# The Spherical Kakeya Problem in Finite Fields 

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#### Abstract

We study subsets of the $n$-dimensional vector space over the finite field $\mathbb{F}_{q}$, for odd $q$, which contain either a sphere for each radius or a sphere for each first coordinate of the center. We call such sets radii spherical Kakeya sets and center spherical Kakeya sets, respectively.

For $n \geq 4$ we prove a general lower bound on the size of any set containing $q-1$ different spheres which applies to both kinds of spherical Kakeya sets. We provide constructions which meet the main terms of this lower bound.

We also give a construction showing that we cannot get a lower bound of order of magnitude $q^{n}$ if we take lower dimensional objects such as circles in $\mathbb{F}_{q}^{3}$ instead of spheres, showing that there are significant differences to the line Kakeya problem.

Finally, we study the case of dimension $n=1$ which is different and equivalent to the study of sum and difference sets that cover $\mathbb{F}_{q}$.


## 1 Introduction

A (line-)Kakeya set $\mathcal{K} \subset \mathbb{F}_{q}^{n}$ of $n$-dimensional vectors over the finite field $\mathbb{F}_{q}$ of $q$ elements is a set containing a line in each direction. It was shown in [3] that every Kakeya set $\mathcal{K}$ satisfies $|\mathcal{K}| \geq c_{n} q^{n}$, where the implied constant $c_{n}$ depends only on the dimension $n$. Later research focused on the constant $c_{n}$, that is, on the one hand improved lower bounds [4] and on the other hand constructions of 'small' Kakeya sets [11, 13, 14].

Several variants of Kakeya sets over finite fields have been studied as well, see for example [5]. In particular the paper [15] deals with conical Kakeya sets over finite fields, that is, subsets of $\mathbb{F}_{q}^{n}$ containing either a parabola or a hyperbola in every direction (ellipses are not used since they do not have a direction). By 'directions' we usually mean points of the hyper-plane at infinity lying on an object. This paper deals with spheres instead of lines. However, since spheres over finite fields have many directions, roughly $q^{n-2}$ for $n \geq 3$, it is not desirable to use directions to define spherical Kakeya sets in finite fields. In analogy with the reals, we can define spherical Kakeya sets with reference to radii (see [2, 10, 16] for real spherical Kakeya sets) or, say, the first coordinates of the centres of the spheres.

[^0]Spheres over finite fields are well-studied objects, see [7, 8, 12, and are defined as follows. Throughout this paper we assume that $q$ is the power of an odd prime. First we define the norm $\|\underline{x}\|$ of a vector in $\underline{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{F}_{q}^{n}$ by

$$
\|\underline{x}\|=x_{1}^{2}+\ldots+x_{n}^{2}
$$

In the finite field case this is more suitable than the square-root of the right hand side as used for the reals. The sphere $\mathcal{S}_{r}(\underline{a})$ of radius $r \in \mathbb{F}_{q}^{*}$ and center $\underline{a}=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{F}_{q}^{n}$ is

$$
\mathcal{S}_{r}(\underline{a})=\left\{\underline{x} \in \mathbb{F}_{q}^{n}:\|\underline{x}-\underline{a}\|=r\right\},
$$

that is the set of solutions $\underline{x}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{F}_{q}^{n}$ of the quadratic diagonal equation

$$
\left(x_{1}-a_{1}\right)^{2}+\ldots+\left(x_{n}-a_{n}\right)^{2}=r .
$$

Again in the finite field case it is more suitable to use $r$ instead of $r^{2}$ as in the real case.
Now a radius spherical Kakeya set in $\mathbb{F}_{q}^{n}, n \geq 2$, contains a sphere for each radius $r \in \mathbb{F}_{q}^{*}$ and a (first coordinate of the) center spherical Kakeya set in $\mathbb{F}_{q}^{n}, n \geq 2$, contains a sphere for each first coordinate $a_{1} \in \mathbb{F}_{q}$ of the center.

For $n \geq 4$ we prove a general lower bound on sets $\mathcal{K} \subset \mathbb{F}_{q}^{n}$ which contain $q-1$ different spheres which is also a lower bound on the size of spherical Kakeya sets. We also provide a slightly different lower bound for $n=2,3$.

Theorem 1.1. Let $q$ be odd and $\mathcal{K} \subset \mathbb{F}_{q}^{n}$ be a set containing at least $q-1$ distinct spheres for $n \geq 4$, or at least $(q-1) / 2$ distinct spheres for $n=2,3$. Then we have

$$
|\mathcal{K}| \geq\left\{\begin{array}{l}
\frac{1}{2} q^{n}+\frac{1}{2} q^{n-1}-q^{n-2}-\frac{1}{2} q^{\left\lfloor\frac{n-1}{2}\right\rfloor+2}+\frac{1}{2} q^{\left\lfloor\frac{n-1}{2}\right\rfloor+1}, \quad n \geq 4 \\
\frac{q^{n}-q^{n-2}}{4}, \quad n=2,3
\end{array}\right.
$$

In Section 2 we prove Theorem 1.1 by combining a well-known result on the number of solutions of quadratic diagonal equations with a simple counting argument.

In Section 3 we provide constructions of both radius spherical Kakeya sets and center spherical Kakeya sets which attain the main terms of this bound. In particular, we construct a radius spherical Kakeya set of size

$$
\frac{1}{2} q^{n}+\frac{1}{2} q^{n-1}-q^{n-2}+O\left(q^{n-3}\right) \quad \text { for } n \geq 8
$$

and a center spherical Kakeya set of size

$$
\frac{1}{2} q^{n}+\frac{1}{2} q^{n-1}+O\left(q^{n-2}\right) \quad \text { for } n \geq 5
$$

(We use the notation $X=O(Y)$ if $|X| \leq c Y$ for some absolute constant $c>0$.)
Now we introduce lower dimensional hyper-spheres, the motivation for which will be given in the next paragraph. Let $\mathcal{V}_{\underline{d}}=\left\{\underline{x} \in \mathbb{F}_{q}^{n}: \underline{d} \cdot \underline{x}=0\right\}$ be a linear subspace of $\mathbb{F}_{q}^{n}$ of dimension $n-1$ for some direction $\underline{d} \in \mathbb{F}_{q}^{n} \backslash\{\underline{0}\}$. (We may assume that the first non-zero coordinate of $\underline{d}$ is 1.) Then the hyper-sphere $\mathcal{H}_{r}(\underline{a}, \underline{d})$ in the hyper-plane $\underline{a}+\mathcal{V}_{\underline{d}}$ of radius $r \in \mathbb{F}_{q}^{*}$, direction $\underline{d} \in \mathbb{F}_{q}^{n}$ and center $\underline{a} \in \mathbb{F}_{q}^{n}$ is given by

$$
\mathcal{H}_{r}(\underline{a}, \underline{d})=\mathcal{S}_{r}(\underline{a}) \cap\left(\underline{a}+\mathcal{V}_{\underline{d}}\right) .
$$

In Section 4 we give a negative answer to the question of whether we could use lowerdimensional objects, for example circles in $\mathbb{F}_{q}^{3}$ instead of spheres, to get lower bounds of order of magnitude $q^{n}$. This question is motivated by the fact that the line Kakeya problem always deals with objects of dimension 1 (lines). However in our case, even hyper-spheres (which are of dimension $n-2)$ are not enough to give asymptotic growth of order $q^{n}$. In particular, we show that in $\mathbb{F}_{q}^{n}$ there is a set of size $q^{n-1}+O\left(q^{n-2}\right), n \geq 3$, which contains a hyper-sphere for each center, direction and radius.

As in the real case [2] our definition for spherical Kakeya sets in $\mathbb{F}_{q}^{n}$ can be adjusted for dimension $n=1$. A circle $\mathcal{C}=\left\{x \in \mathbb{F}_{q}:(x-a)^{2}=r^{2}\right\}$ in $\mathbb{F}_{q}$, for some radius $r \in \mathbb{F}_{q}^{*}$ and center $a \in \mathbb{F}_{q}$, contains exactly two points $a \pm r$. Note that here it is more suitable to use $r^{2}$ instead of $r$ (as for real circles). A radius circular Kakeya set in $\mathbb{F}_{q}$ contains a circle for each radius $r \in \mathbb{F}_{q}^{*}$, or equivalently we have

$$
\mathcal{K}-\mathcal{K}=\mathbb{F}_{q},
$$

where

$$
\begin{equation*}
\mathcal{K}-\mathcal{K}=\left\{x_{1}-x_{2}: x_{1}, x_{2} \in \mathcal{K}\right\} . \tag{1}
\end{equation*}
$$

A center circular Kakeya set in $\mathbb{F}_{q}$ contains a circle for each center $a \in \mathbb{F}_{q}$, or equivalently we have

$$
\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q},
$$

where

$$
\begin{equation*}
\mathcal{K} \oplus \mathcal{K}=\left\{x_{1}+x_{2}: x_{1}, x_{2} \in \mathcal{K}, x_{1} \neq x_{2}\right\} . \tag{2}
\end{equation*}
$$

In Section 5 we provide constructions of both radius circular and center circular Kakeya sets in $\mathbb{F}_{q}$ of optimal order of magnitude $O\left(q^{1 / 2}\right)$.

## 2 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1 which is based on the following lemma.
Lemma 2.1. The intersection of two different spheres $S_{r_{1}}(\underline{a})$ and $S_{r_{2}}(\underline{b}),\left(\underline{a}, r_{1}\right) \neq\left(\underline{b}, r_{2}\right)$, in $\mathbb{F}_{q}^{n}$, where $q$ is odd and $n \geq 2$, contains at most

$$
q^{n-2}+q^{\lfloor(n-1) / 2\rfloor}
$$

points.

Proof. For $n \geq 1, a_{1}, \ldots, a_{n} \in \mathbb{F}_{q}^{*}$ and $r \in \mathbb{F}_{q}$ we recall that the number $N$ of solutions $\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{F}_{q}^{n}$ to the quadratic diagonal equation

$$
a_{1} x_{1}^{2}+\ldots+a_{n} x_{n}^{2}=r
$$

satisfies

$$
\left|N-q^{n-1}\right|=\left\{\begin{array}{cl}
q^{\lfloor(n-1) / 2\rfloor}, & r \neq 0  \tag{3}\\
q^{\lfloor n / 2\rfloor}-q^{\lceil(n-2) / 2\rceil}, & r=0
\end{array}\right.
$$

see for example [1, Theorem 10.5.1] or [12, Theorems 6.26 and 6.27].

For $n \geq 2$ we count the number of joint solutions $\underline{x} \in \mathbb{F}_{q}^{n}$ of the two equations

$$
\begin{equation*}
\|\underline{x}-\underline{a}\|=r_{1} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\underline{x}-\underline{b}\|=r_{2} . \tag{5}
\end{equation*}
$$

Subtracting (5) from (4) we get

$$
\begin{equation*}
2(\underline{b}-\underline{a}) \cdot \underline{x}=2\left(\underline{b^{\prime}}-\underline{a}^{\prime}\right) \cdot \underline{x^{\prime}}+2\left(b_{n}-a_{n}\right) x_{n}=r_{1}-r_{2}-\|\underline{a}\|+\|\underline{b}\|, \tag{6}
\end{equation*}
$$

where $\underline{a}=\left(\underline{a^{\prime}}, a_{n}\right), \underline{b}=\left(\underline{b^{\prime}}, b_{n}\right)$ and $\underline{x}=\left(\underline{x^{\prime}}, x_{n}\right)$ with $\underline{a}^{\prime}, \underline{b^{\prime}}, \underline{x^{\prime}} \in \mathbb{F}_{q}^{n-1}$ and $a_{n}, b_{n}, x_{n} \in \mathbb{F}_{q}$.
If $\underline{a}=\underline{b}$ and thus $r_{1} \neq r_{2}$, then the two spheres are disjoint. Therefore we may assume $\underline{a} \neq \underline{b}$. WLOG we may assume $a_{n} \neq b_{n}$. Then $x_{n}$ is of the form

$$
x_{n}=\underline{u} \cdot \underline{x^{\prime}}+c
$$

by (6), where

$$
\underline{u}=\left(b_{n}-a_{n}\right)^{-1}\left(\underline{a^{\prime}}-\underline{b^{\prime}}\right)
$$

and

$$
c=\left(2\left(b_{n}-a_{n}\right)\right)^{-1}\left(r_{1}-r_{2}-\|\underline{a}\|-\|\underline{b}\|\right) .
$$

Then we substitute $x_{n}$ in (4) and get a quadratic form in at most $n-1$ variables,

$$
\|\underline{x}-\underline{a}\|=\left\|\underline{x^{\prime}}-\underline{a}^{\prime}\right\|+\left(\underline{u} \cdot \underline{x^{\prime}}+c-a_{1}\right)^{2}=r_{1} .
$$

By [12, Theorem 6.21] each quadratic form is equivalent to a diagonal equation, that is, it can be transformed into a diagonal equation by regular linear variable substitution. Hence, it has at most $q^{n-2}+q^{\lfloor(n-1) / 2\rfloor}$ solutions by (3) (applied with $n-1$ instead of $n$ ) and the result follows.

We now prove Theorem 1.1 Let $\mathcal{K} \subset \mathbb{F}_{q}^{n}$ contain at least $M$ different spheres $\mathcal{S}_{1}, \ldots, \mathcal{S}_{M}$. By Lemma 2.1 each pair of spheres intersects in at most $q^{n-2}+q^{\lfloor(n-1) / 2\rfloor}$ points, and each contains at least $q^{n-1}-q^{\lfloor(n-1) / 2\rfloor}$ points by (3). Hence,

$$
\sum_{1 \leq i<j \leq M}\left|S_{i} \cap S_{j}\right| \leq\left(q^{n-2}+q^{\lfloor(n-1) / 2\rfloor}\right) \frac{M(M-1)}{2}
$$

and we get

$$
|\mathcal{K}| \geq\left|\bigcup_{i=1}^{M} \mathcal{S}_{i}\right| \geq M\left(q^{n-1}-q^{\lfloor(n-1) / 2\rfloor}\right)-\left(q^{n-2}+q^{\lfloor(n-1) / 2\rfloor}\right) \frac{M(M-1)}{2} .
$$

Choosing

$$
M=\left\{\begin{array}{cc}
(q-1) / 2, & n=2 \text { or } 3 \\
q-1, & n \geq 4
\end{array}\right.
$$

we get

$$
|\mathcal{K}| \geq \frac{1}{2} q^{n}+\frac{1}{2} q^{n-1}-q^{n-2}-\frac{1}{2} q^{\left\lfloor\frac{n-1}{2}\right\rfloor+2}+\frac{1}{2} q^{\left\lfloor\frac{n-1}{2}\right\rfloor+1} \quad \text { for } n \geq 4
$$

and

$$
|\mathcal{K}| \geq \frac{q^{n}-q^{n-2}}{4} \quad \text { for } n=2,3
$$

which completes the proof.

## 3 Constructions

In this section we give constructions of sets $\mathcal{K} \subset \mathbb{F}_{q}^{n}$ containing either a sphere of every radius, or of $q$ different first coordinates of the centres. In particular, for $n \geq 8$, our construction for radii meets the constants in Theorem 1.1 up to and including the third term, and for $n \geq 5$, our construction for centers meets the first two constants.

### 3.1 Spheres with different radii

First we give a construction for different radii. For $r \in \mathbb{F}_{q}^{*}$ consider the sphere

$$
\mathcal{S}_{r}=\left\{(x, \underline{y}) \in \mathbb{F}_{q}^{n}:(x-r)^{2}+\|\underline{y}\|=r\right\}
$$

The union $\bigcup_{r \in \mathbb{F}_{q}^{*}} \mathcal{S}_{r}$ contains a sphere of every radius. We use the inclusion-exclusion principle to bound the size of this set. We firstly bound the intersection of two different spheres $\mathcal{S}_{r}$ and $\mathcal{S}_{s}$; the intersection points are

$$
\mathcal{S}_{r} \cap \mathcal{S}_{s}=\left\{\left(\frac{r+s-1}{2}, \underline{y}\right):\|\underline{y}\|=r s-\left(\frac{r+s-1}{2}\right)^{2}\right\}, \quad r \neq s, \quad r, s \in \mathbb{F}_{q}^{*}
$$

$\left|\mathcal{S}_{r} \cap \mathcal{S}_{s}\right|$ is precisely the number of solutions $\left(y_{1}, \ldots, y_{n-1}\right)$ to the equation

$$
y_{1}^{2}+\ldots+y_{n-1}^{2}=r s-\left(\frac{r+s-1}{2}\right)^{2}
$$

Therefore, for each valid choice of $(r, s)$, we have $\left|\mathcal{S}_{r} \cap \mathcal{S}_{s}\right|=q^{n-2}+O\left(q^{\frac{n-1}{2}}\right)$ by (3). We can now explicitly find the sum of the size of intersections of any two spheres, as

$$
\sum_{\substack{r, s \in \mathbb{F}_{q}^{*} \\ r \neq s}}\left|\mathcal{S}_{r} \cap \mathcal{S}_{s}\right|=(q-1)(q-2)\left(q^{n-2}+O\left(q^{\frac{n-1}{2}}\right)\right)=q^{n}-3 q^{n-1}+2 q^{n-2}+O\left(q^{\frac{n+3}{2}}\right)
$$

We can see via the $x$ coordinate $\frac{r+s-1}{2}$ that the intersection of any three distinct spheres $\mathcal{S}_{r}, \mathcal{S}_{s}$, and $\mathcal{S}_{t}$ is empty. By the inclusion exclusion principle and (3)

$$
\begin{aligned}
\left|\bigcup_{r \in \mathbb{F}_{q}^{*}} \mathcal{S}_{r}\right| & =\sum_{r \in \mathbb{F}_{q}^{*}}\left|\mathcal{S}_{r}\right|-\frac{1}{2} \sum_{\substack{r, s \in \mathbb{F}_{q}^{*} \\
r \neq s}}\left|\mathcal{S}_{r} \cap \mathcal{S}_{s}\right| \\
& =\frac{1}{2} q^{n}+\frac{1}{2} q^{n-1}-q^{n-2}+O\left(q^{\frac{n+3}{2}}\right)
\end{aligned}
$$

### 3.2 Spheres with different first coordinates of the centres

For a fixed non-square $r \in \mathbb{F}_{q}^{*}$ consider the set

$$
\mathcal{Q}=\left\{(x, \underline{y}) \in \mathbb{F}_{q} \times \mathbb{F}_{q}^{n-1}: r-\|\underline{y}\| \text { is a square in } \mathbb{F}_{q}\right\} .
$$

The $q$ distinct spheres $\mathcal{S}_{r}(a)=\left\{(x, \underline{y}) \in \mathbb{F}_{q} \times \mathbb{F}_{q}^{n-1}:(x-a)^{2}+\|\underline{y}\|=r\right\}, a \in \mathbb{F}_{q}$, are all subsets of $\mathcal{Q}$. However, the size of $\mathcal{Q}$ is

$$
\frac{q^{n}+q^{n-1}}{2}+O\left(q^{n-2}\right) \quad \text { for } n \geq 5
$$

Indeed, by (3) each non-zero value of $\|\underline{y}\|$ is attained

$$
q^{n-2}+O\left(q^{\lfloor n / 2\rfloor-1}\right)
$$

times, that is $q^{n-2}+O\left(q^{n-4}\right)$ for $n \geq 5$. There are $(q+1) / 2$ (non-zero) values $\|y\|$ such that $r-\|\underline{y}\|$ is a square (since $r$ is a non-square) and $x$ can take any value in $\mathbb{F}_{q}$.

For $n=3,4$ we have

$$
|\mathcal{Q}|=\frac{q^{n}}{2}+O\left(q^{n-1}\right)
$$

## 4 Hyper-spheres

In this section we show for $n \geq 3$ that even if a set contains hyper-spheres for all directions, non-zero centers and radii, it may have only $q^{n-1}+O\left(q^{n-2}\right)$ points.

We consider the union

$$
\mathcal{H}=\bigcup_{\underline{a} \in \mathbb{F}_{q}^{n} \backslash\{\underline{0}\}} \mathcal{H}_{-\|a\|}(\underline{a}, \underline{a})
$$

of the hyper-spheres

$$
\mathcal{H}_{-\|a\|}(\underline{a}, \underline{a})=\left\{\underline{x} \in \mathbb{F}_{q}^{n}:\|\underline{x}-\underline{a}\|+\|\underline{a}\|=\underline{a} \cdot(\underline{x}-\underline{a})=0\right\}, \quad \underline{a} \in \mathbb{F}_{q}^{n} \backslash\{\underline{0}\},
$$

with center $\underline{a}$, direction $\underline{a}$ and radius $-\|\underline{a}\|$ (which covers all radii since each element of $\mathbb{F}_{q}^{*}$ is sum of two squares). However, each $\underline{x} \in \mathcal{H}_{-\|a\|}(\underline{a}, \underline{a})$ satisfies

$$
\|\underline{x}\|=\|\underline{x}-\underline{a}+\underline{a}\|=\|\underline{x}-\underline{a}\|+\|\underline{a}\|+2 \underline{a} \cdot(\underline{x}-\underline{a})=0,
$$

which has at most $q^{n-1}+q^{\lfloor n / 2\rfloor}-q^{\lceil(n-2) / 2\rceil}$ solutions by (3) which is an upper bound for $|\mathcal{H}|$.

## 5 One-dimensional circular Kakeya sets

The definitions of circular Kakeya sets in dimension 1 are in fact equivalent to definitions concerning sum and difference sets. More precisely, $\mathcal{K} \subset \mathbb{F}_{q}$ is a radius circular Kakeya set in $\mathbb{F}_{q}$ if and only if

$$
\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}
$$

and $a$ centre circular Kakeya set in $\mathbb{F}_{q}$ if and only if

$$
\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q}
$$

where $\mathcal{K}-\mathcal{K}$ and $\mathcal{K} \oplus \mathcal{K}$ are defined by (11) and (2).

To see the first equivalence, let $\mathcal{K} \subset \mathbb{F}_{q}$ be a set that contains a circle of radius $r$ for each $r \in \mathbb{F}_{q}^{*}$. Therefore there exists $a \in \mathbb{F}_{q}$ such that $\{a+r, a-r\} \subset \mathcal{K}$. We get $a+r-(a-r)=2 r \in \mathcal{K}-\mathcal{K}$. Therefore, since $2 r$ covers all of $\mathbb{F}_{q}^{*}$, we have $\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}(0 \in \mathcal{K}-\mathcal{K}$ trivially). Conversely, suppose that $\mathcal{K} \subset \mathbb{F}_{q}$ is a subset such that $\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}$. Then for each $r \in \mathbb{F}_{q}$, there exist $x_{1}, x_{2} \in \mathcal{K}$, such that $x_{1}-x_{2}=2 r$. By taking $a=\left(x_{1}+x_{2}\right) / 2$ we see $x_{1}=a+r$ and $x_{2}=a-r$ and that the circle $\{a+r, a-r\}$ is in $\mathcal{K}$.

For the second equivalence, let $\mathcal{K} \subset \mathbb{F}_{q}$ be a set containing a circle for any center $a$. Then for all $a \in \mathbb{F}_{q}$, there exists $r \in \mathbb{F}_{q}^{*}$ such that $\{a-r, a+r\} \subset \mathcal{K}$. Then we have $(a-r)+(a+r)=2 a$, and therefore $\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q}$. Conversely, let $\mathcal{K}$ be a subset of $\mathbb{F}_{q}^{*}$ such that $\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q}$. Fix $a \in \mathbb{F}_{q}$. Since $\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}$, there exist $x_{1}, x_{2} \in \mathcal{K}, x_{1} \neq x_{2}$, such that $x_{1}+x_{2}=2 a$. Taking $r=\left(x_{1}-x_{2}\right) / 2$ we can write $x_{1}=a+r$ and $x_{2}=a-r$, so that a circle of centre $a$ is in $\mathcal{K}$.

Since $|\mathcal{K}-\mathcal{K}| \leq|\mathcal{K}|^{2}$, each radius circular Kakeya set in $\mathbb{F}_{q}$ has size at least $\left\lceil q^{1 / 2}\right\rceil$, and since $|\mathcal{K} \oplus \mathcal{K}|<|\mathcal{K}|^{2} / 2$ the size of any center circular Kakeya set $\mathcal{K}$ of $\mathbb{F}_{q}$ is at least $|\mathcal{K}| \geq\lceil\sqrt{2 q}\rceil$. (Keep the condition $x_{1} \neq x_{2}$ in (2) in mind.) In this section we will give constructions of radius circular and center circular Kakeya sets $\mathcal{K}$ in $\mathbb{F}_{q}$ with $|\mathcal{K}|$ of optimal order of magnitude $O\left(q^{1 / 2}\right)$.

For a prime $p>2$ it is easy to find circular Kakeya sets in $\mathbb{F}_{p}$ of size $2\lfloor\sqrt{p}\rfloor+1$,

$$
\begin{equation*}
\mathcal{K}=\mathcal{K}_{p}=\{0,1,2, \ldots,\lfloor\sqrt{p}\rfloor\} \cup-\mathcal{K}_{0}, \tag{7}
\end{equation*}
$$

where

$$
\mathcal{K}_{0}=\{\lceil\sqrt{p}\rceil, 2\lceil\sqrt{p}\rceil, \ldots,\lfloor\sqrt{p}\rfloor\lceil\sqrt{p}\rceil\} .
$$

It is clear that $\mathcal{K}-\mathcal{K}=\mathbb{F}_{p}$. Substituting $-\mathcal{K}_{0}$ by $\mathcal{K}_{0}$ in (7) we get $\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{p}$.
If $q=r^{2}$ is a square and $\alpha$ is a defining element of $\mathbb{F}_{q}$ over $\mathbb{F}_{r}$, that is, $\mathbb{F}_{q}=\mathbb{F}_{r}(\alpha)$, then we can choose

$$
\mathcal{K}=\mathbb{F}_{r} \cup \alpha \mathbb{F}_{r}
$$

of size $|\mathcal{K}|=2 q^{1 / 2}-1$ to get both $\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}$ and $\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q}$.
If $q=p^{2 m+1}$ with a prime $p$ and $\mathbb{F}_{q}=\mathbb{F}_{p}(\beta)$ with a defining element $\beta$ of $\mathbb{F}_{q}$ over $\mathbb{F}_{p}$, then we first choose the construction $\mathcal{K}_{p}$ from (77) and then take

$$
\mathcal{K}=\mathcal{K}_{1} \cup \mathcal{K}_{2},
$$

where

$$
\mathcal{K}_{1}=\left\{a_{0}+a_{1} \beta+\ldots+a_{m} \beta^{m}: a_{0} \in \mathcal{K}_{p}, a_{1}, \ldots, a_{m} \in \mathbb{F}_{p}\right\}
$$

and

$$
\mathcal{K}_{2}=\left\{a_{0}+a_{1} \beta^{m+1}+\ldots+a_{m} \beta^{2 m}: a_{0} \in \mathcal{K}_{p}, a_{1}, \ldots, a_{m} \in \mathbb{F}_{p}\right\} .
$$

It is easy to check that $\mathcal{K}-\mathcal{K}=\mathbb{F}_{q}$ and

$$
|\mathcal{K}|=\left(2 p^{m}-1\right)\left|\mathcal{K}_{p}\right|<4 q^{1 / 2}+2(q / p)^{1 / 2} .
$$

Again substituting $-\mathcal{K}_{0}$ by $\mathcal{K}_{0}$ in (7) we get $\mathcal{K} \oplus \mathcal{K}=\mathbb{F}_{q}$.
Combining all the cases we can formulate a general result.
Theorem 5.1. For a fixed power $q$ of an odd prime let $\mathcal{K} \subset \mathbb{F}_{q}$ be either a radius circular or a center circular Kakeya set in $\mathbb{F}_{q}$ of minimal size. Then we have

$$
q^{1 / 2} \leq|\mathcal{K}|<6 q^{1 / 2}
$$

The constants can be certainly improved using, for example, ideas from [6, 9]. However, we did not calculate these improved constants for the readability of this paper, and since even the improved upper bounds would not be optimal.

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