# Relaxation Oscillations and the Entry-Exit Function in Multi-Dimensional Slow-Fast Systems 

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#### Abstract

For a slow-fast system of the form $\dot{p}=\epsilon f(p, z, \epsilon)+h(p, z, \epsilon), \dot{z}=$ $g(p, z, \epsilon)$ for $(p, z) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$, we consider the scenario that the system has invariant sets $M_{i}=\left\{(p, z): z=z_{i}\right\}, 1 \leq i \leq N$, linked by a singular closed orbit formed by trajectories of the limiting slow and fast systems. Assuming that the stability of $M_{i}$ changes along the slow trajectories at certain turning points, we derive criteria for the existence and stability of relaxation oscillations for the slow-fast system. Our approach is based on a generalization of the entryexit relation to systems with multi-dimensional fast variables. We then apply our criteria to several predator-prey systems with rapid ecological evolutionary dynamics to show the existence of relaxation oscillations in these models.


## 1. Introduction

We consider a system of ordinary differential equations for $(p, z) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$ of the form

$$
\begin{align*}
& \dot{p}=\epsilon f(p, z, \epsilon)+h(p, z, \epsilon), \\
& \dot{z}=g(p, z, \epsilon), \tag{1}
\end{align*}
$$

where - denotes $\frac{d}{d t}$, the functions $f, g$ and $h$ are smooth, and $\epsilon>0$ is a parameter. This system is a generalization of the classical slow-fast systems in Fenichel [15], where the term $h$ was absent. In the scenario that $g$ and $h$ both vanish on some level sets $M_{i}=\left\{(p, z): z=z_{i}\right\}, i=1,2, \ldots, N$, where $z_{i} \in \mathbb{R}^{m}$ are constants, each $M_{i}$ is invariant under (1) since $\dot{z}=0$. System (1) restricted on $M_{i}$ is

$$
\begin{equation*}
p^{\prime}=f\left(p, z_{i}, \epsilon\right), \quad z=z_{i}, \tag{2}
\end{equation*}
$$

where $\boldsymbol{\prime}$ denotes $\frac{d}{d \tau}$ with $\tau=\epsilon$. Hence system (1) has two distinguished limits: The limiting fast system

$$
\begin{equation*}
\dot{p}=h(p, z, 0), \quad \dot{z}=g(p, z, 0), \tag{3}
\end{equation*}
$$

obtained by setting $\epsilon=0$ in system (1), and the limiting slow system

$$
\begin{equation*}
p^{\prime}=f\left(p, z_{i}, 0\right), \quad z=z_{i}, \tag{4}
\end{equation*}
$$

obtained by setting $\epsilon=0$ in (2). When there are trajectories $\gamma_{i}$ of (3) and trajectories $\sigma_{i} \subset M_{i}$ of (4) such that

$$
\begin{equation*}
\gamma_{1} \cup \sigma_{1} \cup \gamma_{2} \cup \sigma_{2} \cup \cdots \cup \gamma_{N} \cup \sigma_{N} \tag{5}
\end{equation*}
$$

forms a closed configuration, in the spirit of Geometric Singular Perturbation Theory (GSPT) (see e.g. Fenichel [15], Jones [28] and Kuehn[34]), there is potentially a periodic orbit of (1) near configuration (5) for all small $\epsilon>0$. However, in the case

[^0]that $\sigma_{i}$ contains turnning points, at which the stability of $M_{i}$ changes, the so-called entry-exit function is needed to determine whether there are trajectories of (1) near the singular orbit. The classical entry-exit function was defined for system (1) with $p$ being a one-dimensional variable (see De Maesschalck [11], De Maesschalck and Schecter [12], Hsu [23], Wang and Zhang [53] and references therein). In the present paper we generalize the entry-exit function (see Section 2.1) for system (1) with a multi-dimensional variable $p$. Using our generalized entry-exit function, we provide criteria under which periodic orbits near the singular orbit exist. Note that if such periodic orbits exist, they must form a relaxation oscillation because the vector field (1) has magnitude of order $O(\epsilon)$ near $\sigma_{i}$ and has magnitude of order $O(1)$ near $\gamma_{i}$.

Our objective is to understand the mechanism of rapid regime shifts in ecological systems. One example is trait oscillations exhibited in an eco-evolutionary system proposed by Cortez and Weitz [10]. The system takes the following form.

$$
\begin{align*}
x^{\prime} & =F(x, \alpha)-G(x, y, \alpha, \beta) \\
y^{\prime} & =H(x, y, \alpha, \beta)-D(y, \beta) \\
\epsilon \alpha^{\prime} & =\alpha(1-\alpha) \frac{\partial}{\partial \alpha}\left(\frac{x^{\prime}}{x}\right),  \tag{6}\\
\epsilon \beta^{\prime} & =\beta(1-\beta) \frac{\partial}{\partial \beta}\left(\frac{y^{\prime}}{y}\right),
\end{align*}
$$

where $x(t)$ and $y(t)$ are the prey and predator densities, respectively, and $\alpha(t)$ and $\beta(t)$ are the average trait values of the prey and predators, respectively, at time $t$. The functions $F$ and $H$ are related to the growth rates of the prey and predators, respectively, $G$ is related to the encounter rate, and $D$ is related to the death rate of predators. The equations of $\alpha$ and $\beta$ were derived from the assumption that the adaptive change in the trait follows fitness-gradient dynamics (see Abrams et al. [1]), i.e., the rate of change of the mean trait value is proportional to the fitness gradient of an individual with this mean trait value. In Cortez and Weitz [10], numerical evidences of periodic orbits oscillating between the level sets, for $(\alpha, \beta)=(0,0),(0,1),(1,1)$ and $(1,0)$, were provided for certain functional responses. A simulation of a periodic orbit with data from that paper is shown in Figure 1. Applying one of our criteria (Theorem 2.4) in Section 4.3, besides confirming the existence of periodic orbits, we determine the limiting configuration (see Figure 2) of the periodic orbit as $\epsilon \rightarrow 0$. This singular orbit can be used to predict the location of periodic orbits.

Another example, proposed by Cortez and Ellner [8], is a predator-prey system with rapid prey evolution:

$$
\begin{align*}
& x^{\prime}=x(\alpha+r-k x)-\frac{x y\left(a \alpha^{2}+b \alpha+c\right)}{1+x} \\
& y^{\prime}=\frac{x y\left(a \alpha^{2}+b \alpha+c\right)}{1+x}-d y  \tag{7}\\
& \epsilon \alpha^{\prime}=\alpha(1-\alpha)\left(1-\frac{y(2 a \alpha+b)}{1+x}\right) \equiv \alpha(1-\alpha) E(x, y, \alpha),
\end{align*}
$$

which can be regarded as a special case of (6) with $\beta$ being constant. Periodic orbits that travel back and forth between the manifolds $M_{0}$ and $M_{1}$ corresponding to $\alpha=0$ and $\alpha=1$, respectively, was discovered numerically by Cortez and Ellner [8] (see Figure 3 for a simulation with data from that paper). Note that the sign of $E(x, y, \alpha)$, where $\alpha=0$ (resp. $\alpha=1$ ), determines whether $M_{0}\left(\right.$ resp. $\left.M_{1}\right)$ is attracting or repelling at that point. It was indicated by those authors that if the trait oscillation occurs,


Figure 1. A periodic orbit for system (6) with $\epsilon=0.25$. (a) On the $(x, y)$-plane the trajectory can roughly be split into four segments. (b) The value of $\alpha$ remains close to 0 along segments i and ii and becomes close to 1 in segments iii and iv. The value of $\beta$ is close to 0 in segments $i$ and iv and is close to 1 in segments ii and iii.


Figure 2. (a) A periodic orbit for system (6) with $\epsilon=0.10$. (b) A singular closed orbit which consists of trajectories of limiting subsystems.
at the landing and jumping points on each $M_{i}$ the values of $E$ has opposite signs. In Section 4.1, applying our criterion (Theorem 2.3) we determine two pairs of the landing and jumping points, $A_{1}, B_{1} \in M_{0}$ and $A_{2}, B_{2} \in M_{1}$, by the equations

$$
\begin{equation*}
\int_{\sigma_{1}} E(x, y, 0) d t=\int_{\sigma_{2}} E(x, y, 1) d t=0 \tag{8}
\end{equation*}
$$

where $\sigma_{1}$ is a trajectory on $M_{0}$ connecting $A_{1}$ and $B_{1}$, and $\sigma_{2}$ is a trajectory on $M_{1}$ connecting $A_{2}$ and $B_{2}$ (see Figure 3). The derivation of (8) is based on the entry-exit functions on $M_{i}$. Also we prove that the corresponding periodic orbits are orbitally locally asymptotically stable.


Figure 3. (a) The trajectory of (7) with $\epsilon=0.1$ and initial data $(x, y, \alpha)=(10,0.5,0.5)$ converges to a periodic orbit. (b) A singular configuration consisting of trajectories of limiting subsystems, and is locally uniquely determined by (8).

The third example is a 1-predator-2-prey system with rapid prey evolution proposed by Piltz et al. [44]:

$$
\begin{align*}
& p_{1}^{\prime}=r_{1} p_{1}-q f_{1}\left(p_{1}\right) z \\
& p_{2}^{\prime}=r_{2} p_{2}-(1-q) f_{2}\left(p_{2}\right) z \\
& z^{\prime}=c_{1} q f_{1}\left(p_{1}\right) z+c_{2}(1-q) f_{2}\left(p_{2}\right) z-m z  \tag{9}\\
& \epsilon q^{\prime}=q(1-q)\left(c_{1} f_{1}\left(p_{1}\right)-c_{2} f_{2}\left(p_{2}\right)\right)
\end{align*}
$$

where $p_{1}$ and $p_{2}$ are population densities of two prey species, $z$ is the population density of predators, and $q$ is the mean trait value of predators. The equation of $q^{\prime}$ is analogous to the equation of $\alpha^{\prime}$ in (6).

A two-parameter family of closed singular configurations formed by trajectories of limiting slow and fast systems of (9) has been derived in Piltz et al. [44]. In Section 4.2, using our criterion (Theorem 2.3) we prove that there is a locally unique closed singular configuration that admits periodic orbits (see Figure 4(a)). Moreover, with parameters adapted from that paper, by computing the linearization of the singular transition maps we prove that the periodic orbits are orbitally unstable (see Figure 4(b)) for all small $\epsilon>0$.

In Section 4.4, we consider the planar system studied by Hsu and Wolkowicz [25]:

$$
\begin{equation*}
\frac{d}{d t} a=\epsilon F(a, b, \epsilon)+b H(a, b, \epsilon), \quad \frac{d}{d t} b=b G(a, b, \epsilon) \tag{10}
\end{equation*}
$$

The $a$-axis is a critical manifold for the limiting fast system of (10). For singular closed orbits for this system, a criterion of the existence and stability of corresponding relaxation oscillations was derived in Hsu and Wolkowicz [25], which generalizes the criterion in Hsu [24]. Using our results (Theorem 2.5), we provide an alternative proof of that result. The derivations in those papers were based on the asymptotic expansion of Floquet exponents for system (10) with $\epsilon>0$. In the present paper, we


Figure 4. (a) A periodic orbit for (9) (red solid curve) with $\epsilon=0.01$ is close to the singular configuration (blue dotted curve) with vertices $A_{i}$ and $B_{i}$. (b) A trajectory for (9) with $\epsilon=0.01$ and initial value (black open circle) close to the periodic orbit leaves the vicinity of the periodic orbit as time evolves, which suggests that the periodic orbit is unstable.
analyze the transition maps for the limiting slow and fast systems with $\epsilon=0$ directly, which provides a better understanding of the slow-fast feature in the system.

The rapid evolution model, i.e., system (6) with $0<\epsilon \ll 1$, has been studied by Cortez [4, 5, 6, 7], Cortez and Ellner [8], Cortez and Patel [9], Cortez and Weitz [10], and Haney and Siepielski [18]. System (6) with slow evolution, i.e. $\epsilon \gg 1$, has been studied by Khibnik and Kondrashov [30], Shen, Hsu, and Yang [49]. Transient behaviors, which are related to regime shifts in ecological systems, have been studied by Hastings [19], Wysham and Hastings [54], and Hastings et al. [20]. Model (9) is a continuous version of the piecewise-smooth model in Piltz, Porter and Maini [43]. A comparison of the numerical solutions of (9) with real data was given in Piltz, Veerman and Maini [42].

Relaxation oscillations for systems with turning points have been studied by Szmolyan and Wechselberger [51], Liu, Xiao and Yi [39]. Our work is complementary to those results since our singular orbit is away from fold points (i.e. singular points of the slow flow). Our result is a generalization of the criterion of relaxation oscillations given by Li et al. [36], Hsu [24], and Hsu and Wolkowicz [25]. Relaxation oscillations in predator-prey systems have been studied by various researchers, including Ghazaryan, Manukian and Schecter [17], Hsu and Shi [21], Huzak [26], Li and Zhu [35] Rinaldi and Muratori [45], and Shen, Hsu and Yang [49]. Relaxation oscillations in multi-dimensional slow-fast systems without turning points have been studied by Soto-Treviño [50]. Boundary value problems for slow-fast systems have been studied by Lin [37] and Tin, Kopell and Jones [52].

The entry-exit function can be traced back to Benoit [2], and is called the way-in way-out function in Diener [13]. This phenomenon that the landing and jumping points satisfy the entry-exit function has been called bifurcation delay in Benoît [3], Pontryagin delay in Mishchenko et al. [41], and delay of instability in Liu [38].

The proof of our criterion is a generalization of the method in Hsu [22, 23], which is a variation of the classical blow-up method. The blow-up method was developed by Dumortier and Roussarie [14] and Krupa and Szmolyan [32, 33], and has been applied extensively to study various problems, including Gasser, Szmolyan and Wächtler [16], Iuorio, Popović and Szmolyan [27], Kosiuk and Szmolyan [31], Manukian and Schecter [40], Schecter [46], and Schecter ans Szmolyan [48].

This paper is organized as follows. In Section 2, we state our criteria for the existence and stability of relaxation oscillations and provide some computable formulas for the criteria. Proofs of the criteria are given in Section 3. In Section 4 we apply our criteria to models described in Section 1.

## 2. Main Theorems

Assumptions needed for our main results are stated in Section 2.1. The criteria for the existence of relaxation oscillations are split into Sections 2.2-2.4, from single to multiple dimensional fast variables. Formulas for computing quantities in the criteria are given in Section 2.5.
2.1. The Assumptions. Let $N$ be a fixed positive integer. Throughout this paper we adopt the notion that $A_{i}=A_{i+N}$ for any integer $i$ and any object $A$. For any vector $z$ in $\mathbb{R}^{m}$, we denote $z^{(j)}$ the $j$-th component of $z$. We denote $\left\{\mathrm{e}_{1}, \mathrm{e}_{2}, \ldots, \mathrm{e}_{m}\right\}$ the standard basis of $\mathbb{R}^{m}$.
Assumption 1. For each $j=1,2, \ldots, m$, there exist $-\infty \leq z_{\min }^{(j)}<z_{\max }^{(j)} \leq \infty$ such that for all sufficiently small $\epsilon \geq 0$,

$$
h(p, z, \epsilon)=0 \quad \text { and } \quad g^{(j)}(p, z, \epsilon)=0
$$

whenever $z^{(j)}=z_{\text {min }}^{(j)}$ or $z=z_{\text {max }}^{(j)}$.
Assumption 2. For each $i=1,2, \ldots, N$, where $N$ is a positive integer, there exist $A_{i}, B_{i} \in \mathbb{R}^{n}, J_{i} \in\{1,2, \ldots, m\}$,

$$
z_{i} \in\left\{z_{\min }^{(1)}, z_{\max }^{(1)}\right\} \times\left\{z_{\min }^{(2)}, z_{\max }^{(2)}\right\} \times \cdots \times\left\{z_{\min }^{(m)}, z_{\max }^{(m)}\right\} \quad \text { with }\left|z_{i}\right|<\infty
$$

and smooth functions $\theta_{i}: \mathbb{R} \rightarrow \mathbb{R}^{n}$ and $\rho_{i}: \mathbb{R} \rightarrow \mathbb{R}$ such that $\rho_{i}$ is non-constant and the curve

$$
\gamma_{i}(t)=\left(\theta_{i}(t), z_{i}+\rho_{i}(t) \mathrm{e}_{J_{i}}\right), \quad-\infty<t<\infty
$$

is a heteroclinic orbit of (3) that connects $\left(B_{i-1}, z_{i-1}\right)$ and $\left(A_{i}, z_{i}\right)$. In additional, for each $j=1,2, \ldots, m$, there exists $i \in\{1,2, \ldots, N\}$ such that $J_{i}=j$.

The expression of the heteroclinic orbit in Assumption 2 implies that $z_{i}$ differs from $z_{i+1}$ at no more than one component. Note that we do not exclude the possibility that $z_{i}=z_{i+1}$.

The assumption of the existence of $i$ such that $J_{i}=j$ means that each component $z^{(j)}$ of ( $p, z$ ) must be non-constant along at least one $\gamma_{i}$. If it is not the case, then we can treat $z^{(j)}$ as a constant and replace the equation of $\dot{z}^{(j)}$ in (1) by $\dot{z}^{(j)}=0$ because the space $\left\{(p, z): z^{(j)}=z_{\min }^{(j)}\right.$ or $\left.z_{\max }^{(j)}\right\}$ is invariant under (1) by Assumption 1.

We define $M_{i}=\left\{(p, z): p \in \mathbb{R}^{n}, z=z_{i}\right\}$ for $i=1,2, \ldots, N$. Then Assumption 1 implies that $M_{i}$ is invariant under (1) for all sufficiently small $\epsilon>0$. The restriction of (1) on $M_{i}$ is (4). We denote the solution operator of (4) by $\Phi_{i}$.
Assumption 3. For each $i=1,2, \ldots, n, f_{i}\left(A_{i}, z_{i}, 0\right) \neq 0$ and there exists $\tau_{i}>0$ such that $\Phi_{i}\left(\tau_{i}, A_{i}\right)=B_{i}$.

Denote $\sigma_{i}=\Phi_{i}\left(\left[0, \tau_{i}\right], A_{i}\right) \times\left\{z_{i}\right\}$. Then by Assumptions $2-3$ the configuration (5) forms a closed orbit. The idea of GSPT is that solutions of the full system can potentially be obtained by joining some trajectories of its limiting systems. The limiting systems (3) and (4) provide a family of uncountably many loops. Our goal is to establish a criterion for the existence of a locally unique periodic orbit near this closed singular orbit.

We impose the following non-degeneracy condition.
Assumption 4. For $i=1,2, \ldots, N$,

$$
\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(A_{i}, z_{i}, 0\right)<0 \quad \text { and } \quad \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)>0
$$

Remark 2.1. By Assumption 1, the linearization of (3) at any point $\left(p, z_{i}\right)$ in $M_{i}$ has the Jacobian matrix

$$
\left(\begin{array}{cc}
0_{n \times n} & * \\
0_{m \times n} & \operatorname{diag}\left(\frac{\partial g^{(1)}}{\partial z^{(1)}}, \ldots, \frac{\partial g^{(m)}}{\partial z^{(m)}}\right)
\end{array}\right)
$$

where the partial derivatives are evaluated at $\left(p, z_{i}, 0\right)$. In the case that $m=1$, the inequalities in Assumption 4 imply that $M_{i}$ is normally hyperbolic at $\left(A_{i}, z_{i}\right)$ and $\left(B_{i}, z_{i}\right)$ and that there is a turning point on $M_{i}$ between these two points.

In the case that $m=1$, where $z$ and $g$ are scalar, the classical entry-exit relation for (1) between $A_{i}$ and $B_{i}$ can be expressed by

$$
\int_{0}^{s} \frac{\partial g}{\partial z}\left(\Phi_{i}\left(\tau, A_{i}\right), z_{i}, 0\right) d \tau \begin{cases}=0, & \text { if } s=\tau_{i}  \tag{11}\\ <0, & \text { if } 0<s<\tau_{i}\end{cases}
$$

Assuming (11) and Assumption 4, on some neighborhood $\mathscr{A}_{i}$ of $A_{i}$ in $\mathbb{R}^{n}$ we can implicitly define $T_{i}: \mathscr{A}_{i} \rightarrow(0, \infty)$ by $T_{i}\left(A_{i}\right)=\tau_{i}$ and

$$
\begin{equation*}
\int_{0}^{T_{i}(p)} \frac{\partial g}{\partial z}\left(\Phi_{i}(\tau, p), z_{i}, 0\right) d \tau=0 \tag{12}
\end{equation*}
$$

The entry-exit function is then defined by

$$
\begin{equation*}
Q_{i}(p)=\Phi_{i}\left(T_{i}(p), p\right) \tag{13}
\end{equation*}
$$

Each pair of points $\left(p, z_{i}\right)$ and $\left(Q_{i}(p), z_{i}\right)$, where $p \in \mathscr{A}_{i}$, is a pair of landing and jumping points on $M_{i}$.

For the general case that $m \geq 1$, we first introduce some notations. Let $J_{i}$, where $i=1,2, \ldots, N$, be the numbers defined in Assumption 1. For each $j=1,2, \ldots, m$, let

$$
I_{j}=\min \left\{i \in\{1,2, \ldots, N\}: J_{i}=j\right\}
$$

This means that $I_{j}$ is the smallest positive $i$ for which the value of $z^{(j)}$ changes along the trajectory $\gamma_{i}$. By Assumption 2, $I_{j}$ is well-defined and is finite. We define

$$
\zeta_{0}^{(j)}=-\sum_{k=1}^{I_{j}}\left(\int_{0}^{\tau_{k}} \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(\Phi_{i}\left(\tau, A_{k}\right), z_{k}, 0\right) d \tau\right)
$$

and

$$
\zeta_{i}^{(j)}=\zeta_{0}^{(j)}+\sum_{k=1}^{i}\left(\int_{0}^{\tau_{k}} \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(\Phi_{i}\left(\tau, A_{k}\right), z_{k}, 0\right) d \tau\right)
$$

for $i=1,2, \ldots, N$ and $j=1,2, \ldots, m$. Also we denote $\zeta_{i}=\left(\zeta_{1}^{(1)}, \ldots, \zeta_{1}^{(m)}\right)$. The following assumption is a generalization of (11).

Assumption 5. For each $i \in\{1,2, \ldots, N\}, j \in\{1,2, \ldots, m\}$ and $s \in\left(0, \tau_{i}\right]$,

$$
\zeta_{i}^{(j)}+\int_{0}^{s} \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(\Phi_{i}\left(\tau, A_{i}\right), z_{i}, 0\right) d \tau \begin{cases}=0, & \text { if } j=J_{i} \text { and } s=\tau_{i} \\ \neq 0, & \text { otherwise }\end{cases}
$$

For each $i=1,2, \ldots, N$, we consider the system

$$
\begin{align*}
& \frac{d}{d \tau} p=f\left(p, z_{i}, 0\right) \\
& \frac{d}{d \tau} \zeta^{(j)}=\frac{\partial g^{(j)}}{\partial z^{(j)}}\left(p, z_{i}, 0\right), \quad j=1,2, \ldots, m \tag{14}
\end{align*}
$$

Let

$$
\begin{equation*}
\Lambda_{i}=\left\{\zeta \in \mathbb{R}^{m}:\left|\zeta-\zeta_{i}\right|<\delta, \zeta^{\left(J_{i}\right)}=\zeta_{i}^{\left(J_{i}\right)}\right\} \tag{15}
\end{equation*}
$$

where $\delta>0$. Let $\widehat{\Phi}_{i}$ be the solution operator for (14). From Assumption 4, by shrinking $\mathscr{A}_{i}$ and $\delta$ if necessary, we can define $\widehat{T}_{i}(p, \zeta)$ on $\mathscr{A}_{i} \times \Lambda_{i}$ implicitly by $\widehat{T}_{i}\left(A_{i}, \zeta_{i}\right)=0$ and

$$
\begin{equation*}
\zeta^{\left(J_{i}\right)}+\int_{0}^{\widehat{T}_{i}(p, \zeta)} \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(\Phi_{i}(\tau, p), z_{i}, 0\right) d \tau=0 \tag{16}
\end{equation*}
$$

Finally, we define the generalized entry-exit function $\widehat{Q}_{i}(p, \zeta)$ on $\mathscr{A}_{i} \times \Lambda_{i}$ by

$$
\begin{equation*}
\widehat{Q}_{i}(p, \zeta)=\widehat{\Phi}_{i}\left((p, \zeta), \widehat{T}_{i}(p, \zeta)\right) \tag{17}
\end{equation*}
$$

Note that $\widehat{T}_{\overparen{ }}\left(p, \zeta_{i}\right)=T_{i}(p)$ and therefore $\widehat{Q}_{i}\left(p, \zeta_{i}\right)=\left(Q_{i}(p), \zeta_{i+1}\right)$ for all $p \in \mathscr{A}_{i}$. In particular, $\widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)=\left(B_{i}, \zeta_{i+1}\right)$.
Remark 2.2. In the case that $m=1$, we have $\zeta_{i}^{(j)}=0$ for all $i$ and $j$, so Assumption 5 is reduced to the classical entry-exit relation (11), and $\widehat{Q}_{i}$ defined by (16)-(17) coincides with $Q_{i}$ defined by (12)-(13).
2.2. Systems with a Single and Simple Fast Variable. First we state our results for system (1) with $m=1$ and $h=0$, which can be applied to models (7) and (9). These restrictions mean that the system has a single variable and that the slow variable is steady in the fast system (3).

Since the slow variable is steady in the fast system (3) in the case that $h=0$, the function $\theta_{i}$ in Assumption 2 is constant for each $i=1,2, \ldots, N$. Hence $B_{i}=A_{i+1}$ for each $i$. Since $Q_{i}\left(A_{i}\right)=B_{i}$, it follows that $Q_{i}\left(A_{i}\right)=A_{i+1}$. Let

$$
\begin{equation*}
P=Q_{N} \circ \cdots \circ Q_{2} \circ Q_{1} \tag{18}
\end{equation*}
$$

Then $P\left(A_{1}\right)=A_{1}$ and $P$ maps a neighborhood of $A_{1}$ in $\mathscr{A}_{1}$ into $\mathscr{A}_{1}$.
Theorem 2.3. Suppose that Assumptions $1-5$ hold for system (1) with $m=1$ and $h=0$. Let $P$ be defined by (18). If

$$
\operatorname{det}\left(D P\left(A_{1}\right)-\mathrm{I}_{n}\right) \neq 0
$$

where $\mathrm{I}_{n}$ is the identify matrix of rank $n$, then the configuration (5) admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if the spectrum radius of $D P\left(A_{1}\right)$ is less than one and orbitally unstable if the spectrum radius of $D P\left(A_{1}\right)$ is greater than one.

The proof of the theorem is shown in Section 3.1.
2.3. Systems with Simple Fast Dynamics. System (1) with $m \geq 1$ and $h=0$ can be applied to (6). For this case, we introduce the following definitions.

Under the assumption that $h=0$, we have $B_{i}=A_{i+1}$. Since $\widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)=$ $\left(B_{i}, \zeta_{i+1}\right)$, it follows that $\widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)=\left(A_{i+1}, \zeta_{i+1}\right)$. Let

$$
\begin{equation*}
\widehat{P}=\widehat{Q}_{N} \circ \cdots \circ \widehat{Q}_{2} \circ \widehat{Q}_{1} \tag{19}
\end{equation*}
$$

Then $\widehat{P}\left(A_{1}, \zeta_{1}\right)=\left(A_{1}, \zeta_{1}\right)$ and $\widehat{P}$ maps a neighborhood of $\left(A_{1}, \zeta_{1}\right)$ in $\mathscr{A}_{1} \times \Lambda_{1}$ into $\mathscr{A}_{1} \times \Lambda_{1}$.
Theorem 2.4. Suppose that Assumptions $1-5$ hold for system (1) with $h=0$. Let $\widehat{P}$ be defined by (19). If

$$
\operatorname{det}\left(D \widehat{P}\left(A_{1}, \zeta_{1}\right)-\mathrm{I}_{n+m-1}\right) \neq 0
$$

where $D \widehat{P}$ is the Jacobian matrix with respect to the standard coordinate of $\mathscr{A}_{1} \times \Lambda_{1}$, then the configuration (5) admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if the spectrum radius of $D \widehat{P}\left(A_{1}, \zeta_{1}\right)$ is less than one and orbitally unstable if the spectrum radius of $D \widehat{P}\left(A_{1}, \zeta_{1}\right)$ is greater than one.

Theorem 2.4 is resulted from a more general theorem, Theorem 2.5, stated below.
2.4. Systems with Multiple Slow and Fast Variables. Now we consider system (1) with general $h$ for treating system (10).

For $i=1,2, \ldots, N$ and $j=1,2, \ldots, m$, let

$$
\omega_{i}^{(j)}= \begin{cases}1, & \text { if } z_{i}^{(j)}=z_{\min }^{(j)}  \tag{20}\\ -1, & \text { if } z_{i}^{(j)}=z_{\max }^{(j)}\end{cases}
$$

Let

$$
\phi_{i}(q)= \begin{cases}\frac{\omega_{i}^{\left(J_{i}\right)}}{q-z_{i}^{\left(J_{i}\right)},} & \text { if } z_{i-1}^{\left(J_{i}\right)}=z_{i-1}^{\left(J_{i}\right)} \\ \frac{\omega_{i}^{\left(J_{i}\right)}}{q-z_{i}^{\left(J_{i}\right)}} \frac{\omega_{i-1}^{\left(J_{i}\right)}}{q-z_{i-1}^{\left(J_{i}\right)}}, & \text { if } z_{i}^{\left(J_{i}\right)} \neq z_{i-1}^{\left(J_{i}\right)}\end{cases}
$$

Note that $\phi_{i}\left(z^{\left(J_{i}\right)}\right)>0$ for all $(p, z)$ on $\gamma_{i}$.
Define functions $g_{i}$ and $h_{i}$ of $(p, q) \in \mathbb{R}^{N} \times \mathbb{R}$ by

$$
\begin{equation*}
\left(g_{i}, h_{i}\right)(p, q)=\phi_{i}(q)\left(g^{\left(J_{i}\right)}, h\right)\left(p, z_{i-1}+q \mathrm{e}_{J_{i}}, 0\right) \quad \text { for } q \neq z_{i}^{\left(J_{i}\right)}, z_{i-1}^{\left(J_{i}\right)} \tag{21}
\end{equation*}
$$

By Assumptions 1, $\left(g_{i}, h_{i}\right)$ can be continuously extended at those singularities. We identify $\left(g_{i}, h_{i}\right)$ with its continuous extension. Thus $g_{i}\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right)$ and $g_{i}\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right)$ are multiples of $\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i-1}, z_{i-1}, 0\right)$ and $\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(A_{i}, z_{i}, 0\right)$, respectively, by nonzero constants. By Assumption 4, it follows that $g_{i}\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right) \neq 0$ and $g_{i}\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right) \neq 0$.

Note that the functions $\theta_{i}$ and $\rho_{i}$ in Assumption 2 satisfy that $\left.\left\{\left(\theta_{i}, \rho_{i}\right)(t)\right): t \in \mathbb{R}\right\}$ is a trajectory of the system

$$
\begin{equation*}
\dot{p}=h_{i}(p, q), \quad \dot{q}=g_{i}(p, q) \tag{22}
\end{equation*}
$$

that connects $\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right)$ and $\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right)$. Since $g_{i}\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right) \neq 0$ and $g_{i}\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right) \neq$ 0 , there exists a neighborhood $\mathscr{B}_{i-1}$ of $B_{i-1}$ such that we can define $\pi_{i}: \mathscr{B}_{i-1} \rightarrow \mathscr{A}_{i}$ implicitly by that

$$
\begin{equation*}
\left(p, z_{i-1}^{\left(J_{i}\right)}\right) \text { and }\left(\pi_{i}(p), z_{i}^{\left(J_{i}\right)}\right) \text { are connected by a trajectory of }(22) \tag{23}
\end{equation*}
$$

Let $\pi_{i} \times$ id be the map from $\mathscr{B}_{i-1} \times \Lambda_{i}$ to $\mathscr{A}_{i} \times \Lambda_{i}$ given by $\left(\pi_{i} \times \mathrm{id}\right)(p, \zeta)=\left(\pi_{i}(p), \zeta\right)$. Define

$$
\begin{equation*}
\widetilde{P}=\left(\pi_{N} \times \mathrm{id}\right) \circ \widehat{Q}_{N} \circ\left(\pi_{N} \times \mathrm{id}\right) \circ \cdots \circ \widehat{Q}_{2} \circ\left(\pi_{2} \times \mathrm{id}\right) \circ \widehat{Q}_{1} \tag{24}
\end{equation*}
$$

Theorem 2.5. Suppose that Assumptions $1-5$ hold for system (1). Let $\widetilde{P}$ be defined by (24). If

$$
\operatorname{det}\left(D \widetilde{P}\left(A_{1}, \zeta_{1}\right)-\mathrm{I}_{n+m-1}\right) \neq 0
$$

where $D \widetilde{P}$ is the Jacobian matrix with respect to the standard coordinate of $\mathscr{A}_{1} \times \Lambda_{1}$, then the configuration (5) admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if the spectrum radius of $D \widetilde{P}\left(A_{1}, \zeta_{1}\right)$ is less than one and orbitally unstable if the spectrum radius of $D \widetilde{P}\left(A_{1}, \zeta_{1}\right)$ is greater than one.

The proof of the theorem is shown in Section 3.2.
2.5. Some Computable Formulas. For each $i=1,2, \ldots, N$, for convenience we define $f_{i}(p)=f\left(p, z_{i}, 0\right)$ and $p_{i}(\tau)=\Phi_{i}\left(\tau, A_{i}\right)$. Let $L_{i}(\tau)$ be the fundamental matrix for the variational equations of (4) along $\sigma_{i}$. This means that for any $v \in \mathbb{R}^{n}$, $w(\tau)=L_{i}(\tau) v$ is the solution of

$$
\begin{equation*}
\frac{d}{d \tau} w=\left[D f_{i}\left(p_{i}(\tau)\right)\right] w, \quad w(0)=v_{0}, \quad \text { for } \quad 0 \leq \tau \leq \tau_{i} \tag{25}
\end{equation*}
$$

It can be shown that, for $v \in \mathbb{R}^{n}$ and $0 \leq \tau \leq \tau_{i}$,

$$
\begin{equation*}
L_{i}(\tau) v=D \Phi\left(\tau, A_{i}\right) v \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{i}(\tau) v=v+\int_{0}^{\tau}\left[D f_{i}\left(p_{i}(s)\right)\right] L_{i}(s) v d s \tag{27}
\end{equation*}
$$

We define the linear functional $\mu_{i}$ on $\mathbb{R}^{n}$ by

$$
\begin{equation*}
\mu_{i}(v)=\int_{0}^{\tau_{i}}\left\langle L_{i}(\tau) v, D \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(p_{i}(\tau), z_{i}, 0\right)\right\rangle d \tau \quad \text { for } v \in \mathbb{R}^{n} \tag{28}
\end{equation*}
$$

where $D$ denotes the derivative with respect to $p$.
Proposition 2.6. Let $Q_{i}$ be defined by (13). Then

$$
\begin{equation*}
D Q_{i}\left(A_{i}, \zeta_{i}\right) v=L_{i}\left(\tau_{i}\right) v-\frac{\mu_{i}(v)}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)} f\left(B_{i}, z_{i}, 0\right) \quad \forall v \in \mathbb{R}^{n} \tag{29}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
D Q_{i}\left(A_{i}, \zeta_{i}\right) f\left(A_{i}, z_{i}, 0\right)=\frac{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(A_{i}, z_{i}, 0\right)}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)} f\left(B_{i}, z_{i}, 0\right) \tag{30}
\end{equation*}
$$

Proof. By differentiating (12) with respect to $p$ we obtain

$$
\begin{aligned}
& \left\langle D T_{i}(p), v\right\rangle \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(\Phi_{i}\left(\tau_{i}, A\right), z_{i}, 0\right) \\
& \quad+\int_{0}^{T_{i}(A)}\left\langle D \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(\Phi_{i}(\tau, A), z_{i}, 0\right), D \Phi_{i}(\tau, A) v\right\rangle d \tau=0 \quad \forall v \in \mathbb{R}^{n}
\end{aligned}
$$

Evaluating this equation at $A=A_{i}$ yields

$$
\left\langle D T_{i}(p), v\right\rangle \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)=-\int_{0}^{\tau_{i}}\left\langle D \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(p_{i}(\tau), z_{i}, 0\right), L_{i}(\tau) v\right\rangle d \tau
$$

By (28) it follows that

$$
\begin{equation*}
\left\langle D T_{i}(p), v\right\rangle=\frac{-\mu_{i}(v)}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)} \tag{31}
\end{equation*}
$$

On the other hand, since $\Phi_{i}$ is the solution operator for (4), the definition of $Q_{i}$ in (13) means that

$$
Q_{i}(p)=p+\int_{0}^{T_{i}(p)} f_{i}\left(\Phi_{i}(\tau, p)\right) d \tau
$$

Differentiating both sides of the equation with respect to $p$ gives

$$
\begin{aligned}
D Q_{i}(p) v=v & +\left\langle D T_{i}(p), v\right\rangle f_{i}\left(\Phi_{i}\left(T_{i}(p), p\right)\right) \\
& +\int_{0}^{T_{i}(p)} D f_{i}\left(\Phi_{i}(\tau, p)\right) D \Phi_{i}(\tau, p) v d \tau \quad \forall v \in \mathbb{R}^{n}
\end{aligned}
$$

Evaluating the equation at $p=A_{i}$ and using (26) we have

$$
D Q_{i}\left(A_{i}\right) v=v+\left\langle D T_{i}\left(A_{i}\right), v\right\rangle f_{i}\left(B_{i}\right)+\int_{0}^{\tau_{i}} D f_{i}\left(p_{i}(\tau)\right) L_{i}(\tau) v d \tau
$$

By (27) it follows that

$$
\begin{equation*}
D Q_{i}\left(A_{i}\right) v=L_{i}\left(\tau_{i}\right) v+\left\langle D T_{i}\left(A_{i}\right), v\right\rangle f_{i}\left(B_{i}\right) \tag{32}
\end{equation*}
$$

Substituting (31) into (32), we then obtain (29).
Since $f_{i}\left(p_{i}(\tau)\right)$ is a solution of (25) with $v_{0}=f_{i}\left(A_{i}\right)$,

$$
\begin{equation*}
L_{i}(\tau) f_{i}\left(A_{i}\right)=f_{i}\left(p_{i}(\tau)\right) \quad \text { for } 0 \leq \tau \leq \tau_{i} \tag{33}
\end{equation*}
$$

Using $\frac{d}{d \tau} p_{i}(\tau)=f_{i}\left(p_{i}(\tau)\right)$ and (33), evaluating (28) at $v=f_{i}(p)$ gives

$$
\begin{equation*}
\mu\left(f_{i}\left(A_{i}\right)\right)=\left.\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(p_{i}(\tau), z_{i}, 0\right)\right|_{\tau=0} ^{\tau_{i}}=\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)-\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(A_{i}, z_{i}, 0\right) \tag{34}
\end{equation*}
$$

Substituting (34) into (29) we obtain (30).
Remark 2.7. Numerical approximations of $L_{i}$ and $\mu_{i}$ can be computed by extending system (3) of $p$ to a system of $(p, w, \mu)$ by appending equations (28) and

$$
\frac{d}{d \tau} \mu_{i}=\left\langle L_{i}(\tau) v, D \frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(p_{i}(\tau), z_{i}, 0\right)\right\rangle
$$

Proposition 2.8. Let $\widehat{Q}_{i}$ be defined by (17). Then

$$
\begin{align*}
& D \widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)(v, 0) \\
& \quad=\left(D Q_{i}\left(A_{i}\right) v, \frac{-\nu_{i}(v)}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)} \sum_{j \neq J_{i}} \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(B_{i}, z_{i}, 0\right) \mathrm{e}_{j}\right) \quad \forall v \in \mathbb{R}^{n} \tag{35}
\end{align*}
$$

where $\nu_{i}(v)$ is defined by (28), and

$$
\begin{align*}
& D \widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)\left(0, \mathrm{e}_{j}\right) \\
& \quad= \begin{cases}\left(0, \mathrm{e}_{j}\right), & \text { if } j \neq J_{i} \\
\frac{1}{\frac{\partial g^{\left(J_{j}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)}\left(f\left(B_{i}, z_{i}, 0\right), \sum_{k \neq J_{i}} \frac{\partial g^{(k)}}{\partial z^{(k)}}\left(B_{i}, z_{i}, 0\right) \mathrm{e}_{\mathrm{k}}\right), & \text { if } j=J_{i}\end{cases} \tag{36}
\end{align*}
$$

Proof. We identify vectors $v \in \mathbb{R}^{n}$ with their images $\left(v, 0_{m}\right) \in \mathbb{R}^{n} \times \mathbb{R}^{m}$, and identify the vector $\mathrm{e}_{j}, j \in\{1,2, \ldots, m\}$, in the standard basis of $\mathbb{R}^{m}$, with the vector $\left(0_{n}, \mathrm{e}_{j}\right)$ in $\mathbb{R}^{n} \times \mathbb{R}^{m}$. The function $\widehat{Q}_{i}(p, \zeta)$ defined by (17) can be written as

$$
\begin{align*}
& \widehat{Q}_{i}(p, \zeta) \\
& \quad=\left(\Phi\left(p, \widehat{T}_{i}(p, \zeta)\right), \sum_{k \neq J_{i}}\left[\zeta^{(k)}+\int_{0}^{\widehat{T}\left(p, \zeta^{\left(J_{i}\right)}\right)} \frac{\partial g^{(k)}}{\partial z^{(k)}}\left(\Phi(p, \tau), z_{i}, 0\right) d \tau\right] \mathrm{e}_{k}\right) . \tag{37}
\end{align*}
$$

Since $\widehat{T}_{i}\left(p, \zeta^{\left(J_{i}\right)}\right)=T_{i}(p)$ and $\Phi\left(p, T_{i}(p)\right)=Q_{i}(p)$ for all $p \in \mathscr{A}_{i}$,

$$
\widehat{Q}_{i}\left(p, \zeta_{i}\right)=\left(Q_{i}(p), \sum_{k \neq J_{i}}\left[\zeta^{(k)}+\int_{0}^{T(p)} \frac{\partial g^{(k)}}{\partial z^{(k)}}\left(\Phi(p, \tau), z_{i}, 0\right) d \tau\right] \mathrm{e}_{k}\right)
$$

Hence

$$
\begin{aligned}
& D \widehat{Q}_{i}\left(p, \zeta_{i}\right)(v, 0) \\
& \quad=\left(D Q_{i}(p) v,\langle D T(p), v\rangle \sum_{j \neq J_{i}} \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(\Phi(p, \tau), z_{i}, 0\right) \mathrm{e}_{j}\right) \quad \forall v \in \mathbb{R}^{n}
\end{aligned}
$$

Evaluating this equation at $p=A_{i}$, by (31) we then obtain (35).
For each $j \in\{1,2, \ldots, m\} \backslash\left\{J_{i}\right\}$, differentiating (37) with respect to $\zeta^{(j)}$ gives $\frac{\partial}{\partial \zeta^{(j)}} \widehat{Q}_{i}(p, \zeta)=\mathrm{e}_{j}$ for all $(p, \zeta)$. On the other hand, by differentiating (37) with respect to $\zeta^{\left(J_{i}\right)}$, from the relation $\frac{\partial}{\partial \tau} \Phi(p, \tau)=f(\Phi(p, \tau))$ we obtain

$$
\begin{align*}
& \frac{\partial}{\partial \zeta^{\left(J_{i}\right)}} \widehat{Q}_{i}(p, \zeta) \\
& \quad=\frac{\partial \widehat{T}\left(p, \zeta^{\left(J_{i}\right)}\right)}{\partial \zeta^{\left(J_{i}\right)}}\left(f\left(\Phi\left(p, \widehat{T}_{i}\left(p, \zeta^{\left(J_{i}\right)}\right)\right), z_{i}, 0\right), \sum_{k \neq J_{i}} \frac{\partial g^{(k)}}{\partial z^{(k)}}\left(B_{i}, z_{i}, 0\right) \mathrm{e}_{\mathrm{k}}\right) \tag{38}
\end{align*}
$$

Note that differentiating (16) with respect to $\zeta^{\left(J_{i}\right)}$ gives

$$
\begin{equation*}
\frac{\partial \widehat{T}_{i}\left(A_{i}, \zeta^{\left(J_{i}\right)}\right)}{\partial \zeta^{\left(J_{i}\right)}}=\frac{-1}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)} \tag{39}
\end{equation*}
$$

By (38) and (39) it follows that

$$
\frac{\partial}{\partial \zeta^{\left(J_{i}\right)}} \widehat{Q}_{i}\left(A_{i}, \zeta\right)=\frac{-1}{\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(B_{i}, z_{i}, 0\right)}\left(f\left(B_{i}, z_{i}, 0\right), \sum_{k \neq J_{i}} \frac{\partial g^{(k)}}{\partial z^{(k)}}\left(B_{i}, z_{i}, 0\right) \mathrm{e}_{\mathrm{k}}\right)
$$

This means that (36) holds.

Let $\Psi_{i}$ be the solution operator for (22). Let $t_{i}$ be the positive number such that

$$
\Psi_{i}\left(t_{i},\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right)\right)=\left(A_{i-1}, z_{i}^{\left(J_{i}\right)}\right)
$$

Let

$$
\bar{\gamma}_{i}(t)=\Psi_{i}\left(t,\left(B_{i-1}, z_{i-1}^{\left(J_{i}\right)}\right)\right), \quad 0 \leq t \leq t_{i}
$$

Thus $\bar{\gamma}$ has the same trajectory as the curve $\gamma$ given in Assumption 2.
We define $R_{i}(t): \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ and $\nu_{i}(t): \mathbb{R}^{n} \rightarrow \mathbb{R}, 0 \leq t \leq t_{i}$, to be the linear operators so that for any $v_{0} \in \mathbb{R}^{n}, R_{i}(t)\left[v_{0}\right]$ and $\nu_{i}\left((t)\left[v_{0}\right]\right.$ are the $v$ - and $w$-components, respectively, of the variational equations of (22) along $\bar{\gamma}_{i}(t)$ with initial data $\left(v_{0}, 0\right)$. This means that for any $\left(v_{0}, w_{0}\right) \in \mathbb{R}^{n} \times \mathbb{R},(v, w)=\left(R_{i}(t)\left[v_{0}\right], \nu_{i}\left((t)\left[v_{0}\right]\right)\right.$ is the solution of

$$
\frac{d}{d t}\binom{v}{w}=\left(\begin{array}{cc}
D_{p} h_{i} & D_{q} h_{i}  \tag{40}\\
D_{p} g_{i} & D_{q} g_{i}
\end{array}\right)_{\bar{\gamma}_{i}(t)}\binom{v}{w}, \quad\binom{v}{w}(0)=\binom{v_{0}}{0}
$$

where $g_{i}$ and $h_{i}$ are defined by (21).
Proposition 2.9. Let $\pi_{i}$ be defined by (23). Then

$$
\begin{equation*}
D \pi_{i}\left(B_{i-1}\right)[v]=R_{i}\left(t_{i}\right)[v]-\nu_{i}\left(t_{i}\right)[v] \frac{h_{i}\left(A_{i}, z_{i}\right)}{g_{i}\left(A_{i}, z_{i}\right)} \quad \forall v \in \mathbb{R}^{n} \tag{41}
\end{equation*}
$$

Moreover, if $n=1$, then

$$
\begin{equation*}
D \pi_{i}\left(B_{i-1}\right)=\frac{g_{i}\left(B_{i-1}, z_{i-1}\right)}{g_{i}\left(A_{i}, z_{i}\right)} \exp \left(\int_{0}^{t_{i}}\left(D_{p} h_{i}+D_{q} g_{i}\right)\left(\widetilde{\gamma}_{i}(t)\right) d t\right) \tag{42}
\end{equation*}
$$

Proof. The first part of the proof is similar to that of Proposition 2.6. Define $S_{i}$ : $\mathscr{B}_{i-1} \rightarrow(0, \infty)$ implicitly by $S_{i}(p)=t_{i}$ and

$$
\begin{equation*}
z_{i-1}^{\left(J_{i}\right)}+\int_{0}^{S_{i}(p)} g_{i}\left(\Psi_{i}\left(t,\left(p, z_{i-1}^{\left(J_{i}\right)}\right)\right) d t=z_{i}^{\left(J_{i}\right)}\right. \tag{43}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left(\pi_{i}(p), z_{i}^{\left(J_{i}\right)}\right)=\Psi_{i}\left(S_{i}(p),\left(p, z_{i-1}^{\left(J_{i}\right)}\right)\right) \tag{44}
\end{equation*}
$$

Differentiating (43) gives (similar to the derivation of (31))

$$
\begin{equation*}
\left\langle D S_{i}(p), v\right\rangle g_{i}\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right)=\nu_{i}\left(t_{i}\right)[v] . \tag{45}
\end{equation*}
$$

Differentiating (44) gives (similar to the derivation of (32))

$$
\begin{equation*}
D \pi_{i}(p)[v]=R_{i}\left(t_{i}\right)[v]-\left\langle D S_{i}(p), v\right\rangle h_{i}\left(A_{i}, z_{i}^{\left(J_{i}\right)}\right) \tag{46}
\end{equation*}
$$

By (45) and (46) we obtain (41).
Now we assume $n=1$. Then (46) gives

$$
\begin{align*}
D \pi_{i}\left(B_{i-1}\right) & =\frac{R_{i}\left(t_{i}\right) g_{i}\left(A_{i}, z_{i}\right)-\nu_{i}\left(t_{i}\right) h_{i}\left(A_{i}, z_{i}\right)}{g_{i}\left(A_{i}, z_{i}\right)} \\
& =\frac{1}{g_{i}\left(A_{i}, z_{i}\right)} \operatorname{det}\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)_{t=t_{i}} \tag{47}
\end{align*}
$$

On the other hand, when $n=1,\left(R_{i}, \nu_{i}\right)(t)$ is the solution of (40) with $v_{0}=1$. Note that $\left(h_{i}, g_{i}\right)\left(\widetilde{\gamma}_{i}(t)\right)$ also satisfies the differential equations in (40). Hence

$$
\frac{d}{d t}\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)=\left(\begin{array}{cc}
D_{p} g & D_{q} g \\
D_{p} h & D_{q} h
\end{array}\right)_{(p, q)=\widetilde{\gamma}_{i}(t)}\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)
$$

and

$$
\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)_{t=0}=\left(\begin{array}{cc}
1 & h_{i}\left(B_{i-1}, z_{i-1}\right) \\
0 & g_{i}\left(B_{i-1}, z_{i-1}\right)
\end{array}\right) .
$$

By Abel's formula, it follows that

$$
\begin{align*}
& \operatorname{det}\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)_{t=t_{i}} \\
& \quad=\operatorname{det}\left(\begin{array}{cc}
R_{i}(t) & h_{i}\left(\widetilde{\gamma}_{i}(t)\right) \\
\nu_{i}(t) & g_{i}\left(\widetilde{\gamma}_{i}(t)\right)
\end{array}\right)_{t=0} \exp \left(\int_{0}^{t_{i}} \operatorname{tr}\left(\begin{array}{ll}
D_{p} g_{i} & D_{q} g_{i} \\
D_{p} h_{i} & D_{q} h_{i}
\end{array}\right)_{(p, q)=\widetilde{\gamma}_{i}(t)} d t\right) \\
& \quad=\operatorname{det}\left(\begin{array}{ll}
1 & h_{i}\left(B_{i-1}, z_{i-1}\right) \\
0 & g_{i}\left(B_{i-1}, z_{i-1}\right)
\end{array}\right) \exp \left(\int_{0}^{t_{i}}\left(D_{p} g_{i}+D_{q} h_{i}\right)\left(\widetilde{\gamma}_{i}(t)\right) d t\right), \\
& \quad=g_{i}\left(B_{i-1}, z_{i-1}\right) \exp \left(\int_{0}^{t_{i}}\left(D_{p} g_{i}+D_{q} h_{i}\right)\left(\widetilde{\gamma}_{i}(t)\right) d t\right) . \tag{48}
\end{align*}
$$

By (47) and (48), we then obtain (42).

## 3. Proofs of the Criteria

Note that Theorem 2.5 is a generalization of Theorems 2.4 and 2.3. While Theorem 2.5 can be proved without relying on the results of the other theorems, for clarity we prove Theorem 2.3 first in Section 3.1, and then prove the general Theorem 2.5 in Section 3.2.
3.1. Proof of Theorem 2.3. In this section we assume $m=1$ for system (1), namely $(p, z) \in \mathbb{R}^{n} \times \mathbb{R}$. For each $i=1,2, \ldots, N$, on curve $\gamma_{i}=\left\{\left(\theta_{i}(t), \rho_{i}(t)\right\}\right.$ from Assumption 2 the function $\rho_{i}$ is non-constant, so we can choose a point $\left(p_{0 i}, q_{0 i}\right) \in \gamma_{i}$ at which $\dot{\rho}_{i} \neq 0$. Let $\Gamma_{i}$ be a cross section of $\gamma_{i}$ at a point $\left(p_{0 i}, z_{0 i}\right)$ of the form

$$
\Gamma_{i}=\left\{(p, z):\left|p-p_{0 i}\right|<\delta_{0}, z=q_{0 i}\right\}
$$

where $\delta_{0}>0$ is to be determined. Our strategy is to track trajectories that evolve from $\Gamma_{i}$ along the flow (1) and reach $\Gamma_{i+1}$ near the configuration $\gamma_{i} \cup \sigma_{i} \cup \gamma_{i+1}$. We set a cross section $\Sigma_{i}$ of $\sigma_{i}$ and analyze the dynamics between $\Gamma_{i}$ and $\Sigma_{i}$. By symmetry, the dynamics between $\Sigma_{i}$ and $\Gamma_{i+1}$ can also be treated. We will choose two cross sections, $\mathscr{A}_{i}^{\text {in }}$ and $\mathscr{A}_{i}^{\text {out }}$, near $A_{i}$ to analyze the transition map from $\Gamma_{i}$ to $\Sigma_{i}$. A list a symbols in this proof is given in Table 1. Note that we use the notation $\kappa_{\epsilon i}^{(j k)}$ for several $\epsilon$ dependent charts. We denote $\kappa_{\epsilon i}^{(k j)}$ the inverse of $\kappa_{\epsilon i}^{(j k)}$, and denote $\kappa_{\epsilon i}^{(j l)}=\kappa_{\epsilon i}^{(j k)} \circ \kappa_{\epsilon i}^{(k l)}$, whenever they are defined.

Table 1. Notations in Section 3.1.

| Variables | Charts | Objects |
| :--- | :--- | :--- |
| $(p, z) \in \Omega$ <br> $=\mathbb{R}^{n} \times\left(z_{\min }, z_{\max }\right)$ | $\kappa_{\epsilon i}^{(12)}(p, z, \zeta)=(p, z)$ | $\Omega, \Gamma_{i}$ |
| $p \in \mathbb{R}^{\text {m }}$ | $\kappa_{\epsilon i}^{(13)}(p, \zeta)=(p, z)$ |  |
| $(p, z, \zeta) \in \Omega \times \mathbb{R}_{+}$ |  | $\kappa_{\epsilon i}^{(21)}(p, z)=(p, z, \zeta)$ |
|  | $\kappa_{\epsilon i}^{(23)}(p, \zeta)=(p, z, \zeta)$ | $\widetilde{\mathscr{A}}_{i}, \widetilde{\mathscr{A}}_{i}^{\text {in }}, \widetilde{\mathscr{A}}_{i}^{\text {out }}$ |
| $(p, \zeta) \in \mathbb{R}^{n} \times \mathbb{R}_{+}$ | $\kappa_{\epsilon i}^{(31)}(p, z)=(p, \zeta)$ | $\widehat{\mathscr{A}}_{i}^{\text {out }}, \widehat{\Sigma}_{i}$ |
|  | $\kappa_{\epsilon i}^{(32)}(p, z, \zeta)=(p, \zeta)$ |  |

Let $\omega_{i}, 1 \leq i \leq N$, be the numbers defined in (20) for $m=1$, which means $\omega_{i}=\omega_{i}^{(1)}$. By Assumption 4, in a neighborhood of $\left(A_{i}, z_{i}\right)$, for $\delta_{1}>0$ sufficiently small, there is a unique point $\left(p_{i}^{\text {in }}, z_{i}+\omega_{i} \delta_{1}\right)$ that lies on the curve $\gamma_{i}$. Here $\mathbb{B}(p, r)$ is the open ball centered at $p$ with radius $r$. Let

$$
\begin{equation*}
\mathscr{A}_{i}^{\text {in }}=\left\{(p, z): p \in \mathbb{B}\left(p_{i}^{\text {in }}, \delta_{2}\right), z=z_{i}+\omega_{i} \delta_{1}\right\} \tag{49}
\end{equation*}
$$

where $\delta_{1}$ and $\delta_{2}$ are positive constants to be determined.
Proposition 3.1. Let $\Gamma_{i}$ and $\mathscr{A}_{i}^{\text {in }}$ be defined as in the preceding paragraphs. For fixed $\delta_{1}>0$ and $\delta_{2}>0$, if $\delta_{0}>0$ is sufficiently small, then the transition map $\Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}{ }^{\text {in }}}$ from $\Gamma_{i}$ to $\mathscr{A}_{i}^{\text {in }}$ for system (1) is well-defined for all small $\epsilon \geq 0$. Moreover,

$$
\left\|\Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}-\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}\right\|_{C^{1}\left(\Gamma_{i}\right)}=O(\epsilon) \quad \text { as } \epsilon \rightarrow 0
$$

that is, $\Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ is $O(\epsilon)$-close to $\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ in the $C^{1}\left(\Gamma_{i}\right)$-norm as $\epsilon \rightarrow 0$.
Proof. Since (1) is a regular perturbation of (3), the results follow directly from regular perturbation theory.

Next we investigate the dynamics near $\sigma_{i}$. Let $\Omega=\mathbb{R}^{n} \times\left(z_{\min }, z_{\max }\right)$. We define an $\epsilon$-dependent chart $\kappa_{\epsilon i}^{(31)}$ on $\Omega$ by

$$
\kappa_{\epsilon i}^{(31)}(p, z)=(p, \zeta) \quad \text { with } \quad \zeta=\epsilon \ln \left(\frac{\omega_{i}}{z-z_{i}}\right)
$$

In this chart system (1) is converted to

$$
\begin{align*}
& p^{\prime}=f(p, z, \epsilon)+h(p, z, \epsilon) / \epsilon \\
& \zeta^{\prime}=-\omega_{i} \frac{g(p, z, \epsilon)}{z-z_{i}}  \tag{50}\\
& \text { where } z=z_{i}+\epsilon \omega_{i} \exp \left(-\zeta_{i} / \epsilon\right)
\end{align*}
$$

Formally, the limit of (50) as $\epsilon \rightarrow 0$ with $z=z_{i}+o(\epsilon)$ is

$$
\begin{align*}
p^{\prime} & =f\left(p, z_{i}, 0\right) \\
\zeta^{\prime} & =-\omega_{i} \frac{\partial g}{\partial z}\left(p, z_{i}, 0\right) \tag{51}
\end{align*}
$$

Let $\widehat{\Phi}_{i}$ to be the solution operator of (51). Let

$$
\begin{equation*}
\mathscr{A}_{i}=\mathbb{B}\left(A_{i}, \delta_{4}\right) \tag{52}
\end{equation*}
$$

and

$$
\begin{equation*}
\widehat{\mathscr{A}}_{i}^{\text {out }}=\widehat{\Phi}_{i}\left(\mathscr{A}_{i} \times\{0\}, \delta_{3}\right), \tag{53}
\end{equation*}
$$

where $\delta_{3}>0$ and $\delta_{4}>0$ are constants to be determined. Let $\widehat{\sigma}_{i}(\tau)=\widehat{\Phi}_{i}\left(\left(A_{i}, \zeta_{i}\right), \tau\right)$, $0 \leq \tau \leq T_{i}$. Let $\widehat{\Sigma}_{i}$ be a cross section of the curve $\widehat{\sigma}_{i}$ at $\widehat{\sigma}_{i}\left(T_{i} / 2\right)$ in $\mathbb{R}^{n} \times \mathbb{R}$. We denote $\Pi_{0 \widehat{\mathscr{A}}_{i} \text { out }}^{\widehat{\widehat{N}}}$ the transition map from $\widehat{\mathscr{A}}_{i}^{\text {out }}$ to $\widehat{\Sigma}_{i}$ following the flow of (51).
Proposition 3.2. Let $\mathscr{A}_{i}$ and $\widehat{\mathscr{A}}_{i}^{\text {out }}$ be defined as in the preceding paragraphs. For fixed $\delta_{3}>0$, if $\delta_{4}>0$ is sufficiently small, then the transition map $\Pi_{\epsilon \widehat{\mathscr{A}}_{i}^{\text {out }}}^{\widehat{\Sigma}}$ from $\widehat{\mathscr{A}}_{i}^{\text {out }}$ to $\widehat{\Sigma}_{i}$ for system (50) is well-defined for all small $\epsilon>0$. Moreover, $\Pi_{\epsilon \widehat{\mathcal{A}}_{i} \text { out }}^{\widehat{\Sigma}}$ is $O(\epsilon)$-close to $\Pi_{0 \widehat{\Sigma} \widehat{\mathscr{A}_{i}} \text { out }}$ in the $C^{1}\left(\widehat{\mathscr{A}}_{i}^{\text {out }}\right)$-norm as $\epsilon \rightarrow 0$.

Proof. Let $\Sigma$ be the image of $\widehat{\Sigma}$ via the projection $(p, \zeta) \mapsto p$. Since the trajectory $\sigma_{i}$ of (4) connects $A_{i}$ and $\Sigma_{i}$, we can choose $\Delta>0$ such that the transition map from $\mathscr{A}_{i}$ to $\Sigma_{i}$ whenever $\delta_{4}>0$ is sufficiently small.

Note that the $p$-component of $\widehat{\Phi}_{i}\left(A_{i}, \tau\right)$ equals $\sigma_{i}(\tau)=\Phi_{i}\left(A_{i}, \tau\right)$ in Assumption 3. Also note that Assumption 5 gives

$$
\inf \left\{\zeta:(p, \zeta) \in \widehat{\Phi}_{i}\left(\left(A_{i}, 0\right), \tau\right), \tau \in\left[\delta_{3}, \tau_{i}-\delta_{3}\right]\right\}>0
$$

Therefore, by decreasing $\Delta$ if necessary, for $\mathscr{A}_{i}$ defined by (52) with $\delta_{3} \in(0, \Delta)$,

$$
\begin{equation*}
\inf \left\{\zeta:(p, \zeta) \in \widehat{\Phi}_{i}\left(\left(p_{0}, 0\right), \tau\right), p_{0} \in \mathscr{A}_{i}, \tau \in\left[\delta_{3}, \tau_{i}-\delta_{3}\right]\right\}>C \tag{54}
\end{equation*}
$$

for some $C>0$. Substituting (54) into (50), we have

$$
\begin{align*}
p^{\prime} & =f\left(p, z_{i}, 0\right)+O\left(\epsilon+e^{-C / \epsilon} / \epsilon\right) \\
\zeta^{\prime} & =-\omega_{i} \frac{\partial g}{\partial z}\left(p, z_{i}, 0\right)+O(\epsilon) \tag{55}
\end{align*}
$$

Hence (50) is a regular perturbation of (51) in a neighborhood of the set

$$
\left\{\widehat{\Phi}(x, \tau): x \in \widehat{\mathscr{A}}_{i}^{\text {out }}, \tau \in\left[0, \tau_{i}-2 \delta_{3}\right]\right\}
$$

Therefore, by regular perturbation theory, $\Pi_{\epsilon \widehat{\mathscr{A}}_{i}^{\text {out }}}^{\widehat{N}}$ is well-defined for small $\epsilon>0$ and is $O(\epsilon)$-close to $\Pi_{0 \widehat{\mathcal{A}}}^{\widehat{\Sigma} \widehat{\mathscr{A}}_{i}^{\text {out }}}$ in the $C^{1}\left(\widehat{\mathscr{A}}_{i}^{\text {out }}\right)$-norm as $\epsilon \rightarrow 0$.

Finally we investigate the dynamics near the joint of $\gamma_{1}$ and $\sigma_{i}$. We define

$$
\kappa_{\epsilon i}^{(21)}(p, z)=(p, z, \zeta) \quad \text { with } \quad \zeta=\epsilon \ln \left(\frac{\omega_{i}}{z-z_{i}}\right) \quad \text { for } \quad(p, z) \in \Omega, \epsilon \geq 0
$$

Note that $\kappa_{\epsilon i}^{(21)}(p, z)=(p, z, \zeta)$ can be obtained by appending $z$ to $\kappa_{\epsilon i}^{(31)}(p, z)=(p, \zeta)$. The transformation $\kappa_{\epsilon i}^{(21)}$ converts system (1) to

$$
\begin{align*}
\dot{p} & =\epsilon f(p, z, \epsilon)+h(p, z, \epsilon), \\
\dot{z} & =g(p, z, \epsilon)  \tag{56}\\
\dot{\zeta} & =-\epsilon \omega_{i} \frac{g(p, z, \epsilon)}{z-z_{i}}
\end{align*}
$$

We define

$$
\begin{equation*}
\tilde{\mathscr{A}}_{\epsilon i}^{\text {in }}=\kappa_{\epsilon i}^{(12)}\left(\mathscr{A}_{i}^{\text {in }}\right) \quad \text { for } \epsilon \geq 0 \tag{57}
\end{equation*}
$$

which means

$$
\widetilde{\mathscr{A}}_{\epsilon i}^{\mathrm{in}}=\left\{(p, z, \zeta): p \in \mathbb{B}\left(p_{0 i}^{\mathrm{in}}, \delta_{2}\right), z=z_{i}+\omega_{i} \delta_{1}, \zeta=\epsilon \ln \delta_{1}\right\} .
$$

Note that $\kappa_{0 i}^{(21)}(p, z)=(p, z, 0)$ for all $(p, z) \in \mathscr{A}_{i}^{\text {in }}$.
Taking $\epsilon \rightarrow 0$ in (56) leads to the system (3) companioned with $\dot{\zeta}=0$. By Assumptions 2 and 4, the projection

$$
\Pi_{0 \mathscr{A}_{i}^{\text {in }}}^{\mathscr{A}_{i}}: \mathscr{A}_{i}^{\text {in }} \rightarrow \mathscr{A}_{i} \times\left\{z_{i}\right\}
$$

following the flow of (3) is well-defined and is a local homeomorphism. We define $\Pi_{0 \tilde{\mathscr{A}}_{0 i} \tilde{\mathscr{A}}_{0}{ }^{\text {in }}}=\Pi_{0 \mathscr{A}_{i}{ }^{\text {in }}}^{\mathscr{A}_{i}} \times$ id, which means

$$
\Pi_{0 \tilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\tilde{\widetilde{A}}_{i}}\left(p, z, \zeta_{i}\right)=\left(\Pi_{0 \mathscr{A}_{i}}^{\mathscr{A}_{i}}(p, z), \zeta_{i}\right) .
$$

In the slow time variable $\tau=\epsilon t$, taking $\epsilon \rightarrow 0$ in (56) with $z=z_{i}+o(\epsilon)$ leads to (51) appended by the equation $z=z_{i}$. We define $\widetilde{\Phi}_{i}\left(\left(p, z_{i}, \zeta\right), \tau\right)$ on $\widetilde{\mathscr{A}}_{0 i} \times\left[0, \tau_{i}\right]$ to be the image of $\widehat{\Phi}((p, \zeta), \tau)$ in the space $\left\{(p, z, \zeta): z=z_{i}\right\}$. Also we define

$$
\widetilde{\mathscr{A}}_{\epsilon i}^{\text {out }}=\kappa_{\epsilon i}^{(23)}\left(\widehat{\mathscr{A}}_{i}^{\text {out }}\right) \quad \text { for } \epsilon>0
$$

Note that

$$
\Pi_{0 \tilde{\mathscr{A}}_{0 i}}^{\tilde{\mathscr{A}}_{0}^{\text {out }}}=\widetilde{\Phi}_{i}\left(\cdot, \delta_{4}\right) .
$$

$\underset{\sim}{\text { Proposition 3.3. There exists }} \Delta>0$ such that the following assertions hold. Let $\widetilde{\mathscr{A}}_{i}^{\text {in }}$ and $\widetilde{\mathscr{A}}_{i}^{\text {out }}$ be defined as in the preceding paragraphs with $\delta_{j}<\Delta, j=1,2,3$, then for all sufficiently small $\delta_{4}>0$, the transition map $\Pi_{\epsilon \tilde{\mathscr{A}}_{\epsilon i} \text { in }}^{\tilde{\mathscr{A}}_{i}^{\text {out }}}$ from $\widetilde{\mathscr{A}}_{\epsilon i}^{\text {in }}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text {out }}$ following the flow of (56) is well-defined for all small $\epsilon>0$. Moreover,

$$
\begin{equation*}
\left\|\Pi_{\epsilon \epsilon \tilde{\mathscr{A}}_{\epsilon i}^{\text {in }}}^{\tilde{\mathscr{A}}_{\text {int }}^{\text {out }}} \circ \kappa_{\epsilon i}^{(21)}-\Pi_{0 \mathscr{\mathscr { A }}_{0 i}}^{\tilde{\mathscr{A}}_{i}^{\text {out }}} \circ \Pi_{0 \tilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\tilde{\mathscr{A}}_{0 i}} \circ \kappa_{0 i}^{(21)}\right\|_{C^{1}\left(\mathscr{A}_{i}^{\text {in }}\right)}=O(\epsilon) \tag{58}
\end{equation*}
$$

as $\epsilon \rightarrow 0$.
A schematic diagram representing Proposition 3.3 is shown in Figure 5. The significance in estimate (58) is that the transition map $\Pi_{\epsilon \mathscr{A}_{\epsilon i}^{\text {in }}}^{\mathscr{A}_{\epsilon i}^{\text {out }}}$ can be approximated by the
 systems. To prove Proposition 3.3, we need the following lemma, which is a variation of the Exchange Lemma in Jones and Tin [29] and Schecter [47].


Figure 5. A schematic diagram representing Proposition 3.3. Here $\hookrightarrow$ indicates injection and $\rightsquigarrow$ indicates the limit as $\epsilon \rightarrow 0$. The transition map from $\widetilde{\mathscr{A}_{\epsilon i}^{\text {in }}}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text {out }}$ along (56) is approximated by the composition function of the transition maps for the limiting systems.

Lemma 3.4. Consider a system for $(a, b) \in \mathbb{R}^{n} \times \mathbb{R}, N \geq 1$, of the form

$$
\begin{align*}
& \dot{a}=\epsilon f(a, b, \epsilon)+b h(a, b, \epsilon) \\
& \dot{b}=b g(a, b, \epsilon) \tag{59}
\end{align*}
$$

where $\cdot$ denotes $\frac{d}{d t}$, and $f, g$ and $h$ are smooth functions. Assume

$$
\begin{equation*}
\sup g(a, b, \epsilon)<0 \tag{60}
\end{equation*}
$$

Assume that for some $\bar{a} \in \mathbb{R}^{N}$ the point $(\bar{a}, 0)$ is the omega limit point of a trajectory $\gamma$ of the system

$$
\begin{align*}
& \dot{a}=b h(b, c, 0) \\
& \dot{b}=b g(b, c, 0), \tag{61}
\end{align*}
$$

Then there exists $\Delta>0$ such that the following assertions hold.

Let $\left\{\mathscr{A}_{\epsilon}^{\mathrm{in}}\right\}_{\epsilon \in\left[0, \epsilon_{0}\right]}$ be a smooth family of $\ell$-dimensional manifolds, $0 \leq \ell \leq N$, that intersects $\gamma$ at a point in $\mathbb{B}\left(\left(a_{0}, 0\right), \Delta\right)$. Let $\Lambda \subset \mathbb{R}^{n}$ be the projection of $\mathscr{A}_{0}^{\text {in }}$ along the flow of system (61). Let $\Phi$ the solution operator for the system

$$
\begin{equation*}
\frac{d}{d \tau} a=f(a, 0,0) \tag{62}
\end{equation*}
$$

Assume the following conditions hold.
(i) $\mathscr{A}_{0}^{\text {in }}$ is non-tangential to the flow of (61);
(ii) $\bar{a} \in \Lambda$ and $\Lambda$ is compact and is non-tangential to the flow of (62);
(iii) The trajectory $\sigma=\Phi\left(\left[0, \tau_{1}\right], \bar{a}\right)$, where $\tau_{1}>0$, lies in $\mathbb{B}\left(a_{0}, \Delta\right)$ and is rectifiable and not self-intersecting.
Let $\iota_{\epsilon}: K \rightarrow \mathscr{A}_{\epsilon}^{\text {in }}$ be a smooth parameterization of $\mathscr{A}_{\epsilon}{ }^{\text {in }}$ for $\epsilon \in\left[0, \epsilon_{0}\right]$, where $K$ is an $\ell$-dimensional manifold. Let $\bar{x} \in \mathscr{A}_{0} \cap \gamma$ be the pre-image of $\bar{a}$ along (61) and $\bar{k} \in K$ be the pre-image of $\bar{x}$ by $\iota_{0}$.

If $\mathscr{A}^{\text {out }}$ is an $n$-dimensional manifold that intersects transversally at an interior point of $\sigma$, then there is an open neighborhood $V$ of $\bar{k}$ in $K$ such that the transition map $\prod_{\epsilon \mathscr{A}_{\epsilon} \mathscr{A}_{\text {in }}^{\text {in }}}^{\text {out }}$ from $\iota_{\epsilon}(V) \subset \mathscr{A}_{\epsilon}^{\text {in }}$ to $\mathscr{A}^{\text {out }}$ following the flow of (59) is well defined for all sufficiently small $\epsilon>0$. Moreover,

$$
\begin{equation*}
\left\|\Pi_{\epsilon \mathscr{A}_{\epsilon}}^{\mathscr{A}_{\epsilon}^{\text {out }}} \circ \iota_{\epsilon}-\Pi_{0 \Lambda}^{\mathscr{A}^{\text {out }}} \circ \Pi_{0 \mathscr{A}_{0}}^{\Lambda} \circ \iota_{0}\right\|_{C^{1}(V)}=O(\epsilon) \tag{63}
\end{equation*}
$$

as $\epsilon \rightarrow 0$, where $\Pi_{0 \mathscr{A}_{0}}^{\Lambda}$ is the transition map from $\mathscr{A}_{0}$ to $\Lambda$ along the flow of (61), and $\Pi_{0 \Lambda}^{\mathscr{A}^{\text {out }}}$ is the transition map from $\Lambda$ to $\mathscr{A}^{\text {out }} \cap\{b=0\}$ along the flow of (62).

Proof of Lemma 3.4. Using a Fenichel type coordinate (see Jones [28]), in the open ball $\mathbb{B}(0,2 \Delta)$ in the $(a, b)$-space, for sufficiently small $\Delta>0$ we can choose an $\epsilon$ dependent change of variable $(a, b) \mapsto(\tilde{a}, \tilde{b})$ with

$$
\left.(\tilde{a}, \tilde{b})\right|_{b=0}=(a, 0)
$$

such that, after dropping the tilde symbol, system (59) is converted to

$$
\begin{align*}
& \dot{a}=\epsilon f(a, \epsilon), \\
& \dot{b}=b g(a, b, \epsilon) . \tag{64}
\end{align*}
$$

We write

$$
\mathscr{A}_{\epsilon}^{\mathrm{in}}=\left\{(a, b): a \in \Lambda, b=\beta_{\epsilon}(a)\right\} .
$$

Since $\mathscr{A}^{\text {out }}$ intersects $\sigma$ transversally, for some neighborhood $U$ of $\bar{a}$ in $\mathbb{R}^{n}$, we can write

$$
\Pi_{0 \Lambda}^{\mathscr{A}^{\text {out }}}(a)=\Phi\left(a, T_{0}(a)\right) \quad \forall a \in \Lambda \cap U
$$

where $T_{0}$ is a smooth function with $\tau_{-}<T_{0}<\tau_{+}$for some $\tau_{-}, \tau_{+} \in\left(0, \tau_{1}\right)$. To prove (63), it suffices to show that

$$
\begin{equation*}
\left\|\Pi_{\epsilon \mathscr{A}_{\epsilon}^{\text {in }}}^{\mathscr{A}^{\text {out }}}\left(a, \beta_{\epsilon}(a)\right)-\left(\Phi\left(a, T_{0}(a)\right), 0\right)\right\|_{C^{1}(\Lambda \cap U)}=O(\epsilon) \tag{65}
\end{equation*}
$$

as $\epsilon \rightarrow 0$. Let $\left(a_{\epsilon}, b_{\epsilon}\right)\left(t ; a_{0}\right)$ be the solution of (64) at time $t$ with initial data $\left(a_{0}, \beta_{\epsilon}\left(a_{0}\right)\right)$. Define

$$
\begin{equation*}
\left(a_{\epsilon 1}, b_{\epsilon 1}\right)\left(a_{0}, \tau\right)=\left(a_{\epsilon}, b_{\epsilon}\right)\left(\tau / \epsilon ; a_{0}\right) \quad \text { for } a_{0} \in \Lambda_{1}, \tau \in\left[\tau_{-}, \tau_{+}\right] . \tag{66}
\end{equation*}
$$

By the General Exchange Lemma (see Schecter [47]),

$$
\begin{equation*}
\left\|\left(a_{\epsilon 1}, b_{\epsilon 1}\right)\left(a_{0}, \tau\right)-\left(\Phi\left(a_{0}, \tau\right), 0\right)\right\|_{C^{1}\left(\Lambda_{1} \times\left[\tau_{-}, \tau_{+}\right]\right)}=O(\epsilon) \tag{67}
\end{equation*}
$$

as $\epsilon \rightarrow 0$. Since the graph of $\left(\Phi\left(a_{0}, \tau\right), 0\right)$ is transversal to $\mathscr{A}^{\text {out }}$, it follows from the Implicit Function Theorem that there exists a function $T_{\epsilon}\left(a_{0}\right)$ defined for all small $\epsilon>0$ such that

$$
\begin{equation*}
\left\|T_{\epsilon}-T_{0}\right\|_{C^{1}(\Lambda \cap U)}=O(\epsilon) \tag{68}
\end{equation*}
$$

and

$$
\left(a_{\epsilon 1}, b_{\epsilon 1}\right)\left(a_{0}, T_{\epsilon}\left(a_{0}\right)\right) \in \mathscr{A}^{\text {out }} \quad \forall a_{0} \in \Lambda \cap U
$$

Note that the last relation means

$$
\begin{equation*}
\Pi_{\epsilon \mathscr{A}}^{\mathscr{A}^{\text {in }}} \stackrel{\text { out }}{ }\left(a_{0}, \delta\right)=\left(a_{\epsilon 1}, b_{\epsilon 1}\right)\left(a_{0}, T_{\epsilon}\left(a_{0}\right)\right) \tag{69}
\end{equation*}
$$

From (67), (68) and (69) we then obtain (65).
Proof of Proposition 3.3. Note that (59) can be written as

$$
\begin{aligned}
& \dot{a}=\epsilon f(a, b, 0)+b h(a, b, 0)+O\left(|(a, b)|^{2}\right) \\
& \dot{b}=b g(a, b, 0)+O\left(|(a, b)|^{2}\right)
\end{aligned}
$$

For system (56), setting $s=z-z_{i}$ yields

$$
\begin{aligned}
\dot{p} & =\epsilon f\left(p, z_{i}+s, 0\right)+O\left(|(\epsilon, s)|^{2}\right) \\
\dot{s} & =s \frac{\partial g}{\partial z}\left(p, z_{i}, 0\right)+O\left(|(\epsilon, s)|^{2}\right) \\
\dot{\zeta} & =\epsilon \frac{\partial g}{\partial z}\left(p, z_{i}, 0\right)+O\left(|(\epsilon, s)|^{2}\right)
\end{aligned}
$$

as $(\epsilon, s) \rightarrow 0$. Since $\frac{\partial g}{\partial z}\left(A_{i}, z_{i}, 0\right)<0$ by Assumption 5, applying Lemma 3.4 with $b=z$ and $a=(p, s)$ we obtain (58).

Also we denote $\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}$ the transition map from $\Gamma_{i}$ to $\mathscr{A}_{i} \times\left\{z_{i}\right\}$ along the flow of (3) and $\Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{\Sigma}_{i}}$ the transition map from $0 \widehat{\mathscr{A}}_{i}$ to $\widehat{\Sigma}_{i}$ along the flow of (51).
Proposition 3.5. There exist $\delta_{j}>0,0 \leq j \leq 4$, such that if $\Gamma_{i}, \mathscr{A}_{i}, \Sigma_{i}$ are defined in the preceding paragraphs, then the transition map $\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}$ from $\Gamma_{i}$ to $\Sigma_{i}$ following the flow of (1) is well-defined for all small $\epsilon>0$, and

$$
\begin{equation*}
\left\|\kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}-\Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}\right\|_{C^{1}\left(\Gamma_{i}\right)}=O(\epsilon) \tag{70}
\end{equation*}
$$

as $\epsilon \rightarrow 0$.
Proof of Proposition 3.5. First we fix constants $\delta_{1}, \delta_{2}$ and $\delta_{3}$ in $(0, \Delta)$, where $\Delta$ is the numbers in Propositions 3.3. Then we choose positive constants $\delta_{0}$ and $\delta_{4}$, such that the results in Propositions 3.1 and 3.2 hold. Then

$$
\begin{aligned}
\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}} & =\Pi_{\epsilon \mathscr{A}_{i}^{\text {out }}}^{\Sigma_{i}} \circ \Pi_{\epsilon \mathscr{A}_{i}^{\text {in }}}^{\mathscr{A}_{i}^{\text {out }}} \circ \Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}} \\
& =\left(\kappa_{\epsilon i}^{(13)} \circ \Pi_{\epsilon \widehat{\mathscr{A}}_{i} \text { out }}^{\widehat{\Sigma}_{i}} \circ \kappa_{\epsilon i}^{(31)}\right) \circ\left(\kappa_{\epsilon i}^{(12)} \circ \Pi_{\epsilon \mathscr{A}_{\epsilon i}^{\text {in }}}^{\tilde{\mathscr{A}}_{i}^{\text {out }}} \circ \kappa_{\epsilon i}^{(21)}\right) \circ \Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}
\end{aligned}
$$

From Propositions 3.1, 3.2 and 3.3, it follows that

$$
\begin{aligned}
& \Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}=\left(\kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\mathscr{A}}_{i}^{\text {out }}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)}\right) \circ\left(\kappa_{0 i}^{(12)} \circ \Pi_{0 \tilde{\mathscr{A}}_{i}}^{\widetilde{\mathscr{A}}_{i}^{\text {out }}} \circ \Pi_{0 \tilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\tilde{\mathscr{A}}_{0}} \circ \kappa_{0 i}^{(21)}\right)+O(\epsilon) \\
& =\kappa_{0 i}^{(13)} \circ\left(\Pi_{0 \widetilde{\mathscr{A}}_{i} \text { out }}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(32)} \circ \Pi_{0 \widetilde{\mathscr{A}}_{i}}^{\widetilde{\mathscr{A}}_{i}^{\text {out }}}\right) \circ\left(\Pi_{0 \widetilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\widetilde{\mathscr{A}}_{0}} \circ \kappa_{0 i}^{(21)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}\right)+O(\epsilon) .
\end{aligned}
$$

Since

$$
\Pi_{0 \widehat{\mathscr{A}}_{i} \text { out }}^{\widehat{S}_{i}} \circ \kappa_{0 i}^{(32)} \circ \Pi_{0 \widetilde{\mathscr{A}}_{i}}^{\widetilde{\mathscr{A}}_{i}^{\text {out }}}=\Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{L}_{i}} \circ \kappa_{0 i}^{(23)}
$$

and

$$
\Pi_{0 . \tilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\tilde{S}_{0}} \circ \kappa_{0 i}^{(21)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{O}_{i}^{\text {in }}}=\kappa_{0 i}^{(21)} \circ \Pi_{0 \Gamma_{i}}^{\alpha_{i}},
$$

it follows that

$$
\begin{aligned}
\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}} & =\kappa_{0 i}^{(13)} \circ\left(\Pi_{0 . \widehat{\mathscr{R}_{i}}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(32)}\right) \circ\left(\kappa_{0 i}^{(21)} \circ \Pi_{0 \Gamma_{i}}^{\left(\mathscr{A}_{i}\right.}\right)+O(\epsilon) \\
& =\kappa_{0 i}^{(13)} \circ \Pi_{0 . \mathscr{A ⿱ 八 厶 ⿻ 丷 木}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{S}_{i}}+O(\epsilon) .
\end{aligned}
$$

Applying both sides of equation by $\kappa_{0 i}^{(31)}$ yields（70）．
Proof of Theorem 2．3．By a reversal of the time variable，applying Proposition 3.5 we obtain

$$
\left\|\kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon \Gamma_{i+1}}^{\Sigma_{i}}-\Pi_{0 \mathscr{\mathscr { B }}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i+1}}^{\mathscr{B}_{i_{i}}}\right\|_{C^{1}\left(\Gamma_{i+1}\right)}=O(\epsilon) .
$$

Taking the inverse of the mappings we obtain

$$
\begin{equation*}
\left\|\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}} \circ \kappa_{\epsilon i}^{(13)}-\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\Sigma}_{i}}^{\widehat{B}_{i}}\right\|_{C^{1}\left(\widehat{\Sigma}_{i}\right)}=O(\epsilon) . \tag{71}
\end{equation*}
$$

By（70）and（71），it follows that

$$
\begin{align*}
& \Pi_{\epsilon \Gamma_{i}}^{\Gamma_{i+1}}=\left(\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}} \circ \kappa_{i \epsilon}^{(13)}\right) \circ\left(\kappa_{i \epsilon}^{(31)} \circ \Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}\right) \\
& =\left(\Pi_{0 \mathscr{\mathscr { B } _ { i }}}^{\Gamma_{i}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\Sigma_{i}}}^{\widehat{\mathscr{S}_{i}}}\right) \circ\left(\Pi_{0 \widehat{\mathscr{A}_{i}}}^{\widehat{\widetilde{\Sigma}}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{S}_{i}}\right)+O(\epsilon)  \tag{72}\\
& =\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \mathscr{\mathscr { A }}_{i}}^{\hat{\mathscr{F}}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}+O(\epsilon) \text {. }
\end{align*}
$$

Define $\varrho(p, z)=p$ ．Since we assumed $h=0$ in（3），

$$
\varrho \circ \Pi_{\epsilon \mathscr{B}_{i}}^{\mathscr{i}_{i+1}}(p, z)=p \quad \forall(p, z) \in \mathscr{B}_{i} .
$$

Hence（72）implies that

$$
\varrho \circ \Pi_{\epsilon \Gamma_{i}}^{\Gamma_{i+1}}=Q_{i}+O(\epsilon) .
$$

Let

$$
P_{\epsilon}=\Pi_{\epsilon \Gamma_{N}}^{\Gamma_{1}} \circ \cdots \circ \Pi_{\epsilon \Gamma_{2}}^{\Gamma_{3}} \circ \Pi_{\epsilon \Gamma_{1}}^{\Gamma_{2}} .
$$

Then

$$
\varrho \circ P_{\epsilon}=Q_{N} \circ \cdots \circ Q_{2} \circ Q_{1}+O(\epsilon)=P+O(\epsilon),
$$

where $P$ is defined by（18）．Since the $z$－component on $\Gamma_{1}$ is a constant，we conclude that

$$
\operatorname{det}\left(P_{\epsilon}-\mathrm{id}\right)=\operatorname{det}(D P-\mathrm{id})+O(\epsilon) .
$$

Hence the linearization of the return map at $p_{01} \in \Gamma_{1}$ does not have a singular value equal to 1 for all small $\epsilon>0$ if $\operatorname{det}(D P-\mathrm{id}) \neq 0$ ．Consequently，for all small $\epsilon>0$ there exists a locally unique fixed point $\left(p_{\epsilon 1}, z_{\epsilon 1}\right) \in \Gamma_{i}$ of $P_{\epsilon}$ ．The trajectory passing through $\left(p_{\epsilon 1}, z_{\epsilon 1}\right)$ is a periodic orbit of system（1）．If the spectrum radius of $D P\left(p_{01}, \zeta_{1}\right)$ is smaller（resp．greater）than 1 ，then $P_{\epsilon}$ is a contraction（resp．expansion），hence the periodic orbit is orbitally asymptotically stable（resp．unstable）．This proves the theorem．

Table 2. Notations in Section 3.2.

| Variables | Charts | Objects |
| :---: | :---: | :---: |
| $\begin{aligned} & (p, z) \in \Omega \subset \mathbb{R}^{n} \times \mathbb{R}^{m} \\ & \quad \text { with } z^{(j)} \in\left(z_{\min }^{(j)}, z_{\max }^{(j)}\right) \end{aligned}$ | $\begin{aligned} & \kappa_{\epsilon i}^{(01)}(p, q, \widehat{\zeta})=(p, z) \\ & \kappa_{\epsilon i}^{(03)}(p, \zeta)=(p, z) \end{aligned}$ | $\Omega, \bar{\Gamma}_{i}$ |
| $p \in \mathbb{R}^{m}$ |  | $\mathscr{A}_{i}, \mathscr{B}_{i}$ |
| $\begin{aligned} & (p, q, \widehat{\zeta}) \\ & \quad \in \mathbb{R}^{n} \times\left(z_{\min }^{(j)}, z_{\max }^{(j)}\right) \times \mathbb{R}_{+}^{m-1} \end{aligned}$ | $\begin{aligned} & \kappa_{\epsilon i}^{(10)}(p, z)=(p, q, \widehat{\zeta}) \\ & \kappa_{\epsilon i}^{(12)}(p, q, \zeta)=(p, z, \widehat{\zeta}) \end{aligned}$ | $\begin{aligned} & \Gamma_{i}, \\ & \mathscr{A}_{i}^{\text {in }}, \mathscr{A}_{i}^{\text {out }} \end{aligned}$ |
| $\begin{aligned} & (p, q, \zeta) \\ & \quad \in \mathbb{R}^{n} \times\left(z_{\min }^{(j)}, z_{\max }^{(j)}\right) \times \mathbb{R}_{+}^{m} \end{aligned}$ | $\begin{aligned} & \kappa_{\epsilon i}^{(21)}(p, q, \widehat{\zeta})=(p, q, \zeta) \\ & \kappa_{\epsilon i}^{(23)}(p, \zeta)=(p, q, \zeta) \end{aligned}$ | $\widetilde{\mathscr{A}}_{i}, \widetilde{\mathscr{A}}_{i}^{\text {in }}, \widetilde{\mathscr{A}}_{i}^{\text {out }}$ |
| $(p, \zeta) \in \mathbb{R}^{n} \times \mathbb{R}_{+}^{m}$ | $\begin{aligned} & \kappa_{\epsilon i}^{(30)}(p, z)=(p, \zeta) \\ & \kappa_{\epsilon i}^{(32)}(p, q, \zeta)=(p, \zeta) \end{aligned}$ | $\widehat{\mathscr{A}}_{i}^{\text {out }}, \widehat{\Sigma}_{i}$ |

3.2. Proof of Theorem 2.5. The approach in this section is to generalize the proof of Theorem 2.3. Some notations to be used are listed in Table 2.

Let

$$
\Omega=\mathbb{R}^{n} \times\left(z_{\min }^{(1)}, z_{\max }^{(1)}\right) \times \cdots \times\left(z_{\min }^{(N)}, z_{\max }^{(N)}\right) \subset \mathbb{R}^{n} \times \mathbb{R}^{m}
$$

We define the $\epsilon$-dependent chart on $\Omega$ by

$$
\kappa_{\epsilon i}^{(10)}(p, z)=\left(p, z^{\left(J_{i}\right)}, \widehat{\zeta}\right) \quad \text { with } \quad \widehat{\zeta}^{(j)}= \begin{cases}\zeta_{i}^{\left(J_{i}\right)}, & \text { if } j=J_{i} \\ \epsilon \ln \frac{\omega_{i}^{(j)}}{z^{(j)}-z_{i}^{(j)}}, & \text { if } j \neq J_{i}\end{cases}
$$

On the curve $\left(p_{i}(t), q_{i}(t)\right) \subset \mathbb{R}^{m} \times \mathbb{R}$ in Assumption 2, since $q_{i}(t)$ is non-constant, we can choose a point $\left(p_{0 i}, q_{0 i}\right)$ at which $q_{i}^{\prime}(t) \neq 0$. Let

$$
\Gamma_{i}=\left\{(p, q, \widehat{\zeta}) \in \mathbb{R}^{n} \times \mathbb{R} \times \Lambda_{i}:\left|p-p_{0 i}\right|<\delta_{0}, q=q_{0 i},\left|\widehat{\zeta}-\zeta_{i}\right|<\delta_{0}\right\}
$$

where $\delta_{0}>0$ is to be determined. Let $\bar{\Gamma}_{i}=\kappa_{\epsilon 1}^{(01)}\left(\Gamma_{i}\right)$. Our strategy is to track the transition map from $\Gamma_{i}$ to $\Gamma_{i+1}$ in the $(p, q, \zeta)$-space to find a fixed point of a composition map from $\Gamma_{1}$ to $\Gamma_{1}$ and then covert it back via $\kappa_{\epsilon 1}^{(01)}$ to obtain a periodic orbit passing through $\bar{\Gamma}_{1}$ in the $(p, z)$-space.

Let

$$
\mathscr{A}_{i}^{\text {in }}=\left\{(p, q, \widehat{\zeta}): p \in \mathbb{B}\left(p_{i}^{\text {in }}, \delta_{2}\right), q=z_{i}^{\left(J_{i}\right)}+\omega_{i} \delta_{1},\left|\widehat{\zeta}-\widehat{\zeta}_{i}\right|<\delta_{2}\right\} .
$$

where $\delta_{1}$ and $\delta_{2}$ are positive constants to be determined.
Proposition 3.6. Let $\Gamma_{i}$ and $\mathscr{A}_{i}^{\text {in }}$ be defined as in the preceding paragraphs. For fixed $\delta_{1}>0$ and $\delta_{2}>0$, if $\delta_{0}>0$ is sufficiently small, then the transition map $\Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ from $\Gamma_{i}$ to $\mathscr{A}_{i}^{\text {in }}$ following the flow of (1) is well-defined for all small $\epsilon \geq 0$ and is $O(\epsilon)$-close to $\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ in the $C^{1}\left(\Gamma_{i}\right)$-norm as $\epsilon \rightarrow 0$.

Proof. Chart $\kappa_{\epsilon i}^{(1)}$ converts system (1) to

$$
\begin{aligned}
& \dot{p}=\epsilon f(p, z, \epsilon)+h(p, z, \epsilon), \\
& \dot{q}=g^{\left(J_{i}\right)}(p, z, \epsilon),
\end{aligned}
$$

$$
\begin{align*}
& \dot{\widehat{\zeta}}^{(j)}=-\epsilon \frac{g^{(j)}(p, z, \epsilon)}{z^{(j)}-z_{i}^{(j)}}, \quad j \in\{1,2, \ldots, m\} \backslash\left\{J_{i}\right\}  \tag{73}\\
& \text { with } \quad z^{\left(J_{i}\right)}=q \quad \text { and } \quad z^{(j)}=z_{i}^{(j)}+\omega_{i}^{(j)} \exp \left(-\widehat{\zeta}^{(j)} / \epsilon\right) \text { for } j \neq J_{i}
\end{align*}
$$

By Assumption 5, all components of $\widehat{\zeta}_{i} \in \Lambda_{i}$ are bounded away from zero. Therefore, for each $j \in\{1,2, \ldots, m\} \backslash\left\{J_{i}\right\}$,

$$
z_{i}^{(j)}+\omega_{i}^{(j)} \exp \left(-\widehat{\zeta}^{(j)} / \epsilon\right) \rightarrow z_{i}^{(j)} \quad \text { as } \epsilon \rightarrow 0
$$

which implies

$$
\frac{g^{(j)}(p, z, 0)}{z^{(j)}-z_{i}^{(j)}} \rightarrow \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(p, z_{i-1}+q \mathrm{e}_{J_{i}}, 0\right) \quad \text { as } \epsilon \rightarrow 0
$$

Hence the expression of $\dot{\hat{\zeta}}^{(j)}$ in (73) tends to zero as $\epsilon \rightarrow 0$. Consequently, (73) is a regular perturbation of the system

$$
\begin{align*}
& \dot{p}=h\left(p, z_{i-1}+q \mathrm{e}_{J_{i-1}}, 0\right), \\
& \dot{q}=g^{\left(J_{i}\right)}\left(p, z_{i-1}+q \mathrm{e}_{J_{i-1}}, 0\right),  \tag{74}\\
& \dot{\hat{\zeta}}^{(j)}=0, \quad j \in\{1,2, \ldots, m\} \backslash\left\{J_{i}\right\} .
\end{align*}
$$

Hence $\Pi_{\epsilon \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ is well-defined and is $O(\epsilon) C^{1}$-close to $\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}^{\text {in }}}$ as $\epsilon \rightarrow 0$.
We define charts $\kappa_{\epsilon i}^{(30)}$ for $(p, z) \in \Omega$ by

$$
\begin{aligned}
\kappa_{\epsilon i}^{(30)}(p, z) & =(p, \zeta) \\
\text { with } \quad \zeta^{(j)} & =\epsilon \ln \frac{\omega_{i}^{(j)}}{z^{(j)}-z_{i}^{(j)}} \text { for } j=1,2 \ldots, m
\end{aligned}
$$

In this chart system (1) is converted to

$$
\begin{align*}
& \frac{d}{d \tau} p=f(p, z, \epsilon)+h(p, z, \epsilon) / \epsilon \\
& \frac{d}{d \tau} \zeta^{(j)}=\frac{-g^{(j)}(p, z, \epsilon)}{z^{(j)}-z_{i}^{(j)}}, \quad j=1,2, \ldots, m  \tag{75}\\
& \quad \text { with } \quad z^{(j)}=z_{i}^{(j)}+\omega_{i}^{(j)} \exp \left(-\zeta^{(j)} / \epsilon\right) \text { for } j=1,2, \ldots, m
\end{align*}
$$

Let $\widehat{\Phi}_{i}$ be the solution operator of

$$
\begin{align*}
& \frac{d}{d \tau} p=f\left(p, z_{i}, 0\right) \\
& \frac{d}{d \tau} \zeta^{(j)}=\frac{-\partial g^{(j)}}{\partial z^{(j)}}\left(p, z_{i}, 0\right) \quad \text { for } j=1,2, \ldots, m \tag{76}
\end{align*}
$$

Let $\mathscr{A}_{i}$ and $\mathscr{A}_{i}^{\text {in }}$ be defined by (52) and (49). We define

$$
\begin{equation*}
\widehat{\mathscr{A}}_{i}=\mathscr{A}_{i} \times \Lambda_{i} \quad \text { and } \quad \hat{\mathscr{A}}_{i}^{\text {out }}=\widehat{\Phi}_{i}\left(\hat{\mathscr{A}}_{i}, \delta_{3}\right) \tag{77}
\end{equation*}
$$

where $\delta_{3}>0$ is a constant to be determined. Let $\widehat{\sigma}_{i}(\tau)=\widehat{\Phi}_{i}\left(\left(A_{i}, \zeta_{i}\right), \tau\right), 0 \leq \tau \leq T_{i}$. Let $\widehat{\Sigma}_{i}$ be a cross section of the curve $\widehat{\sigma}_{i}$ at $\widehat{\sigma}_{i}\left(\tau_{i} / 2\right)$. We denote $\Pi_{0 \widehat{\mathscr{A}}_{i} \text { out }}^{\widehat{\widehat{~}}}$ the transition map from $\widehat{\mathscr{A}}_{i}^{\text {out }}$ to $\widehat{\Sigma}_{i}$ following the flow of (51).
Proposition 3.7. Let $\mathscr{A}_{i}$ and $\widehat{\mathscr{A}}_{i}^{\text {out }}$ be defined as in the preceding paragraphs. For fixed $\delta_{3}>0$, if $\delta_{4}>0$ is sufficiently small, then the transition map $\Pi_{\epsilon \widehat{\mathscr{A}}{ }_{i} \widehat{\mathrm{o}}^{\widehat{\alpha}}}$ from $\widehat{\mathscr{A}}_{i}^{\text {out }}$ to $\widehat{\Sigma}_{i}$ for system (75) is well-defined for all small $\epsilon>0$. Moreover, $\Pi_{\epsilon \widehat{\mathscr{A}}}^{\hat{\mathbb{A}}}{ }_{i}^{\text {out }}$ is $O(\epsilon)$-close to $\Pi_{0 \widehat{\mathscr{A}}}^{\widehat{\widehat{A}}}{ }_{i}^{\text {out }}$ in the $C^{1}\left(\widehat{\mathscr{A}}_{i}^{\text {out }}\right)$-norm as $\epsilon \rightarrow 0$.

Proof. By Assumption 5,

$$
\inf \left\{\zeta^{(j)}:(p, \zeta)=\widehat{\sigma}_{i}(\tau), \tau \in\left[\delta_{3}, \tau_{i}-\delta_{3}\right], j=1,2, \ldots, m\right\}>C
$$

for some $C>0$. Therefore, similar to the proof of Proposition 3.6, system (75) is a regular perturbation of (76), and the desired result follows.

Define chart $\kappa_{\epsilon i}^{(20)}$ for $(p, z) \in \Omega$ by

$$
\begin{aligned}
& \kappa_{\epsilon i}^{(20)}(p, z)=(p, q, \zeta) \\
& \text { with } q=z^{\left(J_{i}\right)} \text { and } z^{(j)}=z_{i}^{(j)}+\omega_{i}^{(j)} \exp \left(-\widehat{\zeta}^{(j)} / \epsilon\right) \text { for } j=1,2 \ldots, m
\end{aligned}
$$

Chart $\kappa_{\epsilon i}^{(20)}$ converts system (1) to

$$
\begin{align*}
& \dot{p}=\epsilon f(p, z, \epsilon)+h(p, z, \epsilon) \\
& \dot{q}=g^{\left(J_{i}\right)}(p, z, \epsilon) \\
& \zeta^{(j)}=\epsilon \frac{-g^{(j)}(p, z, \epsilon)}{z^{(j)}-z_{i}^{(j)}}, \quad j=1,2, \ldots, m  \tag{78}\\
& \text { with } \quad z^{(j)}=z_{i}^{(j)}+\omega_{i}^{(j)} \exp \left(-\widehat{\zeta}^{(j)} / \epsilon\right)
\end{align*}
$$

Here we temporarily ignore the relation $z^{\left(J_{i-1}\right)}=z_{i-1}^{\left(J_{i-1}\right)}+q$. Formally the limiting slow system of (78) at $z=z_{i}$ is

$$
\begin{align*}
& \frac{d}{d \tau} p=f\left(p, z_{i} 0\right) \\
& \frac{d}{d \tau} q=0  \tag{79}\\
& \frac{d}{d \tau} \zeta^{(j)}=\frac{-\partial g^{(j)}}{\partial z^{(j)}}\left(p, z_{i}, 0\right), \quad j=1,2, \ldots, m
\end{align*}
$$

Denote $\widetilde{\Phi}_{i}$ the solution operator for (79). Let $\mathscr{A}_{i}^{\text {in }}$ and $\widehat{\mathscr{A}}_{i}^{\text {out }}$ be the sets defined by (49) and (77). We define

$$
\tilde{\mathscr{A}}_{\epsilon i}^{\text {in }}=\kappa_{\epsilon i}^{(21)}\left(\mathscr{A}_{i}^{\text {in }}\right), \quad \tilde{\mathscr{A}}_{\epsilon i}^{\text {out }}=\kappa_{\epsilon i}^{(23)}\left(\hat{\mathscr{A}}_{i}^{\text {out }}\right) \quad \text { for } \epsilon \geq 0
$$

Note that

$$
\Pi_{0 \tilde{\mathscr{A}}_{0 i}}^{\substack{\tilde{\mathscr{A}}_{0 i}^{\text {out }}}}=\widetilde{\Phi}_{i}\left(\cdot, \delta_{3}\right) .
$$

Proposition 3.8. There exists $\Delta>0$ such that the following assertions hold. Let $\widetilde{\mathscr{A}}_{i}^{\text {in }}$ and $\widetilde{\mathscr{A}}_{i}^{\text {out }}$ be defined as in the preceding paragraphs with $\delta_{j}<\Delta, j=1,2,3$, then for all sufficiently small $\delta_{4}>0$, the transition map $\Pi_{\epsilon \tilde{\mathscr{A}}_{\epsilon i} \text { in }}^{\tilde{\mathscr{A}}_{\epsilon}^{\text {in }} \text { out }}$ from $\widetilde{\mathscr{A}}_{\epsilon i}^{\text {in }}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text {out }}$ following the flow of (78) is well-defined for all small $\epsilon>0$. Moreover,

$$
\begin{equation*}
\left\|\Pi_{\epsilon \mathscr{A}_{\epsilon i}^{\text {in }}}^{\tilde{\mathscr{A}}_{i}^{\text {in }}} \circ \kappa_{\epsilon i}^{(21)}-\Pi_{0 \tilde{\mathscr{A}}_{0 i}}^{\tilde{\mathscr{A}}_{0 i}^{\text {out }}} \circ \Pi_{0 \tilde{\mathscr{A}}_{0 i}^{\text {in }}}^{\tilde{\mathscr{A}}_{0 i}} \circ \kappa_{0 i}^{(21)}\right\|_{C^{1}\left(\mathscr{A}_{i}^{\text {in }}\right)}=O(\epsilon) . \tag{80}
\end{equation*}
$$

Proof. Note that we have $z^{\left(J_{i-1}\right)}=z_{i-1}^{\left(J_{i-1}\right)}+q$ when converting (1) to (78). Let $s=q-z_{i}^{(J-1)}+z_{i-1}^{\left(J_{i-1}\right)}$. By Assumption 1, (78) can be written as

$$
\begin{aligned}
& \dot{p}=\epsilon f\left(p, z_{i}, 0\right)+h\left(p, z_{i}+s \mathrm{e}_{J_{i-1}}, 0\right)+O\left(|(\epsilon, s)|^{2}\right) \\
& \dot{s}=g\left(p, z_{i}+s \mathrm{e}_{J_{i-1}}, 0\right)+O\left(|(\epsilon, s)|^{2}\right) \\
& \dot{\zeta}^{(j)}=-\epsilon \frac{\partial g^{(j)}}{\partial z^{(j)}}\left(p, z_{i}, 0\right)+O\left(|(\epsilon, s)|^{2}\right)
\end{aligned}
$$

as $(\epsilon, s) \rightarrow 0$. Since $\frac{\partial g^{\left(J_{i}\right)}}{\partial z^{\left(J_{i}\right)}}\left(A_{i}, z_{i}, 0\right)<0$ by Assumption 5 , applying Lemma 3.4 with $b=z$ and $a=(p, s)$ we obtain (80).

We denote $\Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}$ the transition map from $\Gamma_{i}$ to $\mathscr{A}_{i} \times\left\{z_{i}^{\left(J_{i}\right)}\right\} \times\left\{\widehat{\zeta}_{i}\right\}$ along the flow of (74) and $\Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{\Sigma}_{i}}$ the transition map from $0 \widehat{\mathscr{A}}_{i}$ to $\widehat{\Sigma}_{i}$ along the flow of (76).
Proposition 3.9. There exist $\delta_{j}>0,0 \leq j \leq 4$, such that if $\Gamma_{i}, \mathscr{A}_{i}, \Sigma_{i}$ are defined in the preceding paragraphs, then the transition map $\Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}$ from $\Gamma_{i}$ to $\Sigma_{i}$ following the flow of (1) is well-defined for all small $\epsilon>0$, and

$$
\begin{equation*}
\left\|\kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}-\Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}\right\|_{C^{1}\left(\Gamma_{i}\right)}=O(\epsilon) \tag{81}
\end{equation*}
$$

as $\epsilon \rightarrow 0$.
Proof. Analogous to the proof of Proposition 3.5, the assertions can be derived from Propositions 3.6, 3.7 and 3.8. We skip it here.

Proof of Theorem 2.5. By a reversal of the time variable, applying Proposition 3.9 we have

$$
\left\|\kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon \Gamma_{i+1}}^{\Sigma_{i}}-\Pi_{0 \widehat{\mathscr{B}}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i+1}}^{\mathscr{B}_{i}}\right\|_{C^{1}\left(\Gamma_{i+1}\right)}=O(\epsilon) .
$$

Taking the inverse of the mappings we obtain

$$
\begin{equation*}
\left\|\Pi_{\epsilon \Sigma_{i}}^{\Gamma_{i+1}} \circ \kappa_{\epsilon i}^{(13)}-\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\Sigma}_{i}}^{\widehat{\mathscr{B}}_{i}}\right\|_{C^{1}\left(\widehat{\Sigma}_{i}\right)}=O(\epsilon) \tag{82}
\end{equation*}
$$

By (81) and (82),

$$
\begin{aligned}
\Pi_{\epsilon \Gamma_{i}}^{\Gamma_{i+1}} & =\left(\Pi_{\epsilon \Sigma_{i}}^{\Gamma_{i+1}} \circ \kappa_{i \epsilon}^{(13)}\right) \circ\left(\kappa_{i \epsilon}^{(31)} \circ \Pi_{\epsilon \Gamma_{i}}^{\Sigma_{i}}\right) \\
& =\left(\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\mathscr{B}}_{i}}^{\widehat{\mathscr{S}}_{i}}\right) \circ\left(\Pi_{0 \mathscr{\mathscr { A }}_{i}}^{\widehat{\Sigma}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}\right)+O(\epsilon) \\
& =\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \kappa_{0 i}^{(13)} \circ \Pi_{0 \widehat{\mathscr{A}}_{i}}^{\widehat{\mathscr{A}}_{i}} \circ \kappa_{0 i}^{(31)} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}+O(\epsilon) \\
& =\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \widehat{Q}_{i} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}+O(\epsilon)
\end{aligned}
$$

Therefore,

$$
\begin{align*}
& \Pi_{\epsilon \Gamma_{i+1}}^{\Gamma_{i+2}} \circ \Pi_{\epsilon \Gamma_{i}}^{\Gamma_{i+1}} \\
& \quad=\left(\Pi_{0 \mathscr{B}_{i+1}}^{\Gamma_{i+2}} \circ \widehat{Q}_{i+1} \circ \Pi_{0 \Gamma_{i+1}}^{\mathscr{A}_{i+1}}\right) \circ\left(\Pi_{0 \mathscr{B}_{i}}^{\Gamma_{i+1}} \circ \widehat{Q}_{i} \circ \Pi_{0 \Gamma_{i}}^{\mathscr{A}_{i}}\right)+O(\epsilon) . \tag{83}
\end{align*}
$$

We denote

$$
P_{\epsilon}=\Pi_{\epsilon \Gamma_{N}}^{\Gamma_{1}} \circ \cdots \circ \Pi_{\epsilon \Gamma_{2}}^{\Gamma_{3}} \circ \Pi_{\epsilon \Gamma_{1}}^{\Gamma_{2}} .
$$

By (83) and the relation that $\Pi_{0 \mathscr{B}_{i-1}}^{\mathscr{A}_{i}}=\pi_{i} \times$ id, we have

$$
P_{\epsilon}=\Pi_{0 \mathscr{B}_{N}}^{\Gamma_{1}} \circ \widehat{Q}_{N} \circ\left(\pi_{N} \times \mathrm{id}\right) \circ \cdots \circ \widehat{Q}_{2} \circ\left(\pi_{2} \times \mathrm{id}\right) \circ \widehat{Q}_{1} \circ \Pi_{0 \Gamma_{1}}^{\mathscr{A}_{1}}+O(\epsilon)
$$

Writing $\Pi_{0 \Gamma_{1}}^{\mathscr{A}_{1}}=\Pi_{0 \mathscr{B}_{N}}^{\mathscr{A}_{1}} \circ \Pi_{0 \Gamma_{1}}^{\mathscr{B}_{N}}=\left(\pi_{1} \times \mathrm{id}\right) \circ\left(\Pi_{\mathscr{B}_{N}}^{\Gamma_{1}}\right)^{-1}$, it follows that

$$
P_{\epsilon}=\Pi_{0 \mathscr{B}_{N}}^{\Gamma_{1}} \circ \widetilde{P} \circ\left(\Pi_{0 \mathscr{B}_{N}}^{\Gamma_{1}}\right)^{-1}+O(\epsilon)
$$

where $\widetilde{P}$ is defined by (24). This implies that

$$
\begin{aligned}
& \operatorname{det}\left(D P_{\epsilon}-\mathrm{id}\right) \\
& \quad=\operatorname{det}\left(D \Pi_{0 \mathscr{B}_{N}}^{\Gamma_{1}} \circ D P \circ\left(D \Pi_{0 \mathscr{B}_{N}}^{\Gamma_{1}}\right)^{-1}-\mathrm{id}\right)+O(\epsilon) \\
& \quad=\operatorname{det}(D P-\mathrm{id})+O(\epsilon)
\end{aligned}
$$

Hence, the linearization of the return map $P_{\epsilon}$ at $\left(p_{01}, q_{01}, \widehat{\zeta}_{1}\right) \in \Gamma_{1}$ does not have a singular value equal to 1 for all small $\epsilon>0$ if $\operatorname{det}(D P-i d) \neq 0$. Consequently, for all small $\epsilon>0$ there exists a locally unique fixed point $\left(p_{\epsilon 1}, q_{\epsilon 1}, \widehat{\zeta}_{\epsilon 1}\right) \in \Gamma_{i}$ of $P_{\epsilon}$. Let $\left(p_{\epsilon 1}, z_{\epsilon 1}\right)=\kappa_{\epsilon 1}^{(01)}\left(p_{\epsilon 1}, q_{\epsilon 1}, \widehat{\zeta}_{\epsilon 1}\right)$. Then the trajectory passing through $\left(p_{\epsilon 1}, z_{\epsilon 1}\right)$ is a periodic orbit of system (1). If the spectrum radius of $D P\left(p_{01}, q_{01}, \widehat{\zeta}_{1}\right)$ is smaller (resp. greater) than 1 , then $P_{\epsilon}$ is a contraction (resp. expansion), hence the periodic orbit is orbitally asymptotically stable (resp. unstable).

## 4. Examples

In this section we apply the main results to study the examples (6), (7), (9) and the planar system (10) mentioned in Section 1.
4.1. Trade-off between Encounter and Growth Rates. Consider system (7), which takes the form

$$
\begin{aligned}
& x^{\prime}=F(x, \alpha)-G(x, y, \alpha) \\
& y^{\prime}=H(x, y, \alpha)-D(y) \\
& \epsilon \alpha^{\prime}=\alpha(1-\alpha) E(x, y, \alpha)
\end{aligned}
$$

with

$$
\begin{aligned}
& F(x, \alpha)=x(\alpha+r-k x) \\
& G(x, y, \alpha)=H(x, y, \alpha)=\frac{x y\left(a \alpha^{2}+b \alpha+c\right)}{1+x} \\
& D(y, \beta)=d y
\end{aligned}
$$

and

$$
E(x, y, \alpha)=\frac{\partial}{\partial \alpha}\left(\frac{x^{\prime}}{x}\right)=1-\frac{y(2 a \alpha+b)}{1+x}
$$

The limiting fast system is

$$
\dot{x}=0, \quad \dot{y}=0, \quad \dot{\alpha}=\alpha(1-\alpha) E(x, y, \alpha)
$$

The critical manifolds are

$$
M_{1}=\{(x, y, \alpha): \alpha=0\} \quad \text { and } \quad M_{2}=\{(x, y, \alpha): \alpha=1\}
$$

On the critical manifolds $M_{i}$, the limiting slow system is

$$
\begin{align*}
& x^{\prime}=F(x, \bar{\alpha})-G(x, y, \bar{\alpha}) \\
& y^{\prime}=H(x, y, \bar{\alpha})-D(y) \tag{84}
\end{align*}
$$

where $\bar{\alpha}=0,1$. Let $\Phi_{1}$ and $\Phi_{2}$ be the solution operators for (84) with $\alpha=0$ and $\alpha=1$, respectively. The transition maps $Q_{1}$ and $Q_{2}$ in Theorem 2.3 are determined by

$$
Q_{1}\left(A_{1}\right)=\Phi_{1}\left(A_{1}, \tau_{1}\right) \text { with }\left.\int_{0}^{\tau_{1}}\left(1-\frac{b y}{1+x}\right)\right|_{(x, y)=\Phi_{1}\left(A_{1}, \tau\right)} d \tau=0
$$

and

$$
Q_{2}\left(A_{2}\right)=\Phi_{2}\left(A_{2}, \tau_{2}\right) \text { with }\left.\int_{0}^{\tau_{2}}\left(1-\frac{y(2 a+b)}{1+x}\right)\right|_{(x, y)=\Phi_{2}\left(A_{2}, \tau\right)} d \tau=0
$$

Following [8], we set $a=-0.1, b=3, c=1, d=2.8, k=1$, and $r=10$. By implementing Newton's iteration, we find points $A_{1}=B_{2} \approx(5.57,11.03)$ and $B_{1}=A_{2} \approx(9.96,0.36)$ satisfying

$$
A_{2}=B_{1}=Q_{1}\left(A_{1}\right) \quad \text { and } \quad A_{1}=B_{2}=Q_{2}\left(A_{2}\right)
$$

This means that $A_{i}$ and $B_{i}$ satisfy the following conditions (see Figure 3(b)):
(i) $A_{1}$ and $B_{1}$ are connected by a trajectory $\sigma_{1}$ of (84) with $\bar{\alpha}=0$;
(ii) $A_{2}$ and $B_{2}$ are connected by a trajectory $\sigma_{2}$ of (84) with $\bar{\alpha}=1$;
(iii) $\int_{\sigma_{1}} E(x, y, 0) d \tau=0$ and $\int_{\sigma_{2}} E(x, y, 1) d \tau=0$.

Using the formulas in Proposition 2.6 and Remark 2.7, we obtain

$$
D Q_{1}\left(A_{1}\right) \approx\left(\begin{array}{cc}
-0.0001 & -0.0029 \\
0.0009 & 0.0258
\end{array}\right) \quad \text { and } \quad D Q_{2}\left(A_{2}\right) \approx\left(\begin{array}{cc}
0.02 & 18.91 \\
-0.02 & -16.95
\end{array}\right)
$$

Hence, the eigenvalues of $D P\left(A_{1}\right)=D Q_{2}\left(A_{2}\right) D Q_{1}\left(A_{1}\right)$ are $\lambda_{1} \approx 2.86 \cdot 10^{-14}$ and $\lambda_{2}=-0.42$, which are both of magnitude less than one. Therefore, by Theorem 2.3, the configuration

$$
\gamma_{1} \cup \sigma_{1} \cup \gamma_{2} \cup \sigma_{2}
$$

corresponds to a relaxation oscillation formed by orbitally locally asymptotically stable periodic orbits.

For system (7) with $\epsilon=0.1$, taking initial data $(x, y, \alpha)=(10,0.5,0.5)$ we find that the trajectory converges to a periodic orbit (see Figure 3(a)) near the singular configuration.
4.2. Prey Switching. Assuming that the response functions $f_{i}\left(p_{i}\right)$ in (9) are linear, after rescaling, the system is converted to

$$
\begin{align*}
& p_{1}^{\prime}=(1-q z) p_{1}, \\
& p_{2}^{\prime}=(r-(1-q) z) p_{2}, \\
& z^{\prime}=\left(q p_{1}+(1-q) p_{2}-1\right) z,  \tag{85}\\
& \epsilon q^{\prime}=q(1-q)\left(p_{1}-p_{2}\right) .
\end{align*}
$$

The critical manifolds for (85) are

$$
M_{1}=\left\{\left(p_{1}, p_{2}, z, q\right): q=0\right\} \quad \text { and } \quad M_{2}=\left\{\left(p_{1}, p_{2}, z, q\right): q=1\right\}
$$

On $M_{1}$, the restriction of (9) is

$$
\begin{align*}
& p_{1}^{\prime}=p_{1} \\
& p_{2}^{\prime}=(r-z) p_{2}  \tag{86}\\
& z^{\prime}=\left(p_{2}-1\right) z
\end{align*}
$$

which means that the predators hunt exclusively only the first prey population. On $M_{2}$, the restriction of (9) is

$$
\begin{align*}
& p_{1}^{\prime}=(1-z) p_{1} \\
& p_{2}^{\prime}=r z p_{2}  \tag{87}\\
& z^{\prime}=\left(p_{1}-1\right) z
\end{align*}
$$

which means that the predators hunt exclusively only the second prey population.

Let $\Phi_{1}$ and $\Phi_{2}$ be the transition maps for (86) and (87), respectively. The transition maps $Q_{1}$ and $Q_{2}$ in Theorem 2.3 are determined by

$$
Q_{1}\left(A_{1}\right)=\Phi_{1}\left(A_{1}, \tau_{1}\right) \text { with }\left.\int_{0}^{\tau_{1}}\left(p_{1}-p_{2}\right)\right|_{\left(p_{1}, p_{2}, z\right)=\Phi_{1}\left(A_{1}, \tau\right)} d \tau=0
$$

and

$$
Q_{2}\left(A_{2}\right)=\Phi_{2}\left(A_{2}, \tau_{2}\right) \text { with }\left.\int_{0}^{\tau_{2}}\left(p_{1}-p_{2}\right)\right|_{\left(p_{1}, p_{2}, z\right)=\Phi_{2}\left(A_{2}, \tau\right)} d \tau=0
$$

With the parameters given in [44], $r=0.5$ and $m=0.4$, we find $A_{1}=B_{2} \approx$ $(0.92,1.08,1.50)$ and $A_{2}=B_{1} \approx(1.08,0.92,1.50)$ such that the transition maps $Q_{i}$ in Theorem 2.3 satisfy $Q_{1}\left(A_{1}\right)=B_{1}$ and $Q_{2}\left(A_{2}\right)=B_{2}$ (see Figure $4(\mathrm{~b})$ ). Using the formulas in Proposition 2.6 and Remark 2.7, we obtain

$$
D Q_{1}\left(A_{1}\right) \approx\left(\begin{array}{ccc}
-6.78 & 5.74 & -1.00 \\
6.77 & -4.03 & 0.70 \\
0.34 & -0.16 & 1.04
\end{array}\right), \quad D Q_{2}\left(A_{2}\right) \approx\left(\begin{array}{ccc}
-1.56 & 3.38 & 0.55 \\
2.80 & -2.80 & -0.99 \\
-0.07 & 0.34 & 1.06
\end{array}\right)
$$

Hence, the eigenvalues of $D P\left(A_{1}\right)=D Q_{2}\left(A_{2}\right) D Q_{1}\left(A_{1}\right)$ are $\lambda_{1} \approx 60.55$ and $\lambda_{2,3} \approx$ $0.97 \pm 0.26 \sqrt{-1}$. Since $\lambda_{1}$ is greater than 1 , by Theorem 2.3, the configuration connecting $A_{i}$ and $B_{i}$ corresponds to a relaxation oscillation formed by orbitally unstable periodic orbits (see Figure 4(b)).
4.3. Coevolution. System (6) has critical manifolds $M_{i}, 1 \leq i \leq 4$, corresponding to $(\alpha, \beta)=\left(\alpha_{i}, \beta_{i}\right)$ with $\left(\alpha_{i}, \beta_{i}\right), i=1,2,3,4$, equal to $(0,0),(0,1),(1,1)$ and $(1,0)$, respectively. The limiting slow system on each $M_{i}$ is

$$
\begin{align*}
\frac{d}{d \tau} x & =F\left(x, \alpha_{i}\right)-G\left(x, y, \alpha_{i}, \beta_{i}\right) \\
\frac{d}{d \tau} y & =H\left(x, y, \alpha_{i}, \beta_{i}\right)-D\left(y, \beta_{i}\right) \tag{88}
\end{align*}
$$

The numbers $\omega_{i}=\left(\omega_{i}^{(1)}, \omega_{i}^{(2)}\right)$ defined by (20) are $\omega_{1}=(1,1), \omega_{2}=(1,-1), \omega_{3}=$ $(-1,-1)$, and $\omega_{4}=(-1,1)$. Equations for $\zeta=\left(\zeta^{(1)}, \zeta^{(2)}\right)$ in $(14)$ on $M_{i}$ are

$$
\begin{align*}
\frac{d}{d \tau} \zeta_{i}^{(1)} & =\omega_{i}^{(1)} E_{1}\left(x, y, \alpha_{i}, \beta_{i}\right) \\
\frac{d}{d \tau} \zeta_{i}^{(2)} & =\omega_{i}^{(2)} E_{2}\left(x, y, \alpha_{i}, \beta_{i}\right) \tag{89}
\end{align*}
$$

where

$$
E_{1}(x, y, \alpha, \beta)=\frac{\partial}{\partial \alpha}\left(\frac{F(x, \alpha)-G(x, y, \alpha, \beta)}{x}\right)
$$

and

$$
E_{2}(x, y, \alpha, \beta)=\frac{\partial}{\partial \beta}\left(\frac{H(x, \alpha)-D(x, y, \alpha, \beta)}{y}\right)
$$

Let $\widehat{\Phi}_{i}, 1 \leq i \leq 4$, be the solution operators for system (88)-(89). Then the transition maps $\widehat{Q}_{i}$ in Theorem 2.4 are determined by

$$
\begin{aligned}
& \widehat{Q}_{1}\left(A_{1}, \zeta\right)=\widehat{\Phi}_{1}\left(\left(A_{1}, \zeta\right), \tau_{1}\right) \text { with } \zeta^{(2)}+\left.\int_{0}^{\tau_{1}} E_{2}(x, y, 0,0)\right|_{(x, y)=\Phi_{1}\left(A_{1}, \tau\right)} d \tau=0 \\
& \widehat{Q}_{2}\left(A_{2}, \zeta\right)=\widehat{\Phi}_{2}\left(\left(A_{2}, \zeta\right), \tau_{2}\right) \text { with } \zeta^{(1)}+\left.\int_{0}^{\tau_{2}} E_{1}(x, y, 0,1)\right|_{(x, y)=\Phi_{2}\left(A_{2}, \tau\right)} d \tau=0 \\
& \widehat{Q}_{3}\left(A_{3}, \zeta\right)=\widehat{\Phi}_{3}\left(\left(A_{3}, \zeta\right), \tau_{3}\right) \text { with } \zeta^{(2)}-\left.\int_{0}^{\tau_{1}} E_{2}(x, y, 1,1)\right|_{(x, y)=\Phi_{1}\left(A_{1}, \tau\right)} d \tau=0 \\
& \widehat{Q}_{4}\left(A_{4}, \zeta\right)=\widehat{\Phi}_{4}\left(\left(A_{4}, \zeta\right), \tau_{4}\right) \text { with } \zeta^{(1)}-\left.\int_{0}^{\tau_{4}} E_{1}(x, y, 1,0)\right|_{(x, y)=\Phi_{4}\left(A_{4}, \tau\right)} d \tau=0
\end{aligned}
$$

Following Cortez and Weitz [10, Supporting Information D], we consider (6) with

$$
\begin{aligned}
& F(x, \alpha)=x\left(s_{0}+s_{1} \alpha\right)\left(1-\frac{x}{k_{0}+k_{1} \alpha}\right) \\
& G(x, y, \alpha, \beta)=\frac{\left(r_{0}+r_{1} \alpha+r_{2} \beta+r_{3} \alpha \beta+r_{4} \beta^{2}\right) x y}{1+h x} \\
& H(x, y, \alpha, \beta)=c_{0} G(x, y, \alpha, \beta) \\
& D(y, \beta)=y^{1.5}\left(\delta_{0}+\delta_{1} \beta\right)
\end{aligned}
$$

and parameters $s_{0}=2.5, s_{1}=3.5, k_{0}=1, k_{1}=0.1, r_{0}=0.65, r_{1}=3, r_{2}=2.3, r_{3}=$ $-0.2, r_{4}=0.01, c_{0}=1.7, \delta_{0}=0.76, \delta_{1}=1.77$ and $h=1$. Implementing Newton's iteration for $\widehat{Q}_{i}\left(A_{i}, \zeta_{i}\right)=\left(A_{i+1}, \zeta_{i+1}\right), 1 \leq i \leq 4$, we find $B_{4}=A_{1} \approx(0.33,1.99)$, $B_{1}=A_{2} \approx(0.92,0.56), B_{2}=A_{3} \approx(0.60,0.55)$ and $B_{3}=A_{4} \approx(0.30,0.93)$ (see Figure $2(\mathrm{~b}))$, and $\zeta_{1} \approx(0,0.98), \zeta_{2} \approx(3.84,0), \zeta_{3} \approx(0,1.12)$ and $\zeta_{4} \approx(0.55,0)$.

Let $\left\{\mathrm{e}_{x}, \mathrm{e}_{y}, \mathrm{e}_{\alpha}, \mathrm{e}_{\beta}\right\}$ be the standard ordered basis of the $(x, y, \alpha, \beta)$-space. Note that the tangent space of $\mathscr{A}_{1} \times \Lambda_{1}$ at $\left(A_{1}, \zeta_{1}\right)$ is spanned by $\left\{\mathrm{e}_{x}, \mathrm{e}_{y}, \mathrm{e}_{\beta}\right\}$, and the tangent space of $\mathscr{B}_{1} \times \Lambda_{2}$ at $\left(B_{1}, \zeta_{2}\right)$ is spanned by $\left\{\mathrm{e}_{x}, \mathrm{e}_{y}, \mathrm{e}_{\alpha}\right\}$. Using formulas in Proposition 2.8, we obtain

$$
D \widehat{Q}_{1}\left(A_{1}, \zeta_{1}\right) \approx\left(\begin{array}{ccc}
\mathrm{e}_{x} & \mathrm{e}_{x} & \mathrm{e}_{\beta} \\
0.013 & 0.004 & -0.007 \\
0.080 & -0.254 & 0.038 \\
-3.29 & -2.42 & 0.67
\end{array}\right) \begin{aligned}
& \mathrm{e}_{x} \\
& \mathrm{e}_{y} \\
& \mathrm{e}_{\alpha}
\end{aligned}
$$

Similarly,

$$
D \widehat{Q}_{2}\left(A_{2}, \zeta_{2}\right) \approx\left(\begin{array}{ccc}
\mathrm{e}_{x} & \mathrm{e}_{x} & \mathrm{e}_{\alpha} \\
-0.00040 & -0.0058 & 0.00024 \\
-0.00003 & 0.00024 & 0.00030 \\
0.37 & -1.44 & -0.26
\end{array}\right) \begin{aligned}
& \mathrm{e}_{x} \\
& \mathrm{e}_{y} \\
& \mathrm{e}_{\beta}
\end{aligned}
$$

and the approximations of $D \widehat{Q}_{3}\left(A_{3}, \zeta_{3}\right)$ and $D \widehat{Q}_{4}\left(A_{4}, \zeta_{4}\right)$ are, respectively,

Hence, the eigenvalues of

$$
D \widehat{P}\left(A_{1}, \zeta_{1}\right)=D \widehat{Q}_{4}\left(A_{4}, \zeta_{4}\right) D \widehat{Q}_{3}\left(A_{3}, \zeta_{3}\right) D \widehat{Q}_{2}\left(A_{2}, \zeta_{2}\right) D \widehat{Q}_{1}\left(A_{1}, \zeta_{1}\right)
$$

are $\lambda_{1} \approx 0.39, \lambda_{2} \approx-6.14 \cdot 10^{-5}$ and $\lambda_{3} \approx-5.11 \cdot 10^{-11}$, which are all of magnitude less than one. Therefore, by Theorem 2.4, this singular configuration corresponds to a relaxation oscillation formed by orbitally locally asymptotically stable periodic orbits.
4.4. A Planar System. The limiting fast system of (10) is

$$
\begin{equation*}
\frac{d}{d t} a=b H(a, b, 0), \quad \frac{d}{d t} b=b G(a, b, 0) \tag{90}
\end{equation*}
$$

On the critical manifold $M=\{(a, b): b=0\}$, the limiting slow system is

$$
a^{\prime}=F(a, 0,0)
$$

We assume that (see Figure 6)


Figure 6. For system (90) with $\epsilon=0$, the $a$-axis is a line of equilibria and $\gamma$ is a heteroclinic orbit connecting $\left(a_{0}, 0\right)$ and $\left(a_{1}, 0\right)$.
(i) There is a trajectory $\gamma$ of (90) satisfying

$$
\lim _{t \rightarrow-\infty} \gamma(t)=\left(a_{0}, 0\right), \quad \lim _{t \rightarrow \infty} \gamma(t)=\left(a_{1}, 0\right)
$$

(ii) $F(a, 0,0)>0$ for all $a \in\left[a_{0}, a_{1}\right]$;
(iii) $G\left(a_{0}, 0,0\right)<0$ and $G\left(a_{1}, 0,0\right)>0$;
(iv) $\int_{a_{0}}^{a_{1}} \frac{G(a, 0,0)}{F(a, 0,0)} d a=0$ and $\int_{a_{0}}^{s} \frac{G(a, 0,0)}{F(a, 0,0)} d a<0 \quad \forall s \in\left(a_{0}, a_{1}\right)$.

We provide an alternative proof of the following theorem from Hsu and Wolkowicz [25].
Theorem 4.1. Consider system (10). Assume (i)-(iv) and let

$$
\lambda=\ln \left|\frac{F\left(a_{1}, 0,0\right)}{F\left(a_{0}, 0,0\right)}\right|+\int_{\gamma} \frac{\partial_{a} H}{H} d a+\int_{\gamma} \frac{\partial_{b} G}{G} d b
$$

If $\lambda \neq 0$, then $\gamma$ admits a relaxation oscillation which is formed by locally unique periodic orbits for small $\epsilon>0$. Moreover, the periodic orbit is orbitally asymptotically stable if $\lambda<0$ and unstable if $\lambda>0$.

Remark 4.2. Assumptions (i) and (iv) are weaker than the conditions assumed in [25]. In that paper, the assumption corresponding to (i) is that there exists a smooth family of heteroclinic orbits; the assumption corresponding to the inequalities in (iii) and (iv) is that $G(a, 0,0)<0$ for $a<\bar{a}$ and $G(a, 0,0)>0$ for $a>\bar{a}$. However, the analysis in that paper is also valid under these weaker assumptions.

Proof. Define a function $Q$ implicitly by $Q\left(a_{0}\right)=a_{1}$ and

$$
\begin{equation*}
\int_{a_{0}}^{Q(a)} \frac{G(r, 0,0)}{F(r, 0,0)} d r=0 \tag{91}
\end{equation*}
$$

By (30) in Proposition 2.6,

$$
\begin{equation*}
\frac{d Q\left(a_{0}\right)}{d a}=\frac{F\left(a_{1}, 0,0\right)}{G\left(a_{1}, 0,0\right)} \frac{G\left(a_{0}, 0,0\right)}{F\left(a_{0}, 0,0\right)} \tag{92}
\end{equation*}
$$

(Note that equation (92) can also be derived directly by differentiating (91).)
Let $\pi$ be the transition map of (90) from a neighborhood of $\left(a_{1}, 0\right)$ to a neighborhood $\left(a_{0}, 0\right)$ in the $a$-axis. By (42) in Proposition 2.9,

$$
\begin{equation*}
\frac{d}{d a} \pi\left(a_{1}\right)=\frac{G\left(a_{1}, 0,0\right)}{G\left(a_{0}, 0,0\right)} \exp \left(\int_{\gamma} \partial_{a} H+\partial_{b} G d t\right) \tag{93}
\end{equation*}
$$

By (93) and (92), we obtain

$$
\begin{aligned}
\frac{d}{d a}(\pi \circ Q) & =\frac{d \pi\left(a_{1}\right)}{d a} \frac{d Q\left(a_{0}\right)}{d a} \\
& =\left(\frac{F\left(a_{1}, 0,0\right)}{G\left(a_{1}, 0,0\right)} \frac{G\left(a_{0}, 0,0\right)}{F\left(a_{0}, 0,0\right)}\right) \frac{G\left(a_{1}, 0,0\right)}{G\left(a_{0}, 0,0\right)} \exp \left(\int_{0}^{T} \partial_{a} H+\partial_{b} G d t\right)
\end{aligned}
$$

Using the relations $d a / d t=H$ and $d a / d t=G$ in (90), it follows that

$$
\frac{d}{d a}(\pi \circ Q)\left(a_{0}\right)=\frac{F\left(a_{1}, 0,0\right)}{F\left(a_{0}, 0,0\right)} \exp \left(\int_{\gamma} \frac{\partial_{a} H}{H} d a+\int_{\gamma} \frac{\partial_{b} G}{H} d b .\right)
$$

Hence

$$
\ln \left|\frac{d}{d a}(\pi \circ P)\left(a_{0}\right)\right|=\ln \left|\frac{F\left(a_{1}, 0,0\right)}{F\left(a_{0}, 0,0\right)}\right|+\int_{\gamma} \frac{\partial_{a} H}{H} d a+\int_{\gamma} \frac{\partial_{b} G}{H} d b
$$

Hence $\lambda<0$ if and only if $\left|\frac{d}{d a}(\pi \circ P)\left(a_{0}\right)\right|<1$. By Theorem 2.5, the desired result follows.

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