected graph H' on $k+1$ vertices, again by (ii).

So we can take

$$f_1(k, n) := 2^{k-1} \left(R(\max(\lceil \frac{(k+2)n-1}{2} \rceil + 1, n+3); (2^{k-1})^2) - 1 \right) + 1. \quad \square$$

Lemma 6.7. *For positive integers k and n , there exists a positive integer $m = f_2(k, n)$ such that if a graph G admits a set $W = \{w_1, \dots, w_m\}$ that is a clique or an independent set satisfying the following two properties, then G has a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices.*

- (i) $G - W = mH$ for some connected graph H on k vertices.
- (ii) For some vertex v of H and its copies v_1, v_2, \dots, v_m in mH , v_i is adjacent to w_j if and only if $i \neq j$. (In other words, the subgraph induced by $W \cup \{v_1, v_2, \dots, v_m\}$ is isomorphic to $\overline{K_m} \boxtimes \overline{K_m}$ or $K_m \boxtimes \overline{K_m}$.)

Proof. Let $f_2(k, n) := f_1(k, n) + 2$ for the function f_1 in Lemma 6.6. Let $G' = (G \wedge v_1 w_2) - V(H_1) - V(H_2) - w_1 - w_2$ where H_1, H_2 are the first and second copies of H . Then $G' - (W \setminus \{w_1, w_2\})$ is isomorphic to $(m-2)H$ and G' satisfies the condition for Lemma 6.6. See Figure 4 for an illustration. \square

Lemma 6.8. *For positive integers k and n , there exists an integer $N := N(k, n)$ with the following property. Let H be a connected graph on k vertices, and G be a graph and $S \subseteq V(G)$ such that $G[S]$ is isomorphic to qH for some integer q and $\rho_G(S) \geq N$. Then G contains a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices.*

Proof. Let f be the function defined in Theorem 6.3. Let f_1, f_2 be the functions defined in Lemmas 6.6 and 6.7. We define that

$$\begin{aligned} n_3(k, n) &:= \max(f_1(k, n), f_2(k, n), \lceil \frac{(k+2)n-1}{2} \rceil), \\ n_2(k, n) &:= \begin{cases} (k-1)n_3(k, n) + 1 & \text{if } k > 1, \\ \max(n+2, \lceil (3n+1)/2 \rceil) & \text{if } k = 1, \end{cases} \\ n_1(k, n) &:= R(n_2(k, n); 2), \\ N(k, n) &:= f(n_1(k, n)). \end{aligned}$$

We shortly denote $n_1(k, n), n_2(k, n), n_3(k, n), N(k, n)$ as n_1, n_2, n_3, N respectively.

Choose $B \subseteq V(G) \setminus S$ such that $|B| = N$ and $\text{rank}(A(G)[S, B]) = N$.

Observe that two distinct vertices in B have distinct sets of neighbors in S . Since $N = f(n_1)$, by Theorem 6.3, there exist $A_1 \subseteq S$ and $B_1 \subseteq B$ with $|A_1| = |B_1| = n_1$ such that $G[A_1, B_1]$ is isomorphic to $\overline{K_{n_1}} \boxminus \overline{K_{n_1}}, \overline{K_{n_1}} \boxtimes \overline{K_{n_1}},$ or $\overline{K_{n_1}} \boxtimes \overline{K_{n_1}}$.

Since $n_1 = R(n_2; 2)$, by Ramsey's theorem, there exists $B_2 \subseteq B_1$ such that $|B_2| = n_2$ and B_2 is a clique or an independent set. Let $A_2 \subseteq A_1$ be the set of vertices matched with vertices in B_2 in the subgraph $G[A_1, B_1]$. Thus, $G[A_2, B_2]$ is isomorphic to $\overline{K_{n_2}} \boxminus \overline{K_{n_2}}, \overline{K_{n_2}} \boxtimes \overline{K_{n_2}},$ or $\overline{K_{n_2}} \boxtimes \overline{K_{n_2}}$.

If $k = 1$, then by Lemma 6.4 or 6.5, $G[A_2 \cup B_2]$ contains a vertex-minor isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, because $n_2 \geq n+2, n_2 \geq (3n+1)/2$, and P_{3n-1} has $\overline{K_n} \boxminus \overline{K_n}$ as an induced subgraph. So, we may assume that $k \geq 2$.

Observe that H has a vertex v such that A_2 has at least $\lceil n_2/k \rceil = n_3$ copies of v . Let A_3 be a set of n_3 copies of v in A_2 , and $B_3 \subseteq B_2$ be the set of vertices matched with vertices in A_3 in the subgraph $G[A_2, B_2]$. Let \mathcal{C} be the set of components of $G[S]$ containing a vertex in A_3 . Clearly, we have

- $|\mathcal{C}| = n_3$,
- $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \square \overline{K_{n_3}}$, $\overline{K_{n_3}} \sqsupset \overline{K_{n_3}}$, or $\overline{K_{n_3}} \boxtimes \overline{K_{n_3}}$,
- A_3 is an independent set,
- B_3 is a clique or an independent set.

If $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \sqsupset \overline{K_{n_3}}$, then $G[A_3 \cup B_3]$ is isomorphic to $\overline{K_{n_3}} \sqsupset \overline{K_{n_3}}$ or $\overline{K_{n_3}} \sqsupset K_{n_3}$, and thus by Lemma 6.5, it is locally equivalent to P_{2n_3} . As $2n_3 \geq (k+2)n - 1$, P_{2n_3} contains an induced subgraph isomorphic to nP_{k+1} . Therefore, we may assume $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \square \overline{K_{n_3}}$ or $\overline{K_{n_3}} \boxtimes \overline{K_{n_3}}$. By Lemmas 6.6 and 6.7, we deduce that G has a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices. \square

From now on, our main focus is to deal with the case that the cut-rank of S is small, where S is the vertex set inducing the disjoint union of many copies of a connected graph H .

Lemma 6.9. *Let k and n be positive integers and let $\ell = k2^{k(N(k,n)-1)} + 1$ for the function N in Lemma 6.8. Let H be a connected graph on k vertices. If G has an induced subgraph isomorphic to ℓH , then at least one of the following holds.*

- (i) G has a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices.
- (ii) There exists $A \subseteq V(G)$ such that $G[A]$ is isomorphic to $(k+1)H$ and for each vertex of H , its copies in $G[A]$ have the same set of neighbors in $V(G) \setminus A$.

Proof. Let $S \subseteq V(G)$ be a vertex set such that $G[S]$ is isomorphic to ℓH .

If $\rho_G(S) \geq N(k, n)$, then by Lemma 6.8, G contains a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices. Therefore, we may assume that $\rho_G(S) < N(k, n)$.

Let $V(H) = \{z_1, z_2, \dots, z_k\}$. For each $i \in \{1, 2, \dots, k\}$, let Z_i be the set of all copies of z_i in S . Since $\rho_G(S) < N(k, n)$,

$$\text{rank } A(G)[Z_i, V(G) \setminus S] \leq N(k, n) - 1$$

for each $i \in \{1, 2, \dots, k\}$ and so $A(G)[Z_i, V(G) \setminus S]$ has at most $2^{N(k,n)-1}$ distinct rows because it is a 0-1 matrix. In other words,

$$|\{N_G(v) \cap (V(G) \setminus S) : v \in Z_i\}| \leq 2^{N(k,n)-1}$$

for each $1 \leq i \leq k$.

Since $\lceil \ell / 2^{N(k,n)-1} \rceil \geq k + 1$, by the pigeon-hole principle, there exists a set \mathcal{C} of at least $k + 1$ components of $G[S]$ such that for each $i \in \{1, 2, \dots, k\}$, vertices in $Z_i \cap (\bigcup_{C \in \mathcal{C}} V(C))$ have the same set of neighbors in $V(G) \setminus S$. It implies (ii). \square

Lemma 6.10. *Let k, n be positive integers. If a graph has more than $(n - 1)2^{\binom{k+1}{2}}$ components having exactly $k + 1$ vertices, then it contains an induced subgraph isomorphic to nH for some connected graph H on $k + 1$ vertices.*

Proof. The number of non-isomorphic graphs on $k + 1$ vertices is at most $2^{\binom{k+1}{2}}$. By the pigeon-hole principle, at least n components are pairwise isomorphic. \square

We will use the following lemma under the condition that $t = k$ but we prove a stronger statement for the convenience of the proof.

Lemma 6.11. *Let k, t be integers such that $1 \leq t \leq k$. Let H be a connected graph on k vertices. Let G be a graph such that every component has more than k vertices and G contains $(t + 1)H$ as an induced subgraph. If*

- *for each vertex of H , their copies in $(t + 1)H$ have the same set of neighbors in $V(G) \setminus V((t + 1)H)$ and*
- *each component of $(t + 1)H$ has at most t vertices having a neighbor in $V(G) \setminus V((t + 1)H)$,*

then there exist a graph G' locally equivalent to G , disjoint subsets S, T of $V(G')$ and a vertex v in S such that

- (i) $G'[S]$ *is a connected graph on $k + 1$ vertices,*
- (ii) $|T| \leq t(k + 1)$, *and*
- (iii) $G'[S \setminus \{v\}]$ *is a component of $G' - (T \cup \{v\})$.*

Proof. Let $A \subseteq V(G)$ such that $G[A]$ is isomorphic to $(t+1)H$. Let $\mathcal{C} := \{C_1, C_2, \dots, C_{t+1}\}$ be the set of components of $G[A]$, and let $V(H) = \{z_1, z_2, \dots, z_k\}$. For each $i \in \{1, 2, \dots, k\}$, let Z_i be the set of all copies of z_i in A . Let U_i be the set of neighbors of vertices of Z_i on $V(G) \setminus A$ in G , that is, $U_i = N_G(r) \cap (V(G) \setminus A)$ for $r \in Z_i$. Let $X \subseteq \{1, 2, \dots, k\}$ be the set of integers i such that U_i is non-empty. By the assumption $|X| \leq t$. Since each component of G has more than k vertices, we have $|X| > 0$. Without loss of generality, we may assume $X = \{1, \dots, |X|\}$.

We proceed by induction on t .

If $t = 1$, then let $x \in Z_1 \cap V(C_1)$ and $y \in U_1$. We obtain a new graph from G by removing vertices of $V(C_1) \setminus \{x\}$ and pivoting xy . Note that the set of neighbors of x in $G - (V(C_1) \setminus \{x\})$ is exactly U_1 . Thus, after pivoting xy , all edges between the vertex z in $Z_1 \cap V(C_2)$ and $U_1 \setminus \{y\}$ are removed and z has exactly one neighbor x on $V(G) \setminus V(C_2)$. Therefore, $(G', S, T, v) = (G \wedge xy, V(C_2) \cup \{x\}, (V(C_1) \setminus \{x\}) \cup \{y\}, x)$ is a required tuple.

Now we assume that $t \geq 2$. We may assume that $|X| = t$ by the induction hypothesis.

Let $x \in Z_1 \cap V(C_1)$ and $y \in U_1$. We obtain G_1 from G by removing vertices of $V(C_1) \setminus \{x\}$ and pivoting xy . Let $A_1 = A \setminus V(C_1)$. Note that in G , the set of neighbors of x in $V(G) \setminus V(C_1)$ is exactly U_1 . Thus,

- the adjacency relations between two vertices in A_1 do not change by pivoting xy ,
- all edges between $Z_1 \setminus \{x\}$ and $U_1 \setminus \{y\}$ are removed by pivoting xy .

Furthermore, as vertices in each Z_i have the same set of neighbors on $V(G) \setminus A$ in G , G_1 has the following properties.

- For all $i' \in \{2, \dots, t\}$, two vertices in $Z_{i'} \cap A_1$ have the same set of neighbors in $V(G_1) \setminus A_1$.
- If $t < k$, then for $i' \in \{t+1, \dots, k\}$, vertices in $Z_{i'} \cap A_1$ have no neighbors in $V(G_1) \setminus A_1$.

If vertices in $Z_j \cap A_1$ have no neighbors on $V(G_1) \setminus (A_1 \cup \{x, y\})$ for all $2 \leq j \leq k$ in G_1 , then $(G', S, T, v) = (G \wedge xy, V(C_2) \cup \{x\}, (V(C_1) \setminus \{x\}) \cup \{y\}, x)$ is a required tuple. Thus, we may assume that there is $j \in \{2, \dots, k\}$ such that vertices in $Z_j \cap A_1$ have a neighbor on $V(G_1) \setminus (A_1 \cup \{x, y\})$ in G_1 .

Note that $G_1 - \{x, y\}$ contains an induced subgraph isomorphic to tH on the vertex set A_1 such that

- for each vertex of H , their copies in tH have the same set of neighbors in $V(G_1 - \{x, y\}) \setminus A_1$,
- each component of tH has at least one and less than t vertices having a neighbor in $V(G_1 - \{x, y\}) \setminus A_1$.

By the induction hypothesis, $G_1 - \{x, y\}$ has the tuple (G', S, T, v) . Let G'' be the graph locally equivalent to G such that $G'' - V(C_1) - y = G'$. Then $(G'', S, T \cup V(C_1) \cup \{y\}, v)$ is a required tuple for G . \square

We prove the main proposition.

Proposition 6.12. *For positive integers k and n , there exists an integer $\ell = \ell(k, n)$ such that every graph with rank k -brittleness more than ℓ contains a vertex-minor isomorphic to nH for some connected graph H on $k + 1$ vertices.*

Proof. Let f, N be the functions defined in Theorem 6.3 and Lemma 6.8, respectively. We define

- $\ell_2(1, n) := \max(n + 2, \lceil (3n + 1)/2 \rceil)$,
- $\ell_1(1, n) := R(\ell_2(1, n); 4)$,
- $\ell(1, n) := f(\ell_1(1, n)) - 1$,

and for $k \geq 2$, let

- $\ell_3(k, n) := k2^{k(N(k, n)-1)} + 1$,
- $\ell_2(k, n) := \max\left((k + 2)n, 2^{\binom{k+1}{2}}(n - 1) + 2\right)$,
- $\ell_1(k, n) := R(\ell_2(k, n); 2^{k+1})$,
- $\ell(k, n) := \ell(k - 1, \ell_3(k, n)) + (k + 1)^2(\ell_1(k, n) - 1)$.

We will prove the statement by induction on k . We shortly denote $\ell_1(k, n)$, $\ell_2(k, n)$, $\ell_3(k, n)$, $\ell(k, n)$ as ℓ_1 , ℓ_2 , ℓ_3 , ℓ , respectively.

Let us first consider the case that $k = 1$. Suppose G has rank 1-brittleness more than ℓ . Then, there exists a vertex set A such that $\rho_G(A) > \ell$. Choose $A_1 \subseteq A$ and $B_1 \subseteq V(G) \setminus A$ such that $|A_1| = |B_1| = \ell + 1$ and $\text{rank}(A(G)[A_1, B_1]) = \ell + 1$. Note that two vertices in B_1 have distinct neighbors on A_1 . Since $\ell + 1 = f(\ell_1)$, by Theorem 6.3, there exist $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2| = \ell_1$ such that $G[A_2, B_2]$ is isomorphic to $\overline{K_{\ell_1}} \boxplus \overline{K_{\ell_1}}$, $\overline{K_{\ell_1}} \boxtimes \overline{K_{\ell_1}}$, or $\overline{K_{\ell_1}} \boxtimes \overline{K_{\ell_1}}$.

As $\ell_1 = R(\ell_2; 4)$, by Ramsey's theorem, there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ such that

- $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \sqcup \overline{K_{\ell_2}}$, $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$, or $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$, and
- each of A_3 and B_3 is a clique or an independent set.

If $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$, then by Lemma 6.5, $G[A_3 \cup B_3]$ contains a vertex-minor isomorphic to $P_{2\ell_2-2}$. As $2\ell_2 - 2 \geq 2(\frac{3n+1}{2}) - 2 \geq 3n - 1$, $P_{2\ell_2-2}$ contains an induced subgraph isomorphic to nK_2 . Therefore we may assume that $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \sqcup \overline{K_{\ell_2}}$ or $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$. Because $\ell_2 \geq n + 2$, by Lemma 6.4, G contains a vertex-minor isomorphic to $\overline{K_n} \sqcup \overline{K_n}$, which is isomorphic to nK_2 , as required.

Now, we prove for $k \geq 2$. Suppose G has rank k -brittleness more than ℓ . Among all graphs G' locally equivalent to G , choose G' admitting a sequence of $m + 1$ tuples

$$(S_0, T_0), (S_1, T_1, v_1), (S_2, T_2, v_2), \dots, (S_m, T_m, v_m)$$

with the maximum m such that

- $S_0 = T_0 = \emptyset$,
- $S_1, S_2, \dots, S_m, T_1, T_2, \dots, T_m$ are pairwise disjoint subsets of $V(G')$,
- for each $i \in \{1, 2, \dots, m\}$,
 - $|S_i| = k + 1$ and $G'[S_i]$ is connected,
 - $|T_i| \leq k(k + 1)$,
 - $v_i \in S_i$,
 - no vertex in $S_i \setminus \{v_i\}$ has a neighbor in $V(G') \setminus (\bigcup_{0 \leq j \leq i} (S_j \cup T_j))$.

Such a graph G' exists trivially because (S_0, T_0) is a valid sequence for G and so $m \geq 0$.

Suppose that $m < \ell_1$. Let $G_1 := G' - (\bigcup_{0 \leq j \leq m} (S_j \cup T_j))$. Since G' is locally equivalent to G , $\beta_k^\rho(G') = \beta_k^\rho(G)$, and therefore,

$$\beta_k^\rho(G') = \beta_k^\rho(G) > \ell(k - 1, \ell_3) + (k + 1)^2(\ell_1 - 1).$$

As $|\bigcup_{0 \leq j \leq m} (S_j \cup T_j)| \leq (k + 1)^2 m \leq (k + 1)^2(\ell_1 - 1)$, by Proposition 6.2, we have that $\beta_k^\rho(G_1) > \ell(k - 1, \ell_3)$. Let G_2 be the graph obtained from G_1 by removing all components having at most k vertices. It is not difficult to observe that $\beta_k^\rho(G_2) = \beta_k^\rho(G_1)$.

As $\beta_{k-1}^\rho(G_2) \geq \beta_k^\rho(G_2)$, by the induction hypothesis, G_2 contains a vertex-minor isomorphic to $\ell_3 F$ for some connected graph F on k vertices. Thus, there exist a graph G_3 locally equivalent to G_2 and a vertex subset A of G_3 such that $G_3[A]$ is isomorphic to $\ell_3 F$.

Note that $\ell_3 = k2^{k(N(k,n)-1)} + 1$. So, by Lemma 6.9, either

- (1) G_3 contains a vertex-minor isomorphic to nH for some connected graph H on $k + 1$ vertices, or
- (2) there exists $A' \subseteq V(G_3)$ such that $G_3[A']$ is isomorphic to $(k + 1)F$ and for each vertex of F , its copies in $G_3[A']$ have the same set of neighbors in $V(G_3) \setminus A'$.

We may assume that (2) holds. Since G_3 is locally equivalent to G_2 , every component of G_3 has more than k vertices. By Lemma 6.11 (with $t := k$), there exist a graph G_4 locally equivalent to G_3 , disjoint subsets S, T of $V(G_4)$, and a vertex v in S such that

- (i) $G_4[S]$ is a connected graph on $k + 1$ vertices,
- (ii) $|T| \leq k(k + 1)$, and
- (iii) $G_4[S \setminus \{v\}]$ is a component of $G_4 - (T \cup \{v\})$.

In G' , no vertex in $S_i \setminus \{v_i\}$ has a neighbor in $V(G') \setminus (\bigcup_{0 \leq j \leq m} (S_j \cup T_j))$. Let G'' be the graph obtained from G' by applying the same sequence of local complementations needed to obtain G_4 from G_2 . Since G_2 has no vertex in $\bigcup_{0 \leq j \leq m} (S_j \cup T_j)$ and at most one vertex of $G'[S_i]$ has a neighbor in $V(G') \setminus \bigcup_{0 \leq j \leq m} (S_j \cup T_j)$, we deduce that $G''[S_i] = G'[S_i]$ for all $i \in \{1, 2, \dots, m\}$. Therefore, G'' admits the sequence $(S_0, T_0), (S_1, T_1, v_1), \dots, (S_m, T_m, v_m), (S, T, v)$, contradicting the assumption on the choice of G' with the maximum m . Thus $m \geq \ell_1$.

In G' , for $i, j \in \{1, 2, \dots, \ell_1\}$ with $i < j$, v_i may have neighbors on S_j , but v_j has no neighbors on $S_i \setminus \{v_i\}$. Let $s_{i,1}, s_{i,2}, \dots, s_{i,k}$ be the vertices in $S_i \setminus \{v_i\}$ for each i .

We construct a complete graph on the vertex set $\{w_1, w_2, \dots, w_{\ell_1}\}$, and for $i, j \in \{1, 2, \dots, \ell_1\}$ with $i < j$, we color the edge $w_i w_j$ by one of 2^{k+1} colors, depending on the adjacency relation between v_i and S_j . As $\ell_1 = R(\ell_2; 2^{k+1})$, there exists a subset $I \subseteq \{1, 2, \dots, \ell_1\}$ such that $|I| = \ell_2$ and edges between two vertices in $\{w_i : i \in I\}$ are monochromatic. This also implies that $\{v_i : i \in I\}$ is a clique or an independent set. Let $i_1 < i_2 < \dots < i_{\ell_2}$ be the elements of I .

For some $i, j \in I$ with $i < j$, if v_i is adjacent to $s_{j,j'}$ for some j' , then for all $i, j \in I$ with $i \neq j$, v_i is adjacent to $s_{j,j'}$ if and only if $i < j$. By taking vertices $v_{i_1}, v_{i_3}, \dots, v_{i_{\lfloor \ell_2/2 \rfloor - 1}}$ and $s_{i_2, j'}, s_{i_4, j'}, \dots, s_{i_{\lfloor \ell_2/2 \rfloor}, j'}$, we obtain an induced subgraph of G' isomorphic to either $\overline{K_{\lfloor \ell_2/2 \rfloor}} \square \overline{K_{\lfloor \ell_2/2 \rfloor}}$ or $\overline{K_{\lfloor \ell_2/2 \rfloor}} \square K_{\lfloor \ell_2/2 \rfloor}$. By Lemma 6.5, G' contains a vertex-minor isomorphic to P_{ℓ_2-1} . As $\ell_2 - 1 \geq (k + 2)n - 1$, P_{ℓ_2-1} contains an induced subgraph

isomorphic to nP_{k+1} . Thus, G contains a vertex-minor isomorphic to nP_{k+1} . Therefore we may assume that for $i, j \in I$ with $i < j$, v_i has no neighbors in $S_j \setminus \{v_j\}$.

If $\{v_i : i \in I\}$ is independent in G' , then $G'[\bigcup_{i \in I} S_i]$ is the disjoint union of ℓ_2 connected graphs, each having exactly $k + 1$ vertices. Since $\ell_2 > 2^{\binom{k+1}{2}}(n - 1)$, by Lemma 6.10, G contains a vertex-minor isomorphic to nH for some connected graph H on $k + 1$ vertices.

If $\{v_i : i \in I\}$ is a clique in G' , then let $i' \in I$ and let $G'' = G' * v_{i'}$. Then $G''[\bigcup_{i \in I, i \neq i'} S_i]$ is the disjoint union of $\ell_2 - 1$ connected graphs, each having exactly $k + 1$ vertices. Since $\ell_2 - 1 > 2^{\binom{k+1}{2}}(n - 1)$, by Lemma 6.10, G contains a vertex-minor isomorphic to nH for some connected graph H on $k + 1$ vertices. \square

Here is the proof of Theorem 1.4. Let \mathcal{C} be a vertex-minor ideal. Suppose \mathcal{C} is rank k -scattered, that is, there exists an integer ℓ such that every graph $G \in \mathcal{C}$ has rank k -brittleness at most ℓ . Then by (3) of Lemma 4.1, for every connected graph H on $k + 1$ vertices, \mathcal{C} does not contain $(2\ell + 1)H$.

For the converse, suppose that for every connected graph H on $k + 1$ vertices, there exists n_H such that $n_H H \notin \mathcal{C}$. Since there are only finitely many non-isomorphic graphs on $k + 1$ vertices, there exists the maximum n among all n_H . Then $nH \notin \mathcal{C}$ for all connected graphs H on $k + 1$ vertices. By Proposition 6.12, all graphs in \mathcal{C} have rank k -brittleness at most $\ell(k, n)$.

7 Comparisons

In this section, we compare our concepts with existing concepts on graphs. See Figure 5 for the relations that we are going to prove.

7.1 Vertex cover number and matching 1-scatteredness

A set S of vertices in G is a *vertex cover* of G if $G - S$ has no edges. Let $\tau(G)$ denote the minimum size of a vertex cover of a graph G , which we call the *vertex cover number* of G .

Proposition 7.1. *A class of graphs has bounded vertex cover number if and only if it is matching 1-scattered.*

Proof. We claim that

$$\beta_1^\nu(G) \leq \tau(G) \leq 4\beta_1^\nu(G).$$

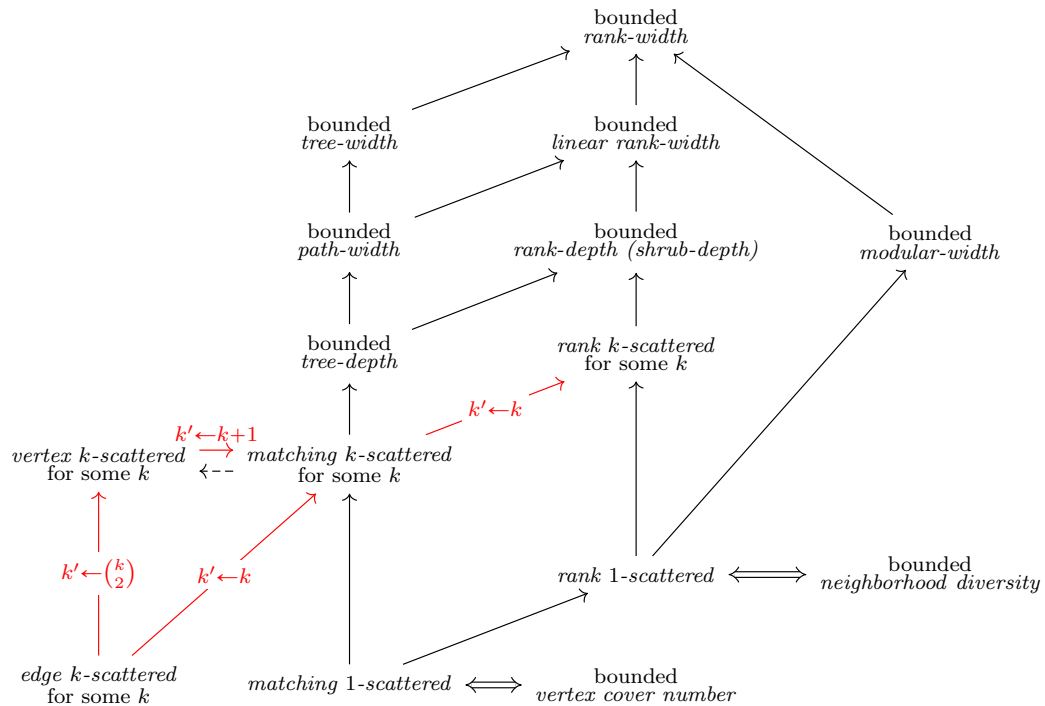


Figure 5: Comparing graph classes. An arrow from A to B means that a class with the property A satisfies the property B . A red solid arrow from A to B with the condition $k' \leftarrow f(k)$ implies that if a class has the property A with k , then it has the property B with $k := k'$. A dashed arrow from A to B means that if a class has the property A with k , then it has the property B with $k := k'$ but we do not have a function for k' depending only on k .

It is easy to see that $\beta_1^\nu(G) \leq \tau(G)$ because G has no matching of size larger than $\tau(G)$.

Let us assume that $\beta_1^\nu(G) \leq m$. Let M be a maximum matching of G . If $|M| > 2m$, then by the probabilistic argument, there is a subset I of $V(G)$ such that at least half of the edges in M joins a vertex in I to a vertex in $V(G) \setminus I$, contradicting the assumption that $\beta_1^\nu(G) \leq m$. So $|M| \leq 2m$. Then the set of all ends of edges in M is a vertex cover of size at most $4m$. \square

Proposition 7.2. *There is a matching 1-scattered class of graphs that is not edge k -scattered for any integer k .*

Proof. The graph $K_{1,n}$ has matching 1-brittleness 1, while it has edge k -brittleness at least $n - k + 1$ by Lemma 4.2. \square

7.2 Neighborhood diversity and rank 1-scatteredness

The neighborhood diversity was introduced by Lampis [22]. Two vertices v and w in a graph G are *twins* if v and w have the same set of neighbors in $V(G) \setminus \{v, w\}$. The *neighborhood diversity* of a graph G is the minimum t such that there is a partition of the vertex set of G into at most t sets, each of which is a set of pairwise twins.

Proposition 7.3. *A class of graphs has bounded neighborhood diversity if and only if it is rank 1-scattered.*

Proof. For the forward direction, we claim that the rank 1-brittleness of a graph is less than or equal to its neighborhood diversity. Let G be a graph of neighborhood diversity at most t . For a set A of vertices, let M_A be the $A \times (V(G) \setminus A)$ submatrix of the adjacency matrix of G over the binary field so that $\text{rank } M_A = \rho_G(A)$. Then M_A has at most t distinct rows and so $\rho_G(A) \leq t$ for all $A \subseteq V(G)$. It implies that $\beta_1^\rho(G) \leq t$.

The backward direction is implied by the lemma of Nguyen and Oum [25, Lemma 5.3], showing that if $\rho_G(X) \leq n$ for all $X \subseteq V(G)$, then the neighborhood diversity is at most 2^{2n+2} . \square

Lampis [22, Lemma 2] shows that if G has a vertex cover of size t , then its neighborhood diversity is at most $2^t + t$.

Proposition 7.4. *There is a class of graphs of neighborhood diversity 1 that has unbounded tree-width.*

Proof. The complete graph K_n has neighborhood diversity 1 and yet its tree-width is $n - 1$. \square

7.3 Modular-width

The modular-width of a graph was defined by Gajarský, Lampis, and Ordyniak [13]. We remark that this modular-width is different from the modular-width defined by Rao [29].

A *module* of a graph G is a set M of vertices such that no vertex in $V(G) \setminus M$ has both a neighbor and a non-neighbor in M . A module M is *trivial* if $|M| \leq 1$ or $M = V(G)$. A graph is *prime* if it has no non-trivial modules.

For a positive integer k , let \mathcal{M}_k be the smallest class of graphs having the following four properties:

1. \mathcal{M}_k contains all graphs of at most 1 vertex.
2. If G and H are in \mathcal{M}_k , then so is the disjoint union of G and H .
3. If G and H are in \mathcal{M}_k , then so is the complete join of G and H , that is the graph obtained from the disjoint union of G and H by adding edges between all pairs of vertices in $V(G)$ and $V(H)$.
4. If G_1, G_2, \dots, G_m are graphs in \mathcal{M}_k for some $m \leq k$ and G is a graph on the vertex set $\{v_1, v_2, \dots, v_m\}$, then \mathcal{M}_k contains the graph obtained from G by substituting v_i with G_i for all $1 \leq i \leq m$.

The *modular-width* of a graph G , denoted by $\text{mw}(G)$, is the minimum positive integer k such that $G \in \mathcal{M}_k$.

We will use the fact that if every prime induced subgraph of a graph G has at most k vertices, then the modular-width of G is at most k .

Proposition 7.5. *Every rank 1-scattered class of graphs has bounded modular-width.*

Proof. We claim that if $m = \beta_1^p(G)$, then every prime induced subgraph of G has less than $R(f(m+2); 2)$ vertices, where f is the function in Theorem 6.3. Suppose for contradiction that a prime induced subgraph H of G has at least $R(f(m+2); 2)$ vertices. Then by Ramsey's theorem, H has a clique or an independent set A of size $f(m+2)$. For two vertices v, w in A , since $\{v, w\}$ is not a module of H , $N_H(v) \setminus A \neq N_H(w) \setminus A$. So, by Theorem 6.3, $H[A, V(H) \setminus A]$ contains an induced subgraph isomorphic to $\overline{K_{m+2}} \square \overline{K_{m+2}}$, $\overline{K_{m+2}} \square K_{m+2}$, or $K_{m+2} \square K_{m+2}$. It implies that the matrix $A_H[A, V(H) \setminus A]$ has rank at least $m+1$, and therefore $\rho_G(A) \geq m+1$, contradicting the assumption that G has rank 1-brittleness m . Thus, every prime induced subgraph of G has less than $R(f(m+2); 2)$ vertices, and so G has modular-width less than $R(f(m+2); 2)$. \square

Proposition 7.6. *There is a rank 2-scattered class of graphs having unbounded modular-width.*

Proof. It is easy to see that $K_n \boxminus K_n$ is prime if $n \geq 3$, and thus it has modular-width $2n$ if $n \geq 3$. But $\beta_2^\rho(K_n \boxminus K_n) \leq 2$ and so $\{K_n \boxminus K_n : n \geq 3\}$ is rank 2-scattered. \square

7.4 Edge k -scatteredness

Proposition 7.7. (1) *Every edge k -scattered class of graphs is vertex $\binom{k}{2}$ -scattered and matching k -scattered.*

(2) *For every integer $k > 1$, there exists an edge k -scattered class of graphs that is neither vertex $(\binom{k}{2} - 1)$ -scattered nor matching $(k - 1)$ -scattered.*

Proof. (1) We claim that

$$\beta_{\binom{k}{2}}^\kappa(G) \leq 4\beta_k^\eta(G) \text{ and } \beta_k^\nu(G) \leq \beta_k^\eta(G).$$

Let $P = (X_1, X_2, \dots, X_t)$ be a partition of $V(G)$ such that $|X_i| \leq k$ for all i and the η_G -width of P is $\beta_k^\eta(G)$. Then, the number of edges meeting two parts of P is at most $2\beta_k^\eta(G)$. Now, we take a partition P' of $E(G)$ such that for each $i \in \{1, 2, \dots, t\}$, all the edges in $G[X_i]$ form one part of P' , and individual edges meeting two parts of X_1, \dots, X_t form individual parts. Then P' has κ_G -width at most $4\beta_k^\eta(G)$ and each part of P' has at most $\binom{k}{2}$ edges. Thus we conclude that $\beta_{\binom{k}{2}}^\kappa(G) \leq 4\beta_k^\eta(G)$.

Note that for every vertex set A of G , $\nu_G(A) \leq \eta_G(A)$. Thus, P has ν_G -width at most $\beta_k^\eta(G)$, which implies that $\beta_k^\nu(G) \leq \beta_k^\eta(G)$.

(2) The graph $(2\ell+1)K_k$ has edge k -brittleness 0, while it has vertex $(\binom{k}{2}-1)$ -brittleness at least $\ell+1$ by Lemma 3.1 and has matching $(k-1)$ -brittleness at least $\ell+1$ by Lemma 4.1. \square

7.5 Vertex k -scatteredness and matching k -scatteredness

Proposition 7.8. (1) *Every vertex k -scattered class of graphs is matching $(k+1)$ -scattered.*

(2) *For every positive integer k , there exists a vertex k -scattered class of graphs that is not matching k -scattered.*

(3) *If a class of graphs is matching k -scattered for some integer k , then there exists an integer k' such that it is vertex k' -scattered.*

Proof. (1) We claim that

$$\beta_{k+1}^\nu(G) \leq 2\beta_k^\kappa(G).$$

Let $P = (X_1, X_2, \dots, X_t)$ be a partition of $E(G)$ such that $|X_i| \leq k$ for all i and the κ_G -width of P is $\beta_k^\kappa(G)$. By the probabilistic argument, there are at most $2\beta_k^\kappa(G)$ vertices meeting at least two parts of P . Let S be the set of vertices incident with edges meeting at least two parts of P . Since no vertex of $G - S$ meets at least two parts of P , each connected component H of $G - S$ has at most k edges and at most $k + 1$ vertices. Now, we take a partition P' of $V(G)$ so that the vertex set of each connected component of $G - S$ forms a part, and vertices in S form individual parts. It is not hard to see that P' has η_G -width at most $|S| \leq 2\beta_k^\kappa(G)$. Thus, $\beta_{k+1}^\nu(G) \leq 2\beta_k^\kappa(G)$.

(2) The graph $(2\ell + 1)P_{k+1}$ has vertex k -brittleness 0 while it has matching k -brittleness at least $\ell + 1$ by Lemma 4.2.

(3) We claim that

$$\text{if } \beta_k^\nu(G) \leq m, \text{ then } \beta_{\binom{4m+k}{2}}^\kappa(G) \leq 4m.$$

Let $P = (X_1, X_2, \dots, X_t)$ be a partition of $V(G)$ such that $|X_i| \leq k$ for all i and the ν_G -width of P is at most m . Let $k' = \binom{4m+k}{2}$. Let H be the subgraph of G consisting of edges meeting two parts of P . Let M be a maximum matching of H . If $|M| > 2m$, then by the probabilistic argument, there is a subset \mathcal{I} of $\{1, 2, \dots, t\}$ such that at least half of the edges in M joins a vertex in X_i for some $i \in \mathcal{I}$ to a vertex in X_j for some $j \notin \mathcal{I}$, contradicting the assumption that ν_G -width of P is at most m .

Thus, $|M| \leq 2m$. Let S be the set of ends of M . Then $|S| \leq 4m$ and S meets every edge of H . Then every component of $G - S$ is a subset of X_i for some i and so has at most k vertices. Now, we take a partition P' of $E(G)$ so that for each component C of $G - S$, the set of edges incident with a vertex in C forms a part, and the edges joining two vertices of S form individual parts. Then each part of P' has at most $\binom{4m+k}{2}$ edges and no vertex outside of S meets more than one part of P' , meaning that κ_G -width of P' is at most $4m$. Thus, $\beta_{\binom{4m+k}{2}}^\kappa(G) \leq 4m$. \square

Proposition 7.9. (1) Every matching k -scattered class of graphs is rank k -scattered.

(2) For every integer $k > 1$, there exists a matching k -scattered class of graphs that is not rank $(k - 1)$ -scattered.

Proof. Observe that if a square 0-1 matrix is non-singular, then the corresponding bipartite graph has a perfect matching. Thus, if a binary matrix M has rank r , then its corresponding bipartite graph has a matching of size r . Thus, for all $S \subseteq V(G)$, $\rho_G(S) \leq \nu_G(S)$. This implies that $\beta_k^\rho(G) \leq \beta_k^\nu(G)$.

It is easy to see (2) from $\{nK_k : n \geq 1\}$ by Lemma 4.1. \square

7.6 Tree-depth

A *rooted forest* is a forest in which every connected component has a specified node called a *root*. The *closure* of a rooted forest T is the graph obtained from T by adding an edge between every vertex and all its ancestors. The *height* of a rooted forest is the number of vertices in a longest path from a root to a leaf. The *tree-depth* of a graph G , denoted by $\text{td}(G)$, is the minimum height of a rooted forest whose closure contains G as a subgraph, see the book [24, Chapter 6].

Let us show that every matching k -scattered class of graphs has bounded tree-depth.

Proposition 7.10. *Every matching k -scattered class of graphs has bounded tree-depth.*

Proof. It is enough to prove that

$$\text{td}(G) \leq 4\beta_k^\nu(G) + k.$$

Let $P = (X_1, X_2, \dots, X_t)$ be a partition of $V(G)$ such that $|X_i| \leq k$ for all i and the ν_G -width of P is $\beta_k^\nu(G)$. Let M be a maximal matching of G such that every edge of M is incident with two sets of $\{X_1, X_2, \dots, X_t\}$. If $|M| \geq 2\beta_k^\nu(G) + 1$, then there exists a subset \mathcal{I} of $\{1, 2, \dots, t\}$ such that at least $\beta_k^\nu(G) + 1$ edges of M are incident with both $\bigcup_{i \in \mathcal{I}} X_i$ and $V(G) \setminus (\bigcup_{i \in \mathcal{I}} X_i)$, which implies that the ν_G -width of P is at least $\beta_k^\nu(G) + 1$, a contradiction. Therefore, $|M| \leq 2\beta_k^\nu(G)$.

Let U be the set of all vertices incident with an edge of M . Then $|U| \leq 4\beta_k^\nu(G)$. By the choice of M , $G - U$ has no edges incident with two parts of P . So, $G - U$ has tree-depth at most k and G has tree-depth at most $4\beta_k^\nu(G) + k$. \square

By Proposition 7.10, every matching k -scattered class of graphs has bounded path-width and bounded tree-width, due to the inequality $\text{tw}(G) \leq \text{pw}(G) \leq \text{td}(G) - 1$ [1], where tw denotes the tree-width and pw denotes the path-width.

Proposition 7.11. *There is a class of graphs of bounded tree-depth that is not rank k -scattered for any integer k .*

Proof. The graph $mK_{1,n}$ has tree-depth 2 and yet its rank k -brittleness is at least $m/2$ when $n \geq k$ by Lemma 4.1. \square

7.7 Shrub-depth and rank-depth

As a dense analogue of tree-depth, Ganian, Hliněný, Nešetřil, Obdržálek, and Ossona de Mendez [15] proposed the notion of shrub-depth. DeVos, Kwon, and Oum [9] introduced the notion of rank-depth of G as the branch-depth of ρ_G , and showed that a class of graphs has bounded rank-depth if and only if it has bounded shrub-depth. So we will omit the definition of shrub-depth and review the definition of branch-depth instead.

A *radius* of a tree is the minimum r such that there is a node having distance at most r from every node. For a function $\lambda : 2^E \rightarrow \mathbb{Z}^{\geq 0}$ on the subsets of a finite set E , a *decomposition* of λ is a pair (T, σ) of a tree T with at least one internal node and a bijection σ from E to the set of leaves of T . The *radius* of a decomposition (T, σ) is defined to be the radius of the tree T . For an internal node $v \in V(T)$, the components of the graph $T - v$ give rise to a partition \mathcal{P}_v of E by σ . The *width* of v is defined to be

$$\max_{\mathcal{P}' \subseteq \mathcal{P}_v} \lambda \left(\bigcup_{X \in \mathcal{P}'} X \right).$$

The *width* of the decomposition (T, σ) is the maximum width of an internal node of T . We say that a decomposition (T, σ) is a (k, r) -*decomposition* of λ if the width is at most k and the radius is at most r . The *branch-depth* of λ is the minimum k such that there exists a (k, k) -decomposition of λ . If $|E| < 2$, then there exists no decomposition and we define λ to have branch-depth $\lambda(\emptyset)$.

We denote by $\text{rd}(G)$ the rank-depth of a graph, that is the branch-depth of ρ_G . We now prove that every rank k -scattered class of graphs has bounded rank-depth.

Proposition 7.12. *Every rank k -scattered class of graphs has bounded rank-depth.*

Proof. We claim that

$$\text{rd}(G) \leq \max(k, \beta_k^\rho(G), 2).$$

Let $P = (X_1, X_2, \dots, X_t)$ be a partition of $V(G)$ such that $|X_i| \leq k$ for all i and the ρ_G -width of P is $\beta_k^\rho(G)$. We can obtain a $(\max(k, \beta_k^\rho(G)), 2)$ -decomposition for ρ_G as follows. Let T be a tree obtained from $K_{1,t}$ with center r and leaves r_1, r_2, \dots, r_t by attaching $|X_i|$ leaves to r_i for each i . We map all vertices of X_i to distinct leaves adjacent to r_i . Then the width of r is $\beta_k^\rho(G)$ and the width of r_i is at most k . \square

7.8 Linear rank-width

Let us present the definition of linear rank-width [14, 18, 28]. For a graph G , an ordering (x_1, \dots, x_n) of the vertex set $V(G)$ is called a *linear layout* of G . If $|V(G)| \geq 2$, then the *width* of a linear layout (x_1, \dots, x_n) of G is defined as $\max_{1 \leq i \leq n-1} \rho_G(\{x_1, \dots, x_i\})$, and if $|V(G)| = 1$, then the width is defined to be 0. The *linear rank-width* of G , denoted by $\text{lrw}(G)$, is defined as the minimum width over all linear layouts of G . For two orderings (x_1, \dots, x_n) , (y_1, \dots, y_m) , we write $(x_1, \dots, x_n) \oplus (y_1, \dots, y_m) := (x_1, \dots, x_n, y_1, \dots, y_m)$ to denote the concatenation of two orderings.

We now aim to obtain an inequality between linear rank-width and rank k -brittleness. Kwon, McCarty, Oum, and Wollan [20] observed that $\text{lrw}(G) \leq \text{rd}(G)^2$, and combining it with Proposition 7.12, we can obtain a quadratic upper bound of linear rank-width in terms of rank k -brittleness. Instead, we will obtain a linear upper bound directly. For that, we use the submodularity of the matrix rank function.

Proposition 7.13 (See [23, Proposition 2.1.9]). *Let M be a matrix over a field \mathbb{F} . Let C be the set of column indexes of M , and R be the set of row indexes of M . Then for all $X_1, X_2 \subseteq R$ and $Y_1, Y_2 \subseteq C$,*

$$\begin{aligned} \text{rank}(M[X_1, Y_1]) + \text{rank}(M[X_2, Y_2]) &\geq \\ \text{rank}(M[X_1 \cap X_2, Y_1 \cup Y_2]) + \text{rank}(M[X_1 \cup X_2, Y_1 \cap Y_2]). \end{aligned}$$

Proposition 7.14. *For every integer $k > 0$, the linear rank-width of a graph G is at most $\beta_k^\rho(G) + \lfloor k/2 \rfloor$.*

Proof. Let $x := \beta_k^\rho(G)$. By the definition of rank k -brittleness, there exists a partition (X_1, X_2, \dots, X_t) of $V(G)$ such that for each $i \in \{1, 2, \dots, t\}$, $|X_i| \leq k$, and for every $I \subseteq \{1, 2, \dots, t\}$, $\rho_G(\bigcup_{i \in I} X_i) \leq x$. For each $i \in \{1, 2, \dots, t\}$, let L_i be any ordering of X_i .

We claim that the ordering $L = L_1 \oplus L_2 \oplus \dots \oplus L_t$ is a linear layout of G having width at most $x + \lfloor k/2 \rfloor$. It suffices to prove that for each $i \in \{1, 2, \dots, t\}$ and a partition (A, B) of X_i , $\rho_G(A \cup \bigcup_{j < i} X_j) \leq x + \lfloor k/2 \rfloor$.

By symmetry, we may assume that $|A| \leq \lfloor k/2 \rfloor$. Let $X = \bigcup_{j < i} X_j$ and $Y = V(G) \setminus X$. Let M be the adjacency matrix of G . By Proposition 7.13,

$$\begin{aligned} \rho_G(A \cup X) &= \text{rank } M[A \cup X, Y \setminus A] + \text{rank } M[\emptyset, Y] \\ &\leq \text{rank } M[X, Y] + \text{rank } M[A, Y \setminus A] \leq x + \lfloor k/2 \rfloor. \end{aligned}$$

This proves the proposition. \square

As the rank-width [26] of a graph is always less than or equal to its linear rank-width, we can deduce that the rank-width of a graph G is at most $\beta_k^\rho(G) + \lfloor k/2 \rfloor$.

Proposition 7.15. *There is a class of graphs of modular-width 1 that has unbounded linear rank-width.*

Proof. Graphs of modular-width 1 are precisely cographs [5] and cographs have unbounded linear rank-width, shown by Gurski and Wanke [17]. \square

8 An application

As an application of Theorem 1.4, we prove that for fixed positive integers m and n , $mK_{1,n}$ -vertex-minor free graphs have bounded linear rank-width. We will use the fact that every sufficiently large connected graph contains either a vertex of large degree or a long induced path.

Proposition 8.1 (See Diestel [10, Proposition 1.3.3]). *For integers $k > 3$ and $\ell > 0$, every connected graph on at least $\frac{k-1}{k-3}(k-2)^{\ell-2}$ vertices contains a vertex of degree at least k or an induced path on ℓ vertices.*

Now we are ready to deduce Theorem 1.6 from Theorem 1.4 and Proposition 7.14.

Theorem 1.6. *For positive integers m and n , the class of graphs having no vertex-minor isomorphic to $mK_{1,n}$ has bounded linear rank-width.*

Proof. We may assume that $n \geq 3$. Trivially $K_{1,n}$ is locally equivalent to K_{n+1} . By Lemma 6.5, P_{2n} is locally equivalent to $\overline{K_n} \square \overline{K_n}$, and a vertex of degree n in $\overline{K_n} \square \overline{K_n}$ gives a vertex-minor isomorphic to $K_{1,n}$. Therefore, by Proposition 8.1, every connected graph on at least $\frac{R(n;2)-1}{R(n;2)-3}(R(n;2)-2)^{2n-2}$ vertices has a vertex-minor isomorphic to $K_{1,n}$.

Let $k := \lceil \frac{R(n;2)-1}{R(n;2)-3}(R(n;2)-2)^{2n-2} \rceil - 1$. Let \mathcal{C} be the class of graphs having no $mK_{1,n}$ as a vertex-minor. Then for every connected graph H on $k+1$ vertices, $mH \notin \mathcal{C}$. Therefore by Theorem 1.4, \mathcal{C} is rank k -scattered. By Proposition 7.14, \mathcal{C} has bounded linear rank-width. \square

Acknowledgement. The authors would like to thank anonymous reviewers for their careful reviews and suggestions.

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