The Spherical Kakeya Problem in Finite Fields

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Abstract

We study subsets of the *n*-dimensional vector space over the finite field \mathbb{F}_q , for odd q, which contain either a sphere for each radius or a sphere for each first coordinate of the center. We call such sets radii spherical Kakeya sets and center spherical Kakeya sets, respectively.

For $n \ge 4$ we prove a general lower bound on the size of any set containing q - 1 different spheres which applies to both kinds of spherical Kakeya sets. We provide constructions which meet the main terms of this lower bound.

We also give a construction showing that we cannot get a lower bound of order of magnitude q^n if we take lower dimensional objects such as circles in \mathbb{F}_q^3 instead of spheres, showing that there are significant differences to the line Kakeya problem.

Finally, we study the case of dimension n = 1 which is different and equivalent to the study of sum and difference sets that cover \mathbb{F}_q .

1 Introduction

A (line-)Kakeya set $\mathcal{K} \subset \mathbb{F}_q^n$ of n-dimensional vectors over the finite field \mathbb{F}_q of q elements is a set containing a line in each direction. It was shown in [3] that every Kakeya set \mathcal{K} satisfies $|\mathcal{K}| \geq c_n q^n$, where the implied constant c_n depends only on the dimension n. Later research focused on the constant c_n , that is, on the one hand improved lower bounds [4] and on the other hand constructions of 'small' Kakeya sets [11, 13, 14].

Several variants of Kakeya sets over finite fields have been studied as well, see for example [5]. In particular the paper [15] deals with *conical Kakeya sets* over finite fields, that is, subsets of \mathbb{F}_q^n containing either a parabola or a hyperbola in every direction (ellipses are not used since they do not have a direction). By 'directions' we usually mean points of the hyper-plane at infinity lying on an object. This paper deals with spheres instead of lines. However, since spheres over finite fields have many directions, roughly q^{n-2} for $n \geq 3$, it is not desirable to use directions to define spherical Kakeya sets in finite fields. In analogy with the reals, we can define spherical Kakeya sets with reference to radii (see [2, 10, 16] for real spherical Kakeya sets) or, say, the first coordinates of the centres of the spheres.

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Spheres over finite fields are well-studied objects, see [7, 8, 12] and are defined as follows. Throughout this paper we assume that q is the power of an odd prime. First we define the norm $||\underline{x}||$ of a vector in $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ by

$$\|\underline{x}\| = x_1^2 + \ldots + x_n^2.$$

In the finite field case this is more suitable than the square-root of the right hand side as used for the reals. The sphere $S_r(\underline{a})$ of radius $r \in \mathbb{F}_q^*$ and center $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$ is

$$\mathcal{S}_r(\underline{a}) = \left\{ \underline{x} \in \mathbb{F}_q^n : \|\underline{x} - \underline{a}\| = r \right\},\$$

that is the set of solutions $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ of the quadratic diagonal equation

$$(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 = r.$$

Again in the finite field case it is more suitable to use r instead of r^2 as in the real case.

Now a radius spherical Kakeya set in \mathbb{F}_q^n , $n \ge 2$, contains a sphere for each radius $r \in \mathbb{F}_q^*$ and a *(first coordinate of the) center spherical Kakeya set* in \mathbb{F}_q^n , $n \ge 2$, contains a sphere for each first coordinate $a_1 \in \mathbb{F}_q$ of the center.

For $n \ge 4$ we prove a general lower bound on sets $\mathcal{K} \subset \mathbb{F}_q^n$ which contain q-1 different spheres which is also a lower bound on the size of spherical Kakeya sets. We also provide a slightly different lower bound for n = 2, 3.

Theorem 1.1. Let q be odd and $\mathcal{K} \subset \mathbb{F}_q^n$ be a set containing at least q-1 distinct spheres for $n \geq 4$, or at least (q-1)/2 distinct spheres for n = 2, 3. Then we have

$$|\mathcal{K}| \ge \begin{cases} \frac{1}{2}q^n + \frac{1}{2}q^{n-1} - q^{n-2} - \frac{1}{2}q^{\lfloor \frac{n-1}{2} \rfloor + 2} + \frac{1}{2}q^{\lfloor \frac{n-1}{2} \rfloor + 1}, & n \ge 4, \\ \frac{q^n - q^{n-2}}{4}, & n = 2, 3. \end{cases}$$

In Section 2 we prove Theorem 1.1 by combining a well-known result on the number of solutions of quadratic diagonal equations with a simple counting argument.

In Section 3 we provide constructions of both radius spherical Kakeya sets and center spherical Kakeya sets which attain the main terms of this bound. In particular, we construct a radius spherical Kakeya set of size

$$\frac{1}{2}q^{n} + \frac{1}{2}q^{n-1} - q^{n-2} + O\left(q^{n-3}\right) \quad \text{for } n \ge 8$$

and a center spherical Kakeya set of size

$$\frac{1}{2}q^{n} + \frac{1}{2}q^{n-1} + O\left(q^{n-2}\right) \quad \text{for } n \ge 5.$$

(We use the notation X = O(Y) if $|X| \le cY$ for some absolute constant c > 0.)

Now we introduce lower dimensional hyper-spheres, the motivation for which will be given in the next paragraph. Let $\mathcal{V}_{\underline{d}} = \{\underline{x} \in \mathbb{F}_q^n : \underline{d} \cdot \underline{x} = 0\}$ be a linear subspace of \mathbb{F}_q^n of dimension n-1 for some direction $\underline{d} \in \mathbb{F}_q^n \setminus \{\underline{0}\}$. (We may assume that the first non-zero coordinate of \underline{d} is 1.) Then the hyper-sphere $\mathcal{H}_r(\underline{a}, \underline{d})$ in the hyper-plane $\underline{a} + \mathcal{V}_{\underline{d}}$ of radius $r \in \mathbb{F}_q^*$, direction $\underline{d} \in \mathbb{F}_q^n$ and center $\underline{a} \in \mathbb{F}_q^n$ is given by

$$\mathcal{H}_r(\underline{a},\underline{d}) = \mathcal{S}_r(\underline{a}) \cap (\underline{a} + \mathcal{V}_{\underline{d}}).$$

In Section 4 we give a negative answer to the question of whether we could use lowerdimensional objects, for example circles in \mathbb{F}_q^3 instead of spheres, to get lower bounds of order of magnitude q^n . This question is motivated by the fact that the line Kakeya problem always deals with objects of dimension 1 (lines). However in our case, even hyper-spheres (which are of dimension n-2) are not enough to give asymptotic growth of order q^n . In particular, we show that in \mathbb{F}_q^n there is a set of size $q^{n-1} + O(q^{n-2})$, $n \geq 3$, which contains a hyper-sphere for each center, direction and radius.

As in the real case [2] our definition for spherical Kakeya sets in \mathbb{F}_q^n can be adjusted for dimension n = 1. A circle $\mathcal{C} = \{x \in \mathbb{F}_q : (x - a)^2 = r^2\}$ in \mathbb{F}_q , for some radius $r \in \mathbb{F}_q^*$ and center $a \in \mathbb{F}_q$, contains exactly two points $a \pm r$. Note that here it is more suitable to use r^2 instead of r(as for real circles). A radius circular Kakeya set in \mathbb{F}_q contains a circle for each radius $r \in \mathbb{F}_q^*$, or equivalently we have

 $\mathcal{K} - \mathcal{K} = \mathbb{F}_a,$

where

$$\mathcal{K} - \mathcal{K} = \{ x_1 - x_2 : x_1, x_2 \in \mathcal{K} \}.$$
(1)

A center circular Kakeya set in \mathbb{F}_q contains a circle for each center $a \in \mathbb{F}_q$, or equivalently we have

$$\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q$$

where

$$\mathcal{K} \oplus \mathcal{K} = \{ x_1 + x_2 : x_1, x_2 \in \mathcal{K}, x_1 \neq x_2 \}.$$
 (2)

In Section 5 we provide constructions of both radius circular and center circular Kakeya sets in \mathbb{F}_q of optimal order of magnitude $O(q^{1/2})$.

2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1 which is based on the following lemma.

Lemma 2.1. The intersection of two different spheres $S_{r_1}(\underline{a})$ and $S_{r_2}(\underline{b})$, $(\underline{a}, r_1) \neq (\underline{b}, r_2)$, in \mathbb{F}_q^n , where q is odd and $n \geq 2$, contains at most

$$q^{n-2} + q^{\lfloor (n-1)/2 \rfloor}$$

points.

Proof. For $n \ge 1$, $a_1, \ldots, a_n \in \mathbb{F}_q^*$ and $r \in \mathbb{F}_q$ we recall that the number N of solutions $(x_1, \ldots, x_n) \in \mathbb{F}_q^n$ to the quadratic diagonal equation

$$a_1x_1^2 + \ldots + a_nx_n^2 = r$$

satisfies

$$N - q^{n-1}| = \begin{cases} q^{\lfloor (n-1)/2 \rfloor}, & r \neq 0, \\ q^{\lfloor n/2 \rfloor} - q^{\lceil (n-2)/2 \rceil}, & r = 0, \end{cases}$$
(3)

see for example [1, Theorem 10.5.1] or [12, Theorems 6.26 and 6.27].

For $n \geq 2$ we count the number of joint solutions $\underline{x} \in \mathbb{F}_q^n$ of the two equations

$$\|\underline{x} - \underline{a}\| = r_1 \tag{4}$$

and

$$\|\underline{x} - \underline{b}\| = r_2. \tag{5}$$

Subtracting (5) from (4) we get

$$2(\underline{b}-\underline{a})\cdot\underline{x} = 2(\underline{b'}-\underline{a'})\cdot\underline{x'} + 2(b_n-a_n)x_n = r_1 - r_2 - ||\underline{a}|| + ||\underline{b}||,$$
(6)

where $\underline{a} = (\underline{a'}, a_n), \underline{b} = (\underline{b'}, b_n)$ and $\underline{x} = (\underline{x'}, x_n)$ with $\underline{a'}, \underline{b'}, \underline{x'} \in \mathbb{F}_q^{n-1}$ and $a_n, b_n, x_n \in \mathbb{F}_q$.

If $\underline{a} = \underline{b}$ and thus $r_1 \neq r_2$, then the two spheres are disjoint. Therefore we may assume $\underline{a} \neq \underline{b}$. WLOG we may assume $a_n \neq b_n$. Then x_n is of the form

$$x_n = \underline{u} \cdot \underline{x'} + c$$

by (6), where

$$\underline{u} = (b_n - a_n)^{-1} (\underline{a'} - \underline{b'})$$

and

$$c = (2(b_n - a_n))^{-1}(r_1 - r_2 - ||\underline{a}|| - ||\underline{b}||).$$

Then we substitute x_n in (4) and get a quadratic form in at most n-1 variables,

$$\|\underline{x} - \underline{a}\| = \|\underline{x'} - \underline{a'}\| + (\underline{u} \cdot \underline{x'} + c - a_1)^2 = r_1.$$

By [12, Theorem 6.21] each quadratic form is equivalent to a diagonal equation, that is, it can be transformed into a diagonal equation by regular linear variable substitution. Hence, it has at most $q^{n-2} + q^{\lfloor (n-1)/2 \rfloor}$ solutions by (3) (applied with n-1 instead of n) and the result follows. \Box

We now prove Theorem 1.1. Let $\mathcal{K} \subset \mathbb{F}_q^n$ contain at least M different spheres $\mathcal{S}_1, \ldots, \mathcal{S}_M$. By Lemma 2.1 each pair of spheres intersects in at most $q^{n-2} + q^{\lfloor (n-1)/2 \rfloor}$ points, and each contains at least $q^{n-1} - q^{\lfloor (n-1)/2 \rfloor}$ points by (3). Hence,

$$\sum_{1 \le i < j \le M} |S_i \cap S_j| \le \left(q^{n-2} + q^{\lfloor (n-1)/2 \rfloor}\right) \frac{M(M-1)}{2}$$

and we get

$$|\mathcal{K}| \ge \left| \bigcup_{i=1}^{M} \mathcal{S}_{i} \right| \ge M \left(q^{n-1} - q^{\lfloor (n-1)/2 \rfloor} \right) - \left(q^{n-2} + q^{\lfloor (n-1)/2 \rfloor} \right) \frac{M(M-1)}{2}.$$

Choosing

$$M = \begin{cases} (q-1)/2, & n = 2 \text{ or } 3, \\ q-1, & n \ge 4, \end{cases}$$

we get

$$|\mathcal{K}| \ge \frac{1}{2}q^n + \frac{1}{2}q^{n-1} - q^{n-2} - \frac{1}{2}q^{\lfloor \frac{n-1}{2} \rfloor + 2} + \frac{1}{2}q^{\lfloor \frac{n-1}{2} \rfloor + 1} \quad \text{for } n \ge 4,$$

and

$$|\mathcal{K}| \ge \frac{q^n - q^{n-2}}{4} \quad \text{for } n = 2,3$$

which completes the proof.

Constructions 3

In this section we give constructions of sets $\mathcal{K} \subset \mathbb{F}_q^n$ containing either a sphere of every radius, or of qdifferent first coordinates of the centres. In particular, for $n \ge 8$, our construction for radii meets the constants in Theorem 1.1 up to and including the third term, and for $n \ge 5$, our construction for centers meets the first two constants.

3.1Spheres with different radii

First we give a construction for different radii. For $r \in \mathbb{F}_q^*$ consider the sphere

$$\mathcal{S}_r = \{ \left(x, \underline{y} \right) \in \mathbb{F}_q^n : (x - r)^2 + \|\underline{y}\| = r \}.$$

The union $\bigcup_{r \in \mathbb{F}_{a}^{*}} S_{r}$ contains a sphere of every radius. We use the inclusion-exclusion principle to bound the size of this set. We firstly bound the intersection of two different spheres S_r and S_s ; the intersection points are

$$S_r \cap S_s = \left\{ \left(\frac{r+s-1}{2}, \underline{y}\right) : \|\underline{y}\| = rs - \left(\frac{r+s-1}{2}\right)^2 \right\}, \quad r \neq s, \quad r, s \in \mathbb{F}_q^*.$$

 $|\mathcal{S}_r \cap \mathcal{S}_s|$ is precisely the number of solutions $(y_1, ..., y_{n-1})$ to the equation

$$y_1^2 + \ldots + y_{n-1}^2 = rs - \left(\frac{r+s-1}{2}\right)^2$$

Therefore, for each valid choice of (r, s), we have $|S_r \cap S_s| = q^{n-2} + O\left(q^{\frac{n-1}{2}}\right)$ by (3). We can now explicitly find the sum of the size of intersections of any two spheres, as

$$\sum_{\substack{r,s \in \mathbb{F}_q^* \\ r \neq s}} |\mathcal{S}_r \cap \mathcal{S}_s| = (q-1)(q-2) \left(q^{n-2} + O\left(q^{\frac{n-1}{2}}\right) \right) = q^n - 3q^{n-1} + 2q^{n-2} + O\left(q^{\frac{n+3}{2}}\right).$$

We can see via the x coordinate $\frac{r+s-1}{2}$ that the intersection of any three distinct spheres S_r , S_s , and S_t is empty. By the inclusion exclusion principle and (3)

$$\left| \bigcup_{r \in \mathbb{F}_q^*} \mathcal{S}_r \right| = \sum_{r \in \mathbb{F}_q^*} |\mathcal{S}_r| - \frac{1}{2} \sum_{\substack{r, s \in \mathbb{F}_q^* \\ r \neq s}} |\mathcal{S}_r \cap \mathcal{S}_s|$$
$$= \frac{1}{2} q^n + \frac{1}{2} q^{n-1} - q^{n-2} + O\left(q^{\frac{n+3}{2}}\right).$$

Spheres with different first coordinates of the centres 3.2

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For a fixed non-square $r\in \mathbb{F}_q^*$ consider the set

$$\mathcal{Q} = \{ (x, \underline{y}) \in \mathbb{F}_q \times \mathbb{F}_q^{n-1} : r - ||\underline{y}|| \text{ is a square in } \mathbb{F}_q \}.$$

The q distinct spheres $S_r(a) = \{(x, \underline{y}) \in \mathbb{F}_q \times \mathbb{F}_q^{n-1} : (x-a)^2 + ||\underline{y}|| = r\}, a \in \mathbb{F}_q$, are all subsets of Q. However, the size of Q is

$$\frac{q^n + q^{n-1}}{2} + O(q^{n-2})$$
 for $n \ge 5$.

Indeed, by (3) each non-zero value of ||y|| is attained

$$q^{n-2} + O\left(q^{\lfloor n/2 \rfloor - 1}\right)$$

times, that is $q^{n-2} + O(q^{n-4})$ for $n \ge 5$. There are (q+1)/2 (non-zero) values ||y|| such that $r - ||\underline{y}||$ is a square (since r is a non-square) and x can take any value in \mathbb{F}_q .

For n = 3, 4 we have

$$|\mathcal{Q}| = \frac{q^n}{2} + O\left(q^{n-1}\right).$$

4 Hyper-spheres

In this section we show for $n \ge 3$ that even if a set contains hyper-spheres for all directions, non-zero centers and radii, it may have only $q^{n-1} + O(q^{n-2})$ points.

We consider the union

$$\mathcal{H} = \bigcup_{\underline{a} \in \mathbb{F}_q^n \setminus \{\underline{0}\}} \mathcal{H}_{-\|a\|}(\underline{a}, \underline{a})$$

of the hyper-spheres

$$\mathcal{H}_{-\|a\|}(\underline{a},\underline{a}) = \{ \underline{x} \in \mathbb{F}_q^n : \|\underline{x} - \underline{a}\| + \|\underline{a}\| = \underline{a} \cdot (\underline{x} - \underline{a}) = 0 \}, \quad \underline{a} \in \mathbb{F}_q^n \setminus \{ \underline{0} \},$$

with center \underline{a} , direction \underline{a} and radius $-||\underline{a}||$ (which covers all radii since each element of \mathbb{F}_q^* is sum of two squares). However, each $\underline{x} \in \mathcal{H}_{-||a||}(\underline{a}, \underline{a})$ satisfies

$$||\underline{x}|| = ||\underline{x} - \underline{a} + \underline{a}|| = ||\underline{x} - \underline{a}|| + ||\underline{a}|| + 2\underline{a} \cdot (\underline{x} - \underline{a}) = 0,$$

which has at most $q^{n-1} + q^{\lfloor n/2 \rfloor} - q^{\lceil (n-2)/2 \rceil}$ solutions by (3) which is an upper bound for $|\mathcal{H}|$.

5 One-dimensional circular Kakeya sets

The definitions of circular Kakeya sets in dimension 1 are in fact equivalent to definitions concerning sum and difference sets. More precisely, $\mathcal{K} \subset \mathbb{F}_q$ is a radius circular Kakeya set in \mathbb{F}_q if and only if

$$\mathcal{K} - \mathcal{K} = \mathbb{F}_q$$

and a centre circular Kakeya set in \mathbb{F}_q if and only if

 $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q,$

where $\mathcal{K} - \mathcal{K}$ and $\mathcal{K} \oplus \mathcal{K}$ are defined by (1) and (2).

To see the first equivalence, let $\mathcal{K} \subset \mathbb{F}_q$ be a set that contains a circle of radius r for each $r \in \mathbb{F}_q^*$. Therefore there exists $a \in \mathbb{F}_q$ such that $\{a + r, a - r\} \subset \mathcal{K}$. We get $a + r - (a - r) = 2r \in \mathcal{K} - \mathcal{K}$. Therefore, since 2r covers all of \mathbb{F}_q^* , we have $\mathcal{K} - \mathcal{K} = \mathbb{F}_q$ ($0 \in \mathcal{K} - \mathcal{K}$ trivially). Conversely, suppose that $\mathcal{K} \subset \mathbb{F}_q$ is a subset such that $\mathcal{K} - \mathcal{K} = \mathbb{F}_q$. Then for each $r \in \mathbb{F}_q$, there exist $x_1, x_2 \in \mathcal{K}$, such that $x_1 - x_2 = 2r$. By taking $a = (x_1 + x_2)/2$ we see $x_1 = a + r$ and $x_2 = a - r$ and that the circle $\{a + r, a - r\}$ is in \mathcal{K} .

For the second equivalence, let $\mathcal{K} \subset \mathbb{F}_q$ be a set containing a circle for any center a. Then for all $a \in \mathbb{F}_q$, there exists $r \in \mathbb{F}_q^*$ such that $\{a - r, a + r\} \subset \mathcal{K}$. Then we have (a - r) + (a + r) = 2a, and therefore $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q$. Conversely, let \mathcal{K} be a subset of \mathbb{F}_q^* such that $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q$. Fix $a \in \mathbb{F}_q$. Since $\mathcal{K} - \mathcal{K} = \mathbb{F}_q$, there exist $x_1, x_2 \in \mathcal{K}, x_1 \neq x_2$, such that $x_1 + x_2 = 2a$. Taking $r = (x_1 - x_2)/2$ we can write $x_1 = a + r$ and $x_2 = a - r$, so that a circle of centre a is in \mathcal{K} .

Since $|\mathcal{K} - \mathcal{K}| \leq |\mathcal{K}|^2$, each radius circular Kakeya set in \mathbb{F}_q has size at least $\lceil q^{1/2} \rceil$, and since $|\mathcal{K} \oplus \mathcal{K}| < |\mathcal{K}|^2/2$ the size of any center circular Kakeya set \mathcal{K} of \mathbb{F}_q is at least $|\mathcal{K}| \geq \lceil \sqrt{2q} \rceil$. (Keep the condition $x_1 \neq x_2$ in (2) in mind.) In this section we will give constructions of radius circular and center circular Kakeya sets \mathcal{K} in \mathbb{F}_q with $|\mathcal{K}|$ of optimal order of magnitude $O(q^{1/2})$.

For a prime p > 2 it is easy to find circular Kakeya sets in \mathbb{F}_p of size $2\lfloor \sqrt{p} \rfloor + 1$,

$$\mathcal{K} = \mathcal{K}_p = \{0, 1, 2, \dots, \lfloor \sqrt{p} \rfloor\} \cup -\mathcal{K}_0, \tag{7}$$

where

$$\mathcal{C}_0 = \{ \lceil \sqrt{p} \rceil, 2 \lceil \sqrt{p} \rceil, ..., \lfloor \sqrt{p} \rfloor \lceil \sqrt{p} \rceil \}$$

It is clear that $\mathcal{K} - \mathcal{K} = \mathbb{F}_p$. Substituting $-\mathcal{K}_0$ by \mathcal{K}_0 in (7) we get $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_p$.

If $q = r^2$ is a square and α is a defining element of \mathbb{F}_q over \mathbb{F}_r , that is, $\mathbb{F}_q = \mathbb{F}_r(\alpha)$, then we can choose

 $\mathcal{K} = \mathbb{F}_r \cup \alpha \mathbb{F}_r$

of size $|\mathcal{K}| = 2q^{1/2} - 1$ to get both $\mathcal{K} - \mathcal{K} = \mathbb{F}_q$ and $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q$.

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If $q = p^{2m+1}$ with a prime p and $\mathbb{F}_q = \mathbb{F}_p(\beta)$ with a defining element β of \mathbb{F}_q over \mathbb{F}_p , then we first choose the construction \mathcal{K}_p from (7) and then take

$$\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2,$$

where

$$\mathcal{K}_1 = \{a_0 + a_1\beta + \ldots + a_m\beta^m : a_0 \in \mathcal{K}_p, a_1, \ldots, a_m \in \mathbb{F}_p\}$$

and

$$\mathcal{K}_2 = \{a_0 + a_1\beta^{m+1} + \ldots + a_m\beta^{2m} : a_0 \in \mathcal{K}_p, a_1, \ldots, a_m \in \mathbb{F}_p\}$$

It is easy to check that $\mathcal{K} - \mathcal{K} = \mathbb{F}_q$ and

$$|\mathcal{K}| = (2p^m - 1)|\mathcal{K}_p| < 4q^{1/2} + 2(q/p)^{1/2}.$$

Again substituting $-\mathcal{K}_0$ by \mathcal{K}_0 in (7) we get $\mathcal{K} \oplus \mathcal{K} = \mathbb{F}_q$.

Combining all the cases we can formulate a general result.

Theorem 5.1. For a fixed power q of an odd prime let $\mathcal{K} \subset \mathbb{F}_q$ be either a radius circular or a center circular Kakeya set in \mathbb{F}_q of minimal size. Then we have

$$q^{1/2} \le |\mathcal{K}| < 6q^{1/2}$$

The constants can be certainly improved using, for example, ideas from [6, 9]. However, we did not calculate these improved constants for the readability of this paper, and since even the improved upper bounds would not be optimal.

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